

# Rectilinear Steiner Trees in Narrow Strips

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## Abstract

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A *rectilinear Steiner tree* for a set  $P$  of points in  $\mathbb{R}^2$  is a tree that connects the points in  $P$  using horizontal and vertical line segments. The goal of MINIMUM RECTILINEAR STEINER TREE is to find a rectilinear Steiner tree with minimal total length. We investigate how the complexity of MINIMUM RECTILINEAR STEINER TREE for point sets  $P$  inside the strip  $(-\infty, +\infty) \times [0, \delta]$  depends on the strip width  $\delta$ . We obtain two main results.

- We present an algorithm with running time  $n^{O(\sqrt{\delta})}$  for sparse point sets, that is, point sets where each  $1 \times \delta$  rectangle inside the strip contains  $O(1)$  points.
- For random point sets, where the points are chosen randomly inside a rectangle of height  $\delta$  and expected width  $n$ , we present an algorithm that is fixed-parameter tractable with respect to  $\delta$  and linear in  $n$ . It has an expected running time of  $2^{O(\delta\sqrt{\delta})}n$ .

**2012 ACM Subject Classification** Theory of computation  $\rightarrow$  Design and analysis of algorithms

**Keywords and phrases** Computational geometry, fixed-parameter tractable algorithms

**Digital Object Identifier** 10.4230/LIPIcs.SoCG.2021.9

**Related Version** *Full Version*: <https://arxiv.org/abs/2103.08354>

**Funding** The work in this paper is supported by the Netherlands Organisation for Scientific Research (NWO) through Gravitation-grant NETWORKS-024.002.003.

**Acknowledgements** We thank Remco van der Hofstad for discussions about the probabilistic analysis.

## 1 Introduction

In the MINIMUM STEINER TREE problem in the plane, we are given as input a set  $P$  of points in the plane, called *terminals*, and the goal is to find a minimum-length tree that connects the terminals in  $P$ . Thus the given terminals must be nodes of the tree, but the tree may also use so-called *Steiner points* as nodes. MINIMUM STEINER TREE is a classic optimization problem. It was among the first problems to be proven NP-hard, not only for the case where the length of the tree is measured using Euclidean metric [13] but also in the rectilinear version [14]. It was also shown to be NP-hard for other metrics [6].

The rectilinear version of the problem, where the edges of the tree must be horizontal or vertical, is one of the most widely studied variants, and it is also the topic of our paper. The MINIMUM RECTILINEAR STEINER TREE problem dates back more than 50 years [15, 16]. Its popularity arises from its many applications, in particular in the design of integrated circuits [7, 4, 5, 23]. The two most important early insights on MINIMUM RECTILINEAR STEINER TREE came from Hanan [16] and Hwang [17]. Hanan observed that any terminal set  $P$  admits a minimum rectilinear Steiner tree (MRST, for short) whose edges lie on the grid formed by all horizontal and vertical lines passing through at least one terminal in  $P$ . This grid is often called the *Hanan grid*. This implies that the MINIMUM RECTILINEAR STEINER TREE problem can be reduced to a purely combinatorial problem – namely, a Steiner-tree problem on graphs – which is not possible for the Euclidean version of the problem. Hwang investigated the structure of optimal MRSTs in more detail, by providing a characterization of the different components of an MRST; see Section 2.



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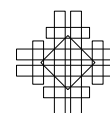
37th International Symposium on Computational Geometry (SoCG 2021).

Editors: Kevin Buchin and Éric Colin de Verdière; Article No. 9; pp. 9:1–9:16

Leibniz International Proceedings in Informatics



LIPICs Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany



As mentioned, MINIMUM RECTILINEAR STEINER TREE can be considered a special case of the Steiner-tree problem on graphs. Here the input is an edge-weighted graph  $G = (V(G), E(G))$  and a terminal set  $P \subseteq V(G)$ , and the goal is to compute a minimum-length subtree of  $G$  that includes all terminals. In 1971 Dreyfus and Wagner [11] gave an algorithm solving the Steiner-Tree problem on graphs in time  $3^n \cdot \log W \cdot |V(G)|^{O(1)}$ , where  $W$  is the maximum edge weight in  $G$ . This was later improved by Björklund *et al.* [3] and Nederlof [19], who gave an algorithm with  $2^n \cdot W \cdot |V(G)|^{O(1)}$  running time. A variant of the Dreyfus-Wagner algorithm for MINIMUM RECTILINEAR STEINER TREE runs in time  $O(n^2 \cdot 3^n)$ . Thobmorsen *et al.* [21] and Deneen *et al.* [10] gave randomized algorithms for the special case of MINIMUM RECTILINEAR STEINER TREE where the terminals are drawn independently and uniformly from a rectangle. Both run in  $2^{O(\sqrt{n} \log n)}$  expected time. Finally, in 2018 Fomin *et al.* [12] presented a  $2^{O(\sqrt{n} \log n)}$  algorithm for general point sets.

Due to the many applications of MINIMUM STEINER TREE variants in the plane, there has also been significant interest in practical implementations. These implementations rely on the insight that a minimum Steiner tree can always be decomposed into so-called *full components*, which are maximal subtrees that do not have any terminals as internal nodes [17]. (This holds for the Euclidean as well as the rectilinear version.) To compute an exact solution, a set of candidate full components is first computed and then it is computed which subset of candidate full components can be concatenated into an MRST. This process was introduced by Winter in 1985 [24], in his software package *GeoSteiner*. Still, only very small data sets could be handled, and even in 1994 the state-of-the-art software could solve the rectilinear variant of the problem for only up to 16 points [20]. Warme’s dissertation [22] significantly improved the process of concatenating the full components, resulting in optimal Steiner trees for up to 1,000 points for the rectilinear version of the problem and up to 2,000 points for the Euclidean version. In 1998 Althaus [2] obtained similar results. Throughout the years, *GeoSteiner*, which had become a collaboration between Warme, Winter and Zachariassen, has remained the fastest publicly available software package for computing minimum Steiner trees in the plane. By 2018, it could solve instances for up to 4,000 points for the rectilinear version, and up to 10,000 points for the Euclidean version [18].

**Our contribution.** The fastest known algorithm for MINIMUM RECTILINEAR STEINER TREE in  $\mathbb{R}^2$  runs in  $2^{O(\sqrt{n} \log n)}$  time [12]. In  $\mathbb{R}$ , on the other hand, the problem can be trivially solved in  $O(n \log n)$  time by just sorting the points. In order to better understand the computational complexity of the classic MINIMUM RECTILINEAR STEINER TREE problem in the plane, we therefore investigate how the complexity depends on the width of the terminal set  $P$ . If the point set in  $P$  is “almost 1-dimensional” in the sense that the points lie in a narrow strip  $\mathbb{R} \times [0, \delta]$ , then can we solve MINIMUM RECTILINEAR STEINER TREE more efficiently than in the general case? And if so, how does the complexity scale with  $\delta$ ? Can we obtain an algorithm that is fixed-parameter tractable with respect to  $\delta$ ? This follows the line of research started recently by Alkema *et al.* [1], who studied these questions for the TRAVELING SALESMAN PROBLEM. We study these questions in the following two scenarios.

