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Hurwitz components of groups with socle PSL$(3, q)$

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Abstract: For a finite group G, the Hurwitz space $\mathcal{H}^{in}_{r,g}(G)$ is the space of genus g covers of the Riemann sphere $\mathbb{P}^1$ with r branch points and the monodromy group G. In this paper, we give a complete list of some almost simple groups of Lie rank two. That is, we assume that G is a primitive almost simple groups of Lie rank two. Under this assumption we determine the braid orbits on the suitable Nielsen classes, which is equivalent to finding connected components in $\mathcal{H}^{in}_{r,g}(G)$.

Key words: Genus zero systems, Braid orbits, Connected components.

MSC (2020): 20B15

1. Introduction

Let Ω be a finite set of order n and G be a transitive subgroup of S_n such that

$$G = \langle x_1, x_2, \ldots, x_r \rangle \,, \tag{1.1}$$

$$\prod_{i=1}^{r} x_i = 1 \,, \quad x_i \in G^{\#} = G \setminus \{1\} \,, \;\; i = 1, \ldots, r \,, \tag{1.2}$$

$$\sum_{i=1}^{r} \operatorname{ind} x_i = 2(n + g - 1) \,, \tag{1.3}$$

where $\operatorname{ind} x_i$ is the minimal number of 2-cycles needed to express x_i as a product. We call G a group of genus g and the triple $(G, \Omega, \langle x_1, x_2, \ldots, x_r \rangle)$ a genus g system. These conditions correspond to the existence of an n sheeted branched covering of Riemann surface X of genus g with r-branch points and monodromy group G [9].

In [9], Guralnick and Thompson have conjectured that the set $\mathcal{E}^*(g)$ of possible isomorphism classes of composition factors of simple groups which

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are neither cyclic nor alternating, is finite for all $g \geq 0$. Furthermore, they have observed that the conjecture reduces to the consideration of the system $(G, \Omega, \langle x_1, x_2, \ldots, x_r \rangle)$ where G is primitive on Ω. A useful reference for more details is [9]. The primitive permutation representations of finite groups are determined by their maximal subgroups whose structure has been described by Aschbacher and O'Nan-Scott Theorem [3].

Proposition 1.1. ([3]) *Suppose that G is a finite group and M is a maximal subgroup of G such that*

$$\bigcap_{g \in G} M^g = 1.$$

Let S be a minimal normal subgroup of G, let L be a minimal normal subgroup of S, and let $\Delta = \{L = L_1, L_2, \ldots, L_m\}$ be the set of the G-conjugates of L. Then L is simple, $S = \langle L_1, \ldots, L_r \rangle$, $G = MS$ and furthermore either

(A) *L is of prime order p;*
or L is non abelian simple group and one of the following hold:

(B) *$F^*(G) = S \times R$, where $S \cong R$ and $M \cap S = 1$;*

(C1) *$F^*(G) = S$ and $M \cap S = 1$;*

(C2) *$F^*(G) = S$ and $M \cap S \neq 1 = M \cap L$;*

(C3) *$F^*(G) = S$ and $M \cap S = M_1 \times M_2 \times \cdots \times M_t$, where $M_i = M \cap L_i$, $1 \leq i \leq t$.*

As far as we know (see [14, 10, 11, 12]), there are four types of classification of genus g system as follows:

1. Up to signature.
2. Up to ramification type.
3. Up to the braid action and diagonal conjugation by $\mathrm{Aut}(G)$.
4. Up to the braid action and diagonal conjugation by $\mathrm{Inn}(G)$.

The weakest classification is up to signature (that is 1.) and the strongest one is up to the braid action and diagonal conjugation by $\mathrm{Inn}(G)$ (that is 4.), because it includes all 1, 2 and 3.

In [14, 15, 9, 1, 5, 4, 2], they have classified these cases (A), (B), (C1), (C2), (C3) up to signatures for genus zero. In [11, 12], they have produced a complete list of affine primitive genus 0, 1 and 2 groups up to the braid action and diagonal conjugation by $\mathrm{Inn}(G)$.

