



Graphs of Neighborhood Metric Dimension Two

Badekara Sooryanarayana^{1*} & Suma Agani Shanmukha²

¹Department of Mathematical and Computational Studies, Dr. Ambedkar Institute of Technology, Bengaluru, Karnataka State, India, Pin 560 056.

²Department of Mathematics, School of Applied Sciences, REVA University, Yelahanka, Bengaluru, Karnataka State, India, Pin 560 064.

*E-mail: dr_bsnrao.mat@drait.edu.in

Abstract. A subset S of vertices of a simple connected graph is a neighborhood set (n -set) of G if G is the union of subgraphs of G induced by the closed neighbors of elements in S . Further, a set S is a resolving set of G if for each pair of distinct vertices x, y of G , there is a vertex $s \in S$ such that $d(s, x) \neq d(s, y)$. An n -set that serves as a resolving set for G is called an nr -set of G . The nr -set with least cardinality is called an nr -metric basis of G and its cardinality is called the neighborhood metric dimension of graph G . In this paper, we characterize graphs of neighborhood metric dimension two.

Keywords: landmarks; metric dimension; neighborhood metric dimension; neighborhood set.

1 Introduction

All graphs considered in this paper are connected, simple, undirected and finite. Let $G(V, E)$ be a graph with vertex set V and edge set E . Let $d(u, v)$ denote the distance between the vertices u and v . Let $N[v]$ denote the closed neighborhood of the vertex $v \in V$, i.e. $N[v] = \{x \in V : d(x, v) \leq 1\}$. A neighborhood set of G is a subset S of the vertex set of G with the property that $G = \bigcup_{v \in S} G_v$ where $G_x = \langle N[x] \rangle$. Further, a subset S of V is called a resolving set of G if for each pair u, v of vertices of G there is a vertex $t \in S$ with the property that $|d(v, t) - d(u, t)| > 0$. A neighboring set of G that also serves as an r -set of G is called a neighborhood resolving set (nr -set) of G . In other words, an nr -set S is an ordered subset $S = (s_1, s_2, s_3, \dots, s_k)$ of V such that $\Gamma(x/S) \neq \Gamma(y/S)$ for all $x, y \in V - S$ and $G = \bigcup_{i=1}^k \langle N[s_i] \rangle$, where $\Gamma(a/S) = (d(a, s_1), d(a, s_2), \dots, d(a, s_k))$ is called the code of vertex a with respect to S .

An nr -set is called a minimal nr -set (mnr -set) if none of its proper subsets is an nr -set. The mnr -set of G with least cardinality is called the neighborhood metric basis (nmb) and its cardinality is called the lower neighborhood metric dimension or simply the neighborhood metric dimension of graph G , denoted by

$nmd(G)$. Similarly, the upper neighborhood metric dimension of G is the greatest cardinality of an mnr -set, denoted by $Nmd(G)$. Further, an nr -set S is called a maximal neighborhood resolving set (Mnr -set) whenever $V - S$ is not an nr -set. The minimum cardinality of an Mnr -set is called the lower maximal neighborhood metric dimension or simply the maximal neighborhood metric dimension of G , denoted by $nMd(G)$. Similarly, the maximum cardinality of an Mnr -set of G is called the upper maximal neighborhood metric dimension of G , denoted by $NMd(G)$. These sets and dimensions have been studied for paths and cycles in [1]. Related work can be found in [2-19]. For the terms not defined here we refer to [20,22].

We recall the following theorem in [23]:

Theorem 1.1. *Let $G(V, E)$ be a graph and $S \subseteq V(G)$. Then, S is an n -set of G if and only if every pair of adjacent vertices in \bar{S} is dominated by a common vertex in S [23].*

Corollary 1.2. *Let G be a triangle free graph. Then, S is an n -set of G if and only if $\langle \bar{S} \rangle \equiv \bar{K}_n$, where $\bar{S} = V(G) - S$ and $n = |V(G)|$ [23].*

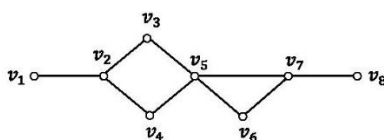


Figure 1 Graph G .

Example: The sets $\{v_3, v_7\}$, $\{v_2, v_5, v_7\}$, and $\{v_2, v_3, v_5, v_8\}$ are an r -set, n -set, and nr -set of graph G in Fig. 1, respectively. The set $\{v_2, v_3, v_4, v_7\}$ is a minimal nr -set.

Theorem 1.3. *The minimum cardinality of an r -set of G is 1 if and only if $G = P_n$ for some $n \geq 2$ [24].*

Theorem 1.4. *Let $T = (V, E)$ be a tree with $\delta(T) \geq 3$. Then the minimum cardinality of an r -set is given by $\sum_{v \in V: l_v > 1} (l_v - 1)$ [24].*

Theorem 1.5. *For any positive integer n , $nmd(P_n) = \begin{cases} \lfloor \frac{n}{2} \rfloor, & \text{if } n \leq 3 \\ \lceil \frac{n}{2} \rceil, & \text{if } n \geq 4 \end{cases}$ [1].*

Remark 1.6. *If graph G contains k pendent vertices adjacent to a vertex v in it, then every resolving set of G should contain at least $k - 1$ of these pendent*

vertices. Otherwise, two of these pendent vertices not in a resolving set receive the same code due to the fact that v is a cut vertex.

We state the following theorems whose proof follows immediately from the definition of an nr -set and Theorem 1.3.