- *Sparse point sets.* In this scenario, for any  $x \in \mathbb{R}$  the rectangle  $[x, x + 1] \times [0, \delta]$  contains  $O(1)$  points. We show that for sparse point sets in  $\mathbb{R}^2$  an MRST must be  $k$ -tonic – an MRST is  $k$ -tonic if it intersects any vertical line at most  $k$ -times – for  $k = O(\sqrt{\delta})$ , and we give a dynamic-programming algorithm which runs in  $n^{O(\sqrt{\delta})}$  time.
- *Random point sets.* Our main result is for point sets  $P$  generated randomly inside a rectangle of height  $\delta$  and expected width  $n$ , as follows. First, we generate  $n$  independent exponentially distributed variables  $\Delta_0, \dots, \Delta_{n-1} \sim \text{Exp}(1)$ . Using these, we compute the

$x$ -coordinates of our points by setting  $x_i$ , the  $x$ -coordinate of the  $i$ -th point from  $P$ , as  $x_i := \sum_{j=0}^{i-1} \Delta_j$  for  $1 \leq i \leq n$ . Next, we generate the  $y$ -coordinates of the points by picking each  $y_i$  uniformly and independently from the interval  $[0, \delta]$ . Thus the points from  $P$  lie inside the rectangle  $[0, x_n] \times [0, \delta]$ . One can show that asymptotically this distribution is essentially the same as the distribution obtained by picking  $n$  points uniformly at random from the rectangle  $[0, n] \times [0, \delta]$  [9]. However, the random point process as just described is somewhat easier to analyze, so we will assume the points are generated according to that process. For this case we provide an FPT algorithm for MINIMUM RECTILINEAR STEINER TREE, which runs in expected time  $2^{O(\delta\sqrt{\delta})}n$ . More precisely, expected running time is  $\min(n^{O(\sqrt{\delta})}, 2^{O(\delta\sqrt{\delta})}n)$ . Note that the running time is linear when  $\delta = O(1)$ .

## 2 Preliminaries

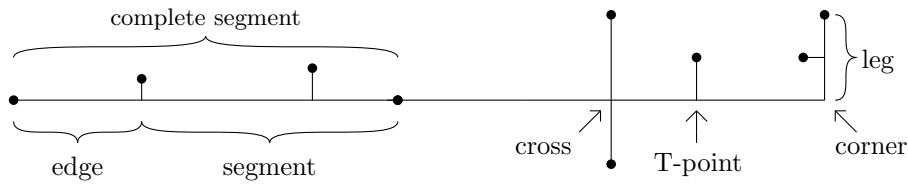
**Notation and terminology.** Let  $P := \{p_1, \dots, p_n\}$  be a set of *terminals* in a 2-dimensional strip with height  $\delta$  – we call such a strip a  $\delta$ -*strip* – which we assume without loss of generality to be  $\mathbb{R} \times [0, \delta]$ . We use  $x_i$  and  $y_i$  to denote the  $x$ - and  $y$ -coordinate of point  $p_i$ , respectively. The points can be easily sorted on their  $x$ -coordinates: this can be done in  $O(n \log n)$  time for sparse point sets, and in  $O(n)$  expected time for random point sets [8]. Therefore, we will from now on assume that  $x_i \leq x_j$  for all  $1 \leq i \leq j \leq n$ . We define the *spacing* of  $p_i$  (in  $P$ ) as  $\Delta_i := x_{i+1} - x_i$ , for all  $1 \leq i \leq n - 1$ . We write  $P[i, j]$  to denote the set  $\{p_i, \dots, p_j\}$ . We denote the vertical distance between two horizontal edges  $e, e'$  (or the horizontal distance between two vertical edges) by  $\text{dist}(e, e')$ .

Next we give some (mostly standard) terminology concerning rectilinear Steiner trees; see also Figure 1. A *rectilinear tree* is a tree structure embedded in the plane whose edges are horizontal or vertical line segments overlapping only at their endpoints. The *length* of a tree  $T$ , or  $\|T\|$ , is the sum of the lengths of its edges. A *rectilinear Steiner tree* for a set  $P$  of terminals is a rectilinear tree such that each terminal  $p \in P$  is an endpoint of an edge in the tree. A *minimal rectilinear Steiner tree* (*MRST*) is such a tree of minimum length.

The *degree* of a (Steiner or terminal) point  $q$  in a tree  $T$  is the number of edges incident on it. We denote the degree of  $q$  in  $T$  by  $\text{degree}_T(q)$ , or simply  $\text{degree}(q)$  when  $T$  is clear from the context. Without loss of generality, if a degree-2 point has collinear (i.e. both horizontal or both vertical) incident edges then that point must be a terminal. Clearly, a point has degree at most 4. A point with degree of at least 3 that is not a terminal is called a *Steiner point*. A *corner* is a degree-2 point with non-collinear incident edges that is not a terminal. Hence, each endpoint of an edge is either a terminal, a Steiner point, or a corner.

A *segment* is defined to be a sequence of one or more adjacent collinear edges, with no terminals in the segments' interior.<sup>1</sup> A *complete segment* is a inclusion-wise maximal segment. Note that a complete segment does not have terminals in its interior. A corner is incident to exactly one horizontal complete segment and exactly one vertical complete segment. These complete segments are the *legs* of the corner. A *T-point* is a degree-3 Steiner point. Finally, a *cross* is a degree-4 Steiner point. Note that the endpoints of a complete segment are T-points, corners or terminals.

<sup>1</sup> When we refer to the “interior” of a segment, we always mean its relative interior, i.e. the segment excluding its endpoints.



■ **Figure 1** Illustration of terminology concerning rectilinear Steiner trees.

Separators will play a crucial role in our algorithms. A *separator* is a vertical line, not containing any of the points in  $P$ , that separates  $P$  into two non-empty subsets. For all  $1 \leq i < n$  such that  $x_i < x_{i+1}$ , we define  $s_i$  to be the separator with  $x$ -coordinate  $(x_i + x_{i+1})/2$ . The *tonicity* of a rectilinear tree  $T$  at a separator  $s$  is the number of times  $T$  crosses  $s$ ; when the tonicity of  $T$  at  $s$  is 1, we call it *monotonic* at  $s$ . The *tonicity* of a rectilinear tree  $T$  is the maximum over the tonicity of  $T$  at all separators. A rectilinear tree is called *monotonic* when its tonicity is 1.

**Characterisation of the MRST.** Over the years, many different properties of the MRST have been proven. One of the most important ones is the following:

► **Observation 1** (Hanan [16]). *Let  $P$  be a set of terminals in  $\mathbb{R}^2$ . Then there exists an MRST on  $P$  that is a subset of the Hanan grid, the grid formed by taking all horizontal and vertical lines which pass through at least one of the points of  $P$ .*

From now on, we will only consider rectilinear Steiner trees that lie on the Hanan grid. Furthermore, we can now directly conclude that the tonicity of an MRST is at most  $n$ .

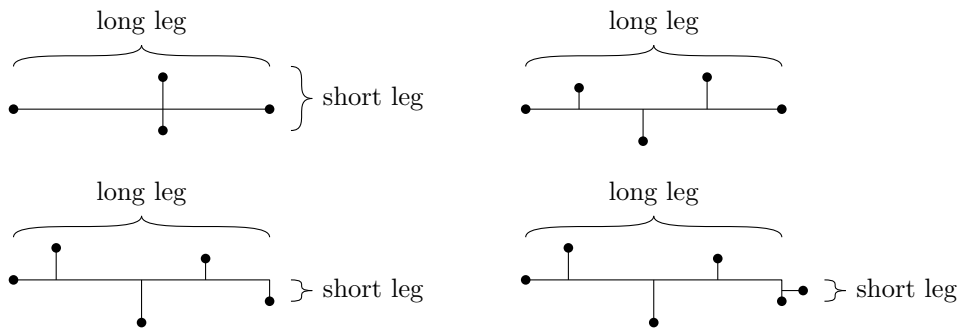
A continuation on this characterisation is given by the Hwang theorem. We define a *full component* of a rectilinear Steiner tree  $T$  to be a maximal subtree that does not have any terminals as internal nodes. Note that a node in a full component of an MRST is a terminal if and only if it is a leaf in that component. Also note that any terminal  $p_i \in P$  will be a leaf in exactly  $\text{degree}(p_i)$  full components. Hwang’s theorem is now given by the following:

