[Applications and Applied Mathematics: An International Journal (AAM)](https://digitalcommons.pvamu.edu/aam)

Volume 16 | Issue 1 Article 15

6-2021

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Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Vol. 16, Issue 1 (June 2021), pp. 289 – 319

Hamacher Operations of Fermatean Fuzzy Matrices

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Received: June 9, 2020; Accepted: January 15, 2021

Abstract

The purpose of this study is to extend the Fermatean fuzzy matrices to the theory of Hamacher operations. In this paper, the concept of Hamacher operations of Fermatean fuzzy matrices are introduced and some desirable properties of these operations, such as commutativity, idempotency, and monotonicity are discussed. Further, we prove DeMorgan's laws over complement for these operations. Furthermore, the scalar multiplication and exponentiation operations of Fermatean fuzzy matrices are constructed and their algebraic properties are investigated. Finally, some properties of necessity and possibility operators of Fermatean fuzzy matrices are proved.

Keywords: Hamacher; Sum; Product; Multiplication; Exponentiation; Necessity; Possibility

MSC 2010 No.: 03E72, 08A72, 15B15

1. Introduction

The concept of an intuitionistic fuzzy matrix (IFM) was introduced by Khan et al. (2002) and simultaneously Im et al. (2001) to generalize the concept of Thomason (1977) fuzzy matrix. Further, Emam and Fndh (2016) defined some kinds of IFMs. Also, they constructed an idempotent IFM from any given one through the min-max composition. Pal (2001) introduced the intuitionistic fuzzy determinant. Khan and Pal (2006) defined some basic operations and relations of IFMs including maxmin, minmax, complement, algebraic sum, algebraic product, etc. and proved equality between IFMs. Mondal and Pal (2013) studied the similarity relations together with invertibility conditions and eigenvalues of IFMs. Zhang and Xu (2012) studied intuitionistic fuzzy value and introduced the concept of composition two intuitionistic fuzzy matrices. Muthuraji et al. (2016) obtained a decomposition of intuitionistic fuzzy matrices. Since the appearance of IFM, several researchers have importantly contributed to the development of IFM theory and its applications, resulting in the greater success from the theoretical and technological points of view.

Yager (2013) introduced the concept of the Pythagorean fuzzy set (PFS) and developed some aggregation operations for PFS. Zhang and Xu (2014) studied various binary operations over PFS and also proposed a decision making algorithm based on PFS. Using the theory of PFS, the Pythagorean fuzzy matrices (PFM) are introduced and its algebraic operations are defined (Silambarasan and Sriram (2018)). We proved the set of all Pythagorean fuzzy matrices form a commutative monoid with respect to algebraic sum and algebraic product (Silambarasan and Sriram (2019a)). We have developed the Hamacher operations on Pythagorean fuzzy matrices and investigated their algebraic properties (Silambarasan and Sriram (2019b)). In 2020, the scalar multiplication and exponentiation operations of a Pythagorean fuzzy matrices are constructed and some desirable properties are investigated. Senapati and Yager (2020) proposed the Fermatean FS (FFS) and the main advantage of FFS is that it can describe more uncertainties than IFS and PFS, which can be applied in many decision making problems. In other words, IFS and PFS are two special forms of FFS, which means that the FFSs are able to handle higher levels of uncertainties. Using the theory of FFS, the Fermatean fuzzy matrices are introduced and its algebraic operations are defined (Silambarasan (2020)). Further, the scalar multiplication and exponentiation operations of Fermatean fuzzy matrices are constructed and their algebraic properties are investigated. In this paper, we develop the Hamacher operations of Fermatean fuzzy matrices and prove their algebraic properties.

This paper is organized as follows. In Section 2, we shall develop the Hamacher operations of Fermatean fuzzy matrices and analyze some desirable properties. In Section 3, we prove on complement of Fermatean fuzzy matrices. In Section 4, we construct the Hamacher scalar multiplication and Hamacher exponentiation operations of a Fermatean fuzzy matrix A and investigate their algebraic properties. In Section 5, we prove some properties of necessity and possibility operators on Fermatean fuzzy matrices. Finally, the conclusions and future studies are provided in Section 6.

2. Hamacher Operations of Fermatean Fuzzy Matrices

In this section, we define the Hamacher operations of Fermatean fuzzy matrices and analyze some desirable properties.

Definition 2.1.

Let $A = \left[\left\langle \zeta_{a_{ij}}, \delta_{a_{ij}} \right\rangle\right]$ and $B = \left[\left\langle \zeta_{b_{ij}}, \delta_{b_{ij}} \right\rangle\right]$ be any two Fermatean fuzzy matrices of the same size, then we have:

(i) The Hamacher sum of A and B is defined by $A \boxplus_h B = (\alpha_{ij})$, where,

$$\alpha_{ij} = \begin{cases} \langle 1, 0 \rangle, & \text{if } \left\langle \zeta_{a_{ij}}, \delta_{a_{ij}} \right\rangle = \langle 1, 0 \rangle, \left\langle \zeta_{b_{ij}}, \delta_{b_{ij}} \right\rangle = \langle 1, 0 \rangle, \\ \left\langle \sqrt[3]{\dfrac{\zeta_{a_{ij}}^3 + \zeta_{b_{ij}}^3 - 2\zeta_{a_{ij}}^3 \zeta_{b_{ij}}^3}{1 - \zeta_{a_{ij}}^3 \zeta_{b_{ij}}^3}}, \sqrt[3]{\dfrac{\delta_{a_{ij}}^3 \delta_{b_{ij}}^3}{\delta_{a_{ij}}^3 + \delta_{b_{ij}}^3 - \delta_{a_{ij}}^3 \delta_{b_{ij}}^3}} \right\rangle, & \text{otherwise,} \end{cases} \tag{1}$$

and

(ii) The Hamacher product of A and B is defined by $A \boxdot_h B = (\beta_{ij})$,

where,

$$\beta_{ij} = \begin{cases} \langle 0, 1 \rangle, & \text{if } \left\langle \zeta_{a_{ij}}, \delta_{a_{ij}} \right\rangle = \langle 0, 1 \rangle, \left\langle \zeta_{b_{ij}}, \delta_{b_{ij}} \right\rangle = \langle 0, 1 \rangle, \\ \left\langle \sqrt[3]{\dfrac{\zeta_{a_{ij}}^3 \zeta_{b_{ij}}^3}{\zeta_{a_{ij}}^3 + \zeta_{b_{ij}}^3 - \zeta_{a_{ij}}^3 \zeta_{b_{ij}}^3}}, \sqrt[3]{\dfrac{\delta_{a_{ij}}^3 + \delta_{b_{ij}}^3 - 2\delta_{a_{ij}}^3 \delta_{b_{ij}}^3}{1 - \delta_{a_{ij}}^3 \delta_{b_{ij}}^3}} \right\rangle, & \text{otherwise,} \end{cases} \tag{2}$$

for all i, j.

Example 2.1.

Any intuitionistic fuzzy matrix $(\zeta_{a_{ij}}, \delta_{a_{ij}})$ that is an IFM is also a PFM and a FFM.

For any two fuzzy matrices $A, B \in [0, 1]$, we get $\zeta_{a_{ij}}^3 \leq \zeta_{a_{ij}}^2 \leq \zeta_{a_{ij}}$ and $\delta_{a_{ij}}^3 \leq \delta_{a_{ij}}^2 \leq \delta_{a_{ij}}$.

Thus $\zeta_{a_{ij}} + \delta_{a_{ij}} \leq 1 \Rightarrow \zeta_{a_{ij}}^2 + \delta_{a_{ij}}^2 \leq 1 \Rightarrow \zeta_{a_{ij}}^3 + \delta_{a_{ij}}^3 \leq 1$.

Consider a point $(0.9, 0.6)$. We see that $(0.9)^3 + (0.6)^3 \leq 1$, and thus, this is an FFM.

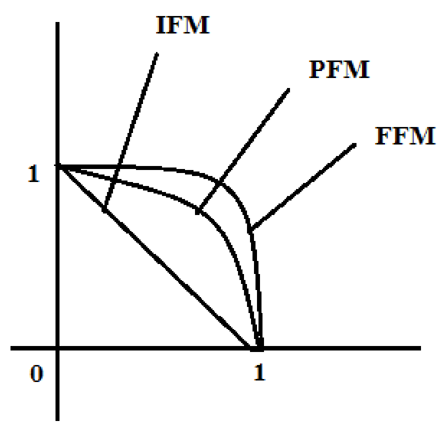

Figure.1 Comparison of space of IFMs, PFMs and FFMs

Since $(0.9)^2 + (0.6)^2 = 0.81 + 0.36 = 1.17 \geq 1$ and $0.9 + 0.6 \geq 1$, therefore, $(0.9, 0.6)$ is neither a PFM nor an IFM.

$\mathbf{A} = \begin{bmatrix} (0.9, 0.6) & (0.2, 0.4) \\ (0.3, 0.4) & (0.4, 0.4) \end{bmatrix}$ is not an IFM and PFM, but it A is a FFM.

This development can be evidently recognized in Figure 1. Here we notice that IFMs are all points beneath the line $\zeta_{a_{ij}} + \delta_{a_{ij}} \leq 1$, the PFMs are all points with $\zeta^2_{a_{ij}} + \delta^2_{a_{ij}} \leq 1$, and the FFMs are all points with $\zeta^3_{a_{ij}} + \delta^3_{a_{ij}} \leq 1$. We see then that the FFMs enable for the presentation of a bigger body of non-standard membership function then IFMs and PFMs.

Lemma 2.1.

For any two real numbers $a, b \in [0, 1]$, the following inequality holds,

$$\sqrt[3]{\frac{a^3b^3}{a^3+b^3-a^3b^3}} \leq \sqrt[3]{\frac{a^3+b^3-2a^3b^3}{1-a^3b^3}}.$$

Proof:

Let $\dfrac{ab}{a+b-ab} \leq \dfrac{a+b-2ab}{1-ab}$.

We know that

$$(a+b)^2 \geq 4ab \tag{2.1}$$

$$a+b-ab \leq 1-(1-a)(1-b) \leq 1$$

$$\Rightarrow 1+3(a+b-ab) \leq 4$$

$$\Rightarrow (1+3(a+b-ab))ab \leq 4ab. \tag{2.2}$$

From the Equations (2.1) and (2.2), we get

$$(1+3(a+b-ab))ab \leq 4ab \leq (a+b)^2$$

$$\Rightarrow 0 \leq (a+b)^2 - (1+3(a+b-ab))ab$$

$$\Rightarrow \quad = (a+b)^2 + 3ab(ab-a-b) - ab$$

$$\Rightarrow ab \leq (a+b)^2 + 3a^2b^2 - 3ab(a+b)$$

$$\Rightarrow ab - a^2b^2 \leq (a+b-2ab)(a+b-ab)$$

$$\Rightarrow \frac{ab}{a+b-ab} \leq \frac{a+b-2ab}{1-ab}$$

$$\Rightarrow \sqrt[3]{\frac{a^3b^3}{a^3+b^3-a^3b^3}} \leq \sqrt[3]{\frac{a^3+b^3-2a^3b^3}{1-a^3b^3}}.$$

■

Lemma 2.2.

For any three real numbers $a, b, c \in [0,1]$.

(i) $\sqrt[3]{\frac{a^3c^3}{a^3+c^3-a^3c^3}} \leq \sqrt[3]{\frac{b^3c^3}{b^3+c^3-b^3c^3}}$, (ii) $\sqrt[3]{\frac{a^3+c^3-2a^3c^3}{1-a^3c^3}} \leq \sqrt[3]{\frac{b^3+c^3-2b^3c^3}{1-b^3c^3}}$.

Proof:

(i) Let $\frac{ac}{a+c-ac} \leq \frac{bc}{b+c-bc}$.

Since

$$a \leq b \Rightarrow ac^2 \leq bc^2$$

$$\Rightarrow ac^2 + abc(1-c) \leq bc^2 + abc(1-c)$$

$$\Rightarrow ac^2 + abc - abc^2 \leq bc^2 + abc - abc^2$$

$$\Rightarrow ac(c+b-bc) \leq bc(c+a-ac)$$

$$\Rightarrow \frac{ac}{a+c-ac} \leq \frac{bc}{b+c-bc}$$

$$\Rightarrow \sqrt[3]{\frac{a^3c^3}{a^3+c^3-a^3c^3}} \leq \sqrt[3]{\frac{b^3c^3}{b^3+c^3-b^3c^3}}.$$

(ii) Let $\frac{a+c-2ac}{1-ac} \leq \frac{b+c-2bc}{1-bc}$.

