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Switching Effects Driven by Predation on Diffusive Predator Prey System

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Abstract

In the present study, a mathematical model of three species dynamical system is formulated with the interaction among prey, middle predator and top predator. Here, Holling II functional response consisting of carrying capacity and constant intrinsic growth rate of prey population is taken into account. Also, we have introduced switching intensity in predator population and studied its impact on the stability behavior. Existence of different bifurcation points is derived by considering carrying capacity of prey population as a bifurcation parameter. Furthermore, analysis of conditions for instability in diffusive system is carried out. Computer simulations are used to support our analytical finding.

Keywords: Predation; Switching; Group defence; Equilibria; Stability; Bifurcation; Diffusion

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1. Introduction

Generally, a ratio of predator-prey population is balanced in an ecosystem. The predator instinctively chooses the most abundant prey in its habitat. Sometimes the balance between predator prey population is tipped over which results in switching. So switching can be described as that natural phenomena where a predator instinctively chooses the second preferential prey as there is lack of first preferential prey. A predator can also change its habitat depending on the availability of the prey. Due to heavy predation, the prey species starts declining. Thus, the change in habitat becomes a natural choice of the predator (Coblentz (2020)).

In the new habitat the prey population is significantly higher compared to the predator and it feeds preferentially on the most numerous prey species. Predator switching is seen not only naturally (Murdoch (1969)), but also in laboratory with Notonecta and Ischnura (Lawton et al. (1974)). Mostly these experiments have been done with mathematical models involving one predator-two prey species (Holling (1961); Takahashi (1964); May (1974); Teramoto et al. (1979); Farhan (2020); Suebcharoen (2017)).

The group defense property in predator-prey system has been discussed by Freedman et al. (1992). Studies have proven that the predator species declines and sometimes also becomes extinct when there is no interaction among them. A significant large carrying capacity of the prey also results in the same. They have shown that the system generates Hopf bifurcations, where environmental carrying capacity represents as a bifurcation parameter. The scientists in Khan et al. (1998) and Khan et al. (2004) have studied an interaction of prey and predator species where prey species exhibits group defense. Group defense of prey have been crosschecked in a mathematical model with a predator and two different habitats of prey (Hashem et al. (2013)). Impact of prey migration and predator switching have been examined in predator-prey models in presence of delay (Xu et al. (2013)). Considering Lotka-Volterra model in omnivory system, Holt et al. (1997) have concluded that the system exhibited both coexistence of all species and a limit cycle.

Tanabe et al. (2005) have found that Lotka-Volterra system creates chaos in Holt and Polis's three species model. Pal et al. (2014) have modified chaotic Hastings and Powell model and observed that chaotic system becomes stable with expansion of omnivory. The same system becomes unstable and can also lead to species extinction due to strong omnivory. The researchers further concluded that for a system being stable it is essential that the top predator switches its feeding habit depending upon the relative abundance between middle predator and prey. This phenomenon further ensures species persistence. Further, Chattopadhyay et al. (2015) have examined that the occurrence of stability and the persistence due to predator switching in a food chain system. Pal et al. (2018) have modified a two prey and one predator Vance's model and observed that chaotic system exhibits stability due to increase in the value of the switching intensity. Recently, Wei (2019) has studied an Intraguild predation (IGP) system with prey switching and suggested that prey switching can stabilize the ecosystem.

A mathematical model of predator prey system is proposed by Tansky (1978), which has switching property of predation in one-predator and two-prey system with switching index. Biologically, these functions signify the decrease in predator rate, i.e., frequency of attack on the prey by the predator. This is usually seen when the prey decreases compared to the predator. The switching functions represent a simple multiplicative effect (Tansky (1978)) for index is equal to one, whereas they exhibit an effect which is stronger than the multiplicative one when switching index is getter than one (Prajneshu et al. (1987)).

In a real ecosystem, the migration of populations of different species are a common phenomenon. These migrations are necessitated by various reasons and can take place covering both large and small geographic locations. Movement could also be influenced by change in requirement for specific nutrients. In a mathematical model, migration of different species are demonstrated by reaction-diffusion equations. These equations have been successfully used by several researchers in different ecological contexts (Bhattacharyya et al. (2006); Petrovskii et al. (2001); Petrovskii et al. (2005); Wang et al. (2020); Bhattacharyya et al. (2013); Chatterjee et al. (2016)).

In our study, we explore the following questions:
(a) How the effect of 'mortality rate of top predator' regulates the non-diffusive and diffusive system dynamics.
(b) How the switching intensity incited the whole dynamical system.
(c) How the carrying capacity of prey population influence the prey-predator system.

These questions act as the main motivator of this study. The novelty of the present study is to incorporate switching of predator in a three species Lotka-Volterra model with functional response of Holling II and investigate its effect. Here the switching mechanism of predator and its impact is taken into account. As per my knowledge the value of switching index as one has being used by various researchers, so a new strategy has been employed here by adopting the value as two. Further, we consider the effects of prey as well as predator dispersal on the model system. Work on the conditions of diffusive instability are also performed. The answers to the above questions also adds to the novelty of this study.

2. The Mathematical Model

In this article, we propose the following model consisting of concentration of prey (x_1), middle predator (x_2) and top predator (y) population, respectively:

$$\begin{cases} \frac{dx_1}{dt} = rx_1(1-\frac{x_1}{K}) - \frac{a_{12}x_1x_2}{K_1+x_1} - \frac{a_{13}x_1y}{1+c(\frac{x_2}{x_1})^2}, \\ \frac{dx_2}{dt} = \frac{a_{21}x_1x_2}{K_1+x_1} - \frac{a_{23}x_2y}{1+c(\frac{x_1}{x_2})^2} - \mu_2x_2, \\ \frac{dy}{dt} = \frac{a_{31}x_1y}{1+c(\frac{x_2}{x_1})^2} + \frac{a_{32}x_2y}{1+c(\frac{x_1}{x_2})^2} - \mu_3y. \end{cases} \tag{1}$$

Clearly, due to switching mechanism in the system (1), it is not well-defined at the origin. To overcome the singularity of the above equations we change them at $(0,0,0)$ with the following:

$$\frac{dx_1}{dt} = \frac{dx_2}{dt} = \frac{dy}{dt} = 0. \tag{2}$$

Let r and K be the intrinsic growth rate and carrying capacity of prey population, respectively. The parameters a_{ij} with $i < j$ denote the prey consumption rates, while a_{ij} for $i > j$ represent the corresponding predator's assimilation rates. K_1 is the half saturation constant for prey population. μ_2 and μ_3 are the natural mortalities of the middle and top predators respectively. Here, all the parameters are non-negative.

Definitely, the prey exhibits logistic growth without any predator which have been indicated by the first equation of system (1). When predators are present, the prey are subject to their trapping. In absence of prey, the middle predator is likely to starve; it also constitutes a possible source of food for the top predator in the food chain. The top predator feeds on both prey and middle predators.

The novel feature of this model resides in the last term of the first equation, middle term of second equation and the corresponding first two terms of the last equation, modelling the feeding switching behavior between the two types of prey for the top predator. This phenomenon is regulated by the relative abundances of the prey and middle predator (Chattopadhyay et al. (2015)). The parameter c denotes the switching intensity. In this study, the system implies no switching behavior for $c = 0$ which allows to regulate the amount of switching feeding that the top predator can avail.

2.1. Equilibria

The system (1) has five steady states while $E_s = (x_{1s}, x_{2s}, y_s)$, $s = 0, ...4$.
(i) The origin $E_0 = (0, 0, 0)$, representing ecosystem disappearance, fortunately cannot be stably attained.
(ii) The prey-only equilibrium $E_1 = (K, 0, 0)$.
(iii) Middle predator free equilibrium $E_2 = (x_{12}, 0, y_2) = \left(\frac{\mu_3}{a_{31}}, 0, \frac{r(a_{31}K-\mu_3)}{a_{13}a_{31}K}\right)$, which exists if $K > \frac{\mu_3}{a_{31}}$.
(iv) Top predator free equilibrium $E_3 = (x_{13}, x_{23}, 0) = \left(\frac{K_1\mu_2}{a_{21}-\mu_2}, \frac{ra_{21}K_1(Ka_{21}-K\mu_2-K_1\mu_2)}{Ka_{12}(a_{21}-\mu_2)^2}, 0\right)$, which exists if $\mu_2 < \min\{a_{21}, \frac{a_{21}K}{K+K_1}\}$.
(v) The interior equilibrium point $E_4 = (x_{14}, x_{24}, y_4)$ with

$$x_{14} = \frac{\mu_3 x_l(1 + cx_l^2)(x_l^2 + c)}{(a_{31}x_l^2(1 + cx_l^2) + a_{32}(x_l^2 + c))}, \; x_{24} = \frac{\mu_3 x_l(1 + cx_l^2)(x_l^2 + c)}{x_{14}(a_{31}x_l^2(1 + cx_l^2) + a_{32}(x_l^2 + c))},$$

and

$$y_4 = \frac{1 + cx_l^2}{a_{23}}\left[\frac{(a_{21} - \mu_2)A - K_1\mu_2}{K_1 + A}\right] = \frac{[r(1 - \frac{A}{K}) - \frac{a_{12}A}{x_l(K_1+A)}](x_l^2 + c)}{a_{13}x_l^2},$$

where $x_l = \frac{x_{14}}{x_{24}}$ is one of the real positive roots of 17th order equation.