► **Theorem 2** (Hwang [17]). *Let  $P$  be a set of terminals in  $\mathbb{R}^2$ . Then there exists an MRST  $T$  on  $P$  with a maximal number of full components, such that each full component  $C$  is of one of the following four types. Let  $m_C$  be the number of terminals in  $C$ . Then  $C$  consists of*

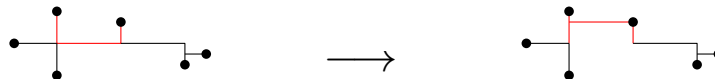
- *four edges, connected in a cross,*
- *a single complete segment with  $m_C - 2$  alternating incident edges,*
- *a corner and its legs, with  $m_C - 2$  alternating edges incident to a single leg, or*
- *a corner and its legs, with  $m_C - 3$  alternating edges incident to a single leg and a single edge incident to the other leg.*

*For all legs, the incident edge closest to the corner must point away from the opposite leg. Furthermore, the edges incident to the long leg on the same side as the short leg are at least as long as the short leg.*

We will call MRSTs which have this property *Hwang trees*. See Figure 2 for an example of each of the four types of full components of Hwang trees. Note that these full components do not contain a U-shape formed by an edge and two adjacent segments lying to the same side of that edge; any component with such a U-shape can be split into two full components by sliding the edge towards the terminals at the end of those segments. See Figure 3 for an example. We will call the complete segment with the  $m_C - 2$  or  $m_C - 3$  incident edges the *long leg*, and the other leg (if any) the *short leg*. If there are two complete segments which



■ **Figure 2** An example of each of the four different types of full components in Hwang trees.



■ **Figure 3** An example showing that Hwang trees do not contain a U-shape. The tree on the left has the same length as the tree on the right, but contains fewer full components. Therefore, the tree on the left is not a Hwang tree.

both have  $m_C - 2$  or  $m_C - 3$  incident edges, we will consider the horizontal one to be the long leg, and the vertical one to be the short leg. If the long leg is horizontal (vertical), we call the full component a *horizontal (vertical) full component*.

### 3 Sparse point sets inside a narrow strip

We say a point set is *sparse* if for all  $x$  the rectangle  $[x, x + 1] \times [0, \delta]$  contains at most  $k$  points for some arbitrary but fixed *sparseness constant*  $k$ . In this section, we will give a  $n^{O(\sqrt{\delta})}$  algorithm for sparse point sets. We will do so in two steps. First, we will show that all separators are crossed at most  $O(\sqrt{\delta})$  times. Then, we will give a dynamic-programming algorithm which sweeps from left to right and runs in the desired time.

First, we will show that parallel edges of an MRST cannot be too close. Recall that  $\Delta_i$  denotes the horizontal spacing between  $p_i$  and  $p_{i+1}$ , and that  $\delta$  denotes the height of the strip containing  $P$ . Also recall that  $s_i$  is the separator in between the points  $p_i$  and  $p_{i+1}$ .

► **Observation 3.**

- (i) Let  $E = \{e_1, \dots, e_m\}$  be a set of  $m$  horizontal edges of an MRST  $T$  which all intersect two vertical lines  $\ell$  and  $\ell'$ . Then  $m \leq 1 + \lfloor \delta / \text{dist}(\ell, \ell') \rfloor$ . A similar statement holds when  $E$  is a set of vertical edges intersecting two horizontal lines.
- (ii) If  $\Delta_i > \delta$ , then the tonicity of any MRST at  $s_i$  is 1.

For the proof, see the full version. We are now ready to bound the tonicity at the separators. The following lemma will also be applicable for randomly generated point sets.

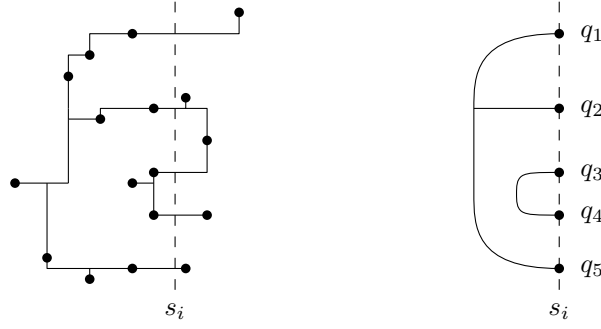
► **Lemma 4.** Let  $T$  be a Hwang tree on  $P$ . Let  $s_i$  be a separator such that

$$x_{i+\lceil \sqrt{\delta} \rceil + c_1} - x_i > c_2 \sqrt{\delta}$$

for an integer constant  $c_1 \geq 0$  and a constant  $c_2 > 0$ . Then the tonicity of  $T$  at  $s_i$  is  $O(\sqrt{\delta} + 1)$ .

The proof of Lemma 4 can be found in the full version.

Using Lemma 4 we can now prove a bound on the tonicity of MRSTs of sparse point sets.



■ **Figure 4** An example of an MRST and its crossing pattern  $C = \{\{q_1, q_2, q_5\}, \{q_3, q_4\}\}$  at  $s_i$ .

► **Corollary 5.** *An MRST on a sparse point set  $P$  in a  $\delta$ -strip is  $\left((9k + 18)(2 + \sqrt{\delta})\right)$ -tonic, where  $k$  is the sparseness constant.*

**Proof.** First, we note that since our point set  $P$  is sparse, we have  $x_j - x_i \geq \lfloor (j - i)/k \rfloor$  for all  $j > i$ . Specifically, for all  $s_i$  such that  $i + \lceil \sqrt{\delta} \rceil + k \leq n$ , we get

$$x_{i+\lceil \sqrt{\delta} \rceil+k} - x_i \geq \left\lfloor \frac{\lceil \sqrt{\delta} \rceil + k}{k} \right\rfloor \geq \left\lfloor \frac{\sqrt{\delta}}{k} + 1 \right\rfloor \geq \frac{\sqrt{\delta}}{k}.$$

Therefore, we can invoke Lemma 4 with  $c_1 = k$  and  $c_2 = 1/k$ , giving us that all these  $s_i$  are crossed at most  $17k + 36 + (4k + 17)\sqrt{\delta}$  times. (These constant follow from the constants in the proof of Lemma 4, see the Appendix.) By symmetry, we can do the same for all  $s_i$  such that  $(i + 1) - \lceil \sqrt{\delta} \rceil - k \geq 1$ . Finally, we note that if this does not cover all  $s_i$ , then we have fewer than  $17k + 36 + (4k + 17)\sqrt{\delta}$  points in total. Since every separator is crossed at most  $n$  times, the statement also holds in this case. We conclude that for sparse terminal sets, all separators are crossed at most  $17k + 36 + (4k + 17)\sqrt{\delta} < (9k + 18)(2 + \sqrt{\delta})$  times. ◀