Theorem 1.7. *For a connected graph G , $nmd(G) = 1$ if and only if $G \simeq K_1$ or K_2 .*

Theorem 1.8. *For any connected graph G , $2 \leq nmd(G) \leq |V(G)| - 1$, whenever $|V(G)| \geq 3$.*

The upper bound in Theorem 1.8 is tight for the complete graph K_n on n vertices. In a later section of this paper we characterize the graphs that satisfy the lower bound and construct a graph of prescribed neighborhood dimension.

2 Application of Neighborhood Resolving Sets

In most safeguard applications of a network model, various types of protection sets have been studied to identify or locate an ‘intruder’ or to check a faulty processor. The locating sets are such a protection sets, not only to identify intruders but also to determine the distance to an intruder. These were introduced by Slater [25] and independently by Harary and Melter [26]. When two intruders are linked (known to each), then it is more efficient to have a common protection device at a vertex v adjacent (known to) both intruders. However, the study of special types of dominating sets, namely neighborhood sets, was introduced in [23]. The neighborhood set is a notion that is somehow in between vertex cover and dominating sets. These sets are considered in [1] to build a powerful locating-dominating set, as they have the additional property of edge covering along with the domination and locating properties. Such sets are useful in the study of the location of vertices as well as edges in a network.

3 Realization

In this section we construct a graph with prescribed parameters.

Theorem 3.1. *For the given positive integers α, β, γ with $\alpha, \beta \leq \gamma$, there is a tree on n vertices such that $l_n(G) = \alpha, l_r(G) = \beta$ and $l_{nr}(G) = \gamma$ whenever $2\gamma - \beta \leq n \leq \gamma + \alpha$.*

Proof. Let $l_n(G) = \alpha, l_r(G) = \beta$ and $l_{nr}(G) = \gamma$. Then $\alpha \leq \gamma, \beta \leq \gamma$ and $\alpha + \beta \geq \gamma$. Let $k = \alpha + \beta - \gamma$; $H_1 = \bar{K}_{\gamma-\alpha}$ and $H_2 = (\alpha + \beta - \gamma) K_2$.

Consider a path $P_{2(\gamma-\beta)}: v_1 - v_2 - \dots - v_{2(\gamma-\beta)}$. Add the edges between one end vertex of each copy of K_2 in H_2 to the vertex v_1 , and each vertex of H_1 to v_1 . The graph G thus obtained satisfies:

1) $l_n(G) = \alpha$.

In fact, if S is any n -set, then S should include $(\gamma - \beta)$ vertices of $P_{2(\gamma-\beta)}$ and $\alpha + \beta - \gamma$ vertices of H_2 , because G is triangular free and by Corollary 1.1, $\langle V - S \rangle$ is totally disconnected and hence $|S| \geq (\gamma - \beta) + (\alpha + \beta - \gamma) = \alpha$.

On the other hand, the set $S = \{u_i, v_j: u_i \text{ is a vertex in the } i^{th} \text{ copy of } K_2 \text{ in } H_2, v_j \in V(P_{2(\gamma-\beta)}) \text{ and } j \text{ is odd}\}$ is an n -set of G with $|S| \leq \alpha$.

2) $l_r(G) = \beta$.

Since G is a tree with $\beta + 1$ legs it follows from Theorem 1.5 that $l_r(G) = \beta + 1 - 1 = \beta$.

3) $l_{nr}(G) = \gamma$.

Let S be an nr -set of G , then at least β pendent vertices of G should be in S being an r -set and $\gamma - \beta$ vertices of a path $P_{2(\gamma-\beta)}(G) = \beta$ being an n -set. So $|S| \geq \beta + \gamma - \beta = \gamma$.

On the other hand, $S = \{v_j: v_j \in V(P_{2(\gamma-\beta)})\} \cup \{x: x \text{ is a pendant vertex of } G \text{ not in } P_n\}$ is an nr -set with $|S| = \gamma$.

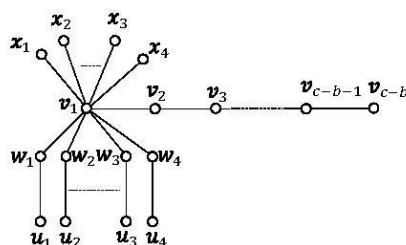


Figure 2 A tree G with $l_n(G) = \alpha$, $l_r(G) = \beta$ and $l_{nr}(G) = \gamma$.

As an immediate consequence of Theorem 3.1, we have:

Corollary 3.2 For the positive integers α, β, γ with $\alpha, \beta \leq \gamma \leq \alpha + \beta$ there is a graph G , see Fig. 2, with $l_n(G) = \alpha$, $l_r(G) = \beta$ and $l_{nr}(G) = \gamma$.

4 Graphs of Neighborhood Metric Dimension Two

We begin this section with the following lemma:

Lemma 4.1. *Let G be a connected graph and $S = \{s_1, s_2\}$ be an independent nr -set of G . Then there is a unique vertex v that is adjacent to both s_1 and s_2 in G .*

Proof. We first prove the existence. Since G is connected and S is an independent n -set, s_1 is adjacent to v , and s_2 is adjacent to u for some $u, v \in V(G)$. The vertex v does not need to be distinct from u . If $v = u$, then v is the desired vertex. Else, if u and v are distinct, then there is a vertex $w_1 \in V(G) - S$ such that w_1 is in the shortest path from v to u adjacent to v . But then, for the edge vw_1 , by Theorem 1.1 we see that w_1 is adjacent to s_1 . If w_1 is adjacent to s_2 , then w_1 is the desired vertex. Otherwise, repeating this argument by replacing v with w_1 , we arrive at k finite steps that there is a vertex $w_k (\neq u)$ adjacent to s_1 as well to s_2 or u . If w_k is adjacent to s_2 , then w_k is the desired vertex. Else, if w_k is not adjacent to s_2 for any k , then for the last possible k , we see that $w_k u \in E(G)$. Hence by Theorem 1.1 we get u is adjacent to s_1 . So u is the desired vertex.

