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Abstract

In this article, in the form of the heat conduction equation with memory-dependent-derivative (MDD), a new model in magneto-thermoelasticity was developed with modified Ohm's law. To obtain the solutions, normal mode analysis is used. The obtained solution is then exposed to time-dependent thermal shock and stress-free boundary conditions. The effect of the modified Ohm's law coefficient, time-delay, and different kernel functions under the magnetic field effect on different quantities are evaluated and observed graphically on all field variables.

Keywords: Generalized thermoelasticity; Time-delay; Magneto-thermoelasticity; Memory-dependent derivative; Modified Ohm's law; Kernel function

MSC 2010 No.: 74F05, 74F15

1. Introduction

The interaction of temperature, strain, and magnetic field deals with magneto-thermoelasticity. The concept for generalized thermoelasticity was introduced by Shulman and Lord (1967) known as the

"LS model." The elastic term is coupled with temperature in this model and introduced relaxation time for the finite speed of propagation of the heatwave. Several generalizations to thermoelasticity theory can be found in literature such as Green and Naghdi (1993), Chandrasekharaiah (1986), Hetnarski (1986), and Green and Lindsay (1972). Lata and Kaur (2019) deal with the study of the transversely isotropic thermoelastic beam in the context of Green-Naghdi's theory of thermoelasticity. The Laplace Transform technique has been used to find expressions for displacement components, lateral thermal moment, deflection, and axial stress. Kumar and Devi (2017) study the thermoelastic beam in modified couple stress theory. It uses the Euler-Bernoulli beam theory to model the vibrations in a homogeneous isotropic thin beam. The Laplace transform technique is applied to solve the system of equations.

The first fractional order generalized thermoelasticity method was formulated by Povstenko (2004), which interpolates classical thermoelasticity, and generalized thermoelasticity by "Green-Naghdi (GN)." Some researchers (Youssef (2010), El-Sayed and Sherief (2010), Ezzat et al. (2011)) developed different problems of fractional thermoelasticity based on the 'LS model'. Dual and three-phase-lag (DPL, TPL) problems of thermoelasticity were discussed by Ezzat et al. (2012) and El-Karamany and Ezzat (2011) using fractional calculus. Yu et al. (2013a), by using a fractional calculus in all models "LS, GL, and DPL," introduced unified fractional order generalized electro-magneto and micro-modeling thermoelasticity. Abbas (2014) developed the model of fractional thermoelasticity theory based on the "GN model."

Some researchers (Nayfeh and Nemat-Nasser (1972) and Agarwal (1979)), based on their applications in many areas such as geophysics and medical sciences, have investigated the effect of magnetic fields on elastic media due to thermal loading. Paria (1966) discussed the effect of the magnetic field in different problems of elasticity and thermoelasticity. Ezzat and Othman (2019) studied a two-dimensional generalized magneto-thermoelastic model under a state-space approach in a perfect conductivity medium. Othman and Song (2009) discussed the two-dimensional thermal shock problem under three different theories with the effect of rotation. The effect of rotation on magneto-thermoelastic waves without energy dissipation was discussed by Othman and Song (2006). The generalized half-space problem with one relaxation time subjected to thermal shock in presence of electromagnetic field for thermoelastic plane waves was discussed by Othman (2005). Yu et al. (2013b) and Ezzat (2011) studied different problems of generalized fractional magneto-thermoelasticity.

In thermoelasticity theory, the use of the temporal and spatial fractional calculus incorporates memory dependence and nonlocality. In generalized fractional thermoelasticity, have no accepted common agreement and require further study. Recently instead of fractional calculus instant rate change depends on its previous change called memory response come into the picture due to its exhaustive force. The influence of MDD is because of its superiority due to the respective kernel function and memory scale parameter with much practical application. In fractional derivative, memory function called as kernel function but lagging in the integer order calculus. The order of the fractional derivative represents the memory index. The memory-dependent derivative is specified in a $[t-\tau, t]$ interval and integral of the kernel function and common derivative. This shows memory response better than fractional one. In MDD, the function in real-time depends on its past time. Li and Wang

(2011) proposed that a memory-dependent derivative is better than a fractional one to represent a memory effect on generalized thermoelasticity. Yu et al. (2014) developed an extension to LS generalized thermoelasticity by using MDD and introduced modified LS theory with the rate of heat flux as MDD.

In the form of heat conduction with MDD, a model of magneto-thermoelasticity theory has been constructed by Ezzat and El-Karamany (2015). Magneto-thermoelastic responses in a perfectly conducting thermoelastic solid half-space based on MDD, investigated by Atwa and Sarkar (2019). In-plane wave, the reflection of memory response in generalized magneto-thermoelasticity theory was studied by Sarkar et al. (2019a). The effect of reflection on thermoelastic waves with memory-dependent heat transfer under isothermal stress-free boundary conditions studied by Sarkar et al. (2019b). By using the memory-dependent derivative, Sarkar et al. (2018) studied the two-dimensional magneto-thermoelastic phenomenon of generalized thermoelasticity for two temperatures. Siddhartha (2019) proposed the effect of the magnetic field on the three-phase lag model of memory-dependent derivative in an orthotropic condition. Othman and Mondal (2019) prepared a two-dimensional Lord-Shulman model of the generalized thermoelastic rotating medium due to the effect of memory-dependent derivative.

The relation between current density and the temperature gradient term gives modified Ohm's law. Inclusion of temperature gradient term in modified Ohm's law improves the strength of the current at each point which is proportional to the gradient of electric potential. Due to this temperature gradient, elastic deformation occurs. So this theory is useful where extremely high temperatures are used such as nuclear devices, heat exchangers, boiler tubes, in the development of highly sensitive magnetometer, electrical power engineering, plasma physics, etc. This flow proportional to the gradient can be more readily tested by modern measurement methods. Abd Elall and Ezzat (2009) explained the effect of modified Ohm's law in the two-dimensional problem of half-space with thermal shock. Sarkar (2014) introduced two-dimensional generalized magneto-thermoelastic problems with modified Ohm's law and discussed the effect of the coefficient of modified Ohm's law on all field variables. We motivated and extend the work of Abd Elall and Ezzat (2009) and Sarkar (2014) by introducing MDD and studied its effects on field variables. This study will be useful to the researchers working on memory response problems of thermoelastic materials under the influence of temperature gradient and magnetic field.

The purpose of this work is to study the influence of modified Ohm's law on two-dimensional magneto-thermoelastic problem with the memory-dependent derivative. Lord Shulman's model of thermoelasticity is considered with the heat equation involves a memory-dependent derivative on a slipping time interval. The novelty of this work is the usage of modified Ohm's law which relates with current density and temperature gradient terms due to the memory-response. By using the memory-dependent derivative the magneto-thermoelastic problem is analyzed in presence of modified Ohm's law. The coupled effect of temperature and the current density changes the nature of displacement and stresses in presence of memory response. Normal mode analysis is used to find the solutions. Complete and comprehensive analysis, of the results with modified Ohm's law and time delay parameter due to memory response, has been presented graphically. Since the numerical analysis has been carried out due to the presence of the memory-dependent derivative, under the effects of the magnetic field upon temperature, displacement, stress components, and presented graphically.