Corollary 5 gives rise to a natural dynamic-programming algorithm, as explained next. Let  $T$  be a rectilinear Steiner tree, and let  $s_i$  be a separator. We define the *crossing pattern* of  $T$  at  $s_i$  as follows. Let  $X(s_i)$  be the set of at most  $n$  points where the Hanan grid crosses  $s_i$ , and let  $X(s_i, T) \subseteq X(s_i)$  be the subset of points where  $T$  crosses  $s_i$ . If  $T$  is an MRST,

$$|X(s_i, T)| \leq (9k + 18)(2 + \sqrt{\delta}) = O(\sqrt{\delta})$$

by Corollary 5. We partition  $X(s_i, T)$  into parts (that is, subsets) such that two points from  $X(s_i, T)$  are in the same part if the path in  $T$  between these points fully lies to the left of  $s_i$ . The resulting partition of  $X(s_i, T)$  is the crossing pattern of  $T$  at  $s_i$ ; see Figure 4 for an example. We will say that a rectilinear forest  $T$  *adheres to*  $C$  at  $s_i$  if  $T$  lies fully to the left of  $s_i$ , and there exists a rectilinear forest  $T'$  which lies fully to the right of  $s_i$  such that  $T \cup T'$  is a rectilinear Steiner tree with crossing pattern  $C$  at  $s_i$ . Note that not all crossing patterns can lead to an MRST: those that require crossing edges on the left-hand side (because they do not have a proper “nesting structure”) can never lead to an MRST. We call the crossing patterns that contain at most  $(9k + 18)(2 + \sqrt{\delta})$  points and do not require crossing edges on the left-hand side *viable crossing patterns*. We will now count the number of viable crossing patterns at  $s_i$ . There are  $n^{O(\sqrt{\delta})}$  possible sets  $X(s_i, T)$  that contain at most  $(9k + 18)(2 + \sqrt{\delta})$  points. The number of viable partitions of these points – also known as the number of non-crossing partitions – follows the Catalan numbers. Hence, there are  $2^{O(\sqrt{\delta})}$  possible viable partitions for each  $X(s_i, T)$ . This implies that the total number of viable crossing patterns for  $s_i$  is  $n^{O(\sqrt{\delta})} \cdot 2^{O(\sqrt{\delta})} = n^{O(\sqrt{\delta})}$ .

**The algorithm.** We can now define a table entry  $A[i, X]$  for each separator  $s_i$  and viable crossing pattern  $X$  at  $s_i$  as follows.

$A[i, X] :=$  the minimum length of a rectilinear forest adhering to  $X$  at  $s_i$ .

Note that the length of an MRST equals  $A[n, \{\emptyset\}]$ . Next we describe a recursive formula to compute the table entries. As a base case, we will use  $A[0, X] = 0$  for  $X = \{\emptyset\}$ , and  $\infty$  for all other  $X$ .

Let  $s_j$  and  $s_i$  be consecutive separators, with  $j < i$ . Note that since the point set is sparse, at most  $k$  points share an  $x$ -coordinate. Therefore,  $j \geq i - k$ . Let  $F(X, s_i)$  be a minimum-length rectilinear forest adhering to  $X$  at  $s_i$ , and let  $X'$  be its (unknown) crossing pattern at  $s_j$ . Then the value of  $A[i, X]$  equals the value of  $A[j, X']$  plus the total length of the edges of  $F(X, s_i)$  between  $s_j$  and  $s_i$ . The total length of  $F(X, s_i)$  between these two separators only depends on  $X'$  and  $X$ . Since this subproblem contains  $O(\sqrt{\delta})$  points with three different  $x$ -coordinates, its Hanan grid contains only  $O(\sqrt{\delta})$  edges. Therefore, its value can be computed in  $2^{O(\sqrt{\delta})}$  time by simply checking every possible subset of edges. Let  $L(X', X)$  denote the total length of the solution to this subproblem. If no solution exists, we define it to be  $\infty$ . Then we get

$$A[i, X] = \min_{\text{viable } X'} A[j, X'] + L(X', X),$$

where  $s_j$  is the separator immediately preceding  $s_i$ , and the sum is over all crossing patterns  $X'$  that are viable at  $s_j$ .

**The running time.** To analyse the running time, we first determine the number of table entries. There are  $O(n)$  separators, and we have already seen for every separator  $s_i$  there are  $n^{O(\sqrt{\delta})}$  possible viable crossing patterns. Hence, the total number of table entries is  $O(n) \cdot n^{O(\sqrt{\delta})} = n^{O(\sqrt{\delta})}$ . Next, we calculate the time needed per table entry. For each of the  $n^{O(\sqrt{\delta})}$  possible viable crossing patterns  $X'$  we compute  $L(X', X)$  in  $2^{O(\sqrt{\delta})}$  time. This brings the total time needed per table entry to  $n^{O(\sqrt{\delta})}$ .

Since we have  $n^{O(\sqrt{\delta})}$  table entries, each needing  $n^{O(\sqrt{\delta})}$  time, we conclude:

► **Theorem 6.** *Let  $P$  be a sparse point set of size  $n$  inside a  $\delta$ -strip. Then we can compute an MRST on  $P$  in  $n^{O(\sqrt{\delta})}$  time.*

## 4 Random point sets inside a narrow rectangle

In this section we give an algorithm with  $\min\{n^{O(\sqrt{\delta})}, 2^{O(\delta\sqrt{\delta})}n\}$  expected running time for points generated randomly inside a rectangle of height  $\delta$  and expected width  $n$ . Specifically, we assume the points in  $P$  are generated as follows. First, we generate  $n$  independent exponentially distributed variables  $\Delta_0, \dots, \Delta_{n-1} \sim \text{Exp}(1)$ . Using these, we compute the  $x$ -coordinates of our points by setting  $x_i := \sum_{j=0}^{i-1} \Delta_j$  for  $1 \leq i \leq n$ . Next, we generate the  $y$ -coordinates of the points by picking each  $y_i$  uniformly and independently from the interval  $[0, \delta]$ . Thus the points from  $P$  lie inside the rectangle  $[0, x_n] \times [0, \delta]$ . Since the spacings  $\Delta_i$  are chosen from an exponential distribution of rate 1, we have  $\mathbb{E}[x_n] = \mathbb{E}[\sum_{i=0}^{n-1} \Delta_i] = n$ . (More precisely,  $\sum_{i=0}^{n-1} \Delta_i$  converges to a normal distribution with mean  $n$  and variance  $\sqrt{n}$ .)

Recall that the algorithm for sparse point sets from the previous section, which had running time  $n^{O(\sqrt{\delta})}$ , was based on the fact that any separator  $s_i$  of a sparse point set is crossed only  $O(\sqrt{\delta})$  times. Thus for each separator there are  $n^{O(\sqrt{\delta})}$  different crossing patterns. Our main goal is now to change this algorithm into an algorithm for random point



sets that is fixed-parameter tractable with parameter  $\delta$ . We face two difficulties. First, unlike in the case of sparse point sets, we cannot guarantee that all separators are crossed only  $O(\sqrt{\delta})$  times. Second, even if a separator is crossed  $O(\sqrt{\delta})$  times, the number of candidate crossing patterns can still be  $n^{\Theta(\sqrt{\delta})}$ , which is too much for an FPT algorithm. We overcome these difficulties as follows.

To deal with the first issue we will define a certain configuration of points and a corresponding separator – we will call such separator a *soft wall* – such that the separator is crossed only  $O(\sqrt{\delta})$  times. Our new dynamic programming algorithm will have table entries for every soft wall instead of for every separator. We will prove that we expect to find sufficiently many soft walls, so that the expected number of points in between two consecutive soft walls only depends on  $\delta$  (and not on  $n$ ). This still leaves the second problem, because where a soft wall is crossed by an MRST may depend on points from  $P$  that are beyond the previous or next soft wall. Thus the number of crossing patterns can still be  $n^{\Theta(\sqrt{\delta})}$ . We therefore also devise a second type of wall, the *hard wall*. This is a vertical line  $\ell$  through an input point  $p_i$  that will not be crossed at all by an edge of an MRST. The MRST will consist of two independent parts: an MRST for the points to the left of  $\ell$  plus  $p_i$  itself, and an MRST for the points to the right of  $\ell$  plus  $p_i$  itself. More generally, if we have a collection of hard walls then the subproblems between any two consecutive hard walls are completely independent. Hard walls will occur much less frequently than soft walls, but still the expected number of points in between two consecutive hard walls will be shown to depend only on  $\delta$ . Hence, the number of crossing patterns we need to consider for the soft walls in between the two hard walls only depends on  $\delta$ , giving us an FPT algorithm.

We first give pseudocode for the global algorithm. Recall that  $P[i, j] := \{p_i, \dots, p_j\}$ . The constant 100 mentioned is not special; it is merely an arbitrary large enough constant.