3. The system general behavior

Let $\overline{E} = (\overline{x}_1, \overline{x}_2, \overline{y})$ be any arbitrary equilibrium. Then the Jacobian matrix about $\overline{E}$ is given by

$$\overline{V} = \begin{bmatrix} \overline{m}_{11} & -\frac{a_{12}\overline{x}_1}{K_1+\overline{x}_1} + \frac{2ca_{13}\overline{x}_1^3\overline{x}_2\overline{y}}{(\overline{x}_1^2+c\overline{x}_2^2)^2} & -\frac{a_{13}\overline{x}_1^3}{(\overline{x}_1^2+c\overline{x}_2^2)} \\ \frac{K_1a_{21}\overline{x}_2}{(K_1+\overline{x}_1)^2} + \frac{2ca_{23}\overline{x}_1\overline{x}_2^3\overline{y}}{(\overline{x}_2^2+c\overline{x}_1^2)^2} & \overline{m}_{22} & -\frac{a_{23}\overline{x}_2^3}{(\overline{x}_2^2+c\overline{x}_1^2)} \\ \frac{a_{31}\overline{x}_1^2\overline{y}(\overline{x}_1^2+3c\overline{x}_2^2)}{(\overline{x}_1^2+c\overline{x}_2^2)^2} - \frac{2ca_{32}\overline{x}_1\overline{x}_2^3\overline{y}}{(\overline{x}_2^2+c\overline{x}_1^2)^2} & \frac{a_{32}\overline{x}_2^2\overline{y}(\overline{x}_2^2+3c\overline{x}_1^2)}{(\overline{x}_2^2+c\overline{x}_1^2)^2} - \frac{2ca_{31}\overline{x}_1^3\overline{x}_2\overline{y}}{(\overline{x}_1^2+c\overline{x}_2^2)^2} & \overline{m}_{33} \end{bmatrix},$$

where

$$\overline{m}_{11} = r - \frac{2r\overline{x}_1}{K} - \frac{K_1a_{21}\overline{x}_2}{(K_1+\overline{x}_1)^2} - \frac{a_{13}\overline{x}_1^2\overline{y}(\overline{x}_1^2+3c\overline{x}_2^2)}{(\overline{x}_1^2+c\overline{x}_2^2)^2},$$

$$\overline{m}_{22} = -\mu_2 + \frac{a_{21}\overline{x}_1}{(K_1+\overline{x}_1)} - \frac{a_{23}\overline{x}_2^2\overline{y}(\overline{x}_2^2+3c\overline{x}_1^2)}{(\overline{x}_2^2+c\overline{x}_1^2)^2}, \overline{m}_{33} = -\mu_3 + \frac{a_{31}\overline{x}_1^3}{\overline{x}_1^2+c\overline{x}_2^2} + \frac{a_{32}\overline{x}_2^3}{\overline{x}_2^2+c\overline{x}_1^2}.$$

The equilibrium point E_0 always exists but is unstable

The prey-only equilibrium $E_1 = (K, 0, 0)$ is locally asymptotically stable if the following conditions hold: $\frac{a_{21}K}{K_1+K} < \mu_2$ and $a_{31}K < \mu_3$ for E_1.

Top predator free equilibrium $E_3 = (x_{13}, x_{23}, 0)$ is locally asymptotically if the following conditions hold: $\mu_3 > \frac{a_{31}x_{13}^3}{x_{13}^2+cx_{23}^2} + \frac{a_{32}x_{23}^3}{x_{23}^2+cx_{13}^2}$ and $\frac{r}{K} > \frac{a_{23}x_{23}}{(K_1+x_{13})^2}$ for E_3.

3.1. Behavior of the system around equilibrium $E_2(x_{12}, 0, y_2)$

The characteristic equation of system (1) near equilibrium point E_2 is:

$$\lambda_1^3 - (n_{11}+n_{22})\lambda_1^2 + (n_{11}n_{22} - n_{13}n_{31})\lambda_1 + n_{13}n_{31}n_{22} = 0,$$

or equivalently,

$$\lambda_1^3 + Q_{11}\lambda_1^2 + Q_{22}\lambda_1 + Q_{33} = 0,$$

where

$$n_{11} = \frac{rx_{12}}{K} > 0, n_{12} = -\frac{a_{12}x_{12}}{K_1+x_{12}} < 0, n_{13} = -a_{13}x_{12} < 0, n_{22} = -\mu_2 + \frac{a_{21}x_{12}}{K_1+x_{12}}, n_{31} = a_{13}y_2.$$

Case 1: If $n_{22} > 0$, then $Q_{33} < 0$, which shows unstable behavior.
Case 2: If $n_{22} < 0$, then $Q_{33} > 0$. Also, $Q_{11} > 0$ if $\mu_2 > \frac{rx_{12}}{K} + \frac{a_{21}x_{12}}{K_1+x_{12}}$ and $Q_{22} > 0$ if $n_{11}n_{22} > n_{13}n_{31}$. Finally, $Q_{11}Q_{22} - Q_{33} > 0$ if $Q_{11}Q_{22} > Q_{33}$, then E_2 is asymptotically stable which depends on system parameters.

3.2. Behavior of the system around equilibrium $E_4(x_{14}, x_{24}, y_4)$

The characteristic equation of system (1) near equilibrium point E_4 is:

$$\lambda^3 + Q_1\lambda^2 + Q_2\lambda + Q_3 = 0, \tag{3}$$

where

$$Q_1 = -(m_{11} + m_{22}),$$

$$Q_2 = (m_{11}m_{22} - m_{23}m_{32} - m_{12}m_{21} - m_{13}m_3),$$

$$Q_3 = (m_{11}m_{23}m_{32} + m_{13}m_{31}m_{22} - m_{12}m_{31}m_{23} - m_{13}m_{21}m_{32}),$$

$$m_{11} = r - \frac{2rx_{14}}{K} - \frac{K_1a_{21}x_{24}}{(K_1 + x_{14})^2} - \frac{a_{13}x_{14}^2y_4(x_{14}^2 + 3cx_{24}^2)}{(x_{14}^2 + cx_{24}^2)^2},\ m_{12} = -\frac{a_{12}x_{14}}{K_1 + x_{14}} + \frac{2ca_{13}x_{14}^3x_{24}y_4}{(x_{14}^2 + cx_{24}^2)^2},$$

$$m_{13} = -\frac{a_{13}x_1^3}{x_{14}^2 + cx_{24}^2},\ m_{21} = \frac{K_1a_{21}x_{24}}{(K_1 + x_{14})^2} + \frac{2ca_{23}x_{14}x_{24}^3y_4}{(x_{24}^2 + cx_{14}^2)^2},$$

$$m_{22} = -\mu_2 + \frac{a_{21}x_{14}}{K_1 + x_{14}} - \frac{a_{23}x_{24}^2y_4(x_{24}^2 + 3cx_{14}^2)}{(x_{24}^2 + cx_{14}^2)^2},\ m_{23} = -\frac{a_{23}x_{24}^3}{x_{24}^2 + cx_{14}^2},$$

$$m_{31} = \frac{a_{31}x_{14}^2y_4(x_{14}^2 + 3cx_{24}^2)}{(x_{14}^2 + cx_{24}^2)^2} - \frac{2ca_{32}x_{14}x_{24}^3y_4}{(x_{24}^2 + cx_{14}^2)^2},$$

$$m_{32} = \frac{a_{32}x_{24}^2y_4(x_{24}^2 + 3cx_{14}^2)}{(x_{24}^2 + cx_{14}^2)^2} - \frac{2ca_{31}x_{14}^3x_{24}y_4}{(x_{14}^2 + cx_{24}^2)^2}.$$

According Routh-Hurwitz criterion, we need to prove the following statements: (i) $Q_i > 0$ for i=1,2,3, (ii) $Q_1Q_2 - Q_3 > 0$. It is clear that some terms are complicated. Therefore, we can not explain the exact sign of all the terms. Thus, it is a very difficult task to study of the stability behavior around interior equilibrium E_4. So, numerical simulations are used to examine the stability behavior around the interior equilibrium by numerical simulation.

To study the oscillatory nature, K is taken as the bifurcation parameter in system (1). Now, if we can find a value of $K = K^*$ such that $Q_1(K^*)Q_2(K^*) - Q_3(K^*) = 0$, then the eigenvalues of (3) represent as:

$$\begin{aligned}\lambda_{1,2} &= \pm\mathbf{i}\sqrt{Q_2(K^*)},\\ \lambda_3 &= -Q_1(K^*).\end{aligned} \tag{4}$$

Let $\lambda = p + \mathbf{i}q$ be a root of (3) then

$$\left[\frac{dp}{dK}\right]_{K^*} = -\left[\frac{(Q_1Q_2)' - Q_3'}{2(Q_1^2 + Q_2^2)}\right]_{K=K^*}. \tag{5}$$

Thus, if $[Q_1Q_2]'_{K=K^*} < [Q'_3]_{K=K^*}$, then $\left[\frac{dp}{dK}\right]_{K^*} > 0$. Consequently, a Hopf-bifurcation will be seen around E_4 and all three species will fluctuate. The oscillation observed in prey species signifies the switching tendency of the predator ecologically (Bhattacharyya et al. (2006)).