Since

$$a \leq b \Rightarrow a(1-c)^2 \leq b(1-c)^2$$

$$\Rightarrow a(1-2c+c^2) \leq b(1-2c+c^2)$$

$$\Rightarrow a - 2ac + ac^2 \leq b - 2bc + bc^2$$

$$\Rightarrow a - 2ac + ac^2 + (c - abc + 2abc^2) \leq b - 2bcbc^2 + (c - abc + 2abc^2)$$

$$\Rightarrow a + c - 2ac - abc - bc^2 + 2abc^2 \leq b + c - 2bc - abc - ac^2 + 2abc^2$$

$$\Rightarrow a + c - 2ac - bc(a+c-2ac) \leq b + c - 2bc - ac(b+c-2bc)$$

$$\Rightarrow (a+c-2ac)(1-bc) \leq (b+c-2bc)(1-ac)$$

$$\Rightarrow \frac{a+c-2ac}{1-ac} \leq \frac{b+c-2bc}{1-bc}$$

$$\Rightarrow \sqrt[3]{\frac{a^3+c^3-2a^3c^3}{1-a^3c^3}} \leq \sqrt[3]{\frac{b^3+c^3-2b^3c^3}{1-b^3c^3}}. \qquad \blacksquare$$

The relation between Hamacher sum and Hamacher product is established by the following theorem.

Theorem 2.1.

Let A, B be any two Fermatean fuzzy matrices of the same size, then

$$A \boxdot_h B \leq A \boxplus_h B.$$

Proof:

By using Lemma 2.1,

$$\sqrt[3]{\frac{\zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}{\zeta^3_{a_{ij}}+\zeta^3_{b_{ij}}-\zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}} \leq \sqrt[3]{\frac{\zeta^3_{a_{ij}}+\zeta^3_{b_{ij}}-2\zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}{1-\zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}},$$

and

$$\sqrt[3]{\frac{\delta^3_{a_{ij}}+\delta^3_{b_{ij}}-2\delta^3_{a_{ij}}\delta^3_{b_{ij}}}{1-\delta^3_{a_{ij}}\delta^3_{b_{ij}}}} \geq \sqrt[3]{\frac{\delta^3_{a_{ij}}\delta^3_{b_{ij}}}{\delta^3_{a_{ij}}+\delta^3_{b_{ij}}-\delta^3_{a_{ij}}\delta^3_{b_{ij}}}}, \quad \text{for all } i,j.$$

By Definition 2.1, it follows that $A \boxdot_h B \leq A \boxplus_h B$. ■

Theorem 2.2.

For any Fermatean fuzzy matrix A,

(i) $A \boxplus_h A \geq A$,

(ii) $A \boxdot_h A \leq A$.

Proof:

$$(i)\ A \boxplus_h A = \left[\left\langle \sqrt[3]{\frac{\zeta^3_{a_{ij}}+\zeta^3_{a_{ij}}-2\zeta^3_{a_{ij}}\zeta^3_{a_{ij}}}{1-\zeta^3_{a_{ij}}\zeta^3_{a_{ij}}}}, \sqrt[3]{\frac{\delta^3_{a_{ij}}\delta^3_{a_{ij}}}{\delta^3_{a_{ij}}+\delta^3_{a_{ij}}-\delta^3_{a_{ij}}\delta^3_{a_{ij}}}} \right\rangle\right]$$

$$= \left[\left\langle \sqrt[3]{\frac{2\zeta^3_{a_{ij}}-2\zeta^6_{a_{ij}}}{1-\zeta^6_{a_{ij}}}}, \sqrt[3]{\frac{\delta^6_{a_{ij}}}{2\delta^3_{a_{ij}}-\delta^6_{a_{ij}}}} \right\rangle\right]$$

$$= \left[\left\langle \sqrt[3]{\frac{2\zeta^3_{a_{ij}}(1-\zeta^3_{a_{ij}})}{(1-\zeta^3_{a_{ij}})(1+\zeta^3_{a_{ij}})}}, \sqrt[3]{\frac{\delta^6_{a_{ij}}}{\delta^3_{a_{ij}}(2-\delta^3_{a_{ij}})}} \right\rangle\right]$$

$$= \left[\left\langle \sqrt[3]{\frac{2\zeta^3_{a_{ij}}}{1+\zeta^3_{a_{ij}}}}, \sqrt[3]{\frac{\delta^3_{a_{ij}}}{2-\delta^3_{a_{ij}}}} \right\rangle\right]$$

$$\geq \left[\left\langle \zeta^3_{a_{ij}}, \delta^3_{a_{ij}} \right\rangle\right] \geq \left[\left\langle \zeta_{a_{ij}}, \delta_{a_{ij}} \right\rangle\right]$$

$$\geq A.$$

(ii) It can be proved similarly. ∎

The operations Hamacher sum and Hamacher product of FFMs are commutative as well as associative, and the identities for $\boxplus_h$ and $\boxdot_h$ exist.

Theorem 2.3.

Let A, B and C be any three Fermatean fuzzy matrices of the same size, then

(i) $A \boxplus_h B = B \boxplus_h A$,

(ii) $(A \boxplus_h B) \boxplus_h C = A \boxplus_h (B \boxplus_h C)$,

(iii) $A \boxdot_h B = B \boxdot_h A$,

(iv) $(A \boxdot_h B) \boxdot_h C = A \boxdot_h (B \boxdot_h C)$.

Proof:

The Proofs (i) and (iii) are straightforward from the Definition 2.1.

(ii) If $(A \boxplus_h B) = (\psi^3_{a_{ij}})$, then

$$(A \boxplus_h B) \boxplus_h C = \left(\sqrt[3]{\frac{\psi^3_{a_{ij}} + \omega^3_{a_{ij}} - 2\psi^3_{a_{ij}}\omega^3_{a_{ij}}}{1-\psi^3_{a_{ij}}\omega^3_{a_{ij}}}} \right).$$

Let $\sqrt[3]{\psi^3_{a_{ij}} + \omega^3_{a_{ij}} - 2\psi^3_{a_{ij}}\omega^3_{a_{ij}}}$

$$= \sqrt[3]{\left(\sqrt[3]{\frac{\zeta^3_{a_{ij}} + \delta^3_{a_{ij}} - 2\zeta^3_{a_{ij}}\delta^3_{a_{ij}}}{1-\zeta^3_{a_{ij}}\delta^3_{a_{ij}}}}\right)^3 + \omega^3_{a_{ij}} - 2\left(\sqrt[3]{\frac{\zeta^3_{a_{ij}} + \delta^3_{a_{ij}} - 2\zeta^3_{a_{ij}}\delta^3_{a_{ij}}}{1-\zeta^3_{a_{ij}}\delta^3_{a_{ij}}}}\right)^3 .\omega^3_{a_{ij}}}$$

$$= \sqrt[3]{\frac{\zeta^3_{a_{ij}} + \delta^3_{a_{ij}} - 2\zeta^3_{a_{ij}}\delta^3_{a_{ij}} + \omega^3_{a_{ij}}(1-\zeta^3_{a_{ij}}\delta^3_{a_{ij}}) - 2\zeta^3_{a_{ij}}(\zeta^3_{a_{ij}} + \delta^3_{a_{ij}} - 2\zeta^3_{a_{ij}}\delta^3_{a_{ij}})}{1-\zeta^3_{a_{ij}}\delta^3_{a_{ij}}}}$$

$$= \sqrt[3]{\frac{\zeta^3_{a_{ij}} + \delta^3_{a_{ij}} - 2\zeta^3_{a_{ij}}\delta^3_{a_{ij}} + \omega^3_{a_{ij}} - \zeta^3_{a_{ij}}\delta^3_{a_{ij}}\omega^3_{a_{ij}} - 2\zeta^3_{a_{ij}}\omega^3_{a_{ij}} - 2\delta^3_{a_{ij}}\omega^3_{a_{ij}}4\zeta^3_{a_{ij}}\delta^3_{a_{ij}}\omega^3_{a_{ij}}}{1-\zeta^3_{a_{ij}}\delta^3_{a_{ij}}}}$$

$$= \sqrt[3]{\frac{\zeta^3_{a_{ij}} + \delta^3_{a_{ij}} + \omega^3_{a_{ij}} - 2\zeta^3_{a_{ij}}\delta^3_{a_{ij}} - 2\zeta^3_{a_{ij}}\omega^3_{a_{ij}} - 2\delta^3_{a_{ij}}\omega^3_{a_{ij}} + 3\zeta^3_{a_{ij}}\delta^3_{a_{ij}}\omega^3_{a_{ij}}}{1 - \zeta^3_{a_{ij}}\omega^3_{a_{ij}}}}.$$

Let $\sqrt[3]{1 - \psi^3_{a_{ij}}\omega^3_{a_{ij}}}$

$$= \sqrt[3]{1 - \frac{\zeta^3_{a_{ij}} + \delta^3_{a_{ij}} - 2\zeta^3_{a_{ij}}\delta^3_{a_{ij}}}{1 - \zeta^3_{a_{ij}}\delta^3_{a_{ij}}}.\omega^3_{a_{ij}}}$$

$$= \sqrt[3]{\frac{1 - \zeta^3_{a_{ij}}\delta^3_{a_{ij}} - \zeta^3_{a_{ij}}\omega^3_{a_{ij}} - \delta^3_{a_{ij}}\omega^3_{a_{ij}} + 2\zeta^3_{a_{ij}}\delta^3_{a_{ij}}\omega^3_{a_{ij}}}{1 - \zeta^3_{a_{ij}}\delta^3_{a_{ij}}}}$$

$$= \sqrt[3]{\frac{\zeta^3_{a_{ij}} + \delta^3_{a_{ij}} + \omega^3_{a_{ij}} - 2\zeta^3_{a_{ij}}\delta^3_{a_{ij}} - 2\zeta^3_{a_{ij}}\omega^3_{a_{ij}} - 2\delta^3_{a_{ij}}\omega^3_{a_{ij}} + 3\zeta^3_{a_{ij}}\delta^3_{a_{ij}}\omega^3_{a_{ij}}}{1 - \zeta^3_{a_{ij}}\delta^3_{a_{ij}} - \zeta^3_{a_{ij}}\omega^3_{a_{ij}} - \delta^3_{a_{ij}}\omega^3_{a_{ij}} + 2\zeta^3_{a_{ij}}\delta^3_{a_{ij}}\omega^3_{a_{ij}}}}. \tag{2.3}$$

If $(B \boxplus_h C) = (\gamma^3_{a_{ij}})$, then

$$A \boxplus_h (B \boxplus_h C) = \left(\sqrt[3]{\frac{\zeta^3_{a_{ij}} + \gamma^3_{a_{ij}} - 2\zeta^3_{a_{ij}}\gamma^3_{a_{ij}}}{1 - \zeta^3_{a_{ij}}\gamma^3_{a_{ij}}}}\right).$$

Let $\sqrt[3]{\zeta^3_{a_{ij}} + \gamma^3_{a_{ij}} - 2\zeta^3_{a_{ij}}\gamma^3_{a_{ij}}}$

$$= \sqrt[3]{\zeta^3_{a_{ij}} + \left(\sqrt[3]{\frac{\delta^3_{a_{ij}} + \omega^3_{a_{ij}} - 2\delta^3_{a_{ij}}\omega^3_{a_{ij}}}{1 - \delta^3_{a_{ij}}\omega^3_{a_{ij}}}}\right)^3 - 2\zeta^3_{a_{ij}}.\left(\sqrt[3]{\frac{\delta^3_{a_{ij}} + \omega^3_{a_{ij}} - 2\delta^3_{a_{ij}}\omega^3_{a_{ij}}}{1 - \delta^3_{a_{ij}}\omega^3_{a_{ij}}}}\right)^3}$$

$$= \sqrt[3]{\frac{\zeta^3_{a_{ij}}(1 - \delta^3_{a_{ij}}\omega^3_{a_{ij}}) + \delta^3_{a_{ij}} + \omega^3_{a_{ij}} - 2\delta^3_{a_{ij}}\omega^3_{a_{ij}} - 2\zeta^3_{a_{ij}}\delta^3_{a_{ij}} - 2\zeta^3_{a_{ij}}\omega^3_{a_{ij}} + 4\zeta^3_{a_{ij}}\delta^3_{a_{ij}}\omega^3_{a_{ij}}}{1 - \delta^3_{a_{ij}}\omega^3_{a_{ij}}}}$$

$$= \sqrt[3]{\frac{\zeta^3_{a_{ij}} + \delta^3_{a_{ij}} + \omega^3_{a_{ij}} - 2\delta^3_{a_{ij}}\omega^3_{a_{ij}} - 2\zeta^3_{a_{ij}}\delta^3_{a_{ij}} - 2\zeta^3_{a_{ij}}\omega^3_{a_{ij}} + \zeta^3_{a_{ij}}\delta^3_{a_{ij}}\omega^3_{a_{ij}}}{1 - \delta^3_{a_{ij}}\omega^3_{a_{ij}}}}.$$

