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Bessel-Maitland Function of Several Variables and its Properties Related to Integral Transforms and Fractional Calculus

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Abstract

In the recent years, various generalizations of Bessel function were introduced and its various properties were investigated by many authors. Bessel-Maitland function is one of the generalizations of Bessel function. The objective of this paper is to establish a new generalization of Bessel-Maitland function using the extension of beta function involving Appell series and Lauricella functions. Some of its properties including recurrence relation, integral representation and differentiation formula are investigated. Moreover, some properties of Riemann-Liouville fractional operator associated with the new generalization of Bessel-Maitland function are also discussed.

Keywords: Generalized Bessel-Maitland function; Extended beta function; Appell series; Lauricella function; Integral transforms; Fractional calculus

MSC 2010 No.: 26A33, 44A10, 44A20 33C10

1. Introduction

Due to the application of Bessel function in a wide range of problems of cylindrical coordinate system, wave propagation, heat conduction in cylindrical object, static potential and various other problems in diverse areas of mathematical physics, it is considered as one of the most important function in special functions. The theory of Bessel function is associated with the theory of various types of differential equations.

Recently, various generalizations of different special functions such as Beta, Gamma, Gauss hypergeometric, confluent hypergeometric, Mittag-Leffler, Bessel and Bessel-Maitland function have been introduced and various properties have been investigated by many researchers (see, e.g., Goswami et al. (2018); Shadab et al. (2018); Chandola et al. (2020b); Agarwal et al. (2020); Ali et al. (2020); Choi et al. (2014); Purohit et al. (2012); Agarwal et al. (2021); Mittal et al. (2016); Mondal and Nisar (2014); Suthar et al. (2017a); Suthar et al. (2016); Suthar et al. (2020a); Suthar et al. (2020b); Suthar et al. (2017b); Suthar et al. (2017c); Tilahun et al. (2020)). The Bessel-Maitland function (Marichev (1983)) is a well known generalization of Bessel function. Pathak (1966) gave a generalization of Bessel-Maitland function using Pochhammer symbol (Rainville (1971)).

Fractional calculus is of immense importance due to its wide applications in science and engineering to solve integral equations, ordinary differential equations, and partial differential equations. For the left and right sided fractional integral and differential operators of Riemann-Liouville type and generalized form of the fractional differential operator (see Kilbas et al. (2006)). The fractional integral and derivative of power function are given by Mathai (2008) and Srivastava and Tomovski (2009). Various types of inequalities and their generalizations have been established using fractional calculus operators (see, e.g., Chandola et al. (2020a); Qi.F et al. (2018); Rahman et al. (2019b); Rahman et al. (2019a); Ravichandran et al. (2020)). The inequalities and their application are of utmost importance in differential equations and applied mathematics.

This paper is organized as follows. In Section 2, we define the new generalization of Bessel-Maitland function using the extension of beta function defined by Chandola et al. (2020b) involving Appell series and Lauricella functions (Exton (1976)). We show that the new generalization reduces to the original Bessel-Maitland function under certain particular conditions. In Section 3, we discuss the recurrence relations for the new generalization of Bessel-Maitland function. Section 4 deals with the integral representation of the new generalization with the help of the generalization of Bessel-Maitland function given by Pathak (1966) and trigonometric and algebraic functions. In Section 5, differentiation formula for the new generalization are established. Section 6 contains certain properties of Riemann-Liouville fractional operator associated with the new generalization of Bessel-Maitland function.

The novelty of the paper is that the authors have established a generalization of Bessel-Maitland function that involves extended beta function with Appell series and Lauricella function. This is a huge variation from the original Bessel-Maitland function, which is clear from Remark 2.1 and 2.2. The manifold generality of the function, its properties and connection with fractional calculus have been clearly discussed in the paper. These generalizations have various applications in the representation of the solutions of various types of engineering and mathematical physics problems.

2. Generalized Bessel-Maitland Function

Here we introduce an interesting generalization of Bessel-Maitland function using extended beta function involving Appell series and Lauricella function.

Definition 2.1.

The new generalization of Bessel-Maitland function using extended beta function involving Appell series $F_1(.)$ is given by

$$\mathfrak{J}_{\tau,\upsilon,s,\omega}^{F_1,\Psi_1,\Psi_2}(z)=\sum_{n=0}^{\infty}\frac{B_{p,q}^{F_1}(\Psi_1+n\omega,s-\Psi_2)}{B(\Psi_1,s-\Psi_2)}\frac{(s)_{n\omega}}{\Gamma(\tau n+\upsilon+1)}\frac{(-z)^n}{n!}, \tag{1}$$

where $\tau,\upsilon,s\in\mathbb{C}$, $\Re(\Psi_1),\Re(\Psi_2),\Re(\tau),\Re(s)>0$, $\Re(\upsilon)\geq -1$ and $\omega=0,1,2,\ldots$.

Remark 2.1.