4. Stability Analysis in the Presence of Diffusion

To study the effects of diffusion on the model system, we assume prey, middle predator and top predator are diffusing in the rectangular domain $\Theta = [0, \mathbf{L}] \times [0, \mathbf{H}] \subseteq \mathbf{R}^2$. Therefore, according to Fick's law, the diffusive system is governed by the system of equations in the domain Θ which are

$$\begin{cases} \frac{\partial x_1}{\partial t} = rx_1(1 - \frac{x_1}{K}) - \frac{a_{12}x_1x_2}{K_1+x_1} - \frac{a_{13}x_1y}{1+c(\frac{x_2}{x_1})^2} + D_1\nabla^2 x_1, & (\mathbf{x}, \mathbf{y}, t) \in \Theta \times (0, \infty), \\ \frac{\partial x_2}{\partial t} = \frac{a_{21}x_1x_2}{K_1+x_1} - \frac{a_{23}x_2y}{1+c(\frac{x_1}{x_2})^2} - \mu_2 x_2 + D_2\nabla^2 x_2, & (\mathbf{x}, \mathbf{y}, t) \in \Theta \times (0, \infty), \\ \frac{\partial y}{\partial t} = \frac{a_{31}x_1y}{1+c(\frac{x_2}{x_1})^2} + \frac{a_{32}x_2y}{1+c(\frac{x_1}{x_2})^2} - \mu_3 y + D_3\nabla^2 y, & (\mathbf{x}, \mathbf{y}, t) \in \Theta \times (0, \infty), \end{cases} \tag{6}$$

with the initial conditions $x_1(\mathbf{x}, \mathbf{y}, 0) \geq 0$, $x_2(\mathbf{x}, \mathbf{y}, 0) \geq 0$, $y(\mathbf{x}, \mathbf{y}, 0) \geq 0$, for all $(\mathbf{x}, \mathbf{y}) \in \Theta$ and satisfying $\frac{\partial x_1}{\partial n_1} = \frac{\partial x_2}{\partial n_1} = \frac{\partial y}{\partial n_1} = 0$, i.e., the zero-flux boundary condition in $\partial\Theta \times (0, \infty)$ (Bhattacharyya et al. (2013)). We assume n_1, the outward unit normal vector is smooth of the boundary $\partial\Theta$. Here, ∇^2 represents the Laplacian operator in two dimensional space and D_1, D_2 and D_3 are the constant diffusion coefficients of prey, middle predator and top predator population.

Now we formulate the linearized form of the system (6) about $(\hat{x_1}, \hat{x_2}, \hat{y})$ as:

$$\begin{cases} \frac{\partial X_1}{\partial t} = \hat{n}_{11}X_1 + \hat{n}_{12}X_2 + \hat{n}_{13}Y + D_1\nabla^2 X_1, \\ \frac{\partial X_2}{\partial t} = \hat{n}_{21}X_1 + \hat{n_{22}}X_2 + \hat{n}_{23}Y + D_2\nabla^2 X_2, \\ \frac{\partial Y}{\partial t} = \hat{n}_{31}X_1 + \hat{n}_{32}X_2 + \hat{n}_{33}Y + D_3\nabla^2 Y, \end{cases} \tag{7}$$

where $x_1 = \hat{x_1} + X_1, x_2 = \hat{x_2} + X_2, y = \hat{y} + Y$ and

$$\hat{n}_{11} = r - \frac{2r\hat{x}_1}{K} - \frac{K_1 a_{21}\hat{x}_2}{(K_1 + \hat{x}_1)^2} - \frac{a_{13}\hat{x}_1^2\hat{y}(\hat{x}_1^2 + 3c\hat{x}_2^2)}{(\hat{x}_1^2 + c\hat{x}_2^2)^2}, \hat{n}_{12} = -\frac{a_{12}\hat{x}_1}{K_1 + \hat{x}_1} + \frac{2ca_{13}\hat{x}_1^3\hat{x}_2\hat{y}}{(\hat{x}_1^2 + c\hat{x}_2^2)^2},$$

$$\hat{n}_{13} = -\frac{a_{13}\hat{x}_1^3}{\hat{x}_1^2 + c\hat{x}_2^2}, \hat{n}_{21} = \frac{K_1 a_{21}\hat{x}_2}{(K_1 + \hat{x}_1)^2} + \frac{2ca_{23}\hat{x}_1\hat{x}_2^3\hat{y}}{(\hat{x}_2^2 + c\hat{x}_1^2)^2}, \hat{n}_{22} = -\mu_2 + \frac{a_{21}\hat{x}_1}{K_1 + \hat{x}_1} - \frac{a_{23}\hat{x}_2^2\hat{y}(\hat{x}_2^2 + 3c\hat{x}_1^2)}{(\hat{x}_2^2 + c\hat{x}_1^2)^2},$$

$$\hat{n}_{23} = -\frac{a_{23}\hat{x}_2^3}{\hat{x}_2^2 + c\hat{x}_1^2}, \hat{n}_{31} = \frac{a_{31}\hat{x}_1^2\hat{y}(\hat{x}_1^2 + 3c\hat{x}_2^2)}{(\hat{x}_1^2 + c\hat{x}_2^2)^2} - \frac{2ca_{32}\hat{x}_1\hat{x}_2^3\hat{y}}{(\hat{x}_2^2 + c\hat{x}_1^2)^2},$$

$$\hat{n}_{32} = \frac{a_{32}\hat{x}_2^2\hat{y}(\hat{x}_2^2 + 3c\hat{x}_1^2)}{(\hat{x}_2^2 + c\hat{x}_1^2)^2} - \frac{2ca_{31}\hat{x}_1^3\hat{x}_2\hat{y}}{(\hat{x}_1^2 + c\hat{x}_2^2)^2}, \hat{n}_{33} = -\mu_3 + \frac{a_{31}\hat{x}_1^3}{\hat{x_1}^2 + c\hat{x_2}^2} + \frac{a_{32}\hat{x_2}^3}{\hat{x_2}^2 + c\hat{x_1}^2}.$$

It is noteworthy that (X_1, X_2, Y) are small perturbations of (x_1, x_2, y) about the equilibrium point $(\hat{x_1}, \hat{x_2}, \hat{y})$. We start by assuming solutions of the form $\begin{pmatrix} X_1 \\ X_2 \\ Y \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} e^{\lambda t + \mathbf{i}(k_x \mathbf{x} + k_y \mathbf{y})}$. Here, $\lambda > 0$

and $v_j > 0$ $(j = 1, 2, 3)$ represent the frequency and the amplitude while $k_{\mathbf{x}}, k_{\mathbf{y}} > 0$ be the wave number of the perturbations in time t. Using $k^2 = k_{\mathbf{x}}^2 + k_{\mathbf{y}}^2$, the system (7) stands as

$$\begin{cases} \frac{\partial X_1}{\partial t} = (\hat{n}_{11} - D_1 k^2)X_1 + \hat{n}_{12}X_2 + \hat{n}_{13}Y, \\ \frac{\partial X_2}{\partial t} = \hat{n}_{21}X_1 + (\hat{n_{22}} - D_2 k^2)X_2 + \hat{n}_{23}Y, \\ \frac{\partial Y}{\partial t} = \hat{n}_{31}X_1 + \hat{n}_{32}X_2 + (\hat{n}_{33} - D_3 k^2)Y. \end{cases} \tag{8}$$

Here, we only analyze the stability behavior at E_4 in diffusive system. The characteristic equation of the linearized system (8) at E_4 is $\lambda^3 + Q_1^*\lambda^2 + Q_2^*\lambda + Q_3^* = 0$, where
$Q_1^* = Q_1 + \zeta_1 k^2$,
$Q_2^* = Q_2 - \zeta_2 k^2 + \zeta_3 k^4$,
$Q_3^* = Q_3 + \zeta_4 k^2 - (m_{11}D_2D_3 + m_{22}D_1D_3)k^4 + D_1D_2D_3k^6$,
$\zeta_1 = (D_1 + D_2 + D_3)$,
$\zeta_2 = [m_{22}D_1 + m_{11}D_2 + (m_{11} + m_{22})D_3]$,
$\zeta_3 = (D_1D_2 + D_2D_3 + D_3D_1)$,
$\zeta_4 = [-m_{23}m_{32}D_1 - m_{13}m_{31}D_2 + (m_{11}m_{22} - m_{12}m_{21})D_3]$.

Here, $Q_1^*Q_2^* - Q_3^* = Q_1Q_2 - Q_3 + k^6H_1 - k^2H_2 + k^2H_3$, where
$H_1 = (D_1 + D_3)(D_1D_2 + D_2D_3 + D_3D_1) + D_2^2(D_1 + D_3) > 0$,
$H_2 = m_{11}D_1(D_2 + D_3) + m_{22}D_2(D_1 + D_3) + \zeta_1\zeta_2$, and
$H_3 = Q_2\zeta_1 + m_{11}^2(D_1 + D_3) + m_{22}^2(D_1 + D_3) + m_{11}m_{22}(D_3 + \zeta_1) - \zeta_4$.

Diffusive instability occurs if $Q_1^*Q_2^* \leq Q_3^*$. This happens if $k^4H_1 - k^2H_2 + H_3 \leq 0$ as well as $k^6H_1 - k^4H_2 + k^2H_3 > Q_3 - Q_1Q_2$. Since $H_1 > 0$ and $H_3 > 0$, it implies that $k^4H_1 - k^2H_2 + H_3 \leq 0$ if $H_2 \geq \frac{k^4H_1 + H_3}{k^2} > 0$. Let $H(\eta) = \eta^2H_1 - \eta H_2 + H_3$, $\eta = k^2$. Since $H'' = 2H_1 = 2(D_1 + D_3)(D_1D_2 + D_2D_3 + D_3D_1) + D_2^2(D_1 + D_3) > 0$, it indicates that H has a minimum at $\eta = k_c^2$, where $k_c^2 = \frac{H_2}{2H_1}$. Thus, $H_{min} = k_c^4H_1 - k_c^2H_2 + H_3 = H_3 - \frac{H_2^2}{4H_1} \leq 0$ if $H_2 \geq 2\sqrt{H_1H_3}$. Therefore, when $k^4H_1 - k^2H_2 + H_3 \leq 0$ and $k^6H_1 - k^4H_2 + k^2H_3 > Q_3 - Q_1Q_2$ are satisfied, diffusive instability will occur.