Let $\sqrt[3]{1 - \zeta^3_{a_{ij}}\gamma^3_{a_{ij}}}$

$$= \sqrt[3]{1 - \zeta^3_{a_{ij}}.\frac{\delta^3_{a_{ij}} + \omega^3_{a_{ij}} - 2\delta^3_{a_{ij}}\omega^3_{a_{ij}}}{1 - \delta^3_{a_{ij}}\omega^3_{a_{ij}}}}$$

$$= \sqrt[3]{\frac{1 - \delta^3_{a_{ij}}\omega^3_{a_{ij}} - \zeta^3_{a_{ij}}\delta^3_{a_{ij}} - \zeta^3_{a_{ij}} + 2\zeta^3_{a_{ij}}\delta^3_{a_{ij}}\omega^3_{a_{ij}}}{1 - \delta^3_{a_{ij}}\omega^3_{a_{ij}}}}$$

$$= \sqrt[3]{\frac{\zeta_{a_{ij}}^3 + \delta_{a_{ij}}^3 + \omega_{a_{ij}}^3 - 2\delta_{a_{ij}}^3\omega_{a_{ij}}^3 - 2\zeta_{a_{ij}}^3\delta_{a_{ij}}^3 - 2\zeta_{a_{ij}}^3\omega_{a_{ij}}^3 + 3\zeta_{a_{ij}}^3\delta_{a_{ij}}^3\omega_{a_{ij}}^3}{1 - \delta_{a_{ij}}^3\omega_{a_{ij}}^3 - \zeta_{a_{ij}}^3\delta_{a_{ij}}^3 - \zeta_{a_{ij}}^3\omega_{a_{ij}}^3 + 2\zeta_{a_{ij}}^3\delta_{a_{ij}}^3\omega_{a_{ij}}^3}}. \tag{2.4}$$

From the Equations (2.3) and (2.4), this implies (ii) holds.

Thus $(A \boxplus_h B) \boxplus_h C = A \boxplus_h (B \boxplus_h C)$.

(iv) It can be proved similarly. ■

Definition 2.2.

The $m \times n$ zero FFM O is a FFM all of whose entries are $\langle 0, 1\rangle$. The $m \times n$ universal FFM J is a FFM all of whose entries are $\langle 1, 0\rangle$.

Theorem 2.4.

For any Fermatean fuzzy matrix A,

(i) $A \boxplus_h O = O \boxplus_h A = A$,

(ii) $A \boxdot_h J = J \boxdot_h A = A$,

(iii) $A \boxplus_h J = J$,

(iv) $A \boxdot_h O = O$.

Proof:

In the following, we shall prove $(i), (iii)$ and $(ii), (iv)$ can be proved analogously.

$$(i)\ A \boxplus_h O = \left[\left\langle \sqrt[3]{\frac{\zeta_{a_{ij}}^3 + 0 - 2\zeta_{a_{ij}}^3.0}{1 - \zeta_{a_{ij}}^3.0}}, \sqrt[3]{\frac{\delta_{a_{ij}}^3.1}{\delta_{a_{ij}}^3 + 1 - \delta_{a_{ij}}^3.1}} \right\rangle\right]$$

$$= \left\langle \sqrt[3]{\zeta_{a_{ij}}^3}, \sqrt[3]{\delta_{a_{ij}}^3} \right\rangle \leq \left\langle \zeta_{a_{ij}}, \delta_{a_{ij}} \right\rangle = A.$$

Similarly, we can prove $O \boxplus_h A = A$.

Thus, $A \boxplus_h O = O \boxplus_h A = A$.

$$(iii)\ A \boxplus_h J = \left[\left\langle \sqrt[3]{\frac{\zeta_{a_{ij}}^3 + 1 - 2\zeta_{a_{ij}}^3.1}{1 - \zeta_{a_{ij}}^3.1}}, \sqrt[3]{\frac{\delta_{a_{ij}}^3.0}{\delta_{a_{ij}}^3 + 0 - \delta_{a_{ij}}^3.0}} \right\rangle\right] = \langle 1, 0\rangle = J.$$ ■

The set of all FFMs with respect to the Hamacher sum and Hamacher product forms a commutative monoid.

Definition 2.3.

Let $A = \left[\left\langle \zeta_{a_{ij}}, \delta_{a_{ij}} \right\rangle\right]$ and $B = \left[\left\langle \zeta_{b_{ij}}, \delta_{b_{ij}} \right\rangle\right]$ be two Fermatean fuzzy matrices of the same size, then we have:

(i) $A \vee B = \left[\left\langle \max\left\{\zeta_{a_{ij}}, \zeta_{b_{ij}}\right\} \min\left\{\delta_{a_{ij}}, \delta_{b_{ij}}\right\}\right\rangle\right]$,

(ii) $A \wedge B = \left[\left\langle \min\left\{\zeta_{a_{ij}}, \zeta_{b_{ij}}\right\} \max\left\{\delta_{a_{ij}}, \delta_{b_{ij}}\right\}\right\rangle\right]$,

(iii) $A^C = \left[\left\langle \delta_{a_{ij}}, \zeta_{a_{ij}} \right\rangle\right]$,

(iv) $A^T = \left[\left\langle \zeta_{a_{ji}}, \delta_{a_{ji}} \right\rangle\right]$, where A^T is the transpose of A.

The Hamacher operations do not obey De Morgan's laws over transpose.

Theorem 2.5.

Let A, B be any two Fermatean fuzzy matrices of the same size, then

(i) $(A \boxplus_h B)^T = A^T \boxplus_h B^T$,

(ii) $(A \boxdot_h B)^T = A^T \boxdot_h B^T$.

Proof:

$$(i)\ A^T \boxplus_h B^T = \left[\left\langle \zeta^3_{a_{ji}}, \delta^3_{a_{ji}} \right\rangle \boxplus_h \left\langle \zeta^3_{b_{ji}}, \delta^3_{b_{ji}} \right\rangle\right]$$

$$= \left[\left\langle \sqrt[3]{\frac{\zeta^3_{a_{ji}} + \zeta^3_{b_{ji}} - 2\zeta^3_{a_{ji}}\zeta^3_{b_{ji}}}{1 - \zeta^3_{a_{ji}}\zeta^3_{b_{ji}}}}, \sqrt[3]{\frac{\delta^3_{a_{ji}}\delta^3_{b_{ji}}}{\delta^3_{a_{ji}} + \delta^3_{b_{ji}} - \delta^3_{a_{ji}}\delta^3_{b_{ji}}}} \right\rangle\right]$$

$$= (A \boxplus_h B)^T.$$

(ii) It can be proved similarly. ■

Theorem 2.6.

Let A, B be any two Fermatean fuzzy matrices of the same size; if $A \leq B$, then $A \boxdot_h C \leq B \boxdot_h C$.

Proof:

Let $\zeta_{a_{ij}} \leq \zeta_{b_{ij}}$ and $\delta_{a_{ij}} \geq \delta_{b_{ij}}$, for all i, j.

By using Lemma 2.2,

$$\sqrt[3]{\frac{\zeta^3_{a_{ij}}\zeta^3_{c_{ij}}}{\zeta^3_{a_{ij}} + \zeta^3_{c_{ij}} - \zeta^3_{a_{ij}}\zeta^3_{c_{ij}}}} \leq \sqrt[3]{\frac{\zeta^3_{b_{ij}}\zeta^3_{c_{ij}}}{\zeta^3_{b_{ij}} + \zeta^3_{c_{ij}} - \zeta^3_{b_{ij}}\zeta^3_{c_{ij}}}},$$

and

$$\sqrt[3]{\frac{\delta_{b_{ij}}^3\delta_{c_{ij}}^3}{\delta_{b_{ij}}^3+\delta_{c_{ij}}^3-\delta_{b_{ij}}^3\delta_{c_{ij}}^3}} \geq \sqrt[3]{\frac{\delta_{a_{ij}}^3\delta_{c_{ij}}^3}{\delta_{a_{ij}}^3+\delta_{c_{ij}}^3-\delta_{a_{ij}}^3\delta_{c_{ij}}^3}}.$$

Therefore, $A \boxdot_h C \leq B \boxdot_h C$. ■

Theorem 2.7.

Let A, B be any two Fermatean fuzzy matrices of the same size; if $A \leq B$, then $A \boxplus_h C \leq B \boxplus_h C$.

Proof:

Let $\zeta_{a_{ij}} \leq \zeta_{b_{ij}}$ and $\delta_{a_{ij}} \geq \delta_{b_{ij}}$, for all i, j.

By using Lemma 2.2,

$$\sqrt[3]{\frac{\zeta_{a_{ij}}^3+\zeta_{c_{ij}}^3-2\zeta_{a_{ij}}^3\zeta_{c_{ij}}^3}{1-\zeta_{a_{ij}}^3\zeta_{c_{ij}}^3}} \leq \sqrt[3]{\frac{\zeta_{b_{ij}}^3+\zeta_{c_{ij}}^3-2\zeta_{b_{ij}}^3\zeta_{c_{ij}}^3}{1-\zeta_{b_{ij}}^3\zeta_{c_{ij}}^3}},$$

and

$$\sqrt[3]{\frac{\delta_{b_{ij}}^3+\delta_{c_{ij}}^3-2\delta_{b_{ij}}^3\delta_{c_{ij}}^3}{1-\delta_{b_{ij}}^3\delta_{c_{ij}}^3}} \geq \sqrt[3]{\frac{\delta_{a_{ij}}^3+\delta_{c_{ij}}^3-2\delta_{a_{ij}}^3\delta_{c_{ij}}^3}{1-\delta_{a_{ij}}^3\delta_{c_{ij}}^3}}.$$

Therefore, $A \boxplus_h C \leq B \boxplus_h C$. ■

Theorem 2.8.

Let A, B be any two Fermatean fuzzy matrices of the same size, then

(i) $(A \wedge B) \boxplus_h (A \vee B) = A \boxplus_h B$,

(ii) $(A \wedge B) \boxdot_h (A \vee B) = A \boxdot_h B$.

Proof:

(i) $(A \wedge B) \boxplus_h (A \vee B)$

$$= \left[\left\langle \min\left\{\zeta_{a_{ij}}, \zeta_{b_{ij}}\right\}, \max\left\{\delta_{a_{ij}}, \delta_{b_{ij}}\right\}\right\rangle \boxplus_h \left\langle \max\left\{\zeta_{a_{ij}}, \zeta_{b_{ij}}\right\}, \min\left\{\delta_{a_{ij}}, \delta_{b_{ij}}\right\}\right\rangle\right]$$

$$= \left[\left\langle \sqrt[3]{\frac{\min\left\{\zeta_{a_{ij}}^3, \zeta_{b_{ij}}^3\right\} + \max\left\{\zeta_{a_{ij}}^3, \zeta_{b_{ij}}^3\right\} - 2\min\left\{\zeta_{a_{ij}}^3, \zeta_{b_{ij}}^3\right\}\max\left\{\zeta_{a_{ij}}^3, \zeta_{b_{ij}}^3\right\}}{1-\min\left\{\zeta_{a_{ij}}^3, \zeta_{b_{ij}}^3\right\}\max\left\{\zeta_{a_{ij}}^3, \zeta_{b_{ij}}^3\right\}}},\right.\right.$$

$$\sqrt[3]{\frac{\max\left\{\delta^3_{a_{ij}},\delta^3_{b_{ij}}\right\}\min\left\{\delta^3_{a_{ij}},\delta^3_{b_{ij}}\right\}}{\max\left\{\delta^3_{a_{ij}},\delta^3_{b_{ij}}\right\}+\min\left\{\delta^3_{a_{ij}},\delta^3_{b_{ij}}\right\}-\max\left\{\delta^3_{a_{ij}},\delta^3_{b_{ij}}\right\}\min\left\{\delta^3_{a_{ij}},\delta^3_{b_{ij}}\right\}}}\Bigg\rangle\Bigg]$$

$$=\left[\left\langle\sqrt[3]{\frac{\zeta^3_{a_{ij}}+\zeta^3_{b_{ij}}-2\zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}{1-\zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}},\sqrt[3]{\frac{\delta^3_{a_{ij}}\delta^3_{b_{ij}}}{\delta^3_{a_{ij}}+\delta^3_{b_{ij}}-\delta^3_{a_{ij}}\delta^3_{b_{ij}}}}\right\rangle\right]$$

$$= A \boxplus_h B.$$

(ii) It can be proved similarly. ■

3. On Complement of Fermatean Fuzzy Matrices

In this section, the complement of a Fermatean fuzzy matrix is used to analyze the complementing nature of any system. Using the following results, we can study the complementing nature of a system with the help of the original Fermatean fuzzy matrix. The operator complement obey De Morgan's law for the operations $\boxplus_h$ and $\boxdot_h$. This is established in the following theorem.