■ **Algorithm 1** COMPUTEMRST( $P$ ).

- 
- 1: Compute a collection  $\mathcal{W}_{\text{hard}} = \{\ell_0, \dots, \ell_m\}$  of hard walls, as described below. The walls in  $\mathcal{W}_{\text{hard}}$  are numbered from left to right, with  $\ell_0$  and  $\ell_m$  being “hard walls” consisting of the leftmost and rightmost points of  $P$ , respectively.
  - 2: **for**  $i \leftarrow 0$  to  $m - 1$  **do**
  - 3:     Let  $p_j$  and  $p_{j'}$  be the middle points of the hard walls  $\ell_i$  and  $\ell_{i+1}$ , respectively.
  - 4:     **if**  $\delta < 100$  **then**
  - 5:         Compute an MRST  $T_i$  for  $P[j, j']$  using the  $2^{O(\sqrt{n} \log n)}$  algorithm by Fomin *et al.*
  - 6:     **else**
  - 7:         Compute a collection  $\mathcal{W}_{\text{soft}} = \{t_1, \dots, t_z\}$  of soft walls for  $P[j, j']$ , as described below.
  - 8:         Compute an MRST  $T_i$  for  $P[j, j']$  using the dynamic-programming algorithm described in Section 3, but using the collection  $\mathcal{W}_{\text{soft}}$  as separators (instead of using all separators between consecutive points), as described below.
  - 9: **return**  $T_0 \cup \dots \cup T_{m-1}$ .
- 

**Computing hard walls.** Let  $P[i, i + 4]$  be a subset of points from  $P$ , and let  $\ell$  be the vertical line through  $p_{i+2}$ . We call  $\ell$  a *hard wall* if  $P[i, i + 4]$  has the following properties:

- $\Delta_j > \delta$  for all  $i \leq j \leq i + 3$
- $y_{i+1} < y_{i+2} < y_{i+3}$

See Figure 5 for an example of a hard wall. A hard wall indeed splits the problem into independent subproblems, as shown in the lemma below.



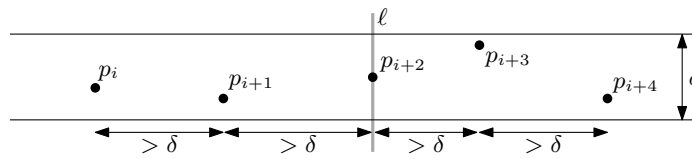


Figure 5 Example of a hard wall.

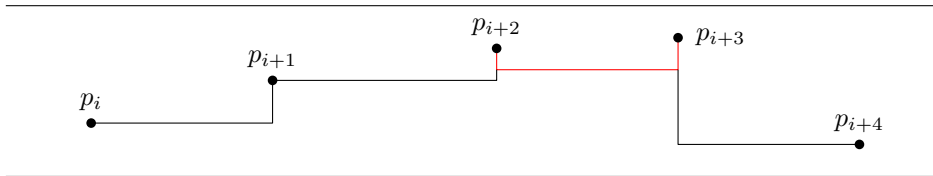


Figure 6 Illustration for the proof of Lemma 7 showing an MRST  $T$  where  $\text{degree}_T(p_{i+2}) = 1$ . In red, the  $U$ -shape showing that  $T$  is not a Hwang tree.

► **Lemma 7.** *Let  $\ell$  be a hard wall, defined by the subset  $P[i, i + 4]$ . Let  $T_1$  be an MRST on  $P[1, i + 2]$  and let  $T_2$  be an MRST on  $P[i + 2, n]$ . Then  $\|T\| = \|T_1\| + \|T_2\|$  and so  $T_1 \cup T_2$  is an MRST on  $P$ .*

**Proof.** Let  $T$  be a Hwang tree on  $P$ . By Observation 3 we know that an MRST on  $P$  is monotonic at  $s_i, \dots, s_{i+3}$ . The monotonicity at  $s_{i+1}$  and  $s_{i+2}$  implies that  $\text{degree}_T(p_{i+2}) \leq 2$ . If  $\text{degree}_T(p_{i+2}) = 2$  then splitting  $T$  at  $p_{i+2}$  results in subtrees on  $P[1, i + 2]$  and  $P[i + 2, n]$  – this follows from the monotonicity at  $s_{i+1}$  and  $s_{i+2}$  – and so we are done. Now assume for a contradiction that  $\text{degree}_T(p_{i+2}) = 1$ . Then the incident edge if  $p_{i+2}$  must be vertical. Assume without loss of generality that  $p_{i+2}$  is the top endpoint of this edge. But then the (single) edge of  $T$  crossing  $s_{i+2}$  must reach the vertical line through  $p_{i+3}$  at a point  $q$  that lies somewhere below  $p_{i+3}$ . The monotonicity at  $s_{i+3}$  then implies that  $q$  must be connected to  $p_{i+3}$  by a vertical segment, thus creating a  $U$ -shape and contradicting that  $T$  is a Hwang tree. See Figure 6 for an example. ◀

The next lemma gives a bound on the probability that  $P[i, i + 4]$  is a hard wall.

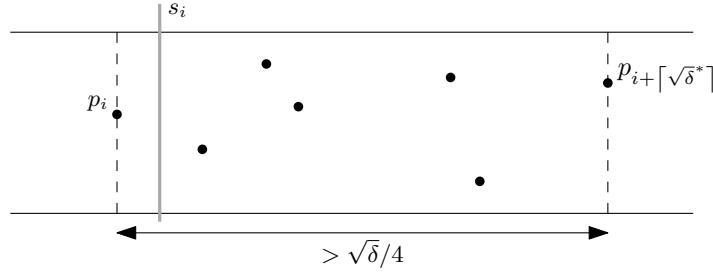
► **Lemma 8.**  $\mathbb{P}\left[ P[i, i + 4] \text{ defines a hard wall} \right] = e^{-4\delta}/6$  for all  $1 \leq i \leq n - 4$ .

**Proof.** Recall that the spacings  $\Delta_j$  are drawn from an exponential distribution with rate 1. Hence,  $\mathbb{P}[\Delta_j > \delta] = e^{-\delta}$  for all  $j$ . Since the spacings are independent, the probability that all four spacings between the points in  $P[i, i + 4]$  are greater than  $\delta$  is  $e^{-4\delta}$ . Finally,  $\mathbb{P}[y_{i+1} < y_{i+2} < y_{i+3}] = 1/6$ , since all  $y$ -coordinates are chosen uniformly at random from  $[0, \delta]$  and so all six orderings of  $y_{i+1} < y_{i+2} < y_{i+3}$  are equally likely. ◀

The set  $\mathcal{W}_{\text{hard}}$  of hard walls is now computed in the following straightforward manner: we check for all  $i := 5j + 1$  with  $j \in \{0, \dots, \lfloor n/5 \rfloor - 1\}$  whether  $P[i, i + 4]$  defines a hard wall; if so, we add the corresponding hard wall to  $\mathcal{W}_{\text{hard}}$ . Note that this takes only  $O(n)$  time in total, as each of the  $O(n)$  candidate hard walls can be checked in  $O(1)$  time.

**Computing soft walls.** Let  $P[i, i + \lceil \sqrt{\delta} \rceil]$  be a subset of  $\lceil \sqrt{\delta} \rceil + 1$  points from  $P$  such that  $x_{i+\lceil \sqrt{\delta} \rceil} - x_i > \lceil \sqrt{\delta} \rceil/4$ . Then we call the separator  $s_i$  – recall that  $s_i$  is the separator between  $p_i$  and  $p_{i+1}$  – a *soft wall*. See Figure 7 for an example.

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■ **Figure 7** Example of a soft wall.