Nomenclature

α_t	Linear thermal expansion coefficient	τ	Delay time parameter
$\mathbf{B}$	Magnetic field induction vector	θ	Thermodynamic temperature
$\mathbf{D}$	Electric displacement vector	C_E	Specific heat at constant strain
$\mathbf{E}$	Induced electric field vector	e	Cubical dilatation
$\mathbf{H}$	Magnetic Intensity vector	H_0	Componenent of initial magnetic field vector
$\mathbf{h}$	Induced magnetic field vector		
$\mathbf{J}$	Current density vector	K	Thermal Conductivity
δ_{ij}	Kronecker delta function	m_0	Magnetic Permeability
ϵ_0	Electric Permeability	T	Absolute temperature
λ, μ	Lame's Constant	t	Time
ρ	Density	T_0	Reference temperature chosen so that $\mid (T - T_0)/T_0 \mid << 1$
σ_0	Electric Conductivity		
α_t	Linear thermal expansion coefficient	τ	Delay time parameter
σ_{ij}	Stress tensor components	u, v	Components of displacement

2. Governing Equations

In the generalized thermoelasticity theory, we consider the two-dimensional problem of homogeneous, isotropic, electrically, and thermally conducting half-space with memory-response. The magnetic field operates parallel to the bounding plane with constant strength H = (0; 0; H_0) and is applied to the medium that generates induced electric field E, and induced magnetic-field h which represent in Figure 1. The governing Maxwell's equation for a homogeneous conducting solid are as follows (see Paria (1966)),

$$\mathbf{J} + \dot{\mathbf{D}} = \text{rot } \mathbf{h}, \tag{1}$$

$$-\dot{\mathbf{B}} = \text{rot } \mathbf{E}, \tag{2}$$

$$div\, \mathbf{B} = div\, \mathbf{D} = 0, \tag{3}$$

$$\mathbf{B} = m_0(\mathbf{h} + \mathbf{H_0}), \qquad \mathbf{D} = \epsilon_0\, \mathbf{E}. \tag{4}$$

We consider an equation that relates current density and temperature gradient known as modified Ohm's law (see Abd Elall and Ezzat (2009)),

$$\mathbf{J} + \pi_\mathbf{0} \nabla \mathbf{T} = \sigma_\mathbf{0}(\mathbf{E} + \mathbf{m_0} \dot{\mathbf{u}} \times \mathbf{H}), \tag{5}$$

where π_0 is the modified Ohm's law coefficient.

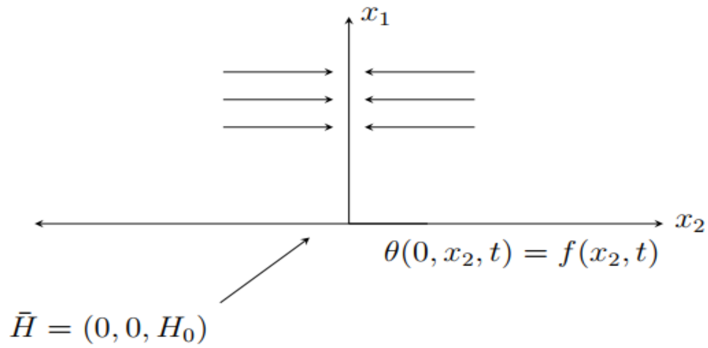

Figure 1. The geometry of the problem

The temperature deviation with the reference temperature is considered to be infinitesimal in the coupled and generalized thermoelasticity theory. The stress-strain relationship equation is given by Hooke-Duhamel-Neumann law (see Noda (1993)).

The equation of motion is:

$$\rho \ddot{u}_i = \mathbf{F_i} + \sigma_{ij,j}, \tag{6}$$

where $\mathbf{F_i} = (\mathbf{J} \times \mathbf{B})_i$ is the Lorentz force.

The heat conduction equation with MDD in absence of heat sources is:

$$K \nabla^2 T = (\rho C_E \dot{T} + \gamma T_0 \dot{e}) + \int_{t-\tau}^{t} K(t-\zeta) \left(\rho C_E \frac{\partial^2 T}{\partial \zeta^2} + \gamma T_0 \frac{\partial^2 e}{\partial \zeta^2} \right) d\zeta, \tag{7}$$

where superposed dot denotes the derivative with respect to time t.

3. Problem Formulation

We consider the magneto-thermoelastic half-space whose state depends on the space variable x_1, x_2 and time t. The components of displacement are $u_{x_1} = u(x_1, x_2, t),\ u_{x_2} = v(x_1, x_2, t), u_{x_3} = 0$. The constant magnetic field produces the medium in absence of external magnetic field. The components of strain are given by,

$$e_{x_1 x_2} = \frac{1}{2}(u_{,x_2} + v_{,x_1}), \quad e_{x_1 x_3} = e_{x_2 x_3} = e_{x_3 x_3} = 0. \tag{8}$$

From the equation of stress, the stress components are:

$$\sigma_{x_1 x_1} = (2\mu + \lambda)\, u_{,x_1} + \lambda\, v_{,x_2} - \gamma\, (T - T_0), \tag{9}$$

$$\sigma_{x_2 x_2} = (2\mu + \lambda)\, v_{,x_2} + \lambda\, u_{,x_1} - \gamma\, (T - T_0), \tag{10}$$

$$\sigma_{x_3 x_3} = \lambda\, (u_{,x_1} + v_{,x_2}) - \gamma\, (T - T_0), \tag{11}$$

$$\sigma_{x_1x_2} = \mu\,(u_{,x_2} + v_{,x_1}). \tag{12}$$

The elements of the magnetic field H are:

$$H_{x_1} = 0, \qquad H_{x_2} = 0, \qquad H_{x_3} = H_0 + h(x_1, x_2, t). \tag{13}$$

The vectors of displacement and the magnetic intensity are normal to the electric strength vector, so the components are:

$$E_{x_1} = E_1, \qquad E_{x_2} = E_2, \qquad E_{x_3} = 0. \tag{14}$$

The current density components of vector J which is parallel to the intensity of electric vector E are given by

$$J_{x_1} = J_1, \qquad J_{x_2} = J_2, \qquad J_{x_3} = 0. \tag{15}$$

From Equation (5) the components of current density after linearization become:

$$J_1 + \pi_0\,T_{,x_1} = \sigma_0(E_1 +\ m_0\,H_0\,\dot{v}), \tag{16}$$

$$J_2 + \pi_0\,T_{,x_2} = \sigma_0(E_2 -\ m_0\,H_0\,\dot{u}). \tag{17}$$

We obtain the following equations using Equations (16) and (17) in Equation (1). We get:

$$h_{,x_1} = -\sigma_0[E_2 - m_0\,H_0\,\dot{u}] - \pi_0\,T_{,x_2} - \epsilon_0\,\dot{E}_2, \tag{18}$$

$$h_{,x_2} = \sigma_0[E_1 + m_0\,H_0\,\dot{v}] - \pi_0\,T_{,x_1} + \epsilon_0\,\dot{E}_1. \tag{19}$$

From Equations (1) to (4), we get the relation as follows:

$$m_0\,\dot{h} = (E_1)_{,x_2} - (E_2)_{,x_1}. \tag{20}$$

The component of Lorentz force is obtained from Equations (16) and (17) as:

$$F_{x_1} + \pi_0\,T_{,x_2} = m_0\,H_0\,\sigma_0\,(E_2 -\ m_0\,H_0\,\dot{u}), \tag{21}$$

$$F_{x_2} + \pi_0\,T_{,x_1} = -m_0\,H_0\,\sigma_0\,(E_1 +\ m_0\,H_0\,\dot{v}), \tag{22}$$

$$F_{x_3} = 0. \tag{23}$$

Using Equations (9) to (12) and Equations (21) to (23) in Equation (6), we can write:

$$(\mu +\ \lambda)\,e_{,x_1} - \gamma\,T_{,x_1} + \mu\nabla^2 u\ + m_0\,H_0\,\sigma_0\,(E_2 -\ m_0\,H_0\,\dot{u}) = \rho\,\ddot{u} + \pi_0\,T_{,x_2}, \tag{24}$$

$$(\mu +\ \lambda)\,e_{,x_2} - \gamma\,T_{,x_2} + \mu\nabla^2 v\ - m_0\,H_0\,\sigma_0\,(E_1 +\ m_0\,H_0\,\dot{v}) = \rho\,\ddot{v} - \pi_0\,T_{,x_1}. \tag{25}$$