A group G is said to be almost simple if it contains a non-abelian simple group S and $S \leq G \leq \mathrm{Aut}(S)$. In [10], Kong works on almost simple groups of type projective special linear group $\mathrm{PSL}(3,q)$. Let G be a group such that $\mathrm{PSL}(3,q) \leq G \leq \mathrm{P\Gamma L}(3,q)$ where $\mathrm{P\Gamma L}(3,q)$ is the projective semilinear group. G acts on points in the natural module, that is the set of projective points of 2-dimensional projective geometry $PG(2,q)$. She gave a complete list for some almost simple groups of Lie rank 2 up to ramification type in her PhD thesis for a genus 0, 1 and 2 system.

In this paper, we consider almost simple groups of Lie rank 2 for genus zero and classify them up to the braid action and diagonal conjugation by $\mathrm{Inn}(G)$.

The equivalence classes of G-covers X of $\mathbb{P}^1$ with r branched points are called a Hurwitz space and denoted by $\mathcal{H}^{in}_{r,g}(G)$ where in denotes an inner automorphism of G. Note that X is a Riemann surface of genus g.

Let C_i be the conjugacy class of x_i. Then the multi set of non trivial conjugacy classes $C = \{C_1, \ldots, C_r\}$ in G is called the ramification type of the G-covers X. For any r-tuple $(x_1, \ldots, x_r)$ gives a ramification type $\bar{C}$ with $x_i \in C_i$ for $i = 1, \ldots, r$. Let $\bar{C}$ be a fixed ramification type, then the subset $\mathcal{H}^{in}_r(G, \bar{C})$ of $\mathcal{H}^{in}_r(G)$ consists of all $[P, \phi]$ with admissible surjective map $\phi\colon \pi(\mathbb{P}^1 \setminus P, p) \to G$ sends the conjugacy class $\sum_{p_i}$ to the conjugacy class C_i for $i = 1, \ldots, r$. It is a union of connected components in $\mathcal{H}^{in}_r(G)$.

In this paper, we study the Hurwitz space $\mathcal{H}^{in}_r(G)$. In particular we focus on the subset $\mathcal{H}^{in}_r(G, \bar{C})$ of $\mathcal{H}^{in}_r(G)$. We try to find the connected components $\mathcal{H}_r(G, \bar{C})$ of G-curves X of genus 0 such that $g(X/G) = g(\mathbb{P}^1) = 0$. To do this, one needs to find corresponding braid orbits. The our main result is Theorem 1.2, which gives the complete classification of primitive genus 0 systems of almost simple group of Lie rank two.

Theorem 1.2. *Up to isomorphism, there exist exactly seven primitive genus zero groups with socle* $\mathrm{PSL}(3,q)$ *for some* q, $3 \leq q \leq 13$. *The corresponding primitive genus zero groups are enumerated in Table* 2 *and Table* 3.

In our situation, the computation shows that there is exactly 514 braid orbits of primitive genus 0 systems for some almost simple groups of Lie rank two. The degree and the number of the branch points are given in Table 1.

Table 1: Primitive Genus Zero Systems: Number of Components

Degree	Number of Group Iso. Types	Number of Ramification Types	Number of Components $r=3$	Number of Comp. $r=4$	Number of Comp. $r=5$	Number of Comp. Total
13	1	45	93	14	5	112
21	4	76	204	26	4	234
31	1	15	92	2	-	94
57	1	8	72	2	-	74
Totals	7	144	461	44	9	514

This paper consists of four sections as follows. Section 2 sets out some notation and results that will be needed throughout the paper. We then discuss the relationship between connected components of Hurwitz spaces and braid orbits on Nielsen classes. In Section 3, we describe our methodology which will be used to obtain the ramification types and braid orbits. Furthermore, we give a particular example to explain this methodology. Finally, several results are given about Hurwitz spaces.

2. Braid action on Nielsen classes

We begin this section with a formal definition of the Artin braid group.

Definition 2.1. For $r \geq 2$, the Artin braid group B_r is generated by $r-1$ elements $\sigma_1, \sigma_2, \ldots, \sigma_{r-1}$ that satisfy the following relations: $\sigma_i\sigma_j = \sigma_j\sigma_i$ for all $i, j = 1, 2, \ldots, r-1$ with $|i-j| \geq 2$, and $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $i = 1, 2, \ldots, r-2$. These relations are known as the braid relations.