When $\omega,q=0$ in $B_{p,q}^{F_1}(\Psi_1+n\omega,s-\Psi_2)$ given by Chandola et al. (2020b), the beta function involving Appell series reduces to the beta function containing Gauss hypergeometric function ${}_2F_1(a_1,a_2;a_3;\frac{p}{t^r})$.

Subsequently, for $p=1,r=0$ and $a_1,\ a_2,\ a_3=1$, the function $B_{p,q}^{F_1}(\Psi_1+n\omega,s-\Psi_2)$ reduces to the classical beta function $B(\Psi_1,s-\Psi_2)$ (Rainville (1971)).

Hence, the new generalization of Bessel-Maitland function reduces to the Bessel-Maitland function (Marichev (1983)) $J_\upsilon^\tau(z)$.

Similarly, we can define the generalization of Bessel-Maitland function using extended beta function involving Appell series $F_2(.)$, $F_3(.)$ and $F_4(.)$.

Definition 2.2.

The new generalization of Bessel-Maitland function using extended beta function involving Lauricella function $F_A^{(m)}(.)$ is given by

$$\mathfrak{J}_{\tau,\upsilon,s,\omega}^{F_A^{(m)},\Psi_1,\Psi_2}(z)=\sum_{n=0}^{\infty}\frac{B_{p_1,\ldots p_m}^{F_A^{(m)}}(\Psi_1+n\omega,s-\Psi_2)}{B(\Psi_1,s-\Psi_2)}\frac{(s)_{n\omega}}{\Gamma(\tau n+\upsilon+1)}\frac{(-z)^n}{n!}, \tag{2}$$

where $\tau,\upsilon,s\in\mathbb{C}$, $\Re(\Psi_1),\Re(\Psi_2),\Re(\tau),\Re(s)>0$, $\Re(\upsilon)\geq -1$ and $\omega=0,1,2,\ldots$.

Remark 2.2.

For $m=1$, $\omega=0$ in $B_{p_1,\ldots,p_m}^{F_A^{(m)}}(\Psi_1+n\omega,s-\Psi_2)$ given by Chandola et al. (2020b), the beta function involving Lauricella function reduces to the beta function involving Gauss hypergeometric function ${}_2F_1(a_1, a_1'; a_1''; \frac{p_1}{t^r})$, and subsequently, if $p_1 = 1, r = 0$ and $a_1, a_1^{'}, a_1^{''} = 1$, the function $B^{F_A^{(m)}}_{p_1,\ldots,p_m}(\Psi_1 + n\omega, s - \Psi_2)$ reduces to the classical Beta function $B(\Psi_1, s - \Psi_2)$.

Hence, the new generalization of Bessel-Maitland function (2) reduces to the Bessel-Maitland function (Marichev (1983)) $J_\upsilon^\tau(z)$.

Similarly, we can define the generalization of Bessel-Maitland function using extended beta function involving Lauricella function $F_B^{(m)}(.)$, $F_C^{(m)}(.)$ and $F_D^{(m)}(.)$.

Remark 2.3.

(1) The Euler-Beta transform, which when applied on the function $\mathfrak{J}^{F_1,\Psi_1,\Psi_2}_{\tau,\upsilon,s,\omega}(\sigma z^\tau)$ from $\upsilon + 1$ to 1, gives the result in terms of the new generalization of Bessel-Maitland function involving Appell series with parameters $\upsilon + 1$ instead of υ and σ in place of σz^τ.

(2) The Laplace transform, which when applied on the function $z^{\upsilon-1}\mathfrak{J}^{F_1,\Psi_1,\Psi_2}_{\tau,\upsilon,s,\omega}(\sigma z^\tau)$, gives the result in terms of classical beta function, extended beta function involving Appell series $F_1(.)$ and Wright hypergeometric function.

3. Recurrence Relation

In this section, we derive the recurrence relation of the newly defined generalization of Bessel-Maitland function.

Theorem 3.1.

Let $\tau, \upsilon, s \in \mathbb{C}$, $\Re(\Psi_1), \Re(\Psi_2), \Re(\tau), \Re(s) > 0$, $\Re(\upsilon) \geq -1$ and $\omega = 0, 1, 2, \ldots$, then

$$\mathfrak{J}^{F_1,\Psi_1,\Psi_2}_{\tau,\upsilon,s,\omega}(z) = (\upsilon + 1)\mathfrak{J}^{F_1,\Psi_1,\Psi_2}_{\tau,\upsilon+1,s,\omega}(z) + \tau z \frac{d}{dz}\mathfrak{J}^{F_1,\Psi_1,\Psi_2}_{\tau,\upsilon+1,s,\omega}(z). \tag{3}$$

Proof:

Using Definition 2.1 on the right hand side of equation (3), we get

$$\begin{aligned}
&(\upsilon + 1)\mathfrak{J}^{F_1,\Psi_1,\Psi_2}_{\tau,\upsilon+1,s,\omega}(z) + \tau z \frac{d}{dz}\mathfrak{J}^{F_1,\Psi_1,\Psi_2}_{\tau,\upsilon+1,s,\omega}(z) \\
=&(\upsilon + 1)\left\{\sum_{n=0}^{\infty} \frac{B^{F_1}_{p,q}(\Psi_1 + n\omega, s - \Psi_2)}{B(\Psi_1, s - \Psi_2)} \frac{(s)_{n\omega}}{\Gamma(\tau n + \upsilon + 2)} \frac{(-z)^n}{n!}\right\} \\
&+ \tau z \frac{d}{dz}\left\{\sum_{n=0}^{\infty} \frac{B^{F_1}_{p,q}(\Psi_1 + n\omega, s - \Psi_2)}{B(\Psi_1, s - \Psi_2)} \frac{(s)_{n\omega}}{\Gamma(\tau n + \upsilon + 2)} \frac{(-z)^n}{n!}\right\}
\end{aligned}$$

$$
\begin{aligned}
=&(\upsilon+1)\left\{\sum_{n=0}^{\infty}\frac{B_{p,q}^{F_1}(\Psi_1+n\omega,s-\Psi_2)}{B(\Psi_1,s-\Psi_2)}\frac{(s)_{n\omega}}{\Gamma(\tau n+\upsilon+2)}\frac{(-z)^n}{n!}\right\}\\
&+\tau z\left\{\sum_{n=1}^{\infty}\frac{B_{p,q}^{F_1}(\Psi_1+n\omega,s-\Psi_2)}{B(\Psi_1,s-\Psi_2)}\frac{(s)_{n\omega}}{\Gamma(\tau n+\upsilon+2)}\frac{(-1)^n(z)^{n-1}}{(n-1)!}\right\}\\
=&\sum_{n=0}^{\infty}\frac{B_{p,q}^{F_1}(\Psi_1+n\omega,s-\Psi_2)}{B(\Psi_1,s-\Psi_2)}\frac{(s)_{n\omega}(\upsilon+1+\tau n)}{\Gamma(\tau n+\upsilon+2)}\frac{(-z)^n}{n!}\\
=&\mathfrak{J}_{\tau,\upsilon,s,\omega}^{F_1,\Psi_1,\Psi_2}(z).
\end{aligned}
$$

Hence we get our desired result (3). ■

Theorem 3.2.

Let $\tau,\upsilon,s\in\mathbb{C}$, $\Re(\Psi_1),\Re(\Psi_2),\Re(\tau),\Re(s)>0$, $\Re(\upsilon)\geq -1$ and $\omega=0,1,2,\ldots$, then

$$
\mathfrak{J}_{\tau,\upsilon,s,\omega}^{F_1,\Psi_1,\Psi_2}(z)-\mathfrak{J}_{\tau,\upsilon,s-1,\omega}^{F_1,\Psi_1,\Psi_2}(z)=\omega\sum_{n=0}^{\infty}n\frac{B_{F_1}^{p,q}(\Psi_1+n\omega,s-\Psi_2)}{B(\Psi_1,s-\Psi_2)}\frac{(s)_{n\omega-1}}{\Gamma(\tau n+\upsilon+1)}\frac{(-z)^n}{n!}. \tag{4}
$$

Proof:

Using Definition 2.1 on the left hand side of Equation (4), we get

$$
\begin{aligned}
\mathfrak{J}_{\tau,\upsilon,s,\omega}^{F_1,\Psi_1,\Psi_2}(z)-\mathfrak{J}_{\tau,\upsilon,s-1,\omega}^{F_1,\Psi_1,\Psi_2}(z)=&\sum_{n=0}^{\infty}\frac{B_{p,q}^{F_1}(\Psi_1+n\omega,s-\Psi_2)}{B(\Psi_1,s-\Psi_2)}\frac{(s)_{n\omega}}{\Gamma(\tau n+\upsilon+1)}\frac{(-z)^n}{n!}\\
&-\sum_{n=0}^{\infty}\frac{B_{p,q}^{F_1}(\Psi_1+n\omega,s-\Psi_2)}{B(\Psi_1,s-\Psi_2)}\frac{(s-1)_{n\omega}}{\Gamma(\tau n+\upsilon+1)}\frac{(-z)^n}{n!}\\
=&\sum_{n=0}^{\infty}\frac{B_{p,q}^{F_1}(\Psi_1+n\omega,s-\Psi_2)}{B(\Psi_1,s-\Psi_2)}\frac{(-z)^n}{\Gamma(\tau n+\upsilon+1)n!}\left[(s)_{n\omega}-(s-1)_{n\omega}\right]\\
=&\omega\sum_{n=0}^{\infty}n\frac{B_{F_1}^{p,q}(\Psi_1+n\omega,s-\Psi_2)}{B(\Psi_1,s-\Psi_2)}\frac{(s)_{n\omega-1}}{\Gamma(\tau n+\upsilon+1)}\frac{(-z)^n}{n!}.
\end{aligned}
$$

Hence we get our desired result (4). ■

4. Integral Representation

In this section, we have made efforts to discuss the various integral representations of the generalized Bessel-Maitland function.