We introduce the following non-dimensional variables as:

$$\begin{aligned}
&(x_1^\diamond, x_2^\diamond) = c_1\,\eta\,(x_1, x_2),\ \ (t^\diamond, \tau^\diamond) = c_1^2\,\eta\,(t, \tau)\ \ ,(u^\diamond, v^\diamond) = c_1\,\eta\,(u, v),\ \ \theta^\diamond = \frac{\theta}{T_0},\\
&\sigma_{ij}^\diamond = \frac{\sigma_{ij}}{\mu},\ \ h^\diamond = \frac{\eta\,h}{m_0\,H_0\,\sigma_0}\ \ ,(E_1^\diamond, E_2^\diamond) = \frac{\eta\,(E_1, E_2)}{\sigma_0\,m_0^2\,H_0\,c_1},\ \ (J_1^\diamond, J_2^\diamond) = \frac{\eta\,(J_1, J_2)}{\sigma_0^2\,m_0^2\,H_0\,c_1},\\
&\pi_0^\diamond = \frac{m_0\,H_0\,\pi_0}{\gamma},\ \ \theta = T - T_0,\ \ \eta = \frac{\rho C_E}{K},\ \ c_1^2 = \frac{\lambda + 2\mu}{\rho}.
\end{aligned} \tag{26}$$

Using Equation (26) in Equations (24) and (25), we obtain the dimensionless equations as:

$$(\beta_2^2 - 1)e_{,x_1} + \nabla^2 u - \beta_2^2 \theta_{,x_1} + \beta_2^2 \alpha \beta_1 (\beta_1 E_2 - \dot{u}) = \beta_2^2 \ddot{u} + \pi_0 \theta_{,x_2}, \tag{27}$$

$$(\beta_2^2 - 1)e_{,x_2} + \nabla^2 v - \beta_2^2 \theta_{,x_2} - \beta_2^2 \alpha \beta_1 (\beta_1 E_1 + \dot{v}) = \beta_2^2 \ddot{v} - \pi_0 \theta_{,x_1}. \tag{28}$$

This first-order memory-dependent derivative has the form:

$$D_\tau^1 f(t) = \frac{1}{\tau} \int_{t-\tau}^{t} f^{'}(\zeta) K(t-\zeta) d\zeta, \tag{29}$$

where τ stands for memory size, $D\tau$ is a non-local operator and integration is a local operator. The kernel function can be selected to be suitable for the problem, and the function should be monotonically increased within the $[t-\tau, t]$ interval.

The non-dimensional heat conduction can be reduced as:

$$\nabla^2 \theta = (1 + \tau D_\tau)(\dot{\theta} + \epsilon_1 \dot{e}), \tag{30}$$

where ϵ_1 is the parameter for dimensionless thermoelastic coupling, $\nabla^2 = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right)$, and the kernel function is $K(t-\zeta)$. The kernel function can freely be selected as:

$$K(t-\zeta) = 1 - \frac{2s(t-\zeta)}{\tau} + \frac{r^2(t-\zeta)^2}{\tau^2} = \begin{cases} 1, & \text{if } r = s = 0, \\ 1 - \frac{(t-\zeta)}{\tau}, & \text{if } r = 0,\ s = 1/2, \\ 1 - (t-\zeta), & \text{if } r = 0,\ s = \tau/2, \\ (1 - \frac{(t-\zeta)}{\tau})^2, & \text{if } r = s = 1, \end{cases} \tag{31}$$

where r, and s are constants.

From Equation (26), Equations (18) and (19) become:

$$h_{,x_1} - \pi_0 \, \theta_{,x_2} = -\beta_1 E_2 - \epsilon_2 \, \dot{E}_2 + \dot{u}, \tag{32}$$

$$h_{,x_2} + \pi_0 \, \theta_{,x_1} = \beta_1 E_1 + \epsilon_2 \, \dot{E}_1 + \dot{v}, \tag{33}$$

$$\dot{h} = (E_1)_{,x_2} - (E_2)_{,x_1}. \tag{34}$$

The stress components in non-dimensionless form by using Equation (26) to Equations (9) to (12) are:

$$\sigma_{x_1 x_1} = (\beta_2^2 - 2)\, e + 2\, u_{,x_1} - \beta_2^2\, \theta, \tag{35}$$

$$\sigma_{x_2 x_2} = (\beta_2^2 - 2)\, e + 2\, v_{,x_2} - \beta_2^2\, \theta, \tag{36}$$

$$\sigma_{x_3 x_3} = (\beta_2^2 - 2)\, e - \beta_2^2\, \theta, \tag{37}$$

$$\sigma_{x_1 x_2} = (u_{,x_2} + v_{,x_1}), \tag{38}$$

where constants are defined (see Appendix B).

We obtain by differentiating Equation (24) with x_1, and Equation (25) with x_2 and then adding:

$$\nabla^2 e - \beta_1 \alpha \dot{e} - \ddot{e} - \nabla^2 \theta - \beta_1^2 \alpha \dot{h} = 0. \tag{39}$$

We obtain by differentiating Equation (32) with x_1, and Equation (33) to x_2 and then adding:

$$\nabla^2 h - \beta_1 \dot{h} - \epsilon_2 \ddot{h} - \dot{e} = 0. \tag{40}$$

4. Problem Solution

The solution to the problem can be preferred by using the normal mode analysis in the following form (see Sarkar (2014)),

$$[\theta, e, u, v, h, E_1, E_2, \sigma_{ij}][x_1, x_2, t] = [\theta^\star, e^\star, u^\star, v^\star, h^\star, E_1^\star, E_2^\star, \sigma_{ij}^\star](x_1)e^{(\varpi t + i a_1 x_2)}, \tag{41}$$

where the angular-frequency is ϖ, $i = \sqrt{-1}$, and the wave number a_1 is in the direction of x_2.

Using Equation (41) in Equations (30), (39), and (40), we get:

$$(D^2 - a_1^2 - d_1)\theta^\star = d_2 e^\star, \tag{42}$$

$$(D^2 - a_1^2 - \beta_1 \varpi - \epsilon_2 \varpi^2)h^\star = \varpi e^\star, \tag{43}$$

$$(D^2 - a_1^2 - \beta_1 \alpha \varpi - \varpi^2)e^\star = (D^2 - a_1^2)\theta^\star + \beta_1^2 \alpha \varpi h^\star. \tag{44}$$

where d_1 and d_2 are constants (see Appendix B).

Eliminating $\theta^\star$ and $h^\star$ from Equations (42) to (44), we get the following six order PDE:

$$(D^6 - BD^2 + AD^4 - C)e^\star(x_1) = 0. \tag{45}$$

Similarly, we get:

$$(D^6 - AD^4 + BD^2 - C)(\theta^\star, h^\star)(x_1) = 0. \tag{46}$$

The roots of Equation (45) are:

$$(D^2 - m_1^2)(D^2 - m_2^2)(D^2 - m_3^2)e^\star(x_1) = 0. \tag{47}$$

The characteristic equation has roots m_i^2 given by

$$m^6 - Am^4 + Bm^2 - C = 0. \tag{48}$$

The roots of Equation (48) are obtained (see Appendix B).

The solution of Equation (45) is given by

$$e^\star(x_1) = \sum_{n=1}^{3} R_n(a_1, \varpi)e^{-m_n x_1}. \tag{49}$$

Similarly, the solution of Equation (46) is obtained as:

$$\theta^\star(x_1) = \sum_{n=1}^{3} R_n^{'}(a_1, \varpi)e^{-m_n x_1}, \tag{50}$$

$$h^\star(x_1) = \sum_{n=1}^{3} R_n^{''}(a_1, \varpi)e^{-m_n x_1}, \tag{51}$$

where $R_n(a_1, \varpi), R_n^{'}(a_1, \varpi), R_n^{''}(a_1, \varpi)$ are some parameters depending on a_1 and ϖ.

The solution of Equations (50) to (51) can be written as:

$$\theta^\star(x_1) = \sum_{n=1}^{3} \frac{d_2}{(m_n^2 - a_1^2 - d_1)} R_n(a_1, \varpi) e^{-m_n x_1}, \tag{52}$$

$$h^\star(x_1) = \sum_{n=1}^{3} \frac{\varpi}{(m_n^2 - a_1^2 - \beta_1 \varpi - \epsilon_2 \varpi^2)} R_n(a_1, \varpi) e^{-m_n x_1}. \tag{53}$$

To get the solution for displacement $u^\star(x_1)$, we use Equation (41) in Equation (27) satisfied as:

$$u^\star(x_1) = G e^{-k x_1} - A \sum_{n=1}^{3} \frac{m_n}{(m_n^2 - k^2)} R_n(a_1, \varpi) e^{-m_n x_1}, \tag{54}$$

where depending on a_1, and ϖ, $G = G(a_1, \varpi)$ is a certain parameter.