The braid σ_i acts on generating tuples $x = (x_1, \ldots, x_r)$ of a finite group G with $\prod_{i=1}^{r} x_i = 1$ as follows:

$$(x_1, \ldots, x_i, x_{i+1}, \ldots, x_r)\sigma_i = (x_1, \ldots, x_{i+1}, x_{i+1}^{-1}x_ix_{i+1}, \ldots, x_r) \tag{2.1}$$

for $i = 1, \ldots, r-1$. The braid orbit of x is the smallest set of tuples which contains x and is closed under the operations (2.1).

Applying $\phi\colon \pi(\mathbb{P}^1 \setminus P, p) \to G$ to the canonical generators of $\pi_1(\mathbb{P}^1 \setminus P, p)$ gives the generators of a product one generating tuple in G that is, $\phi(\lambda_i) = x_i$. We define

$$\epsilon_r(G) = \left\{ (x_1, \ldots, x_r) \,:\, G = \langle x_1, \ldots, x_r \rangle \,,\, \prod_{i=1}^{r} x_i = 1 \,,\, x_i \in G^{\#} \,,\, i = 1, \ldots, r \right\}.$$

Let $A \leq \mathrm{Aut}(G)$. Then the subgroup A acts on $\epsilon_r(G)$ via sending $(x_1, \ldots, x_r)$ to $(a(x_1), \ldots, a(x_r))$, for $a \in$ A, which is known as the diagonal conjugation. This action commutes with the operations (2.1). Thus A permutes the braid orbits. If $A = \mathrm{Inn}(G)$, then it leaves each braid orbit invariant [16]. Let $\epsilon_r^{in}(G) = \epsilon_r(G)/\mathrm{Inn}(G)$.

For a ramification type $\bar{C}$, we define the subset $\mathcal{N}(\bar{C}) = \big\{(x_1, \ldots, x_r) : G = \langle x_1, \ldots, x_r\rangle, \prod_{i=1}^r x_i = 1, \exists \sigma \in S_n$ such that $x_i \in C_{i\sigma}$ for all $i\big\}$ which is called the Nielsen class of $\bar{C}$.

The topology on $\mathcal{H}_r^{\mathrm{A}}(G)$ is well defined. Let O_r be the set of all r-tuples of distinct elements in $\mathbb{P}^1$, equipped with the product topology [16].

For the remaining of this section, we collect few results which will be used to explain the relationship between the braid orbits and their corresponding covers.

Lemma 2.2. ([16]) *The map* $\Psi_{\mathrm{A}} : \mathcal{H}_r^{\mathrm{A}}(G) \to O_r$, $\Psi_{\mathrm{A}}([P, \phi])) = P$, *is covering.*

The fundamental group $\pi_1(O_r, P_0) = B_r$ acts on $\Psi_{\mathrm{A}}^{-1}(P_0)$ where $P_0 = \{1, \ldots, r\}$ is the base point in O_r via path lifting where the fiber is

$$\Psi_{\mathrm{A}}^{-1}(P_0) = \big\{[P_0, \phi]_{\mathrm{A}} \ : \ \phi : \pi_1(\mathbb{P}^1 \setminus P_0, \infty) \to G \text{ is admissible } \}.$$

This ϕ gives as product one generating tuple $(x_1, \ldots, x_r)$ of G.

Lemma 2.3. ([16]) *We obtain a bijection* $\Psi_{\mathrm{A}}^{-1}(P_0) \to \epsilon_r^{\mathrm{A}}(G)$ *by sending* $[P_0, \phi]_{\mathrm{A}}$ *to the generators* $(x_1, \ldots, x_r)$ *where* $x_i = \phi([\gamma_i])$ *for* $i = 1, \ldots, r$.

The image $\mathcal{N}^A(\bar{C})$ of $\mathcal{N}(\bar{C})$ in $\epsilon_r^{\mathrm{A}}(G)$ is the union of braid orbits. If Ψ_{A} in Lemma 2.2 restricts to a connected component $\mathcal{H}$ of $\mathcal{H}_r^{\mathrm{A}}(G)$, then Lemma 2.3 implies that the fiber in $\mathcal{H}$ over P_0 corresponds to the set $\mathcal{N}^A(\bar{C})$.