Theorem 4.1.

Let $\tau,\upsilon,s\in\mathbb{C}$, $\Re(\Psi_1),\Re(\Psi_2),\Re(\tau),\Re(s)>0$, $\Re(\upsilon)\geq -1$, $\Re(p),\Re(q)\geq 0$, $\Re(a_1),\Re(a_2),\Re(a_2'),\Re(a_3)>0$, $r\geq 0$ and $\omega=0,1,2,\ldots$. Then, the following integral representation holds true:

$$\mathfrak{J}_{\tau,\upsilon,s,\omega}^{F_1,\Psi_1,\Psi_2}(z) = \frac{1}{B(\Psi_1, s-\Psi_2)} \int_0^1 t^{\Psi_1-1}(1-t)^{s-\Psi_2-1} F_1\left(a_1, a_2, a_2'; a_3; \frac{p}{t^r}, \frac{q}{(1-t)^r}\right) \mathfrak{J}_{\upsilon,\omega}^{\tau,s}(zt^\omega). \tag{5}$$

Proof:

From Equation (1), we have

$$\begin{aligned}\mathfrak{J}_{\tau,\upsilon,s,\omega}^{F_1,\Psi_1,\Psi_2}(z) =& \sum_{n=0}^{\infty}\left\{\int_0^1 t^{\Psi_1+n\omega-1}(1-t)^{s-\Psi_2-1} F_1\left(a_1, a_2, a_2'; a_3; \frac{p}{t^r}, \frac{q}{(1-t)^r}\right) dt\right\} \\ &\times \frac{1}{B(\Psi_1, s-\Psi_2)} \frac{(s)_{n\omega}}{\Gamma(\tau n+\upsilon+1)} \frac{(-z)^n}{n!} \\ =& \frac{1}{B(\Psi_1, s-\Psi_2)} \int_0^1 t^{\Psi_1-1}(1-t)^{s-\Psi_2-1} F_1\left(a_1, a_2, a_2'; a_3; \frac{p}{t^r}, \frac{q}{(1-t)^r}\right) \\ &\times \sum_{n=0}^{\infty} \frac{(s)_{n\omega}}{\Gamma(\tau n+\upsilon+1)} \frac{(-zt^\omega)^n}{n!} dt. \end{aligned} \tag{6}$$

Using the formula for the generalized Bessel-Maitland function $\mathfrak{J}_{\upsilon,\omega}^{\tau,s}(zt^\omega)$ given by Pathak (1966) in (6), we get the desired result (5). ■

Theorem 4.2.

The following integral representations for the generalized Bessel-Maitland function hold true:

$$\begin{aligned}(i)\quad \mathfrak{J}_{\tau,\upsilon,s,\omega}^{F_1,\Psi_1,\Psi_2}(z) =& \frac{2}{B(\Psi_1, s-\Psi_2)} \int_0^{\frac{\pi}{2}} \cos^{2\Psi_1-1}\theta \sin^{2s-2\Psi_2-1}\theta \\ & F_1\left(a_1, a_2, a_2'; a_3; \frac{p}{\cos^{2r}\theta}, \frac{q}{\sin^{2r}\theta}\right) \mathfrak{J}_{\upsilon,\omega}^{\tau,s}(z cos^{2\omega}\theta) d\theta, \end{aligned} \tag{7}$$

$$\begin{aligned}(ii)\quad \mathfrak{J}_{\tau,\upsilon,s,\omega}^{F_1,\Psi_1,\Psi_2}(z) =& \frac{1}{B(\Psi_1, s-\Psi_2)} \int_0^{\infty} \frac{u^{\Psi_1-1}}{(1+u)^{\Psi_1+s-\Psi_2}} \\ & F_1\left(a_1, a_2, a_2'; a_3; \frac{p(1+u)^r}{u^r}, q(1+u)^r\right) \mathfrak{J}_{\upsilon,\omega}^{\tau,s}\left(z\left(\frac{u}{1+u}\right)^{\omega}\right) du, \end{aligned} \tag{8}$$

$$\begin{aligned}(iii)\quad \mathfrak{J}_{\tau,\upsilon,s,\omega}^{F_1,\Psi_1,\Psi_2}(z) =& \frac{2^{1-\Psi_1-s+\Psi_2}}{B(\Psi_1, s-\Psi_2)} \int_{-1}^{1} (1+u)^{\Psi_1-1}(1-u)^{s-\Psi_2-1} \\ & F_1\left(a_1, a_2, a_2'; a_3; \frac{2^r p}{(1+u)^r}, \frac{2^r q}{(1-u)^r}\right) \mathfrak{J}_{\upsilon,\omega}^{\tau,s}\left(z\left(\frac{1+u}{2}\right)^{\omega}\right) du, \end{aligned} \tag{9}$$