Substituting Equation (41) into Equation (28), we obtain displacement component as:

$$v^\star(x_1) = \frac{i}{a} \left[k G e^{-k x_1} + A \sum_{n=1}^{3} \frac{(1 - m_n^2 A)}{(m_n^2 - k^2)} R_n(a_1, \varpi) e^{-m_n x_1} \right]. \tag{55}$$

Using Equations (41), (52), (53), and (55) into Equation (33), we deduce that:

$$\begin{aligned} E_1^\star(x_1) = \frac{1}{\beta_1 + \epsilon_2 \varpi} \Bigg\{ & \frac{i a_1 \varpi R_n e^{-m_n x_1}}{(m_n^2 - a_1^2 - \beta_1 \varpi - \epsilon_2 \varpi^2)} + \frac{i \varpi}{a_1} \Bigg[k G e^{-k x_1} + A \sum_{n=1}^{3} \frac{(1 - m_n^2 A)}{(m_n^2 - k^2)} \\ & R_n(a_1, \varpi) e^{-m_n x_1} \Bigg] - \frac{\pi_0 m_n d_2}{(m_n^2 - a_1^2 - d_1)} R_n(a_1, \varpi) e^{-m_n x_1} \Bigg\}. \end{aligned} \tag{56}$$

Substituting Equations (41), (52), (53), and (55) into Equation (32), we get:

$$\begin{aligned} E_2^\star(x_1) = \frac{\varpi}{\beta_1 + \epsilon_2 \varpi} \Bigg\{ & G e^{-k x_1} + m_n \left[\frac{(1 - A) m_n^2 + [A(a_1^2 + \beta_1 \varpi + \epsilon_2 \varpi^2) - k^2]}{(m_n^2 - k^2)(m_n^2 - a_1^2 - \beta_1 \varpi - \epsilon_2 \varpi^2)} \right] \\ & + \frac{i a_1 \pi_0 d_2}{\varpi (m_n^2 - a_1^2 - d_1)} \Bigg\} R_n(a_1, \varpi) e^{-m_n x_1}. \end{aligned} \tag{57}$$

Similarly using Equations (41), (49), (52), (54), and (55) into Equations (35), (36), (37), and (38), the stress components are:

$$\sigma_{x_1 x_1}^\star = -2 k G e^{-k x_1} + \sum_{n=1}^{3} \left[A \frac{2 m_n^2}{(m_n^2 - k^2)} + (\beta_2^2 - 2) - \frac{\beta_2^2 d_2}{(m_n^2 - a_1^2 - d_1)} \right] R_n(\varpi) e^{-m_n x_1}, \tag{58}$$

$$\sigma_{x_2 x_2}^\star = 2 k G e^{-k x_1} + \sum_{n=1}^{3} \left[A \frac{\beta_2^2 (m_n^2 - a_1^2 - d_1 - d_2)}{(m_n^2 - a_1^2 - d_1)} + \frac{2[1 - m_n^2 (1 + A) + k^2]}{(m_n^2 - k^2)} \right] R_n(\varpi) e^{-m_n x_1}, \tag{59}$$

$$\sigma_{x_3 x_3}^\star = \sum_{n=1}^{3} \left[\frac{\beta_2^2 (m_n^2 - a_1^2 - d_1 - d_2) - 2(m_n^2 - a_1^2 - d_1)}{(m_n^2 - a_1^2 - d_1)} \right] R_n(\varpi) e^{-m_n x_1}, \tag{60}$$

$$\sigma^{\star}_{x_1x_2} = ia_1\left[\frac{k^2+a_1^2}{a_1^2}Ge^{-kx_1} - \sum_{n=1}^{3} A\frac{(a_1^2+m_n^2)A-1}{a_1^2(m_n^2-k^2)}\right]R_n(\varpi)e^{-m_nx_1}. \tag{61}$$

Now, taking into account the free space, electric and magnetic-field intensities denoted by $h_1,\ E_{10},\ E_{20}$, respectively.

The non-dimensional field equations that these variables satisfy are given by

$$(h_1)_{,x_2} = \epsilon_2(\dot{E}_{10}), \tag{62}$$

$$(h_1)_{,x_1} = -\epsilon_2(\dot{E}_{20}). \tag{63}$$

Equation (20) becomes:

$$\dot{h}_1 = (E_{10})_{,x_2} - (E_{20})_{,x_1}, \tag{64}$$

where $h_1,\ E_{10}$, and E_{20} is decomposed as follows:

$$[h_1,\ E_{10},\ E_{20}]\ =\ [h_1^{\star},\ E_{10}^{\star},\ E_{20}^{\star}](x_1)e^{(\varpi t+ia_1x_2)}. \tag{65}$$

Using Equations (62) and (63) to Equation (65), and then solving the solution obtained for $x_1 < 0$ as:

$$h_1^{\star}\ =\ Q(a_1,\varpi)e^{(n_1x_1)}, \tag{66}$$

$$E_{10}^{\star}\ =\ \left(\frac{ia_1}{\epsilon_2\ \varpi}\right)Q(a_1,\varpi)\ e^{(n_1x_1)}, \tag{67}$$

$$E_{20}^{\star}\ =\ \left(\frac{-n}{\epsilon_2\ \varpi}\right)Q(a_1,\varpi)\ e^{(n_1x_1)}, \tag{68}$$

where $Q(a_1,\varpi)$ is the parameter depends on a_1,ϖ , and $n_1 = \sqrt{a_1^2+\epsilon_2\varpi^2}$.

5. Application

To evaluate the R_i, G, and Q parameters, the following boundary condition must be considered at $x_1 = 0$. In non-dimensional form, and half-space at $x_1 = 0$, the boundary condition is taken into account.

(1) Thermal Boundary condition: The $x_1 = 0$ surface is exposed in the form of time-dependent thermal shock,

$$\theta(0,x_2,t) = f(x_2,t). \tag{69}$$

(2) Mechaincal Boundary condition: The half-space surface is free of traction,

$$\sigma_{x_1x_1}(0,x_2,t) = \sigma_{x_2x_2}(0,x_2,t) = \sigma_{x_3x_3}(0,x_2,t) = 0. \tag{70}$$

(3) Electric Boundary condition: The vector component of the electrical-field is continuous over the half space $x_1 = 0$,

$$E_1(0,x_2,t) = E_{10}(0,x_2,t), \qquad E_2(0,x_2,t) = E_{20}(0,x_2,t). \tag{71}$$

(4) Magnetic Boundary Condition: The magnetic-field strength vector is continuous across half space for $x_1 = 0$,

$$h(0, x_2, t) = h_1(0, x_2, t). \tag{72}$$

Using Equation (41), and obtained solutions in Equations (69) to (72), we get:

$$\sum_{n=1}^{3} \frac{d_2 R_i(a_1, \varpi)}{(m_n^2 - a_1^2 - d_1)} G_n(\varpi) = f^{\star}(a_1, \varpi), \tag{73}$$

$$-2kG + \sum_{n=1}^{3} \left[\frac{2Am_n^2}{m_n^2 - k^2} + (\beta_2^2 - 2) - \frac{\beta_2^2 d_2}{(m_n^2 - a_1^2 - d_1)} \right] R_i(a_1, \varpi) = 0, \tag{74}$$

$$ia_1 \sum_{n=1}^{3} \left[\frac{m_n^2 + a_1^2}{a_1^2} G(a_1, \varpi) - \frac{[(a_1^2 + m_n^2)A - 1]}{a_1^2(m_n^2 - k^2)} m_n R_i(a_1, \varpi) \right] = 0, \tag{75}$$

$$\begin{aligned} \frac{\varpi}{(\beta_1 + \epsilon_2 \varpi)} \sum_{n=1}^{3} \Bigg\{ G(a_1, \varpi) + m_n \left[\frac{(1 - A_1)m_n^2 + [A_1(a_1^2 + \beta_1 \varpi + \epsilon_2 \varpi^2) - k^2]}{(m_n^2 - k^2)(m_n^2 - a_1^2 - \beta_1 \varpi - \epsilon_2 \varpi^2)} \right] \\ + \frac{ia_1 \pi_0 d_2}{\varpi(m_n^2 - a_1^2 - d_1)} \Bigg\} R_i(a_1, \varpi) = \frac{-n_1}{\epsilon_2 \varpi} Q(a_1, \varpi), \end{aligned} \tag{76}$$

$$\sum_{n=1}^{3} \frac{\varpi}{(m_n^2 - a_1^2 - \beta_1 \varpi - \epsilon_2 \varpi^2)} R_i(a_1, \varpi) = Q(a_1, \varpi). \tag{77}$$