Theorem 3.1.

Let A, B be any two Fermatean fuzzy matrices of the same size, then

(i) $(A \boxplus_h B)^C = A^C \boxdot_h B^C$,

(ii) $(A \boxdot_h B)^C = A^C \boxplus_h B^C$,

(iii) $(A \boxplus_h B)^C \leq A^C \boxplus_h B^C$,

(iv) $(A \boxdot_h B)^C \geq A^C \boxdot_h B^C$,

(v) $(A^C \boxdot_h B^C)^C = A \boxplus_h B$,

(vi) $(A^C \boxplus_h B^C)^C = A \boxdot_h B$.

Proof:

In the following, we shall prove $(i), (ii), (iii)$ and $(iv), (v), (vi)$ can be proved analogously.

$$(i)\ A^C \boxdot_h B^C = \left[\left\langle\sqrt[3]{\frac{\delta^3_{a_{ij}}\delta^3_{b_{ij}}}{\delta^3_{a_{ij}}+\delta^3_{b_{ij}}-\delta^3_{a_{ij}}\delta^3_{b_{ij}}}},\sqrt[3]{\frac{\zeta^3_{a_{ij}}+\zeta^3_{b_{ij}}-2\zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}{1-\zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}}\right\rangle\right]$$

$$= (A \boxplus_h B)^C.$$

$$(ii)\ A^C \boxplus_h B^C = \left[\left\langle\sqrt[3]{\frac{\delta^3_{a_{ij}}+\delta^3_{b_{ij}}-2\delta^3_{a_{ij}}\delta^3_{b_{ij}}}{1-\delta^3_{a_{ij}}\delta^3_{b_{ij}}}},\sqrt[3]{\frac{\zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}{\zeta^3_{a_{ij}}+\zeta^3_{b_{ij}}-\zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}}\right\rangle\right]$$

$$= (A \boxdot_h B)^C.$$

$$(iii)\ (A \boxplus_h B)^C = \left[\left\langle \sqrt[3]{\frac{\delta^3_{a_{ij}}\delta^3_{b_{ij}}}{\delta^3_{a_{ij}}+\delta^3_{b_{ij}}-\delta^3_{a_{ij}}\delta^3_{b_{ij}}}}, \sqrt[3]{\frac{\zeta^3_{a_{ij}}+\zeta^3_{b_{ij}}-2\zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}{1-\zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}} \right\rangle\right]$$

$$A^C \boxplus_h B^C = \left[\left\langle \sqrt[3]{\frac{\delta^3_{a_{ij}}+\delta^3_{b_{ij}}-2\delta^3_{a_{ij}}\delta^3_{b_{ij}}}{1-\delta^3_{a_{ij}}\delta^3_{b_{ij}}}}, \sqrt[3]{\frac{\zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}{\zeta^3_{a_{ij}}+\zeta^3_{b_{ij}}-\zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}} \right\rangle\right].$$

By Lemma 2.1,

$$\sqrt[3]{\frac{\delta^3_{a_{ij}}\delta^3_{b_{ij}}}{\delta^3_{a_{ij}}+\delta^3_{b_{ij}}-\delta^3_{a_{ij}}\delta^3_{b_{ij}}}} \leq \sqrt[3]{\frac{\delta^3_{a_{ij}}+\delta^3_{b_{ij}}-2\delta^3_{a_{ij}}\delta^3_{b_{ij}}}{1-\delta^3_{a_{ij}}\delta^3_{b_{ij}}}},$$

and

$$\sqrt[3]{\frac{\zeta^3_{a_{ij}}+\zeta^3_{b_{ij}}-2\zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}{1-\zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}} \geq \sqrt[3]{\frac{\zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}{\zeta^3_{a_{ij}}+\zeta^3_{b_{ij}}-\zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}}, \quad \text{for all } i, j.$$

Hence, $(A \boxplus_h B)^C \leq A^C \boxplus_h B^C$. ■

4. Scalar Multiplication and Exponentiation Operations of Fermatean Fuzzy Matrices

We defined the following operations over Hamacher operations on FFMs. In this section, we construct Hamacher scalar multiplication $(n._h A)$ and Hamacher exponentiation $(A^{\wedge_h n})$ operations on Fermatean fuzzy matrix A and investigate their algebraic properties.

Based on Equation (1), further indicated as the following operations.

Theorem 4.1.

If n is any positive integer and A is a FFM, then the Hamacher scalar multiplication operation $(._h)$ is

$$n._h A = \underbrace{A \boxplus_h ... \boxplus_h A}_{n} = \left[\left\langle \sqrt[3]{\frac{n\zeta^3_{a_{ij}}}{1+(n-1)\zeta^3_{a_{ij}}}}, \sqrt[3]{\frac{\delta^3_{a_{ij}}}{n-(n-1)\delta^3_{a_{ij}}}} \right\rangle\right]. \tag{3}$$

Proof:

Mathematical induction can be used to prove that the above equation (3) holds for all positive integer n. The equation (3) is called *P(n)*.

$$A._h A = \left[\left\langle \sqrt[3]{\frac{\zeta^3_{a_{ij}}+\zeta^3_{a_{ij}}-2\zeta^3_{a_{ij}}\zeta^3_{a_{ij}}}{1-\zeta^3_{a_{ij}}\zeta^3_{a_{ij}}}}, \sqrt[3]{\frac{\delta^3_{a_{ij}}\delta^3_{a_{ij}}}{\delta^3_{a_{ij}}+\delta^3_{a_{ij}}-\delta^3_{a_{ij}}\delta^3_{a_{ij}}}} \right\rangle\right]$$

$$= \left[\left\langle \sqrt[3]{\frac{2\zeta_{a_{ij}}^3 - 2\zeta_{a_{ij}}^6}{1-\zeta_{a_{ij}}^6}}, \sqrt[3]{\frac{\delta_{a_{ij}}^6}{2\delta_{a_{ij}}^3 - \delta_{a_{ij}}^6}} \right\rangle\right]$$

$$= \left[\left\langle \sqrt[3]{\frac{2\zeta_{a_{ij}}^3(1-\zeta_{a_{ij}}^3)}{1-\zeta_{a_{ij}}^6}}, \sqrt[3]{\frac{\delta_{a_{ij}}^6}{\delta_{a_{ij}}^3(2-\delta_{a_{ij}}^3)}} \right\rangle\right]$$

$$= \left[\left\langle \sqrt[3]{\frac{2\zeta_{a_{ij}}^3(1-\zeta_{a_{ij}}^3)}{(1-\zeta_{a_{ij}}^3)(1+\zeta_{a_{ij}}^3)}}, \sqrt[3]{\frac{\delta_{a_{ij}}^3}{(2-\delta_{a_{ij}}^3)}} \right\rangle\right]$$

$$= \left[\left\langle \sqrt[3]{\frac{2\zeta_{a_{ij}}^3}{1+\zeta_{a_{ij}}^3}}, \sqrt[3]{\frac{\delta_{a_{ij}}^3}{(2-\delta_{a_{ij}}^3)}} \right\rangle\right],$$

$$2._h A = \left[\left\langle \sqrt[3]{\frac{2\zeta_{a_{ij}}^3}{1+(2-1)\zeta_{a_{ij}}^3}}, \sqrt[3]{\frac{\delta_{a_{ij}}^3}{2-(2-1)\delta_{a_{ij}}^3}} \right\rangle\right], \quad \text{since } \zeta_{a_{ij}}^3 = (2-1)\zeta_{a_{ij}}^3,$$

$$n._h A = \left[\left\langle \sqrt[3]{\frac{n\zeta_{a_{ij}}^3}{1+(n-1)\zeta_{a_{ij}}^3}}, \sqrt[3]{\frac{\delta_{a_{ij}}^3}{n-(n-1)\delta_{a_{ij}}^3}} \right\rangle\right].$$

P(n) holds.

Suppose that Equation (3) holds for $n = m$,

i.e., $m._h A = \underbrace{A \boxplus_h \ldots \boxplus_h A}_{m} = \left[\left\langle \sqrt[3]{\frac{m\zeta_{a_{ij}}^3}{1+(m-1)\zeta_{a_{ij}}^3}}, \sqrt[3]{\frac{\delta_{a_{ij}}^3}{m-(m-1)\delta_{a_{ij}}^3}} \right\rangle\right].$

Then,

$$(m+1)._h A = ((m._h A) \boxplus_h A)$$

$$= \left[\left\langle \sqrt[3]{\frac{\zeta_{a_{ij}}^3(m+1)(1-\zeta_{a_{ij}}^3)}{(1+m\zeta_{a_{ij}}^3)(1-\zeta_{a_{ij}}^3)}}, \sqrt[3]{\frac{(\delta_{a_{ij}}^3)^2}{\delta_{a_{ij}}^3(m+1-m\delta_{a_{ij}}^3)}} \right\rangle\right]$$

$$= \left[\left\langle \sqrt[3]{\frac{(m+1)\zeta_{a_{ij}}^3}{1+m\zeta_{a_{ij}}^3}}, \sqrt[3]{\frac{\delta_{a_{ij}}^3}{m+1-m\delta_{a_{ij}}^3}} \right\rangle\right]$$

$$= \left[\left\langle \sqrt[3]{\frac{(m+1)\zeta_{a_{ij}}^3}{1+[(m+1)-1]\zeta_{a_{ij}}^3}}, \sqrt[3]{\frac{\delta_{a_{ij}}^3}{m+1-[(m+1)-1]\delta_{a_{ij}}^3}} \right\rangle\right].$$

So, when $n = m + 1$,

$$n \cdot_h A = \underbrace{A \boxplus_h \ldots \boxplus_h A}_{n} = \left[\left\langle \sqrt[3]{\frac{n\zeta^3_{a_{ij}}}{1+(n-1)\zeta^3_{a_{ij}}}} \sqrt[3]{\frac{\delta^3_{a_{ij}}}{n-(n-1)\delta^3_{a_{ij}}}} \right\rangle \right],$$

also holds.

Use the induction hypothesis that *P(n)* holds for any positive integer n. ■

Based on Equation (2), further indicated as the following operations.

Theorem 4.2.

If n is any positive integer and A is a FFM, then the Hamacher exponentiation operation $(\wedge_h)$ is

$$A^{\wedge_h n} = \overbrace{A \boxdot_h \ldots \boxdot_h A}^{n} = \left[\left\langle \sqrt[3]{\frac{\zeta^3_{a_{ij}}}{n-(n-1)\zeta^3_{a_{ij}}}}, \sqrt[3]{\frac{n\delta^3_{a_{ij}}}{1+(n-1)\delta^3_{a_{ij}}}} \right\rangle \right]. \tag{4}$$

Proof:

Mathematical induction can be used to prove that the above Equation (4) holds for all positive integer n. The equation (4) is called *P(n)*.

$$A^{\wedge_h A} = \left[\left\langle \sqrt[3]{\frac{\zeta^3_{a_{ij}}}{\zeta^3_{a_{ij}} + \zeta^3_{a_{ij}} - \zeta^3_{a_{ij}}\zeta^3_{a_{ij}}}}, \sqrt[3]{\frac{\delta^3_{a_{ij}} + \delta^3_{a_{ij}} - 2\delta^3_{a_{ij}}\delta^3_{a_{ij}}}{1-\delta^3_{a_{ij}}\delta^3_{a_{ij}}}} \right\rangle \right]$$

$$= \left[\left\langle \sqrt[3]{\frac{\zeta^3_{a_{ij}}}{2\zeta^3_{a_{ij}} - \zeta^6_{a_{ij}}}}, \sqrt[3]{\frac{2\delta^3_{a_{ij}} - 2\delta^6_{a_{ij}}}{1-\delta^6_{a_{ij}}}} \right\rangle \right]$$

$$= \left[\left\langle \sqrt[3]{\frac{\zeta^3_{a_{ij}}}{\zeta^3_{a_{ij}}(2-\zeta^3_{a_{ij}})}}, \sqrt[3]{\frac{2\delta^3_{a_{ij}}(1-\delta^3_{a_{ij}})}{1-\delta^6_{a_{ij}}}} \right\rangle \right]$$

$$= \left[\left\langle \sqrt[3]{\frac{\zeta^3_{a_{ij}}}{(2-\zeta^3_{a_{ij}})}}, \sqrt[3]{\frac{2\delta^3_{a_{ij}}(1-\delta^3_{a_{ij}})}{(1-\delta^3_{a_{ij}})(1+\delta^3_{a_{ij}})}} \right\rangle \right]$$

$$= \left[\left\langle \sqrt[3]{\frac{\zeta^3_{a_{ij}}}{(2-\zeta^3_{a_{ij}})}}, \sqrt[3]{\frac{2\delta^3_{a_{ij}}}{1+\delta^3_{a_{ij}}}} \right\rangle \right],$$

$$A^{\wedge_h 3} = \left[\left\langle \sqrt[3]{\frac{\zeta^3_{a_{ij}}}{2-(2-1)\zeta^3_{a_{ij}}}}, \sqrt[3]{\frac{2\delta^3_{a_{ij}}}{1+(2-1)\delta^3_{a_{ij}}}} \right\rangle \right], \quad \text{since } \zeta^3_{a_{ij}} = (2-1)\zeta^3_{a_{ij}},$$

$$A^{\wedge_h n} = \left[\left\langle \sqrt[3]{\frac{\zeta^3_{a_{ij}}}{n-(n-1)\zeta^3_{a_{ij}}}}, \sqrt[3]{\frac{n\delta^3_{a_{ij}}}{1+(n-1)\delta^3_{a_{ij}}}} \right\rangle \right].$$

P(n) holds.