To prove uniqueness: Suppose that v_1 and v_2 are two distinct vertices adjacent to both s_1 and s_2 . However, then $d(v_1, s_1) = d(v_2, s_1)$ and $d(v_1, s_2) = d(v_2, s_2)$ imply that S has no vertex that resolves v_1 and v_2 , a contradiction ($\because S$ is an r -set).

Remark 4.2. *If S is a non-independent nr -set, then G does not need to have a common vertex adjacent to both elements in S as in Lemma 4.1. Thus, S can have at most one vertex in $V(G) - S$ that is adjacent to both s_1 and s_2 for any general nr -set S of cardinality 2.*

Lemma 4.3. *Let G be a connected graph and $S = \{s_1, s_2\}$ be an nr -set of G . Then, $1 \leq \deg(s_i) \leq 3$, for each $i = 1, 2$.*

Proof. Let us assume to contrary that $\deg(s_i) \geq 4$ for some $i = 1, 2$. Without loss of generality we take $\deg(s_1) \geq 4$. Let v_1, v_2, v_3, v_4 be the vertices adjacent to s_1 . Then:

Case 1: $\langle S \rangle$ is independent.

In this case, by Lemma 4.1, there is a unique vertex x adjacent to both s_1 and s_2 , and hence at least three of the vertices in $\{v_1, v_2, v_3, v_4\}$ are not adjacent to s_2 . Without loss of generality, let v_1, v_2, v_3 be the vertices not adjacent to s_2 .

But then, $d(v_i, s_1) = 1$ and $d(v_i, s_2) = \begin{cases} 3 & \text{if } v_i \notin E(G) \\ 2 & \text{if } v_i \in E(G) \end{cases}$. In either of these cases there are at least two vertices $v_l, v_m \in \{v_1, v_2, v_3\}$ that satisfy $d(v_l, s_2) = d(v_m, s_2)$, a contradiction to the fact that S resolves G .
 Case 2: $\langle S \rangle$ is connected.

In this case there are at least three vertices v_1, v_2, v_3 not in S adjacent to s_1 in G . But then, by Remark 4.2, at least two of these, say v_1 and v_2 , are not adjacent to s_2 . So, $d(v_1, s_1) = d(v_2, s_1)$ and $d(v_1, s_2) = d(v_2, s_2) = 2$, a contradiction to the fact that S resolves G .

The above Lemma 4.1 and Lemma 4.3, together with the domination property of n -sets, yield the following theorem:

Lemma 4.4. *If G is a connected graph of order n with $nmd(G) = 2$, then $3 \leq n \leq 7$.*

Lemma 4.5. *Let G be a connected graph and $S = \{s_1, s_2\}$ be the neighborhood metric basis of G . Let x be the vertex adjacent to both s_1, s_2 . If u, v are vertices adjacent to s_1 , then either u is adjacent to x or v is adjacent to x in G .*

Proof. If not, then there are two possible cases: (i) both u and v are not adjacent to x , and (ii) both u and v are adjacent to x . By Lemma 4.1, neither u nor v is adjacent to s_2 . So, $\Gamma(u/S) = \Gamma(v/S) = (1,3)$ and $\Gamma\left(\frac{v}{S}\right) = \Gamma\left(\frac{u}{S}\right) = (1,2)$ in the cases (i) and (ii) respectively, a contradiction to the fact that S is an r -set.

Lemma 4.6. *If G is a connected graph of size m and $nmd(G) = 2$, then $2 \leq m \leq 10$.*

Proof. Let $nmd(G) = 2$ and $S = \{s_1, s_2\}$ be the neighborhood metric basis of G . The lower bound is trivial, as $nmd(K_2) = 1$. To prove the upper bound, let $s_{l,i}$ be the i^{th} vertex adjacent to s_1 and $s_{2,j}$ be the j^{th} vertex adjacent to s_2 . Then $1 \leq i, j \leq 3$ (by Lemma 4.3), $s_{1,i} = s_{2,j}$ for exactly one (i, j) (by Lemma 4.1) and $s_{1,i}$ is adjacent to $s_{1,1}$ for exactly one $i, i \geq 2$ (by Lemma 4.5). Without loss of generality we take $s_{1,1} = s_{2,1}$ as the common vertex adjacent to both s_1 and s_2 . Similarly, $s_{2,j}$ is adjacent to $s_{1,1}$ for exactly one $j, j \geq 2$. Without loss of generality, we assume that $s_{1,2}$ is adjacent to $s_{1,1}$, $s_{2,2}$ is adjacent to $s_{2,1}$ (note that $s_{1,1} = s_{2,1}$). Further, by Lemma 4.4, G has at most $s_1, s_2, s_{1,i}, s_{2,j}$, for $1 \leq i, j \leq 3$ vertices. Therefore, $\sum_{v \in V} \deg(v) \leq \sum_{i=1}^2 \deg(s_i) +$

$$\sum_{i=2}^3 \deg(s_{1,i}) + \sum_{j=2}^3 \deg(s_{2,j}) + \deg(s_{1,1}) \leq (3+3) + (3+2) + (3+2) + 4 = 20. \text{ Hence, } m = \frac{1}{2} \sum_{v \in V} \deg(v) \leq \frac{1}{2}(20) = 10.$$

Lemma 4.7. *A graph G with $nmd(G) = 2$ cannot have an induced cycle C_n for any $n \geq 4$.*

Proof. Suppose to contrary that $nmd(G)=2$ and G has an induced cycle C_n for some $n \geq 4$. Then C_n has no induced triangle. Let $S = \{s_1, s_2\}$ be the neighborhood metric basis for G . We have only the following three cases:

Case (1): $s_1, s_2 \in V(C_n)$

In this case, S is not an independent set (by Lemma 4.1). Since $n \geq 4$, there is an edge $uv \in \langle V(C_n) - S \rangle$. However, then by Theorem 1.1 either $\langle \{u, v, s_1\} \rangle$ or $\langle \{u, v, s_2\} \rangle$ is an induced triangle of C_n , a contradiction to the fact that C_n has no induced triangle.