6. Discussions and Numerical Results

Copper material, and $\varpi = \varpi_0 + i\xi$, are chosen for the numerical calculations, but we can take $\varpi = \varpi_0$ (real) for small values of "t." The numerical values of constant are referred as follows (see Sarkar (2014)),

$$\epsilon_1 = 0.0168,\ \ \epsilon_2 = 1.921 \times 10^{-10},\ \ \beta_1 = 0.008,\ \ \beta_2 = 2.01,\ \ \alpha = 0.249,\ \ \theta_0 = 1,\ \ \tau = 0.02.$$

Values of some constants can be chosen as:

$$L = 4, a_1 = 2,\ \ \varpi_0 = 3,\ \ \varsigma = 1.$$

The $f(x_2, t)$ function applied to the boundary surface can be taken as:

$$f(x_2, t) = \ \theta_0\ H(|L| - x_2)\ e^{-\varsigma t},$$

where θ_0 is constant, the displaced Heaviside unit step function $H(|L| - x_2)$ indicates that the heat along the x_2 axis has a $2L$ width to sustain a θ_0 temperature while it has no temperature on the rest of the surface.

By using Equation (41) to $f(x_2, t)$ we obtain:

$$f^*(a_1, \varpi) = \frac{\sqrt{2}\theta_0 \, sin(a_1 L) \, (1 + i a_1 \pi \delta(a_1))}{\sqrt{\pi} \, a_1(\varpi + \varsigma)}.$$

The numerical computations are done and the results presented in graphical form. The graphs are seen with $t = 0.1$, and $x_2 = 0.1$ for a space value and time. Kernel function can be chosen for the variations with time-delay parameter as $K(t, \zeta) = \left(1 - \frac{t-\zeta}{\tau}\right)^2$, $r = s = 1$ for all the figures. As well as suitable kernel can be consider for variations with respect to modified Ohm's law and time delay parameter as $K(t, \zeta) = 1 - (t - \zeta)/\tau$ i.e. $r = 0, s = 0.5$, and for the variation with time can be chosen as $K(t, \zeta) = 1 - (t - \zeta)$, $r = 0, s = \tau/2$ in all figures. Figure 2 illustrates the temperature distribution for the various values of the modified Ohm's law coefficient and time delay parameter. The values of the coefficient of modified Ohms' law is taken as $\pi_0 = 5, 10$, and time delay equal to $\tau = 0.02, 0.002$. Figure 3 describes a distribution of temperature with the different kernel functions. Figure 4 shows the distribution for displacement with values of time delay, and coefficient of modified Ohm's law.

The effects of various kernel functions on displacement are expressed by Figure 5. Here we consider the kernel $K(t, \zeta) = 1$, i.e., $r = s = 0$ given by dashed and dotted line, $K(t, \zeta) = 1 - (t - \zeta)/\tau$, i.e., $r = 0, s = 0.5$ represented by dashed line. Dotted line represented by $K(t, \zeta) = 1 - (t - \zeta)$, i.e., $r = 0, s = \tau/2$. Solid line shows for the case of kernel $K(t, \zeta) = \left(1 - \frac{t-\zeta}{\tau}\right)^2$, i.e., $r = s = 1$. Displacement has significant change, i.e., increases initially and then decreases slowly and formed a peak. The variation in the stress of the different time-delay parameter values that correspond to the modified Ohm's law parameter is seen in Figure 6 and 8. Figures 7 and 9 shows the stress distribution for different kernel function. Stress distribution $\sigma_{x_1 x_1}$, has maximum value in the beginning but changes slowly, i.e. tensile nature but $\sigma_{x_2 x_2}$ is compressing in nature.

Figure 10 specifies the variation of temperature with different time-delay parameter $\tau = 0.02, 0.002,$ and 0.0002. Displacement shift for different time-delay values $\tau = 0.02, 0.002,$ and 0.0002 is indicated in Figure 11. Variation of stress for several values of time delay parameter is present in Figures 12 and 13. Figure 14 is deviation of temperature for various time values $t = 0.05, 0.1,$ and 0.2. A significant difference between the time delay parameter, the modified ohm's law coefficient, and the time values are observed. In all the figures, the temperature rises and the peak developed at the various magnitude and then gradually decreases over time. The displacement variation with many time value $t = 0.05, 0.1,$ and 0.2 shows in Figure 15. The stress distribution for the various time values is indicated in Figures 16 and 17.

The three-dimension representation of the temperature, stress, magnetic field and displacement ranges from $x_1$to x_2 shows in Figures 18 to 21. Different values for the modified Ohm's law coefficient are seen in the graph. Values of the coefficient of modified Ohm's law are taken as $\pi_0 = 5$. The significant change is examined in all the field variables. All figures showed, that in an enclosed space region the field variables were obtained and disappear outside of the region. The waves are propagated with finite speed. The boundary status indicates that the response varies over time in a small area of space. Thus the main shift is noted in the use of MDD with modified Ohm's law in a double dimension model. The boundary condition shows the response in a bounded region of space varies with time. Hence a significant change is observed due to modified Ohm's law in a two-dimensional model with the memory response.

