Escaping the Curse of Spatial Partitioning: Matchings with Low Crossing Numbers and Their Applications

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Abstract

Given a set system $(X, \mathcal{S})$, constructing a matching of X with low crossing number is a key tool in combinatorics and algorithms. In this paper we present a new sampling-based algorithm which is applicable to finite set systems. Let $n = |X|$, $m = |\mathcal{S}|$ and assume that X has a perfect matching M such that any set in $\mathcal{S}$ crosses at most $\kappa = \Theta(n^\gamma)$ edges of M. In the case $\gamma = 1 - 1/d$, our algorithm computes a perfect matching of X with expected crossing number at most 10κ, in expected time $\tilde{O}\left(n^{2+2/d} + mn^{2/d}\right)$.

As an immediate consequence, we get improved bounds for constructing low-crossing matchings for a slew of both abstract and geometric problems, including many basic geometric set systems (e.g., balls in $\mathbb{R}^d$). This further implies improved algorithms for many well-studied problems such as construction of ϵ-approximations. Our work is related to two earlier themes: the work of Varadarajan (STOC '10) / Chan et al. (SODA '12) that avoids spatial partitionings for constructing ϵ-nets, and of Chan (DCG '12) that gives an optimal algorithm for matchings with respect to hyperplanes in $\mathbb{R}^d$.

Another major advantage of our method is its simplicity. An implementation of a variant of our algorithm in `C++` is available on Github; it is approximately 200 lines of basic code without any non-trivial data-structure. Since the start of the study of matchings with low-crossing numbers with respect to half-spaces in the 1980s, this is the first implementation made possible for dimensions larger than 2.

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Supplementary Material *Software (Source Code)*:
`https://github.com/csikosm/LowCrossingMatchings`
archived at `swh:1:dir:527290e997b57b959e879363b26dd9218906f3a8`

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1 Introduction

Given a set system $(X, \mathcal{S})$, we say that a set $S \in \mathcal{S}$ *crosses* a pair $\{x, y\} \subseteq X$ iff $|S \cap \{x, y\}| = 1$. Define the *crossing number* of a perfect matching (resp. a spanning tree) G of X with respect to $\mathcal{S}$ as the maximum number of edges of G crossed by any $S \in \mathcal{S}$. The focus of this paper is on constructing perfect matchings of X with low crossing numbers with respect to $\mathcal{S}$.

Matchings with low crossing numbers were originally introduced by Welzl [31, 32] for the special case where X is a set of points in $\mathbb{R}^d$ and $\mathcal{S}$ is induced on X by half-spaces. His result was then generalized by Chazelle and Welzl [10] to a broader class of set systems, which together with an improvement due to Haussler [20], gives the following general theorem.

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▶ **Theorem A.** *Let $(X, \mathcal{S})$ be a set system with $n = |X|$, and dual shatter function*[1] $\pi^*_{\mathcal{S}}(k) = O(k^d)$. *Then there exists a perfect matching of X with crossing number $O\left(n^{1-1/d}\right)$.*

Previous constructions. Let $(X, \mathcal{S})$ be a set system with $n = |X|$, $m = |\mathcal{S}|$, and let κ denote the smallest integer such that X has a perfect matching (resp. spanning tree) with crossing number κ with respect to $\mathcal{S}$. We review previous constructions in two separate settings.

- **Abstract set systems.** The original method of Welzl [31, 32, 10] builds a perfect matching using the multiplicative weight update (MWU) method. Briefly, the algorithm maintains a weight function π on $\mathcal{S}$, with initial weights set to 1. It selects edges iteratively, always choosing an edge that is guaranteed to be crossed by sets of low total weight in π; it then updates π based on the chosen edge. The algorithmic bottleneck is in finding such an edge: for an abstract set system without additional structure, this takes $O(n^2 m)$ time for each of the $n/2$ iterations.
  Another approach for the abstract case was proposed by Har-Peled [19] (see also [14]). His result implies that if $\kappa = \Theta(n^\gamma)$ for some $\gamma \in [1/\log n, 1]$, then a spanning tree of crossing number $O(\kappa/\gamma)$ can be found by solving an LP on $\binom{n}{2}$ variables and $m + n$ constraints. Combining this with an efficient approximate LP solver (e.g., [11]) leads to a randomized $\tilde{O}(mn^2)$ time algorithm. The approximation factor can be further improved using a general framework of rounding fractional solutions of minimax integer programs with matroid constraints. This method gives a randomized algorithm that constructs a spanning tree with expected crossing number at most $\kappa + O(\sqrt{\kappa \log m})$ in time $\tilde{O}(mn^4 + n^8)$ [12].
- **Geometric set systems.** Now we turn to the case where X is a set of n points in $\mathbb{R}^d$ and $\mathcal{S}$ consists of subsets of X that are induced by geometric objects. In this setting, improved bounds are made possible using spatial partitioning. The current-best algorithms for geometric set systems induced by half-spaces recursively construct simplicial partitions, stored in a hierarchical structure called the partition tree, which then at its base level gives a matching with low crossing number. This approach is used in the breakthrough result of Chan [8] who gave an $O(n \log n)$ time algorithm to build partition trees with respect to half-spaces in $\mathbb{R}^d$, which then implies the same for computing matchings with crossing number $O(n^{1-1/d})$.

While the use of cuttings – and more generally, spatial partitioning – gives $o(mn^2)$ running times, progress remains blocked in several ways:

a) Simplicial partitions only exist in certain geometric settings. Indeed, as shown by Alon et al.[6], they do not always exist in settings satisfying the requirements of Theorem A (e.g., the projective plane). Furthermore, spatial partitioning is not possible when dealing with abstract set systems such as those arising in graph theory or learning theory.

b) Optimal bounds for constructing simplicial partitions are only known for the case of half-spaces; this is one of the main problems left open by Chan [8]. Despite a series of research for semi-algebraic set systems (using linearization, cuttings, and more recently, polynomial partitioning [3]), the bounds are still sub-optimal for polynomials of degree larger than 2, with exponential dependence on the dimension.

c) There are large constants in the asymptotic notation depending on the dimension d both in the running time as well as the crossing number bounds, due to the use of cuttings (see [13]). For instance, in Chan's algorithm the constants are quite large – Theorem 3.2 [8]

[1] The *dual shatter function* $\pi^*_{\mathcal{S}}$ of $(X, \mathcal{S})$ is defined as follows. For any $k \leq |\mathcal{S}|$, $\pi^*_{\mathcal{S}}(k)$ is the maximum number of equivalence classes on X defined by a k-element subfamily $\mathcal{R} \subseteq \mathcal{S}$, where $x, y \in X$ are equivalent with respect to $\mathcal{R}$ if x belongs to the same sets of $\mathcal{R}$ as y.

requires $\delta \leq \frac{1}{d^2}$, $b = 22$ (see [21]), which then implies that it constructs a spanning tree with a guaranteed crossing number no better than $12 \cdot 22 \cdot d^4 n^{1-1/d}$; this is at least $20000 \cdot n^{1-1/d}$ even for $d = 3$. Furthermore, the actual construction running time is at least $264 \cdot d^2 n \log n$, not counting the typically large constants in the several complex data structures that the algorithm needs (simplex range searching in $\mathbb{R}^d$ with dynamic insertion; see [21] for a discussion of its practical aspects in $\mathbb{R}^2$).

d) Practical implementation of spatial partitioning in $\mathbb{R}^d$, $d > 2$, even cuttings for hyperplanes, remains an open problem in geometric computing. Cuttings have been implemented in the planar case [18], which have then been used recently for computing ε-approximations w.r.t. half-spaces in $\mathbb{R}^2$ [21]. In particular, for $d > 2$, we know of no implementations for low-crossing matchings.

Recently there have been algorithms proposed for ε-nets and ε-approximations that avoid spatial partitioning [29, 9, 27, 26]. Our work can be considered another step along this theme.

2 Our Results

We state our main result assuming that we have access to a membership Oracle of $(X, \mathcal{S})$, which for a given element $x \in X$ and a set $S \in \mathcal{S}$ returns whether $x \in S$. Our main result is the following.

▶ **Theorem 1.** *Let $(X, \mathcal{S})$ be a set system, $n = |X|$, $m = |\mathcal{S}|$ with $m \geq n$. Let $a > 0$, b and $\gamma \in [1/\log n, 1]$ be constants such that any $Y \subseteq X$ has a perfect matching with crossing number at most $a|Y|^\gamma + b$. Then* BuildMatching$\big((X, \mathcal{S}), a, b, \gamma\big)$ *computes a perfect matching of X with expected crossing number at most $\frac{5a}{\gamma} n^\gamma + (3b + 8 \ln m) \log n$, and with an expected $O\left(\min\left\{n^{4-2\gamma} \ln n + mn^{2-2\gamma} \ln m \ln n,\ n^3 + mn\right\}\right)$ calls to the membership Oracle of $(X, \mathcal{S})$.*

Remarks:

- The algorithm BuildMatching is presented in Section 3.
- Our method can easily be modified to construct a spanning tree or a spanning path with the same guarantees up to a constant factor.