Suppose that Equation (4) holds for $n = m$,

i.e., $$A^{\wedge_h m} = \overbrace{A \boxdot_h \ldots \boxdot_h A}^{m} = \left[\left\langle \sqrt[3]{\frac{\zeta^3_{a_{ij}}}{m-(m-1)\zeta^3_{a_{ij}}}}, \sqrt[3]{\frac{m\delta^3_{a_{ij}}}{1+(m-1)\delta^3_{a_{ij}}}} \right\rangle\right].$$

So, when $n = m+1$,

$$A^{\wedge_h m+1} = \left[\left\langle \sqrt[3]{\frac{{\zeta_{a_{ij}}}^2}{m+1-[(m+1)-1]{\zeta_{a_{ij}}}^2}}, \sqrt[3]{\frac{(m+1)\delta^3_{a_{ij}}}{1+[(m+1)-1]\delta^3_{a_{ij}}}} \right\rangle\right],$$

$$A^{\wedge_h n} = \overbrace{A \boxdot_h \ldots \boxdot_h A}^{n} = \left[\left\langle \sqrt[3]{\frac{\zeta^3_{a_{ij}}}{n-(n-1)\zeta^3_{a_{ij}}}}, \sqrt[3]{\frac{n\delta^3_{a_{ij}}}{1+(n-1)\delta^3_{a_{ij}}}} \right\rangle\right],$$

also holds.

Use the induction hypothesis that *P(n)* holds for any positive integer n. ■

Next, we prove the result of $(n._h A)$ and $(A^{\wedge_h n})$ are also FFMs.

Theorem 4.3.

For any FFM A and for any positive integer n, then $(n._h A)$ and $(A^{\wedge_h n})$ are FFMs.

Proof:

Since $0 \leq \zeta^3_{a_{ij}} \leq 1$, $0 \leq \delta^3_{a_{ij}} \leq 1$, $0 \leq \zeta^3_{a_{ij}} + \delta^3_{a_{ij}} \leq 1$, and $n > 1$, we have:

$$(n-1)\zeta^3_{a_{ij}} > -1, \quad 1+(n-1)\zeta^3_{a_{ij}} > 0,$$

$$n-(n-1)\delta^3_{a_{ij}} = (1-\delta^3_{a_{ij}})n + \delta^3_{a_{ij}} > \delta^3_{a_{ij}} \geq 0.$$

Then, it is easy to get that $\sqrt[3]{\frac{n\zeta^3_{a_{ij}}}{1+(n-1)\zeta^3_{a_{ij}}}} \geq 0$, $\sqrt[3]{\frac{\delta^3_{a_{ij}}}{n-(n-1)\delta^3_{a_{ij}}}} \geq 0$.

Considering that $1+(n-1)\zeta^3_{a_{ij}} = n\zeta^3_{a_{ij}} + 1 - \zeta^3_{a_{ij}} \geq n\zeta^3_{a_{ij}}$ and

$$n-(n-1)\delta^3_{a_{ij}} = \delta^3_{a_{ij}} + n(1-\delta^3_{a_{ij}}) \geq \delta^3_{a_{ij}}.$$

We get:

$$\sqrt[3]{\frac{n\zeta^3_{a_{ij}}}{1+(n-1)\zeta^3_{a_{ij}}}} \leq 1, \quad \sqrt[3]{\frac{\delta^3_{a_{ij}}}{n-(n-1)\delta^3_{a_{ij}}}} \leq 1.$$

For $\zeta^3_{a_{ij}} + \delta^3_{a_{ij}} \leq 1, \quad 0 \leq \delta^3_{a_{ij}} \leq 1 - \zeta^3_{a_{ij}}$,

$$\Rightarrow \sqrt[3]{\frac{n\zeta^3_{a_{ij}}}{1+(n-1)\zeta^3_{a_{ij}}}} + \sqrt[3]{\frac{\delta^3_{a_{ij}}}{n-(n-1)\delta^3_{a_{ij}}}},$$

$$= \sqrt[3]{\frac{n\zeta^3_{a_{ij}}}{1+(n-1)\zeta^3_{a_{ij}}}} + \sqrt[3]{\frac{1}{\frac{n}{\delta^3_{a_{ij}}}-(n-1)}},$$

$$\leq \sqrt[3]{\frac{n\zeta^3_{a_{ij}}}{1+(n-1)\zeta^3_{a_{ij}}}} + \sqrt[3]{\frac{1}{\frac{n}{1-\zeta^3_{a_{ij}}}-(n-1)}} = 1.$$

Thus,

$$0 \leq \sqrt[3]{\frac{n\zeta^3_{a_{ij}}}{1+(n-1)\zeta^3_{a_{ij}}}} \leq 1, \quad 0 \leq \sqrt[3]{\frac{\delta^3_{a_{ij}}}{n-(n-1)\delta^3_{a_{ij}}}} \leq 1,$$

$$\Rightarrow \sqrt[3]{\frac{n\zeta^3_{a_{ij}}}{1+(n-1)\zeta^3_{a_{ij}}}} + \sqrt[3]{\frac{\delta^3_{a_{ij}}}{n-(n-1)\delta^3_{a_{ij}}}} \leq 1.$$

Similarly, we can also get

$$0 \leq \sqrt[3]{\frac{\zeta^3_{a_{ij}}}{n-(n-1)\zeta^3_{a_{ij}}}} \leq 1, \quad 0 \leq \sqrt[3]{\frac{n\delta^3_{a_{ij}}}{1+(n-1)\delta^3_{a_{ij}}}} \leq 1,$$

$$\Rightarrow \sqrt[3]{\frac{\zeta^3_{a_{ij}}}{n-(n-1)\zeta^3_{a_{ij}}}} + \sqrt[3]{\frac{n\delta^3_{a_{ij}}}{1+(n-1)\delta^3_{a_{ij}}}} \leq 1.$$

Hence, $(n._h)A$ and $(A^{\wedge_h n})$ are FFMs. ■

Theorem 4.4.

Let A, B be any two Fermatean fuzzy matrices of the same size and for any positive integers n, n_1, n_2.

(i) $\quad (n_1._h A) \boxplus_h (n_2._h A) = (n_1+n_2)._h A$,

(ii) $\quad (n._h A) \boxplus_h (n._h B) = n._h(A \boxplus_h B)$,

(iii) $\quad A^{\wedge_h n_1} \boxdot_h A^{\wedge_h n_2} = A^{\wedge_h (n_1+n_2)}$,

(iv) $\quad A^{\wedge_h n} \boxdot_h B^{\wedge_h n} = (A \boxdot_h B)^{\wedge_h n}$,

(v) $\quad n_2._h(n_1._h A) = (n_1 n_2)._h A$,

$(vi)\quad (A^{\wedge_h n_1})^{\wedge_h n_2} = A^{\wedge_h (n_1 n_2)}$.

Proof:

In the following, we shall prove $(i), (ii), (v)$ and $(iii), (iv), (vi)$ can be proved analogously.

(i) By Equations (3) and (4), we have:

$$n_1 \cdot_h A = \left[\left\langle \sqrt[3]{\frac{n_1\zeta_{a_{ij}}^3}{1+(n_1-1)\zeta_{a_{ij}}^3}}, \sqrt[3]{\frac{\delta_{a_{ij}}^3}{n_1-(n_1-1)\delta_{a_{ij}}^3}} \right\rangle\right] = \left[\left\langle \zeta_{b_{ij}}, \delta_{b_{ij}} \right\rangle\right],$$

$$n_2 \cdot_h A = \left[\left\langle \sqrt[3]{\frac{n_2\zeta_{a_{ij}}^3}{1+(n_2-1)\zeta_{a_{ij}}^3}}, \sqrt[3]{\frac{\delta_{a_{ij}}^3}{n_2-(n_2-1)\delta_{a_{ij}}^3}} \right\rangle\right] = \left[\left\langle \zeta_{c_{ij}}, \delta_{c_{ij}} \right\rangle\right],$$

$$B \boxplus_h C = (n_1 \cdot_h A) \boxplus_h (n_2 \cdot_h A),$$

$$B \boxplus_h C = \left[\left\langle \sqrt[3]{\frac{\zeta_{b_{ij}}^3+\zeta_{c_{ij}}^3-2\zeta_{b_{ij}}^3\zeta_{c_{ij}}^3}{1-\zeta_{b_{ij}}^3\zeta_{c_{ij}}^3}}, \sqrt[3]{\frac{\delta_{b_{ij}}^3\delta_{c_{ij}}^3}{\delta_{b_{ij}}^3+\delta_{c_{ij}}^3-\delta_{b_{ij}}^3\delta_{c_{ij}}^3}} \right\rangle\right].$$

We can further get:

$$\text{Let } \sqrt[3]{\frac{\zeta_{b_{ij}}^3+\zeta_{c_{ij}}^3-2\zeta_{b_{ij}}^3\zeta_{c_{ij}}^3}{1-\zeta_{b_{ij}}^3\zeta_{c_{ij}}^3}}$$

$$= \sqrt[3]{\frac{\frac{n_1\zeta_{a_{ij}}^3}{1+(n_1-1)\zeta_{a_{ij}}^3}+\frac{n_2\zeta_{a_{ij}}^3}{1+(n_2-1)\zeta_{a_{ij}}^3}-2\frac{n_1\zeta_{a_{ij}}^3}{1+(n_1-1)\zeta_{a_{ij}}^3}\frac{n_2\zeta_{a_{ij}}^3}{1+(n_2-1)\zeta_{a_{ij}}^3}}{1-\frac{n_1\zeta_{a_{ij}}^3}{1+(n_1-1)\zeta_{a_{ij}}^3}\frac{n_2\zeta_{a_{ij}}^3}{1+(n_2-1)\zeta_{a_{ij}}^3}}}$$

$$= \sqrt[3]{\frac{n_1\zeta_{a_{ij}}^3(1+n_2\zeta_{a_{ij}}^3-\zeta_{a_{ij}}^3)+n_2\zeta_{a_{ij}}^3(1+n_1\zeta_{a_{ij}}^3-\zeta_{a_{ij}}^3)-2n_1\zeta_{a_{ij}}^3n_2\zeta_{a_{ij}}^3}{(1+n_1\zeta_{a_{ij}}^3-\zeta_{a_{ij}}^3)(1+n_2\zeta_{a_{ij}}^3-\zeta_{a_{ij}}^3)-n_1\zeta_{a_{ij}}^3n_2\zeta_{a_{ij}}^3}}$$

$$= \sqrt[3]{\frac{(n_1+n_2)\zeta_{a_{ij}}^3-(n_1+n_2)a_{ij}^4}{1+(n_1+n_2-2)\zeta_{a_{ij}}^3-(n_1+n_2-1)a_{ij}^4}}$$

$$= \sqrt[3]{\frac{(n_1+n_2)\zeta_{a_{ij}}^3(1-\zeta_{a_{ij}}^3)}{(1+(n_1+n_2-1)\zeta_{a_{ij}}^3)(1-\zeta_{a_{ij}}^3)}} = \sqrt[3]{\frac{(n_1+n_2)\zeta_{a_{ij}}^3}{1+(n_1+n_2-1)\zeta_{a_{ij}}^3}} = n_1 \cdot_h A,$$

