Minimal Delaunay Triangulations of Hyperbolic Surfaces

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Abstract

Motivated by recent work on Delaunay triangulations of hyperbolic surfaces, we consider the minimal number of vertices of such triangulations. First, we show that every hyperbolic surface of genus g has a simplicial Delaunay triangulation with $O(g)$ vertices, where edges are given by distance paths. Then, we construct a class of hyperbolic surfaces for which the order of this bound is optimal. Finally, to give a general lower bound, we show that the $\Omega(\sqrt{g})$ lower bound for the number of vertices of a simplicial triangulation of a topological surface of genus g is tight for hyperbolic surfaces as well.

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1 Introduction

The classical topic of Delaunay triangulations has recently been studied in the context of hyperbolic surfaces. Bowyer's incremental algorithm for computing simplicial Delaunay triangulations in the Euclidean plane [5] has been generalized to orientable hyperbolic surfaces and implemented for some specific cases [4, 11]. Moreover, it has been shown that the flip graph of geometric (but not necessarily simplicial) Delaunay triangulations on a hyperbolic surface is connected [7].

In this work, we consider the minimal number of vertices of a simplicial Delaunay triangulation of a closed hyperbolic surface of genus g. Motivated by the interest in embeddings where edges are shortest paths between their endpoints [8, 10], which have applications in for example the field of graph drawing [17], we restrict ourselves to *distance* Delaunay triangulations, where edges are distance paths.

Our main result is the upper bound on the number of vertices with sharp order of growth:

▶ **Theorem 1.** *An orientable closed hyperbolic surface of genus $g \geq 2$ has a distance Delaunay triangulation with at most $O(g)$ vertices. There exists a family of surfaces, X_g, $g \geq 2$, such that the number of vertices of any distance Delaunay triangulation of them grows like $\Omega(g)$.*

The above result is a compilation of Theorems 4 and 18 where explicit upper and lower bounds are given.

Another reason to study triangulations whose edges are distance paths, comes from the study of moduli spaces $\mathcal{M}_g$, which we can think of as a space of all hyperbolic surfaces of genus $g \geq 2$ up to isometry. These spaces admit natural coordinates associated to pants decompositions (the so-called Fenchel-Nielsen coordinates, see Section 2 for details). It is a classical theorem of Bers [2] that any surface admits a short pants decomposition, meaning that the length of each of its simple closed geodesics is bounded by a function that only depends on the topology of the surface (but not its geometry). As these curves provide a local description of the surface, one might hope that they are also geodesically convex, meaning that the shortest distance path between any two points of a given curve is contained in the curve. It is perhaps surprising that most surfaces admit no short pants decompositions with geodesically convex curves. Indeed it is known that *any* pants decomposition of a random surface (chosen with respect to a natural probability measure on $\mathcal{M}_g$) has at least one curve of length on the order of $g^{\frac{1}{6}-\varepsilon}$ as g grows (for any fixed $\varepsilon > 0$) [9]. And it is a theorem of Mirzakhani that these same random surfaces are also of *diameter* on the order of $\log(g)$ [13]. Hence the longest curve of any pants decomposition of a random surface is not convex.

The lengths of edges in a given triangulation are another parameter set for $\mathcal{M}_g$. By the theorem above, such a parameter set can be chosen with a reasonable number of vertices such that the edges are all convex. Using the moduli space point of view, one has a function $\omega : \mathcal{M}_g \to \mathbb{N}$ which associates to a surface the minimal number of vertices of any of its distance Delaunay triangulations. The above result implies that

$$\limsup_{g\to\infty} \max_{X\in\mathcal{M}_g} \frac{\omega(X)}{g}$$

is finite and strictly positive, but for instance we do not know whether the actual limit exists.

The examples we exhibit are geometrically quite simple, as they are made by gluing hyperbolic pants, with bounded cuff lengths, in something that resembles a line as the genus grows. One might wonder whether all surfaces have this property, but we show this is not the case by exploring the quantity $\min_{X\in\mathcal{M}_g} \omega(X)$. This quantity has a precise lower bound on the order of $\Theta(\sqrt{g})$ because we ask that our triangulations be simplicial [12]. We show how to use the celebrated Ringel-Youngs construction [15] to construct a family of hyperbolic surfaces that attain this bound for infinitely many genera (Theorem 26), showing that one cannot hope for better than the simplicial lower bound in general.

This paper is structured as follows. In Section 2, we introduce our notation and give some preliminaries on hyperbolic surface theory and triangulations. In Section 3, we prove our linear upper bound for the number of vertices of a minimal distance Delaunay triangulation. In Section 4, we construct classes of hyperbolic surfaces attaining the order of this linear upper bound. Finally, in Section 5, we construct a family of hyperbolic surfaces attaining the general $\Theta(\sqrt{g})$ lower bound.

2 Preliminaries

We will start by recalling some hyperbolic geometry. There are several models for the hyperbolic plane [1]. In the Poincaré disk model, the hyperbolic plane is represented by the unit disk $\mathbb{D}$ in the complex plane equipped with a specific Riemannian metric of constant Gaussian curvature -1. With respect to this metric, hyperbolic lines, i.e., geodesics are given by diameters of $\mathbb{D}$ or circle segments intersecting $\partial\mathbb{D}$ orthogonally. A hyperbolic circle is a Euclidean circle contained in $\mathbb{D}$. However, in general the centre and radius of a hyperbolic circle are different from the Euclidean centre and radius.

A *hyperbolic surface* is a 2-dimensional Riemannian manifold that is locally isometric to an open subset of the hyperbolic plane [6, 16], thus of constant curvature -1. Our surfaces are assumed throughout to be *closed* and *orientable*, and because they are hyperbolic, via Gauss-Bonnet, their genus g satisfies $g \geq 2$ and their area is $4\pi(g-1)$. Note that we will frequently be interested in subsurfaces of a closed surface which we think of as compact surfaces with boundary consisting of a collection of simple closed geodesics. The signature of such a subsurface is (g', k) where g' is its genus and k is the number of boundary geodesics.

Via the uniformization theorem, any hyperbolic surface X can be written as a quotient space $X = \mathbb{D}/\Gamma$ of the hyperbolic plane under the action of a Fuchsian group Γ (a discrete subgroup of the group of orientation-preserving isometries of $\mathbb{D}$). The hyperbolic plane $\mathbb{D}$ is the *universal cover* of X and is equipped with a projection $\pi : \mathbb{D} \to \mathbb{D}/\Gamma$.

In the free homotopy class of any non-contractible closed curve on a hyperbolic surface lies a unique closed geodesic. If the curve is simple, then the corresponding geodesic is simple, and hence it is a straightforward topological exercise to decompose a hyperbolic surface into $2g-2$ pairs of pants by cutting along $3g-3$ disjoint simple closed geodesics (Figure 1). A *pair of pants* is a surface homeomorphic to a three times punctured sphere but we generally think of its closure, and thus of a hyperbolic pair of pants as being a surface of genus 0 with three simple closed geodesics as boundary, i.e., a surface of signature $(0, 3)$.