5. Numerical Simulations

In this section, the dynamic characteristics of predator-prey species is being emphasized with the help of numerical simulations. We begin with a set of parametric values,

$$\begin{aligned} &r = 1.1,\ K = 8,\ a_{12} = 0.8,\ a_{13} = 0.6,\ a_{21} = 0.6,\ a_{23} = 0.9, \\ &a_{31} = 0.2,\ a_{32} = 0.6,\ c = 1,\ K_1 = 0.3,\ \mu_2 = o.2,\ \mu_3 = 0.5, \end{aligned} \tag{9}$$

for which the existing condition of the coexistence equilibrium point $E_4 = (1.674, 1.291, 0.919)$ is satisfied and E_4 is locally asymptotically stable (cf. Figure 1). Now by varying the different parametric values we study the changes of the dynamic system (1).

5.1. Dynamical changes due to K

Taking $K = 4$, the system exhibits oscillations around E_4 (cf. Figure 2(a)). Figures 3(a-c) illustrate the different steady state behaviour of prey, middle predator and top predator in the system (1) for the parameter K. Here we see two Hopf bifurcation points at $K = 4.538705$ and 21.111319 (red star (H)) with first Lyapunov coefficient is $-2.068938e^{-001}$ and $-8.370017e^{-003}$, respectively. In both cases, we note that the two complex eigenvalues of real part ≈ 0. Clearly the negative sign of the first Lyapunov coefficient indicates that a stable limit cycle bifurcates from the equilibrium and looses stability. Here LP denotes limit point bifurcation of the curve while the LP-coefficient is nonzero with eigenvalues $0, 0.0489 \pm 0.5451i$. To proceed further, a family of stable limit cycles bifurcating from Hopf points is plotted (cf. Figure 3(d)).

5.2. Dynamical changes due to μ_3

From Figure 2(b) it follows high mortality rate of top predator (viz. $\mu_3 = 0.8$), for which the system (1) can switch to oscillatory behavior around E_4. Figures 4(a-c) represent the different steady state behaviour of each species for the parameter μ_3. These show that Hopf bifurcation point at $\mu_3 = 0.668829$ (red star (H)) with first Lyapunov coefficient is $-2.978162e^{-001}$ which represents that a stable limit cycle bifurcates from the equilibrium and it looses stability. To proceed further, we have a limit point (LP) at $\mu_3 = 0.674112$ with eigenvalues $0, 0.0186689 \pm 0.603271i$ while the LP-coefficient is nonzero. Here a branch point (BP) is located at $\mu_3 = 0.345$ with eigenvalues $0, 0.169583 \pm 0.33935i$ and the transcritical bifurcation occur. We are drawn a family of stable limit cycles bifurcating from H points (cf. Figure 4(d)).

5.3. Dynamical changes due to r

Figure 2(c) follows that the system switches from stable to oscillatory behavior due to low value of intrinsic growth rate (viz. $r = 0.9$). In the Figures 5(a-c), we see one Hopf point (H) at $r = 0.981204$, one limit point (LP) at $r = 0.976078$ and one branch point (BP) at $r = 1.553236$ of the system (1) with respect to r. The value of first Lyapunov coefficient is $-3.889208e^{-01}$ at Hopf point which indicates a supercritical bifurcation. Further, a family of stable limit cycles produce a period-doubling bifurcation (PD) at $r = 0.8272320$ (cf. Figure 5(d)).

5.4. Dynamical changes due to c

Figure 2(d) depicts the oscillatory behavior of system (1) for high value of switching intensity (viz. $c = 2$). It is evident that we have one limit point at $c = 1.976826$ and one Hopf point at $c = 1.929548$ with first Lyapunov coefficient $-3.522685e^{-01}$ (cf. Figures 6(a-c)). We are now simulated a family of stable limit cycles bifurcating from Hopf point when c as the free parameter (cf. Figure 6(d)). Summarizing the observations, natures of equilibrium points are displayed in Table 1.

Table 1. Natures of equilibrium points

Parameters	Values	Eigenvalues	Equilibrium points
K	21.111319	$(-0.509533, \pm 0.531116i)$	Hopf (H)
	4.538705	$(-0.06726, \pm 0.553698i)$	Hopf (H)
	4.304647	$(0, 0.0489 \pm 0.5451i)$	Limit Point (LP)
μ_3	0.668829	$(-0.0474, \pm .60164i)$	Hopf (H)
	0.674112	$(0, 0.0187 \pm 0.6033i)$	Limit Point (LP)
	0.345	$(0, 0.169583 \pm 0.33935i)$	Branch Point (BP)
c	1.976826	$(0, 0.036698 \pm 0.606217i)$	Limit Point (LP)
	1.929548	$(-0.0435022, \pm 0.603951i)$	Hopf (H)
r	0.981204	$(-0.0497222, \pm 0.525789i)$	Hopf (H)
	0.976078	$(0, 0.0240483 \pm 0.527178i)$	Limit Point (LP)
	1.553236	$(0, 0.239457 \pm 0.381936)$	Branch Point (BP)

5.5. Bifurcation diagram

For a better understanding of the dynamical implication of changes in some parameters, we have displayed four suitable bifurcation diagrams when K, μ_3, r and c are treated as free parameters simultaneously (cf. Figures 7(a-d)).

Finally, two parameter bifurcation diagrams $K-r$, $K-c$ and μ_3-c (cf. Figures 8(a,b,c)) respectively are generated. Here we see a generalized Hopf (GH) point, where the first Lyapunov coefficient vanishes but the second lyapounov coefficient is non-zero. In the parameter plain, GH point splits into two branches of sub and supercritical Andronov-Hopf bifurcations. The Bogdanov-Takens (BT) point is a common intersecting point between the limit point curve and the curve corresponding to equilibria. Here ZH represents zero-Hopf bifurcation point at which we see one zero eigenvalue and pair of purely imaginary eigenvalues.

To identify the effects of diffusion coefficients we have plotted the Figures 9(a-c) (viz. $D_1 = 0.5$, $D_2 = 0.5$ and $D_3 = 0.5$). These figures show that biomass distribution of all species is spatially stable. Again for $K = 4$ with $D_1 = 0.5$, $D_2 = 0.5$ and $D_3 = 0.5$, diffusive system shows oscillatory behavior at E_4 (cf. Figures 10(a-c)). Finally, we have noted that by incorporating the high value of switching intensity (viz. $c = 2$), spatial system becomes unstable (cf. Figures 11 (a-c)).

6. Conclusion

We have formulated a mathematical model sketching the interaction of prey-middle predator-top predator species which has switching property of predation. In our study, due to predation when the middle predator declines, the top predator switches its preference to the next easily available species which is the prey. As a result, the middle predator which act as the primary prey of the top predator is not radically reduced because hunting stops or reduces when its numbers decline. In this scenario, the predation will resume after some time such that no prey population becomes too much abundant. Therefore, the stability and the persistence of the system occur due to such mechanism in a prey predator system. We have shown numerically that if we decrease the prey carrying capacity significantly by enrichment of the environment and increase of the mortality rate of top predator simultaneously it could lead to oscillation of all population. It is also observed that the system tends to top predator free equilibrium for high value of prey carrying capacity. Our observation have been that all the population fluctuate for high value of switching intensity of predator population. Further, our analysis suggest that when the carrying capacity of prey is high, the system (6) undergo diffusive instability. Also, we have found that the systems (1) and (6) can exhibit periodic fluctuations of all species due to higher values of switching intensity. So, we can conclude that whenever the non-diffusive system becomes unstable, the diffusive system also follows it which satisfy our analytical results. Thus, in order to maintain stable coexistence between all species establishment of moderate parametric values is essential.

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Appendix

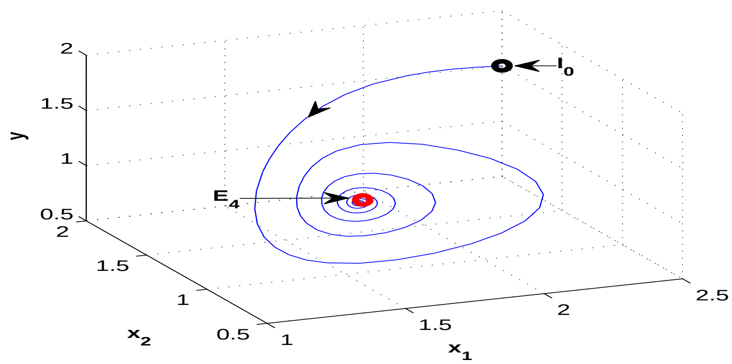

Figure 1. Phase plane diagram shows local stability of E_4 for the given set of parametric values