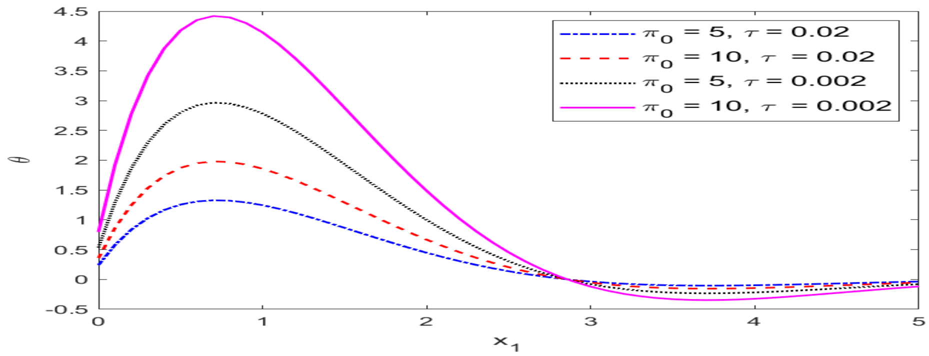

Figure 2. Temperature variation (θ) for several π_0 and τ values

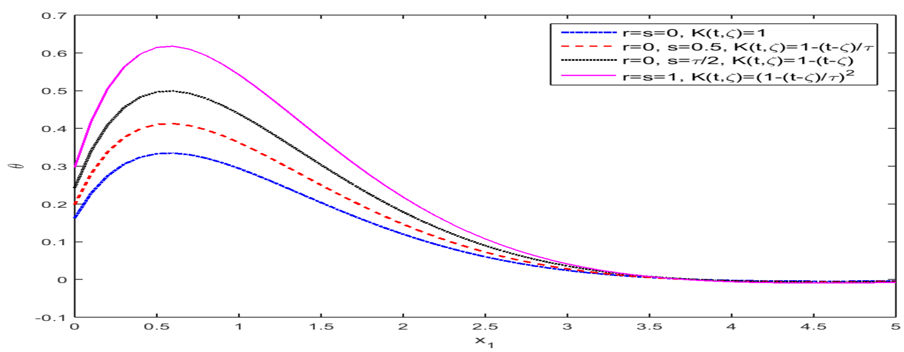

Figure 3. Temperature variation (θ) for several kernel function

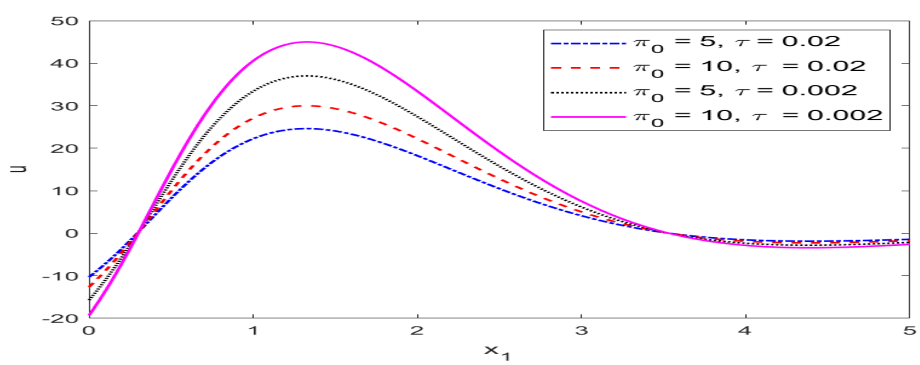

Figure 4. Displacement variation (u) for several π_0 and τ values

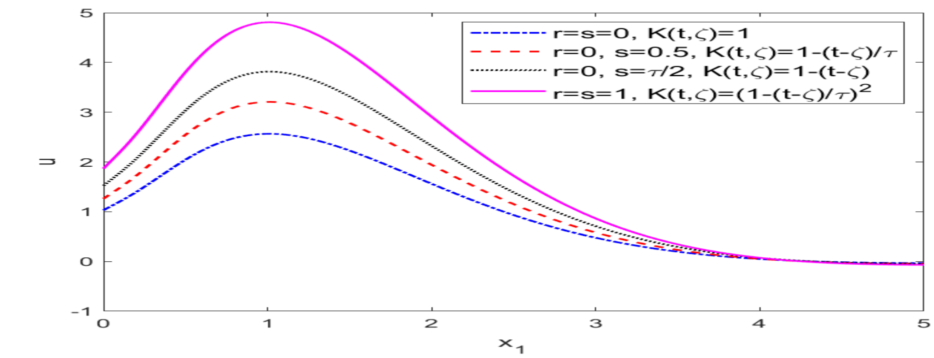

Figure 5. Displacement variation (u) for several kernel functions

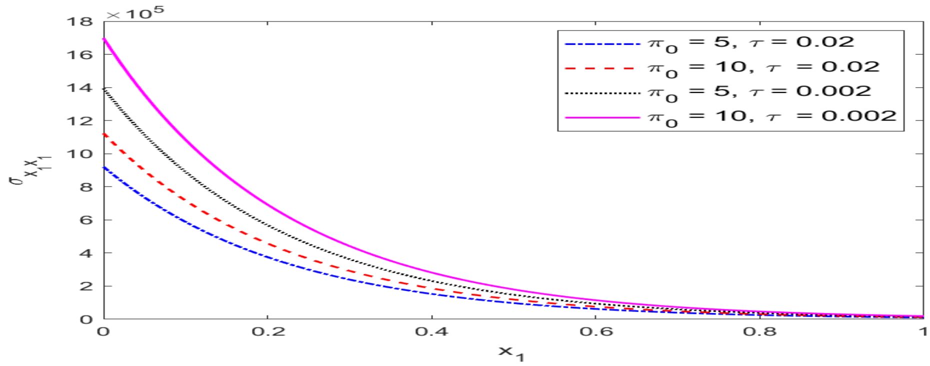

Figure 6. Stress variation ($\sigma_{x_1x_1}$) for several π_0 and τ values

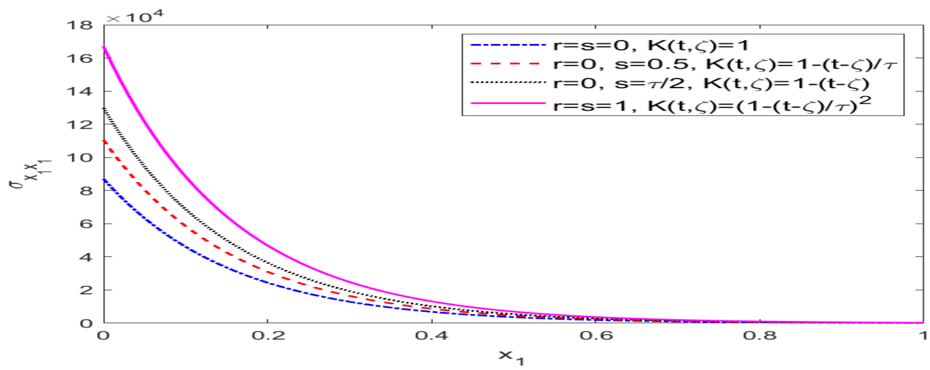

Figure 7. Stress variation ($\sigma_{x_1x_1}$) for several kernel function

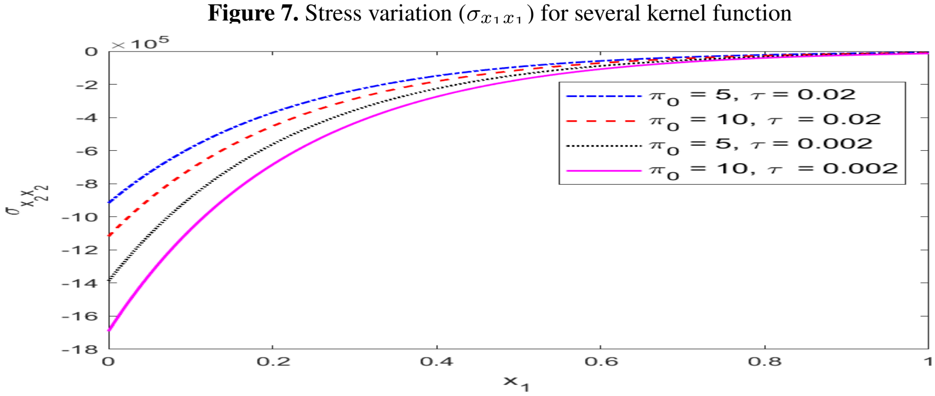

Figure 8. Stress variation ($\sigma_{x_2x_2}$) for several π_0 and τ values

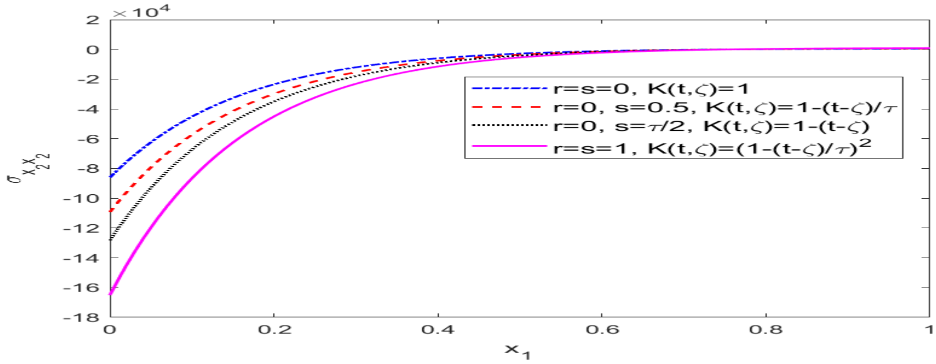

Figure 9. Variation of stress ($\sigma_{x_2x_2}$) with various kernel functions

7. Conclusion

This paper represents a two-dimensional magneto-thermoelastic problem due to the effect of modified Ohm's law with memory response. Using normal mode analysis, the problem is solved, and the effects of time delay, modified Ohm's law coefficient, time, and various kernel functions are observed. We infer from the graphical illustration that the following facts help design new material in the development of magneto-thermoelasticity theory.

(1) The kernel function and the time delay play an important role in the heat conduction equation due to the finite speed of wave propagation.