It is a short but useful exercise in hyperbolic trigonometry to show that a hyperbolic pair of pants is determined by its three boundary lengths. Hence, the lengths of the $3g-3$ geodesics determine the geometry of each of the $2g-2$ pairs of pants, but to determine X, one needs to add twist parameters that control how the pants are pasted together. How one computes the twist coordinate is at least partially a matter of taste, and although we will not make much use of it, for completeness we follow [6], where the twist is the signed distance between marked points on the boundary curves.

The length and twist parameters determine X and are called Fenchel-Nielsen coordinates. These parameters can be chosen freely in the set $(\mathbb{R}^{>0})^{3g-3} \times \mathbb{R}^{3g-3}$. What they determine is more than just an isometry class of a surface: they determine a *marked* hyperbolic surface, homeomorphic to a base topological surface Σ. As the lengths and twists change, the marked surface changes, and the Fenchel-Nielsen coordinates provide a parameter set for the space of marked hyperbolic surfaces of genus g, called *Teichmüller space* $\mathcal{T}_g$. The underlying *moduli space* $\mathcal{M}_g$ can be thought of as the space of hyperbolic surfaces up to isometry, obtained from $\mathcal{T}_g$ by "forgetting" the marking.

Throughout the paper, lengths of closed geodesics will play an important role. As mentioned above, in the free homotopy class of a non-contractible closed curve lies a unique geodesic representative, and as the metric changes, the length of the geodesic changes, but the free homotopy class does not. Generally we will be dealing with a fixed surface $X \in \mathcal{T}_g$, and the length of a geodesic γ will be denoted by $\ell(\gamma)$. Nonetheless, it is sometimes useful to think of the length of the corresponding homotopy class as a function over $\mathcal{T}_g$ which associates to X the length of the geodesic corresponding to γ.

To a pair of pants decomposition, we associate a 3-regular graph, where each pair of pants is represented by a vertex and two vertices share an edge if the corresponding pairs of pants share a boundary geodesic (see Figure 2). As our parametrization of $\mathcal{T}_g$ depends on a choice of pair of pants decomposition, one can think of the Fenchel-Nielsen coordinates as associating a length and a twist to each edge.

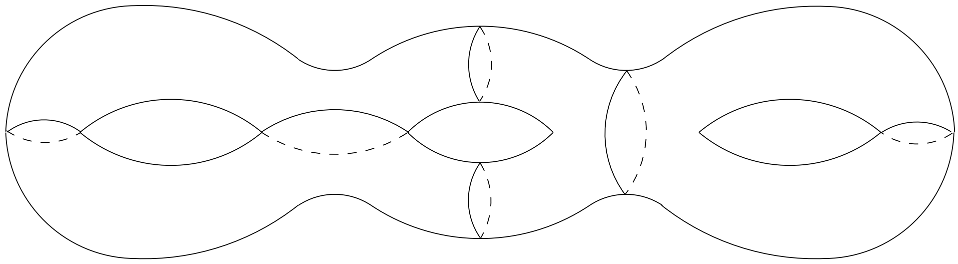

Figure 1 Decomposition of a genus 3 surface into 4 pair of pants using 6 disjoint simple closed geodesics.

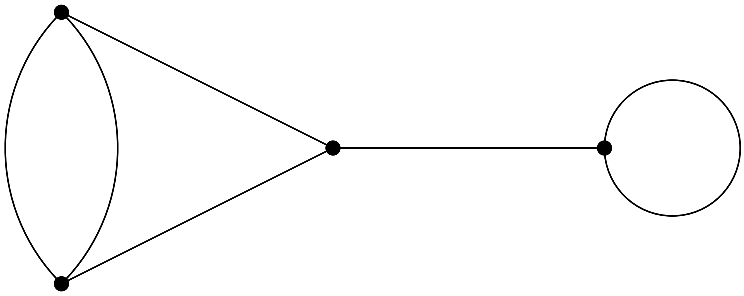

Figure 2 3-regular graph corresponding to the pair of pants decomposition shown in Figure 1.

Around a simple closed geodesic γ, the local geometry of a surface is given by its so-called *collar*. Roughly speaking, for small enough r, the set $C_\gamma(r) = \{x \in X \mid d(x, \gamma) \leq r\}$ is an embedded cylinder. A bound on how large one can take the r to be while retaining the cylinder topology is given by the Collar Lemma:

▶ **Lemma 2** ([6, Theorem 4.1.1]). *Let γ by a simple closed geodesic on a closed hyperbolic surface X. The collar $C_\gamma(w(\gamma))$ of width $w(\gamma)$ given by*

$$w(\gamma) = \operatorname{arcsinh}\left(\frac{1}{\sinh(\frac{1}{2}\ell(\gamma))}\right) \tag{1}$$

is an embedded hyperbolic cylinder isometric to $[-w(\gamma), w(\gamma)] \times \mathbb{S}^1$ with the Riemannian metric $ds^2 = d\rho^2 + \ell^2(\gamma)\cosh^2(\rho)dt^2$ at (ρ, t). Furthermore, if two simple closed geodesics γ and γ' are disjoint, then the collars $C_\gamma(w(\gamma))$ and $C_{\gamma'}(w(\gamma'))$ are disjoint as well.

This paper is about *distance Delaunay triangulations* on closed hyperbolic surfaces.

▶ **Definition 3.** *A* distance Delaunay triangulation *is a triangulation satisfying the following three properties:*

1. *it is a simplicial complex,*
2. *it is a Delaunay triangulation,*
3. *its edges are distance paths.*

The set of all distance Delaunay triangulations of a closed hyperbolic surface X is denoted by $\mathcal{D}(X)$.

We will describe each of these three properties in more detail below.

Simplicial complexes. We will use the standard definition of a simplicial complex. In our case, an embedding of a graph into a surface is a simplicial complex if and only if it does not contain any 1- or 2-cycles. In particular, a geodesic triangulation of a point set in the Euclidean or hyperbolic planes is always a simplicial complex. This is because there are no geodesic monogons or bigons.

Delaunay triangulations. Given a set of vertices in the Euclidean plane, a triangle is called a *Delaunay triangle* if its circumscribed disk does not contain any vertex in its interior. A triangulation of a set of vertices in the Euclidean plane is a *Delaunay triangulation* if all triangles are Delaunay triangles. Using the correspondence between hyperbolic and Euclidean circles, we define Delaunay triangulations in the hyperbolic plane similarly.