and

$$\text{let } \sqrt[3]{\frac{\delta_{b_{ij}}^3\delta_{c_{ij}}^3}{\delta_{b_{ij}}^3+\delta_{c_{ij}}^3-\delta_{b_{ij}}^3\delta_{c_{ij}}^3}}$$

$$= \sqrt[3]{\frac{\frac{\delta^3_{a_{ij}}}{n_1-(n_1-1)\delta^3_{a_{ij}}}\frac{\delta^3_{a_{ij}}}{n_2-(n_2-1)\delta^3_{a_{ij}}}}{\frac{\delta^3_{a_{ij}}}{n_1-(n_1-1)\delta^3_{a_{ij}}}+\frac{\delta^3_{a_{ij}}}{n_2-(n_2-1)\delta^3_{a_{ij}}}-\frac{\delta^3_{a_{ij}}}{n_1-(n_1-1)\delta^3_{a_{ij}}}\frac{\delta^3_{a_{ij}}}{n_2-(n_2-1)\delta^3_{a_{ij}}}}}$$

$$= \sqrt[3]{\frac{\delta^3_{a_{ij}}\delta^3_{a_{ij}}}{\delta^3_{a_{ij}}(n_2+\delta^3_{a_{ij}}-n_2\delta^3_{a_{ij}})+\delta^3_{a_{ij}}(n_1+\delta^3_{a_{ij}}-n_1\delta^3_{a_{ij}})-\delta^3_{a_{ij}}\delta^3_{a_{ij}}}}$$

$$= \sqrt[3]{\frac{\delta^3_{a_{ij}}}{(n_2+\delta^3_{a_{ij}}-n_2\delta^3_{a_{ij}})+(n_1+\delta^3_{a_{ij}}-n_1\delta^3_{a_{ij}})-\delta^3_{a_{ij}}}} = \sqrt[3]{\frac{\delta^3_{a_{ij}}}{(n_1+n_2)-(n_1+n_2-1)\delta^3_{a_{ij}}}}$$

$$= n_2 \cdot_h A.$$

Since $(n_1+n_2)\cdot_h A = \left[\left\langle \sqrt[3]{\frac{(n_1+n_2)\zeta^3_{a_{ij}}}{1+(n_1+n_2-1)\zeta^3_{a_{ij}}}}, \sqrt[3]{\frac{\delta^3_{a_{ij}}}{(n_1+n_2)-(n_1+n_2-1)\delta^3_{a_{ij}}}} \right\rangle\right]$,

we can finally get $(n_1 \cdot_h A) \boxplus_h (n_2 \cdot_h A) = (n_1+n_2)\cdot_h A$.

(ii) By Equations (3) and (4), we have:

$$n\cdot_h A = \left[\left\langle \sqrt[3]{\frac{n\zeta^3_{a_{ij}}}{1+(n-1)\zeta^3_{a_{ij}}}}, \sqrt[3]{\frac{\delta^3_{a_{ij}}}{n-(n-1)\delta^3_{a_{ij}}}} \right\rangle\right] = \left[\left\langle \zeta_{b_{ij}}, \delta_{b_{ij}} \right\rangle\right],$$

$$n\cdot_h B = \left[\left\langle \sqrt[3]{\frac{n\zeta^3_{b_{ij}}}{1+(n-1)\zeta^3_{b_{ij}}}}, \sqrt[3]{\frac{\delta^3_{b_{ij}}}{n-(n-1)\delta^3_{b_{ij}}}} \right\rangle\right] = \left[\left\langle \zeta_{c_{ij}}, \delta_{c_{ij}} \right\rangle\right],$$

$$B \boxplus_h C = (n\cdot_h A) \boxplus_h (n\cdot_h B),$$

$$B \boxplus_h C = \left[\left\langle \sqrt[3]{\frac{\zeta^3_{b_{ij}}+\zeta^3_{c_{ij}}-2\zeta^3_{b_{ij}}\zeta^3_{c_{ij}}}{1-\zeta^3_{b_{ij}}\zeta^3_{c_{ij}}}}, \sqrt[3]{\frac{\delta^3_{b_{ij}}\delta^3_{c_{ij}}}{\delta^3_{b_{ij}}+\delta^3_{c_{ij}}-\delta^3_{b_{ij}}\delta^3_{c_{ij}}}} \right\rangle\right].$$

We can further get the following.

Let $\sqrt[3]{\frac{\zeta^3_{b_{ij}}+\zeta^3_{c_{ij}}-2\zeta^3_{b_{ij}}\zeta^3_{c_{ij}}}{1-\zeta^3_{b_{ij}}\zeta^3_{c_{ij}}}}$

$$= \sqrt[3]{\frac{\frac{n\zeta^3_{a_{ij}}}{1+(n-1)\zeta^3_{a_{ij}}}+\frac{n\zeta^3_{b_{ij}}}{1+(n-1)\zeta^3_{b_{ij}}}-2\frac{n\zeta^3_{a_{ij}}}{1+(n-1)\zeta^3_{a_{ij}}}\frac{n\zeta^3_{b_{ij}}}{1+(n-1)\zeta^3_{b_{ij}}}}{1-\frac{n\zeta^3_{a_{ij}}}{1+(n-1)\zeta^3_{a_{ij}}}\frac{n\zeta^3_{b_{ij}}}{1+(n-1)\zeta^3_{b_{ij}}}}}$$

$$= \sqrt[3]{\frac{n\zeta_{a_{ij}}^3(1+n\zeta_{b_{ij}}^3-\zeta_{b_{ij}}^3)+n\zeta_{b_{ij}}^3(1+n\zeta_{a_{ij}}^3-\zeta_{a_{ij}}^3)-2n\zeta_{a_{ij}}^3n\zeta_{b_{ij}}^3}{(1+n\zeta_{b_{ij}}^3-\zeta_{b_{ij}}^3)(1+n\zeta_{a_{ij}}^3-\zeta_{a_{ij}}^3)-n\zeta_{a_{ij}}^3n\zeta_{b_{ij}}^3}}$$

$$= \sqrt[3]{\frac{n\zeta_{a_{ij}}^3+n\zeta_{b_{ij}}^3-2n\zeta_{a_{ij}}^3\zeta_{b_{ij}}^3}{1+(n-1)(\zeta_{a_{ij}}^3+\zeta_{b_{ij}}^3)-(2n-1)\zeta_{a_{ij}}^3\zeta_{b_{ij}}^3}} = (n._h A),$$

and

$$\text{let } \sqrt[3]{\frac{\delta_{b_{ij}}^3\delta_{c_{ij}}^3}{\delta_{b_{ij}}^3+\delta_{c_{ij}}^3-\delta_{b_{ij}}^3\delta_{c_{ij}}^3}}$$

$$= \sqrt[3]{\frac{\frac{\delta_{a_{ij}}^3}{n-(n-1)\delta_{a_{ij}}^3}\frac{\delta_{b_{ij}}^3}{n-(n-1)\delta_{b_{ij}}^3}}{\frac{\delta_{a_{ij}}^3}{n-(n-1)\delta_{a_{ij}}^3}+\frac{\delta_{b_{ij}}^3}{n-(n-1)\delta_{b_{ij}}^3}-\frac{\delta_{a_{ij}}^3}{n-(n-1)\delta_{a_{ij}}^3}\frac{\delta_{b_{ij}}^3}{n-(n-1)\delta_{b_{ij}}^3}}}$$

$$= \sqrt[3]{\frac{\delta_{a_{ij}}^3\delta_{b_{ij}}^3}{\delta_{a_{ij}}^3(n+\delta_{b_{ij}}^3-n\delta_{b_{ij}}^3)+\delta_{b_{ij}}^3(n+\delta_{a_{ij}}^3-n\delta_{a_{ij}}^3)-\delta_{a_{ij}}^3\delta_{b_{ij}}^3}}$$

$$\sqrt[3]{\frac{\delta_{a_{ij}}^3\delta_{b_{ij}}^3}{n(\delta_{a_{ij}}^3+\delta_{b_{ij}}^3)-(2n-1)\delta_{a_{ij}}^3\delta_{b_{ij}}^3}} = (n._h B).$$

Thus,

$$n._h(A \boxplus_h B)$$

$$= \left[\left\langle \sqrt[3]{\frac{n\frac{\zeta_{a_{ij}}^3+\zeta_{b_{ij}}^3-2\zeta_{a_{ij}}^3\zeta_{b_{ij}}^3}{1-\zeta_{a_{ij}}^3\zeta_{b_{ij}}^3}}{1+(n-1)\frac{\zeta_{a_{ij}}^3+\zeta_{b_{ij}}^3-2\zeta_{a_{ij}}^3\zeta_{b_{ij}}^3}{1-\zeta_{a_{ij}}^3\zeta_{b_{ij}}^3}}}, \sqrt[3]{\frac{\frac{\delta_{a_{ij}}^3\delta_{b_{ij}}^3}{\delta_{a_{ij}}^3+\delta_{b_{ij}}^3-\delta_{a_{ij}}^3\delta_{b_{ij}}^3}}{n-(n-1)\frac{\delta_{a_{ij}}^3\delta_{b_{ij}}^3}{\delta_{a_{ij}}^3+\delta_{b_{ij}}^3-\delta_{a_{ij}}^3\delta_{b_{ij}}^3}}} \right\rangle\right]$$

$$= \left[\left\langle \sqrt[3]{\frac{n\zeta_{a_{ij}}^3+n\zeta_{b_{ij}}^3-2n\zeta_{a_{ij}}^3\zeta_{b_{ij}}^3}{1+(n-1)(\zeta_{a_{ij}}^3+\zeta_{b_{ij}}^3)-(2n-1)\zeta_{a_{ij}}^3\zeta_{b_{ij}}^3}}, \sqrt[3]{\frac{\delta_{a_{ij}}^3\delta_{b_{ij}}^3}{n(\delta_{a_{ij}}^3+\delta_{b_{ij}}^3)-(2n-1)\delta_{a_{ij}}^3\delta_{b_{ij}}^3}} \right\rangle\right].$$

Comparing the above results, we can finally get

$$(n._h A) \boxplus_h (n._h B) = n._h(A \boxplus_h B).$$

(v) By Equations (3) and (4), we have:

$$n_1._h A = \left[\left\langle \sqrt[3]{\frac{n_1\zeta_{a_{ij}}^3}{1+(n_1-1)\zeta_{a_{ij}}^3}}, \sqrt[3]{\frac{\delta_{a_{ij}}^3}{n_1-(n_1-1)\delta_{a_{ij}}^3}} \right\rangle\right] = [\langle \zeta_{b_{ij}}, \delta_{b_{ij}} \rangle],$$

$$n_{2\cdot_h}(n_{1\cdot_h}A)=\left[\left\langle \sqrt[3]{\frac{n_2\zeta_{b_{ij}}}{1+(n_2-1)\zeta_{b_{ij}}}}, \sqrt[3]{\frac{\delta_{b_{ij}}}{n_2-(n_2-1)\delta_{b_{ij}}}} \right\rangle\right].$$

We can further get the following.

$$\text{Let } \sqrt[3]{\frac{n_2\zeta_{b_{ij}}}{1+(n_2-1)\zeta_{b_{ij}}}}$$

$$=\sqrt[3]{\frac{n_2\dfrac{n_1\zeta^3_{a_{ij}}}{1+(n_1-1)\zeta^3_{a_{ij}}}}{1+(n_2-1)\dfrac{n_1\zeta^3_{a_{ij}}}{1+(n_1-1)\zeta^3_{a_{ij}}}}}=\sqrt[3]{\frac{n_1n_2\zeta^3_{a_{ij}}}{1+(n_1n_2-1)\zeta^3_{a_{ij}}}},$$

and

$$\text{let } \sqrt[3]{\frac{\delta_{b_{ij}}}{n_2-(n_2-1)\delta_{b_{ij}}}}$$

$$=\sqrt[3]{\frac{\dfrac{\delta^3_{a_{ij}}}{n_1-(n_1-1)\delta^3_{a_{ij}}}}{n_2-(n_2-1)\dfrac{\delta^3_{a_{ij}}}{n_1-(n_1-1)\delta^3_{a_{ij}}}}}=\sqrt[3]{\frac{\delta^3_{a_{ij}}}{n_1n_2-(n_1n_2-1)\delta^3_{a_{ij}}}}.$$

Since $n_1(n_{2\cdot_h})A=\left[\left\langle \sqrt[3]{\frac{n_1n_2\zeta^3_{a_{ij}}}{1+(n_1n_2-1)\zeta^3_{a_{ij}}}}, \sqrt[3]{\frac{\delta^3_{a_{ij}}}{n_1n_2-(n_1n_2-1)\delta^3_{a_{ij}}}}\right\rangle\right]$,

we can finally get $n_{2\cdot_h}(n_{1\cdot_h}A)=(n_1n_2)_{\cdot_h}A$. ■

Theorem 4.5.