Now we give a list of consequences of Theorem 1, divided into three topics. All stated crossing number and running time bounds are in expectation.

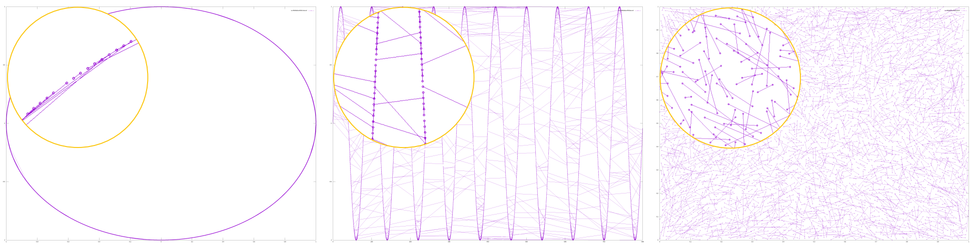
1. **Low-crossing matchings.** Our results improve upon several previous constructions, see Table 1 (the precise guarantees and their proofs are presented in Section 4). For abstract set systems with dual shatter function $\pi^*_{\mathcal{S}}(k) = O(k^d)$, we improve the running time from $\tilde{O}\left(mn^2\right)$ to $\tilde{O}\left(mn^{2/d} + n^{2+2/d}\right)$. This further implies a sub-cubic time construction for matchings with asymptotically-optimal crossing number with respect to balls in $\mathbb{R}^d$ for $d \geq 3$. For set systems induced by semialgebraic sets in $\mathbb{R}^d$ (each set defined by at most s polynomial inequalities of degree at most Δ), we significantly improve the crossing number guarantee by removing the exponential dependence on d. However in contrast to the previous best algorithm for this setup [3], our running time depends on m.
   Importantly, our method does not use spatial partitioning, which makes it possible to handle abstract set-systems, and geometric set systems in $\mathbb{R}^d$ (not only in $\mathbb{R}^2$) without additional complications.
2. **Practical aspects.** Our algorithm consists of $\frac{n}{2}$ iterations, where each iteration adjusts the weight of a random subset of $\binom{X}{2}$ and $\mathcal{S}$ and adds a randomly picked edge to the matching. The only black-box needed is the membership Oracle that returns for a given

Table 1 Summary of our results for set systems $(X, \mathcal{S})$ with $n = |X|$, $m = |\mathcal{S}|$, $n \le m$, and $d \ge 2$. We use the notation $\pi^*_{\mathcal{S}}(\cdot)$ for the dual shatter function of $(X, \mathcal{S})$, $\mathcal{H}_d$ for half-spaces in $\mathbb{R}^d$, $\mathcal{B}_d$ for balls in $\mathbb{R}^d$, and $\Gamma_{d,\Delta,s}$ for semialgebraic ranges in $\mathbb{R}^d$ described by at most s equations of degree at most Δ (see Sec. 4).

	Matchings / Spanning trees Our method		Previous-best	
Set system	Crossing number	time	Crossing number	time
arbitrary with $\pi^*_{\mathcal{S}}(k) \le ck^d$	$\left(\frac{5c^{1/d}d}{d-1} + o(1)\right) n^{1-1/d}$	$\boldsymbol{\tilde{O}\left(mn^{2/d}+n^{2+2/d}\right)}$ (Corollary 12)	$O\left(n^{1-1/d}\right)$	$\tilde{O}(mn^2)$ [19, 11]
geometric induced by $\mathcal{B}_d$	$\left(6d^2 + o(d^2)\right) \cdot n^{1-1/d}$	$\boldsymbol{\tilde{O}\left(dn^{2+2/d}\right)}$ (Corollary 18)	$O\left(n^{1-1/d}\right)$	$O\left(n^{3+1/d}\right)$ [19, 11]
geometric induced by $\Gamma_{d,\Delta,s}$	$\frac{20e\Delta sd}{d-1} n^{1-1/d} + o\left(n^{1-1/d}\right)$	$\boldsymbol{\tilde{O}\left(s\Delta^d\left(mn^{2/d}+n^{2+2/d}\right)\right)}$ (Corollary 14)	$O\left(10^d s\Delta n^{1-1/d}\right)$	$O\left(n^{O(d^3)}\right)$ [3]
			$O\left(\Delta s n^{1-1/d}\right)$	$\tilde{O}(s\Delta^d mn^2)$ [19, 11]
geometric induced by $\mathcal{H}_d$	$\left(6d^2 + o(d^2)\right) \cdot n^{1-1/d}$	$\tilde{O}\left(dn^{2+2/d}\right)$ (Corollary 16)	$\ge 264d^4n^{1-1/d}$	$\tilde{O}(d^2n)$ [8]

$x \in X$ and $S \in \mathcal{S}$, if $x \in S$. The time complexity of this operation depends on the precise way $(X, \mathcal{S})$ is given; typically this is independent of $|X|$ and $|\mathcal{S}|$ (using indexing, hashing). A preliminary multi-threaded implementation of a variant of our algorithm in `C++` for set systems induced on points by half-spaces in $\mathbb{R}^d$ is available on Github. It is approximately 200 lines of basic code without any non-trivial data-structures, being the first such implementation for $d > 2$.

The figures below show the matchings with respect to half-planes returned by our algorithm for $5,000$ points in $\mathbb{R}^2$ uniformly placed on a circle (in 17.39s), sine curve (in 17.17s), and randomly perturbed in a uniform grid (in 17.41s), each with a zoomed-in region. We find it surprising that our method, that is based only on random sampling, gives a matching that adapts so well to each specific instance.



This makes progress towards the goals expressed at the end of the survey on range searching and its applications [1]: "...an interesting open question is to develop simple data structures that work well under some assumptions on input points and query ranges".

3. **Discrepancy and approximations.** By plugging in various upper-bounds on crossing numbers given by Theorem 1 and using techniques in Matoušek et al.[25], we immediately get improved construction bounds for discrepancy and ε-approximations. In particular, if d is a constant such that $(X, \mathcal{S})$ has dual shatter function $\pi^*_{\mathcal{S}}(k) = O(k^d)$, then we improve the running time of computing colorings with expected discrepancy $O\left(\sqrt{n^{1-1/d} \ln m}\right)$ from $O(mn^2)$ to $\tilde{O}(mn^{2/d}+n^{2+2/d})$. Moreover if in addition, $(X, \mathcal{S})$ has VC dimension bounded by a constant $D \ge 2$, then our method can be used to compute an ε-approximations of

expected size $\tilde{O}\left(\left(\frac{D}{\varepsilon^2}\right)^{\frac{d}{d+1}}\right)$ in expected time $\tilde{O}\left(n+\left(D\left(\frac{D}{\varepsilon^2}\right)^{D+2/d}\right)\right)$, improving upon the previous-best time $O\left(n+\left(\frac{D}{\varepsilon^2}\right)^{D+2}\right)$. As these are standard applications of matchings with low crossing number, the proofs are omitted (see the survey [28]).

Organization. In Section 3, we describe our algorithm and prove Theorem 1. In Section 4, we show how Theorem 1 implies the bounds stated in Table 1. In Section 5, we present our experiments. In Section 6, we give an application in learning theory.

3 Proof of Theorem 1

The proof rests on the following three key ideas:

1. We replace the bottleneck algorithmic step of finding a light edge in the multiplicative weights update technique by simply sampling an edge according to a carefully maintained distribution. In particular, we maintain weights not only on the sets in $\mathcal{S}$, but also on $\binom{X}{2}$. At each iteration we sample an edge e and a set S according to the current weights. Then we add e to our matching and update the weights by *doubling* the weight of each set that crosses e and *halving* the weight of each edge that is crossed by S. The idea of maintaining "primal-dual" weights has been used earlier to approximately solve matrix games [17] and in geometric optimization [4].
2. In our case, the process is more elaborate as we are constructing a matching M at the same time as reweighing. Therefore, at the end of each iteration, as we add e to M, we are forced to set the weights of e and all edges adjacent to e to 0. This breaks down the reweighing scheme, as the removal of the edges amplifies the error introduced in later iterations and thus our maintained weights degrade over time. However, we prove that restarting the algorithm by "resetting" all the weights a logarithmic number of times suffices to ensure the required low crossing numbers.
3. This still does not get us to our goal as updating the weights of *all* edges and sets crossing the randomly picked set and edge would be too expensive. Instead, we show that if $\gamma \geq 1/2$, then updating the weights of a *uniform* sample of $\tilde{O}\left(n^{3-2\gamma}\right)$ edges and $\tilde{O}(mn^{1-2\gamma})$ sets at each iteration is sufficient for our purposes.

The main algorithm BuildMatching is given below, followed by the presentation of the subroutine MatchHalf.