$$(iv)\quad \mathfrak{J}_{\tau,\upsilon,s,\omega}^{F_1,\Psi_1,\Psi_2}(z) = \frac{(c-a)^{1-\Psi_1-s+\Psi_2}}{B(\Psi_1, s-\Psi_2)} \int_a^c (u-a)^{\Psi_1-1}(c-u)^{s-\Psi_2-1} F_1\left(a_1, a_2, a_2^{'}; a_3; \frac{p(c-a)^r}{(u-a)^r}, \frac{q(c-a)^r}{(c-u)^r}\right) \mathfrak{J}_{\upsilon,\omega}^{\tau,s}\left(z\left(\frac{u-a}{c-a}\right)^{\omega}\right) du, \tag{10}$$

$$(v)\quad \mathfrak{J}_{\tau,\upsilon,s,\omega}^{F_1,\Psi_1,\Psi_2}(z) = \frac{1}{B(\Psi_1, s-\Psi_2)} \int_0^{\frac{\pi}{4}} \tanh^{2\Psi_1-2}\theta \operatorname{sech}^{2s-2\Psi_2}\theta \, F_1\left(a_1, a_2, a_2^{'}; a_3; \frac{p}{\tanh^{2r}\theta}, \frac{q}{\operatorname{sech}^{2r}\theta}\right) \mathfrak{J}_{\upsilon,\omega}^{\tau,s}(z \tanh^{2\omega}\theta) d\theta, \tag{11}$$

where $\tau, \upsilon, s \in \mathbb{C}$, $\Re(\Psi_1), \Re(\Psi_2), \Re(\tau), \Re(s) > 0$, $\Re(\upsilon) \geq -1$, $r \geq 0$, $\Re(p), \Re(q) \geq 0$, $\Re(a_1), \Re(a_2), \Re(a_2^{'}), \Re(a_3) > 0$ and $\omega = 0, 1, 2, \ldots$

Proof:

Substitute $t = \cos^2\theta, t = \frac{u}{(1+u)}, t = \frac{(1+u)}{2}, t = \frac{(u-a)}{(c-a)}$ and $t = \tanh^2\theta$ in Equation (5) to get Equations (7) - (11), respectively. ■

5. Differentiation Formula

Here we discuss the k-th differentiation of the newly generalized Bessel-Maitland function.

Theorem 5.1.

Let $\tau, \upsilon, s \in \mathbb{C}$, $\Re(\Psi_1), \Re(\Psi_2), \Re(\tau), \Re(s) > 0$, $\Re(\upsilon) \geq -1$ and $\omega = 0, 1, 2, \ldots$. Then,

$$\frac{d^k}{dz^k}\left\{\mathfrak{J}_{\tau,\upsilon,s,\omega}^{F_1,\Psi_1,\Psi_2}(z)\right\} = \left\{(-1)^k (s)_\omega (s+\omega)_\omega \ldots (s+(k-1)\omega)_\omega\right\} \mathfrak{J}_{\tau,\tau k+\upsilon,s+k\omega,\omega}^{F_1,\Psi_1+k\omega,\Psi_2}(z). \tag{12}$$

Proof:

Consider Equation (1). Taking the derivative with respect to z, we have

$$\begin{aligned} \frac{d}{dz}\left\{\mathfrak{J}_{\tau,\upsilon,s,\omega}^{F_1,\Psi_1,\Psi_2}(z)\right\} &= \sum_{n=1}^{\infty} \frac{B_{p,q}^{F_1}(\Psi_1+n\omega, s-\Psi_2)}{B(\Psi_1, s-\Psi_2)} \frac{(s)_{n\omega}}{\Gamma(\tau n+\upsilon+1)} \frac{(-1)^n (z)^{n-1}}{(n-1)!} \\ &= \sum_{n=0}^{\infty} \frac{B_{p,q}^{F_1}(\Psi_1+n\omega+\omega, s-\Psi_2)}{B(\Psi_1, s-\Psi_2)} \frac{(s)_{n\omega+\omega}}{\Gamma(\tau n+\tau+\upsilon+1)} \frac{(-1)^{n+1}(z)^n}{n!} \\ &= -(s)_\omega \sum_{n=0}^{\infty} \frac{B_{p,q}^{F_1}(\Psi_1+n\omega+\omega, s-\Psi_2)}{B(\Psi_1, s-\Psi_2)} \frac{(s+\omega)_{n\omega}}{\Gamma(\tau n+\tau+\upsilon+1)} \frac{(-z)^n}{n!} \\ &= \{-(s)_\omega\} \mathfrak{J}_{\tau,\tau+\upsilon,s+\omega,\omega}^{F_1,\Psi_1+\omega,\Psi_2}(z). \end{aligned} \tag{13}$$