► **Lemma 9.** Let  $\delta \geq 100$ . Let  $s_i$  be a soft wall, defined by  $P[i, i + \lceil\sqrt{\delta}\rceil]$ . Then  $s_i$  is crossed  $O(\sqrt{\delta})$  times by an MRST. Furthermore, even under the assumption that  $\Delta_j < \delta$  for all  $1 \leq j \leq n - 1$ , we have

$$\mathbb{P} \left[ P \left[ i, i + \lceil\sqrt{\delta}\rceil \right] \text{ defines a soft wall} \right] \geq 1 - 2^{3 - \lceil\sqrt{\delta}\rceil/2} \text{ for all } 1 \leq i \leq n - \lceil\sqrt{\delta}\rceil.$$

**Proof.** The fact that  $s_i$  is crossed at most  $O(1 + \lceil\sqrt{\delta}\rceil) = O(\lceil\sqrt{\delta}\rceil)$  times follows immediately from Lemma 4. To be precise,  $s_i$  is crossed at most  $18(2 + \sqrt{\delta})$  times. It remains to derive a lower bound on the probability that  $P[i, i + \lceil\sqrt{\delta}\rceil]$  is a soft wall, given that  $\Delta_j < \delta$  for all  $1 \leq j \leq n - 1$ . We have

$$\begin{aligned} & \mathbb{P} \left[ x_{i+\lceil\sqrt{\delta}\rceil} - x_i > \lceil\sqrt{\delta}\rceil/4 \right] \\ &= 1 - \mathbb{P} \left[ \sum_{j=i}^{i+\lceil\sqrt{\delta}\rceil-1} \Delta_j \leq \lceil\sqrt{\delta}\rceil/4 \right] \\ &\geq 1 - \min_{t>0} e^{t\lceil\sqrt{\delta}\rceil/4} (\mathbb{E}[e^{-t\Delta_1}])^{\lceil\sqrt{\delta}\rceil-1} && \text{by the Chernoff bound} \\ &\geq 1 - \min_{t>0} e^{t\lceil\sqrt{\delta}\rceil/4} \left( \int_{x=0}^{\delta} e^{-tx} e^{-x} \frac{1}{1 - e^{-\delta}} \right)^{\lceil\sqrt{\delta}\rceil-1} && \Delta_1 \sim \text{Exp}(1) \text{ and } \Delta_1 \leq \delta \end{aligned}$$

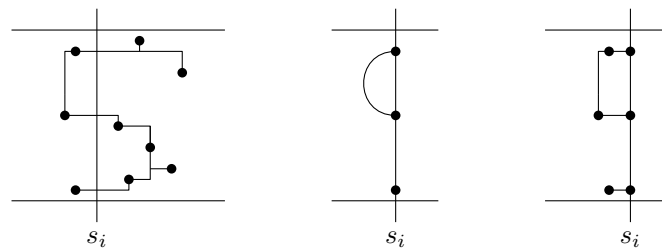
By taking  $t = 3$ , it can be shown that this is at least  $1 - 2^{3 - \lceil\sqrt{\delta}\rceil/2}$ . ◀

Recall that in Algorithm COMPUTEMRST we need to compute soft walls for every subset  $P[j, j']$  between two consecutive hard walls (including the points on those two hard walls). To this end we check whether  $P[p_r, \dots, p_{r+\lceil\sqrt{\delta}\rceil}]$  forms a soft wall for all  $r := j + i\lceil\sqrt{\delta}\rceil$  with  $0 \leq i \leq j' - \lceil\sqrt{\delta}\rceil$ .

**The dynamic-programming algorithm between two hard walls.** Recall that  $X(s_i)$  is the set of points where the Hanan grid crosses  $s_i$ . Let  $\mathcal{X}_i$  denote the family of subsets of  $X(s_i)$  of size at most  $18(2 + \sqrt{\delta})$ . We can now define a table entry  $A[i]$  for each soft wall  $s_i$  as follows.

$$A[i] := \text{a representative set of pairs } (X, l) \text{ where } l \text{ is the minimum length of a rectilinear forest adhering to } X \in \mathcal{X}_i \text{ at } s_i.$$

Here, “representative” means that for every soft wall  $s_i$  there exists an MRST  $T$  and  $(X, l) \in A[i]$  such that  $T$  adheres to  $X$  at  $s_i$ . We will call this  $T$  an MRST represented in  $A[i]$ . See Figure 8 for an example. Note that  $A[n]$  contains one element, of which  $l$  equals



■ **Figure 8** An example of an element of a table entry  $A[i]$ . On the left, an MRST  $T$  represented in  $A[i]$ .  $T$  is represented in  $A[i]$  by a pair  $(X_i, l)$ . In the middle, we have  $X_i$ . On the right, the edges contributing to the length  $l$ . Note that since  $T$  is an MRST, these edges indeed form a minimal length rectilinear forest adhering to  $X_i$  at  $s_i$ .

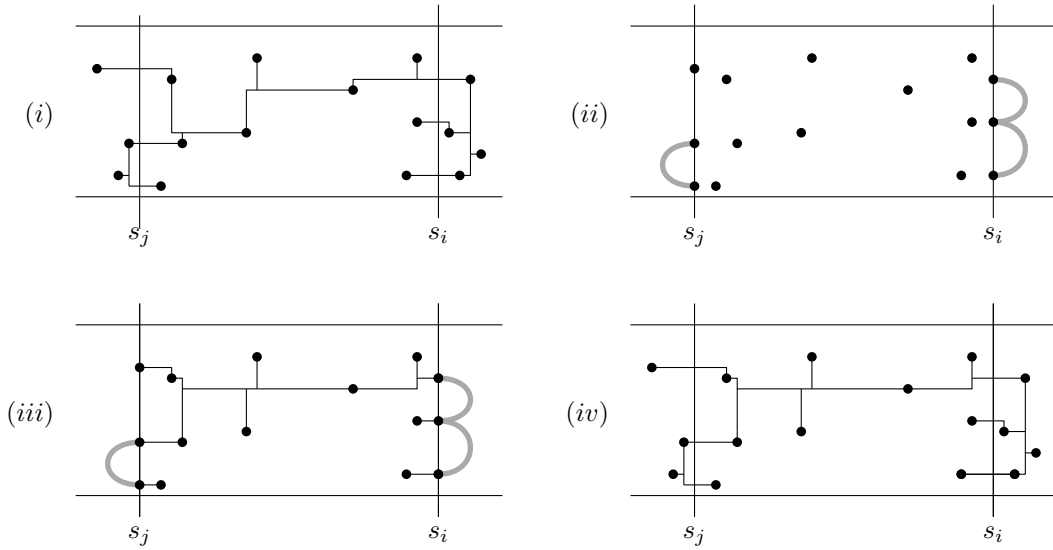
the length of an MRST on  $P$ . Next we describe a recursive formula to compute the table entries. As base case, we have  $A[0] = \{\{\emptyset\}, 0\}$ . We first give pseudocode for this part of the algorithm.

■ **Algorithm 2** COMPUTEA( $i$ ).