Algorithm 1 BuildMatching$\big((X, \mathcal{S}), a, b, \gamma\big)$.

```
M ← ∅
while |X| ≥ 4 do
    M′ ← MatchHalf((X, S), a|X|^γ + b)      // M′ covers |X|/2 elements
    M ← M ∪ M′
    X ← X \ vertices(M′)                    // remove elements covered by M′
M ← M ∪ {edge connecting the remaining two elements of X}
return M
```

Algorithm 2 $\textsc{MatchHalf}\big((X,\mathcal{S}),\kappa\big)$.

```
ω_1(e) ← 1,   π_1(S) ← 1      ∀e ∈ E, S ∈ 𝒮                  // E denotes (X choose 2)
p ← min{106 · |X|/κ² · ln(|E| · |X|/4), 1}
q ← min{39 · |X|/κ² · ln(|𝒮| · |X|/4), 1}
for i = 1, ..., |X|/4 do
    ω_i(E) ← Σ_{e∈E} ω_i(e)
    π_i(𝒮) ← Σ_{S∈𝒮} π_i(S)
    choose e_i ~ ω_i                        // ℙ[e_i = e] = ω_i(e)/ω_i(E)   ∀e ∈ E
    choose S_i ~ π_i                        // ℙ[S_i = S] = π_i(S)/π_i(𝒮)   ∀S ∈ 𝒮
    E_i ← sample from E with probability p  // ℙ[e ∈ E_i] = p   ∀e ∈ E
    𝒮_i ← sample from 𝒮 with probability q  // ℙ[S ∈ 𝒮_i] = q   ∀S ∈ 𝒮
                       // I(e,S) = 1 if e crosses S, I(e,S) = 0 otherwise
    for e ∈ E_i do
        ω_{i+1}(e) ← ω_i(e)(1 − ½ I(e, S_i))      // halve weight if S_i crosses e
    for S ∈ 𝒮_i do
        π_{i+1}(S) ← π_i(S)(1 + I(e_i, S))        // double weight if S crosses e_i
    set the weight in ω_{i+1} of e_i and of each edge adjacent to e_i to zero
return {e_1, ..., e_{|X|/4}}
```

Proof of Theorem 1. Later, we will prove the following statement for $\textsc{MatchHalf}$.

▶ **Theorem 2.** *Let $(X,\mathcal{S})$ be a set system, $n=|X|$, $m=|\mathcal{S}|$ with $m\ge n$, and let κ be such that any $Y\subseteq X$ of size $|Y|=n/2$ has a perfect matching of crossing number at most κ with respect to $\mathcal{S}$. Then $\textsc{MatchHalf}\big((X,\mathcal{S}),\kappa\big)$ returns a matching of size $n/4$ with expected crossing number at most $5\kappa/2+8\ln m$, with expected $O\left(\min\left\{n^4\ln(n)/\kappa^2+mn^2\ln(m)/\kappa^2,\ n^3+mn\right\}\right)$ calls to the membership Oracle of $(X,\mathcal{S})$.*

The proof of Theorem 1 follows by applying Theorem 2 to each of the $\log n$ calls of $\textsc{MatchHalf}$. We get that the expected crossing number of the matching returned by $\textsc{BuildMatching}\big((X,\mathcal{S}),a,b,\gamma\big)$ is at most

$$\sum_{i=1}^{\log n}\left[\frac{5a}{2}\left(\frac{n}{2^i}\right)^{\gamma}+\frac{5b}{2}+8\ln m\right]<\left(\frac{5b}{2}+8\ln m\right)\log n+\frac{5an^{\gamma}}{2}\sum_{i=1}^{\infty}\left(\frac{1}{2^{\gamma}}\right)^{i}$$
$$<(3b+8\ln m)\log n+\frac{5a}{\gamma}n^{\gamma}.$$

If $1/2<\gamma\le 1$, then the overall expected number of calls to the membership Oracle is

$$\sum_{i=0}^{\log n}O\left(\frac{\left(\frac{n}{2^i}\right)^4\ln\left(\frac{n}{2^i}\right)}{\left(a\left(\frac{n}{2^i}\right)^{\gamma}+b\right)^2}+\frac{m\left(\frac{n}{2^i}\right)^2\ln m}{\left(a\left(\frac{n}{2^i}\right)^{\gamma}+b\right)^2}\right)=O\left(n^{4-2\gamma}\ln n+mn^{2-2\gamma}\ln m\log n\right),$$

and if $\gamma\le 1/2$, the overall expected number of calls to the membership Oracle is

$$\sum_{i=0}^{\log n}O\left(\left(\frac{n}{2^i}\right)^3+m\left(\frac{n}{2^i}\right)\right)=O\left(n^3+mn\right).$$

◀

Proof of Theorem 2. The proof relies on the following technical lemma, whose proof is presented later in this section. For an edge e and a set S, we define $\mathrm{I}(e, S)$ to be 1 if S crosses e and 0 otherwise.

▶ **Lemma 3** (Main Lemma). *Let $\tilde{E}$ denote the set of edges that have non-zero weight when* $\textsc{MatchHalf}\big((X, \mathcal{S}), \kappa\big)$ *terminates. Then*

$$\mathbb{E}\left[\max_{S \in \mathcal{S}} \sum_{i=1}^{n/4} \mathrm{I}(e_i, S)\right] \le 2 \cdot \mathbb{E}\left[\min_{e \in \tilde{E}} \sum_{i=1}^{n/4} \mathrm{I}(e, S_i)\right] + \frac{\kappa}{2} + \frac{2 \ln |E| + \ln |\mathcal{S}|}{\ln 2}. \tag{1}$$

The left-hand side of Equation (1) is precisely the expected crossing number of the edges returned by $\textsc{MatchHalf}$. We use the following "short-edge" lemma.

▶ **Lemma 4.** *Let $(Y, \mathcal{R})$ be a set system and κ be such that Y has a perfect matching with crossing number at most κ with respect to $\mathcal{R}$. Then there is an edge spanned by the points of Y that is crossed by at most $\frac{2|\mathcal{R}|\kappa}{|Y|}$ sets of $\mathcal{R}$.*

Proof. Let M be a matching of Y such that any set of $\mathcal{R}$ crosses at most κ edges of M. Then there are at most $|\mathcal{R}| \cdot \kappa$ crossings between the edges of M and sets in $\mathcal{R}$. By the pigeonhole principle, there is an edge in M that is crossed by at most

$$\frac{|\mathcal{R}| \cdot \kappa}{|M|} = \frac{|\mathcal{R}| \cdot \kappa}{|Y|/2} = \frac{2|\mathcal{R}|\kappa}{|Y|}$$

sets of $\mathcal{R}$. ◀

Let $\tilde{X} \subset X$ denote the set of points that are not covered by the edges $\{e_1, \ldots, e_{n/4}\}$ returned by $\textsc{MatchHalf}\big((X, \mathcal{S}), \kappa\big)$. Note that $\tilde{E} = \binom{\tilde{X}}{2}$. Applying Lemma 4 to $Y = \tilde{X}$ and $\mathcal{R} = \{S_1, \ldots, S_{n/4}\}$, we get that there is an edge $e \in \tilde{E}$ that satisfies

$$\sum_{i=1}^{n/4} \mathrm{I}(e, S_i) \le \frac{2 \cdot |\mathcal{R}| \cdot \kappa}{|\tilde{X}|} = \frac{2 \cdot n/4 \cdot \kappa}{n/2} = \kappa. \tag{2}$$

Thus, by using the Main Lemma, the expected crossing number of the edges $\{e_1, \ldots, e_{n/4}\}$ with respect to $\mathcal{S}$ can be bounded as

$$\mathbb{E}\left[\max_{S \in \mathcal{S}} \sum_{i=1}^{n/4} \mathrm{I}(e_i, S)\right] \le 2 \cdot \mathbb{E}\left[\min_{e \in \tilde{E}} \sum_{i=1}^{n/4} \mathrm{I}(e, S_i)\right] + \frac{\kappa}{2} + \frac{2 \ln n^2 + \ln m}{\ln 2} \le \frac{5\kappa}{2} + 8 \ln m.$$

Finally, we bound the number of membership Oracle calls. At each iteration $i = 1, \ldots, n/4$, we update the weights of $|E_i| + |\mathcal{S}_i| = O\left(n^2 \mathbf{p} + m\mathbf{q}\right)$ elements in expectation, each requiring one call to the membership Oracle. Thus in expectation, the total number of membership Oracle calls is $O\left(n\left(n^2 \cdot \min\{n/\kappa^2 \ln n,\, 1\} + m \cdot \min\{n/\kappa^2 \ln m,\, 1\}\right)\right)$. ◀

Proof of Main Lemma. The proof is subdivided into three lemmas. For brevity, we set $t = n/4$. The first lemma is proved by examining the total weight of the sets of $\mathcal{S}$ in π_{t+1}.