Case (2): $s_1 \in V(C_n)$ and $s_2 \notin V(C_n)$

Let x and u be the vertices of C_n adjacent to s_1 . By Lemma 4.1 the vertex s_2 can be adjacent to at most one of these vertices, say $u \notin N(s_2)$. Then, by Theorem 1.1, the vertex $u' (\neq s_1)$ adjacent to u in C_n should be in $N(s_1)$ and hence $\langle \{s_1, u, u'\} \rangle$ is an induced triangle of C_n , a contradiction.

Case (3): $s_1, s_2 \notin V(C_n)$

In this case, by Lemma 4.5, $n = 4$ or $n = 5$. By Lemma 4.1, s_1 and s_2 are adjacent to at most one common vertex $x \in V - S$. Therefore, there exist two adjacent vertices $u, v \in V(C_n)$ such that $u, v \notin N(s_1) \cap N(s_2)$. In view of Theorem 1.1, without loss of generality we take $u, v \notin N(s_1)$ (hence $u, v \in N(s_2)$). Since $n \geq 4$, there are two distinct vertices u' and v' adjacent to respectively u and v in C_n . Let $v' = x \in N(s_1) \cap N(s_2)$ and from Remark 4.2 s_1 is adjacent to only u' . Then an edge $uu' \notin \cup_{v \in S} \langle N[v] \rangle$, a contradiction to an n -set S . And also $\Gamma(u/S) = \Gamma(v/S) = (1,2)$ implies S is not an r -set.

Lemma 4.8. *A graph G with $nmd(G) = 2$ cannot have an induced subgraph isomorphic to P_6 .*

Proof. Let $nmd(G) = 2$ and S be any nmb of G . If possible, let $P_6: v_1, v_2, \dots, v_6$ be an induced path of G with $v_i v_{i+1} \in E$ for each $i, 1 \leq i \leq 5$. If G has no other vertices, then $G \cong P_6$, a contradiction to the fact that $nmd(P_6) = 3$ (by Theorem 1.5). Further, as G can have at most 7 vertices (by Lemma 4.4),

G has exactly one new vertex v . The graph $G \neq P_7$ (since $nmd(P_7) = 3 > 2$, by Theorem 1.5) and hence v cannot be adjacent to only v_1 or only v_6 . Also, if v is not adjacent to both v_1 and v_6 , then by Theorem 1.1, every nmr -set S should include at least two vertices from the set $\{v_1, v_2, v_5, v_6\}$ for the edges v_1v_2 and v_5v_6 , and a new vertex for the edge v_3v_4 , so $|S| \geq 3 \Rightarrow nmd(G) \geq 3$, a contradiction. Finally if v should be adjacent to both v_1 and v_6 , then by Lemma 4.7, the vertex v is adjacent to every v_i , $2 \leq i \leq 5$ and hence G has 11 edges, a contradiction by Lemma 4.6. Thus, v should be adjacent to exactly v_1 (or v_6 but not both) and a vertex v_i , $2 \leq i \leq 5$. Without loss of generality we assume $vv_1 \in E$.

If $v \notin S$, then by Theorem 1.1 for the edge vv_1 , either $v_1 \in S$, or $v_2 \in S$ and $vv_2 \in E(G)$. Let $S = \{v_i, v_j\}$ where $i = 1$ or $i = 2$, and $i + 1 \leq j \leq 5$. Then by Lemma 4.7, we get $j = i + 1$. But then for the edge v_5v_6 , by Theorem 1.1, either v_i or v_{i+1} adjacent to both v_5 and v_6 , a contradiction to the fact that P_6 is an induced path. Therefore, $v \in S$ and hence $S = \{v, v_1\}$. But then, again for the edge v_5v_6 , by Theorem 1.1 only v should be adjacent to both v_5 and v_6 (since P_6 is an induced path), a contraction to the fact that v is not adjacent to v_6 . Hence the theorem is proved.

Lemma 4.9. *Let G be a graph of order n and S be a neighborhood metric basis of cardinality 2 in G . If S has a pendent vertex of G , then $n \leq 5$.*

Proof. Let $S = \{s_1, s_2\}$ be the neighborhood metric basis and s_1 be a pendent vertex in S . If s_2 is adjacent to s_1 , then $d(s_1, x) = d(s_2, x)$ for every $x \in V(G)$ and hence s_2 may be adjacent to at most one more vertex in G , which implies that $n \leq 3$. If s_2 is not adjacent to s_1 , then S is an independent set. Hence by Lemma 4.1, G has exactly one vertex adjacent to both s_1 and s_2 . Finally, as each vertex of G is adjacent to either s_1 or s_2 , the order of $G = |N[s_1] \cup N[s_2]| = |N[s_1]| + |N[s_2]| - |N[s_1] \cap N[s_2]| = 2 + 4 - 1 = 5$.

Lemma 4.10. *A graph G with $nmd(G) = 2$ cannot have a subgraph isomorphic to K_4 .*

Proof. Let $nmd(G) = 2$ and $S = \{s_1, s_2\}$ be an nmr -set of G . If possible, suppose to contrary that G has a subgraph H isomorphic to K_4 . Let v_1, v_2, v_3, v_4 be the vertices of H . Then we have the following cases:

Case (i): $S \subseteq H$

In this case, for the vertices, $x, y \in H - S$, we get $d(s_1, x) = d(s_2, x)$; $d(s_1, y) = d(s_2, y) = 1$, a contradiction to the fact that S is an r -set.