- 
- 1: Let  $s_j$  be the rightmost soft wall to the left of  $s_i$ .
  - 2: **for**  $(X_j, l) \in A[j]$  **do**
  - 3:     **for** all viable mirrored crossing patterns  $X'_i$  at  $s_i$  **do**
  - 4:         Compute an MRST  $T'$  for the subproblem given by  $X_j$  and  $X'_i$ .
  - 5:         Add  $(X_i, l + \|T'\|)$  to  $A[i]$ , where  $X_i$  is the crossing pattern of  $T'$  at  $s_i$ .
  - 6: Remove unviable pairs from  $A[i]$ .
- 

Let  $s_j$  be a soft wall, and let  $s_i$  be the rightmost soft wall to the left of  $s_j$ . We define a mirrored crossing pattern to be a crossing pattern where the partition denotes on how the rectilinear Steiner tree is connected on the *right* hand side. Let  $(X_j, l)$  be a pair in  $A[j]$ . Let  $T$  be an MRST adhering to  $X_j$  at  $s_j$ , and adhering to some (unknown) mirrored crossing pattern  $X'_i$  at  $s_i$ . Then there is a pair  $(X_i, l')$  in  $A[i]$ , where  $l'$  equals  $l$  plus the total length of the edges of  $T$  between  $s_j$  and  $s_i$ . The total length of  $T$  between these two separators only depends on  $X_j$  and  $X'_i$ . Let  $L(X_j, X'_i)$  denote the total length of the solution to this subproblem. Now, to compute the value of  $L(X_j, X'_i)$ , we use the  $2^{O(\sqrt{n} \log n)}$  algorithm by Fomin *et al.* [12]. Since this algorithm only computes Steiner trees (not forests adhering to some crossing pattern), we need to adapt our subproblem. To ensure the crossing pattern  $X_j$ , we mimic the edges on the left of  $X_j$ . For every part of  $X_j$ , we add a path of “virtual” edges of length 0, connecting the points in that part. These are automatically added to the so-called shortest path RST found by the first part of the algorithm by Fomin *et al.* Since the number of virtual edges added is constant in  $n$ , it does not affect its running time. We ensure the crossing pattern  $X'_i$  analogously. Given the output  $T'$  of the algorithm, we remove its virtual edges, and analyse its (non-mirrored) crossing pattern  $X_i$  at  $s_i$ . We then add the pair  $(X_i, l + \|T'\|)$  to  $A[i]$ . After doing so for all pairs in  $A[j]$  and viable mirrored crossing patterns  $X'_i$ , we may be able to remove some elements from  $A[i]$ . First, we remove any duplicates. Then, if two pairs have the same crossing pattern  $X_i$ , we need only the one with the smallest  $l$ .

We will now prove that  $A[i]$  is indeed a representative set by induction on  $i$ . Clearly,  $A[0]$  is a representative set. Now, suppose  $A[j]$  is a representative set. We will now show that after performing the above,  $A[i]$  is a representative set. See Figure 9 for an example. Since



■ **Figure 9** Illustration for the correctness proof of Algorithm 2. At (i), we have an MRST  $T$  which adheres to  $X_j$  at  $s_j$ , and to  $X'_i$  at  $s_i$ . At (ii), we have the corresponding subproblem. The thick grey edges denote the virtual edges of length 0. At (iii), a solution  $T'$  to the subproblem. Note that the total length of  $T'$  equals the total length of  $T$  between  $s_j$  and  $s_i$ . At (iv), the new MRST  $T''$  represented in  $A[i]$ , obtained by combining  $T$  and  $T'$ .

$A[j]$  is a representative set, there exists a pair  $(X_j, l) \in A[j]$  and an MRST  $T$  such that  $T$  adheres to  $X_j$  at  $s_j$ . Now,  $T$  adheres to some mirrored crossing pattern  $X'_i$  at  $s_i$ . Therefore, we will find an MRST  $T'$  on the subproblem defined by  $X_j$  and  $X'_i$ , and add a pair  $(X_i, l')$  to  $A[i]$ . Let  $T''$  be the MRST on  $P$  obtained by exchanging the part of  $T$  between  $s_j$  and  $s_i$  for  $T'$ . Note that we can do that, since  $T$  and  $T'$  adhere to the same crossing patterns  $X_j$  and  $X'_i$ . Now,  $T''$  is represented in  $A[i]$  by  $(X_i, l')$ .

**Analysis of the running time.** We now analyze the expected running time of Algorithm COMPUTEMRST. To do so, we will bound certain distributions by other distributions. To be precise, we bound the expected running time of any algorithm on a point set with a random number of points following a certain distribution by the expected running time of the algorithm on a point set with a differently distributed random number of points.

► **Observation 10** ([1]). *Let  $Y_1, Y_2$  be two discrete nonnegative random variables, such that for all  $k \geq 0$ , the equation  $\mathbb{P}[Y_1 \leq k] \geq \mathbb{P}[Y_2 \leq k]$  holds. Let  $f(k)$  be an increasing nonnegative function such that  $\mathbb{E}[f(Y_2)] < \infty$ . Then*

$$\mathbb{E}[f(Y_1)] = \sum_{k=0}^{\infty} f(k)\mathbb{P}[Y_1 = k] \leq \sum_{k=0}^{\infty} f(k)\mathbb{P}[Y_2 = k] = \mathbb{E}[f(Y_2)].$$

We write  $Y_1 \preceq Y_2$  to denote that for all  $k \geq 0$ , the equation  $\mathbb{P}[Y_1 \leq k] \geq \mathbb{P}[Y_2 \leq k]$  holds.

Let us take a look at the sizes of the subproblems defined by the hard walls. Suppose we are computing  $\mathcal{W}_{hard}$  and have just found a hard wall  $\ell_i$ . Let the random variable  $X_1$  denote the number of points in the subproblem  $P[j, j']$  between the two hard walls  $\ell_i$  and the unknown  $\ell_{i+1}$ . Note that  $X_1$  is at most  $m := n - j + 1$ , and that  $X_1$  only depends on  $m$ . Therefore, we will write  $X_{1,m}$ . Now,  $X_{1,m}$  is almost geometrically distributed. There are two differences: we only check whether  $P[i, i + 4]$  defines a hard wall for  $i$  of the form  $5j + 1$ , and

$X_{1,m}$  is at most  $m$ . Since the probability that  $P[i, i + 4]$  defines a hard wall is  $e^{-4\delta}/6$ , we have  $X_{1,m} \sim \min\{m, 1 + 5 \cdot \text{Geom}(e^{-4\delta}/6)\}$ . Here, the probability mass function of  $\text{Geom}(p)$  is  $(1 - p)^{k-1}p$ . Let  $X_2$  be the same distribution, but where we ignore the maximum number of points,  $X_2 \sim 1 + 5 \cdot \text{Geom}(e^{-4\delta}/6)$ . Then,  $X_{1,m} \preceq X_2$  for all  $m$ .

We are now ready to calculate the expected running time of COMPUTEMRST if  $\delta < 100$ . We have already seen that we can find  $\mathcal{W}_{\text{hard}}$  in  $O(n)$  time. Since  $X_{1,m} \preceq X_2$  for all  $m$ , the expected time needed per subproblem is bounded by the expected time needed to run the  $2^{O(\sqrt{n} \log n)}$  algorithm by Fomin *et al.* on a point set with  $X_2$  points. We get:

$$\sum_{k=1}^{\infty} (1 - e^{-4\delta}/6)^{k-1} \cdot (e^{-4\delta}/6) \cdot 2^{O(\sqrt{5k+1} \log(5k+1))} < \sum_{k=1}^{\infty} 2^{-\Theta(k)} \cdot 2^{O(\sqrt{5k} \log(5k))} = O(1)$$

Since there are  $O(n)$  subproblems, this finishes the case  $\delta < 100$ .

We can use the same trick for the distribution of the number of points between soft walls. Here, we let the random variable  $Y_{1,m}$  denote the number of points between two consecutive soft walls, given that we have found no hard walls between the hard walls defining our subproblem and where  $m$  is once more the maximum number of points. We can bound  $Y_{1,m}$  in three steps. First, note that the condition that no  $\Delta_j$  is larger than  $\delta$  is stronger than the condition that there are no hard walls between the hard walls defining our subproblem. Let  $Y_{2,m}$  denote the number of points between the soft walls, given that no  $\Delta_j$  is larger than  $\delta$ . Then  $Y_{1,m} \preceq Y_{2,m}$ . Next, recall that if  $\delta \geq 100$ , by Lemma 9 the probability that  $s_i$  is a soft wall is at least  $1 - 2^{3-\lceil\sqrt{\delta}\rceil/2}$ , even if all  $\Delta_j < \delta$ . Define  $Y_{3,m} \sim \min\{m, \lceil\sqrt{\delta}\rceil \cdot \text{Geom}(1 - 2^{3-\lceil\sqrt{\delta}\rceil/2})\}$ . Then  $Y_{2,m} \preceq Y_{3,m}$ . Finally, analogously to the hard walls, we can remove the maximum number of points. Define  $Y_4 \sim \lceil\sqrt{\delta}\rceil \cdot \text{Geom}(1 - 2^{3-\lceil\sqrt{\delta}\rceil/2})$ . We conclude that  $Y_{1,m} \preceq Y_4$  for all  $m$ .