Delaunay triangulations on hyperbolic surfaces can be defined by lifting vertices on a hyperbolic surface X to the universal cover $\mathbb{D}$ [3, 7]. More specifically, let $\mathcal{P}$ be a set of vertices on X and let $\pi : \mathbb{D} \to \mathbb{D}/\Gamma$ be the projection of the hyperbolic plane $\mathbb{D}$ to the hyperbolic surface $X = \mathbb{D}/\Gamma$. A triangle (v_1, v_2, v_3) with $v_i \in \mathcal{P}$ is called a *Delaunay triangle* if there exist pre-images $v_i' \in \pi^{-1}(\{v_i\})$ such that the circumscribed disk of the triangle (v_1', v_2', v_3') in the hyperbolic plane does not contain any point of $\pi^{-1}(\mathcal{P})$ in its interior. A triangulation of $\mathcal{P}$ on X is a *Delaunay triangulation* if all triangles are Delaunay triangles.

A Delaunay triangulation of a point set on a hyperbolic surface X is related to a Delaunay triangulation in $\mathbb{D}$ as follows [3]. Given a point set $\mathcal{P}$ on X, we consider a Delaunay triangulation T' of the infinite point set $\pi^{-1}(\mathcal{P})$. Then, we let $T = \pi(T')$. By definition, T is a Delaunay triangulation. Moreover, because every triangulation in $\mathbb{D}$ is a simplicial complex, T' is a simplicial complex. However, T is not necessarily a simplicial complex, because projecting T' to X might introduce 1- or 2-cycles. We will use the correspondence between Delaunay triangulations in $\mathbb{D}$ and in X in Definition 11 and the proof of Theorem 4 and show explicitly that in these cases the result after projecting to X is simplicial.

To make sure that $T = \pi(T')$ is a well-defined triangulation, we will assume without loss of generality that T' is Γ-*invariant*, i.e., the image of any Delaunay triangle in T' under an element of Γ is a Delaunay triangle. Otherwise, it is possible that in so-called *degenerate cases* T contains edges that intersect in a point that is not a vertex [4]. Namely, suppose that T' contains a polygon $P = \{p_1, p_2, \ldots, p_k\}$ consisting of $k \geq 4$ concircular vertices and let T_P be the Delaunay triangulation of P in T'. Because the Delaunay triangulation of a set of at least four concircular vertices is not uniquely defined, assume that there exists $A \in \Gamma$ such that the Delaunay triangulation $T_{A(P)}$ of $A(P)$ in T' is not equal to $A(T_P)$. Because $\pi(P) = \pi(A(P))$, there exists an edge of $\pi(T_{A(P)})$ and an edge of $\pi(A(T_P))$ that intersect in a point that is not a vertex.

Distance paths. Suppose we are given an edge (u, v) in a triangulation of a hyperbolic surface X. Because (u, v) is embedded in X, there exists a geodesic $\gamma : [0, 1] \to X$ with $\gamma(0) = u$ and $\gamma(1) = v$. We say that (u, v) is a *distance path* if $\ell(\gamma) = d(u, v)$, where $d(u, v)$ is the infimum of the lengths of all curves joining u to v.

3 Linear upper bound for the number of vertices of a minimal distance Delaunay triangulation

As our first result, we prove that for every hyperbolic surface there exists a distance Delaunay triangulation with $O(g)$ vertices. Note that the constant 151 is certainly not optimal.

▶ **Theorem 4.** *For every closed hyperbolic surface X of genus g there exists a distance Delaunay triangulation $T \in \mathcal{D}(X)$ with at most $151g$ vertices.*

The idea of the proof is the following. Given a hyperbolic surface X, we construct a vertex set $\mathcal{P}$ on X consisting of at most $151g$ vertices such that the projection T of a Delaunay triangulation of $\pi^{-1}(\mathcal{P})$ in $\mathbb{D}$ to X is a distance Delaunay triangulation of X.

It is known that T is a simplicial complex if $\mathcal{P}$ is sufficiently dense and well-distributed [3]. More precisely, there are no 1- or 2-cycles in T if the diameter of the largest disk in $\mathbb{D}$ not containing any points of $\pi^{-1}(\mathcal{P})$ is less than $\frac{1}{2}\operatorname{sys}(X)$, where $\operatorname{sys}(X)$ is the systole of X, i.e. the length of the shortest homotopically non-trivial closed curve. However, the systole of a hyperbolic surface can be arbitrarily close to zero, which means that we would need an arbitrarily dense set $\mathcal{P}$ to satisfy this condition.

Instead, for a constant $\varepsilon > 0$ we subdivide X into its ε-*thick* part $X^{\varepsilon}_{\text{thick}} = \{x \in X \mid \operatorname{injrad}(x) > \varepsilon\}$ and its ε-*thin* part $X^{\varepsilon}_{\text{thin}} = X \setminus X^{\varepsilon}_{\text{thick}}$, where $\operatorname{injrad}(x)$ is the injectivity radius at x, i.e., the radius of the largest embedded open disk centered at x. Note that the minimum of $\operatorname{injrad}(x)$ over all $x \in X$ is given by $\frac{1}{2}\operatorname{sys}(X)$. We will see in Section 3.1 that, for sufficiently small ε, $X^{\varepsilon}_{\text{thin}}$ is a collection of hyperbolic cylinders (see Figure 3). In these hyperbolic cylinders we want to construct a set of vertices the cardinality of which does not depend on $\operatorname{sys}(X)$. To do this, we put three vertices on the "waist" and each of the two boundary components of the cylinders that are "long and narrow" (for definitions of "waist" and "long and narrow", see Section 3.1). In the cylinders that are not "long and narrow" it suffices to place three vertices on its waist only. Because $\operatorname{injrad}(x) > \varepsilon$ for all $x \in X^{\varepsilon}_{\text{thick}}$, we can construct a sufficiently dense and well-distributed point set in $X^{\varepsilon}_{\text{thick}}$ whose cardinality does not depend on $\operatorname{sys}(X)$ but only on ε. In Section 3.2 we will describe how we combine the vertices placed in the hyperbolic cylinders with the dense and well-distributed set of vertices in $X^{\varepsilon}_{\text{thick}}$. Finally, the proof of Theorem 4 is given in Section 3.3.

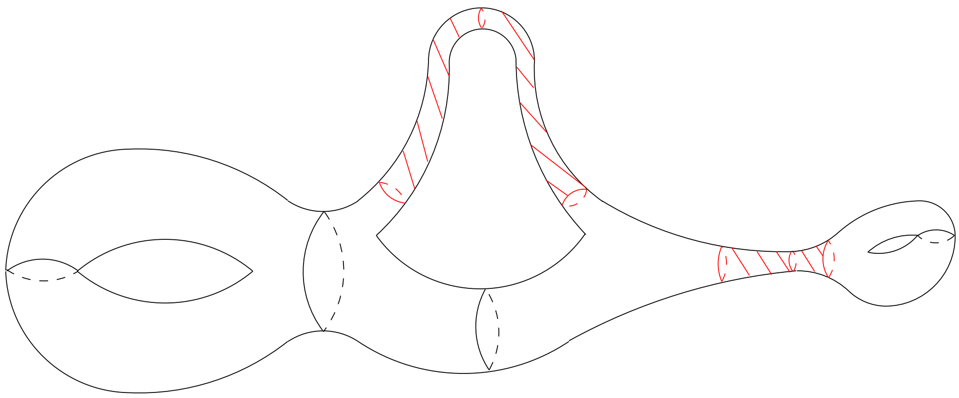

Figure 3 Decomposition of a hyperbolic surface into a thick part consisting of two connected components and two narrow hyperbolic cylinders (in red).

