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Simplified Intuitionistic Neutrosophic Soft Set and its Application on Diagnosing Psychological Disorder by Using Similarity Measure

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Abstract

The primary focus of this manuscript comprises three sections. Initially, we introduce the concept of a simplified intuitionistic neutrosophic soft set. We impose an intuitionistic condition between the membership values of truth and falsity such that their sum does not exceed unity. Similarly, for indeterminacy, the membership value is a real number from the closed interval $[0, 1]$. Hence, the sum of membership values of truth, indeterminacy, and falsity does not exceed two. We present the notion of necessity, possibility, concentration, and dilation operators and establish some of its properties. Second, we define the similarity measure between two simplified intuitionistic neutrosophic soft sets. Also, we discuss its superiority by comparing it with existing methods. Finally, we develop an algorithm and illustrate with an example of diagnosing psychological disorders. Even though the similarity measure plays a vital role in diagnosing psychological disorders, existing methods deal hardly in diagnosing psychological disorders. By nature, most of the psychological disorder behaviors are ambivalence. Hence, it is vital to capture the membership values by using simplified intuitionistic neutrosophic soft set. In this manuscript, we provide a solution in diagnosing psychological disorders, and the proposed similarity measure is valuable and compatible in diagnosing psychological disorders in any neutrosophic environment.

Keywords: Intuitionistic neutrosophic set; Necessity operator; Possibility operator; Concentration operator; Dilation operator; Similarity measures; Decision making; Diagnosing psychological disorders

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1. Introduction

Zadeh (1965) introduced the concept of a fuzzy set (FS) to the world. In FS theory, the membership value of each element in a set is specified by a real number from the closed interval of $[0, 1]$. Later, Atanassov (1986) defined the notion of an intuitionistic fuzzy set (IFS) as an extension of FS. In IFS theory, the elements are assumed to posses both membership and non-membership values with the condition that their sum does not exceed unity. Also, Atanassov (1994) established some properties of IFS. Smarandache (1999) presented the concept of neutrosophic set (NS), characterized by the values of truth, indeterminacy and falsity membership for each element of the set. Decision-makers (DMs) applied this concept widely to show the importance of neutrosophic theory. Wang et al. (2010) defined the notion of single-valued NS (SVNS) with restricted conditions for the membership values to facilitate the real-life applications and to overcome the constraints faced in NS theory. Smarandache (2018b) studied the concept of soul in psychology by using neutrosophic theory. Christianto and Smarandache (2019) reviewed the concept of cultural psychology as one of the seven applications using neutrosophic logic. Chicaiza et al. (2020) studied the concept of emotional intelligence of the students using neutrosophic psychology. Smarandache (2016) introduced the concept of degree of dependence and the degree of independence between the components of the FS, and also between the components of the NS. Also, Smarandache (2018a) discussed the concept of hypersoft set an extension of soft set. The domination of NS and SVNS in psychology is clear from the above-cited published papers and books.

Shahzadi et al. (2017) diagnosed the medical symptoms using SVNSs. Broumi et al. (2016) presented the concept of single valued neutrosophic graph a generalization of fuzzy graph and intuitionistic fuzzy graph. Hamidi and Borumand Saeid (2018) developed the concept of neutrosophic graphs to analyze the sensor networks. Beg and Tabasam (2015) introduced the concept of comparative linguistic expression based on hesitant IFSs. Anita et al. (2016) developed an application to solve multi-criteria decision-making (MCDM) problems using interval-valued IFS of root type. Jianming Zhan et al. (2017) defined the concepts of the weighted aggregation operators in neutrosophic cubic sets (NCSs) and provided applications in MCDM. Majid Khan et al. (2019) presented the notions of algebraic and Einstein operators on NCSs and developed an MCDM application based on these operators. Chinnadurai et al. (2020) presented the concept of unique ranking by using the parameters in a neutrosophic environment. Chinnadurai and Bobin (2020) used prospect theory in real-life applications to solve MCDM problems. Chahhtterjee et al. (2019) presented various concepts of similarity measures (SMs) in neutrosophic environment. Abdel-Basset et al. (2019) used cosine SMs in bipolar and interval-valued bipolar SVNS to diagnose bipolar disorder behaviors. Saranya et al. (2020) introduced an application for finding the similarity value of any two NSs in the neutrosophic environment by using programming language. Broumi and Smarandache (2013b) developed SMs using Hausdorff distance. Liu et al. (2018) introduced the concept of SMs using Euclidean distance. Hashim et al. (2018) introduced SMs in neutrosophic bipolar FS with a purpose to build a children's hospital with the help of the HOPE foundation.

Bhowmik and Pal (2008) presented the concept of intuitionistic neutrosophic set (INS) and studied its properties. Broumi and Smarandache (2013a) defined the concept of intuitionistic neutrosophic soft set (INSS) and established some of its properties. Both INS and INSS have a significant role in handling decision-making problems. They defined the restricted conditions as i) the minimum of membership values between truth and indeterminacy to be less than or equal to 0.5, ii) the minimum of membership values between truth and falsity to be less than or equal to 0.5, and iii) the minimum of membership values between falsity and indeterminacy to be less than or equal to 0.5, such that the sum of membership values of truth, indeterminacy, and falsity cannot exceed two. Let us consider an example $\mathcal{A} = \langle 0.3, 0.8, 0.7\rangle$. Now according to INS definition, we have $\min\{0.3, 0.8\} < 0.5,\ \min\{0.3, 0.7\} < 0.5$ and $\min\{0.7, 0.8\} \nless 0.5$ but satisfies the condition $0 < 0.3+0.8+0.7 < 2$. Similarly, let us consider another example $\mathcal{A} = \langle 0.6, 0.8, 0.4\rangle$. Now according to INS definition, we have $\min\{0.6, 0.8\} \nless 0.5,\ \min\{0.6, 0.4\} < 0.5$ and $\min\{0.4, 0.8\} < 0.5$ but satisfies the condition $0 < 0.3 + 0.8 + 0.7 < 2$. The given examples show that they aren't INS. However, the DM may have a situation where the membership grades of falsity and indeterminacy or truth and indeterminacy are greater than 0.5. Therefore DM may have some constraints while handling this information in the INS environment. Now, when the membership values of true and false are in continuum and the membership value of indeterminacy is independent, it becomes a challenge to input the values during decision-making with the help of INS and INSS. Hence, there is a need for a new INS with simplified conditions. So, we introduce a simplified INS (SINS) and simplified INSS (SINSS) to effectively handle the decision-making problems.

The purpose of this study is to bring out the importance of SINSS when experts provide membership values in truth, indeterminacy, and falsity in a restricted environment. In recent years, human beings face many decisions-making problems in multiple fields and analyzing the psychological disorder of the subject is one of them. Similarly, SM plays a significant factor in diagnosing psychological disorders, but hardly no existing methods deal with it. Therefore, it is necessary to provide a working model for determining the same.

2. Simplified intuitionistic neutrosophic soft set

In this section, we introduce the notion of SINS, SINSS and establish some of its properties. Let $\mathcal{V}$ be the universe and $u \in \mathcal{V}$, $\mathcal{P}$ be a set of parameters, $\mathcal{E} \subseteq \mathcal{P}$ and $\mathcal{S}^I$ be the set of all SINS over $\mathcal{V}$. We generalize these operations and properties on SINSS by the concepts discussed in Atanassov (1986) and Jiang et al. (2010). Let us consider the following notations throughout this manuscript unless otherwise specified .

Definition 2.1.

A SINS in $\mathcal{V}$ is of the form $\mathcal{S} = \{\langle u, \mathcal{T}_{\mathcal{S}}(u), \mathcal{I}_{\mathcal{S}}(u), \mathcal{F}_{\mathcal{S}}(u)\rangle\}$, where $\mathcal{T}_{\mathcal{S}}(u) : \mathcal{V} \to [0, 1]$, $\mathcal{I}_{\mathcal{S}}(u) : \mathcal{V} \to [0, 1]$ and $\mathcal{F}_{\mathcal{S}}(u) : \mathcal{V} \to [0, 1]$ are the membership values of truth, indeterminacy and falsity of the element $u \in \mathcal{V}$ respectively, satisfying the conditions $0 \le \mathcal{T}_{\mathcal{S}}(u) + \mathcal{F}_{\mathcal{S}}(u) \le 1$ and $0 \le \mathcal{T}_{\mathcal{S}}(u) + \mathcal{I}_{\mathcal{S}}(u) + \mathcal{F}_{\mathcal{S}}(u) \le 2$.

Example 2.1.

Let $\mathcal{V} = \{c_1, c_2, c_3\}$ be a non-empty set. Then a SINS on $\mathcal{V}$ can be represented as,

$$\mathcal{S} = \{\langle c_1, 0.4, 0.7, 0.3\rangle, \langle c_2, 0.6, 1, 0.3\rangle, \langle c_3, 0.7, 0.9, 0.3\rangle\}.$$

Definition 2.2.

A pair $(\Psi, \mathcal{E})$ is called SINSS over $\mathcal{V}$, where $\Psi : \mathcal{E} \to \mathcal{S}^I$. Thus, for any parameter $p \in \mathcal{E}$, $\Psi(p)$ is a SINSS.

Example 2.2.

Let $\mathcal{V} = \{c_1, c_2, c_3, c_4\}$ be a set of clients with cognitive disorders and $\mathcal{E} = \{p_1, p_2, p_3, p_4\}$ be the set of symptoms which stand for inability of motor coordination (IMC), loss of memory (LM), identity confusion (IC) and impaired judgment (IJ) respectively. A SINSS $(\Psi, \mathcal{E})$ is a collection of subsets of $\mathcal{V}$, given by a psychiatrist based on the description in Table 1.

Table 1. Representation of clients with cognitive disorders in SINSS $(\Psi, \mathcal{E})$ form.

$\mathcal{V}$	IMC(p_1)	LM(p_2)	IC(p_3)	IJ(p_4)
c_1	$\langle 0.5, 0.7, 0.3\rangle$	$\langle 0.4, 0.5, 0.6\rangle$	$\langle 0.2, 0.3, 0.5\rangle$	$\langle 0.6, 0.7, 0.2\rangle$
c_2	$\langle 0.6, 0.4, 0.3\rangle$	$\langle 0.7, 0.8, 0.2\rangle$	$\langle 0.8, 0.9, 0.1\rangle$	$\langle 0.1, 0.7, 0.6\rangle$
c_3	$\langle 0.3, 0.5, 0.6\rangle$	$\langle 0.6, 0.9, 0.3\rangle$	$\langle 0.9, 0.9, 0.1\rangle$	$\langle 0.2, 0.3, 0.7\rangle$
c_4	$\langle 0.4, 0.2, 0.5\rangle$	$\langle 0.5, 0.5, 0.5\rangle$	$\langle 0.4, 0.7, 0.5\rangle$	$\langle 0.5, 0.4, 0.4\rangle$

Definition 2.3.

Let $(\Psi_1, \mathcal{E}_1)$ and $(\Psi_2, \mathcal{E}_2)$ be two SINSS over $\mathcal{V}$. Then,
(i) $(\Psi_1, \mathcal{E}_1)$ AND $(\Psi_2, \mathcal{E}_2)$ is a SINSS represented as $(\Psi_1, \mathcal{E}_1) \wedge (\Psi_2, \mathcal{E}_2) = (\Psi_\wedge, \mathcal{E}_1 \times \mathcal{E}_2)$, where $\Psi_\wedge(p_1, p_2) = \Psi_1(p_1) \cap \Psi_2(p_2)$, $\forall\ (p_1, p_2) \in \mathcal{E}_1 \times \mathcal{E}_2$.

$$\begin{aligned}\Psi_\wedge(p_1, p_2) =& \Big\langle \min(\mathcal{T}_{\Psi_1(p_1)}(u), \mathcal{T}_{\Psi_2(p_2)}(u)), \min(\mathcal{I}_{\Psi_1(p_1)}(u), \mathcal{I}_{\Psi_2(p_2)}(u)),\\ & \max(\mathcal{F}_{\Psi_1(p_1)}(u), \mathcal{F}_{\Psi_2(p_2)}(u))\Big\rangle, \forall\ (p_1, p_2) \in \mathcal{E}_1 \times \mathcal{E}_2;\end{aligned}$$

(ii) $(\Psi_1, \mathcal{E}_1)$ OR $(\Psi_2, \mathcal{E}_2)$ is a SINSS represented as $(\Psi_1, \mathcal{E}_1) \vee (\Psi_2, \mathcal{E}_2) = (\Psi_\vee, \mathcal{E}_1 \times \mathcal{E}_2)$, where $\Psi_\vee(p_1, p_2) = \Psi_1(p_1) \cup \Psi_2(p_2)$, $\forall\ (p_1, p_2) \in \mathcal{E}_1 \times \mathcal{E}_2$.

$$\begin{aligned}\Psi_\vee(p_1, p_2) =& \Big\langle \max(\mathcal{T}_{\Psi_1(p_1)}(u), \mathcal{T}_{\Psi_2(p_2)}(u)), \max(\mathcal{I}_{\Psi_1(p_1)}(u), \mathcal{I}_{\Psi_2(p_2)}(u)),\\ & \min(\mathcal{F}_{\Psi_1(p_1)}(u), \mathcal{F}_{\Psi_2(p_2)}(u))\Big\rangle, \forall\ (p_1, p_2) \in \mathcal{E}_1 \times \mathcal{E}_2.\end{aligned}$$

Example 2.3.

Let $\Psi_1 = \langle 0.2, 0.5, 0.6\rangle$ and $\Psi_1 = \langle 0.1, 0.7, 0.2\rangle$ be two SINSS. Then
(i) $\Psi_1 \wedge \Psi_2 = \langle 0.1, 0.5, 0.6\rangle$;
(ii) $\Psi_1 \vee \Psi_2 = \langle 0.2, 0.7, 0.2\rangle$.

Definition 2.4.

Let $(\Psi_1, \mathcal{E}_1)$ and $(\Psi_2, \mathcal{E}_2)$ be two SINSS over $\mathcal{V}$. Then,

(i) $(\Psi_1, \mathcal{E}_1)$ union $(\Psi_2, \mathcal{E}_2)$ is a SINSS represented as $(\Psi_1, \mathcal{E}_1) \Cup (\Psi_2, \mathcal{E}_2) = (\Psi_{\Cup}, \mathcal{E}_{\Cup})$, where $\mathcal{E}_{\Cup} = \mathcal{E}_1 \cup \mathcal{E}_2$ and $\forall\, p \in \mathcal{E}_{\Cup}$,

$$\Psi_{\Cup}(p) = \begin{cases} \left\{\left\langle u, (\mathcal{T}_{\Psi_1(p)}(u), \mathcal{I}_{\Psi_1(p)}(u), \mathcal{F}_{\Psi_1(p)}(u))\right\rangle; & \text{if } p \in \mathcal{E}_1 - \mathcal{E}_2\right\}, \\ \left\{\left\langle u, (\mathcal{T}_{\Psi_2(p)}(u), \mathcal{I}_{\Psi_2(p)}(u), \mathcal{F}_{\Psi_2(p)}(u))\right\rangle; & \text{if } p \in \mathcal{E}_2 - \mathcal{E}_1\right\}, \\ \left\{\left\langle u, \max(\mathcal{T}_{\Psi_1(p)}(u), \mathcal{T}_{\Psi_2(p)}(u)), \max(\mathcal{I}_{\Psi_1(p)}(u), \mathcal{I}_{\Psi_2(p)}(u)), \right.\right. & \\ \qquad\qquad \left.\left. \min(\mathcal{F}_{\Psi_1(p)}(u), \mathcal{F}_{\Psi_2(p)}(u))\right\rangle; & \text{if } p \in \mathcal{E}_1 \cap \mathcal{E}_2\right\}. \end{cases}$$

(ii) $(\Psi_1, \mathcal{E}_1)$ intersection $(\Psi_2, \mathcal{E}_2)$ is a SINSS represented as $(\Psi_1, \mathcal{E}_1) \Cap (\Psi_2, \mathcal{E}_2) = (\Psi_{\Cap}, \mathcal{E}_{\Cap})$, where $\mathcal{E}_{\Cap} = \mathcal{E}_1 \cup \mathcal{E}_2$ and $\forall\, p \in \mathcal{E}_{\Cap}$,

$$\Psi_{\Cap}(p) = \begin{cases} \left\{\left\langle u, (\mathcal{T}_{\Psi_1(p)}(u), \mathcal{I}_{\Psi_1(p)}(u), \mathcal{F}_{\Psi_1(p)}(u))\right\rangle; & \text{if } p \in \mathcal{E}_1 - \mathcal{E}_2\right\}, \\ \left\{\left\langle u, (\mathcal{T}_{\Psi_2(p)}(u), \mathcal{I}_{\Psi_2(p)}(u), \mathcal{F}_{\Psi_2(p)}(u))\right\rangle; & \text{if } p \in \mathcal{E}_2 - \mathcal{E}_1\right\}, \\ \left\{\left\langle u, \min(\mathcal{T}_{\Psi_1(p)}(u), \mathcal{T}_{\Psi_2(p)}(u)), \min(\mathcal{I}_{\Psi_1(p)}(u), \mathcal{I}_{\Psi_2(p)}(u)), \right.\right. & \\ \qquad\qquad \left.\left. \max(\mathcal{F}_{\Psi_1(p)}(u), \mathcal{F}_{\Psi_2(p)}(u))\right\rangle; & \text{if } p \in \mathcal{E}_1 \cap \mathcal{E}_2\right\}. \end{cases}$$

Definition 2.5.