Again, differentiating Equation (13) with respect to z, we get

$$\begin{aligned}\frac{d^2}{dz^2}\left\{\mathfrak{J}^{F_1,\Psi_1,\Psi_2}_{\tau,\upsilon,s,\omega}(z)\right\} &= \frac{d}{dz}\left\{\sum_{n=1}^{\infty}\frac{B^{F_1}_{p,q}(\Psi_1+n\omega,s-\Psi_2)}{B(\Psi_1,s-\Psi_2)}\frac{(s)_{n\omega}}{\Gamma(\tau n+\upsilon+1)}\frac{(-1)^n(z)^{n-1}}{(n-1)!}\right\}\\
&= \sum_{n=2}^{\infty}\frac{B^{F_1}_{p,q}(\Psi_1+n\omega,s-\Psi_2)}{B(\Psi_1,s-\Psi_2)}\frac{(s)_{n\omega}}{\Gamma(\tau n+\upsilon+1)}\frac{(-1)^n(z)^{n-2}}{(n-2)!}\\
&= \sum_{n=0}^{\infty}\frac{B^{F_1}_{p,q}(\Psi_1+n\omega+2\omega,s-\Psi_2)}{B(\Psi_1,s-\Psi_2)}\frac{(s)_{n\omega+2\omega}}{\Gamma(\tau n+2\tau+\upsilon+1)}\frac{(-1)^{n+2}(z)^n}{n!}\\
&= (-1)^2(s)_\omega(s+\omega)_\omega\sum_{n=0}^{\infty}\frac{B^{F_1}_{p,q}(\Psi_1+n\omega+2\omega,s-\Psi_2)(s+2\omega)_{n\omega}}{B(\Psi_1,s-\Psi_2)\Gamma(\tau n+2\tau+\upsilon+1)}\frac{(-z)^n}{n!}\\
&= \left\{(-1)^2(s)_\omega(s+\omega)_\omega\right\}\mathfrak{J}^{F_1,\Psi_1+2\omega,\Psi_2}_{\tau,2\tau+\upsilon,s+2\omega,\omega}(z).\end{aligned}$$

Repeating this process k times, we get the desired result (12). ■

Theorem 5.2.

Let $\tau,\upsilon,s\in\mathbb{C}$, $\Re(\Psi_1),\Re(\Psi_2),\Re(\tau),\Re(s)>0$, $\Re(\upsilon)\geq -1$ and $\omega=0,1,2,\ldots$. Then,

$$\frac{d^k}{dz^k}\left\{z^\upsilon\mathfrak{J}^{F_1,\Psi_1,\Psi_2}_{\tau,\upsilon,s,\omega}(\sigma z^\tau)\right\} = z^{\upsilon-k}\mathfrak{J}^{F_1,\Psi_1,\Psi_2}_{\tau,\upsilon-k,s,\omega}(\sigma z^\tau). \tag{14}$$

Proof:

From Equation (1), we have

$$z^\upsilon\mathfrak{J}^{F_1,\Psi_1,\Psi_2}_{\tau,\upsilon,s,\omega}(\sigma z^\tau) = z^\upsilon\sum_{n=0}^{\infty}\frac{B^{F_1}_{p,q}(\Psi_1+n\omega,s-\Psi_2)}{B(\Psi_1,s-\Psi_2)}\frac{(s)_{n\omega}}{\Gamma(\tau n+\upsilon+1)}\frac{(-\sigma z^\tau)^n}{n!}. \tag{15}$$

Differentiating Equation (15) with respect z, we get

$$\begin{aligned}\frac{d}{dz}\left\{z^\upsilon\mathfrak{J}^{F_1,\Psi_1,\Psi_2}_{\tau,\upsilon,s,\omega}(\sigma z^\tau)\right\} &= \frac{d}{dz}\left\{\sum_{n=0}^{\infty}\frac{B^{F_1}_{p,q}(\Psi_1+n\omega,s-\Psi_2)}{B(\Psi_1,s-\Psi_2)}\frac{(s)_{n\omega}}{\Gamma(\tau n+\upsilon+1)}\frac{(-1)^n\sigma^n z^{\upsilon+\tau n}}{n!}\right\}\\
&= \sum_{n=0}^{\infty}\frac{B^{F_1}_{p,q}(\Psi_1+n\omega,s-\Psi_2)}{B(\Psi_1,s-\Psi_2)}\frac{(s)_{n\omega}}{\Gamma(\tau n+\upsilon+1)}\frac{(-1)^n\sigma^n(\upsilon+\tau n)z^{\upsilon+\tau n-1}}{n!}\\
&= z^{\upsilon-1}\sum_{n=0}^{\infty}\frac{B^{F_1}_{p,q}(\Psi_1+n\omega,s-\Psi_2)}{B(\Psi_1,s-\Psi_2)}\frac{(s)_{n\omega}}{\Gamma(\tau n+(\upsilon-1)+1)}\frac{(-\sigma z^\tau)^n}{n!}\\
&= z^{\upsilon-1}\mathfrak{J}^{F_1,\Psi_1,\Psi_2}_{\tau,\upsilon-1,s,\omega}(\sigma z^\tau). \end{aligned}\tag{16}$$