3.1 Distance Delaunay triangulations of hyperbolic cylinders

We now describe our construction of a set of vertices for the ε-thin part $X^{\varepsilon}_{\text{thin}}$ of the hyperbolic surface X. The following lemma describes $X^{\varepsilon}_{\text{thin}}$ in more detail.

▶ **Lemma 5** ([6, Theorem 4.1.6]). *If $\varepsilon < \operatorname{arcsinh}(1)$ then $X^{\varepsilon}_{\text{thin}}$ is a collection of at most $3g-3$ pairwise disjoint hyperbolic cylinders.*

Following the description of the geometry of the hyperbolic cylinders in [14], each hyperbolic cylinder C in $X^{\varepsilon}_{\mathrm{thin}}$ consists of points with injectivity radius at most ε and the boundary curves γ^+ and γ^- consist of all points with injectivity radius equal to ε. Every point on the boundary curves is the base point of an embedded geodesic loop of length 2ε (Figure 4), which is completely contained in the hyperbolic cylinder. All points on the boundary curves have the same distance K_C to a closed geodesic γ (called the *waist* of C), where K_C only depends on ε and the length $\ell(\gamma)$ of γ. To see this, fix a point p on γ^+ and consider a distance path ξ from p to γ (Figure 4). Cutting along γ, ξ and the loop of length 2ε with base point p yields a hyperbolic quadrilateral. The common orthogonal of γ and the geodesic loop subdivides this quadrilateral into two congruent quadrilaterals, each with three right angles. Applying a standard result from hyperbolic trigonometry yields $\sinh(\varepsilon) = \sinh(\frac{1}{2}\ell(\gamma))\cosh(\ell(\xi))$ (see, e.g., [6, Formula Glossary 2.3.1(v)]). Because $K_C = \ell(\xi)$, it follows that

$$K_C = \operatorname{arccosh}\left(\frac{\sinh(\varepsilon)}{\sinh(\frac{1}{2}\ell(\gamma))}\right). \tag{2}$$

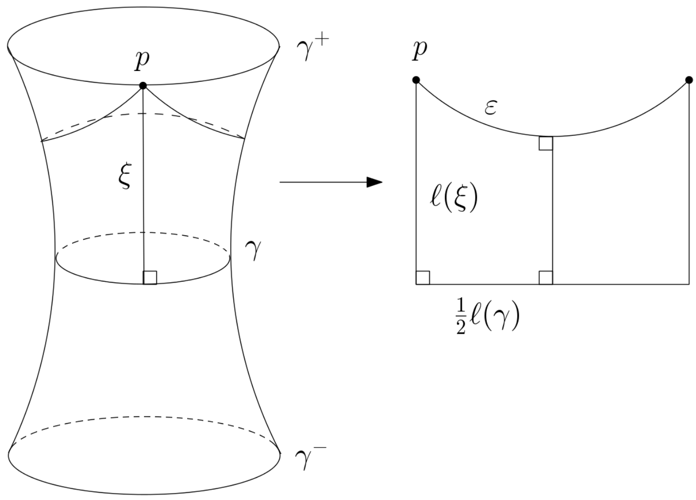

Figure 4 Computing K_C.

We see that γ^+ and, by symmetry, γ^- consist of points that are equidistant to γ with distance K_C. Moreover, γ^+ and γ^- are smooth.

Recall the notion of a collar from Section 2. In particular, each hyperbolic cylinder C in $X^{\varepsilon}_{\mathrm{thin}}$ is a collar of width K_C, i.e., $C = C_\gamma(K_C)$. Comparing equation (2) for K_C with equation (1) in the statement of the Collar Lemma, we see that $w(\gamma) > K_C$, because $\sinh\varepsilon < 1$. This inequality will be used in the proof of Lemma 7 to give a lower bound for the distance between distinct hyperbolic cylinders in $X^{\varepsilon}_{\mathrm{thin}}$.

We distinguish between two kinds of hyperbolic cylinders in $X^{\varepsilon}_{\mathrm{thin}}$, namely ε'-thin cylinders and ε'-thick cylinders, where $\varepsilon' = 0.99\varepsilon$. An ε'-thick cylinder with waist γ satisfies $2\varepsilon' \leq \ell(\gamma) \leq 2\varepsilon$, since γ is contained in the ε-thin part. An ε'-thin cylinder satisfies $\ell(\gamma) < 2\varepsilon'$.

Lemma 14 in Section 3.2 states that the triangulation depicted in Figure 5 is a Delaunay triangulation for ε'-thin cylinders. We call this triangulation a *standard triangulation* and describe it in more detail in the following definition. For ε'-thick cylinders we use a different construction defined in Definition 10.

▶ **Definition 6.** *Let X be a closed hyperbolic surface. Let C be an ε'-thin hyperbolic cylinder in $X^{\varepsilon}_{\text{thin}}$ with waist γ and boundary curves γ^+, γ^-. Place three equally-spaced points $x_i, i = 1, 2, 3$ on γ (see Figure 5). Then, place three points $x_i^+, i = 1, 2, 3$ on γ^+ and three points $x_i^-, i = 1, 2, 3$ on γ^- such that the projection of $x_i^{\pm}$ on γ is equal to x_i for $i = 1, 2, 3$. Let V be the set consisting of x_i, x_i^- and x_i^+ for $i = 1, 2, 3$. Let E be the set of edges of one of the forms $(x_i^-, x_{i+1}^-), (x_i^-, x_i), (x_i^-, x_{i+1}), (x_i, x_{i+1}), (x_i, x_i^+), (x_i, x_{i+1}^+), (x_i^+, x_{i+1}^+)$ for $i = 1, 2, 3$ (counting modulo 3), where the embedding of an edge in C is as shown in Figure 5. We call (V, E) a* standard triangulation *of C.*

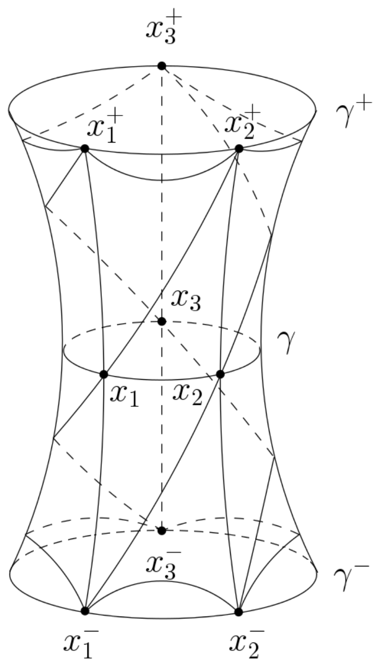

Figure 5 Standard triangulation of an ε'-thin cylinder.