(2) With the change in the time delay parameter, the significant result of the coefficient of the modified Ohm's law is observed.

(3) The strong effect of the kernel function and time delay on all field quantities is observed, indicating heat passing through the medium.

The numerical results presented here should prove useful to improve the efficiency of a thermoelastic material. Introducing modified Ohm's law in magneto-thermoelastic problems with the memory response gives a novel contribution to this field.

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Appendix A: Graphical Results

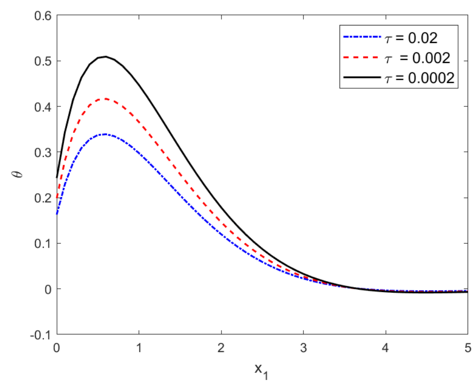

Figure 10. Temperature variation (θ) for several τ values

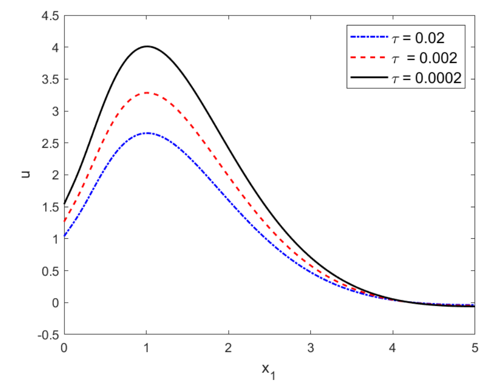

Figure 11. Displacement (u) variation for several τ values

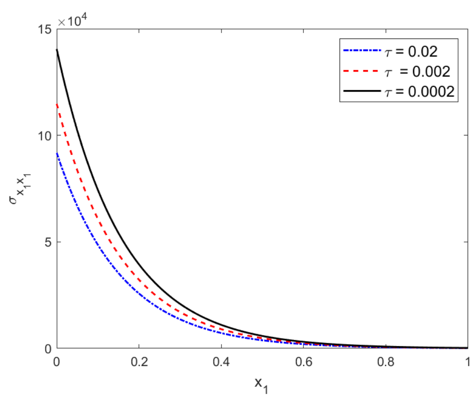

Figure 12. Stress variation ($\sigma_{x_1x_1}$) for several τ values

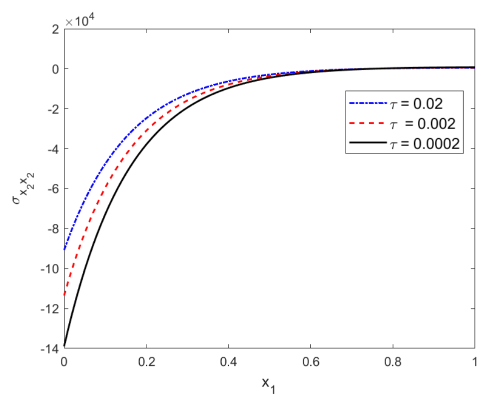

Figure 13. Stress variation ($\sigma_{x_2x_2}$) for several τ values

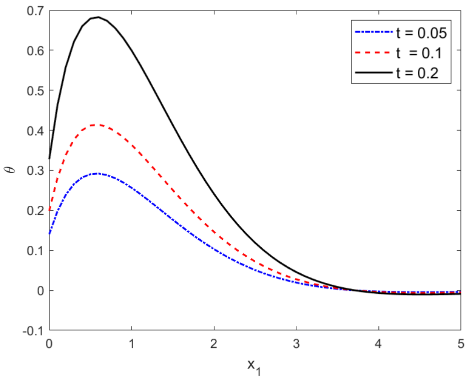

Figure 14. Temperature variation (θ) for several t values when $k(t,\zeta) = 1 - (t - \zeta)$

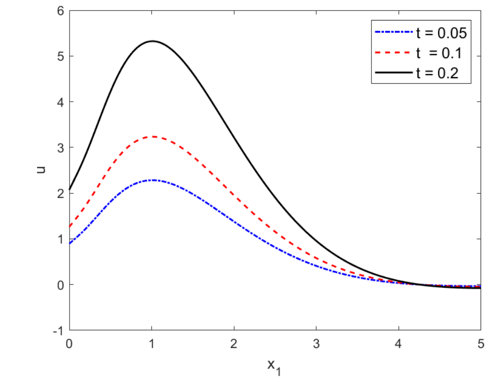

Figure 15. Displacement (u) variation for several t values when $k(t,\zeta) = 1 - (t - \zeta)$

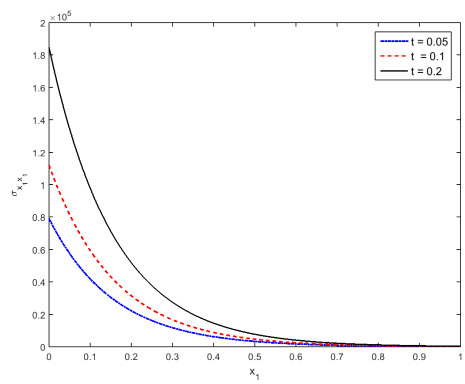

Figure 16. Stress variation $(\sigma_{x_1x_1})$ for several t values as $k(t,\zeta) = 1 - (t - \zeta)$

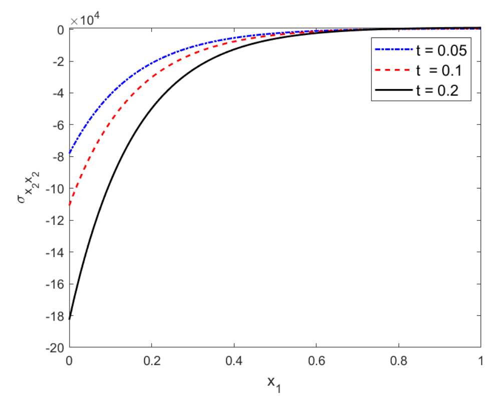

Figure 17. Stress variation $(\sigma_{x_2x_2})$ for several t values as $k(t,\zeta) = 1 - (t - \zeta)$

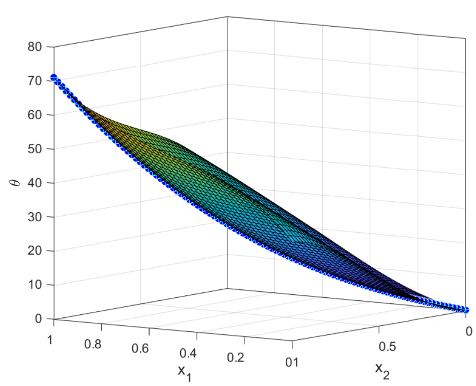

Figure 18. Three-dimensional temperature variation (θ) with modified Ohm's law coefficients (π_0)

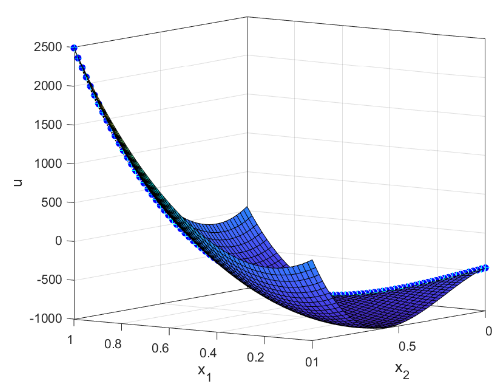

Figure 19. Three-dimensional displacement variants (u) with modified Ohm's law coefficients (π_0)