▶ **Lemma 5.**

$$\mathbb{E}\left[\max_{S \in \mathcal{S}} \sum_{i=1}^{t} \mathrm{I}(e_i, S)\right] \le \frac{1}{\ln 2} \sum_{i=1}^{t} \mathbb{E}\left[\sum_{S \in \mathcal{S}} \frac{\pi_i(S)}{\pi_i(\mathcal{S})} \mathrm{I}(e_i, S)\right] + \frac{\kappa}{4} + \frac{\ln |\mathcal{S}|}{\ln 2}.$$

Proof. Let $\pi_{t+1}(\mathcal{S})$ denote the total weight of the sets of $\mathcal{S}$ in π_{t+1}. We bound $\pi_{t+1}(\mathcal{S})$ in two different ways. On the one hand, $\pi_{t+1}(\mathcal{S})$ is clearly lower-bounded by the weight of the set of maximum weight in π_{t+1}. Recall that the weight of a set S is doubled in iteration i if and only if S crosses e_i, therefore

$$\pi_{t+1}(\mathcal{S}) \geq \max_{S\in\mathcal{S}} \pi_{t+1}(S) = 2^{\max\limits_{S\in\mathcal{S}} \sum\limits_{i=1}^{t} \mathrm{I}\,(e_i,S)\cdot \mathbf{1}_{\{S\in\mathcal{S}_i\}}},$$

where $\mathbf{1}_{\mathcal{A}}$ denotes the indicator whether an event $\mathcal{A}$ happens. On the other hand, we can express $\pi_{t+1}(\mathcal{S})$ using the update rule of the algorithm

$$\begin{aligned}
\pi_{t+1}(\mathcal{S}) &= \sum_{S\in\mathcal{S}} \pi_{t+1}(S) = \sum_{S\in\mathcal{S}} \pi_t(S)\left(1+\mathrm{I}\,(e_t,S)\cdot \mathbf{1}_{\{S\in\mathcal{S}_t\}}\right)\\
&= \sum_{S\in\mathcal{S}} \pi_t(S) + \sum_{S\in\mathcal{S}} \pi_t(S)\mathrm{I}\,(e_t,S)\cdot \mathbf{1}_{\{S\in\mathcal{S}_t\}}\\
&= \pi_t(\mathcal{S}) + \pi_t(\mathcal{S}) \sum_{S\in\mathcal{S}} \frac{\pi_t(S)}{\pi_t(\mathcal{S})}\mathrm{I}\,(e_t,S)\cdot \mathbf{1}_{\{S\in\mathcal{S}_t\}}\\
&= \pi_t(\mathcal{S})\left(1+ \sum_{S\in\mathcal{S}} \frac{\pi_t(S)}{\pi_t(\mathcal{S})}\mathrm{I}\,(e_t,S)\cdot \mathbf{1}_{\{S\in\mathcal{S}_t\}}\right).
\end{aligned}$$

Unfolding this recursion and using the fact that $1+a \leq \exp(a)$, we get

$$\begin{aligned}
\pi_{t+1}(\mathcal{S}) &= \pi_1(\mathcal{S}) \prod_{i=1}^{t}\left(1+ \sum_{S\in\mathcal{S}} \frac{\pi_i(S)}{\pi_i(\mathcal{S})}\mathrm{I}\,(e_i,S)\cdot \mathbf{1}_{\{S\in\mathcal{S}_i\}}\right)\\
&\leq |\mathcal{S}|\cdot \exp\left(\sum_{i=1}^{t}\sum_{S\in\mathcal{S}} \frac{\pi_i(S)}{\pi_i(\mathcal{S})}\mathrm{I}\,(e_i,S)\cdot \mathbf{1}_{\{S\in\mathcal{S}_i\}}\right).
\end{aligned}$$

Putting together the obtained upper and lower bounds on $\pi_{t+1}(\mathcal{S})$, we get

$$2^{\max\limits_{S\in\mathcal{S}} \sum\limits_{i=1}^{t} \mathrm{I}\,(e_i,S)\cdot \mathbf{1}_{\{S\in\mathcal{S}_i\}}} \leq |\mathcal{S}|\cdot \exp\left(\sum_{i=1}^{t}\sum_{S\in\mathcal{S}} \frac{\pi_i(S)}{\pi_i(\mathcal{S})}\mathrm{I}\,(e_i,S)\cdot \mathbf{1}_{\{S\in\mathcal{S}_i\}}\right).$$

Taking the logarithm of each side yields

$$\ln(2)\cdot \max_{S\in\mathcal{S}} \sum_{i=1}^{t} \mathrm{I}\,(e_i,S)\cdot \mathbf{1}_{\{S\in\mathcal{S}_i\}} \leq \sum_{i=1}^{t}\sum_{S\in\mathcal{S}} \frac{\pi_i(S)}{\pi_i(\mathcal{S})}\mathrm{I}\,(e_i,S)\cdot \mathbf{1}_{\{S\in\mathcal{S}_i\}} + \ln|\mathcal{S}|\,. \tag{3}$$

If $\mathbf{q}=1$, then $\mathbf{1}_{\{S\in\mathcal{S}_i\}}=1$ for all i and $S\in\mathcal{S}$, thus taking total expectation we conclude

$$\mathbb{E}\left[\max_{S\in\mathcal{S}} \sum_{i=1}^{t} \mathrm{I}\,(e_i,S)\right] \leq \frac{1}{\ln 2}\sum_{i=1}^{t} \mathbb{E}\left[\sum_{S\in\mathcal{S}} \frac{\pi_i(S)}{\pi_i(\mathcal{S})}\mathrm{I}\,(e_i,S)\right] + \frac{\ln|\mathcal{S}|}{\ln 2}.$$

Assume that $\mathbf{q}<1$. Since $\max f(x) - \max g(x) \leq \max(f(x)-g(x))$, Equation (3) implies

$$\begin{aligned}
\ln(2)\cdot \mathbf{q}\cdot \max_{S\in\mathcal{S}} \sum_{i=1}^{t} \mathrm{I}\,(e_i,S) \leq{}& \ln(2)\cdot \max_{S\in\mathcal{S}} \sum_{i=1}^{t} \mathrm{I}\,(e_i,S)\cdot \left(\mathbf{q}-\mathbf{1}_{\{S\in\mathcal{S}_i\}}\right)\\
&+ \sum_{i=1}^{t}\sum_{S\in\mathcal{S}} \frac{\pi_i(S)}{\pi_i(\mathcal{S})}\mathrm{I}\,(e_i,S)\cdot \mathbf{1}_{\{S\in\mathcal{S}_i\}} + \ln|\mathcal{S}|\,.
\end{aligned}$$

Taking total expectation of each side, we obtain

$$\begin{aligned}\mathbf{q}\ln(2)\cdot\mathbb{E}\left[\max_{S\in\mathcal{S}}\sum_{i=1}^{t}\mathrm{I}\left(e_i,S\right)\right] \leq \ln(2)\cdot\mathbb{E}\left[\max_{S\in\mathcal{S}}\sum_{i=1}^{t}\mathrm{I}\left(e_i,S\right)\cdot\left(\mathbf{q}-\mathbf{1}_{\{S\in\mathcal{S}_i\}}\right)\right] \\ +\sum_{i=1}^{t}\sum_{S\in\mathcal{S}}\mathbb{E}\left[\frac{\pi_i(S)}{\pi_i(\mathcal{S})}\mathrm{I}\left(e_i,S\right)\cdot\mathbf{1}_{\{S\in\mathcal{S}_i\}}\right]+\ln|\mathcal{S}|\,. \end{aligned} \tag{4}$$

Since for each fixed i, the random variables $\{\pi_i, e_i\}$ and $\mathcal{S}_i$ are independent, we get that

$$\sum_{i=1}^{t}\sum_{S\in\mathcal{S}}\mathbb{E}\left[\frac{\pi_i(S)}{\pi_i(\mathcal{S})}\mathrm{I}\left(e_i,S\right)\cdot\mathbf{1}_{\{S\in\mathcal{S}_i\}}\right]=\mathbf{q}\cdot\sum_{i=1}^{t}\sum_{S\in\mathcal{S}}\mathbb{E}\left[\frac{\pi_i(S)}{\pi_i(\mathcal{S})}\mathrm{I}\left(e_i,S\right)\right].$$

To bound the expectation of $\max\limits_{S\in\mathcal{S}}\sum\limits_{i=1}^{t}\mathrm{I}\left(e_i,S\right)\cdot\left(\mathbf{q}-\mathbf{1}_{\{S\in\mathcal{S}_i\}}\right)$, we will need the following concentration bound for martingales.