We not only have to prove that a standard triangulation of an ε'-thin cylinder is a Delaunay triangulation, we also have to show that its edges are distance paths. Corollary 9 states that all edges in a standard triangulation are distance paths if $\varepsilon \leq 0.72$. Before we can prove Corollary 9, we first need the following lemma.

▶ **Lemma 7.** *Let X be a closed hyperbolic surface and let $\varepsilon \leq 0.72$. For each pair of distinct closed geodesics γ_1 and γ_2 in $X^{\varepsilon}_{\text{thin}}$ the collars $C_{\gamma_1}(K_{C_1} + \frac{1}{3}\varepsilon)$ and $C_{\gamma_2}(K_{C_2} + \frac{1}{3}\varepsilon)$ are embedded and disjoint.*

▶ Remark 8. The value 0.72 was found experimentally and is optimal up to two decimal digits, i.e., the statement is not true for $\varepsilon = 0.73$. More specifically, if $\varepsilon \geq 0.73$ then there exists a closed hyperbolic surface X with disjoint closed geodesics γ_1 and γ_2 in $X^{\varepsilon}_{\text{thin}}$ such that $C_{\gamma_1}(K_{C_1} + \frac{1}{3}\varepsilon)$ and $C_{\gamma_2}(K_{C_2} + \frac{1}{3}\varepsilon)$ are not disjoint.

Proof. See Figure 6. We will show that $w(\gamma_i) - K_{C_i} \geq \frac{1}{3}\varepsilon$ for $i = 1, 2$. Namely, this implies that $C_{\gamma_i}(K_{C_i} + \frac{1}{3}\varepsilon) \subseteq C_{\gamma_i}(w(\gamma_i))$. Because $C_{\gamma_1}(w(\gamma_1))$ and $C_{\gamma_2}(w(\gamma_2))$ are embedded and disjoint by the Collar Lemma, it follows that $C_{\gamma_1}(K_{C_1} + \frac{1}{3}\varepsilon)$ and $C_{\gamma_2}(K_{C_2} + \frac{1}{3}\varepsilon)$ are embedded and disjoint as well. Comparing expression (2) for K_{C_i} and expression (1) for $w(\gamma_i)$, we see that $w(\gamma_i) - K_{C_i}$ is a positive number, with infimum when $\ell(\gamma_i) \to 0$ [14]. A straightforward computation shows that for $\varepsilon = 0.72$ this infimum is equal to $0.24\ldots > \frac{1}{3}\varepsilon$. Since $w(\gamma_i) - K_{C_i}$ is decreasing as a function of ε, it follows that $w(\gamma_i) - K_{C_i} \geq \frac{1}{3}\varepsilon$ for all $\varepsilon \leq 0.72$. ◀

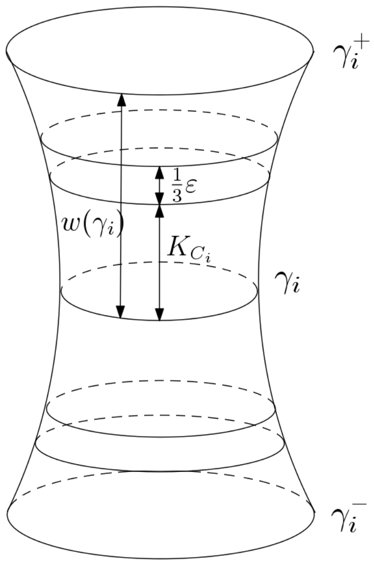

Figure 6 Illustration of the collars $C_{\gamma_i}(K_{C_i}) \subset C_{\gamma_i}(K_{C_i} + \frac{1}{3}\varepsilon) \subseteq C_{\gamma_i}(w(\gamma_i))$.

▶ **Corollary 9** (Proof in full version). *Let X be a closed hyperbolic surface and let $\varepsilon \leq 0.72$. All edges in a standard triangulation of an ε'-thin cylinder in $X^{\varepsilon}_{\text{thin}}$ are distance paths.*

For ε'-thick cylinders, we see from Equation (2) for K_C that the width K_C is close to zero. It turns out that we only need to place three points on its waist.

▶ **Definition 10.** *Let X be a closed hyperbolic surface. Let C be a ε'-thick hyperbolic cylinder in $X^{\varepsilon}_{\text{thin}}$ with waist γ. Place three equally-spaced points $x_i, i = 1, 2, 3$ on γ. Let $V = \{x_i \mid i = 1, 2, 3\}$ and $E = \{(x_1, x_2), (x_2, x_3), (x_3, x_1)\}$. We call (V, E) a* standard cycle *of C.*

3.2 Constructing a distance Delaunay triangulation of X with few vertices

After placing points in $X^{\varepsilon}_{\text{thin}}$, we construct a sufficiently dense and well-distributed set of vertices in the remainder of the surface. The following definition shows more precisely how we construct a set of vertices in $X^{\varepsilon}_{\text{thick}}$ and a corresponding Delaunay triangulation.

▶ **Definition 11.** *Set $\varepsilon = 0.72$ and $\varepsilon' = 0.99\varepsilon$. Let X be a closed hyperbolic surface. Let $\mathcal{P}_1$ be the set consisting of the vertices of a standard triangulation of every ε'-thin cylinder in $X^{\varepsilon}_{\text{thin}}$ together with the vertices of a standard cycle for every ε'-thick cylinder in $X^{\varepsilon}_{\text{thin}}$. Let T_j be the union of triangles in a standard triangulation (V_j, E_j) of an ε'-thin cylinder C_j. For every ε'-thick cylinder C_j, set $T_j = \emptyset$. Define $\mathcal{P}_2$ to be a maximal set in $X \setminus \cup_j T_j$ such that $d(p, q) \geq \frac{1}{2}\varepsilon$ for all distinct $p \in \mathcal{P}_1 \cup \mathcal{P}_2, q \in \mathcal{P}_2$. Denote the union $\mathcal{P}_1 \cup \mathcal{P}_2$ by $\mathcal{P}$ and let T be the Delaunay triangulation of $\mathcal{P}$ on X obtained after projecting a Delaunay triangulation of $\pi^{-1}(\mathcal{P})$ in $\mathbb{D}$ to X. We call T a* thick-thin Delaunay triangulation *of X. The vertices in $\mathcal{P}_1$ and $\mathcal{P}_2$ are called the* cylinder vertices *and* non-cylinder vertices *of T, respectively.*

▶ Remark 12. Because by Corollary 9 all edges in a standard triangulation of any ε'-thin cylinder are distance paths if we choose $\varepsilon \leq 0.72$, we have chosen $\varepsilon = 0.72$ in Definition 11. Namely, we will see in the proof of Theorem 4 that the larger we choose ε, the smaller the constant (in our case 151) in the upper bound for the number of vertices. As in Section 3.1 we will fix $\varepsilon = 0.72$ and $\varepsilon' = 0.99\varepsilon$ throughout this subsection.