Case (ii): $|S \cap V(H)| = 1$.

Without loss of generality we take $s_1 = v_1 \in S \cap V$, then in this case $s_2 \neq v_i$, for any i , $2 \leq i \leq 4$ and s_1 is adjacent to each v_j , $2 \leq j \leq 4$. Hence by Lemma 4.3, s_1 is not adjacent to s_2 as well no other vertex in $V(G) - V(H)$. So, by Lemma 4.1, there is a vertex v_j for exactly one j , $2 \leq j \leq 4$, adjacent to both s_1 and s_2 . Without loss of generality, we take v_2 is adjacent to s_1 and s_2 . But then, $\Gamma(v_3|S) = \Gamma(v_4|S) = (1,2)$, a contradiction to the fact that S is an r -set.

Case (iii): $|S \cap V(H)| = \emptyset$.

By Theorem 1.1, for the edge v_1v_2 , we have either $v_1, v_2 \in N(s_1)$ or $v_1, v_2 \in N(s_2)$. Without loss of generality we take $v_1, v_2 \in N(s_1)$. By Lemma 4.3, s_1 can be adjacent to at most one of the vertices in $\{v_3, v_4\}$, and by Lemma 4.1, s_2 can be adjacent to at most one vertices in $\{v_1, v_2\}$. Without loss of generality, we take $v_3 \notin N(s_1)$ and $v_1 \notin N(s_2)$. However, then G has no triangle containing the edge v_1v_3 , one of its vertices in S , a contradiction (by Theorem 1.1) to the fact that S is an n -set.

Corollary 4.11. *If S is a neighborhood metric basis of cardinality 2 of a graph G , then the graph $\langle V - S \rangle$ is acyclic.*

Proof. If not, then by Lemma 4.7, $\langle V - S \rangle$ has a triangle. Let v_1, v_2, v_3 be vertices in a cycle of $\langle V - S \rangle$. Then, by Theorem 1.1 both end vertices of the edge v_1v_2 are adjacent to one of the vertices in S , say s_1 . If v_3 is adjacent to s_1 , then $\langle \{v_1, v_2, v_3, s_1\} \rangle$ is an induced K_4 , a contradiction to Lemma 4.10. So, v_3 is not adjacent to s_1 .

Now, by Theorem 1.1 the end vertices v_2, v_3 of the edge v_2v_3 should be adjacent to s_2 . But then v_2 is adjacent to both s_1 and s_2 and hence by Lemma 4.1, v_1 cannot be adjacent to s_2 . Therefore none of the end vertices of edge v_1v_3 are adjacent to s_1 or s_2 , a contradiction to Theorem 1.1.

We now prove our main theorem of this section.

Theorem 4.12. For a graph G , the neighborhood metric dimension, $nmd(G) = 2$ if and only if G is isomorphic to one of the graphs in Fig. 3.

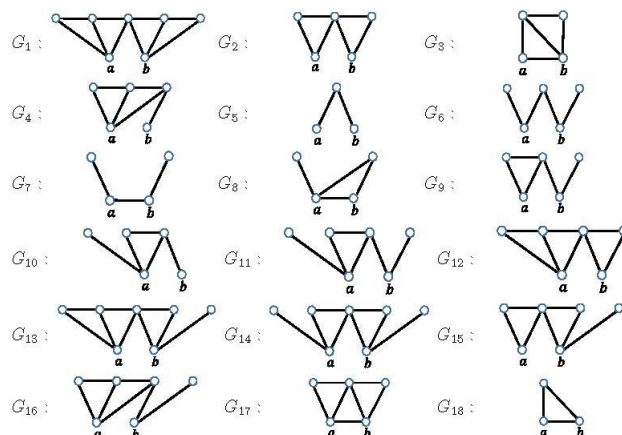


Figure 3 Graphs of neighborhood metric dimension two.

Proof. For the graphs G_6 , G_7 , and G_5 , an n -set S should contain a non-pendent vertex. But such a set with cardinality less than 2 is not a resolving set. Hence $nr(G) \geq 2$. Other graphs in the Figure are not paths, hence by Theorem 1.3 it follows that $nr(G) \geq 2$. The reverse inequality follows by noting that the set $S = \{a, b\}$ is an nr -set for each of the graphs.

Conversely, let G be any connected graph with $nmd(G) = 2$ and let $S = \{s_1, s_2\}$ be a neighborhood resolving basis of G . Then, by Corollary 4.11 $\langle V - S \rangle$ is a forest. Therefore we have only the following cases:

Case 1: $\langle S \rangle$ is connected.

By Lemma 4.1 and Lemma 4.3, $\langle V - S \rangle$ has at most two edges.

Subcase 1: $\langle V - S \rangle \simeq P_3 \cup mK_1$.

By Theorem 1.1 and Lemma 4.1, both end vertices of one of the edges of P_3 should be adjacent to s_1 and both end vertices of other edges to s_2 . Therefore the center of P_3 is adjacent to both s_1 and s_2 . Thus by Lemma 4.3, neither s_1 nor s_2 is adjacent to any more vertex, which implies that $m = 0$. Hence $G \simeq G_{17}$ in this case.

Subcase 2: $\langle V - S \rangle \simeq 2P_2 \cup mK_1$.

In this case both vertices of one of the copies of P_2 are adjacent to one of the vertices in S , say s_1 , and both vertices of the other copy of P_2 should be

adjacent to only s_2 . Further, by Lemma 4.3, $m = 0$. We observe that $G \simeq P_2 \odot P_2$, so the end vertices in each copy of P_2 receive the same code with respect to S . Therefore, no graph G exists with $nmd(G) = 2$ in this case.