Again, differentiating Equation (16), we get

$$\begin{aligned}\frac{d^2}{dz^2}\left\{z^\upsilon\mathfrak{J}^{F_1,\Psi_1,\Psi_2}_{\tau,\upsilon,s,\omega}(\sigma z^\tau)\right\} &= z^{\upsilon-2}\sum_{n=0}^{\infty}\frac{B^{F_1}_{p,q}(\Psi_1+n\omega,s-\Psi_2)}{B(\Psi_1,s-\Psi_2)}\frac{(s)_{n\omega}}{\Gamma(\tau n+(\upsilon-2)+1)}\frac{(-\sigma z^\tau)^n}{n!}\\
&= z^{\upsilon-2}\mathfrak{J}^{F_1,\Psi_1,\Psi_2}_{\tau,\upsilon-2,s,\omega}(\sigma z^\tau). \end{aligned}\tag{17}$$

Repeating this process k times, we get the desired result (14). ■

6. Fractional Operators

In this section, we discuss the relation of the generalized Bessel-Maitland function with fractional calculus operators.

Theorem 6.1.

Let $\tau, \upsilon, s \in \mathbb{C}$, $\Re(\Psi_1), \Re(\Psi_2), \Re(\tau), \Re(s) > 0$, $\Re(\upsilon) \geq -1$ and $\omega = 0, 1, 2, \ldots$. Then for $x > a$, the following result holds true:

$$\left(I_{a+}^{\xi}\left\{(z-a)^{\upsilon}\mathfrak{J}_{\tau,\upsilon,s,\omega}^{F_1,\Psi_1,\Psi_2}(\sigma(z-a)^{\tau})\right\}\right)(x) = (x-a)^{\xi+\upsilon}\mathfrak{J}_{\tau,\upsilon+\xi,s,\omega}^{F_1,\Psi_1,\Psi_2}(\sigma(x-a)^{\tau}). \tag{18}$$

Proof:

Using Definition 2.1 on the left hand side of Equation (18), we get

$$\begin{aligned}&\left(I_{a+}^{\xi}\left\{(z-a)^{\upsilon}\mathfrak{J}_{\tau,\upsilon,s,\omega}^{F_1,\Psi_1,\Psi_2}(\sigma(z-a)^{\tau})\right\}\right)(x)\\ &= \sum_{n=0}^{\infty}\frac{B_{p,q}^{F_1}(\Psi_1+n\omega, s-\Psi_2)}{B(\Psi_1, s-\Psi_2)}\frac{(s)_{n\omega}}{\Gamma(\tau n+\upsilon+1)}\frac{(-\sigma)^n}{n!}\left(I_{a+}^{\xi}(z-a)^{\tau n+\upsilon}\right)(x).\end{aligned} \tag{19}$$

Using the result for $I_{a+}^{\xi}\left[(z-a)^{\upsilon}\right]$ given in Mathai (2008) in (19), we get

$$\begin{aligned}&\left(I_{a+}^{\xi}\left\{(z-a)^{\upsilon}\mathfrak{J}_{\tau,\upsilon,s,\omega}^{F_1,\Psi_1,\Psi_2}(\sigma(z-a)^{\tau})\right\}\right)(x)\\ &= (x-a)^{\xi+\upsilon}\sum_{n=0}^{\infty}\frac{B_{p,q}^{F_1}(\Psi_1+n\omega, s-\Psi_2)}{B(\Psi_1, s-\Psi_2)}\frac{(s)_{n\omega}}{\Gamma(\tau n+\upsilon+\xi+1)}\frac{(-\sigma(x-a)^{\tau})^n}{n!}\\ &= (x-a)^{\xi+\upsilon}\mathfrak{J}_{\tau,\upsilon+\xi,s,\omega}^{F_1,\Psi_1,\Psi_2}(\sigma(x-a)^{\tau}).\end{aligned}$$

Hence, we get the desired result (18). ■

Theorem 6.2.

Let $\tau, \upsilon, s \in \mathbb{C}$, $\Re(\Psi_1), \Re(\Psi_2), \Re(\tau), \Re(s) > 0$, $\Re(\upsilon) \geq -1$ and $\omega = 0, 1, 2, \ldots$. Then for $x > a$, the following result holds true:

$$\left(D_{a+}^{\xi}\left\{(z-a)^{\upsilon}\mathfrak{J}_{\tau,\upsilon,s,\omega}^{F_1,\Psi_1,\Psi_2}(\sigma(z-a)^{\tau})\right\}\right)(x) = (x-a)^{\upsilon-\xi}\mathfrak{J}_{\tau,\upsilon-\xi,s,\omega}^{F_1,\Psi_1,\Psi_2}(\sigma(x-a)^{\tau}). \tag{20}$$

Proof:

Using Kilbas et al. (2006) for D_{a+}^{ξ}, we get

$$\left(D_{a+}^{\xi}\left\{(z-a)^{\upsilon}\mathfrak{J}_{\tau,\upsilon,s,\omega}^{F_1,\Psi_1,\Psi_2}(\sigma(z-a)^{\tau})\right\}\right)(x) = \frac{d^k}{dx^k}\left(I_{a+}^{k-\xi}\left\{(z-a)^{\upsilon}\mathfrak{J}_{\tau,\upsilon,s,\omega}^{F_1,\Psi_1,\Psi_2}(\sigma(z-a)^{\tau})\right\}\right)(x). \tag{21}$$