The complement of a SINSS $(\Psi, \mathcal{E})$ is represented as,

$$(\Psi, \mathcal{E})^c = \left\{\left\langle u, \mathcal{F}_{\Psi(p)}(u), (1 - \mathcal{I}_{\Psi(p)})(u), \mathcal{T}_{\Psi(p)}(u)\right\rangle; \text{and } p \in \mathcal{E}\right\}.$$

Example 2.4.

Let $\Psi = \langle 0.2, 0.7, 0.6\rangle$ be a SINSS. Then $\Psi^c = \langle 0.6, 0.3, 0.2\rangle$.

Theorem 2.1.

Let $(\Psi_1, \mathcal{E}_1)$ and $(\Psi_2, \mathcal{E}_2)$ be two SINSS over $\mathcal{V}$. Then,
(i) $\left\langle(\Psi_1, \mathcal{E}_1) \wedge (\Psi_2, \mathcal{E}_2)\right\rangle^c = (\Psi_1, \mathcal{E}_1)^c \vee (\Psi_2, \mathcal{E}_2)^c$;
(ii) $\left\langle(\Psi_1, \mathcal{E}_1) \vee (\Psi_2, \mathcal{E}_2)\right\rangle^c = (\Psi_1, \mathcal{E}_1)^c \wedge (\Psi_2, \mathcal{E}_2)^c$.

Proof:

We give the prove of (i), and the proof of (ii) is analogous.

(i) $(\Psi_1, \mathcal{E}_1) \wedge (\Psi_2, \mathcal{E}_2) = (\Psi_\wedge, \mathcal{E}_1 \times \mathcal{E}_2)$.

$\left\langle(\Psi_1, \mathcal{E}_1) \wedge (\Psi_2, \mathcal{E}_2)\right\rangle^c = (\Psi_\wedge, \mathcal{E}_1 \times \mathcal{E}_2)^c$.

$$\Psi^c_\wedge(p_1, p_2) = \Big\langle \max(\mathcal{F}_{\Psi_1(p_1)}(u), \mathcal{F}_{\Psi_2(p_2)}(u)), \max((1 - \mathcal{I}_{\Psi_1(p_1)}(u)), (1 - \mathcal{I}_{\Psi_2(p_2)}(u))), \\ \min(\mathcal{T}_{\Psi_1(p_1)}(u), \mathcal{T}_{\Psi_2(p_2)}(u))\Big\rangle, \forall (p_1, p_2) \in \mathcal{E}_1 \times \mathcal{E}_2.$$

$(\Psi_1, \mathcal{E}_1)^c \vee (\Psi_2, \mathcal{E}_2)^c = (\Psi_\vee, \mathcal{E}_1 \times \mathcal{E}_2)$.

$$\Psi_\vee(p_1, p_2) = \Big\langle \max(\mathcal{F}_{\Psi_1(p_1)}(u), \mathcal{F}_{\Psi_2(p_2)}(u)), \max((1 - \mathcal{I}_{\Psi_1(p_1)}(u)), (1 - \mathcal{I}_{\Psi_2(p_2)}(u))), \\ \min(\mathcal{T}_{\Psi_1(p_1)}(u), \mathcal{T}_{\Psi_2(p_2)}(u))\Big\rangle, \forall (p_1, p_2) \in \mathcal{E}_1 \times \mathcal{E}_2.$$

Thus, $\left\langle(\Psi_1, \mathcal{E}_1) \wedge (\Psi_2, \mathcal{E}_2)\right\rangle^c = (\Psi_1, \mathcal{E}_1)^c \vee (\Psi_2, \mathcal{E}_2)^c$. ∎

Definition 2.6.

Let $\mathcal{E}_1, \mathcal{E}_2 \subseteq \mathcal{P}$. $(\Psi_1, \mathcal{E}_1)$ is a simplified intuitionistic neutrosophic soft subset of $(\Psi_2, \mathcal{E}_2)$ denoted by $(\Psi_1, \mathcal{E}_1) \Subset (\Psi_2, \mathcal{E}_2)$ if and only if
(i) $\mathcal{E}_1 \subseteq \mathcal{E}_2$;
(ii) $\Psi_1(p)$ is a simplified intuitionistic neutrosophic soft subset of $\Psi_2(p)$ that is for all $p \in \mathcal{E}_1$, $\mathcal{T}_{\Psi_1(p)}(u) \leq \mathcal{T}_{\Psi_2(p)}(u)$, $\mathcal{I}_{\Psi_1(p)}(u) \leq \mathcal{I}_{\Psi_2(p)}(u)$ and $\mathcal{F}_{\Psi_1(p)}(u) \geq \mathcal{F}_{\Psi_2(p)}(u)$.
Also, $(\Psi_2, \mathcal{E}_2)$ is called a simplified intuitionistic neutrosophic soft superset of $(\Psi_1, \mathcal{E}_1)$ and represented as $(\Psi_2, \mathcal{E}_2) \Supset (\Psi_1, \mathcal{E}_1)$.

Definition 2.7.

If $(\Psi_1, \mathcal{E}_1)$ and $(\Psi_2, \mathcal{E}_2)$ are two SINSS, then $(\Psi_1, \mathcal{E}_1) = (\Psi_2, \mathcal{E}_2)$ if and only if $(\Psi_1, \mathcal{E}_1) \Subset (\Psi_2, \mathcal{E}_2)$ and $(\Psi_2, \mathcal{E}_2) \Subset (\Psi_1, \mathcal{E}_1)$.

3. Necessity $(\oplus)$ and possibility $(\ominus)$ operators on SINSS

In this section, we define $\oplus$ and $\ominus$ operators on SINSS and establish some of their properties. We generalize these operations and properties on SINSS using the concepts discussed in Atanassov (1986) and Jiang et al. (2010).

Definition 3.1.

If $(\Psi, \mathcal{E})$ is a SINSS over $\mathcal{V}$ and $\Psi : \mathcal{E} \to \mathcal{S}^I$, then,
(i) the necessity operator $(\oplus)$ is represented as,

$$\oplus(\Psi, \mathcal{E}) = \left\{ \left\langle u, \mathcal{T}_{\oplus\Psi(p)}(u), \mathcal{I}_{\oplus\Psi(p)}(u), \mathcal{F}_{\oplus\Psi(p)}(u) \right\rangle;\ p \in \mathcal{E} \right\}.$$

Here, $\mathcal{T}_{\oplus\Psi(p)}(u) = \mathcal{T}_{\Psi(p)}(u)$, $\mathcal{I}_{\oplus\Psi(p)}(u) = \mathcal{I}_{\Psi(p)}(u)$ and $\mathcal{F}_{\oplus\Psi(p)}(u) = (1 - \mathcal{T}_{\Psi(p)}(u))$, are the membership values of truth, indeterminacy and falsity for the object u on the parameter p.

(ii) the possibility operator $(\ominus)$ is represented as,

$$\ominus(\Psi, \mathcal{E}) = \left\{ \left\langle u, \mathcal{T}_{\ominus\Psi(p)}(u), \mathcal{I}_{\ominus\Psi(p)}(u), \mathcal{F}_{\ominus\Psi(p)}(u) \right\rangle;\ p \in \mathcal{E} \right\}.$$

Here, $\mathcal{T}_{\ominus\Psi(p)}(u) = (1 - \mathcal{F}_{\Psi(p)}(u))$, $\mathcal{I}_{\ominus\Psi(p)}(u) = \mathcal{I}_{\Psi(p)}(u)$ $\mathcal{F}_{\ominus\Psi(p)}(u) = \mathcal{F}_{\Psi(p)}(u)$, are the membership values of truth, indeterminacy and falsity for the object u on the parameter p.

Example 3.1.

(i) The SINSS $\oplus(\Psi, \mathcal{E})$ for Example 2.2 is given in Table 2.

(ii) The SINSS $\ominus(\Psi, \mathcal{E})$ for Example 2.2 is given in Table 3.

Theorem 3.1.

Let $(\Psi_1, \mathcal{E}_1)$ and $(\Psi_2, \mathcal{E}_2)$ be two SINSS over $\mathcal{V}$. Then,
(i) $\oplus \langle (\Psi_1, \mathcal{E}_1) \Cup (\Psi_2, \mathcal{E}_2) \rangle = \oplus(\Psi_1, \mathcal{E}_1) \Cup \oplus(\Psi_2, \mathcal{E}_2)$;

Table 2. Representation of clients with cognitive disorders using $\oplus$ operator.

$\mathcal{U}$	IMC(p_1)	LM(p_2)	IC(p_3)	IJ(p_4)
c_1	$\langle 0.5, 0.7, 0.5\rangle$	$\langle 0.4, 0.5, 0.6\rangle$	$\langle 0.2, 0.3, 0.8\rangle$	$\langle 0.6, 0.7, 0.4\rangle$
c_2	$\langle 0.6, 0.4, 0.4\rangle$	$\langle 0.7, 0.8, 0.3\rangle$	$\langle 0.8, 0.9, 0.2\rangle$	$\langle 0.1, 0.7, 0.9\rangle$
c_3	$\langle 0.3, 0.5, 0.7\rangle$	$\langle 0.6, 0.9, 0.4\rangle$	$\langle 0.9, 0.9, 0.1\rangle$	$\langle 0.2, 0.3, 0.8\rangle$
c_4	$\langle 0.4, 0.2, 0.6\rangle$	$\langle 0.5, 0.5, 0.5\rangle$	$\langle 0.4, 0.7, 0.6\rangle$	$\langle 0.5, 0.4, 0.5\rangle$

Table 3. Representation of clients with cognitive disorders using $\ominus$ operator.

$\mathcal{U}$	IMC(p_1)	LM(p_2)	IC(p_3)	IJ(p_4)
c_1	$\langle 0.7, 0.7, 0.3\rangle$	$\langle 0.4, 0.5, 0.6\rangle$	$\langle 0.5, 0.3, 0.5\rangle$	$\langle 0.8, 0.7, 0.2\rangle$
c_2	$\langle 0.7, 0.4, 0.3\rangle$	$\langle 0.8, 0.8, 0.2\rangle$	$\langle 0.9, 0.9, 0.1\rangle$	$\langle 0.4, 0.7, 0.6\rangle$
c_3	$\langle 0.4, 0.5, 0.6\rangle$	$\langle 0.7, 0.9, 0.3\rangle$	$\langle 0.9, 0.9, 0.1\rangle$	$\langle 0.3, 0.3, 0.7\rangle$
c_4	$\langle 0.5, 0.2, 0.5\rangle$	$\langle 0.5, 0.5, 0.5\rangle$	$\langle 0.5, 0.7, 0.5\rangle$	$\langle 0.6, 0.4, 0.4\rangle$

(ii) $\oplus\left\langle(\Psi_1,\mathcal{E}_1)\Cap(\Psi_2,\mathcal{E}_2)\right\rangle = \oplus(\Psi_1,\mathcal{E}_1)\Cap\oplus(\Psi_2,\mathcal{E}_2)$;
(iii) $\oplus\oplus(\Psi_1,\mathcal{E}_1) = \oplus(\Psi_1,\mathcal{E}_1)$.

Proof:

We present the proof of (i), and the proof of (ii) is analogous.

(i) Let $(\Psi_1,\mathcal{E}_1)\Cup(\Psi_2,\mathcal{E}_2) = (\Psi_{\Cup},\mathcal{E}_{\Cup})$, where $\mathcal{E}_{\Cup}=\mathcal{E}_1\cup\mathcal{E}_2, \forall\, p\in\mathcal{E}_{\Cup}$.

Consider,

$$\oplus\Psi_{\Cup}(p)=\begin{cases}\left\{\left\langle u,\mathcal{T}_{\Psi_1(p)}(u),\mathcal{I}_{\Psi_1(p)}(u),(1-\mathcal{T}_{\Psi_1(p)}(u))\right\rangle; & \text{if } p\in\mathcal{E}_1-\mathcal{E}_2\right\},\\ \left\{\left\langle u,\mathcal{T}_{\Psi_2(p)}(u),\mathcal{I}_{\Psi_2(p)}(u),(1-\mathcal{T}_{\Psi_2(p)}(u))\right\rangle; & \text{if } p\in\mathcal{E}_2-\mathcal{E}_1\right\},\\ \left\{\left\langle u,\max(\mathcal{T}_{\Psi_1(p)}(u),\mathcal{T}_{\Psi_2(p)}(u)),\max(\mathcal{I}_{\Psi_1(p)}(u),\mathcal{I}_{\Psi_2(p)}(u)),\right.\right.\\ \qquad\qquad \left.\left.(1-\max(\mathcal{T}_{\Psi_1(p)}(u),\mathcal{T}_{\Psi_2(p)}(u)))\right\rangle; & \text{if } p\in\mathcal{E}_1\cap\mathcal{E}_2\right\},\end{cases}$$

$$=\begin{cases}\left\{\left\langle u,\mathcal{T}_{\Psi_1(p)}(u),\mathcal{I}_{\Psi_1(p)}(u),(1-\mathcal{T}_{\Psi_1(p)}(u))\right\rangle; & \text{if } p\in\mathcal{E}_1-\mathcal{E}_2\right\},\\ \left\{\left\langle u,\mathcal{T}_{\Psi_2(p)}(u),\mathcal{I}_{\Psi_2(p)}(u),(1-\mathcal{T}_{\Psi_2(p)}(u))\right\rangle; & \text{if } p\in\mathcal{E}_2-\mathcal{E}_1\right\},\\ \left\{\left\langle u,\max(\mathcal{T}_{\Psi_1(p)}(u),\mathcal{T}_{\Psi_2(p)}(u)),\max(\mathcal{I}_{\Psi_1(p)}(u),\mathcal{I}_{\Psi_2(p)}(u)),\right.\right.\\ \qquad\qquad \left.\left.\min((1-\mathcal{T}_{\Psi_1(p)}(u)),(1-\mathcal{T}_{\Psi_2(p)}(u)))\right\rangle; & \text{if } p\in\mathcal{E}_1\cap\mathcal{E}_2\right\},\end{cases}$$

$$\oplus(\Psi_1,\mathcal{E}_1)=\left\{\left\langle u,\mathcal{T}_{\Psi_1(p)}(u),\mathcal{I}_{\Psi_1(p)}(u),(1-\mathcal{T}_{\Psi_1(p)}(u))\right\rangle; p\in\mathcal{E}_1\right\},$$

$$\oplus(\Psi_2,\mathcal{E}_2)=\left\{\left\langle u,\mathcal{T}_{\Psi_2(p)}(u),\mathcal{I}_{\Psi_2(p)}(u),(1-\mathcal{T}_{\Psi_2(p)}(u))\right\rangle; p\in\mathcal{E}_2\right\}.$$

Let $\oplus(\Psi_1,\mathcal{E}_1)\Cup\oplus(\Psi_2,\mathcal{E}_2) = (\Psi_{\oplus\Cup},\mathcal{E}_{\oplus\Cup})$, where $\mathcal{E}_{\oplus\Cup}=\mathcal{E}_1\cup\mathcal{E}_2$.

For $p\in\mathcal{E}_{\oplus\Cup}$,

$$\Psi_{\oplus \Cup}(p)=\begin{cases}\left\{\left\langle u,\mathcal{T}_{\Psi_1(p)}(u),\mathcal{I}_{\Psi_1(p)}(u),(1-\mathcal{T}_{\Psi_1(p)}(u))\right\rangle; & \text{if } p\in\mathcal{E}_1-\mathcal{E}_2\right\},\\ \left\{\left\langle u,\mathcal{T}_{\Psi_2(p)}(u),\mathcal{I}_{\Psi_2(p)}(u),(1-\mathcal{T}_{\Psi_2(p)}(u))\right\rangle; & \text{if } p\in\mathcal{E}_2-\mathcal{E}_1\right\},\\ \left\{\left\langle u,\max(\mathcal{T}_{\Psi_1(p)}(u),\mathcal{T}_{\Psi_2(p)}(u)),\max(\mathcal{I}_{\Psi_1(p)}(u),\mathcal{I}_{\Psi_2(p)}(u)),\right.\right. & \\ \qquad \left.\left.\min((1-\mathcal{T}_{\Psi_1(p)}(u)),(1-\mathcal{T}_{\Psi_2(p)}(u)))\right\rangle; & \text{if } p\in\mathcal{E}_1\cap\mathcal{E}_2\right\}.\end{cases}$$

Thus, $\oplus\langle(\Psi_1,\mathcal{E}_1)\Cup(\Psi_2,\mathcal{E}_2)\rangle=\oplus(\Psi_1,\mathcal{E}_1)\Cup\oplus(\Psi_2,\mathcal{E}_2)$.