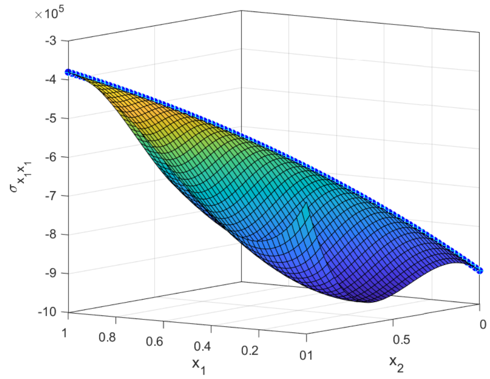

Figure 20. Three-dimensional stress combinations of $(\sigma_{x_1x_1})$ with modified Ohm's Law coefficients of (π_0)

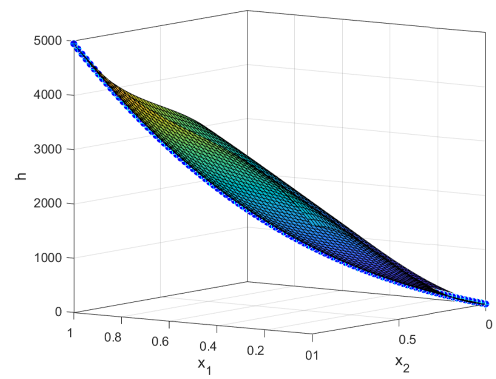

Figure 21. Three-dimensional variants of the (h) magnetic field with modified Ohm's law coefficients (π_0)

Appendix B: Equations

$$c^2 = \frac{1}{\epsilon_0 m_0}, \quad \epsilon_1 = \frac{\gamma^2 T_0}{\rho C_E(\lambda + 2\mu)}, \quad \gamma = (3\lambda + 2\mu)\alpha_t, \quad \alpha = \frac{\mu\, H_0^2}{\lambda + 2\mu}, \quad \beta_1 = \frac{\sigma_0\, m_0}{\eta},$$

$$A = (3a_1^2 - d_1 + d_2) + \beta_1\varpi(1 + \alpha) + (1 + \epsilon_2)\varpi^2,$$

$$B = (\beta_1\alpha\epsilon_2 + \beta_1)\varpi^3 + \epsilon_2\varpi^4 + ((\epsilon_2 + 1)d_1 - \epsilon_2 d_2)\varpi^2 + \beta_1[(1 + \alpha)d_1 - d_2]\varpi$$
$$+ 2(d_1 a_1^2 - a_1^4 + a_1^2 d_2),$$

$$C = [\epsilon_2(a_1^2 - d_1)\varpi^4 + \beta_1(a_1^2 - d_1)(\alpha\epsilon_2 + 1)\varpi^3 + (a_1^4\epsilon_2 + \beta_1^2\alpha a_1^2 - a_1^2\epsilon_2 d_1 - 2\beta_1^2\alpha d_1$$
$$+ a_1^4 - a_1^2 d_1 + \epsilon_2 a_1^2 - \beta_1^2\alpha a_1^2)\varpi^2 + (a_1^4\beta_1(1 + \alpha) - \beta_1 d_1(a_1^2 + \alpha) + \beta_1 a_1^2)\varpi + a_1^4],$$

$$d_1 = \varpi[1 + M(\tau, \varpi)], \;\; d_2 = \epsilon_1\varpi[1 + M(\tau, \varpi)], \;\; \beta_2^2 = \frac{\lambda + 2\mu}{\mu}, \;\; \epsilon_2 = \frac{c_1^2}{c^2},$$

$$M(\tau, \varpi) = \left\{\frac{(\tau^2\varpi^2 - 2s\tau\varpi + 2r^2) + e^{-\varpi\tau}[\tau^2\varpi^2(2s - 1 - r^2) + 2\tau\varpi(s - r^2) - 2r^2]}{\tau^2\varpi^3}\right\},$$

$$m_1^2 = \frac{1}{3}[A + 2\, N_1\, \sin(\Lambda)], \;\; m_2^2 = \frac{1}{3}\left[A - 2\, N_1\, \sin\left(\Lambda + \frac{\pi}{3}\right)\right],$$

$$m_3^2 = \frac{1}{3}\left[A + 2\, N_1\, \cos\left(\Lambda + \frac{\pi}{6}\right)\right],$$

$$N_1 = \sqrt{A^2 - 3\, B}, \;\; N_2 = \frac{\sqrt{3}}{9}\,(9AB - 2\, A^3 - 27\, C),$$

$$N_3 = \sqrt{A^2B^2 + 18ABC - 4A^3C - 4B^3 - 27C^2}, \;\; \Lambda = \frac{1}{3}\, \tan^{-1}\left(\frac{N_2}{N_3}\right),$$

$$k^2 = a_1^2 + \varpi\beta_2^2\alpha\beta_1 + \beta_2^2\varpi^2 - \frac{\varpi\beta_2^2\alpha\beta_1^2}{(\beta_1 + \epsilon_2\varpi)},$$

$$
\begin{aligned}
&A = 1 - \beta_2^2 + \sum_{n=1}^{3} \left\{ \frac{d_2[\beta_2^2 m_n(\beta_1 + \epsilon_1 \varpi)]}{m_n(m_n^2 - a_1^2 - d_1)} - \frac{d_2[\pi_0 i a_1(\beta_1 + \epsilon_1 \varpi) + \pi_0 i a_1 \beta_2^2 \alpha \beta_1^2]}{m_n(m_n^2 - a_1^2 - d_1)} \right. \\
&\left. + \frac{d_2^2 \beta_2^2 \alpha \beta_1^2 \varpi m_n(\beta_1 + \epsilon_2 \varpi)}{m_n(m_n^2 - a_1^2 - \beta_1 \varpi - \epsilon_2 \varpi^2)} \right\}, \\
&\Delta_1 = f^\star(a_1, \varpi)\,[Y_3\, Z_2 - Y_2\, Z_3], \quad \Delta_2 = f^\star(a_1, \varpi)\,[Y_1\, Z_3 - Y_3\, Z_1], \\
&\Delta_3 = f^\star(a_1, \varpi)\,[Y_2\, Z_1 - Y_1\, Z_2], \quad \Delta = X_1\,[-Y_2\, Z_3 + Y_3\, Z_2] \\
&\quad + X_2\,[-Y_3\, Z_1 + Y_1\, Z_3] + X_3\,[-Y_1\, Z_2 + Y_2\, Z_1], \\
&R_i(a_1, \varpi) = \sum_{n=1}^{3} \frac{\Delta_n}{\Delta}, \quad G(a_1, \varpi) = \frac{1}{k^2 + a_1^2} \sum_{n=1}^{3} \frac{[(a_1^2 + m_n^2)A_1 - 1]}{(m_n^2 - k^2)} m_n R_i(a_1, \varpi), \\
&X_i = \sum_{n=1}^{3} \frac{d_2}{(m_n^2 - a_1^2 - d_1)}, \quad Z_i = V_i + N_i, \\
&Y_i = \sum_{n=1}^{3} \left[\frac{[(a_1^2 + m_n^2)A_1 - 1]}{(a_1^2 + k^2)(m_n^2 - k^2)} m_n + m_n \left[\frac{(1 - A_1)m_n^2 + [A_1(a_1^2 + \beta_1 \varpi + \epsilon_2 \varpi^2) - k^2]}{(m_n^2 - k^2)(m_n^2 - a_1^2 - \beta_1 \varpi - \epsilon_2 \varpi^2)} \right] \right. \\
&\left. \quad + \frac{i a_1 \pi_0 d_2}{\varpi(m_n^2 - a_1^2 - d_1)} + \frac{n_1(\beta_1 + \epsilon_2 \varpi)}{\epsilon_2 \varpi(m_n^2 - a_1^2 - \beta_1 \varpi - \epsilon_2 \varpi^2)} \right], \\
&V_i = \sum_{n=1}^{3} \left[\frac{-2k[(a_1^2 + m_n^2)A_1 - 1]}{(a_1^2 + k^2)(m_n^2 - k^2)} m_n + 2A_1 m_n^2 \right], \\
&N_i = \sum_{n=1}^{3} \left[\frac{[\beta_2^2 - 2](m_n^2 - a_1^2 + k^2) - \beta_2^2 d_2}{(m_n^2 - a_1^2 + k^2)} \right].
\end{aligned}
$$