Using Equation (18) in (21)

$$\left(D_{a+}^{\xi}\left\{(z-a)^{\upsilon}\mathfrak{J}_{\tau,\upsilon,s,\omega}^{F_1,\Psi_1,\Psi_2}(\sigma(z-a)^{\tau})\right\}\right)(x) = \frac{d^k}{dx^k}\left\{(x-a)^{k-\xi+\upsilon}\mathfrak{J}_{\tau,\upsilon+k-\xi,s,\omega}^{F_1,\Psi_1,\Psi_2}(\sigma(x-a)^{\tau})\right\}. \tag{22}$$

Using equation (14) in (22), we get the desired result (20). ■

Theorem 6.3.

Let $\tau, \upsilon, s \in \mathbb{C}$, $\Re(\Psi_1), \Re(\Psi_2), \Re(\tau), \Re(s) > 0$, $\Re(\upsilon) \geq -1$ and $\omega = 0, 1, 2, \ldots$. Then for $x > a$, the following result holds true:

$$\left(D_{a+}^{\xi,\lambda}\left\{(z-a)^{\upsilon}\mathfrak{J}_{\tau,\upsilon,s,\omega}^{F_1,\Psi_1,\Psi_2}(\sigma(z-a)^{\tau})\right\}\right)(x) = (x-a)^{\upsilon-\xi}\mathfrak{J}_{\tau,\upsilon-\xi,s,\omega}^{F_1,\Psi_1,\Psi_2}(\sigma(x-a)^{\tau}). \tag{23}$$

Proof:

Using Definition 2.1 on the left hand side of Equation (23), we get

$$\begin{aligned}&\left(D_{a+}^{\xi.\lambda}\left\{(z-a)^{\upsilon}\mathfrak{J}_{\tau,\upsilon,s,\omega}^{F_1,\Psi_1,\Psi_2}(\sigma(z-a)^{\tau})\right\}\right)(x)\\&=\sum_{n=0}^{\infty}\frac{B_{p,q}^{F_1}(\Psi_1+n\omega, s-\Psi_2)}{B(\Psi_1, s-\Psi_2)}\frac{(s)_{n\omega}}{\Gamma(\tau n+\upsilon+1)}\frac{(-\sigma)^n}{n!}\left(D_{a+}^{\xi,\lambda}(z-a)^{\tau n+\upsilon}\right)(x).\end{aligned} \tag{24}$$

Using the result for $D_{a+}^{\xi,\lambda}\left[(z-a)^{\upsilon}\right]$ given in Srivastava and Tomovski (2009) in (24), we get

$$\begin{aligned}&\left(D_{a+}^{\xi,\lambda}\left\{(z-a)^{\upsilon}\mathfrak{J}_{\tau,\upsilon,s,\omega}^{F_1,\Psi_1,\Psi_2}(\sigma(z-a)^{\tau})\right\}\right)(x)\\&=(x-a)^{\upsilon-\xi}\sum_{n=0}^{\infty}\frac{B_{p,q}^{F_1}(\Psi_1+n\omega, s-\Psi_2)}{B(\Psi_1, s-\Psi_2)}\frac{(s)_{n\omega}}{\Gamma(\tau n+\upsilon-\xi+1)}\frac{(-\sigma(x-a)^{\tau})^n}{n!}\\&=(x-a)^{\upsilon-\xi}\mathfrak{J}_{\tau,\upsilon-\xi,s,\omega}^{F_1,\Psi_1,\Psi_2}(\sigma(x-a)^{\tau}).\end{aligned}$$

Hence, we get the desired result (23). ■

Similarly, we can obtain all the above results (from Section $3-6$) for generalized Bessel-Maitland function involving Appell series $F_2(.), F_3(.), F_4(.)$ and Lauricella function $F_A^{(m)}(.), F_B^{(m)}(.), F_C^{(m)}(.), F_D^{(m)}(.)$.

7. Conclusion

The Bessel-Maitland function has many applications in the diverse fields of engineering, biological science and mathematical physics. Various generalizations of Bessel-Maitland functions have been introduced and examined with different applications. In this paper, we have defined a new generalization of Bessel-Maitland function using the extension of beta function involving Appell series and Lauricella functions, which reduces to the Bessel-Maitland function under certain particular values of parameters. Particular properties such as recurrence relation, integral representation and differentiation formula of the new generalization of Bessel-Maitland function have been investigated. We have also discussed some properties of the generalized Bessel-Maitland function connected with Riemann-Liouville fractional integral and differential operator. The results obtained can be extended for other fractional operators of significance like Caputo fractional derivative, Erdelyi Kober operators, Caputo-Fabrizio operators among various others.

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