Subcase 3: $\langle V - S \rangle \simeq P_2 \cup mK_1$.

As above, P_2 should be adjacent to one of the vertices in S , say s_1 . But then, by Lemma 4.3 the vertices of mK_1 are adjacent to only s_2 and $m \leq 2$. Further, none of these pendent vertices are in S and hence by Remark 1.6 it follows that $m \neq 2$. Therefore $m \leq 1$.

If $m = 0$, then by Lemma 4.10, exactly one of the end vertices of P_2 should be adjacent to s_2 (else the end vertices of P_n receive the same code or induce K_4). Thus, $G \simeq G_3$ in this case.

If $m = 1$, then as above exactly one of the vertices of P_2 should be adjacent to s_2 (to resolve the vertices of P_2). Hence graph $G \simeq G_4$ in this case.

Subcase 4: $\langle V - S \rangle \simeq mK_1$.

In this case, one of the isolated vertices in $\langle V - S \rangle$ may be common to both elements in S (by Lemma 4.1) and each vertex in S may be adjacent to at most one isolated vertex in G (by Remark 1.6). Therefore $m \leq 3$ and $G \simeq G_5, G_7, G_8, G_{10}, G_{18}$.

Case (ii): $\langle S \rangle$ is disconnected.

In this case, in view of Lemma 4.1 and Lemma 4.3, the graph $\langle V - S \rangle$ can have at most four edges.

Subcase (i): $\langle V - S \rangle$ is totally disconnected.

Let $\langle V - S \rangle = mK_1$. Then $m \geq 1$, else G is disconnected. Suppose that u_1, u_2, \dots, u_m are the vertices in $\overline{K_m} = mK_1$. By Lemma 4.1, without loss of generality we take u_1 is the only vertex that is adjacent to both s_1 and s_2 .

If $m = 1$, then $G \simeq G_5$ in this case.

If $m \geq 2$, then u_2 is adjacent to exactly one of the vertices in S , say s_1 . Now, $G \simeq G_7$ if $m = 2$. Else if $m \geq 3$, then by Remark 1.6 u_3 is adjacent to only s_2 . Therefore $G \simeq G_6$ if $m = 3$. Finally, if $m > 3$ then by the pigeonhole principle, at least one of the vertices in S is a support for at least two pendent vertices not in S . By Remark 1.6 no such graph with $nmd(G) = 2$ exists.

Subcase (ii): $\langle V - S \rangle$ has exactly one edge.

In this case, $\langle V - S \rangle \simeq P_2 \cup mK_1$ and by Theorem 1.1 both vertices of P_2 are adjacent to exactly one of the vertices of S , say s_1 . By Lemma 4.1 exactly one of the vertices of P_2 is adjacent to s_2 . Therefore, similar to Subcase (i) it follows that $G \simeq G_8$ if $m = 0$; $G \simeq G_9$ or G_{10} if $m = 1$; $G \simeq G_{11}$ if $m = 2$; and no graph exists if $m \geq 3$.

Subcase (iii): $\langle V - S \rangle$ has exactly two edges.

Subsubcase (i): $\langle V - S \rangle \simeq 2P_2 \cup mK_1$.

In this case, since S is an n -set, both vertices of one of the copies of P_2 are adjacent to exactly one of the vertices in S , say s_1 . Also exactly one of the vertices of this copy of P_2 is adjacent to s_2 . Similarly, both vertices of the other copies of P_2 are adjacent to only s_2 and exactly one of them is adjacent to s_1 . Hence there are two vertices, one from each copy of P_2 , that are adjacent to both s_1 and s_2 , which is not possible by Lemma 4.1. Therefore, no graph exists in this case with $nmd(G) = 2$.

Subsubcase (ii): $\langle V - S \rangle \simeq P_3 \cup mK_1$.

If all vertices of P_3 are adjacent to a vertex in S , say s_1 , then exactly one of the end vertices of P_3 should be adjacent to s_2 (by Lemma 4.1) to resolve all vertices of P_3 by s_2 . Further, no vertex of mK_1 is adjacent to s_1 (by Lemma 4.3) and at most one vertex is adjacent to s_2 (by Remark 1.6). Therefore, $m \leq 1$, and $G \simeq G_4$ if $m = 0$, and $G \simeq G_{16}$ if $m = 1$.

If all vertices of P_3 are not adjacent to a vertex in S , then the end vertices of the edges are adjacent to s_1 and the end vertices of the other edges are adjacent to s_2 , and hence the central vertex of P_3 is adjacent to both s_1 and s_2 . Further, by Lemma 4.1 and Remark 1.6 vertex s_1 may be adjacent to a pendent vertex of mK_1 and s_2 may be adjacent to at most one other pendent vertex in of mK_1 , which implies that $m \leq 2$. Hence $G \simeq G_2$ if $m = 0$; $G \simeq G_{15}$ if $m = 1$; and $G \simeq G_{14}$ if $m = 2$.

Subcase (iv): $\langle V - S \rangle$ has exactly three edges.

Subsubcase (i): $\langle V - S \rangle \simeq P_4 \cup mK_1$.

Let v_1, v_2, v_3, v_4 be the vertices of P_4 . Then, as above, if the vertices v_1, v_2 are adjacent to say s_1 , the vertices v_3, v_4 are adjacent to s_2 (by Theorem 1.1 and

Lemma 4.3). Further, for the edge v_2v_3 , by Theorem 1.1 either v_2 is adjacent to s_2 , or v_3 is adjacent to s_1 . Thus, similar to the previous arguments on mK_1 , $G \simeq G_{12}$ if $m = 0$; $G \simeq G_{13}$ if $m = 1$; and no graph exists if $m > 1$ (since $\deg(s_i) = 3$, for one $i = 1, 2$).