The edges between vertices on the same boundary curve of C_j are not equal to the boundary curves of C_j (because the latter are not geodesics), so T_j is strictly contained in C_j. We define $\mathcal{P}_2$ as a point set in $X \setminus \cup_j T_j$ instead of in $X \setminus \cup_j C_j$ to simplify our proof of Lemma 16, where we show that a thick-thin Delaunay triangulation of X is a simplicial complex. The definition of $\mathcal{P}$ does not explicitly forbid placing vertices of $\mathcal{P}_2$ in ε'-thick cylinders. However, we will see in the next lemma that there are no vertices of $\mathcal{P}_2$ in ε'-thick cylinders, because then they would be too close to the vertices of a standard cycle.

▶ **Lemma 13** (Proof in full version)**.** *Let X be a closed hyperbolic surface and let T be a thick-thin Delaunay triangulation of X. Every vertex of T contained in an ε'-thick cylinder in $X^{\varepsilon}_{\text{thin}}$ is a cylinder vertex.*

Even though the set of vertices of a thick-thin Delaunay triangulation of X contains the vertices of a standard triangulation (V_j, E_j) for every ε'-thin cylinder C_j, a priori it is not clear that the edges in E_j are edges in T as well. In the next lemma, we will show that for every ε'-thin cylinder the triangles in a standard triangulation are Delaunay triangles with respect to the set of vertices of any thick-thin Delaunay triangulation of X. Namely, if this holds, then there exists a Delaunay triangulation of $\mathcal{P}$ on X containing a standard triangulation of every ε'-thin cylinder in $X^{\varepsilon}_{\text{thin}}$.

▶ **Lemma 14** (Proof in full version)**.** *Let X be a closed hyperbolic surface. Let T be a thick-thin Delaunay triangulation of X with vertex set $\mathcal{P}$ and let C be an ε'-thin cylinder in $X^{\varepsilon}_{\text{thin}}$ with waist γ. Let (V, E) be a standard triangulation of C such that $V \subset \mathcal{P}$. Then all triangles of (V, E) are Delaunay triangles with respect to the point set $\mathcal{P}$.*

Henceforth, we will assume that for each ε'-thin cylinder the vertices and edges of a standard triangulation are contained in a thick-thin Delaunay triangulation of X. To show that $T \in \mathcal{D}(X)$, we must show that T is a simplicial complex, i.e. it does not contain any 1- or 2-cycles, and that its edges are distance paths. This is stated in Lemma 16, for which we need the following preliminary lemma.

▶ **Lemma 15** (Proof in full version)**.** *Let X be a closed hyperbolic surface and let T be a thick-thin Delaunay triangulation of X. Any edge of T that intersects $X^{\varepsilon}_{\text{thick}}$ has length smaller than ε and is a distance path. Moreover, there are no 1- or 2-cycles that intersect $X^{\varepsilon}_{\text{thick}}$ and consist of edges of length smaller than ε.*

▶ **Lemma 16** (Proof in full version)**.** *Every thick-thin Delaunay triangulation of a closed hyperbolic surface is a distance Delaunay triangulation.*

3.3 Proof of Theorem 4

Proof (Theorem 4). Let X be an arbitrary hyperbolic surface of genus g and let T be a thick-thin Delaunay triangulation of X. By definition, T is a Delaunay triangulation. By Lemma 16, T is a simplicial complex and all edges of T are distance paths. Hence, $T \in \mathcal{D}(X)$.

We will show here that the number of vertices of T is smaller than $151g$. By Lemma 5, $X^{\varepsilon}_{\text{thin}}$ consists of at most $3g-3$ cylinders and each of these cylinders contains either 9 vertices (if it is ε'-thin) or 3 vertices (if it is ε'-thick). Therefore, $|\mathcal{P}_1| \leq 27g - 27$.

To find an upper bound for the cardinality of $\mathcal{P}_2$, observe that for distinct $p, q \in \mathcal{P}_2$ the disks $B_p(\frac{1}{4}\varepsilon)$ and $B_q(\frac{1}{4}\varepsilon)$ of radius $\frac{1}{4}\varepsilon$ centered at p and q, respectively, are embedded and disjoint. Therefore, the cardinality of $\mathcal{P}_2$ is bounded above by the number of disjoint, embedded disks of radius $\frac{1}{4}\varepsilon$ that we can fit in X. Because the area of a hyperbolic disk of radius $\frac{1}{4}\varepsilon$ is $2\pi(\cosh(\frac{1}{4}\varepsilon)-1)$ [1] and because the area of X is $4\pi(g-1)$ [16], we obtain

$$|\mathcal{P}_2| \leq \frac{2(g-1)}{\cosh(\frac{1}{4}\varepsilon)-1}.$$

Combining the upper bounds for $|\mathcal{P}_1|$ and $|\mathcal{P}_2|$ and plugging in $\varepsilon = 0.72$ yields the result. ◀

▶ Remark 17. The constant 151 is not optimal. We can obtain the stronger upper bound $|\mathcal{P}| \leq 124g$ by looking more precisely at the upper bounds of $|\mathcal{P}_1|$ and $|\mathcal{P}_2|$ but because we are mainly interested in the the order of growth, we will not provide any details.

4 Classes of hyperbolic surfaces attaining the order of the upper bound

As our second result, we show that there exists a class of hyperbolic surfaces which attains the order of the upper bound presented in Theorem 4. We will first introduce this class of hyperbolic surfaces and then state the precise result in Theorem 18.

Using the idea of a trivalent graph associated to a pair of pants decomposition discussed in Section 2, define L_g as the trivalent graph depicted in Figure 7. Here, every vertex v_i corresponds to a pair of pants Y_i. There is one edge from v_1 to itself and similarly from v_{2g-2} to itself. Moreover, for $1 \leq i \leq 2g-3$ there is one edge between v_i and v_{i+1} if i is odd and there are two edges if i is even.

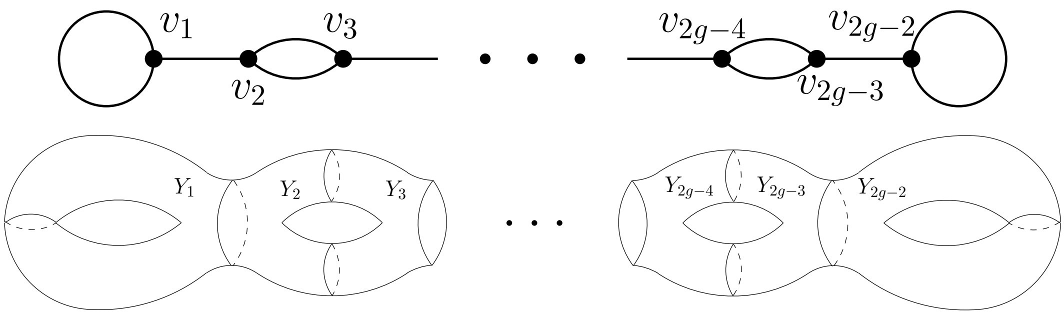

Figure 7 Trivalent graph L_g (top) with corresponding pair of pants decomposition (bottom).