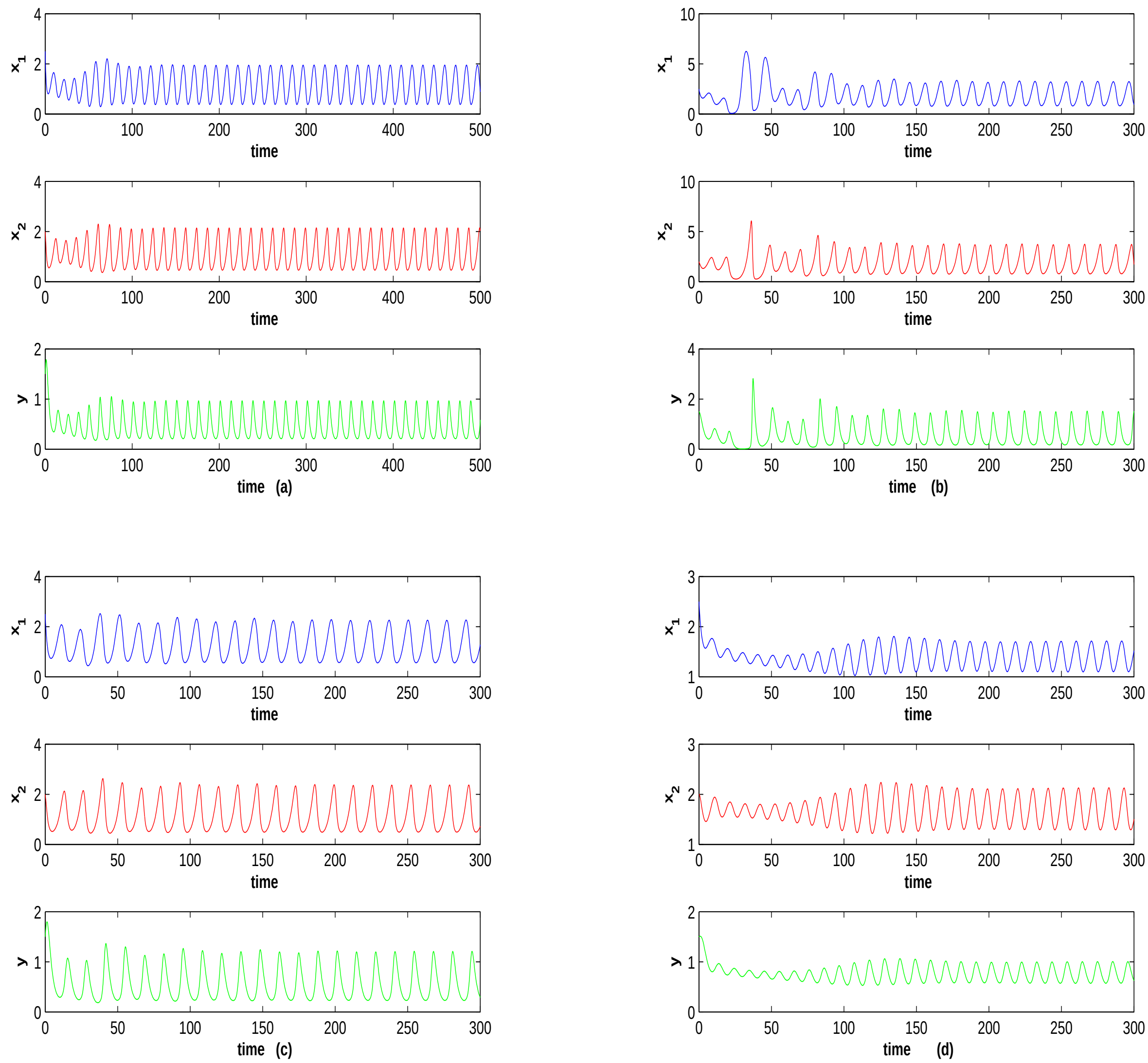

Figure 2. The time series depicts oscillatory behavior around the positive interior equilibrium point E_4 for the set of parameters (9) with $K = 4$ (Figure 2(a)), $\mu_3 = 0.8$ (Figure 2(b)), $r = 0.9$ (Figure 2(c)) and $c = 2$ (Figure 2(d))

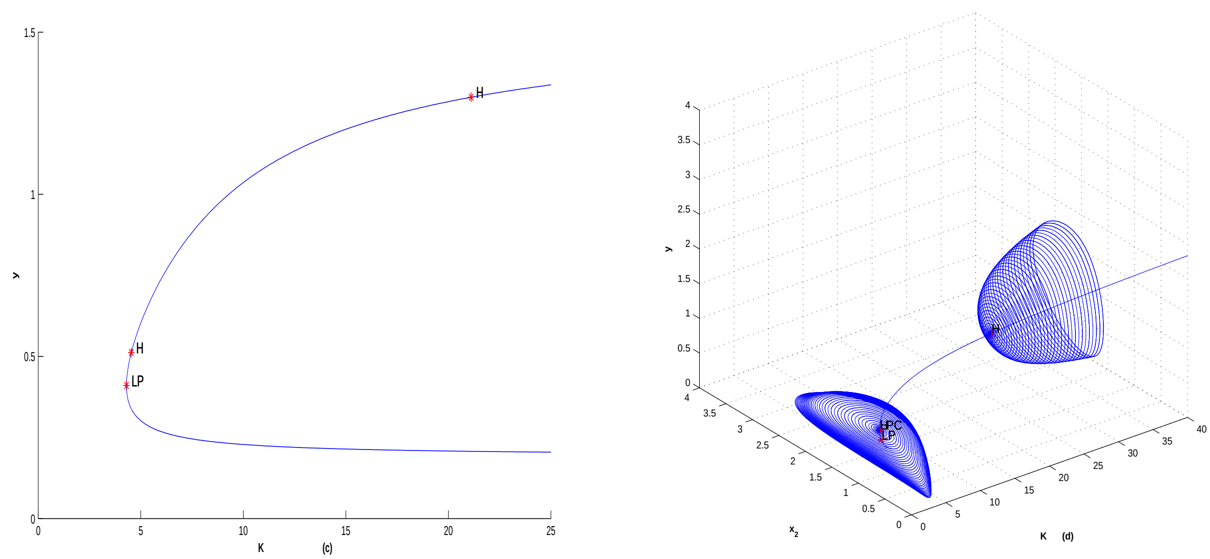

Figure 3. **(a)** Different steady state behavior of prey for the effect of K **(b)** Different steady state behavior of middle predator for the effect of K **(c)** Different steady state behavior of top predator for the effect of K **(d)** The family of limit cycles bifurcating from the Hopf point H for K

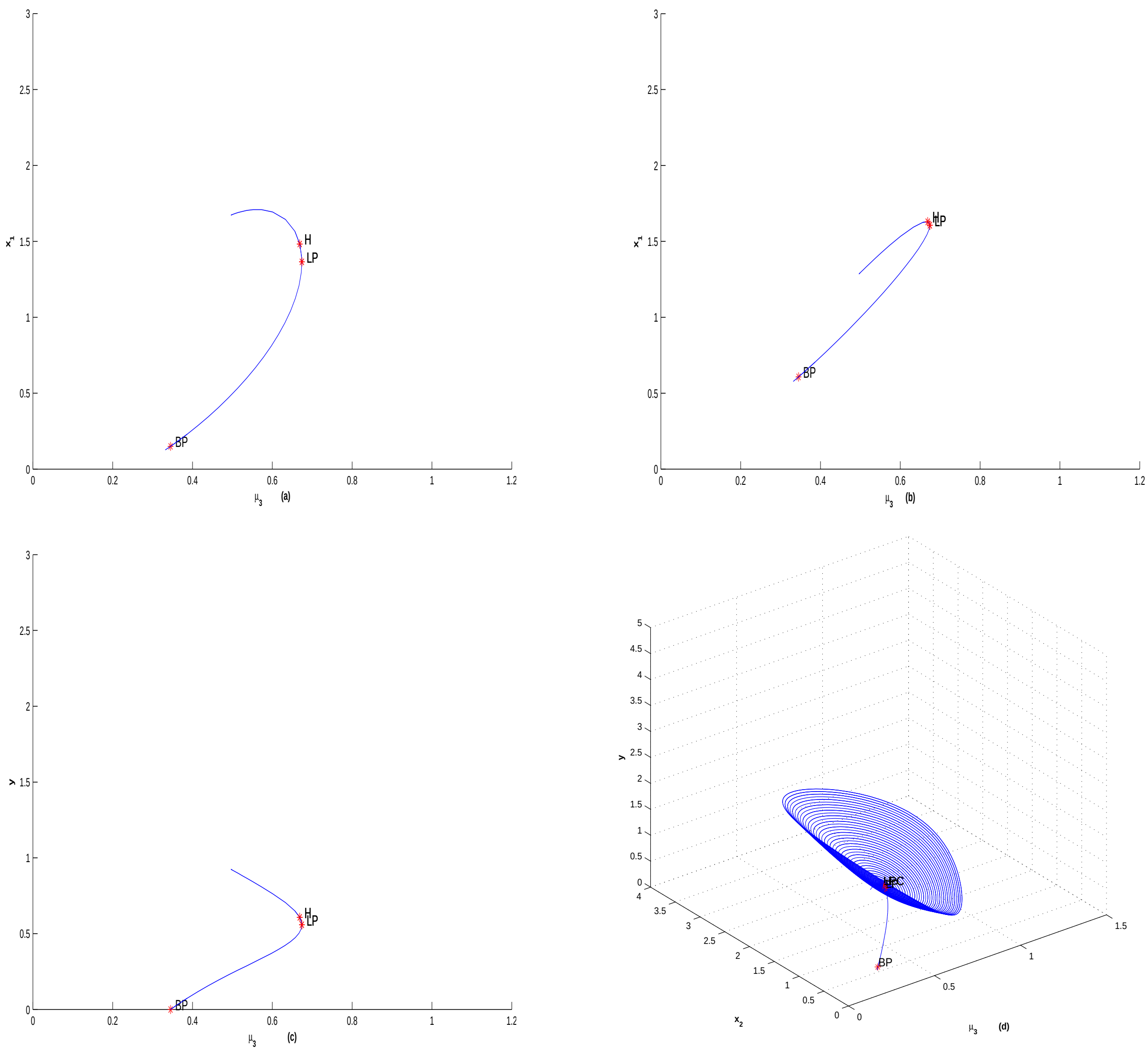

Figure 4. **(a)** Different steady state behavior of prey population for the effect of μ_3 **(b)** Different steady state behavior of middle predator population for the effect of μ_3 **(c)** Different steady state behavior of top predator population for the effect of μ_3 **(d)** The family of limit cycles bifurcating from the Hopf point H for μ_3