(iii) $$\oplus\oplus(\Psi_1,\mathcal{E}_1)=\oplus\left\{\left\langle u,\mathcal{T}_{\Psi_1(p)}(u),\mathcal{I}_{\Psi_1(p)}(u),(1-\mathcal{T}_{\Psi_1(p)}(u))\right\rangle;p\in\mathcal{E}_1\right\}$$
$$=\left\{\left\langle u,\mathcal{T}_{\Psi_1(p)}(u),\mathcal{I}_{\Psi_1(p)}(u),(1-\mathcal{T}_{\Psi_1(p)}(u))\right\rangle;p\in\mathcal{E}_1\right\}$$
$$=\oplus(\Psi_1,\mathcal{E}_1).$$

■

Theorem 3.2.

Let $(\Psi_1,\mathcal{E}_1)$ and $(\Psi_2,\mathcal{E}_2)$ be two SINSS over $\mathcal{V}$. Then,
(i) $\ominus\langle(\Psi_1,\mathcal{E}_1)\Cup(\Psi_2,\mathcal{E}_2)\rangle=\ominus(\Psi_1,\mathcal{E}_1)\Cup\ominus(\Psi_2,\mathcal{E}_2)$;
(ii) $\ominus\langle(\Psi_1,\mathcal{E}_1)\Cap(\Psi_2,\mathcal{E}_2)\rangle=\ominus(\Psi_1,\mathcal{E}_1)\Cap\ominus(\Psi_2,\mathcal{E}_2)$;
(iii) $\ominus\ominus(\Psi_1,\mathcal{E}_1)=\ominus(\Psi_1,\mathcal{E}_1)$.

Proof:

We give the proof of (i), and the proof of (ii) is analogous.

(i) Let $(\Psi_1,\mathcal{E}_1)\Cup(\Psi_2,\mathcal{E}_2)=(\Psi_{\Cup},\mathcal{E}_{\Cup})$, where $\mathcal{E}_{\Cup}=\mathcal{E}_1\cup\mathcal{E}_2$, $\forall\, p\in\mathcal{E}_{\Cup}$.

Consider,

$$\ominus\Psi_{\Cup}(p)=\begin{cases}\left\{\left\langle u,(1-\mathcal{F}_{\Psi_1(p)}(u)),\mathcal{I}_{\Psi_1(p)}(u),\mathcal{F}_{\Psi_1(p)}(u)\right\rangle; & \text{if } p\in\mathcal{E}_1-\mathcal{E}_2\right\},\\ \left\{\left\langle u,(1-\mathcal{F}_{\Psi_2(p)}(u)),\mathcal{I}_{\Psi_2(p)}(u),\mathcal{F}_{\Psi_2(p)}(u)\right\rangle; & \text{if } p\in\mathcal{E}_2-\mathcal{E}_1\right\},\\ \left\{\left\langle u,(1-\min(\mathcal{F}_{\Psi_1(p)}(u),\mathcal{F}_{\Psi_2(p)}(u))),\right.\right. & \\ \left.\left.\max(\mathcal{I}_{\Psi_1(p)}(u),\mathcal{I}_{\Psi_2(p)}(u)),\min(\mathcal{F}_{\Psi_1(p)}(u),\mathcal{F}_{\Psi_2(p)}(u))\right\rangle; \right. & \left.\text{if } p\in\mathcal{E}_1\cap\mathcal{E}_2\right\},\end{cases}$$

$$=\begin{cases}\left\{\left\langle u,(1-\mathcal{F}_{\Psi_1(p)}(u)),\mathcal{I}_{\Psi_1(p)}(u),\mathcal{F}_{\Psi_1(p)}(u)\right\rangle; & \text{if } p\in\mathcal{E}_1-\mathcal{E}_2\right\},\\ \left\{\left\langle u,(1-\mathcal{F}_{\Psi_2(p)}(u)),\mathcal{I}_{\Psi_2(p)}(u),\mathcal{F}_{\Psi_2(p)}(u)\right\rangle; & \text{if } p\in\mathcal{E}_2-\mathcal{E}_1\right\},\\ \left\{\left\langle u,\max((1-\mathcal{F}_{\Psi_1(p)}(u)),(1-\mathcal{F}_{\Psi_2(p)}(u))),\right.\right. & \\ \left.\left.\max(\mathcal{I}_{\Psi_1(p)}(u),\mathcal{I}_{\Psi_2(p)}(u)),\min(\mathcal{F}_{\Psi_1(p)}(u),\mathcal{F}_{\Psi_2(p)}(u))\right\rangle; \right. & \left.\text{if } p\in\mathcal{E}_1\cap\mathcal{E}_2\right\}.\end{cases}$$

$$\ominus(\Psi_1,\mathcal{E}_1)=\left\{\left\langle u,(1-\mathcal{F}_{\Psi_1(p)}(u)),\mathcal{I}_{\Psi_1(p)}(u),\mathcal{F}_{\Psi_1(p)}(u)\right\rangle;p\in\mathcal{E}_1\right\},$$

$$\ominus(\Psi_2,\mathcal{E}_2)=\left\{\left\langle u,(1-\mathcal{F}_{\Psi_2(p)}(u)),\mathcal{I}_{\Psi_2(p)}(u),\mathcal{F}_{\Psi_2(p)}(u)\right\rangle;p\in\mathcal{E}_2\right\}.$$

Let $\ominus(\Psi_1,\mathcal{E}_1)\Cup\ominus(\Psi_2,\mathcal{E}_2)=(\Psi_{\ominus\Cup},\mathcal{E}_{\ominus\Cup})$, where $\mathcal{E}_{\ominus\Cup}=\mathcal{E}_1\cup\mathcal{E}_2$.

For $p \in \mathcal{E}_{\ominus\Cup}$,

$$\Psi_{\ominus\Cup}(p) = \begin{cases} \left\{\left\langle u, (1-\mathcal{F}_{\Psi_1(p)}(u)), \mathcal{I}_{\Psi_1(p)}(u), \mathcal{F}_{\Psi_1(p)}(u)\right\rangle; & \text{if } p \in \mathcal{E}_1 - \mathcal{E}_2\right\}, \\ \left\{\left\langle u, (1-\mathcal{F}_{\Psi_2(p)}(u)), \mathcal{I}_{\Psi_2(p)}(u), \mathcal{F}_{\Psi_2(p)}(u)\right\rangle; & \text{if } p \in \mathcal{E}_2 - \mathcal{E}_1\right\}, \\ \left\{\left\langle u, \max((1-\mathcal{F}_{\Psi_1(p)}(u)), (1-\mathcal{F}_{\Psi_2(p)}(u))),\right.\right. & \\ \left.\left.\max(\mathcal{I}_{\Psi_1(p)}(u), \mathcal{I}_{\Psi_2(p)}(u)), \min(\mathcal{F}_{\Psi_1(p)}(u), \mathcal{F}_{\Psi_2(p)}(u))\right\rangle; \text{ if } p \in \mathcal{E}_1 \cap \mathcal{E}_2\right\}. & \end{cases}$$

Thus, $\ominus\left\langle(\Psi_1, \mathcal{E}_1) \Cup (\Psi_2, \mathcal{E}_2)\right\rangle = \ominus(\Psi_1, \mathcal{E}_1) \Cup \ominus(\Psi_2, \mathcal{E}_2)$.

(iii) $\ominus\ominus(\Psi_1, \mathcal{E}_1) = \ominus\left\{\left\langle u, (1-\mathcal{F}_{\Psi_1(p)}(u)), \mathcal{I}_{\Psi_1(p)}(u), \mathcal{F}_{\Psi_1(p)}(u)\right\rangle; p \in \mathcal{E}_1\right\}$

$$= \left\{\left\langle u, (1-\mathcal{F}_{\Psi_1(p)}(u)), \mathcal{I}_{\Psi_1(p)}(u), \mathcal{F}_{\Psi_1(p)}(u)\right\rangle; p \in \mathcal{E}_1\right\}$$

$$= \ominus(\Psi_1, \mathcal{E}_1).$$

■

Theorem 3.3.

Let $(\Psi, \mathcal{E})$ be a SINSS over $\mathcal{V}$. Then,
(i) $\ominus\oplus(\Psi, \mathcal{E}) = \oplus(\Psi, \mathcal{E})$;
(ii) $\oplus\ominus(\Psi, \mathcal{E}) = \ominus(\Psi, \mathcal{E})$.

Proof:

(i) $\ominus\oplus(\Psi, \mathcal{E}) = \left\{\left\langle u, (1-(1-\mathcal{T}_{\Psi(p)}(u))), \mathcal{I}_{\Psi(p)}(u), (1-\mathcal{T}_{\Psi(p)}(u))\right\rangle; p \in \mathcal{E}\right\}$

$$= \left\{\left\langle u, \mathcal{T}_{\Psi(p)}(u), \mathcal{I}_{\Psi(p)}(u), (1-\mathcal{T}_{\Psi(p)}(u))\right\rangle; p \in \mathcal{E}\right\}$$

$\ominus\oplus(\Psi, \mathcal{E}) = \oplus(\Psi, \mathcal{E})$.

(ii) $\oplus\ominus(\Psi, \mathcal{E}) = \left\{\left\langle u, (1-\mathcal{F}_{\Psi(p)}(u)), \mathcal{I}_{\Psi(p)}(u), (1-(1-\mathcal{F}_{\Psi(p)}(u)))\right\rangle; p \in \mathcal{E}\right\}$

$$= \left\{\left\langle u, (1-\mathcal{F}_{\Psi(p)}(u)), \mathcal{I}_{\Psi(p)}(u), \mathcal{F}_{\Psi(p)}(u)\right\rangle; p \in \mathcal{E}\right\}$$

$\oplus\ominus(\Psi, \mathcal{E}) = \ominus(\Psi, \mathcal{E})$. ■

Theorem 3.4.

Let $(\Psi_1, \mathcal{E}_1)$ and $(\Psi_2, \mathcal{E}_2)$ be two SINSS over $\mathcal{V}$. Then
(i) $\oplus\left\langle(\Psi_1, \mathcal{E}_1) \wedge (\Psi_2, \mathcal{E}_2)\right\rangle = \oplus(\Psi_1, \mathcal{E}_1) \wedge \oplus(\Psi_2, \mathcal{E}_2)$;
(ii) $\oplus\left\langle(\Psi_1, \mathcal{E}_1) \vee (\Psi_2, \mathcal{E}_2)\right\rangle = \oplus(\Psi_1, \mathcal{E}_1) \vee \oplus(\Psi_2, \mathcal{E}_2)$;
(iii) $\ominus\left\langle(\Psi_1, \mathcal{E}_1) \wedge (\Psi_2, \mathcal{E}_2)\right\rangle = \ominus(\Psi_1, \mathcal{E}_1) \wedge \oplus(\Psi_2, \mathcal{E}_2)$;
(iv) $\ominus\left\langle(\Psi_1, \mathcal{E}_1) \vee (\Psi_2, \mathcal{E}_2)\right\rangle = \ominus(\Psi_1, \mathcal{E}_1) \vee \oplus(\Psi_2, \mathcal{E}_2)$.

Proof:

We present the proofs of (i) and (iii), and the proofs of (ii) and (iv) are analogous.

(i) $\oplus \langle (\Psi_1, \mathcal{E}_1) \wedge (\Psi_2, \mathcal{E}_2) \rangle$

$$= \Big\{ \Big\langle \min(\mathcal{T}_{\Psi_1(p_1)}(u), \mathcal{T}_{\Psi_2(p_2)}(u)), \min(\mathcal{I}_{\Psi_1(p_1)}(u), \mathcal{I}_{\Psi_2(p_2)}(u)), (1 - \min(\mathcal{T}_{\Psi_1(p_1)}(u), \mathcal{T}_{\Psi_2(p_2)}(u))) \Big\rangle, \forall (p_1, p_2) \in \mathcal{E}_1 \times \mathcal{E}_2 \Big\}$$

$$= \Big\{ \Big\langle \min(\mathcal{T}_{\Psi_1(p_1)}(u), \mathcal{T}_{\Psi_2(p_2)}(u)), \min(\mathcal{I}_{\Psi_1(p_1)}(u), \mathcal{I}_{\Psi_2(p_2)}(u)), (\max(1 - (\mathcal{T}_{\Psi_1(p_1)}(u))), (1 - (\mathcal{T}_{\Psi_2(p_2)}(u)))) \Big\rangle, \forall (p_1, p_2) \in \mathcal{E}_1 \times \mathcal{E}_2 \Big\}.$$

Also, $\oplus(\Psi_1, \mathcal{E}_1) = \Big\{ \Big\langle u, \mathcal{T}_{\Psi_1(p_1)}(u), \mathcal{I}_{\Psi_1(p_1)}(u), (1 - \mathcal{T}_{\Psi_1(p_1)}(u)) \Big\rangle ;\ p_1 \in \mathcal{E}_1 \Big\}$,

$\oplus(\Psi_2, \mathcal{E}_2) = \Big\{ \Big\langle u, \mathcal{T}_{\Psi_2(p_2)}(u), \mathcal{I}_{\Psi_2(p_2)}(u), (1 - \mathcal{T}_{\Psi_2(p_2)}(u)) \Big\rangle ;\ p_2 \in \mathcal{E}_2 \Big\}$.

Therefore, we have

$$\oplus(\Psi_1, \mathcal{E}_1) \wedge \oplus(\Psi_2, \mathcal{E}_2)$$
$$= \Big\{ \Big\langle \min(\mathcal{T}_{\Psi_1(p_1)}(u), \mathcal{T}_{\Psi_2(p_2)}(u)), \min(\mathcal{I}_{\Psi_1(p_1)}(u), \mathcal{I}_{\Psi_2(p_2)}(u)), (\max(1 - (\mathcal{T}_{\Psi_1(p_1)}(u))), (1 - (\mathcal{T}_{\Psi_2(p_2)}(u)))) \Big\rangle, \forall (p_1, p_2) \in \mathcal{E}_1 \times \mathcal{E}_2 \Big\}$$
$$= \oplus \langle (\Psi_1, \mathcal{E}_1) \wedge (\Psi_2, \mathcal{E}_2) \rangle .$$

(iii) $\ominus \langle (\Psi_1, \mathcal{E}_1) \wedge (\Psi_2, \mathcal{E}_2) \rangle$

$$= \Big\{ \Big\langle (1 - \max(\mathcal{F}_{\Psi_1(p_1)}(u), \mathcal{F}_{\Psi_2(p_2)}(u))), \max(\mathcal{I}_{\Psi_1(p_1)}(u), \mathcal{I}_{\Psi_2(p_2)}(u)), \max(\mathcal{F}_{\Psi_1(p_1)}(u), \mathcal{F}_{\Psi_2(p_2)}(u)) \Big\rangle, \forall (p_1, p_2) \in \mathcal{E}_1 \times \mathcal{E}_2 \Big\}$$

$$= \Big\{ \Big\langle (\min(1 - (\mathcal{F}_{\Psi_1(p_1)}(u))), (1 - (\mathcal{F}_{\Psi_2(p_2)}(u)))), \max(\mathcal{I}_{\Psi_1(p_1)}(u), \mathcal{I}_{\Psi_2(p_2)}(u)), \max(\mathcal{F}_{\Psi_1(p_1)}(u), \mathcal{F}_{\Psi_2(p_2)}(u)) \Big\rangle, \forall (p_1, p_2) \in \mathcal{E}_1 \times \mathcal{E}_2 \Big\}.$$

Also, $\ominus(\Psi_1, \mathcal{E}_1) = \Big\{ \Big\langle u, (1 - \mathcal{F}_{\Psi_1(p_1)}(u)), \mathcal{I}_{\Psi_1(p_1)}(u), \mathcal{F}_{\Psi_1(p_1)}(u) \Big\rangle ;\ p_1 \in \mathcal{E}_1 \Big\}$,

$\ominus(\Psi_2, \mathcal{E}_2) = \Big\{ \Big\langle u, (1 - \mathcal{F}_{\Psi_2(p_2)}(u)), \mathcal{I}_{\Psi_2(p_2)}(u), \mathcal{F}_{\Psi_2(p_2)}(u) \Big\rangle ;\ p_2 \in \mathcal{E}_2 \Big\}$.

Therefore, we have

$$\ominus(\Psi_1, \mathcal{E}_1) \wedge \ominus(\Psi_2, \mathcal{E}_2)$$
$$= \Big\{ \Big\langle (\min(1 - (\mathcal{F}_{\Psi_1(p_1)}(u))), (1 - (\mathcal{F}_{\Psi_2(p_2)}(u)))), \max(\mathcal{I}_{\Psi_1(p_1)}(u), \mathcal{I}_{\Psi_2(p_2)}(u)), \max(\mathcal{F}_{\Psi_1(p_1)}(u), \mathcal{F}_{\Psi_2(p_2)}(u)), \Big\rangle \forall (p_1, p_2) \in \mathcal{E}_1 \times \mathcal{E}_2 \Big\}$$
$$= \oplus \langle (\Psi_1, \mathcal{E}_1) \wedge (\Psi_2, \mathcal{E}_2) \rangle .$$

■

4. $\pm$ and $\mp$ operators on SINSS

In this section, we define two new operators ($\pm$ and $\mp$) on SINSS and discuss some of their properties. We generalize these operations and properties on SINSS using the concepts given in Atanassov (1994).

Definition 4.1.