Now, fix some interval $[a, b] \subset \mathbb{R}$ with $0 < a < b$. Let $S_g(a, b)$ be the subset of $\mathcal{T}_g$ with underlying graph L_g such that all length parameters are contained in $[a, b]$. In particular, $S_g(a, b)$ contains an open subset of $\mathcal{T}_g$. The following result thus shows that having a linear number of vertices in terms of the genus is relatively stable in this part of Teichmüller space.

▶ **Theorem 18.** *There exists a constant $B > 0$ depending only on a, b such that a minimal distance Delaunay triangulation of any hyperbolic surface in $S_g(a, b)$ has at least Bg vertices.*

The idea of the proof is the following. Let a hyperbolic surface $X \in S_g(a, b)$ and a triangulation $T \in \mathcal{D}(X)$ be given. Euler's formula implies $v - \frac{1}{3}e = 2 - 2g$ for triangulations of a surface of genus g, where v and e are the number of vertices and edges of the triangulation. We prove that $e \leq B'v$ for some constant $B' > 3$ only depending on a, b, which implies that

$$v \geq \frac{6g-6}{B'-3}.$$

This implies the result of Theorem 18. Hence, the argument consists mostly in finding an upper bound for the number of edges in terms of the number of vertices.

Before we continue with the proof of Theorem 18, we will look at our construction of $S_g(a,b)$ in more detail. By definition, every boundary geodesic of a pair of pants in the pair of pants decomposition of $X \in S_g(a,b)$ with respect to L_g has length in $[a,b]$. As explained in Section 2, the geometry of a pair of pants depends continuously on the lengths of its three boundary geodesics. In particular, the diameter $\mathrm{diam}(Y)$ of a pair of pants Y as well as the minimal distance $\mathrm{mindist}(Y)$ between any two of its boundary geodesics depend continuously on the lengths of its boundary geodesics. Because $[a,b]$ is a compact set, we obtain as an immediate consequence the following lemma.

▶ **Lemma 19.** *There exist positive numbers $m(a,b)$ and $M(a,b)$ depending on a and b such that $m(a,b) \le \mathrm{mindist}(Y) < \mathrm{diam}(Y) \le M(a,b)$ for every pair of pants Y whose boundary geodesics have length in $[a,b]$.*

▶ Remark 20. It is not too difficult to compute bounds for $\mathrm{mindist}(Y)$ and $\mathrm{diam}(Y)$ in terms of the lengths of the boundary geodesics of Y. This would give explicit expressions for $m(a,b)$ and $M(a,b)$ in terms of a and b. As we are only interested in the order of growth, to avoid further technical details, we do not provide details.

From now on, the numbers $m = m(a,b)$ and $M = M(a,b)$ will be fixed. Furthermore, a *cluster* in a hyperbolic surface X is a subset of X consisting of a number of consecutive pairs of pants, where *consecutive* is with respect to the ordering of L_g. Consider $T \in \mathcal{D}(X)$. A *k-gap* is a cluster consisting of k consecutive empty pairs of pants, where *empty* means that the pairs of pants do not contain any vertices of T. If a vertex of T is contained in two pairs of pants, i.e., if the vertex lies on a boundary geodesic, then we only count it as a vertex of the pair of pants with the lowest index in L_g. We say that an edge of T *crosses* a cluster if the pairs of pants containing its endpoints are separated by the cluster. Note that the cluster need not contain all pairs of pants which separate the two endpoints.

The next lemma states that if an edge of a distance Delaunay triangulation crosses many pairs of pants, then "many" of these pairs of pants are empty.

▶ **Lemma 21** (Proof in full version)**.** *Let $X \in S_g(a,b)$ and define $N = N(a,b)$ as*

$$N(a,b) := \left\lceil \frac{M(a,b)}{m(a,b)} \right\rceil + 1.$$

Then, for every $T \in \mathcal{D}(X)$ the following statements hold:

1. *If an edge of T crosses a cluster consisting of at least $3N$ pairs of pants, this cluster contains an N-gap.*
2. *If an edge of T crosses a cluster in which the first N and the last N pairs of pants are empty, then all pairs of pants in the cluster are empty.*

The following lemma states that we can construct a set of clusters which has as one of its properties that every edge of the distance Delaunay triangulation has its endpoints in the same cluster, or in two consecutive clusters.

▶ **Lemma 22** (Proof in full version)**.** *Let $X \in S_g(a,b)$ be a hyperbolic surface and let $N = N(a,b)$ be as defined in Lemma 21. Let $T \in \mathcal{D}(X)$. There are interior-disjoint clusters with the following properties:*

1. *Each cluster consists of at most $6N$ consecutive pairs of pants;*
2. *Every cluster contains at least one vertex of $\mathcal{T}$, and every vertex of $\mathcal{T}$ belongs to exactly one cluster;*
3. *Every edge of $\mathcal{T}$ has its endpoints in the same cluster, or in two consecutive clusters.*

In the following corollary, we denote the number of vertices of $T \in \mathcal{D}(X)$ contained in a subset U of X by $v(U)$. Likewise, let $e(U, W)$ be the number of edges of T with one endpoint in $U \subset X$ and one endpoint in $W \subset X$.

▶ **Corollary 23.** *Let $X \in S_g(a, b)$ be a hyperbolic surface and let $T \in \mathcal{D}(X)$. Let $\{\Gamma_i \mid i = 1, \ldots, n\}$ be a collection of clusters satisfying the properties of Lemma 22 for some $n \in \mathbb{N}$. If v and e are the number of vertices and edges of T, respectively, then*

$$n \leq v,$$
$$v = \sum_{i=1}^{n} v(\Gamma_i),$$
$$e = \sum_{i=1}^{n} e(\Gamma_i, \Gamma_i) + \sum_{i=1}^{n-1} e(\Gamma_i, \Gamma_{i+1}).$$

Proof. Because every cluster contains at least one vertex, the number of clusters is at most the number of vertices, which proves the first equation. The second equation follows from the property that every vertex is contained in a cluster. Because every edge has its endpoints in the same cluster, or in two consecutive clusters, the third equation holds. ◀

Recall that we want to find a linear upper bound for the number of edges of a distance Delaunay triangulation in terms of the number of vertices. By Corollary 23, it suffices to find upper bounds for $e(\Gamma_i, \Gamma_i)$ and $e(\Gamma_i, \Gamma_{i+1})$ for clusters Γ_i satisfying the properties of Lemma 22. We will do this in the next lemma.