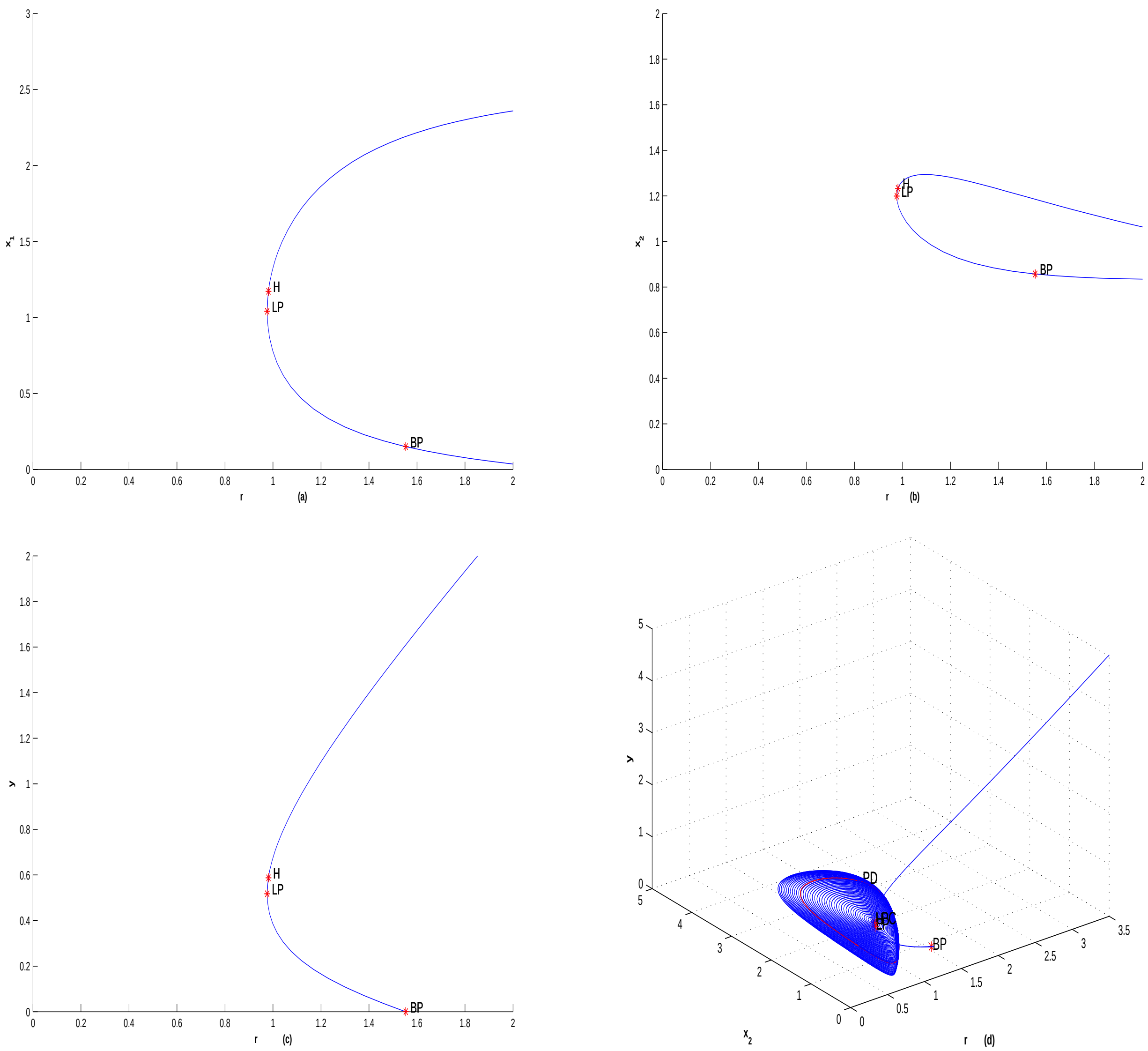

Figure 5. **(a)** Different steady state behavior of prey population for the effect of r **(b)** Different steady state behavior of middle predator population for the effect of r **(c)** Different steady state behavior of top predator population for the effect of r **(d)** The family of limit cycles bifurcating from the Hopf point H for r

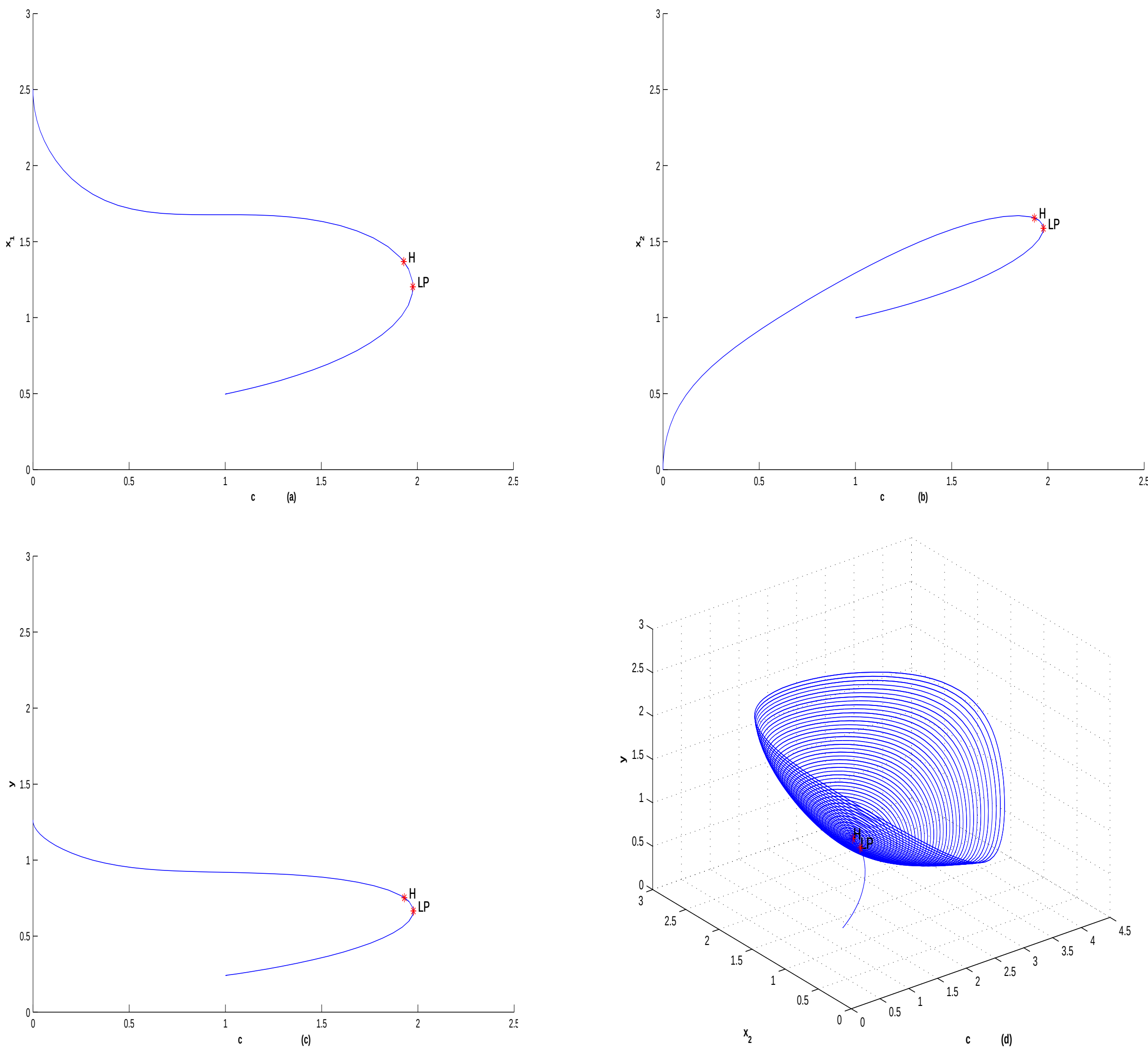

Figure 6. **(a)** Different steady state behavior of prey population for the effect of c **(b)** Different steady state behavior of middle predator population for the effect of c **(c)** Different steady state behavior of top predator population for the effect of c **(d)** The family of limit cycles bifurcating from the Hopf point H for c