Proposition 2.4. ([16]) *Let* $\bar{C}$ *be a fixed ramification type in* G, *and the subset* $\mathcal{H}_r^A(G, \bar{C})$ *of* $\mathcal{H}_r^A(G)$ *consists of all* $[B, \phi]_A$ *with* $B = \{b_1, \ldots, b_r\}$, $\phi \colon \pi_1(\mathbb{P}^1 \setminus P, \infty) \to G$ *and* $\phi(\theta_{b_i})) \in C_i$ *for* $i = 1, \ldots, r$. *Then* $\mathcal{H}_r^A(G, \bar{C})$ *is a union of connected components in* $\mathcal{H}_r^A(G)$. *Under the bijection from Lemma 2.3, the fiber in* $\mathcal{H}_r^A(G, \bar{C})$ *over* B_0 *corresponds the set* $\mathcal{N}^A(\bar{C})$. *This yields a one to one correspondence between components of* $\mathcal{H}_r^A(C)$ *and the braid orbits on* $\mathcal{N}^A(\bar{C})$. *In particular,* $\mathcal{H}_r^{in}(G, C)$ *is connected if and only if there is only one braid orbit.*

The following Riemann Existence Theorem tells us there is a one to one correspondence between the equivalence classes of product one generating tuples $(x_1, \ldots, x_r)$ of G and the equivalence classes of G-covers of type $\bar{C}$ such that $x_i \in C_i$ for $i = 1, \ldots, r$.

PROPOSITION 2.5. ([8]) *Let G be a finite group and $\bar{C} = \{C_1, \ldots, C_r\}$ be a ramification type. Then there exists a G-cover of type $\bar{C}$ if and only if there exists a generating tuple $(x_1, \ldots, x_r)$ of G with $\prod_{i=1}^{r} x_i = 1$ and $x_i \in C_i$, for $i = 1, \ldots, r$.*

DEFINITION 2.6. ([8]) Two generating tuples are braid equivalent if they lie in the same orbit under the group generated by the braid action and diagonal conjugation by $\mathrm{Inn}(G)$.

This means that if two generating tuples lie in the same braid orbit under either the diagonal conjugation or the braid action, then the corresponding covers are equivalent by Riemann's Existence Theorem.

DEFINITION 2.7. Two coverings $\mu_1 \colon X_1 \to \mathbb{P}^1$ and $\mu_2 \colon X_2 \to \mathbb{P}^1$ are equivalent if there exists a homeomorphism $\alpha \colon X_1 \to X_2$ with $\mu_2 \alpha = \mu_1$.

As a consequence we have the following result.

PROPOSITION 2.8. ([16]) *Two generating tuples are braid equivalent if and only if their corresponding covers are equivalent.*

To answer whether or not $\mathcal{H}_r(G, \bar{C})$ is connected which is still an open problem, both computationally and theoretically. The MAPCLASS package of James, Magaard, Shpectorov and Volklein, is designed to perform braid orbit computations for a given finite group and given type. Few results were known about it such as in [11] and [13].

3. METHODOLOGY AND EXAMPLE: LISTING PRIMITIVE GENUS ZERO SYSTEMS

The theory introduced in the previous section provides reformation of the geometric problem into the language of permutation groups. This leads us to work with permutation groups rather than with G-covers (see Proposition 2.5). The following method shows that the existence primitive genus 0 system for a given group G and type $\bar{C}$, and then computing braid orbits on the set of Nielsen class $\mathcal{N}^A(\bar{C})$. Proposition 2.4 yields a one to one correspondence between the braid orbits on $\mathcal{N}^A(\bar{C})$ and connected components of $\mathcal{H}_r^A(G,\bar{C})$. Now we can decide whether or not $\mathcal{H}_r^A(G,\bar{C})$ connected, when G is a primitive almost simple groups of Lie rank two and given type $\bar{C}$.

We are presenting our computations in Tables 2 and 3. To obtain these tables we needed to do the following steps:

- We extract all primitive permutation group G by using the GAP function AllPrimitiveGroups(DegreeOperation,n).
- For every almost simple group G, compute the conjugacy class representatives and permutation indices on n points.
- For given n, g and G we use the GAP function RestrictedPartions to compute all possible ramification types satisfying the Riemann-Hurwitz formula.
- Compute the character table of G if possible and remove those types which have zero structure constant.
- For each of the remaining types of length greater than or equal to 4, we use MAPCLASS package to compute braid orbits, especially by using the function `GeneratingMCOrbits(G,0,tuple)`. For tuples of length 3 determine braid orbits via double cosets [8].
- We use the same rules for labeling and ordering conjugacy classes of G as in [13].