▶ **Lemma 6** (Freedman's inequality [15, 5])**.** *Let $Y_0,\ldots,Y_n$ be a martingale adapted to the filtration $F_0,\ldots,F_n$ such that $|Y_i-Y_{i-1}|<M$ for all i and $\sum_{i=1}^{n}\mathbb{E}\left[\,(Y_i-Y_{i-1})^2\mid F_{i-1}\,\right]\leq s$ almost surely for some $s>0$. Then for any $\varepsilon>0$,*

$$\mathbb{P}\left[\,Y_n-Y_0\geq\varepsilon\,\right]\leq\exp\left(-\frac{\varepsilon^2}{2(s+M\varepsilon)}\right).$$

▷ Claim 7.

$$\mathbb{P}\left[\max_{S\in\mathcal{S}}\sum_{i=1}^{t}\mathrm{I}\left(e_i,S\right)\cdot\left(\mathbf{q}-\mathbf{1}_{\{S\in\mathcal{S}_i\}}\right)\geq 2\sqrt{\mathbf{q}t\ln(|\mathcal{S}|t)}\right]\leq\frac{1}{t}.$$

Proof. For each $i\in[1,t]$ and $S\in\mathcal{S}$, consider the random variable $X_i(S)=\mathrm{I}\left(e_i,S\right)\cdot\left(\mathbf{q}-\mathbf{1}_{\{S\in\mathcal{S}_i\}}\right)$, which is measurable with respect to e_i and $\mathcal{S}_i$.
Let $F_i=\sigma\left(e_1,\ldots,e_i,S_1,\ldots,S_i,E_1,\ldots,E_i,\mathcal{S}_1,\ldots\mathcal{S}_i\right)$. Observe that conditioned on F_{i-1}, e_i and $\mathcal{S}_i$ are independent, and thus $\mathbb{E}\left[\,X_i(S)\mid F_{i-1}\,\right]=0$ for all $i\in[1,t]$ and $S\in\mathcal{S}$, as $\mathbb{E}\left[\,\mathbf{q}-\mathbf{1}_{\{S\in\mathcal{S}_i\}}\,\right]=0$. Therefore, $Y_0(S)=0$, $Y_k(S)=\sum_{i=1}^{k}X_i(S)$ is a martingale adapted to the filtration $F_0,\ldots,F_{k-1}$. Notice that for any $S\in\mathcal{S}$, $\sum_{i=1}^{t}\mathbb{E}[(Y_i(S)-Y_{i-1}(S))^2\mid F_{i-1}]\leq\mathbf{q}t$ and $|Y_i(S)-Y_{i-1}(S)|\leq 1$.
Thus Lemma 6 combined with the union bound implies for any $\varepsilon\leq\mathbf{q}t$,

$$\mathbb{P}\left(\max_{S\in\mathcal{S}}Y_t(S)\geq\varepsilon\right)\leq|\mathcal{S}|\exp\left(-\frac{\varepsilon^2}{4\mathbf{q}t}\right).$$

Setting $\varepsilon=2\sqrt{\mathbf{q}t\ln(|\mathcal{S}|t)}$, we conclude the proof of Claim 7. ◁

Applying Claim 7 and using that $\sum_{i=1}^{t}\mathrm{I}\left(e_i,S\right)\cdot\left(\mathbf{q}-\mathbf{1}_{\{S\in\mathcal{S}_i\}}\right)\leq t$ always holds, we get

$$\mathbb{E}\left[\max_{S\in\mathcal{S}}\sum_{i=1}^{t}\mathrm{I}\left(e_i,S\right)\cdot\left(\mathbf{q}-\mathbf{1}_{\{S\in\mathcal{S}_i\}}\right)\right]\leq 2\sqrt{\mathbf{q}t\ln(|\mathcal{S}|t)}+t\cdot\frac{1}{t}\leq 2\sqrt{\mathbf{q}t\ln(|\mathcal{S}|t)}+1.$$

Hence Equation (4) implies

$$\ln(2)\cdot\mathbf{q}\cdot\mathbb{E}\left[\max_{S\in\mathcal{S}}\sum_{i=1}^{t}\mathrm{I}\left(e_i,S\right)\right]$$

$$\leq \mathbf{q} \cdot \sum_{i=1}^{t} \mathbb{E}\left[\sum_{S \in \mathcal{S}} \frac{\pi_i(S)}{\pi_i(\mathcal{S})} \mathrm{I}\left(e_i, S\right) \right] + 2\sqrt{\mathbf{q} t \ln(|\mathcal{S}|t)} + 1 + \ln|\mathcal{S}|.$$

Dividing both sides by $\mathbf{q} \ln 2$ and substituting $\mathbf{q} = 39n \log(|\mathcal{S}|n/4)/\kappa^2 = 156t \log(|\mathcal{S}|t)/\kappa^2$,

$$\begin{aligned}
&\mathbb{E}\left[\max_{S \in \mathcal{S}} \sum_{i=1}^{t} \mathrm{I}\left(e_i, S\right) \right] \\
&\leq \frac{1}{\ln 2} \cdot \sum_{i=1}^{t} \mathbb{E}\left[\sum_{S \in \mathcal{S}} \frac{\pi_i(S)}{\pi_i(\mathcal{S})} \mathrm{I}\left(e_i, S\right) \right] + \frac{2}{\ln 2}\sqrt{\frac{t \ln(|\mathcal{S}|t)}{\mathbf{q}}} + \frac{1 + \ln|\mathcal{S}|}{\mathbf{q} \ln 2} \\
&\leq \frac{1}{\ln 2} \cdot \sum_{i=1}^{t} \mathbb{E}\left[\sum_{S \in \mathcal{S}} \frac{\pi_i(S)}{\pi_i(\mathcal{S})} \mathrm{I}\left(e_i, S\right) \right] + \frac{2\kappa}{\sqrt{156} \ln 2} + \frac{1}{\ln 2} \cdot \frac{1 + \ln|\mathcal{S}|}{\log(|\mathcal{S}|n/4)} \cdot \frac{\kappa^2}{78(n/2)} \\
&\leq \frac{1}{\ln 2} \cdot \sum_{i=1}^{t} \mathbb{E}\left[\sum_{S \in \mathcal{S}} \frac{\pi_i(S)}{\pi_i(\mathcal{S})} \mathrm{I}\left(e_i, S\right) \right] + \frac{\kappa}{4},
\end{aligned}$$

where we used that $\kappa \leq n/2$. This concludes the proof of Lemma 5. ◀

The next lemma is proven by applying analogous arguments for the total weight of edges in ω_{t+1} with a small adjustment as in each iteration we set some edge weights to zero. Recall that $\tilde{E}$ denotes the set of edges that have non-zero weight in ω_{t+1}.

▶ **Lemma 8.**

$$\sum_{i=1}^{t} \mathbb{E}\left[\sum_{e \in E} \frac{\omega_i(e)}{\omega_i(E)} \mathrm{I}\left(e, S_i\right) \right] < 2\ln(2) \cdot \mathbb{E}\left[\min_{e \in \tilde{E}} \sum_{i=1}^{t} \mathrm{I}\left(e, S_i\right) \right] + \frac{\kappa \ln 2}{4} + 2\ln|E|.$$

Proof. Let $\omega_{t+1}(E)$ denote the total weight of edges in ω_{t+1}. Again, we lower-bound $\omega_{t+1}(E)$ by the largest edge-weight in ω_{t+1}, which is now attained at some edge of $\tilde{E}$:

$$\omega_{t+1}(E) \geq \max_{e \in E} \omega_{t+1}(e) = \max_{e \in \tilde{E}} \omega_{t+1}(e) = \left(\frac{1}{2}\right)^{\min_{e \in \tilde{E}} \sum_{i=1}^{t} \mathrm{I}\left(e, S_i\right) \cdot \mathbf{1}_{\{e \in E_i\}}}.$$

The upper bound is obtained by using the algorithm's weight update rule. Since e_t has positive weight in ω_t, but its weight in ω_{t+1} is set to 0, we have a strict inequality

$$\begin{aligned}
\omega_{t+1}(E) &= \sum_{e \in E} \omega_{t+1}(e) < \sum_{e \in E} \omega_t(e) \left(1 - \frac{1}{2} \mathrm{I}\left(e, S_t\right) \cdot \mathbf{1}_{\{e \in E_t\}}\right) \\
&= \sum_{e \in E} \omega_t(e) - \frac{1}{2} \sum_{e \in E} \omega_t(e) \mathrm{I}\left(e, S_t\right) \cdot \mathbf{1}_{\{e \in E_t\}} \\
&= \omega_t(E) \left(1 - \frac{1}{2} \sum_{e \in E} \frac{\omega_t(e)}{\omega_t(E)} \mathrm{I}\left(e, S_t\right) \cdot \mathbf{1}_{\{e \in E_t\}}\right).
\end{aligned}$$

Unfolding this recursion and using the fact that $1 + a \leq \exp(a)$, we get

$$\omega_{t+1}(E) < |E| \cdot \exp\left(-\frac{1}{2} \sum_{i=1}^{t} \sum_{e \in E} \frac{\omega_i(e)}{\omega_i(E)} \mathrm{I}\left(e, S_i\right) \cdot \mathbf{1}_{\{e \in E_i\}}\right).$$