Subsubcase (ii): $\langle V - S \rangle \simeq (P_3 \cup P_2) \cup mK_1$.

In this case, both the end vertices of P_2 should be adjacent to one of the vertices of S say s_1 . But then by Theorem 1.1 and Lemma 4.3 all the vertices of P_3 should be adjacent to s_2 , and by Lemma 4.1 exactly one of the end vertices of P_3 should be adjacent to s_1 (to resolve all the vertices of P_3). But then $m = 0$ (since already $\deg(s_1) = \deg(s_2) = 3$). Now, the end vertices of P_2 are equidistant from s_1 (at a distance 1) as well as equidistant from s_2 (at a distance 2). Therefore, no graph G exists in this case with $nmd(G) = 2$.

Subsubcase (iii): $\langle V - S \rangle \simeq 3P_2 \cup mK_1$.

In this case $n \geq |S| + 3|P_2| + m \geq 2 + 6 + 0 = 8$. Hence, by Lemma 4.4, no graphs exist in this case.

Subcase (v): $\langle V - S \rangle$ has exactly four edges.

By Lemma 4.3 the maximum number of vertices in $\langle V - S \rangle$ is 5. Therefore the only the possible forests are $\langle V - S \rangle \simeq$ a path P_5 or a Bistar $B_{1,2}$ or a star $K_{1,4}$ or $(K_{1,3} + e) \cup mK_1$ with $m \in \{0, 1\}$.

Subsubcase (i): $\langle V - S \rangle \simeq P_5$.

Let v_1, v_2, v_3, v_4, v_5 be the vertices of P_5 . Then by Theorem 1.1, Lemma 4.1 and Lemma 4.3 the only possibility is that s_1 is adjacent to v_1, v_2, v_3 , and s_2 is adjacent to v_3, v_4, v_5 . The graph $G \simeq G_1$ exists in this case.

Subsubcase (ii): $\langle V - S \rangle \simeq B_{1,2}$.

Let v_1, v_2, v_3, v_4, v_5 be the vertices of $B_{1,2}$ and $v_1v_2, v_2v_3, v_3v_4, v_3v_5$ be its edges. By Theorem 1.1 and without loss of generality we assume that $v_3, v_5 \in N(s_2)$.

If $v_3, v_4 \notin N(s_2)$, then by Theorem 1.1, $v_3, v_4 \in N(s_1)$ and hence v_3 is the vertex adjacent to both s_1 and s_2 . So for the edge v_2v_3 , by Theorem 1.1 by symmetry we assume that $v_2 \in N(s_1)$. But then by Lemma 4.1 $v_2 \notin N(s_2)$ and hence for the edge v_1v_2 by Theorem 1.1 $v_1 \in N(s_1)$, therefore $\deg(s_1) \geq 4$, a

contradiction to Lemma 4.3. Thus, $v_3, v_4 \in N(s_2)$, and v_3 is adjacent to both s_1 and s_2 . But then, $\Gamma(v_4|S) = \Gamma(v_5|S) = (1,2)$, so S cannot be an r -set and hence there is no graph G with $nmd(G) = 2$ in this case.

Subsubcase (iii): $\langle V - S \rangle \simeq K_{1,4}$.

Let v be the central vertex and v_1, v_2, v_3, v_4 be the end vertices of $K_{1,4}$. Then by Lemma 4.1, Lemma 4.3 and Theorem 1.1, without loss of generality we assume only the possibilities that $\{v_1, v_2, v\} = N(s_1)$ and $\{v_3, v_4, v\} = N(s_2)$. Now, for the vertices v_3, v_4 , we get $\Gamma(v_3|S) = \Gamma(v_4|S) = (1,2)$, and hence S cannot be an r -set. Therefore, there is no graph G with $nmd(G) = 2$ in this case.

Subsubcase (iv): $\langle V - S \rangle \simeq (K_{1,3} + e) \cup mK_1, 0 \leq m \leq 1$.

Let v_1, v_2, v_3, v_4 be the vertices of $(K_{1,3} + e)$ and $v_1v_2, v_1v_3, v_2v_3, v_3v_4$ be its edges. By Theorem 1.1 without loss of generality we assume that $v_3, v_4 \in N(s_1)$. From Lemma 4.10, $\{v_1, v_2, v_3\} \not\subseteq N(s_1)$. Without loss of generality we assume that v_1 is adjacent to s_1 , and v_2 is adjacent to s_2 . But then by Theorem 1.1, $v_3 \in N(s_2)$ and hence by Lemma 4.1 $v_4 \notin N(s_2)$. Thus, for the vertices v_1, v_4 , $\Gamma(v_1|S) = \Gamma(v_4|S) = (1,2)$ implies that S cannot be an r -set. Therefore, there is no graph G with $nmd(G) = 2$ in this case. Hence the theorem is proved.

5 Conclusions and Open Problems

An NP -complete problem of uniquely determining the location of an intruder in a network was the principal motivation behind introducing the concept of metric dimension in graphs by P.J. Slater.