This will be done by both the proof in algebraic topology and calculations of GAP (Groups, Algorithms, Programming) software. Also genus 0 generating tuples for almost simple groups of type PSL$(3,q)$ on their other primitive actions and genus 0 are given.

The next example show that how to compute the ramification types and braid orbits for the group PSL$(3,3)$.

EXAMPLE 3.1. Suppose that $G = \mathrm{PSL}(3,3)$ and $|\Omega| = n = \frac{3^3-1}{3-1} = 13$.

```
gap> a:=AllPrimitiveGroups(DegreeOperation,13);
[ C(13), D(2*13), 13:3, 13:4, 13:6, AGL(1, 13), L(3, 3), A(13),
  S(13) ]
gap> List(a,x->ONanScottType(x));
[ "1", "1", "1", "1", "1", "1", "2", "2", "2" ]
gap> LoadPackage("mapclass");;
gap> Read("qu1.g");
```

```
gap> CheckingTheGroup(k);
gap> k:=a[7];
L(3, 3)
gap> CheckingTheGroup(k);
gap> gt:=GeneratingType(k,13,0);
Checking the ramification type 66 with 0 remaining
[ [ 7, 8, 8 ], [ 7, 7, 8 ], [ 7, 7, 7 ], [ 6, 8, 5 ],
  [ 6, 8, 4 ], [ 6, 3, 5 ], [ 6, 3, 4 ], [ 3, 8, 8 ],
  [ 3, 7, 8 ], [ 3, 7, 7 ], [ 3, 3, 8 ], [ 3, 3, 7 ],
  [ 3, 3, 3 ], [ 2, 8, 12 ], [ 2, 8, 11 ], [ 2, 8, 10],
  [ 2, 8, 9 ], [ 2, 7, 12 ], [ 2, 7, 11 ], [ 2, 7, 10 ],
  [ 2, 7, 9 ], [ 2, 6, 6, 8 ], [ 2, 6, 6, 3 ], [ 2, 4, 5 ],
  [ 2, 3, 12 ], [ 2, 3, 11 ], [ 2, 3, 10 ], [ 2, 3, 9 ],
  [ 2, 2, 8, 8 ], [ 2, 2, 7, 8 ], [ 2, 2, 7, 7 ],
  [ 2, 2, 6, 5 ], [ 2, 2, 6, 4 ], [ 2, 2, 3, 8 ],
  [ 2, 2, 3, 7 ], [ 2, 2, 3, 3 ], [ 2, 2, 2, 12 ],
  [ 2, 2, 2, 11 ], [ 2, 2, 2, 10 ], [ 2, 2, 2, 9 ],
  [ 2, 2, 2, 6, 6 ], [ 2, 2, 2, 2, 8 ], [ 2, 2, 2, 2, 7 ],
  [ 2, 2, 2, 2, 3 ], [ 2, 2, 2, 2, 2, 2 ] ]
gap> Length(gt);
45
```

We can pick one of the generating tuple t and compute braid orbits as follows:

```
gap> t:=List(gt[45],x->CC[x]);
[ (2,13)(3,10)(4,9)(5,6), (2,13)(3,10)(4,9)(5,6),
(2,13)(3,10)(4,9)(5,6), (2,13)(3,10)(4,9)(5,6),
(2,13)(3,10)(4,9)(5,6), (2,13)(3,10)(4,9)(5,6) ]
gap> orb:=GeneratingMCOrbits(k,0,t);;
Total Number of Tuples: 183980160
Collecting 20 generating tuples .. done
Cleaning done; 20 random tuples remaining
Orbit1:
Length=32760
Generating Tuple  =[ ( 2,11)( 4,12)( 7,10)( 9,13),
( 1, 9)( 4, 7)( 6,10)( 8,12), ( 1, 6)( 2, 3)( 7,13)(10,12),
( 1, 9)( 2, 8)( 5, 7)(10,11), ( 2,10)( 5, 7)( 6,12)( 8,11),
( 3,11)( 4,10)( 7, 9)(12,13) ]
Centralizer size=1
0 tuples remaining
Cleaning a list of 20 tuples
Random Tuples Remaining: 0
Cleaning done; 0 random tuples remaining
Computation complete : 1 orbits found.
```

4. Results

In this section, we present some results which related to the connectedness of the Hurwitz space for some almost simple groups of Lie rank two for genus zero.