Combining the obtained upper and the lower bounds on $\omega_{t+1}(E)$ and taking the logarithm of each side, we get

$$\ln\left(\frac{1}{2}\right) \cdot \min_{e \in \tilde{E}} \sum_{i=1}^{t} \mathrm{I}\left(e, S_i\right) \cdot \mathbf{1}_{\{e \in E_i\}} < -\frac{1}{2} \sum_{i=1}^{t} \sum_{e \in E} \frac{\omega_i(e)}{\omega_i(E)} \mathrm{I}\left(e, S_i\right) \cdot \mathbf{1}_{\{e \in E_i\}} + \ln |E|,$$

which is equivalent to

$$\sum_{i=1}^{t} \sum_{e \in E} \frac{\omega_i(e)}{\omega_i(E)} \mathrm{I}\left(e, S_i\right) \cdot \mathbf{1}_{\{e \in E_i\}} < 2\ln(2) \cdot \min_{e \in \tilde{E}} \sum_{i=1}^{t} \mathrm{I}\left(e, S_i\right) \cdot \mathbf{1}_{\{e \in E_i\}} + 2\ln |E|. \tag{5}$$

If $\mathbf{p} = 1$, then $\mathbf{1}_{\{e \in E_i\}} = 1$ for all i and $e \in E$, thus taking total expectation we conclude

$$\sum_{i=1}^{t} \mathbb{E}\left[\sum_{e \in E} \frac{\omega_i(e)}{\omega_i(E)} \mathrm{I}\left(e, S_i\right)\right] < 2\ln(2) \cdot \mathbb{E}\left[\min_{e \in \tilde{E}} \sum_{i=1}^{t} \mathrm{I}\left(e, S_i\right)\right] + 2\ln |E|.$$

Assume that $\mathbf{p} < 1$. Since $\min f(x) - \min g(x) \leq \max(f(x) - g(x))$, Equation (5) implies

$$\begin{aligned}\sum_{i=1}^{t} \sum_{e \in E} \frac{\omega_i(e)}{\omega_i(E)} \mathrm{I}\left(e, S_i\right) \cdot \mathbf{1}_{\{e \in E_i\}} < \ & 2\ln(2) \cdot \max_{e \in \tilde{E}} \sum_{i=1}^{t} \mathrm{I}\left(e, S_i\right) \cdot \left(\mathbf{1}_{\{e \in E_i\}} - \mathbf{p}\right) \\ & + 2\ln(2) \cdot \min_{e \in \tilde{E}} \sum_{i=1}^{t} \mathrm{I}\left(e, S_i\right) \cdot \mathbf{p} + 2\ln |E|\,.\end{aligned}$$

Taking total expectation of each side, and using that for each fixed i, the random variables $\{\omega_i, S_i\}$ and E_i are independent, we get

$$\begin{aligned}\mathbf{p} \cdot \sum_{i=1}^{t} \sum_{e \in E} \mathbb{E}\left[\frac{\omega_i(e)}{\omega_i(E)} \mathrm{I}\left(e, S_i\right)\right] < \ & 2\ln(2) \cdot \mathbb{E}\left[\max_{e \in \tilde{E}} \sum_{i=1}^{t} \mathrm{I}\left(e, S_i\right) \cdot \left(\mathbf{1}_{\{e \in E_i\}} - \mathbf{p}\right)\right] \\ & + 2\ln(2)\mathbf{p} \cdot \mathbb{E}\left[\min_{e \in \tilde{E}} \sum_{i=1}^{t} \mathrm{I}\left(e, S_i\right)\right] + 2\ln |E|\,.\end{aligned} \tag{6}$$

We need the following claim whose proof uses Lemma 6 and is similar to Claim 7.

▷ Claim 9.

$$\mathbb{P}\left[\max_{e \in \tilde{E}} \sum_{i=1}^{t} \mathrm{I}\left(e, S_i\right) \cdot \left(\mathbf{1}_{\{e \in E_i\} - \mathbf{p}}\right) \geq 2\sqrt{\mathbf{p}t \ln(|E|t)}\right] \leq \frac{1}{t}.$$

This, together with the fact that $\sum_{i=1}^{t} \mathrm{I}\left(e, S_i\right) \cdot \left(\mathbf{1}_{\{e \in E_i\}} - \mathbf{p}\right) \leq t$ always holds imply

$$\mathbb{E}\left[\max_{e \in \tilde{E}} \sum_{i=1}^{t} \mathrm{I}\left(e, S_i\right) \cdot \left(\mathbf{1}_{\{e \in E_i\}}\right) - \mathbf{p}\right] \leq 2\sqrt{\mathbf{p}t \ln(|E|t)} + t \cdot \frac{1}{t} \leq 2\sqrt{\mathbf{p}t \ln(|E|t)} + 1.$$

Hence Equation (6) yields

$$\begin{aligned}&\mathbf{p} \cdot \sum_{i=1}^{t} \sum_{e \in E} \mathbb{E}\left[\frac{\omega_i(e)}{\omega_i(E)} \mathrm{I}\left(e, S_i\right)\right] \\ &\quad < 2\ln(2)\mathbf{p} \cdot \mathbb{E}\left[\min_{e \in \tilde{E}} \sum_{i=1}^{t} \mathrm{I}\left(e, S_i\right)\right] + 4\ln(2) \cdot \sqrt{\mathbf{p}t \ln(|E|t)} + 2\ln 2 + 2\ln |E|.\end{aligned}$$

Dividing both sides by $\mathbf{p} = 106n\log(|E|n/4)/\kappa^2 = 424t\log(|E|t)/\kappa^2$, we get

$$\sum_{i=1}^{t}\sum_{e\in E}\mathbb{E}\left[\frac{\omega_i(e)}{\omega_i(E)}\mathrm{I}\left(e,S_i\right)\right]$$
$$< 2\ln(2)\cdot\mathbb{E}\left[\min_{e\in\tilde{E}}\sum_{i=1}^{t}\mathrm{I}\left(e,S_i\right)\right] + 4\ln(2)\cdot\sqrt{\frac{t\ln(|E|t)}{\mathbf{p}}} + \frac{2\ln 2+2\ln|E|}{\mathbf{p}}$$
$$= 2\ln(2)\cdot\mathbb{E}\left[\min_{e\in\tilde{E}}\sum_{i=1}^{t}\mathrm{I}\left(e,S_i\right)\right] + \frac{4\ln(2)\cdot\kappa}{\sqrt{424}} + \frac{2\ln 2+2\ln|E|}{2\log(|E|t)}\cdot\frac{\kappa^2}{n/2}\cdot\frac{1}{106}$$
$$\le 2\ln(2)\cdot\mathbb{E}\left[\min_{e\in\tilde{E}}\sum_{i=1}^{t}\mathrm{I}\left(e,S_i\right)\right] + \frac{\kappa\ln 2}{4},$$

where we used that $\kappa \le n/2$. This concludes the proof of Lemma 8. ◀

We need one more lemma to tie the previous two together.

▶ **Lemma 10.** *For any $i \in [1,t]$, we have*

$$\mathbb{E}\left[\sum_{S\in\mathcal{S}}\frac{\pi_i(S)}{\pi_i(\mathcal{S})}\mathrm{I}\left(e_i,S\right)\right] = \mathbb{E}\left[\sum_{e\in E}\frac{\omega_i(e)}{\omega_i(E)}\mathrm{I}\left(e,S_i\right)\right].$$

Proof. Let $F_i = \sigma\left(e_1,\ldots,e_i,S_1,\ldots,S_i,E_1,\ldots,E_i,\mathcal{S}_1,\ldots\mathcal{S}_i\right)$. We have

$$\mathbb{E}\left[\sum_{S\in\mathcal{S}}\frac{\pi_i(S)}{\pi_i(\mathcal{S})}\mathrm{I}\left(e_i,S\right)\right] = \mathbb{E}\left[\mathbb{E}\left[\sum_{S\in\mathcal{S}}\frac{\pi_i(S)}{\pi_i(\mathcal{S})}\mathrm{I}\left(e_i,S\right)\,\middle|\,F_{i-1}\right]\right] \quad \text{and}$$
$$\mathbb{E}\left[\sum_{e\in E}\frac{\omega_i(e)}{\omega_i(E)}\mathrm{I}\left(e,S_i\right)\right] = \mathbb{E}\left[\mathbb{E}\left[\sum_{e\in E}\frac{\omega_i(e)}{\omega_i(E)}\mathrm{I}\left(e,S_i\right)\,\middle|\,F_{i-1}\right]\right].$$