In many practical situations, the role of a vertex in a network depends on its neighborhood. Since every neighborhood set is a dominating set, the concept of the neighborhood resolving set can be related to connected dominating sets, which has wide applications in building algorithms for wireless sensor networks (WSNs). The common approach to constructing a backbone for a WSN is to build a set of nodes such that every other node is close to a node in the given set. Such a set is known as a dominating set. The concept of the nr -set studied in this paper not only covers neighboring nodes but also distinguishes them through their resolving property. The idea of connected nr -sets can also be used in ad-hoc networks. The main result of this paper is a nice characterization of graphs of nr -dimension two. In general, it is natural to ask for a characterization of graph classes with respect to the nature of their neighborhood metric dimension. Thus, we end this section with some open problems:

1. Find a vertex transitive graph $G(V, E)$ of regularity r with $nmd(G) = k$ for every possible $2 \leq k \leq |V| - 1$.
2. Find an algorithm to execute a minimal (optimal) connected nr -set of a given connected graph $G(V, E)$.
3. Solve the graph equation $nmd(G) = |V| - 1$ for $G(V, E)$.

6 Acknowledgments

The authors wish to express sincere thanks to the referees for their careful reading of the manuscript and helpful suggestions.

References

- [1] Sooryanarayana, B. & Suma, A.S., *On Classes of Neighborhood Resolving Sets of a Graph*, Electron. J. Graph Theory Appl., 6(1) pp. 29-36, 2018.
- [2] Carmen Hern, A. & Mora, M., *Metric-Locating-Dominating Sets of Graphs for Constructing Related Subsets of Vertices*, Applied Mathematics and Computation, 332, pp. 449-456, 2018.
- [3] Chartrand, G., Eroh, L., Johnson, M.A. & Oellermann, O., *Resolvability in Graphs and the Metric Dimension of a Graph*, Discrete Appl. Math., 105, pp. 99-113, 2000.
- [4] Geetha, K.N., Narahari, N., Meera, K.N. & Sooryanarayana, B., *Open Neighborhood Coloring of Prisms*, J. Math. Fund. Sci., 45(3), pp. 245-262, 2013.
- [5] Jayalakshmi, M. & Padma, M.M., *Variety of Rational Resolving Sets of Corona Product of Graphs*, Advances in Mathematics: Scientific Journal, 9(10), pp. 8367-8374, 2020.
- [6] Cong, L.E., Kang, X. & Yi, E. *The Connected Metric Dimension at a Vertex of a Graph*, Theoretical Computer Science, 806, pp. 53-69, 2020.
- [7] Padma, M.M. & Jayalakshmi, M., *Variety of Rational Resolving Sets of Power of a Cycle*, TEST: Engineering and Managements, (July-August), pp. 4162-4167, 2020.
- [8] Raghunath, P., Sooryanarayana, B. & Siddaraju, B., *Metro Domination in Graphs*, International Journal of Mathematics and Computations, 7(10), pp. 147-160, 2010.
- [9] Reshma, Lamani, L.S. & Sooryanarayana, B., *Accurate Neighborhood Resolving Sets of a Graph*, International Journal of Applied Engineering Research, 14(15), pp. 3460-3463, 2019.
- [10] Saenpholphat, V. & Zhang, P., *Connected Resolvability of Graphs*, Czech Math. J., 53, pp. 827-840, 2003.

- [11] Sampathkumar E. & Neeralagi, P.S., *The Independent, Perfect and Connected neighborhood Numbers of a Graph*, J. Comb. Inf. Syst. Sci., 19, pp. 139-145, 1994.
- [12] Shanmukha, B, Sooryanarayana B. & Harinath, K.S., *Metric Dimension of Wheels*, Far East J. Appl. Math., 8(3), pp. 217-229, 2002.
- [13] Silvia, L.S., Sooryanarayana, B. & Hegde, C., *Neighborhood Alliance in Join of a Graph with K_1* , J. Math. Compt. Sci., 11(3), pp. 2624-2649, 2021.
- [14] Slater, P.J., *Fault-Tolerant Locating-Dominating Sets*, Discrete Math., 249, pp. 179-189, 2002.
- [15] Sooryanarayana, B., *On the Metric Dimension of Graph*, Indian J. Pure Appl. Math., 29(4), pp. 413- 415, 1998.
- [16] Sooryanarayana, B., Hebbar, R. & Lamani, L.S., *Accurate Neighborhood Resolving Number of a Graph*, Advances in Mathematics: Scientific Journal, 9(9), pp. 7201-7210, 2020.
- [17] Sooryanarayana, B., Kunikullaya, S. & Swamy, N.N., *Metric Dimension of Generalized Wheels*, Arab J. Math. Sci., 25(2), pp. 131-144, 2019.
- [18] Sooryanarayana, B. & Shanmukha, B., *A Note on Metric Dimension*, Far East J. Appl. Math., 5(3), pp. 331-339, 2001
- [19] Sooryanarayana, B., Suma, A.S. & Chandrakala, S.B., *Certain Varieties of Resolving Sets of a Graph*, J. Indones. Math. Soc., 27(1), pp. 103-114, 2021.
- [20] Suma A.S., Lamani, L.S., Silvia, L.S. & Sooryanarayana, B., *Neighborhood Resolving Sets of a Graph*, International Journal of Applied Engineering Research, 15(8), pp. 778-782, 2020.
- [21] Buckley, F. & Harary, F., *Distance in Graphs*, 3rd ed. Addison-Wesley, 1990.
- [22] Hartsfield, N. & Ringel, G., *Pearls in Graph Theory*, Academic Press, USA, 1994.
- [23] Sampathkumar E. & Neeralagi, P.S., *The Neighborhood Number of a Graph*, Indian J. Pure Appl. Math., 16(2), pp. 126-132, 1985.
- [24] Khuller, S., Raghavachari, B. & Rosenfeld, A., *Land Marks in Graphs*, Disc. Appl. Math., 70, pp. 217-229, 1996.
- [25] Slater, P.J., *Leaves of Trees*, Congr. Numer., 14, pp. 549-559, 1975.
- [26] Harary, F., & Melter, R.A., *On the Metric Dimension of a Graph*, Ars Combin., 2, pp. 191-195, 1976.