Let $(\Psi_1, \mathcal{E}_1)$ and $(\Psi_2, \mathcal{E}_2)$ be two SINSS over $\mathcal{V}$. Then,
(i) the operator $\pm$ is represented as $(\Psi_1, \mathcal{E}_1) \pm (\Psi_2, \mathcal{E}_2)$ = $(\Psi_\pm, \mathcal{E}_\pm)$, where $\mathcal{E}_\pm = \mathcal{E}_1 \cup \mathcal{E}_2$. $\forall\, p \in \mathcal{E}_\pm$,

$$\Psi_\pm(p) = \begin{cases} \left\{\left\langle u, (\mathcal{T}_{\Psi_1(p)}(u), \mathcal{I}_{\Psi_1(p)}(u), \mathcal{F}_{\Psi_1(p)}(u))\right\rangle; & \text{if } p \in \mathcal{E}_1 - \mathcal{E}_2\right\}, \\ \left\{\left\langle u, (\mathcal{T}_{\Psi_2(p)}(u), \mathcal{I}_{\Psi_2(p)}(u), \mathcal{F}_{\Psi_2(p)}(u))\right\rangle; & \text{if } p \in \mathcal{E}_2 - \mathcal{E}_1\right\}, \\ \left\{\left\langle u, \frac{(\mathcal{T}_{\Psi_1(p)}(u)+\mathcal{T}_{\Psi_2(p)}(u))}{2}, \frac{(\mathcal{I}_{\Psi_1(p)}(u)+\mathcal{I}_{\Psi_2(p)}(u))}{2}, \right.\right. & \\ \qquad\qquad \left.\left. \frac{(\mathcal{F}_{\Psi_1(p)}(u)+\mathcal{F}_{\Psi_2(p)}(u))}{2}\right\rangle; & \text{if } p \in \mathcal{E}_1 \cap \mathcal{E}_2\right\}. \end{cases}$$

(ii) the operator $\mp$ is represented as $(\Psi_1, \mathcal{E}_1) \mp (\Psi_2, \mathcal{E}_2)$ = $(\Psi_\mp, \mathcal{E}_\mp)$, where $\mathcal{E}_\mp = \mathcal{E}_1 \cup \mathcal{E}_2$. $\forall\, p \in \mathcal{E}_\mp$,

$$\Psi_\mp(p) = \begin{cases} \left\{\left\langle u, (\mathcal{T}_{\Psi_1(p)}(u), \mathcal{I}_{\Psi_1(p)}(u), \mathcal{F}_{\Psi_1(p)}(u))\right\rangle; & \text{if } p \in \mathcal{E}_1 - \mathcal{E}_2\right\}, \\ \left\{\left\langle u, (\mathcal{T}_{\Psi_2(p)}(u), \mathcal{I}_{\Psi_2(p)}(u), \mathcal{F}_{\Psi_2(p)}(u))\right\rangle; & \text{if } p \in \mathcal{E}_2 - \mathcal{E}_1\right\}, \\ \left\{\left\langle u, \frac{2*(\mathcal{T}_{\Psi_1(p)}(u)*\mathcal{T}_{\Psi_2(p)}(u))}{\mathcal{T}_{\Psi_1(p)}(u)+\mathcal{T}_{\Psi_2(p)}(u)}, \frac{(\mathcal{I}_{\Psi_1(p)}(u)+\mathcal{I}_{\Psi_2(p)}(u))}{2}, \right.\right. & \\ \qquad\qquad \left.\left. \frac{2*(\mathcal{F}_{\Psi_1(p)}(u)*\mathcal{F}_{\Psi_2(p)}(u))}{\mathcal{F}_{\Psi_1(p)}(u)+\mathcal{F}_{\Psi_2(p)}(u)}\right\rangle; & \text{if } p \in \mathcal{E}_1 \cap \mathcal{E}_2\right\}. \end{cases}$$

Example 4.1.

Consider that a psychiatrist has conducted two counseling sessions for his clients. Let's assume that the psychiatrist has given the values in the SINSS form for the first session $(\Psi_1, \mathcal{E}_1)$, as in Table 1 and for the second session $(\Psi_2, \mathcal{E}_2)$ in Table 4. Now we calculate the combined results of the two sessions using $(\Psi_1, \mathcal{E}_1) \pm (\Psi_2, \mathcal{E}_2)$, $(\Psi_1, \mathcal{E}_1) \mp (\Psi_2, \mathcal{E}_2)$ and present the results in Table 5 and 6 respectively.

Table 4. Representation of clients with cognitive disorders in SINSS $(\Psi_2, \mathcal{E}_2)$ form.

$\mathcal{U}$	IMC(p_1)	LM(p_2)	IC(p_3)	IJ(p_4)
c_1	$\langle 0.6, 0.8, 0.2\rangle$	$\langle 0.5, 0.3, 0.3\rangle$	$\langle 0.6, 0.7, 0.1\rangle$	$\langle 0.5, 0.8, 0.3\rangle$
c_2	$\langle 0.5, 0.3, 0.2\rangle$	$\langle 0.6, 0.6, 0.3\rangle$	$\langle 0.7, 0.9, 0.2\rangle$	$\langle 0.5, 0.8, 0.4\rangle$
c_3	$\langle 0.4, 0.2, 0.6\rangle$	$\langle 0.5, 0.9, 0.4\rangle$	$\langle 0.6, 0.5, 0.2\rangle$	$\langle 0.7, 0.3, 0.2\rangle$
c_4	$\langle 0.3, 0.2, 0.4\rangle$	$\langle 0.4, 0.5, 0.4\rangle$	$\langle 0.5, 0.6, 0.5\rangle$	$\langle 0.6, 0.6, 0.2\rangle$

(i) The SINSS $(\Psi_1, \mathcal{E}_1) \pm (\Psi_2, \mathcal{E}_2)$ is shown in Table 5.

Table 5. Representation of clients with cognitive disorders in SINSS $(\Psi_1, \mathcal{E}_1) \pm (\Psi_2, \mathcal{E}_2)$ form.

$\mathcal{U}$	IMC(p_1)	LM(p_2)	IC(p_3)	IJ(p_4)
c_1	$\langle 0.55, 0.75, 0.25\rangle$	$\langle 0.45, 0.40, 0.45\rangle$	$\langle 0.40, 0.50, 0.30\rangle$	$\langle 0.55, 0.75, 0.25\rangle$
c_2	$\langle 0.55, 0.35, 0.25\rangle$	$\langle 0.65, 0.70, 0.25\rangle$	$\langle 0.75, 0.90, 0.15\rangle$	$\langle 0.30, 0.75, 0.50\rangle$
c_3	$\langle 0.35, 0.35, 0.60\rangle$	$\langle 0.55, 0.90, 0.35\rangle$	$\langle 0.75, 0.70, 0.15\rangle$	$\langle 0.45, 0.30, 0.45\rangle$
c_4	$\langle 0.35, 0.20, 0.45\rangle$	$\langle 0.45, 0.50, 0.45\rangle$	$\langle 0.45, 0.65, 0.50\rangle$	$\langle 0.55, 0.50, 0.30\rangle$

(ii) The SINSS $(\Psi_1, \mathcal{E}_1) \mp (\Psi_2, \mathcal{E}_2)$ is shown in Table 6.

Table 6. Representation of clients with cognitive disorders in SINSS $(\Psi_1, \mathcal{E}_1) \mp (\Psi_2, \mathcal{E}_2)$ form.

$\mathcal{U}$	IMC(p_1)	LM(p_2)	IC(p_3)	IJ(p_4)
c_1	$\langle 0.55, 0.75, 0.24\rangle$	$\langle 0.44, 0.40, 0.40\rangle$	$\langle 0.30, 0.50, 0.17\rangle$	$\langle 0.55, 0.75, 0.24\rangle$
c_2	$\langle 0.55, 0.35, 0.24\rangle$	$\langle 0.65, 0.70, 0.24\rangle$	$\langle 0.75, 0.90, 0.13\rangle$	$\langle 0.17, 0.75, 0.48\rangle$
c_3	$\langle 0.34, 0.35, 0.60\rangle$	$\langle 0.55, 0.90, 0.34\rangle$	$\langle 0.72, 0.70, 0.13\rangle$	$\langle 0.31, 0.30, 0.31\rangle$
c_4	$\langle 0.34, 0.20, 0.44\rangle$	$\langle 0.44, 0.50, 0.44\rangle$	$\langle 0.44, 0.65, 0.50\rangle$	$\langle 0.55, 0.50, 0.27\rangle$

Proposition 4.1.

Let $(\Psi_1, \mathcal{E}_1)$ and $(\Psi_2, \mathcal{E}_2)$ be non-empty over $\mathcal{V}$. Then,
(i) $(\Psi_1, \mathcal{E}_1) \pm (\Psi_2, \mathcal{E}_2) = (\Psi_2, \mathcal{E}_2) \pm (\Psi_1, \mathcal{E}_1)$;
(ii) $[(\Psi_1, \mathcal{E}_1)^c \pm (\Psi_2, \mathcal{E}_2)^c]^c = (\Psi_1, \mathcal{E}_1) \pm (\Psi_1, \mathcal{E}_1)$.

Proof:

(i) The proof is straightforward.

(ii) Let

$$(\Psi_1, \mathcal{E}_1) = \left\{ \left\langle u, \mathcal{T}_{\Psi_1(p)}(u), \mathcal{I}_{\Psi_1(p)}(u), \mathcal{F}_{\Psi_1(p)}(u) \right\rangle ; p \in \mathcal{E}_1 \right\}, \text{and}$$

$$(\Psi_2, \mathcal{E}_2) = \left\{ \left\langle u, \mathcal{T}_{\Psi_2(p)}(u), \mathcal{I}_{\Psi_2(p)}(u), \mathcal{I}_{\Psi_2(p)}(u) \right\rangle ; p \in \mathcal{E}_2 \right\} \text{ be two SINSS.}$$

Then,

$$[(\Psi_1, \mathcal{E}_1)^c \pm (\Psi_2, \mathcal{E}_2)^c] = \begin{cases} \left\{ \left\langle u, (\mathcal{F}_{\Psi_1(p)}(u), (1 - \mathcal{I}_{\Psi_1(p)}(u)), \mathcal{T}_{\Psi_1(p)}(u)) \right\rangle ; \quad \text{if } p \in \mathcal{E}_1 - \mathcal{E}_2 \right\}, \\ \left\{ \left\langle u, (\mathcal{F}_{\Psi_2(p)}(u), (1 - \mathcal{I}_{\Psi_2(p)}(u)), \mathcal{T}_{\Psi_2(p)}(u)) \right\rangle ; \quad \text{if } p \in \mathcal{E}_2 - \mathcal{E}_1 \right\}, \\ \left\{ \left\langle u, \frac{(\mathcal{F}_{\Psi_1(p)}(u) + \mathcal{F}_{\Psi_2(p)}(u))}{2}, \frac{((1 - \mathcal{I}_{\Psi_1(p)}(u)) + (1 - \mathcal{I}_{\Psi_2(p)}(u)))}{2}, \right.\right. \\ \left.\left. \qquad \frac{(\mathcal{T}_{\Psi_1(p)}(u) + \mathcal{T}_{\Psi_2(p)}(u))}{2} \right\rangle ; \quad \text{if } p \in \mathcal{E}_1 \cap \mathcal{E}_2 \right\}. \end{cases}$$

Now consider,

$$[(\Psi_1, \mathcal{E}_1)^c \pm (\Psi_2, \mathcal{E}_2)^c]^c = \begin{cases} \left\{ \left\langle u, (\mathcal{T}_{\Psi_1(p)}(u), \mathcal{I}_{\Psi_1(p)}(u), \mathcal{F}_{\Psi_1(p)}(u)) \right\rangle ; \quad \text{if } p \in \mathcal{E}_1 - \mathcal{E}_2 \right\}, \\ \left\{ \left\langle u, (\mathcal{T}_{\Psi_2(p)}(u), \mathcal{I}_{\Psi_2(p)}(u), \mathcal{F}_{\Psi_2(p)}(u)) \right\rangle ; \quad \text{if } p \in \mathcal{E}_2 - \mathcal{E}_1 \right\}, \\ \left\{ \left\langle u, \frac{(\mathcal{T}_{\Psi_1(p)}(u) + \mathcal{T}_{\Psi_2(p)}(u))}{2}, \frac{(\mathcal{I}_{\Psi_1(p)}(u) + \mathcal{I}_{\Psi_2(p)}(u))}{2}, \right.\right. \\ \left.\left. \qquad \frac{(\mathcal{F}_{\Psi_1(p)}(u) + \mathcal{F}_{\Psi_2(p)}(u))}{2} \right\rangle ; \quad \text{if } p \in \mathcal{E}_1 \cap \mathcal{E}_2 \right\}. \end{cases}$$

Hence, $[(\Psi_1, \mathcal{E}_1)^c \pm (\Psi_2, \mathcal{E}_2)^c]^c = (\Psi_1, \mathcal{E}_1) \pm (\Psi_2, \mathcal{E}_2)$. ■

Proposition 4.2.

Let $(\Psi_1, \mathcal{E}_1)$ and $(\Psi_2, \mathcal{E}_2)$ be non-empty over $\mathcal{V}$. Then,
(i) $(\Psi_1, \mathcal{E}_1) \mp (\Psi_2, \mathcal{E}_2) = (\Psi_2, \mathcal{E}_2) \mp (\Psi_1, \mathcal{E}_1)$;

(ii) $[(\Psi_1,\mathcal{E}_1)^c \mp (\Psi_2,\mathcal{E}_2)^c]^c = (\Psi_1,\mathcal{E}_1) \mp (\Psi_1,\mathcal{E}_1)$.

Proof:

(i) Consider,

$$(\Psi_1,\mathcal{E}_1) \mp (\Psi_2,\mathcal{E}_2) = \begin{cases} \left\{\left\langle u, (\mathcal{T}_{\Psi_1(p)}(u), \mathcal{I}_{\Psi_1(p)}(u), \mathcal{F}_{\Psi_1(p)}(u))\right\rangle; & \text{if } p \in \mathcal{E}_1 - \mathcal{E}_2\right\}, \\ \left\{\left\langle u, (\mathcal{T}_{\Psi_2(p)}(u), \mathcal{I}_{\Psi_2(p)}(u), \mathcal{F}_{\Psi_2(p)}(u))\right\rangle; & \text{if } p \in \mathcal{E}_2 - \mathcal{E}_1\right\}, \\ \left\{\left\langle u, \frac{2*(\mathcal{T}_{\Psi_1(p)}(u)*\mathcal{T}_{\Psi_2(p)}(u))}{\mathcal{T}_{\Psi_1(p)}(u)+\mathcal{T}_{\Psi_2(p)}(u)}, \frac{(\mathcal{I}_{\Psi_1(p)}(u)+\mathcal{I}_{\Psi_2(p)}(u))}{2}, \right.\right. & \\ \left.\left. \qquad \frac{2*(\mathcal{F}_{\Psi_1(p)}(u)*\mathcal{F}_{\Psi_2(p)}(u))}{\mathcal{F}_{\Psi_1(p)}(u)+\mathcal{F}_{\Psi_2(p)}(u)}\right\rangle; & \text{if } p \in \mathcal{E}_1 \cap \mathcal{E}_2\right\}, \end{cases}$$

$$= \begin{cases} \left\{\left\langle u, (\mathcal{T}_{\Psi_1(p)}(u), \mathcal{I}_{\Psi_1(p)}(u), \mathcal{F}_{\Psi_1(p)}(u))\right\rangle; & \text{if } p \in \mathcal{E}_1 - \mathcal{E}_2\right\}, \\ \left\{\left\langle u, (\mathcal{T}_{\Psi_2(p)}(u), \mathcal{I}_{\Psi_2(p)}(u), \mathcal{F}_{\Psi_2(p)}(u))\right\rangle; & \text{if } p \in \mathcal{E}_2 - \mathcal{E}_1\right\}, \\ \left\{\left\langle u, \frac{2*(\mathcal{T}_{\Psi_2(p)}(u)*\mathcal{T}_{\Psi_1(p)}(u))}{\mathcal{T}_{\Psi_2(p)}(u)+\mathcal{T}_{\Psi_1(p)}(u)}, \frac{(\mathcal{I}_{\Psi_2(p)}(u)+\mathcal{I}_{\Psi_1(p)}(u))}{2}, \right.\right. & \\ \left.\left. \qquad \frac{2*(\mathcal{F}_{\Psi_2(p)}(u)*\mathcal{F}_{\Psi_1(p)}(u))}{\mathcal{F}_{\Psi_2(p)}(u)+\mathcal{F}_{\Psi_1(p)}(u)}\right\rangle; & \text{if } p \in \mathcal{E}_1 \cap \mathcal{E}_2\right\}. \end{cases}$$

Hence, $(\Psi_1,\mathcal{E}_1) \mp (\Psi_2,\mathcal{E}_2) = (\Psi_2,\mathcal{E}_2) \mp (\Psi_1,\mathcal{E}_1)$.