Observe that ω_i and π_i are measurable with respect to F_{i-1}, thus

$$\mathbb{E}\left[\sum_{S\in\mathcal{S}}\frac{\pi_i(S)}{\pi_i(\mathcal{S})}\mathrm{I}\left(e_i,S\right)\,\middle|\,F_{i-1}\right] = \sum_{e\in E}\frac{\omega_i(e)}{\omega_i(E)}\cdot\left(\sum_{S\in\mathcal{S}}\frac{\pi_i(S)}{\pi_i(\mathcal{S})}\mathrm{I}\left(e,S\right)\right)$$
$$= \sum_{e\in E}\sum_{S\in\mathcal{S}}\frac{\omega_i(e)}{\omega_i(E)}\cdot\frac{\pi_i(S)}{\pi_i(\mathcal{S})}\mathrm{I}\left(e,S\right)$$
$$= \sum_{S\in\mathcal{S}}\frac{\pi_i(S)}{\pi_i(\mathcal{S})}\cdot\left(\sum_{e\in E}\frac{\omega_i(e)}{\omega_i(E)}\mathrm{I}\left(e,S\right)\right) = \mathbb{E}\left[\sum_{e\in E}\frac{\omega_i(e)}{\omega_i(E)}\mathrm{I}\left(e,S_i\right)\,\middle|\,F_{i-1}\right].$$ ◀

Finally, we combine Lemmas 5, 8, and 10 in the following way

$$\mathbb{E}\left[\max_{S\in\mathcal{S}}\sum_{i=1}^{t}\mathrm{I}\left(e_i,S\right)\right] \le \frac{1}{\ln 2}\sum_{i=1}^{t}\mathbb{E}\left[\sum_{S\in\mathcal{S}}\frac{\pi_i(S)}{\pi_i(\mathcal{S})}\mathrm{I}\left(e_i,S\right)\right] + \frac{\kappa}{4} + \frac{\ln|\mathcal{S}|}{\ln 2} \qquad \text{(Lemma 5)}$$
$$= \frac{1}{\ln 2}\sum_{i=1}^{t}\mathbb{E}\left[\sum_{e\in E}\frac{\omega_i(e)}{\omega_i(E)}\mathrm{I}\left(e,S_i\right)\right] + \frac{\kappa}{4} + \frac{\ln|\mathcal{S}|}{\ln 2} \qquad \text{(Lemma 10)}$$
$$< \frac{1}{\ln 2}\left(2\ln(2)\cdot\mathbb{E}\left[\min_{e\in\tilde{E}}\sum_{i=1}^{t}\mathrm{I}\left(e,S_i\right)\right] + \frac{\kappa\ln 2}{4} + 2\ln|E|\right) + \frac{\kappa}{4} + \frac{\ln|\mathcal{S}|}{\ln 2} \qquad \text{(Lemma 8)}$$

$$\leq 2 \cdot \mathbb{E}\left[\min_{e \in \tilde{E}} \sum_{i=1}^{t} \mathrm{I}\left(e, S_i\right)\right] + \frac{\kappa}{2} + \frac{2\ln|E| + \ln|\mathcal{S}|}{\ln 2}.$$

This completes the proof of the Main Lemma and thus of Theorem 2. ◀

4 Corollaries of Theorem 1

Set systems with bounded dual shatter function. As before, let $(X, \mathcal{S})$ be a set system, $n = |X|$ and $m = |\mathcal{S}|$. We first recall the definition of the *dual shatter function* $\pi^*_{\mathcal{S}}$ of $(X, \mathcal{S})$. For any $\mathcal{R} \subseteq \mathcal{S}$, we say that the elements $x, y \in X$ are equivalent with respect to $\mathcal{R}$ if x belongs to the same sets of $\mathcal{R}$ as y. Then $\pi^*_{\mathcal{S}}(k)$ is defined as the maximum number of equivalence classes on X defined by a k-element subfamily $\mathcal{R} \subseteq \mathcal{S}$. The following theorem shows that set systems with polynomially bounded dual shatter function possess matchings with sublinear crossing number [23, Chap. 5.4].

▶ **Lemma 11.** *Let $(X, \mathcal{S})$ be a set system and c, d be constants such that $\pi^*_{\mathcal{S}}(k) \leq ck^d$ for all $k \in [1, n]$. Then there is a perfect matching of X such that any set $S \in \mathcal{S}$ crosses at most $c^{1/d}n^{1-1/d} + \ln m$ edges of the matching.*

Observe that by definition, the dual shatter function of $(Y, \mathcal{S}|_Y)$ is upper-bounded by the dual shatter function of $(X, \mathcal{S})$ for any $Y \subseteq X$. Thus Lemma 11 implies that any $Y \subseteq X$ has a perfect matching with crossing number at most $c^{1/d}|Y|^{1-1/d} + \ln m$ with respect to $\mathcal{S}$. Applying Theorem 1, we get the following corollary.

▶ **Corollary 12.** *Let $(X, \mathcal{S})$ be a set system and c, d be constants such that $\pi^*_{\mathcal{S}}(k) \leq ck^d$ for all $k \in [1, n]$. Then* BuildMatching$\big((X, \mathcal{S}),\ c^{1/d},\ \ln m,\ 1 - \frac{1}{d}\big)$ *returns a perfect matching of X with expected crossing number at most $\frac{5c^{1/d}d}{d-1} \cdot n^{1-1/d} + 11 \ln m \log n$ with an expected $\tilde{O}\left(mn^{2/d} + n^{2+2/d}\right)$ calls to the membership Oracle of $(X, \mathcal{S})$.*

Semialgebraic set systems. Let $\Gamma_{d,\Delta,s}$ denote the collection of semialgebraic sets in $\mathbb{R}^d$ that can be defined as the solution set of a Boolean combination of at most s polynomial inequalities of degree at most Δ. First, we give a bound on its dual shatter function.

▶ **Lemma 13.** *Let $(X, \mathcal{S})$ be a set system such that X is a set of points in $\mathbb{R}^d$ and each set in $\mathcal{S}$ is induced by an element $\Gamma_{d,\Delta,s}$. Then the dual shatter function of $(X, \mathcal{S})$ can be upper-bounded as $\pi^*_{\mathcal{S}}(k) \leq (4e\Delta s)^d \cdot k^d$.*

Proof. Let $\mathcal{R} \subseteq \Gamma_{d,\Delta,s}$ be a set of k ranges, defined by $\mathcal{P} = \{p_{ij} : 1 \leq i \leq k, 1 \leq j \leq s\}$, where each element is a d-variate polynomial of degree at most Δ. Observe that if $\mathrm{sign}\,[\,p(x)\,] = \mathrm{sign}\,[\,p(y)\,]$ for all $p \in \mathcal{P}$, then x, y are equivalent with respect to $\mathcal{R}$. Therefore, $\pi^*_{\Gamma_{d,\Delta,s}}(k)$ can be upper-bounded by the number of different sign patterns in $\{-1, 1\}^{ks}$ induced by ks d-variate polynomials of degree at most Δ. This quantity is bounded by $(4e\Delta s)^d \cdot k^d$, see [30, Theorem 3]. ◀

Now we can apply Corollary 12 and obtain the following.

▶ **Corollary 14.** *Let $(X, \mathcal{S})$ be a set system such that X is a set of n points in $\mathbb{R}^d$ and $\mathcal{S}$ consists of m subsets of X, each induced by an element of $\Gamma_{d,\Delta,s}$. Then* BuildMatching$\big((X, \mathcal{S}),\ 4e\Delta s,\ \ln m, 1 - \frac{1}{d}\big)$ *returns a perfect matching of X with expected crossing number at most $\frac{20e\Delta sd}{d-1} \cdot n^{1-1/d} + 11 \ln m \log n$ in expected time $\tilde{O}\left(s\Delta^d\left(mn^{2/d} + n^{2+2/d}\right)\right)$.*

Half-spaces. Let $\mathcal{H}_d$ denote the set of all half-spaces in $\mathbb{R}^d$ and consider set systems induced by $\mathcal{H}_d$. For this setting, a typical pre-processing step is constructing a small-sized subfamily of $\mathcal{H}_d$ – called a *test-set* – such that it suffices to construct a low-crossing matching with respect to this subfamily. We use a result of Matoušek [22] on test-sets, with a small addition:

▶ **Lemma 15** (Test set lemma [22]). *Let X be a set of n points in $\mathbb{R}^d$, $\mathcal{H}_d$ be the set of all half-spaces in $\mathbb{R}^d$, and t be a parameter. There exists a set $\mathcal{T}(t)$ of at most $(d+1)t^d$ hyperplanes such that if a perfect matching of X has crossing number κ with respect to $\mathcal{T}(t)$, then its crossing number with respect to $\mathcal{H}_d$ is at most $(d+1)\kappa + \frac{6d^2 n}{t}$.*

Now let X be a set of n points in $\mathbb{R}^d$ and $\mathcal{T} = \mathcal{T}(n^{1/d})$ be the set of $(d+1)n$ half-spaces in $\mathbb{R}^d$ provided by Lemma 15. Notice that $\mathcal{T} \subset \mathcal{H}_d = \Gamma_{d,1,1}$, thus by Lemma 13, $\pi^*_{\mathcal{T}}(k) \leq (4e)^d k^d$. We apply Corollary 12 for $(X, \mathcal{T})$ and obtain the following.