Let A, B be any two Fermatean fuzzy matrices of the same size and for any positive integer n.

(i) $n_{\cdot_h}(A\wedge B)=(n_{\cdot_h}A)\wedge(n_{\cdot_h}B)$,

(ii) $n_{\cdot_h}(A\vee B)=(n_{\cdot_h}A)\vee(n_{\cdot_h}B)$,

(iii) $(A\wedge B)^{\wedge_h n}=A^{\wedge_h n}\wedge B^{\wedge_h n}$,

(iv) $(A\vee B)^{\wedge_h n}=A^{\wedge_h n}\vee B^{\wedge_h n}$.

Proof:

In the following, we shall prove $(ii), (iv)$ and $(i), (iii)$ can be proved analogously.

(i) Since $(A \wedge B) = \left[\left\langle \min\left\{\zeta_{a_{ij}}, \zeta_{b_{ij}}\right\}, \max\left\{\delta_{a_{ij}}, \delta_{b_{ij}}\right\}\right\rangle\right]$,

$$n._h(A \wedge B) = \left[\left\langle \zeta_{c_{ij}}, \delta_{c_{ij}}\right\rangle\right], n._h A = \left[\left\langle \zeta_{d_{ij}}, \delta_{d_{ij}}\right\rangle\right], n._h B = \left[\left\langle \zeta_{e_{ij}}, \delta_{e_{ij}}\right\rangle\right],$$

where,

$$\zeta_{c_{ij}} = \sqrt[3]{\frac{n(\min\left\{\zeta^3_{a_{ij}}, \zeta^3_{b_{ij}}\right\})}{1+(n-1)(\min\left\{\zeta^3_{a_{ij}}, \zeta^3_{b_{ij}}\right\})}}, \quad \text{and} \quad \delta_{c_{ij}} = \sqrt[3]{\frac{(\max\left\{\delta^3_{a_{ij}}, \delta^3_{b_{ij}}\right\})}{n-(n-1)(\max\left\{\delta^3_{a_{ij}}, \delta^3_{b_{ij}}\right\})}},$$

$$\begin{aligned} \zeta_{c_{ij}} &= \sqrt[3]{\frac{n(\min\left\{\zeta^3_{a_{ij}}, \zeta^3_{b_{ij}}\right\})}{1+(n-1)(\min\left\{\zeta^3_{a_{ij}}, \zeta^3_{b_{ij}}\right\})}} \\ &= \min\left\{\sqrt[3]{\frac{n\zeta^3_{a_{ij}}}{1+(n-1)\zeta^3_{a_{ij}}}}, \sqrt[3]{\frac{n\zeta^3_{b_{ij}}}{1+(n-1)\zeta^3_{b_{ij}}}}\right\} \\ &= \min\left\{\zeta_{d_{ij}}, \zeta_{e_{ij}}\right\}, \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} \delta_{c_{ij}} &= \sqrt[3]{\frac{(\max\left\{\delta^3_{a_{ij}}, \delta^3_{b_{ij}}\right\})}{n-(n-1)(\max\left\{\delta^3_{a_{ij}}, \delta^3_{b_{ij}}\right\})}} \\ &= \max\left\{\sqrt[3]{\frac{\delta^3_{a_{ij}}}{n-(n-1)\delta^3_{a_{ij}}}}, \sqrt[3]{\frac{\delta^3_{b_{ij}}}{n-(n-1)\delta^3_{b_{ij}}}}\right\} \\ &= \max\left\{\delta_{\zeta_{d_{ij}}}, \delta_{e_{ij}}\right\}. \end{aligned} \tag{4.2}$$

Comparing the Equations (4.1) and (4.2), we get:

$$\begin{aligned} (n._h A) \wedge (n._h B) &= \left[\left\langle \min\left\{\zeta_{d_{ij}}, \delta_{d_{ij}}\right\}, \max\left\{\zeta_{e_{ij}}, \delta_{e_{ij}}\right\}\right\rangle\right] \\ &= \left[\min\left\{\sqrt[3]{\frac{n\zeta^3_{a_{ij}}}{1+(n-1)\zeta^3_{a_{ij}}}}, \sqrt[3]{\frac{\delta^3_{a_{ij}}}{n-(n-1)\delta^3_{a_{ij}}}}\right\},\right. \\ &\qquad \left.\max\left\{\sqrt[3]{\frac{n\zeta^3_{b_{ij}}}{1+(n-1)\zeta^3_{b_{ij}}}}, \sqrt[3]{\frac{\delta^3_{b_{ij}}}{n-(n-1)\delta^3_{b_{ij}}}}\right\}\right]. \end{aligned}$$

Thus, we have $n._h(A \wedge B) = (n._h A) \wedge (n._h B)$.

(iii) Since $(A \wedge B) = \left[\left\langle \min\left\{\zeta_{a_{ij}}, \zeta_{b_{ij}}\right\}, \max\left\{\delta_{a_{ij}}, \delta_{b_{ij}}\right\}\right\rangle\right]$, then

$$(A \wedge B)^{\wedge_h n} = \left[\left\langle \zeta_{c_{ij}}, \delta_{c_{ij}} \right\rangle\right], A^{\wedge_h n} = \left[\left\langle \zeta_{d_{ij}}, \delta_{d_{ij}} \right\rangle\right], B^{\wedge_h n} = \left[\left\langle \zeta_{e_{ij}}, \delta_{e_{ij}} \right\rangle\right],$$

where,

$$\zeta_{c_{ij}} = \sqrt[3]{\frac{(\min\left\{\zeta^3_{a_{ij}}, \zeta^3_{b_{ij}}\right\})}{n-(n-1)(\min\left\{\zeta^3_{a_{ij}}, \zeta^3_{b_{ij}}\right\})}}, \quad \text{and} \quad \delta_{c_{ij}} = \sqrt[3]{\frac{n(\max\left\{\delta^3_{a_{ij}}, \delta^3_{b_{ij}}\right\})}{1+(n-1)(\max\left\{\delta^3_{a_{ij}}, \delta^3_{b_{ij}}\right\})}},$$

$$\begin{aligned}
\zeta_{c_{ij}} &= \sqrt[3]{\frac{(\min\left\{\zeta^3_{a_{ij}}, \zeta^3_{b_{ij}}\right\})}{n-(n-1)(\min\left\{\zeta^3_{a_{ij}}, \zeta^3_{b_{ij}}\right\})}} \\
&= \min\left\{\sqrt[3]{\frac{\zeta^3_{a_{ij}}}{n-(n-1)\zeta^3_{a_{ij}}}}, \sqrt[3]{\frac{\zeta^3_{b_{ij}}}{n-(n-1)\zeta^3_{b_{ij}}}}\right\} \\
&= \min\left\{\zeta_{d_{ij}}, \zeta_{e_{ij}}\right\}, \qquad (4.3)
\end{aligned}$$

and

$$\begin{aligned}
\delta_{c_{ij}} &= \sqrt[3]{\frac{n(\max\left\{\delta^3_{a_{ij}}, \delta^3_{b_{ij}}\right\})}{1+(n-1)(\max\left\{\delta^3_{a_{ij}}, \delta^3_{b_{ij}}\right\})}} \\
&= \max\left\{\sqrt[3]{\frac{n\delta^3_{a_{ij}}}{1+(n-1)\delta^3_{a_{ij}}}}, \sqrt[3]{\frac{n\delta^3_{b_{ij}}}{1+(n-1)\delta^3_{b_{ij}}}}\right\} \\
&= \max\left\{\delta_{d_{ij}}, \delta_{e_{ij}}\right\}. \qquad (4.4)
\end{aligned}$$

Comparing the Equations (4.3) and (4.4), we get:

$$\begin{aligned}
A^{\wedge_h n} \wedge B^{\wedge_h n} &= \left[\left\langle \min\left\{\zeta_{d_{ij}}, \delta_{d_{ij}}\right\}, \max\left\{\zeta_{e_{ij}}, \delta_{e_{ij}}\right\}\right\rangle\right] \\
&= \left[\min\left\{\sqrt[3]{\frac{\zeta^3_{a_{ij}}}{n-(n-1)\zeta^3_{a_{ij}}}}, \sqrt[3]{\frac{n\delta^3_{a_{ij}}}{1+(n-1)\delta^3_{a_{ij}}}}\right\},\right. \\
&\qquad \left.\max\left\{\sqrt[3]{\frac{\zeta^3_{b_{ij}}}{n-(n-1)\zeta^3_{b_{ij}}}}, \sqrt[3]{\frac{n\delta^3_{b_{ij}}}{1+(n-1)\delta^3_{b_{ij}}}}\right\}\right].
\end{aligned}$$

Hence, $(A \wedge B)^{\wedge_h n} = A^{\wedge_h n} \wedge B^{\wedge_h n}$. ■

5. Necessity and Possibility Operators on Fermatean Fuzzy Matrices

In this section, some properties of necessity and possibility operators on Fermatean fuzzy matrices are verify.

Definition 5.1.

For any Fermatean fuzzy matrix A, the necessity $(\Box)$ and the possibility $(\Diamond)$ operators are defined as follows:

$$\Box A = \left[\left\langle \zeta_{a_{ij}}, \sqrt[3]{1-\zeta^3_{a_{ij}}} \right\rangle\right],$$

$$\Diamond A = \left[\left\langle \sqrt[3]{1-\delta^3_{a_{ij}}}, \delta_{a_{ij}} \right\rangle\right].$$

Theorem 5.1.

For A, B be any two Fermatean fuzzy matrices of the same size, then

(i) $\Box(A \boxplus_h B) = \Box A \boxplus_h \Box B$,

(ii) $\Diamond(A \boxplus_h B) = \Diamond A \boxplus_h \Diamond B$.

Proof:

$$(i)\ \Box(A \boxplus_h B) = \left[\left\langle \sqrt[3]{\frac{\zeta^3_{a_{ij}} + \zeta^3_{b_{ij}} - 2\zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}{1-\zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}}, \sqrt[3]{1 - \frac{\zeta^3_{a_{ij}} + \zeta^3_{b_{ij}} - 2\zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}{1-\zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}} \right\rangle\right]$$

$$= \left[\left\langle \sqrt[3]{\frac{\zeta^3_{a_{ij}} + \zeta^3_{b_{ij}} - 2\zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}{1-\zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}}, \sqrt[3]{\frac{(1-\zeta^3_{a_{ij}})(1-\zeta^3_{b_{ij}})}{1-\zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}} \right\rangle\right]$$

$$= \Box A \boxplus_h \Box B.$$

$$(ii)\ \Diamond(A \boxplus_h B) = \left[\left\langle \sqrt[3]{1 - \frac{\delta^3_{a_{ij}}\delta^3_{b_{ij}}}{\delta^3_{a_{ij}} + \delta^3_{b_{ij}} - \delta^3_{a_{ij}}\delta^3_{b_{ij}}}}, \sqrt[3]{\frac{\delta^3_{a_{ij}}\delta^3_{b_{ij}}}{\delta^3_{a_{ij}} + \delta^3_{b_{ij}} - \delta^3_{a_{ij}}\delta^3_{b_{ij}}}} \right\rangle\right]$$

$$= \left[\left\langle \sqrt[3]{\frac{\delta^3_{a_{ij}} + \delta^3_{b_{ij}} - 2\delta^3_{a_{ij}}\delta^3_{b_{ij}}}{\delta^3_{a_{ij}} + \delta^3_{b_{ij}} - \delta^3_{a_{ij}}\delta^3_{b_{ij}}}}, \sqrt[3]{\frac{\delta^3_{a_{ij}}\delta^3_{b_{ij}}}{\delta^3_{a_{ij}} + \delta^3_{b_{ij}} - \delta^3_{a_{ij}}\delta^3_{b_{ij}}}} \right\rangle\right]$$

$$= \Diamond A \boxplus_h \Diamond B.$$

■

Theorem 5.2.

For A, B be any two Fermatean fuzzy matrices of the same size, then

(i) $\Box(A \boxdot_h B) = \Box A \boxdot_h \Box B$,

(ii) $\Diamond(A \boxdot_h B) = \Diamond A \boxdot_h \Diamond B$.