(ii) Consider,

$$(\Psi_1,\mathcal{E}_1)^c \mp (\Psi_2,\mathcal{E}_2)^c = \begin{cases} \left\{\left\langle u, (\mathcal{F}_{\Psi_1(p)}(u), (1-\mathcal{I}_{\Psi_1(p)}(u)), \mathcal{T}_{\Psi_1(p)}(u))\right\rangle; & \text{if } p \in \mathcal{E}_1 - \mathcal{E}_2\right\}, \\ \left\{\left\langle u, (\mathcal{F}_{\Psi_2(p)}(u), (1-\mathcal{I}_{\Psi_2(p)}(u)), \mathcal{T}_{\Psi_2(p)}(u))\right\rangle; & \text{if } p \in \mathcal{E}_2 - \mathcal{E}_1\right\}, \\ \left\{\left\langle u, \frac{2*(\mathcal{F}_{\Psi_1(p)}(u)*\mathcal{F}_{\Psi_2(p)}(u))}{\mathcal{F}_{\Psi_1(p)}(u)+\mathcal{F}_{\Psi_2(p)}(u)}, \frac{((1-\mathcal{I}_{\Psi_1(p)}(u))+(1-\mathcal{I}_{\Psi_2(p)}(u)))}{2}, \right.\right. & \\ \left.\left. \qquad \frac{2*(\mathcal{T}_{\Psi_1(p)}(u)*\mathcal{T}_{\Psi_2(p)}(u))}{\mathcal{T}_{\Psi_1(p)}(u)+\mathcal{T}_{\Psi_2(p)}(u)}\right\rangle; & \text{if } p \in \mathcal{E}_1 \cap \mathcal{E}_2\right\}. \end{cases}$$

Then,

$$[(\Psi_1,\mathcal{E}_1)^c \mp (\Psi_2,\mathcal{E}_2)^c]^c = \begin{cases} \left\{\left\langle u, (\mathcal{T}_{\Psi_1(p)}(u), \mathcal{I}_{\Psi_1(p)}(u), \mathcal{F}_{\Psi_1(p)}(u))\right\rangle; & \text{if } p \in \mathcal{E}_1 - \mathcal{E}_2\right\}, \\ \left\{\left\langle u, (\mathcal{T}_{\Psi_2(p)}(u), \mathcal{I}_{\Psi_2(p)}(u), \mathcal{F}_{\Psi_2(p)}(u))\right\rangle; & \text{if } p \in \mathcal{E}_2 - \mathcal{E}_1\right\}, \\ \left\{\left\langle u, \frac{2*(\mathcal{T}_{\Psi_1(p)}(u)*\mathcal{T}_{\Psi_2(p)}(u))}{\mathcal{T}_{\Psi_1(p)}(u)+\mathcal{T}_{\Psi_2(p)}(u)}, \frac{(\mathcal{I}_{\Psi_1(p)}(u)+\mathcal{I}_{\Psi_2(p)}(u))}{2}, \right.\right. & \\ \left.\left. \qquad \frac{2*(\mathcal{F}_{\Psi_1(p)}(u)*\mathcal{F}_{\Psi_2(p)}(u))}{\mathcal{F}_{\Psi_1(p)}(u)+\mathcal{F}_{\Psi_2(p)}(u)}\right\rangle; & \text{if } p \in \mathcal{E}_1 \cap \mathcal{E}_2\right\}. \end{cases}$$

Hence, $[(\Psi_1,\mathcal{E}_1)^c \mp (\Psi_2,\mathcal{E}_2)^c]^c = (\Psi_1,\mathcal{E}_1) \mp (\Psi_2,\mathcal{E}_2)$. ■

5. $\mathcal{S}_\alpha$, $\mathcal{S}_{\alpha,\beta}$ and $\mathcal{I}_{\alpha,\beta}$ operators on SINSS

In this section, we define the operators $\mathcal{S}_\alpha$, $\mathcal{S}_{\alpha,\beta}$ and $\mathcal{I}_{\alpha,\beta}$ on SINSS and discuss some of their properties in detail. We generalize these operations and properties on SINSS by the concepts discussed in Atanassov (1989).

Definition 5.1.

Let $\alpha \in [0,1]$. Then the operator $\mathcal{S}_\alpha(\Psi,\mathcal{E})$ is represented as,

$$\begin{aligned}\mathcal{S}_\alpha(\Psi,\mathcal{E}) = \big\{\big\langle u, \big(\mathcal{T}_{\psi(p)}(u) + \alpha\big(1-\mathcal{T}_{\psi(p)}(u)-\mathcal{F}_{\psi(p)}(u)\big)\big), \mathcal{I}_{\psi(p)}(u),\\ \big(\mathcal{F}_{\psi(p)}(u) + (1-\alpha)\big(1-\mathcal{T}_{\psi(p)}(u)-\mathcal{F}_{\psi(p)}(u)\big)\big)\big\rangle; p\in\mathcal{E}\big\}\\ =\big\{\big\langle u, \big(\mathcal{T}_{\psi(p)}(u) + \alpha\big(\pi_{\psi(p)}(u)\big)\big), \mathcal{I}_{\psi(p)}(u), \big(\mathcal{F}_{\psi(p)}(u) + (1-\alpha)\big(\pi_{\psi(p)}(u)\big)\big)\big\rangle; p\in\mathcal{E}\big\},\\ \text{where } \pi_{\psi(p)}(u) = (1-\mathcal{T}_{\psi(p)}(u)-\mathcal{F}_{\psi(p)}(u)).\end{aligned}$$

Proposition 5.1.

Let $\alpha,\beta \in [0,1]$ and $\alpha \le \beta$. Then for every SINSS $(\Psi,\mathcal{E})$ the following hold:
(i) $\mathcal{S}_\alpha(\Psi,\mathcal{E}) \Subset \mathcal{S}_\beta(\Psi,\mathcal{E})$;
(ii) $\mathcal{S}_0(\Psi,\mathcal{E}) = \oplus(\Psi,\mathcal{E})$;
(iii) $\mathcal{S}_1(\Psi,\mathcal{E}) = \ominus(\Psi,\mathcal{E})$.

Proof:

(i) $\mathcal{S}_\alpha(\Psi,\mathcal{E}) = \big\{\big\langle u, \big(\mathcal{T}_{\psi(p)}(u) + \alpha\big(\pi_{\psi(p)}(u)\big)\big), \mathcal{I}_{\psi(p)}(u), \big(\mathcal{F}_{\psi(p)}(u) + (1-\alpha)\big(\pi_{\psi(p)}(u)\big)\big)\big\rangle; p\in\mathcal{E}\big\}$,
and $\mathcal{S}_\beta(\Psi,\mathcal{E}) = \big\{\big\langle u, \big(\mathcal{T}_{\psi(p)}(u) + \beta\big(\pi_{\psi(p)}(u)\big)\big), \mathcal{I}_{\psi(p)}(u), \big(\mathcal{F}_{\psi(p)}(u) + (1-\beta)\big(\pi_{\psi(p)}(u)\big)\big)\big\rangle; p\in\mathcal{E}\big\}$.

Since $\alpha \le \beta$, we have

$$(\mathcal{T}_{\psi(p)}(u) + \alpha(\pi_{\psi(p)}(u))) \le (\mathcal{T}_{\psi(p)}(u) + \beta(\pi_{\psi(p)}(u))).$$

Also, $(1-\beta) \le (1-\alpha)$, we have

$$(\mathcal{F}_{\psi(p)}(u) + (1-\beta)(\pi_{\psi(p)}(u))) \le (\mathcal{F}_{\psi(p)}(u) + (1-\alpha)(\pi_{\psi(p)}(u))).$$

Hence, $\mathcal{S}_\alpha(\Psi,\mathcal{E}) \Subset \mathcal{S}_\beta(\Psi,\mathcal{E})$.

(ii) Consider, $\alpha = 0$

$$\begin{aligned}\mathcal{S}_0(\Psi,\mathcal{E}) &= \big\{\big\langle u, \big(\mathcal{T}_{\psi(p)}(u) + 0\big(\pi_{\psi(p)}(u)\big)\big), \mathcal{I}_{\psi(p)}(u), \big(\mathcal{F}_{\psi(p)}(u) + (1-0)\big(\pi_{\psi(p)}(u)\big)\big)\big\rangle; p\in\mathcal{E}\big\}\\ &=\big\{\big\langle u, \big(\mathcal{T}_{\psi(p)}(u)\big), \big(\mathcal{I}_{\psi(p)}(u)\big), \big(1-\mathcal{T}_{\psi(p)}(u)\big)\big\rangle; p\in\mathcal{E}\big\}\\ &= \oplus\,(\Psi,\mathcal{E}).\end{aligned}$$

Hence, $\mathcal{S}_0(\Psi,\mathcal{E}) = \oplus(\Psi,\mathcal{E})$.

(iii) Consider, α = 1

$$\begin{aligned}\mathcal{S}_1(\Psi,\mathcal{E}) &= \big\{\big\langle u, \big(\mathcal{T}_{\psi(p)}(u) + \big(\pi_{\psi(p)}(u)\big)\big), \mathcal{I}_{\psi(p)}(u), \big(\mathcal{F}_{\psi(p)}(u)\big)\big\rangle; p \in \mathcal{E}\big\}\\ &=\big\{\big\langle u, \big(1 - \mathcal{F}_{\psi(p)}(u)\big), \mathcal{I}_{\psi(p)}(u), \mathcal{F}_{\psi(p)}(u)\big\rangle; p \in \mathcal{E}\big\}\\ &= \oplus\,(\Psi,\mathcal{E}).\end{aligned}$$

Hence, $\mathcal{S}_1(\Psi,\mathcal{E}) = \ominus(\Psi,\mathcal{E})$. ■

Remark 5.1.

The operator $\mathcal{S}_\alpha$ is an extension of $\oplus$ and $\ominus$ operators.

Definition 5.2.

Let $\alpha, \beta \in [0,1]$ and $\alpha + \beta \leq 1$. Then the operator $\mathcal{S}_{\alpha,\beta}(\Psi,\mathcal{E})$ is represented as,

$$\mathcal{S}_{\alpha,\beta}(\Psi,\mathcal{E}) = \big\{\big\langle u, \big(\mathcal{T}_{\psi(p)}(u) + \alpha\big(\pi_{\psi(p)}(u)\big)\big), \mathcal{I}_{\psi(p)}(u), \big(\mathcal{F}_{\psi(p)}(u) + \beta\big(\pi_{\psi(p)}(u)\big)\big)\big\rangle; p \in \mathcal{E}\big\},$$
$$\text{where } \pi_{\psi(p)}(u) = (1 - \mathcal{T}_{\psi(p)}(u) - \mathcal{F}_{\psi(p)}(u)).$$

Theorem 5.1.

Let $\alpha, \beta, \gamma \in [0,1]$ and $\alpha + \beta \leq 1$. Then for every SINSS $(\Psi,\mathcal{E})$ the following hold:
(i) $\mathcal{S}_{\alpha,\beta}(\Psi,\mathcal{E})$ is a SINSS;
(ii) If $0 \leq \gamma \leq \alpha$ then $\mathcal{S}_{\gamma,\beta}(\Psi,\mathcal{E}) \Subset \mathcal{S}_{\alpha,\beta}(\Psi,\mathcal{E})$;
(iii) If $0 \leq \gamma \leq \beta$ then $\mathcal{S}_{\alpha,\beta}(\Psi,\mathcal{E}) \Subset \mathcal{S}_{\alpha,\gamma}(\Psi,\mathcal{E})$;
(iv) $\mathcal{S}_\alpha(\Psi,\mathcal{E}) = \mathcal{S}_{\alpha,(1-\alpha)}(\Psi,\mathcal{E})$;
(v) $\oplus(\Psi,\mathcal{E}) = \mathcal{S}_{0,1}(\Psi,\mathcal{E})$;
(vi) $\ominus(\Psi,\mathcal{E}) = \mathcal{S}_{1,0}(\Psi,\mathcal{E})$;
(vii) $(\mathcal{S}_{\alpha,\beta}(\Psi,\mathcal{E})^c)^c = (\mathcal{S}_{\beta,\alpha}(\Psi,\mathcal{E}))$.

Proof:

(i) Consider,

$$\begin{aligned}&\frac{(\mathcal{T}_{\psi(p)}(u) + \alpha(\pi_{\psi(p)}(u)))}{2} + (\mathcal{I}_{\psi(p)}(u)) + \frac{(\mathcal{F}_{\psi(p)}(u) + \beta(\pi_{\psi(p)}(u)))}{2}\\ &\qquad= \frac{(\mathcal{T}_{\psi(p)}(u) + \mathcal{F}_{\psi(p)}(u)}{2} + (\mathcal{I}_{\psi(p)}(u)) + (\alpha+\beta)\frac{(\pi_{\psi(p)}(u))}{2}\\ &\qquad\leq \frac{(\mathcal{T}_{\psi(p)}(u) + \mathcal{F}_{\psi(p)}(u)}{2} + (\mathcal{I}_{\psi(p)}(u)) + \frac{(1 - \mathcal{T}_{\psi(p)}(u) - \mathcal{F}_{\psi(p)}(u)}{2}\\ &\qquad\leq \frac{1}{2} + 1 < 2. \qquad\qquad (\text{Since, } \alpha+\beta \leq 1 \text{ and } \mathcal{I}_{\psi(p)}(u) \leq 1)\end{aligned}$$

Hence, $\mathcal{S}_{\alpha,\beta}(\Psi,\mathcal{E})$ is a SINSS.

(ii) Consider,

$$\mathcal{S}_{\gamma,\beta}(\Psi,\mathcal{E}) = \big\{\big\langle u, \big(\mathcal{T}_{\psi(p)}(u) + \gamma\big(\pi_{\psi(p)}(u)\big)\big), \mathcal{I}_{\psi(p)}(u), \big(\mathcal{F}_{\psi(p)}(u) + \beta\big(\pi_{\psi(p)}(u)\big)\big)\big\rangle; p \in \mathcal{E}\big\},$$

$$\mathcal{S}_{\alpha,\beta}(\Psi,\mathcal{E}) = \big\{\big\langle u, \big(\mathcal{T}_{\psi(p)}(u) + \alpha\big(\pi_{\psi(p)}(u)\big)\big), \mathcal{I}_{\psi(p)}(u), \big(\mathcal{F}_{\psi(p)}(u) + \beta\big(\pi_{\psi(p)}(u)\big)\big)\big\rangle; p \in \mathcal{E}\big\}.$$

Now

$$(\mathcal{T}_{\psi(p)}(u) + \gamma(\pi_{\psi(p)}(u)) \leq (\mathcal{T}_{\psi(p)}(u) + \alpha(\pi_{\psi(p)}(u)). \qquad \text{(Since, } \gamma \leq \alpha)$$

Hence, $\mathcal{S}_{\gamma,\beta}(\Psi, \mathcal{E}) \Subset \mathcal{S}_{\alpha,\beta}(\Psi, \mathcal{E})$.

(iii) Similar to proof (ii).

(iv) Consider,

$$\begin{aligned}\mathcal{S}_{\alpha,1-\alpha}(\Psi, \mathcal{E}) =& \big\{\big\langle u, \big(\mathcal{T}_{\psi(p)}(u) + \alpha\big(\pi_{\psi(p)}(u)\big)\big), \mathcal{I}_{\psi(p)}(u), \big(\mathcal{F}_{\psi(p)}(u) + 1 - \alpha\big(\pi_{\psi(p)}(u)\big)\big)\big\rangle; p \in \mathcal{E}\big\}\\ =& \mathcal{S}_{\alpha}(\Psi, \mathcal{E}).\end{aligned}$$

Hence, $\mathcal{S}_{\alpha}(\Psi, \mathcal{E}) = \mathcal{S}_{\alpha,(1-\alpha)}(\Psi, \mathcal{E})$.

(v) Let α = 0 and β = 1,

$$\begin{aligned}\mathcal{S}_{0,1}(\Psi, \mathcal{E}) =& \big\{\big\langle u, \big(\mathcal{T}_{\psi(p)}(u)\big), \mathcal{I}_{\psi(p)}(u), \big(\mathcal{F}_{\psi(p)}(u) + \big(\pi_{\psi(p)}(u)\big)\big)\big\rangle; p \in \mathcal{E}\big\}\\ =& \big\{\big\langle u, \big(\mathcal{T}_{\psi(p)}(u)\big), \mathcal{I}_{\psi(p)}(u), \big(1 - \mathcal{T}_{\psi(p)}(u)\big)\big)\big\rangle; p \in \mathcal{E}\big\}\\ =& \oplus (\Psi, \mathcal{E}).\end{aligned}$$

Hence, $\oplus(\Psi, \mathcal{E}) = \mathcal{S}_{0,1}(\Psi, \mathcal{E})$.

(vi)Let α = 1 and β = 0,

$$\begin{aligned}\mathcal{S}_{1,0}(\Psi, \mathcal{E}) =& \big\{\big\langle u, \big(\mathcal{T}_{\psi(p)}(u) + \big(\pi_{\psi(p)}(u)\big)\big), \mathcal{I}_{\psi(p)}(u), \big(\mathcal{F}_{\psi(p)}(u)\big)\big\rangle; p \in \mathcal{E}\big\}\\ =& \big\{\big\langle u, \big(1 - \mathcal{F}_{\psi(p)}(u)\big), \mathcal{I}_{\psi(p)}(u), \big(\mathcal{F}_{\psi(p)}(u)\big)\big)\big\rangle; p \in \mathcal{E}\big\}\\ =& \ominus (\Psi, \mathcal{E}).\end{aligned}$$

Hence, $\ominus(\Psi, \mathcal{E}) = \mathcal{S}_{1,0}(\Psi, \mathcal{E})$.