▶ **Corollary 16.** *Let X be a set of n points in $\mathbb{R}^d$ and $\mathcal{T} = \mathcal{T}(n^{1/d})$ be the set of half-spaces provided by Lemma 15. Then* BuildMatching$\big((X, \mathcal{T}),\ 4e,\ \ln n, 1 - \frac{1}{d}\big)$ *returns a perfect matching of X with expected crossing number at most $\left[\ 6d^2 + (d+1) \cdot \frac{20ed}{d-1}\ \right] n^{1-1/d} + \frac{11}{\ln 2} \ln^2 n$ with respect to half-spaces in $\mathbb{R}^d$, in expected time $O\left(dn^{2+2/d} \ln n\right)$.*

Balls. Let $\mathcal{B}_d$ denote the subsets of X that are induced by balls in $\mathbb{R}^d$. It is well known that there are mappings $\alpha : X \to \mathbb{R}^{d+1}$ and $\beta : \mathcal{B}_d \to \mathcal{H}_{d+1}$ such that for any $x \in X$ and $B \in \mathcal{B}_d$, we have $x \in B$ iff $\alpha(x) \in \beta(B)$, see eg. [24, Chap. 10]. This mapping and Lemma 15 applied in $\mathbb{R}^{d+1}$ with $t = n^{1/d}$ give the following test set lemma for $\mathcal{B}_d$.

▶ **Lemma 17.** *Let X be a set of n points in $\mathbb{R}^d$. There exists a set $\mathcal{Q}$ of at most $(d+2)n^{1+1/d}$ balls such that if a perfect matching of X has crossing number κ with respect to $\mathcal{Q}$, then its crossing number with respect to $\mathcal{B}_d$ is at most $(d+2)\kappa + 6(d+1)^2 n^{1-1/d}$.*

Given a set X of n points in $\mathbb{R}^d$, let $\mathcal{Q}$ be the set of balls provided by Lemma 15. As $\mathcal{Q} \subset \mathcal{B}_d \subset \Gamma_{d,2,1}$, the dual shatter function of $\mathcal{Q}$ can be bounded as $\pi^*_{\mathcal{Q}}(k) \leq (8e)^d k^d$ (Lemma 13). We apply Corollary 12 for $(X, \mathcal{Q})$, and obtain the following corollary.

▶ **Corollary 18.** *Let X be a set of n points in $\mathbb{R}^d$ and let $\mathcal{Q}$ be the set of balls provided by Lemma 17. Then* BuildMatching$\big((X, \mathcal{Q}),\ 8e,\ \ln(n^{1+1/d}),\ 1 - \frac{1}{d}\big)$ *returns a perfect matching of X with expected crossing number at most $\left[\ 6(d+1)^2 + (d+2) \cdot \frac{40ed}{d-1}\ \right] n^{1-1/d} + \frac{11(d+1)}{d \ln 2} \ln^2 n$ with respect to balls in $\mathbb{R}^d$, in expected time $\tilde{O}(dn^{2+2/d})$.*

▶ Remark. The previous-best algorithm to construct spanning trees with crossing number $O(n^{1-1/d})$ with respect to $\mathcal{B}_d$ is based on randomized LP rounding and has time complexity $\tilde{O}(mn^2)$ [19, 11], which combined with Lemma 17 yields an $\tilde{O}\left(n^{3+1/d}\right)$ time algorithm. Alternatively, one can obtain a matching with suboptimal crossing number $O\left(n^{1-1/(d+1)}\right)$ by lifting X into $\mathbb{R}^{d+1}$, where the image of each range in $\mathcal{B}_d$ can be represented by a range in $\mathcal{H}_{d+1}$ and applying Chan's algorithm [8] with time complexity $\tilde{O}(n)$.

5 Empirical Aspects

In this section we present preliminary experimental results and provide some implementation details. We conducted our experiments on an accelerated version of BuildMatching (available on Github):

- instead of maintaining the weights on all the $O(n^2)$ edges, we work with an initial uniform random sample of $O(n \ln n)$ edges;
- at each iteration, we set $\mathbf{p} = \Theta\left(\frac{\ln n}{n^{1-1/d}}\right)$ and $\mathbf{q} = \Theta\left(\frac{\ln m}{n^{1-1/d}}\right)$ instead of $\mathbf{p} = \Theta\left(\frac{\ln n}{n^{1-2/d}}\right)$ and $\mathbf{q} = \Theta\left(\frac{\ln m}{n^{1-2/d}}\right)$.

Despite restricting ourselves to pick matching edges only from the inital sample of $O(n \log n)$ edges, we still obtain matchings with relatively low crossing numbers (see the table below). Incorporating this pre-sampling idea to the theoretical analysis of the algorithm is an interesting direction for future study.

Experimental setup. We apply the algorithm for set systems induced by half-spaces in dimensions $2, 4, 6, 8$, and 10. We consider two different types of input point sets:

`Grid`: each point is picked randomly in a cell of the uniform grid;

`Moment Curve`: each point is a slightly perturbed element of the moment curve.

All the experiments are performed with dual Xeon E5-2643 v3 processors, each with 6 cores, 12 threads, at 3.4 GHz.

`Grid`

Input size	$d=2$ cr #	$d=2$ time (s)	$d=4$ cr #	$d=4$ time (s)	$d=6$ cr #	$d=6$ time (s)	$d=8$ cr #	$d=8$ time (s)	$d=10$ cr #	$d=10$ time (s)
10000	162	58.89	699	11.84	1238	8.07	1639	6.38	1863	6.73
25000	330	279.82	1509	37.33	2804	26.49	3912	20.32	4525	20.76
50000	630	918.26	2732	99.62	5251	61.21	7387	47.02	8797	48.66
100000	1170	3001.16	5040	271.29	9774	147.91	13683	120.53	16754	110.48

`Moment Curve`

Input size	$d=2$ cr #	$d=2$ time (s)	$d=4$ cr #	$d=4$ time (s)	$d=6$ cr #	$d=6$ time (s)	$d=8$ cr #	$d=8$ time (s)	$d=10$ cr #	$d=10$ time (s)
10000	57	58.51	324	11.68	807	7.9	1028	6.47	1354	6.12
25000	89	275.96	706	37.35	1698	24.08	2642	22.79	3411	20.62
50000	132	916.39	1151	98.25	2608	61.06	4836	52.4	6263	44.79
100000	209	2978.21	2797	268.95	5502	161.1	7743	133.25	10713	113.01

Evaluation. We present our experimental results in the table below. It shows the observed crossing numbers and running times on inputs of size up to 100000. We see that the algorithm becomes faster as the dimension increases (note that the crossing number increases with dimension). For example, in dimension 6, it takes only around 160 seconds to create a matching for 100000 points. Previous experimental results only considered inputs of size at most 159, see [16].

Test set generation. Linear-sized test set that achieves the guarantee of Lemma 15 can be constructed via cuttings, which are impractical in higher dimensions. Since the study of test-sets is not the main focus of this work and to speed-up the computations, our implementation, builds the test set by $n \log n$ random d-tuples of the input points; we report the crossing numbers with respect to this particular test set. We refer to [2] for a detailed overview on constructions and sizes of test-sets for various geometric objects.

6 Applications

We present an application of spanning path with low crossing number from learning theory. Further applications will be provided in the full version of the paper.

Approximating sign rank. Let $(X, \mathcal{S})$ be a set system and let $A \in \mathbb{R}^{n\times m}$ be its signed membership matrix, that is, $(A)_{x,S} = 1$ if $x \in S$ and $(A)_{x,S} = -1$ otherwise. The *sign rank* of $(X, \mathcal{S})$ is defined as the minimum rank of a matrix having the same sign pattern as A. Geometrically, it captures the minimum dimension of a Euclidean space in which $(X, \mathcal{S})$ can be embedded and realized by half-spaces through the origin. Using a connection between the sign-rank and the crossing number of a spanning path established in Alon et al.[7], we get the following corollary.

▶ **Corollary 19.** *Let $(X, \mathcal{S})$ be a set system and let $a > 0, b$ and $\gamma \in [1/\log n, 1]$ such that any $Y \subseteq X$ has a spanning path with crossing number at most $a|Y|^\gamma + b$. Then there is a randomized algorithm that constructs an embedding of X into $\mathbb{R}^D$ with $D \leq \frac{5}{\gamma}n^\gamma + (3b + 8\ln m)\log n$ in expectation such that each $S \in \mathcal{S}$ can be represented with a half-space in $\mathbb{R}^D$. The algorithm makes $O\left(\min\left\{n^{4-2\gamma}\ln n + mn^{2-2\gamma}\ln m \ln n,\ n^3 + mn\right\}\right)$ calls to the membership Oracle of $(X, \mathcal{S})$.*

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