Let  $c, \lambda > 0$  be such that the algorithm by Fomin *et al.* runs in under  $c \cdot 2^{\lambda\sqrt{n} \log n}$  time. For the case  $\delta \geq 100$ , the total expected time needed per subsubproblem is then bounded by

$$\begin{aligned} & \sum_{i=1}^{\infty} c \cdot 2^{\lambda\sqrt{i\lceil\sqrt{\delta}\rceil+1} \log(i\lceil\sqrt{\delta}\rceil+1)} \cdot \left(2^{3-\lceil\sqrt{\delta}\rceil/2}\right)^{i-1} \cdot \left(1 - 2^{3-\lceil\sqrt{\delta}\rceil/2}\right) \\ & < c \cdot 2^{\lceil\sqrt{\delta}\rceil/2-3} \sum_{i=1}^{\infty} 2^{\lambda\sqrt{2i\lceil\sqrt{\delta}\rceil} \log(2i\lceil\sqrt{\delta}\rceil)} \cdot 2^{(3-\lceil\sqrt{\delta}\rceil/2)i} \cdot 1 \\ & < c \cdot 2^{\lceil\sqrt{\delta}\rceil} \sum_{i=1}^{\infty} 2^{4\lambda(2i\lceil\sqrt{\delta}\rceil)^{3/4}} \cdot 2^{-i\lceil\sqrt{\delta}\rceil/5} \qquad \text{since } \delta \geq 100 \\ & < c \cdot 2^{\lceil\sqrt{\delta}\rceil} \sum_{i=1}^{\infty} 2^{\lceil\sqrt{\delta}\rceil(8\lambda i^{3/4} - i/5)} \end{aligned}$$

Now, for a sufficiently large  $M$ , we have  $8\lambda i^{3/4} - i/5 < -i/10$  for all  $i \geq M$ . We get:

$$\begin{aligned} & c \cdot 2^{\lceil\sqrt{\delta}\rceil} \sum_{i=1}^{\infty} 2^{\lceil\sqrt{\delta}\rceil(8\lambda i^{3/4} - i/5)} \\ & = c \cdot 2^{\lceil\sqrt{\delta}\rceil} \left( \sum_{i=1}^M 2^{\lceil\sqrt{\delta}\rceil(8\lambda i^{3/4} - i/5)} + \sum_{i=M+1}^{\infty} 2^{\lceil\sqrt{\delta}\rceil(8\lambda i^{3/4} - i/5)} \right) \\ & < c \cdot 2^{\lceil\sqrt{\delta}\rceil} \left( M \cdot 2^{\lceil\sqrt{\delta}\rceil \max_{i \geq 1} (8\lambda i^{3/4} - i/5)} + \sum_{i=M+1}^{\infty} 2^{-i/10} \right) = 2^{O(\lceil\sqrt{\delta}\rceil)}. \end{aligned}$$

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Let  $m$  be the number of points in the corresponding subproblem defined by two hard walls. Note that the above bound is independent of  $m$ . Analogously to the original sparse point-set algorithm, there are  $m^{O(\lceil\sqrt{\delta}\rceil)}$  possible crossing patterns per separator. In total, this algorithm therefore takes  $m \cdot m^{O(\lceil\sqrt{\delta}\rceil)} \cdot 2^{O(\lceil\sqrt{\delta}\rceil)} = m^{O(\sqrt{\delta})}$  expected time.

Now, all that remains is calculating the expected running time of our main random point set algorithm in this case. Clearly, it runs in at most expected  $n^{O(\sqrt{\delta})}$  time, since splitting up the problem using the hard walls can only speed up the algorithm.

Let  $\lambda, \mu \geq 1$  be such that the  $m^{O(\sqrt{\delta})}$  expected running time algorithm runs in at most  $m^{\lambda\sqrt{\delta}}$  expected time, and that the probability of a hard wall is  $2^{-\mu\delta}$ . Let  $Y_1$  be the distribution of the number of points of a subproblem. Recall that  $Y_1 \preceq 1 + 5 \cdot \text{Geom}(e^{-4\delta}/6)$ . Then the total expected time needed per subproblem is bounded by

$$\mathbb{E} \left[ Y_1^{\lambda\sqrt{\delta}} \right] \leq \sum_{k=1}^{\infty} (1 - 2^{-\mu\delta})^{k-1} \cdot 2^{-\mu\delta} \cdot (5k+1)^{\lambda\sqrt{\delta}} < O(1) + 2^{-\mu\delta} \sum_{k=2}^{\infty} 2^{4\lambda\sqrt{\delta} \log k - 2^{-\mu\delta} k}.$$

Splitting this sum as well (see the full version for details), it can be shown that:

$$2^{-\mu\delta} \sum_{k=2}^{\infty} 2^{4\lambda\sqrt{\delta} \log k - 2^{-\mu\delta} k} < 2^{O(\delta\sqrt{\delta})}.$$

This brings the total expected running time to  $O(n) \cdot (O(1) + 2^{O(\delta\sqrt{\delta})}) = 2^{O(\delta\sqrt{\delta})}n$ .

All in all, our main random point set algorithm run in  $O(n)$  expected time if  $\delta < 100$ , and in  $\min\{n^{O(\sqrt{\delta})}, 2^{O(\delta\sqrt{\delta})}n\}$  expected time if  $\delta \geq 100$ . We conclude:

► **Theorem 11.** *Let  $P$  be a set of  $n$  points generated randomly inside a rectangle of height  $\delta$  and expected width  $n$ , generated according to the procedure described earlier. Then an MRST on  $P$  can be found in  $\min\{n^{O(\sqrt{\delta})}, 2^{O(\delta\sqrt{\delta})}n\}$  expected time.*

## 5 Concluding remarks

Our paper contains two main results on MINIMUM RECTILINEAR STEINER TREE. First, we proved that for sparse point sets in a strip of width  $\delta$ , an MRST can be found in  $n^{O(\sqrt{\delta})}$  time. Second, we gave a  $\min\{n^{O(\sqrt{\delta})}, 2^{O(\delta\sqrt{\delta})}n\}$  expected running time algorithm for random point sets. For  $\delta = \Theta(n)$  the running time equals the  $2^{O(\sqrt{n} \log n)}$  of the algorithm for arbitrary point sets in the plane [12]. A challenging open problem is to see if an algorithm with running time  $2^{O(\sqrt{\delta} \log \delta)} \text{poly}(n)$  is possible. Another direction for future research is to study the problem in higher dimensions. We believe that our algorithmic results may carry over to  $\mathbb{R}^d$  to points that are almost collinear, that is, that lie in a narrow cylinder. Generalizing the results to, say, points lying in a narrow slab will most likely be more challenging.

More generally, we believe that it is interesting to study the parameterized complexity of geometric problems using a “geometric parameter”. For problems involving planar point sets, the strip width  $\delta$  is a natural parameter, which is interesting because it explores the boundary between the 1-dimensional and 2-dimensional version of the problem. We have studied this for TSP in a previous paper [1] and for MINIMUM RECTILINEAR STEINER TREE in the current paper, but many other problems can be studied from this perspective as well.

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