(vii) Consider,

$$\begin{aligned}\mathcal{S}_{\alpha,\beta}(\Psi, \mathcal{E})^c =& \big\{\big\langle u, \big(\mathcal{F}_{\psi(p)}(u) + \alpha\big(\pi_{\psi(p)}(u)\big)\big), (1 - \mathcal{I}_{\psi(p)}(u)),\\ & \big(\mathcal{T}_{\psi(p)}(u) + \beta\big(\pi_{\psi(p)}(u)\big)\big)\big\rangle; p \in \mathcal{E}\big\}.\\ (\mathcal{S}_{\alpha,\beta}(\Psi, \mathcal{E})^c)^c =& \big\{\big\langle u, \big(\mathcal{T}_{\psi(p)}(u) + \beta\big(\pi_{\psi(p)}(u)\big)\big), \mathcal{I}_{\psi(p)}(u), \big(\mathcal{F}_{\psi(p)}(u) + \alpha\big(\pi_{\psi(p)}(u)\big)\big)\big\rangle; p \in \mathcal{E}\big\}\\ =& (\mathcal{S}_{\beta,\alpha}(\Psi, \mathcal{E})).\end{aligned}$$

Hence, $(\mathcal{S}_{\alpha,\beta}(\Psi, \mathcal{E})^c)^c = (\mathcal{S}_{\beta,\alpha}(\Psi, \mathcal{E}))$. ■

Remark 5.2.

If $\alpha + \beta$ = 1, then $\mathcal{S}_{\alpha,\beta}(\Psi, \mathcal{E}) = \mathcal{S}_{\alpha}(\Psi, \mathcal{E})$.

Definition 5.3.

Let $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \leq 1$. Then the operator $\mathcal{I}_{\alpha,\beta}(\Psi, \mathcal{E})$ is represented as,

$$\mathcal{I}_{\alpha,\beta}(\Psi, \mathcal{E}) = \big\{\big\langle u, \alpha\big(\mathcal{T}_{\psi(p)}(u)\big), \mathcal{I}_{\psi(p)}(u), \beta\big(\mathcal{F}_{\psi(p)}(u)\big)\big\rangle; p \in \mathcal{E}\big\}.$$

Theorem 5.2.

Let $\alpha, \beta, \gamma \in [0,1]$. Then for every SINSS $(\Psi, \mathcal{E})$ the following hold:
(i) $\mathcal{I}_{\alpha,\beta}(\Psi, \mathcal{E})$ is a SINSS;
(ii) If $\alpha \leq \gamma$ then $\mathcal{I}_{\alpha,\beta}(\Psi, \mathcal{E}) \Subset \mathcal{I}_{\gamma,\beta}(\Psi, \mathcal{E})$;
(iii) If $\beta \leq \gamma$ then $\mathcal{I}_{\alpha,\beta}(\Psi, \mathcal{E}) \Supset \mathcal{I}_{\alpha,\gamma}(\Psi, \mathcal{E})$;
(iv) If $\delta \in [0,1]$ then $\mathcal{I}_{\alpha,\beta}(\mathcal{I}_{\gamma,\delta}(\Psi, \mathcal{E})) = \mathcal{I}_{\alpha\gamma,\beta\delta}(\Psi, \mathcal{E}) = \mathcal{I}_{\gamma,\delta}(\mathcal{I}_{\alpha,\beta}(\Psi, \mathcal{E}))$;
(v)$(\mathcal{I}_{\alpha,\beta}(\Psi, \mathcal{E})^c)^c = (\mathcal{I}_{\beta,\alpha}(\Psi, \mathcal{E}))$.

Proof:

(i) The proof is straightforward.
(ii) Consider,

$$\begin{aligned} \mathcal{I}_{\alpha,\beta}(\Psi, \mathcal{E}) &= \big\{\big\langle u, \alpha\big(\mathcal{T}_{\psi(p)}(u)\big), \mathcal{I}_{\psi(p)}(u), \beta\big(\mathcal{F}_{\psi(p)}(u)\big)\big\rangle; p \in \mathcal{E}\big\}, \\ \mathcal{I}_{\gamma,\beta}(\Psi, \mathcal{E}) &= \big\{\big\langle u, \gamma\big(\mathcal{T}_{\psi(p)}(u)\big), \mathcal{I}_{\psi(p)}(u), \beta\big(\mathcal{F}_{\psi(p)}(u)\big)\big\rangle; p \in \mathcal{E}\big\}. \end{aligned}$$

Since, $\alpha \leq \gamma$, $\alpha\big(\mathcal{T}_{\psi(p)}(u)\big) \leq \gamma\big(\mathcal{T}_{\psi(p)}(u)\big)$.
Hence, $\mathcal{I}_{\alpha,\beta}(\Psi, \mathcal{E}) \Subset \mathcal{I}_{\gamma,\beta}(\Psi, \mathcal{E})$.

(iii) Consider,

$$\begin{aligned} \mathcal{I}_{\alpha,\beta}(\Psi, \mathcal{E}) &= \big\{\big\langle u, \alpha\big(\mathcal{T}_{\psi(p)}(u)\big), \mathcal{I}_{\psi(p)}(u), \beta\big(\mathcal{F}_{\psi(p)}(u)\big)\big\rangle; p \in \mathcal{E}\big\}, \\ \mathcal{I}_{\alpha,\gamma}(\Psi, \mathcal{E}) &= \big\{\big\langle u, \alpha\big(\mathcal{T}_{\psi(p)}(u)\big), \mathcal{I}_{\psi(p)}(u), \gamma\big(\mathcal{F}_{\psi(p)}(u)\big)\big\rangle; p \in \mathcal{E}\big\}. \end{aligned}$$

Since, $\beta \leq \gamma$, $\beta\big(\mathcal{F}_{\psi(p)}(u)\big) \leq \gamma\big(\mathcal{T}_{\psi(p)}(u)\big)$.
Hence, $\mathcal{I}_{\alpha,\beta}(\Psi, \mathcal{E}) \Supset \mathcal{I}_{\alpha,\gamma}(\Psi, \mathcal{E})$.

(iv) Consider,

$$\begin{aligned} \mathcal{I}_{\alpha,\beta}(\mathcal{I}_{\gamma,\delta}(\Psi, \mathcal{E})) &= \big\{\big\langle u, \alpha\gamma\big(\mathcal{T}_{\psi(p)}(u)\big), \mathcal{I}_{\psi(p)}(u), \beta\delta\big(\mathcal{F}_{\psi(p)}(u)\big)\big\rangle; p \in \mathcal{E}\big\} \\ &= \mathcal{I}_{\alpha\gamma,\beta\delta}(\Psi, \mathcal{E}). \\ \mathcal{I}_{\gamma,\delta}(\mathcal{I}_{\alpha,\beta}(\Psi, \mathcal{E})) &= \big\{\big\langle u, \gamma\alpha\big(\mathcal{T}_{\psi(p)}(u)\big), \mathcal{I}_{\psi(p)}(u), \delta\beta\big(\mathcal{F}_{\psi(p)}(u)\big)\big\rangle; p \in \mathcal{E}\big\} \\ &= \big\{\big\langle u, \alpha\gamma\big(\mathcal{T}_{\psi(p)}(u)\big), \mathcal{I}_{\psi(p)}(u), \beta\delta\big(\mathcal{F}_{\psi(p)}(u)\big)\big\rangle; p \in \mathcal{E}\big\} \\ &= \mathcal{I}_{\alpha\gamma,\beta\delta}(\Psi, \mathcal{E})). \end{aligned}$$

Hence, $\mathcal{I}_{\alpha,\beta}(\mathcal{I}_{\gamma,\delta}(\Psi, \mathcal{E})) = \mathcal{I}_{\alpha\gamma,\beta\delta}(\Psi, \mathcal{E}) = \mathcal{I}_{\gamma,\delta}(\mathcal{I}_{\alpha,\beta}(\Psi, \mathcal{E}))$.

(v) Consider,

$$\begin{aligned} (\Psi, \mathcal{E})^c &= \big\{\big\langle u, \big(\mathcal{F}_{\psi(p)}(u)\big), (1 - \mathcal{I}_{\psi(p)}(u)), \big(\mathcal{T}_{\psi(p)}(u)\big)\big\rangle; p \in \mathcal{E}\big\}. \\ \mathcal{I}_{\alpha,\beta}(\Psi, \mathcal{E})^c &= \big\{\big\langle u, \alpha\big(\mathcal{F}_{\psi(p)}(u)\big), (1 - \mathcal{I}_{\psi(p)}(u)), \beta\big(\mathcal{T}_{\psi(p)}(u)\big)\big\rangle; p \in \mathcal{E}\big\}. \\ (\mathcal{I}_{\alpha,\beta}(\Psi, \mathcal{E})^c)^c &= \big\{\big\langle u, \beta\big(\mathcal{T}_{\psi(p)}(u)\big), \mathcal{I}_{\psi(p)}(u), \alpha\big(\mathcal{F}_{\psi(p)}(u)\big)\big\rangle; p \in \mathcal{E}\big\} \\ &= \mathcal{I}_{\beta,\alpha}(\Psi, \mathcal{E}). \end{aligned}$$

Hence, $(\mathcal{I}_{\alpha,\beta}(\Psi, \mathcal{E})^c)^c = \mathcal{I}_{\beta,\alpha}(\Psi, \mathcal{E})$. ■

6. Concentration $(\mathcal{CO})$ and dilation $(\mathcal{DO})$ operators on SINSS

In this section, we define $(\mathcal{CO})$ and $(\mathcal{DO})$ on SINSS and discuss their properties in detail. We generalize these operations and properties on SINSS by the concepts discussed in Wang and Xinwang (2013), De et al. (2000), and Anita et al. (2016).

Definition 6.1.

Let $(\Psi, \mathcal{E})$ be a SINSS over $\mathcal{V}$. Then,
(i) the $\mathcal{CO}$ of $(\Psi, \mathcal{E})$ is represented as,

$$\mathcal{C}(\Psi, \mathcal{E}) = \big\{\big\langle u, (\mathcal{T}_{\psi(p)}(u), \mathcal{I}_{\psi(p)}(u), (1-(1-\mathcal{F}_{\psi(p)}(u))^2)\big\rangle; p \in \mathcal{E}\big\};$$

(ii) the $\mathcal{DO}$ of $(\Psi, \mathcal{E})$ is represented as,

$$\mathcal{D}(\Psi, \mathcal{E}) = \big\{\big\langle u, ((\mathcal{T}_{\psi(p)}(u),)^{\frac{1}{2}}, \mathcal{I}_{\psi(p)}(u), (1-(1-\mathcal{F}_{\psi(p)}(u))^{\frac{1}{4}})\big\rangle; p \in \mathcal{E}\big\}.$$

Proposition 6.1.

Let $\mathcal{V}$ denote a non-empty set and $(\Psi, \mathcal{E})$ be a SINSS over $\mathcal{V}$.
(i) If $\pi_{\psi(p)}(u) = 0$, then $\pi_{\mathcal{C}\psi(p)}(u) = 0$ if and only if $\mathcal{T}_{\psi(p)}(u) = 0$ or $\mathcal{T}_{\psi(p)}(u) = 1$;
(ii) $\oplus[\mathcal{C}(\psi, \mathcal{E})] = \mathcal{C}[\oplus(\psi, \mathcal{E})]$ if and only if $\mathcal{T}_{\psi(p)}(u) = 0$ or $\mathcal{T}_{\psi(p)}(u) = 1$;
(iii) $\ominus[\mathcal{C}(\psi, \mathcal{E})] = \mathcal{C}[\ominus(\psi, \mathcal{E})]$ if and only if $\mathcal{F}_{\psi(p)}(u) = 0$ or $\mathcal{F}_{\psi(p)}(u) = 1$.

Proof:

(i) If $\pi_{\psi(p)}(u) = 0 \Leftrightarrow 1 - \mathcal{T}_{\psi(p)}(u) - \mathcal{F}_{\psi(p)}(u) = 0$.

$$\begin{aligned}\mathcal{C}(\Psi, \mathcal{E}) &= \big\{\big\langle u, (\mathcal{T}_{\psi(p)}(u), \mathcal{I}_{\psi(p)}(u), 1-(1-\mathcal{F}_{\psi(p)}(u))^2)\big\rangle; p \in \mathcal{E}\big\} \\ &= \big\{\big\langle u, (\mathcal{T}_{\psi(p)}(u), \mathcal{I}_{\psi(p)}(u), 1-\mathcal{T}^2_{\psi(p)}(u))\big\rangle; p \in \mathcal{E}\big\}.\end{aligned}$$

If $\pi_{\mathcal{C}\psi(p)}(u) = 0 \Leftrightarrow 1 - \mathcal{T}_{\psi(p)}(u) - (1 - \mathcal{T}^2_{\psi(p)}(u)) = 0$.
Then, $\mathcal{T}_{\psi(p)}(u)(1 - \mathcal{T}_{\psi(p)}(u)) = 0 \Leftrightarrow \mathcal{T}_{\psi(p)}(u) = 0$ or $\mathcal{T}_{\psi(p)}(u) = 1$.

(ii)

$$\oplus[\mathcal{C}(\psi, \mathcal{E})] = \big\{\big\langle u, (\mathcal{T}_{\psi(p)}(u), \mathcal{I}_{\psi(p)}(u), 1-\mathcal{T}_{\psi(p)}(u))\big\rangle; p \in \mathcal{E}\big\}. \tag{1}$$

Also,

$$\begin{aligned}\mathcal{C}[\oplus(\psi, \mathcal{E})] &= \big\{\big\langle u, (\mathcal{T}_{\psi(p)}(u), \mathcal{I}_{\psi(p)}(u), 1-(1-(1-\mathcal{T}_{\psi(p)}(u)))^2)\big\rangle; p \in \mathcal{E}\big\} \\ &= \big\{\big\langle u, (\mathcal{T}_{\psi(p)}(u), \mathcal{I}_{\psi(p)}(u), 1-\mathcal{T}^2_{\psi(p)}(u))\big\rangle; p \in \mathcal{E}\big\}.\end{aligned} \tag{2}$$

From (1) and (2), we conclude that

$$\begin{aligned}\oplus[\mathcal{C}(\psi, \mathcal{E})] = \mathcal{C}[\oplus(\psi, \mathcal{E})] &\Leftrightarrow 1 - \mathcal{T}_{\psi(p)}(u) = 1 - \mathcal{T}^2_{\psi(p)}(u). \\ &\Leftrightarrow \mathcal{T}_{\psi(p)}(u)(1-\mathcal{T}_{\psi(p)}(u)) = 0. \\ &\Leftrightarrow \mathcal{T}_{\psi(p)}(u) = 0 \text{ or } \mathcal{T}_{\psi(p)}(u) = 1.\end{aligned}$$

(iii)

$$\ominus[\mathcal{C}(\psi,\mathcal{E})]=\big\{\big\langle u,(1-(1-(1-\mathcal{F}_{\psi(p)}(u))^2),\mathcal{I}_{\psi(p)}(u),1-(1-\mathcal{F}_{\psi(p)}(u))^2)\big\rangle;p\in\mathcal{E}\big\}. \quad (3)$$

Also,

$$\mathcal{C}[\ominus(\psi,\mathcal{E})]=\big\{\big\langle u,(1-\mathcal{F}_{\psi(p)}(u),\mathcal{I}_{\psi(p)}(u),1-(1-\mathcal{F}_{\psi(p)}(u))^2)\big\rangle;p\in\mathcal{E}\big\}. \quad (4)$$

From (3) and (4), we conclude that

$$\begin{aligned}\ominus[\mathcal{C}(\psi,\mathcal{E})]=\mathcal{C}[\ominus(\psi,\mathcal{E})] &\Leftrightarrow 1-(1-(1-\mathcal{F}_{\psi(p)}(u))^2)=1-\mathcal{F}_{\psi(p)}(u).\\ &\Leftrightarrow (1-\mathcal{F}_{\psi(p)}(u))^2=1-\mathcal{F}_{\psi(p)}(u).\\ &\Leftrightarrow \mathcal{F}_{\psi(p)}(u)(1-\mathcal{F}_{\psi(p)}(u))=0.\\ &\Leftrightarrow \mathcal{F}_{\psi(p)}(u)=0 \text{ or } \mathcal{F}_{\psi(p)}(u)=1.\end{aligned}$$

■

Proposition 6.2.

Let $\mathcal{V}$ denote a non-empty set and $(\Psi,\mathcal{E})$ be a SINSS over $\mathcal{V}$.
(i) If $\pi_{\psi(p)}(u)$ = 0, then $\pi_{\mathcal{D}\psi(p)}(u)$ = 0 if and only if $\mathcal{T}_{\psi(p)}(u)$ = 0 or $\mathcal{T}_{\psi(p)}(u)$ = 1;
(ii) $\oplus[\mathcal{D}(\psi,\mathcal{E})]=\mathcal{D}[\oplus(\psi,\mathcal{E})]$ if and only if $\mathcal{T}_{\psi(p)}(u)$ = 0 or $\mathcal{T}_{\psi(p)}(u)$ = 1;
(iii) $\ominus[\mathcal{D}(\psi,\mathcal{E})]=\mathcal{D}[\ominus(\psi,\mathcal{E})]$ if and only if $\mathcal{F}_{\psi(p)}(u)$ = 0 or $\mathcal{F}_{\psi(p)}(u)$ = 1.