PROPOSITION 4.1. *If* $r \geq 4$ *and* $G = \mathrm{PSL}(3,q)$ *where* $q = 3,5$*, then* $\mathcal{H}_r^{in}(G,\bar{C})$ *is connected.*

Proof. Since we have just one braid orbit for all types $\bar{C}$ and the Nielsen classes $\mathcal{N}(\bar{C})$ are the disjoint union of braid orbits. From Proposition 2.4, we obtain that the Hurwitz spaces $\mathcal{H}_r^{in}(G,C)$ are disconnected. ∎

PROPOSITION 4.2. *If* $G = \mathrm{PSL}(3,4).2$ *or* $G = \mathrm{PSL}(3,q)$ *where* $q = 4,9$*, then* $\mathcal{H}_r^{in}(G,\bar{C})$ *is disconnected.*

Proof. Since we have at least two braid orbits for some type $\bar{C}$ and the Nielsen classes $\mathcal{N}(\bar{C})$ are the disjoint union of braid orbits. From Proposition 2.4, we obtain that the Hurwitz spaces $\mathcal{H}_r^{in}(G,C)$ are disconnected. ∎

The proof of the following is analogous to the proof of Proposition 4.1.

PROPOSITION 4.3. *If* $G = \mathrm{PGL}(3,q)$ *where* $q = 4,7$*, then* $\mathcal{H}_r^{in}(G,C)$ *is connected.*

PROPOSITION 4.4. *If* $G = \mathrm{P\Gamma L}(3,4)$ *where* $r \geq 4$*, then* $\mathcal{H}_r^{in}(G,C)$ *is connected.*

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5. Appendix

Note that N.O means number of orbits, L.O means largest length of the orbit and GZS means Genus 0 System. The following Tables represent almost simple groups of Lie rank two.

Table 2: Part1: GZSs for Almost Simple Groups of Lie Rank Two

group	ramification type	N.O	L.O	ramification type	N.O	L.O
	(3E,3B,6B)	18	1	(3E,3D,6B)	8	1
	(3E,3C,6A)	8	1	(3B,6B,5A)	1	1
	(3B,6B,5B)	1	1	(3B,3C,5A)	1	1
	(3B,3C,5B)	1	1	(3A,6A,5A)	1	1
	(3A,6A,5B)	1	1	(3A,3D,5A)	1	1
PGL$(3,4)$	(2A,6B,15A)	1	1	(3A,3D,5B)	1	1
	(2A,6B,15C)	1	1	(2A,6A,15D)	1	1
	(2A,6A,15B)	1	1	(2A,3D,15D)	1	1
	(2A,3C,15A)	1	1	(2A,3D,15B)	1	1
	(2A,3C,15C)	1	1			
	(2A,2A,3B,6B)	1	24	(2A,2A,3B,3C)	1	18
	(2A,2A,3A,6A)	1	24	(2A,2A,3A,3D)	1	18

Table 3: Part1: GZSs for Almost Simple Groups of Lie Rank Two

group	ramification type	N.O	L.O	ramification type	N.O	L.O
	(6A,3B,3B)	8	1	(6A,6A,3B)	12	1
	(6A,6A,6A)	8	1	(2A,8A,8B)	1	1
	(3A,3B,8A)	1	1	(3A,3B,8B)	1	1
	(3A,4A,8A)	1	1	(3A,4B,8B)	6	1
	(4A,3B,3B)	8	1	(4A,6A,3B)	8	1
	(4A,6A,6A)	8	1	(4A,4A,3B)	12	1
	(4A,4A,6A)	6	1	(4A,4A,4A)	1	1
	(2A,3B,13A)	1	1	(2A,3B,13B)	1	1
	(2A,3B,13C)	1	1	(2A,3B,13D)	1	1
	(2A,6A,13A)	1	1	(2A,6A,13B)	1	1
	(2A,6A,13C)	1	1	(2A,6A,13D)	1	1
	(2A,4A,13A)	1	1	(2A,4A,13B)	1	1
PSL$(3,3)$	(2A,4A,13C)	1	1	(2A,4A,13D)	1	1
	(2A,3A,3A,4A)	1	12	(2A,3A,3A,3B)	1	12
	(2A,2A,4A,4A)	1	124	(2A,2A,3B,4A)	1	120
	(2A,2A,3B,3B)	1	108	(2A,2A,4A,6A)	1	144
	(2A,2A,3B,6A)	1	144	(2A,2A,6A,6A)	1	132
	(2A,2A,3A,8A)	1	8	(2A,2A,2A,8B)	1	8
	(2A,2A,2A,13A)	1	13	(2A,2A,2A,13B)	1	13
	(2A,2A,2A,13C)	1	13	(2A,2A,2A,13D)	1	13
	(2A,2A,2A,3A,3A)	1	120	(2A,2A,2A,2A,4A)	1	2016
	(2A,2A,2A,2A,3B)	1	1944	(2A,2A,2A,2A,6A)	1	2160
	(2A,2A,2A,2A,2A)	1	32760			