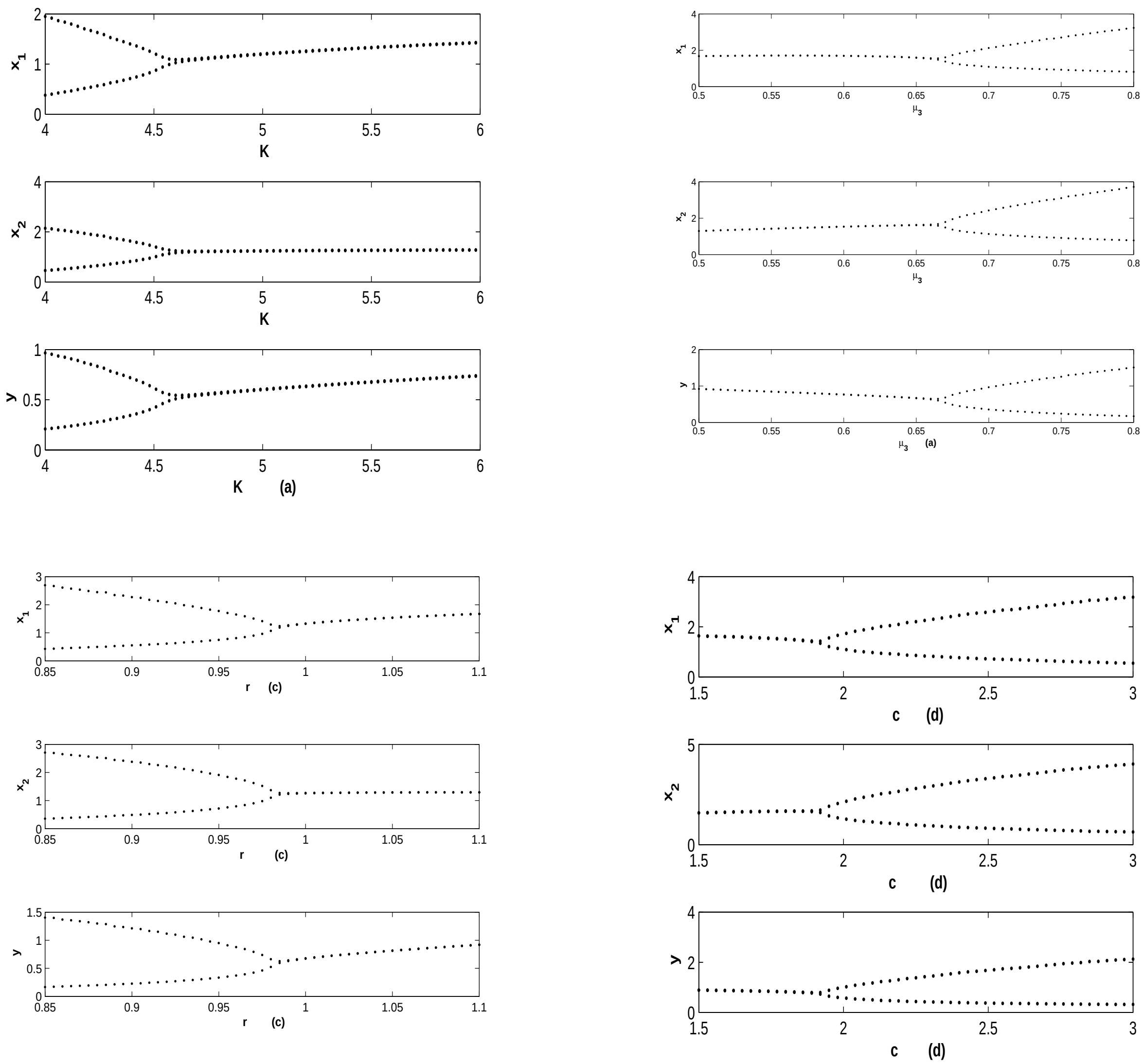

Figure 7. **(a)** Bifurcation diagram for K **(b)** Bifurcation diagram for μ_3 **(c)** Bifurcation diagram for r **(d)** Bifurcation diagram for c

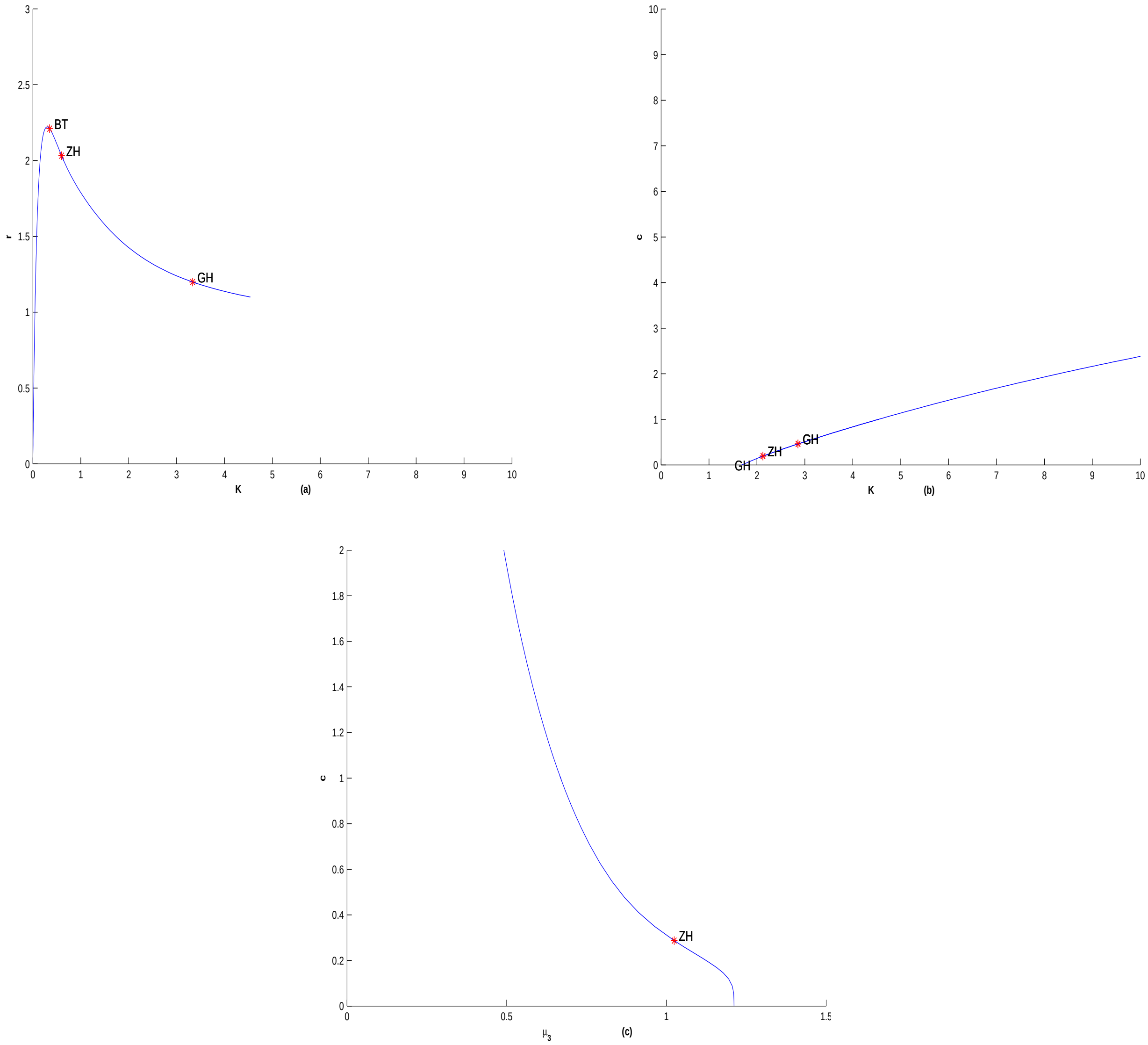

Figure 8. **(a)** The two parameters bifurcation diagram for $K - r$ **(b)** The two parameters bifurcation diagram for $K - c$ **(c)** The two parameters bifurcation diagram for $\mu_3 - c$

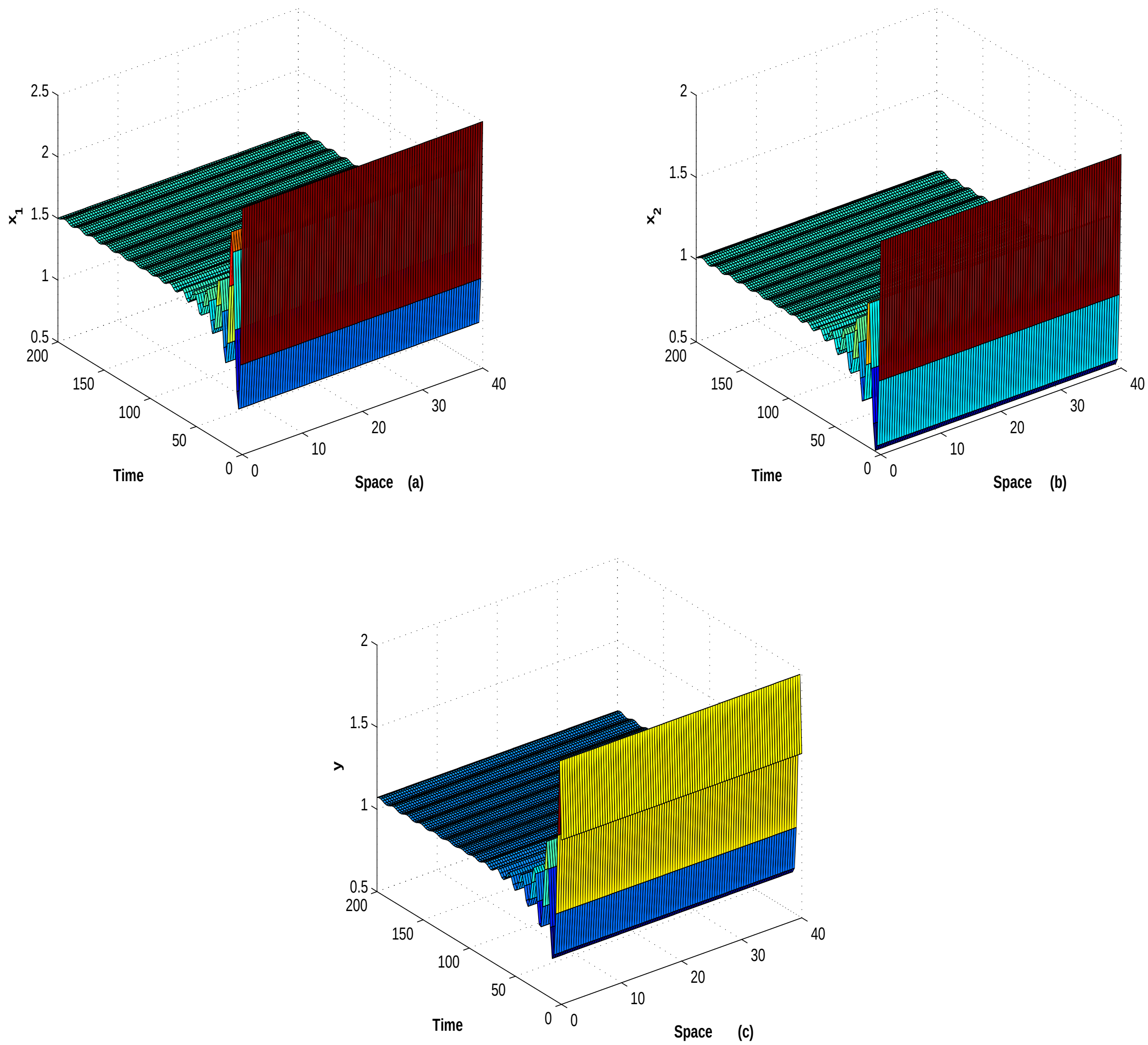

Figure 9. Biomass distribution of prey, middle predator, top predator species over time and space of system (6) for set of parameter values (9) with $D_1 = 0.5$, $D_2 = 0.5$ and $D_3 = 0.5$ showing in Figures 9(a-c) respectively