Proof:

If $\pi_{\psi(p)}(u)$ = 0 $\Leftrightarrow 1-\mathcal{T}_{\psi(p)}(u)-\mathcal{F}_{\psi(p)}(u)$ = 0.

$$\begin{aligned}\mathcal{D}(\Psi,\mathcal{E})=&\big\{\big\langle u,((\mathcal{T}_{\psi(p)}(u))^{\frac{1}{2}},\mathcal{I}_{\psi(p)}(u),1-(1-\mathcal{F}_{\psi(p)}(u))^{\frac{1}{4}})\big\rangle;p\in\mathcal{E}\big\}\\ =&\big\{\big\langle u,((\mathcal{T}_{\psi(p)}(u))^{\frac{1}{2}},\mathcal{I}_{\psi(p)}(u),1-(\mathcal{T}_{\psi(p)}(u))^{\frac{1}{4}})\big\rangle;p\in\mathcal{E}\big\}.\end{aligned}$$

If $\pi_{\mathcal{D}\psi(p)}(u)$ = 0 $\Leftrightarrow 1-(\mathcal{T}_{\psi(p)}(u))^{\frac{1}{2}}-1-(\mathcal{T}_{\psi(p)}(u))^{\frac{1}{4}}$ = 0 $\Leftrightarrow (\mathcal{T}_{\psi(p)}(u))^{\frac{1}{2}}=(\mathcal{T}_{\psi(p)}(u))^{\frac{1}{4}}$.
Then, $\mathcal{T}_{\psi(p)}(u)(1-\mathcal{T}_{\psi(p)}(u))$ = 0 $\Leftrightarrow \mathcal{T}_{\psi(p)}(u)=0$ or $\mathcal{T}_{\psi(p)}(u)=1$.

(ii)

$$\oplus[\mathcal{D}(\psi,\mathcal{E})]=\big\{\big\langle u,(\mathcal{T}_{\psi(p)}(u))^{\frac{1}{2}},\mathcal{I}_{\psi(p)}(u),1-(\mathcal{T}_{\psi(p)}(u))^{\frac{1}{2}})\big\rangle;p\in\mathcal{E}\big\}. \quad (5)$$

Also,

$$\begin{aligned}\mathcal{D}[\oplus(\psi,\mathcal{E})]&=\big\{\big\langle u,((\mathcal{T}_{\psi(p)}(u))^{\frac{1}{2}},\mathcal{I}_{\psi(p)}(u),1-(1-(1-\mathcal{T}_{\psi(p)}(u)))^{\frac{1}{4}})\big\rangle;p\in\mathcal{E}\big\}\\ &=\big\{\big\langle u,((\mathcal{T}_{\psi(p)}(u))^{\frac{1}{2}},\mathcal{I}_{\psi(p)}(u),1-(\mathcal{T}_{\psi(p)}(u))^{\frac{1}{4}})\big\rangle;p\in\mathcal{E}\big\}.\end{aligned} \quad (6)$$

From (5) and (6), we conclude that

$$\begin{aligned}\oplus[\mathcal{D}(\psi,\mathcal{E})]=\mathcal{D}[\oplus(\psi,\mathcal{E})] &\Leftrightarrow 1-(\mathcal{T}_{\psi(p)}(u))^{\frac{1}{2}}=1-(\mathcal{T}_{\psi(p)}(u))^{\frac{1}{4}}.\\ &\Leftrightarrow \mathcal{T}_{\psi(p)}(u)(1-\mathcal{T}_{\psi(p)}(u))=0.\\ &\Leftrightarrow \mathcal{T}_{\psi(p)}(u)=0 \text{ or } \mathcal{T}_{\psi(p)}(u)=1.\end{aligned}$$

(iii)

$$\ominus[\mathcal{D}(\psi,\mathcal{E})]=\big\{\big\langle u,(1-(1-(1-\mathcal{F}_{\psi(p)}(u)))^{\frac{1}{4}},\mathcal{I}_{\psi(p)}(u),1-(1-\mathcal{F}_{\psi(p)}(u))^{\frac{1}{4}})\big\rangle;p\in\mathcal{E}\big\}. \quad (7)$$

Also,

$$\mathcal{D}[\ominus(\psi,\mathcal{E})]=\left\{\left\langle u,(1-\mathcal{F}_{\psi(p)}(u)^{\frac{1}{2}},\mathcal{I}_{\psi(p)}(u),1-(1-\mathcal{F}_{\psi(p)}(u))^{\frac{1}{4}}\right\rangle;p\in\mathcal{E}\right\}. \tag{8}$$

From (7) and (8), we conclude that

$$\begin{aligned}\ominus[\mathcal{D}(\psi,\mathcal{E})]=\mathcal{D}[\ominus(\psi,\mathcal{E})] &\Leftrightarrow 1-(\mathcal{F}_{\psi(p)}(u))^{\frac{1}{4}}=1-(\mathcal{F}_{\psi(p)}(u))^{\frac{1}{2}}.\\ &\Leftrightarrow \mathcal{F}_{\psi(p)}(u)(1-\mathcal{F}_{\psi(p)}(u))=0.\\ &\Leftrightarrow \mathcal{F}_{\psi(p)}(u)=0 \text{ or } \mathcal{F}_{\psi(p)}(u)=1.\end{aligned}$$

∎

Proposition 6.3.

For any SINSS $(\psi,\mathcal{E})$, $\mathcal{C}(\psi,\mathcal{E})\Subset(\psi,\mathcal{E})\Subset\mathcal{D}(\psi,\mathcal{E})$.

Proof:

Consider,

$$(\psi,\mathcal{E})=\left\{\left\langle u,\mathcal{T}_{\Psi(p)}(u),\mathcal{I}_{\Psi(p)}(u),\mathcal{F}_{\Psi(p)}(u)\right\rangle;\ p\in\mathcal{E}\right\}.$$

$$\mathcal{C}(\Psi,\mathcal{E})=\left\{\left\langle u,(\mathcal{T}_{\psi(p)}(u),\mathcal{I}_{\psi(p)}(u),(1-(1-\mathcal{F}_{\psi(p)}(u))^2))\right\rangle;p\in\mathcal{E}\right\}.$$

Since, $\mathcal{F}_{\Psi(p)}(u)\in[0,1],(1-(1-\mathcal{F}_{\psi(p)}(u))^2)\geq\mathcal{F}_{\Psi(p)}(u)$.

$$\text{Hence, } \mathcal{C}(\Psi,\mathcal{E})\Subset(\psi,\mathcal{E}). \tag{9}$$

$$\mathcal{D}(\Psi,\mathcal{E})=\left\{\left\langle u,((\mathcal{T}_{\psi(p)}(u),)^{\frac{1}{2}},\mathcal{I}_{\psi(p)}(u),(1-(1-\mathcal{F}_{\psi(p)}(u))^{\frac{1}{4}}))\right\rangle;p\in\mathcal{E}\right\}.$$

Since, $\mathcal{T}_{\psi(p)}(u),\mathcal{F}_{\psi(p)}(u)\in[0,1]$,

$\mathcal{T}_{\psi(p)}(u)\leq(\mathcal{T}_{\psi(p)}(u),)^{\frac{1}{2}},\mathcal{F}_{\psi(p)}(u)\geq(1-(1-\mathcal{F}_{\psi(p)}(u))^{\frac{1}{4}})$.

$$\text{Hence, } (\psi,\mathcal{E})\Subset\mathcal{D}(\Psi,\mathcal{E}). \tag{10}$$

From (9) and (10), we get $\mathcal{C}(\psi,\mathcal{E})\Subset(\psi,\mathcal{E})\Subset\mathcal{D}(\psi,\mathcal{E})$. ∎

7. Similarity measures between SINSS

In this section, we define a new similarity measure (SM) between SINSS and explain its use with an application. We illustrate the working model with an algorithm and examples. Also, we bring out the importance of the proposed SM by comparing with existing SMs.

Definition 7.1.

Let $\mathcal{V}=\{u_1,u_2,...,u_n\}$ be the universe and $\mathcal{E}=\{p_1,p_2,....p_m\}$ be the parameters. Then the SM between SINSS $(\psi_1,\mathcal{E})$ and $(\psi_2,\mathcal{E})$ is represented as,

$$\begin{aligned}S_M\left\langle(\psi_1,\mathcal{E}),(\psi_2,\mathcal{E})\right\rangle=1-\frac{1}{2m}\sum_{i=1}^{m}\sum_{j=1}^{n}\Bigg[&\frac{|\mathcal{T}_{\psi_1(p_i)}(u_j)-\mathcal{T}_{\psi_2(p_i)}(u_j)|}{2+\mathcal{T}_{\psi_1(p_i)}(u_j)+\mathcal{T}_{\psi_2(p_i)}(u_j)}+\frac{|\mathcal{I}_{\psi_1(p_i)}(u_j)-\mathcal{I}_{\psi_2(p_i)}(u_j)|}{2+\mathcal{I}_{\psi_1(p_i)}(u_j)+\mathcal{I}_{\psi_2(p_i)}(u_j)}\\ &+\frac{|\mathcal{F}_{\psi_1(p_i)}(u_j)-\mathcal{F}_{\psi_2(p_i)}(u_j)|}{2+\mathcal{F}_{\psi_1(p_i)}(u_j)+\mathcal{F}_{\psi_2(p_i)}(u_j)}+\left|\frac{(\mathcal{T}_{\psi_1(p_i)}(u_j)-\mathcal{F}_{\psi_1(p_i)}(u_j))}{2}-\frac{(\mathcal{T}_{\psi_2(p_i)}(u_j)-\mathcal{F}_{\psi_2(p_i)}(u_j))}{2}\right|\Bigg].\end{aligned}$$

7.1. Comparison analysis with existing SMs

In this section, we analyze some existing SMs in neutrosophic environment. DMs apply SM to identify the most similar pattern between the precise (ψ) and imprecise (ψ_i), $(i = 1, 2, ..., t)$ values. When DMs apply to determine the SM, they chose (ψ_i) such that $S(\psi, \psi_i)$ is the largest among all. Table 7 shows the framework of existing measures.

Table 7. Framework of existing similarity measures.

Author details	Existing similarity measures
Ye (2014)	$\mathcal{S}_J(\psi_1, \psi_2) = \frac{1}{n}\sum_{i=1}^{n} \frac{\tilde{\mathcal{J}}}{(\mathcal{T}^2_{\psi_1}(u_i)+\mathcal{I}^2_{\psi_1}(u_i)+\mathcal{F}^2_{\psi_1}(u_i))+(\mathcal{T}^2_{\psi_2}(u_i)+\mathcal{I}^2_{\psi_2}(u_i)+\mathcal{F}^2_{\psi_2}(u_i))-\tilde{\mathcal{J}}}$, where $\tilde{\mathcal{J}} = \mathcal{T}_{\psi_1}(u_i)\mathcal{T}_{\psi_2}(u_i) + \mathcal{I}_{\psi_1}(u_i)\mathcal{I}_{\psi_2}(u_i) + \mathcal{F}_{\psi_1}(u_i)\mathcal{F}_{\psi_2}(u_i)$.
Ye (2014)	$\mathcal{S}_D(\psi_1, \psi_2) = \frac{1}{n}\sum_{i=1}^{n} \frac{2(\mathcal{T}_{\psi_1}(u_i)\mathcal{T}_{\psi_2}(u_i)+\mathcal{I}_{\psi_1}(u_i)\mathcal{I}_{\psi_2}(u_i)+\mathcal{F}_{\psi_1}(u_i)\mathcal{F}_{\psi_2}(u_i))}{(\mathcal{T}^2_{\psi_1}(u_i)+\mathcal{I}^2_{\psi_1}(u_i)+\mathcal{F}^2_{\psi_1}(u_i))+(\mathcal{T}^2_{\psi_2}(u_i)+\mathcal{I}^2_{\psi_2}(u_i)+\mathcal{F}^2_{\psi_2}(u_i))}$.
Ye (2014)	$\mathcal{S}_C(\psi_1, \psi_2) = \frac{1}{n}\sum_{i=1}^{n} \frac{\mathcal{T}_{\psi_1}(u_i)\mathcal{T}_{\psi_2}(u_i)+\mathcal{I}_{\psi_1}(u_i)\mathcal{I}_{\psi_2}(u_i)+\mathcal{F}_{\psi_1}(u_i)\mathcal{F}_{\psi_2}(u_i)}{\sqrt{\mathcal{T}^2_{\psi_1}(u_i)+\mathcal{I}^2_{\psi_1}(u_i)+\mathcal{F}^2_{\psi_1}(u_i)}\sqrt{\mathcal{T}^2_{\psi_2}(u_i)+\mathcal{I}^2_{\psi_2}(u_i)+\mathcal{F}^2_{\psi_2}(u_i)}}$.
Ye (2015a)	$\mathcal{S}_1(\psi_1, \psi_2) = \frac{1}{n}\sum_{i=1}^{n} \cos\left[\frac{\pi}{2}\max(\lvert\mathcal{T}_{\psi_1}(u_i) - \mathcal{T}_{\psi_2}(u_i)\rvert, \lvert\mathcal{I}_{\psi_1}(u_i) - \mathcal{I}_{\psi_2}(u_i)\rvert, \lvert\mathcal{F}_{\psi_1}(u_i) - \mathcal{F}_{\psi_2}(u_i)\rvert)\right]$.
Ye (2015a)	$\mathcal{S}_2(\psi_1, \psi_2) = \frac{1}{n}\sum_{i=1}^{n} \cos\left[\frac{\pi}{6}(\lvert\mathcal{T}_{\psi_1}(u_i) - \mathcal{T}_{\psi_2}(u_i)\rvert + \lvert\mathcal{I}_{\psi_1}(u_i) - \mathcal{I}_{\psi_2}(u_i)\rvert + \lvert\mathcal{F}_{\psi_1}(u_i) - \mathcal{F}_{\psi_2}(u_i)\rvert)\right]$.
Ye and Fu (2016)	$\mathcal{S}_3(\psi_1, \psi_2) = \frac{1}{n}\sum_{i=1}^{n} \tan\left[\frac{\pi}{4}\max(\lvert\mathcal{T}_{\psi_1}(u_i) - \mathcal{T}_{\psi_2}(u_i)\rvert, \lvert\mathcal{I}_{\psi_1}(u_i) - \mathcal{I}_{\psi_2}(u_i)\rvert, \lvert\mathcal{F}_{\psi_1}(u_i) - \mathcal{F}_{\psi_2}(u_i)\rvert)\right]$.
Ye and Fu (2016)	$\mathcal{S}_4(\psi_1, \psi_2) = \frac{1}{n}\sum_{i=1}^{n} \tan\left[\frac{\pi}{12}(\lvert\mathcal{T}_{\psi_1}(u_i) - \mathcal{T}_{\psi_2}(u_i)\rvert + \lvert\mathcal{I}_{\psi_1}(u_i) - \mathcal{I}_{\psi_2}(u_i)\rvert + \lvert\mathcal{F}_{\psi_1}(u_i) - \mathcal{F}_{\psi_2}(u_i)\rvert)\right]$.
Ye (2015b)	$\mathcal{S}_5(\psi_1, \psi_2) = \frac{1}{n}\sum_{i=1}^{n} \cot\left[\frac{\pi}{4} + \frac{\pi}{4}\max(\lvert\mathcal{T}_{\psi_1}(u_i) - \mathcal{T}_{\psi_2}(u_i)\rvert, \lvert\mathcal{I}_{\psi_1}(u_i) - \mathcal{I}_{\psi_2}(u_i)\rvert, \lvert\mathcal{F}_{\psi_1}(u_i) - \mathcal{F}_{\psi_2}(u_i)\rvert)\right]$.
Ye (2015b)	$\mathcal{S}_6(\psi_1, \psi_2) = \frac{1}{n}\sum_{i=1}^{n} \cot\left[\frac{\pi}{4} + \frac{\pi}{6}(\lvert\mathcal{T}_{\psi_1}(u_i) - \mathcal{T}_{\psi_2}(u_i)\rvert + \lvert\mathcal{I}_{\psi_1}(u_i) - \mathcal{I}_{\psi_2}(u_i)\rvert + \lvert\mathcal{F}_{\psi_1}(u_i) - \mathcal{F}_{\psi_2}(u_i)\rvert)\right]$.

Example 7.1.

Consider the following values, as in Table 8, which shows the superiority of the proposed SM than the existing SMs. It illustrates that the proposed SM can identify similar patterns even when the existing SMs have some limitations.

Theorem 7.1.