Table 3 (continued): Part1: GZSs for Almost Simple Groups of Lie Rank Two

group	ramification type	N.O	L.O	ramification type	N.O	L.O
PSL(3, 4)	(2A,4C,7B)	2	1	(2A,4C,7A)	2	1
	(2A,4B,7B)	2	1	(2A,4B,7A)	2	1
	(2A,4A,7B)	2	1	(2A,4A,7A)	2	1
	(2A,5A,5B)	6	1	(3A,4B,4C)	8	1
	(3A,4A,4C)	8	1	(3A,4A,4B)	8	1
	(3A,3A,5B)	12	1	(3A,3A,5A)	12	1
	(2A,2A,2A,2A,2A)	2	756	(2A,2A,2A,5B)	2	30
PSL(3, 5)	(2A,8A,5B)	1	1	(2A,8A,10A)	1	1
	(2A,8B,5B)	1	1	(2A,8B,10A)	1	1
	(2A,6A,8A)	1	1	(2A,6A,8B)	1	1
	(2A,3A,24B)	1	1	(2A,3A,24A)	1	1
	(2A,3A,24C)	1	1	(2A,3A,24D)	1	1
	(4C,4C,4C)	28	1	(3A,4C,4C)	26	1
	(3A,3A,4C)	28	1			
	(2A,2A,2A,8A)	1	32	(2A,2A,2A,8B)	1	32
PSL(3, 7)	(2A,2A,2A,4A)	2	180	(3A,3A,4A)	48	1
	(2A,4A,8B)	6	1	(2A,4A,8A)	6	1
	(2A,4A,7B)	2	1	(2A,4A,7A)	2	1
	(2A,4A,7C)	2	1	(2A,4A,14A)	6	1
PΣL(3, 4)	(4B,4B,4C)	8	1	(3A,4B,6A)	10	1
	(2B,4B,14A)	1	1	(2B,4B,14B)	1	1
	(2A,4C,14A)	1	1	(2A,4C,14B)	1	1
	(2A,6A,7A)	2	1	(2A,6A,7B)	2	1
	(2B,6A,8A)	8	1	(2A,5A,8A)	2	1
	(2A,2B,3A,6A)	1	42	(2A,2B,2B,8A)	2	16
	(2A,2A,3A,4C)	1	64	(2A,2A,2B,7A)	1	7
	(2A,2A,2B,7B)	1	7			
PΓL(3, 4)	(2B,4B,21A)	1	1	(2B,4B,21B)	1	1
	(2B,6A,14A)	3	1	(2B,6A,14B)	3	1
	(2B,3B,21A)	1	1	(2B,3B,21B)	1	1
	(2B,6B,15A)	2	1	(2B,6B,15B)	2	1
	(4B,4B,6A)	12	1	(4B,4B,3B)	14	1
	(3A,6B,6B)	12	1			
	(2B,2B,3C,3B)	1	58	(2B,2B,3C,6A)	1	156
	(2B,2B,3A,5A)	1	20	(2B,2B,4B,4B)	1	192
	(2B,2B,2B,14A)	1	28	(2B,2B,2B,14B)	1	28
	(2B,2B,2A,15B)	1	264	(2B,2B,2A,15A)	1	10
	(2B,2A,3A,6B)	1	54			
	(2B,2B,2B,2B,3A)	1	1824	(2B,2B,2A,2A,3A)	1	192