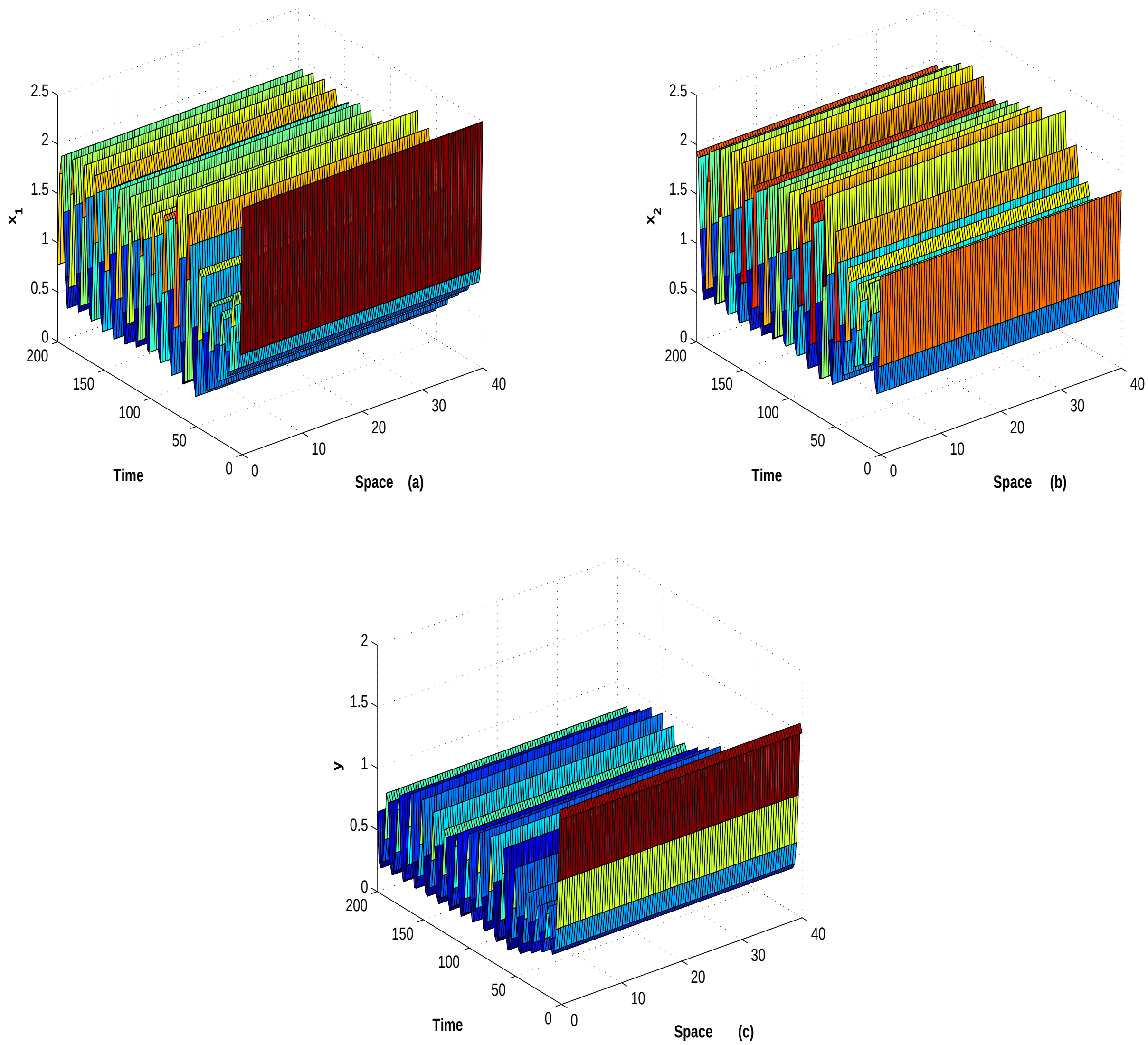

Figure 10. Biomass distribution of prey, middle predator and top predator species over time and space of system (6) for set of parameter values (9) with $K = 4$, $D_1 = 0.5$, $D_2 = 0.5$ and $D_3 = 0.5$ showing in Figures 10(a-c) respectively

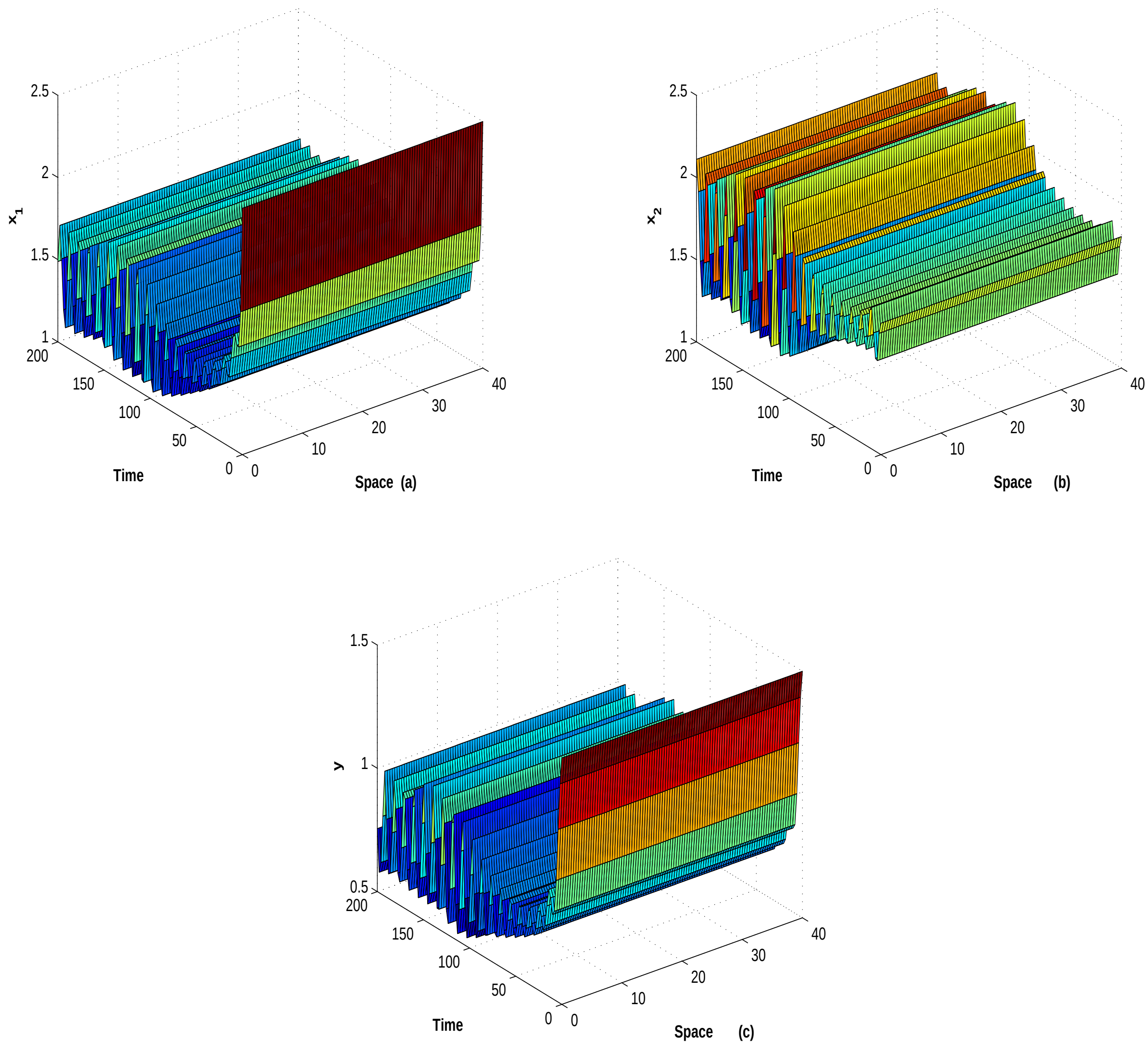

Figure 11. Biomass distribution of prey, middle predator, top predator species over time and space of system (6) for set of parameter values (9) with $c = 2$, $D_1 = 0.5$, $D_2 = 0.5$ and $D_3 = 0.5$ showing in Figures 11(a-c) respectively