Let $(\psi_1, \mathcal{E})$ and $(\psi_2, \mathcal{E})$ be two SINSS over $\mathcal{V}$. Then,
(i) $0 \leq S_M((\psi_1, \mathcal{E}), (\psi_2, \mathcal{E})) \leq 1$;
(ii) $S_M((\psi_1, \mathcal{E}), (\psi_2, \mathcal{E})) = S_M((\psi_2, \mathcal{E}), (\psi_1, \mathcal{E}))$;
(iii) $S_M((\psi_1, \mathcal{E}), (\psi_2, \mathcal{E})) = 1$ if and only if $(\psi_1, \mathcal{E}) = (\psi_2, \mathcal{E})$.

Proof:

The proof is straightforward. ■

Table 8. Analysis of existing SMs.

Precise value	Imprecise values	Existing SMs	Proposed SMs
$\psi = \langle 0.20, 0.90, 0.13\rangle$	$\psi_1 = \langle 0.50, 0.77, 0.20\rangle$, $\psi_2 = \langle 0.51, 0.78, 0.18\rangle$.	$S_J(\psi,\psi_1) = S_J(\psi,\psi_2) = 0.879$, $S_D(\psi,\psi_1) = S_D(\psi,\psi_2) = 0.936$, $S_C(\psi,\psi_1) = S_C(\psi,\psi_2) = 0.936$.	$S_M(\psi,\psi_1) = 0.854, S_M(\psi,\psi_2) = 0.850$. $S_M(\psi,\psi_1) > S_M(\psi,\psi_2) \Rightarrow \psi_1$.
$\psi = \langle 0.60, 0.80, 0.10\rangle$	$\psi_1 = \langle 0.70, 0.50, 0.20\rangle$, $\psi_2 = \langle 0.60, 0.50, 0.21\rangle$.	$S_J(\psi,\psi_1) = S_J(\psi,\psi_2) = 0.884$, $S_D(\psi,\psi_1) = S_D(\psi,\psi_2) = 0.938$, $S_1(\psi,\psi_1) = S_1(\psi,\psi_2) = 0.891$, $S_3(\psi,\psi_1) = S_3(\psi,\psi_2) = 0.760$, $S_5(\psi,\psi_1) = S_5(\psi,\psi_2) = 0.613$.	$S_M(\psi,\psi_1) = 0.917, S_M(\psi,\psi_2) = 0.903$. $S_M(\psi,\psi_1) > S_M(\psi,\psi_2) \Rightarrow \psi_1$.
$\psi = \langle 0.56, 0.90, 0.13\rangle$	$\psi_1 = \langle 0.51, 0.80, 0.20\rangle$, $\psi_2 = \langle 0.65, 0.80, 0.15\rangle$.	$S_J(\psi,\psi_1) = S_J(\psi,\psi_2) = 0.983$, $S_D(\psi,\psi_1) = S_D(\psi,\psi_2) = 0.991$, $S_1(\psi,\psi_1) = S_1(\psi,\psi_2) = 0.987$, $S_3(\psi,\psi_1) = S_3(\psi,\psi_2) = 0.921$, $S_5(\psi,\psi_1) = S_5(\psi,\psi_2) = 0.854$.	$S_M(\psi,\psi_1) = 0.933, S_M(\psi,\psi_2) = 0.950$. $S_M(\psi,\psi_2) > S_M(\psi,\psi_1) \Rightarrow \psi_2$.
$\psi = \langle 0.70, 0.90, 0.15\rangle$	$\psi_1 = \langle 0.77, 0.90, 0.10\rangle$, $\psi_2 = \langle 0.77, 0.90, 0.20\rangle$.	$S_J(\psi,\psi_1) = S_J(\psi,\psi_2) = 0.994$, $S_D(\psi,\psi_1) = S_D(\psi,\psi_2) = 0.997$, $S_1(\psi,\psi_1) = S_1(\psi,\psi_2) = 0.994$, $S_2(\psi,\psi_1) = S_2(\psi,\psi_2) = 0.998$, $S_3(\psi,\psi_1) = S_3(\psi,\psi_2) = 0.945$, $S_4(\psi,\psi_1) = S_4(\psi,\psi_2) = 0.968$, $S_5(\psi,\psi_1) = S_5(\psi,\psi_2) = 0.896$, $S_6(\psi,\psi_1) = S_6(\psi,\psi_2) = 0.939$.	$S_M(\psi,\psi_1) = 0.948, S_M(\psi,\psi_2) = 0.974$. $S_M(\psi,\psi_2) > S_M(\psi,\psi_1) \Rightarrow \psi_2$.
$\psi = \langle 0.80, 0.90, 0.15\rangle$	$\psi_1 = \langle 0.65, 0.80, 0.20\rangle$, $\psi_2 = \langle 0.70, 0.75, 0.20\rangle$.	$S_1(\psi,\psi_1) = S_1(\psi,\psi_2) = 0.972$, $S_2(\psi,\psi_1) = S_2(\psi,\psi_2) = 0.987$, $S_3(\psi,\psi_1) = S_3(\psi,\psi_2) = 0.881$, $S_4(\psi,\psi_1) = S_4(\psi,\psi_2) = 0.921$, $S_5(\psi,\psi_1) = S_5(\psi,\psi_2) = 0.789$, $S_6(\psi,\psi_1) = S_6(\psi,\psi_2) = 0.854$.	$S_M(\psi,\psi_1) = 0.904, S_M(\psi,\psi_2) = 0.917$. $S_M(\psi,\psi_2) > S_M(\psi,\psi_1) \Rightarrow \psi_2$.
$\psi = \langle 0.45, 0.45, 0.30\rangle$	$\psi_1 = \langle 0.60, 0.40, 0.40\rangle$, $\psi_2 = \langle 0.50, 0.30, 0.20\rangle$	$S_1(\psi,\psi_1) = S_1(\psi,\psi_2) = 0.972$, $S_2(\psi,\psi_1) = S_2(\psi,\psi_2) = 0.987$, $S_3(\psi,\psi_1) = S_3(\psi,\psi_2) = 0.881$, $S_4(\psi,\psi_1) = S_4(\psi,\psi_2) = 0.921$, $S_5(\psi,\psi_1) = S_5(\psi,\psi_2) = 0.789$, $S_6(\psi,\psi_1) = S_6(\psi,\psi_2) = 0.854$.	$S_M(\psi,\psi_1) = 0.935, S_M(\psi,\psi_2) = 0.906$. $S_M(\psi,\psi_1) > S_M(\psi,\psi_2) \Rightarrow \psi_1$.

7.2. Diagnosing narcissistic personality disorder

In this section, we present an application on diagnosing narcissistic personality disorder (NPD) using SINSS. Let us consider the SM between two SINSS over different universes with the same set of parameters. We use this to analyze the NPD problem. We have proposed an algorithm and illustrated the technique with a suitable example.

7.3. Description of the problem

Let $\mathcal{V} = \{u_1, u_2, ..., u_n\}$ be the universe and $\mathcal{E} = \{p_1, p_2, ..., p_m\}$ be the set of parameters. Let the precise values $(\psi, \mathcal{E})$ describe the elements of the universe in SINSS form given by the psychiatrist for each stage. Let the psychiatrist define the norms to identify the levels (low or moderate or high) associated with NPD as in Table 10. Let $(\psi_i, \mathcal{E})$, $(i = 1, 2, ..., t)$ denote the imprecise values. Each $(\psi_i, \mathcal{E})$ is in SINSS form representing the alternatives based on the observations on the subject by the psychiatrist made in relation to each element of the universe and for each element of the parameter set. Now the problem is to identify the level associated with $(\psi_i, \mathcal{E})$ to the precise information $(\psi, \mathcal{E})$.

7.4. A new method to diagnose NPD

Let's assume that $(\psi, \mathcal{E})$ and $(\psi_i, \mathcal{E})$ represent the precise and imprecise values, respectively in SINSS form. By using Definition 7.1, the psychiatrist identifies the SM value associated with $(\psi_i, \mathcal{E})$ $(i = 1, 2, ..., t)$ to the precise information $(\psi, \mathcal{E})$. Now, the psychiatrist compares the obtained SM value with the norms (Table 10) and interprets on the level of NPD for each subject.

7.5. Algorithm for diagnosing NPD

In this section, we develop an algorithm for diagnosing NPD based on SM between SINSS.

Step 1: Construct the precise values $(\psi, \mathcal{E})$ and the norms based on the evaluation of psychiatrist for diagnosing NPD.
Step 2: Construct the imprecise values $(\psi_i, \mathcal{E})$, $(i = 1, 2, ..., t)$ by observing the behavior of the subjects.
Step 3: Compute the SM between $(\psi, \mathcal{E})$ and $(\psi_i, \mathcal{E})$.
Step 4: Compare the calculated SM value between $(\psi, \mathcal{E})$ and $(\psi_i, \mathcal{E})$ with the norms.
Step 5: Identify the level associated with each subject to diagnose the NPD problem.

Example 7.2.

Let $\mathcal{V} = \{s_1, s_2, s_3, s_4\}$ represent the sessions conducted by a psychiatrist. Let C_1, C_2 and C_3 represent the subjects and $\mathcal{E} = \{p_1, p_2, p_3, p_4, p_5\}$ be the parameters where p_1 = exaggerated self-importance, p_2 = excessive self admiration, p_3 = exaggerated achievements and talents, p_4 = preoccupied with fantasies and p_5 = arrogant behavior. The psychiatrist has to diagnose the NPD based on the norms associated with each subject.

Step 1. Construct the precise values $(\psi, \mathcal{E})$ as in Table 9 and the norms as in Table 10 based on the evaluation of psychiatrist for diagnosing NPD.

Table 9. Representation of precise values $(\psi, \mathcal{E})$ in SINSS form for each session.

$\mathcal{V}$	s_1	s_2	s_3	s_4
p_1	$\langle 0.8, 0.8, 0.1\rangle$	$\langle 0.7, 0.7, 0.2\rangle$	$\langle 0.6, 0.8, 0.3\rangle$	$\langle 0.7, 0.7, 0.2\rangle$
p_2	$\langle 0.7, 0.9, 0.2\rangle$	$\langle 0.8, 0.8, 0.1\rangle$	$\langle 0.5, 0.9, 0.4\rangle$	$\langle 0.5, 0.8, 0.4\rangle$
p_3	$\langle 0.6, 0.8, 0.3\rangle$	$\langle 0.5, 0.9, 0.4\rangle$	$\langle 0.4, 0.8, 0.4\rangle$	$\langle 0.6, 0.9, 0.2\rangle$
p_4	$\langle 0.5, 0.9, 0.4\rangle$	$\langle 0.6, 0.7, 0.3\rangle$	$\langle 0.8, 0.8, 0.1\rangle$	$\langle 0.4, 0.8, 0.5\rangle$
p_5	$\langle 0.4, 0.8, 0.4\rangle$	$\langle 0.5, 0.7, 0.4\rangle$	$\langle 0.5, 0.9, 0.4\rangle$	$\langle 0.6, 0.9, 0.2\rangle$

Table 10. Norms for NPD.

Range of SM values	Levels of NPD
$0.00 \leq S_M \langle(\psi, \mathcal{E}), (\psi_i, \mathcal{E})\rangle < 0.45$	Low
$0.45 \leq S_M \langle(\psi, \mathcal{E}), (\psi_i, \mathcal{E})\rangle < 0.80$	Moderate
$0.80 \leq S_M \langle(\psi, \mathcal{E}), (\psi_i, \mathcal{E})\rangle \leq 1.00$	High

Step 2. Now construct the imprecise values $(\psi_i, \mathcal{E})$, $(i = 1, 2, ..., t)$ by observing the behavior of the subjects C_1, C_2 and C_3 respectively, as in Table 11, 12 and 13.

Step 3. By using Definition 7.1, calculate the $S_M \langle(\psi, \mathcal{E}), (\psi_i, \mathcal{E})\rangle$.

Table 11. Representation of imprecise values $(\psi_1, \mathcal{E})$ for the first subject in SINSS form for each session.

$\mathcal{V}$	s_1	s_2	s_3	s_4
p_1	$\langle 0.2, 0.7, 0.2\rangle$	$\langle 0.8, 0.8, 0.1\rangle$	$\langle 0.7, 0.7, 0.2\rangle$	$\langle 0.5, 0.8, 0.4\rangle$
p_2	$\langle 0.8, 0.8, 0.1\rangle$	$\langle 0.7, 0.9, 0.2\rangle$	$\langle 0.8, 0.8, 0.1\rangle$	$\langle 0.6, 0.9, 0.2\rangle$
p_3	$\langle 0.5, 0.9, 0.4\rangle$	$\langle 0.6, 0.8, 0.3\rangle$	$\langle 0.5, 0.9, 0.4\rangle$	$\langle 0.4, 0.8, 0.5\rangle$
p_4	$\langle 0.6, 0.7, 0.3\rangle$	$\langle 0.5, 0.9, 0.4\rangle$	$\langle 0.7, 0.9, 0.2\rangle$	$\langle 0.1, 0.9, 0.6\rangle$
p_5	$\langle 0.5, 0.7, 0.4\rangle$	$\langle 0.2, 0.8, 0.1\rangle$	$\langle 0.6, 0.8, 0.3\rangle$	$\langle 0.9, 0.5, 0.1\rangle$

Table 12. Representation of imprecise values $(\psi_2, \mathcal{E})$ for the second subject in SINSS form for each session.

$\mathcal{V}$	s_1	s_2	s_3	s_4
p_1	$\langle 0.7, 0.8, 0.2\rangle$	$\langle 0.7, 0.4, 0.1\rangle$	$\langle 0.6, 0.9, 0.2\rangle$	$\langle 0.8, 0.9, 0.1\rangle$
p_2	$\langle 0.6, 0.9, 0.2\rangle$	$\langle 0.6, 0.7, 0.1\rangle$	$\langle 0.5, 0.7, 0.4\rangle$	$\langle 0.7, 0.9, 0.2\rangle$
p_3	$\langle 0.4, 0.7, 0.1\rangle$	$\langle 0.3, 0.8, 0.1\rangle$	$\langle 0.4, 0.2, 0.1\rangle$	$\langle 0.6, 0.8, 0.2\rangle$
p_4	$\langle 0.3, 0.8, 0.1\rangle$	$\langle 0.6, 0.8, 0.2\rangle$	$\langle 0.7, 0.8, 0.2\rangle$	$\langle 0.7, 0.7, 0.2\rangle$
p_5	$\langle 0.2, 0.8, 0.3\rangle$	$\langle 0.4, 0.9, 0.2\rangle$	$\langle 0.5, 0.8, 0.4\rangle$	$\langle 0.8, 0.8, 0.1\rangle$

Table 13. Representation of imprecise values $(\psi_3, \mathcal{E})$ for the third subject in SINSS form for each session.

$\mathcal{V}$	s_1	s_2	s_3	s_4
p_1	$\langle 0.8, 0.8, 0.1\rangle$	$\langle 0.7, 0.7, 0.1\rangle$	$\langle 0.5, 0.8, 0.2\rangle$	$\langle 0.7, 0.6, 0.2\rangle$
p_2	$\langle 0.5, 0.7, 0.2\rangle$	$\langle 0.7, 0.8, 0.1\rangle$	$\langle 0.5, 0.7, 0.4\rangle$	$\langle 0.4, 0.8, 0.3\rangle$
p_3	$\langle 0.6, 0.8, 0.3\rangle$	$\langle 0.4, 0.9, 0.1\rangle$	$\langle 0.4, 0.8, 0.1\rangle$	$\langle 0.6, 0.8, 0.2\rangle$
p_4	$\langle 0.3, 0.8, 0.1\rangle$	$\langle 0.6, 0.7, 0.2\rangle$	$\langle 0.7, 0.8, 0.1\rangle$	$\langle 0.4, 0.8, 0.4\rangle$
p_5	$\langle 0.4, 0.8, 0.4\rangle$	$\langle 0.5, 0.7, 0.4\rangle$	$\langle 0.5, 0.8, 0.4\rangle$	$\langle 0.6, 0.8, 0.1\rangle$

The values are as below:
$S_M \langle(\psi, \mathcal{E}), (\psi_1, \mathcal{E})\rangle = 0.438$, $S_M \langle(\psi, \mathcal{E}), (\psi_2, \mathcal{E})\rangle = 0.587$, $S_M \langle(\psi, \mathcal{E}), (\psi_3, \mathcal{E})\rangle = 0.815$.

Step 4. Now compare the calculated values of $S_M \langle(\psi, \mathcal{E}), (\psi_i, \mathcal{E})\rangle$ with Table 10.

The level of NPD for the first subject shows low, for the second average and the third high.

Step 5. We can conclude from the above observation that the psychiatrist to start the next set of treatment sessions for the subjects C_2 and C_3 to lower the level of NPD.

8. Conclusion

In this manuscript, we outline the notions of SINS, SINSS, and establish some of their properties. The outcome of this study is to overcome the recited limitations confronted by the experts while handling the values of truth, indeterminacy, and falsity in a restricted environment. Also, we propose a better SM to overcome the existing drawbacks in the neutrosophic environment. We discuss a comparative study between the proposed SM and existing SMs to show the reliability and validity of the diagnosis method. In today's complicated psychological disorder behaviors, SM plays a significant role in diagnosing the same. So, we propose a diagnosing method based on the SM for diagnosing NPD with SINSSs. In this method, we predict the psychological behavior of the subjects represented in the SINSS form. In the future, we can apply the proposed concept with hypersoft set for diagnosing psychological disorders.

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