▶ **Lemma 24** (Proof in full version). *With notation as in Corollary 23, the following upper bounds hold:*

1. $e(\Gamma_i, \Gamma_i) \leq 3v(\Gamma_i) + 18N(N + 1)$ *for all* $i = 1, \ldots, n$,
2. $e(\Gamma_i, \Gamma_{i+1}) \leq 18v(\Gamma_i \cup \Gamma_{i+1}) + 216N(N + 1)$ *for all* $i = 1, \ldots, i - 1$.

We can now commence with the proof of Theorem 18.

Proof (Theorem 18). Take $X \in S_g(a, b)$ arbitrary and let $T \in \mathcal{D}(X)$ be arbitrary. Let $\{\Gamma_i \mid i = 1, \ldots, n\}$ be a collection of clusters satisfying the properties of Lemma 22. By Corollary 23,

$$e = \sum_{i=1}^{n} e(\Gamma_i, \Gamma_i) + \sum_{i=1}^{n-1} e(\Gamma_i, \Gamma_{i+1}).$$

Substituting the upper bounds for $e(\Gamma_i, \Gamma_i)$ and $e(\Gamma_i, \Gamma_{i+1})$ from Lemma 24, we obtain

$$e \leq 39 \sum_{i=1}^{n} \big(v(\Gamma_i) + 6N(N + 1)\big).$$

From Corollary 23, we know that $\sum_{i=1}^{n} v(\Gamma_i) = v$ and $n \leq v$. Hence, $e \leq 39(1 + 6N(N + 1))v$. Euler's formula for triangulations $v - \frac{1}{3}e = 2 - 2g$ implies that

$$39(1 + 6N(N + 1))v \geq e = 3v + 6g - 6,$$
$$v \geq \frac{g - 1}{6 + 39N(N + 1)},$$

which finishes the proof. ◀

5 Lower bound

In this section, we will look at a general lower bound for the minimal number of vertices of a distance Delaunay triangulation of a hyperbolic surface of genus g.

In the more general situation of a simplicial triangulation of a topological surface of genus g, one has an immediate lower bound on the minimal number of vertices. The fact that this lower bound is sharp is the following classical theorem of Jungerman and Ringel:

▶ **Theorem 25** ([12, Theorem 1.1])**.** *The minimal number of vertices of a simplicial triangulation of a topological surface of genus g is*

$$\left\lceil \frac{7+\sqrt{1+48g}}{2} \right\rceil.$$

We show that the same result holds for the minimal number of vertices of a distance Delaunay triangulation of a hyperbolic surface of genus g for infinitely many values of g.

▶ **Theorem 26.** *For any $g \geq 2$ of the form $g = \frac{1}{12}(n-3)(n-4)$ for some $n \equiv 0 \mod 12$, the minimal number of vertices of a distance Delaunay triangulation of a hyperbolic surface of genus g is*

$$n = \frac{7+\sqrt{1+48g}}{2}.$$

Proof. Because there are no distance Delaunay triangulations with fewer than the stated number of vertices by Theorem 25, it is sufficient to construct for a given hyperbolic surface a distance Delaunay triangulation with the stated number of vertices.

Our construction is inspired by a similar construction in the context of the chromatic number of hyperbolic surfaces [14]. Let $n \equiv 0 \mod 12$ and assume that $n \neq 0$. The complete graph K_n on n vertices can be embedded in a topological surface S_g of genus $g = \frac{1}{12}(n-3)(n-4)$ which is the smallest possible genus [15]. Because we have assumed that $n \equiv 0 \mod 12$, we know that the embedding of K_n into S_g is a triangulation T [18]. To turn T into a distance Delaunay triangulation, we will add a hyperbolic metric to the topological surface as follows. Every triangle in T is replaced by the unique equilateral hyperbolic triangle with all three angles equal to $\frac{2\pi}{n-1}$. In the complete graph K_n every vertex has $n-1$ neighbouring vertices. This means that in every vertex $n-1$ equilateral triangles meet, so the total angle at each vertex is 2π. Therefore, the result after replacing all triangles in T by hyperbolic triangles is a smooth hyperbolic surface Z_g.

It remains to be shown that $T \in \mathcal{D}(Z)$. By construction, T is a simplicial complex. It has also been shown that all edges are distance paths [14]. We will show here that T is a Delaunay triangulation of Z_g. Consider an arbitrary triangle (u, v, w) in T with circumcenter c and let $p \notin \{u, v, w\}$ be an arbitrary vertex of T (Figure 8). Consider a distance path γ from c to p. We can regard γ as the concatenation of simple segments that each pass through an individual triangle.

The first of these simple segments starts from c and leaves the triangle (u, v, w), so its length is at least the distance between c and a side of (u, v, w). Therefore, denoting by x the projection of c on one of the edges as shown in Figure 8, the length of the first segment is at least $d(c, x)$. The last of the simple segments passes through a triangle, say Δ, before arriving at p, so it has to pass through the side of Δ opposite to p. Therefore, its length is at least the distance between p and the opposite side of Δ. It is known that the distance between a vertex and the opposite side of an equilateral triangle is at least $\frac{1}{2}\ell$, where ℓ denotes the length of the sides of the equilateral triangle [14]. Hence, $d(c, p) = \ell(\gamma) \geq d(c, x) + \frac{1}{2}\ell$. By the

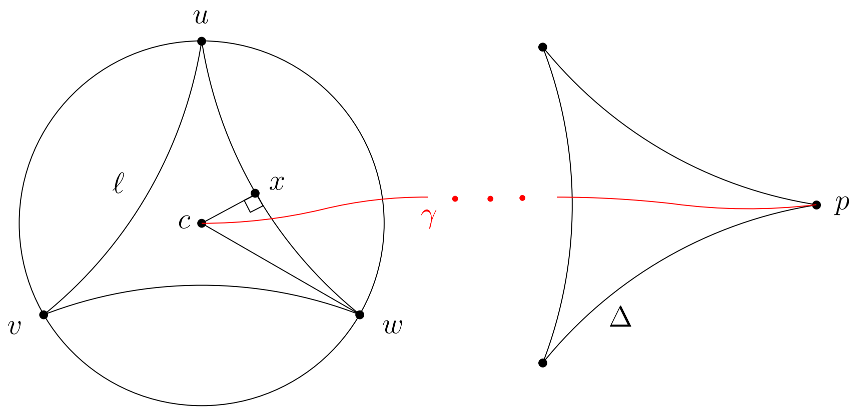

Figure 8 Schematic overview of the proof of T being a Delaunay triangulation.

triangle inequality in triangle (c, w, x) we see that $d(c, w) \leq d(c, x) + d(x, w) = d(c, x) + \frac{1}{2}\ell$, so we conclude that $d(c, p) \geq d(c, w)$. This means that p is not contained in the interior of the circumcircle of (u, v, w), which shows that (u, v, w) is a Delaunay triangle. By symmetry, it follows that all triangles are Delaunay triangles, which finishes the proof. ◀

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