Proof:

$$(i)\ \Box(A \boxdot_h B) = \left[\left\langle \sqrt[3]{\frac{\zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}{\zeta^3_{a_{ij}} + \zeta^3_{b_{ij}} - \zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}}, \sqrt[3]{1 - \frac{\zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}{\zeta^3_{a_{ij}} + \zeta^3_{b_{ij}} - \zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}} \right\rangle\right]$$

$$= \left[\left\langle \sqrt[3]{\frac{\zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}{\zeta^3_{a_{ij}} + \zeta^3_{b_{ij}} - \zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}}, \sqrt[3]{\frac{\zeta^3_{a_{ij}} + \zeta^3_{b_{ij}} - 2\zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}{\zeta^3_{a_{ij}} + \zeta^3_{b_{ij}} - \zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}} \right\rangle\right]$$

$$= \Box A \boxdot_h \Box B.$$

$$(ii)\ \Diamond(A \boxdot_h B) = \left[\left\langle \sqrt[3]{1 - \frac{\delta^3_{a_{ij}} + \delta^3_{b_{ij}} - 2\delta^3_{a_{ij}}\delta^3_{b_{ij}}}{1 - \delta^3_{a_{ij}}\delta^3_{b_{ij}}}}, \sqrt[3]{\frac{\delta^3_{a_{ij}} + \delta^3_{b_{ij}} - 2\delta^3_{a_{ij}}\delta^3_{b_{ij}}}{1 - \delta^3_{a_{ij}}\delta^3_{b_{ij}}}} \right\rangle\right]$$

$$= \left[\left\langle \sqrt[3]{\frac{(1-\delta^3_{a_{ij}})(1-\delta^3_{b_{ij}})}{1 - \delta^3_{a_{ij}}\delta^3_{b_{ij}}}}, \sqrt[3]{\frac{\delta^3_{a_{ij}} + \delta^3_{b_{ij}} - 2\delta^3_{a_{ij}}\delta^3_{b_{ij}}}{1 - \delta^3_{a_{ij}}\delta^3_{b_{ij}}}} \right\rangle\right]$$

$$= \Diamond A \boxdot_h \Diamond B.$$ ∎

Theorem 5.3.

For A, B be any two Fermatean fuzzy matrices of the same size, then

$(i)\ \left(\Box(A^C \boxplus_h B^C)\right)^C = \Diamond A \boxdot_h \Diamond B,$

$(ii)\ \left(\Box(A^C \boxdot_h B^C)\right)^C = \Diamond A \boxplus_h \Diamond B.$

Proof:

$$(i)\ \Box(A^C \boxplus_h B^C) = \left[\left\langle \sqrt[3]{\frac{\delta^3_{a_{ij}} + \delta^3_{b_{ij}} - 2\delta^3_{a_{ij}}\delta^3_{b_{ij}}}{1 - \delta^3_{a_{ij}}\delta^3_{b_{ij}}}}, \sqrt[3]{1 - \frac{\delta^3_{a_{ij}} + \delta^3_{b_{ij}} - 2\delta^3_{a_{ij}}\delta^3_{b_{ij}}}{1 - \delta^3_{a_{ij}}\delta^3_{b_{ij}}}} \right\rangle\right]$$

$$= \left[\left\langle \sqrt[3]{\frac{\delta^3_{a_{ij}} + \delta^3_{b_{ij}} - 2\delta^3_{a_{ij}}\delta^3_{b_{ij}}}{1 - \delta^3_{a_{ij}}\delta^3_{b_{ij}}}}, \sqrt[3]{\frac{(1-\delta^3_{a_{ij}})(1-\delta^3_{b_{ij}})}{1 - \delta^3_{a_{ij}}\delta^3_{b_{ij}}}} \right\rangle\right],$$

$$\left(\Box(A^C \boxplus_h B^C)\right)^C = \left[\left\langle \sqrt[3]{\frac{(1-\delta^3_{a_{ij}})(1-\delta^3_{b_{ij}})}{1 - \delta^3_{a_{ij}}\delta^3_{b_{ij}}}}, \sqrt[3]{\frac{\delta^3_{a_{ij}} + \delta^3_{b_{ij}} - 2\delta^3_{a_{ij}}\delta^3_{b_{ij}}}{1 - \delta^3_{a_{ij}}\delta^3_{b_{ij}}}} \right\rangle\right]$$

$$= \Diamond A \boxdot_h \Diamond B.$$

$$(ii)\ \Box(A^C \boxdot_h B^C) = \left[\left\langle \sqrt[3]{\frac{\delta^3_{a_{ij}}\delta^3_{b_{ij}}}{\delta^3_{a_{ij}} + \delta^3_{b_{ij}} - \delta^3_{a_{ij}}\delta^3_{b_{ij}}}}, \sqrt[3]{1 - \frac{\delta^3_{a_{ij}}\delta^3_{b_{ij}}}{\delta^3_{a_{ij}} + \delta^3_{b_{ij}} - \delta^3_{a_{ij}}\delta^3_{b_{ij}}}} \right\rangle\right]$$

$$= \left[\left\langle \sqrt[3]{\frac{\delta^3_{a_{ij}}\delta^3_{b_{ij}}}{\delta^3_{a_{ij}} + \delta^3_{b_{ij}} - \delta^3_{a_{ij}}\delta^3_{b_{ij}}}}, \sqrt[3]{\frac{\delta^3_{a_{ij}} + \delta^3_{b_{ij}} - 2\delta^3_{a_{ij}}\delta^3_{b_{ij}}}{\delta^3_{a_{ij}} + \delta^3_{b_{ij}} - \delta^3_{a_{ij}}\delta^3_{b_{ij}}}} \right\rangle\right],$$

$$\big(\Box(A^C \boxdot_h B^C)\big)^C = \left[\left\langle \sqrt[3]{\frac{\delta^3_{a_{ij}}+\delta^3_{b_{ij}}-2\delta^3_{a_{ij}}\delta^3_{b_{ij}}}{\delta^3_{a_{ij}}+\delta^3_{b_{ij}}-\delta^3_{a_{ij}}\delta^3_{b_{ij}}}}, \sqrt[3]{\frac{\delta^3_{a_{ij}}\delta^3_{b_{ij}}}{\delta^3_{a_{ij}}+\delta^3_{b_{ij}}-\delta^3_{a_{ij}}\delta^3_{b_{ij}}}} \right\rangle\right]$$

$$= \Diamond A \boxplus_h \Diamond B. \qquad \blacksquare$$

Theorem 5.4.

For A, B be any two Fermatean fuzzy matrices of the same size, then

(i) $\big(\Diamond(A^C \boxplus_h B^C)\big)^C = \Box A \boxdot_h \Box B$,

(ii) $\big(\Diamond(A^C \boxdot_h B^C)\big)^C = \Box A \boxplus_h \Box B$.

Proof:

$$(i)\ \Diamond(A^C \boxplus_h B^C) = \left[\left\langle \sqrt[3]{1-\frac{\zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}{\zeta^3_{a_{ij}}+\zeta^3_{b_{ij}}-\zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}}, \sqrt[3]{\frac{\zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}{\zeta^3_{a_{ij}}+\zeta^3_{b_{ij}}-\zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}} \right\rangle\right]$$

$$= \left[\left\langle \sqrt[3]{\frac{\zeta^3_{a_{ij}}+\zeta^3_{b_{ij}}-2\zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}{\zeta^3_{a_{ij}}+\zeta^3_{b_{ij}}-\zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}}, \sqrt[3]{\frac{\zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}{\zeta^3_{a_{ij}}+\zeta^3_{b_{ij}}-\zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}} \right\rangle\right],$$

$$(\Diamond(A^C \boxplus_h B^C))^C = \left[\left\langle \sqrt[3]{\frac{\zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}{\zeta^3_{a_{ij}}+\zeta^3_{b_{ij}}-\zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}}, \sqrt[3]{\frac{\zeta^3_{a_{ij}}+\zeta^3_{b_{ij}}-2\zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}{\zeta^3_{a_{ij}}+\zeta^3_{b_{ij}}-\zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}} \right\rangle\right]$$

$$= \Box A \boxdot_h \Box B.$$

$$(ii)\ \Diamond(A^C \boxdot_h B^C) = \left[\left\langle \sqrt[3]{1-\frac{\zeta^3_{a_{ij}}+\zeta^3_{b_{ij}}-2\zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}{1-\zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}}, \sqrt[3]{\frac{\zeta^3_{a_{ij}}+\zeta^3_{b_{ij}}-2\zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}{1-\zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}} \right\rangle\right]$$

$$= \left[\left\langle \sqrt[3]{\frac{(1-\zeta^3_{a_{ij}})(1-\zeta^3_{b_{ij}})}{1-\zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}}, \sqrt[3]{\frac{\zeta^3_{a_{ij}}+\zeta^3_{b_{ij}}-2\zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}{1-\zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}} \right\rangle\right],$$

$$(\Diamond(A^C \boxdot_h B^C))^C = \left[\left\langle \sqrt[3]{\frac{\zeta^3_{a_{ij}}+\zeta^3_{b_{ij}}-2\zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}{1-\zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}}, \sqrt[3]{\frac{(1-\zeta^3_{a_{ij}})(1-\zeta^3_{b_{ij}})}{1-\zeta^3_{a_{ij}}\zeta^3_{b_{ij}}}} \right\rangle\right]$$

$$= \Box A \boxplus_h \Box B. \qquad \blacksquare$$

Theorem 5.5.

For any FFM A and for any positive integer n.

(i) $\Box(n._h A) = n._h(\Box A)$,

(ii) $\Diamond(n._h A) = n._h(\Diamond A)$,

(iii) $\Box A^{\wedge_h n} = (\Box A)^{\wedge_h n}$,

$(iv)\quad \Diamond A^{\wedge_h n} = (\Diamond A)^{\wedge_h n}$.

Proof:

$$(i)\ \Box(n._h A) = \left[\left\langle \sqrt[3]{\frac{n\zeta^3_{a_{ij}}}{1+(n-1)\zeta^3_{a_{ij}}}}, \sqrt[3]{1-\left(\sqrt[3]{\frac{n\zeta^3_{a_{ij}}}{1+(n-1)\zeta^3_{a_{ij}}}}\right)^3} \right\rangle\right]$$

$$= \left[\left\langle \sqrt[3]{\frac{n\zeta^3_{a_{ij}}}{1+(n-1)\zeta^3_{a_{ij}}}}, \sqrt[3]{1-\frac{n\zeta^3_{a_{ij}}}{1+(n-1)\zeta^3_{a_{ij}}}} \right\rangle\right]$$

$$= \left[\left\langle \sqrt[3]{\frac{n\zeta^3_{a_{ij}}}{1+(n-1)\zeta^3_{a_{ij}}}}, \sqrt[3]{\frac{1+(n-1)\zeta^3_{a_{ij}}-n\zeta^3_{a_{ij}}}{1+(n-1)\zeta^3_{a_{ij}}}} \right\rangle\right]$$

$$= \left[\left\langle \sqrt[3]{\frac{n\zeta^3_{a_{ij}}}{1+(n-1)\zeta^3_{a_{ij}}}}, \sqrt[3]{\frac{1+n\zeta^3_{a_{ij}}-\zeta^3_{a_{ij}}-n\zeta^3_{a_{ij}}}{1+(n-1)\zeta^3_{a_{ij}}}} \right\rangle\right],$$

$$\Box(n._h A) = \left[\left\langle \sqrt[3]{\frac{n\zeta^3_{a_{ij}}}{1+(n-1)\zeta^3_{a_{ij}}}}, \sqrt[3]{\frac{1-\zeta^3_{a_{ij}}}{1+(n-1)\zeta^3_{a_{ij}}}} \right\rangle\right].$$

$$n._h(\Box A) = \left[\left\langle \sqrt[3]{\frac{n\zeta^3_{a_{ij}}}{1+(n-1)\zeta^3_{a_{ij}}}}, \sqrt[3]{\frac{\left(\sqrt[3]{1-\zeta^3_{a_{ij}}}\right)^3}{n-(n-1)\left(\sqrt[3]{1-\zeta^3_{a_{ij}}}\right)^3}} \right\rangle\right]$$

$$= \left[\left\langle \sqrt[3]{\frac{n\zeta^3_{a_{ij}}}{1+(n-1)\zeta^3_{a_{ij}}}}, \sqrt[3]{\frac{1-\zeta^3_{a_{ij}}}{n-(n-1)\left(1-\zeta^3_{a_{ij}}\right)}} \right\rangle\right],$$

$$n._h(\Box A) = \left[\left\langle \sqrt[3]{\frac{n\zeta^3_{a_{ij}}}{1+(n-1)\zeta^3_{a_{ij}}}}, \sqrt[3]{\frac{1-\zeta^3_{a_{ij}}}{1+(n-1)\zeta^3_{a_{ij}}}} \right\rangle\right].$$

Hence, $\Box(n._h A) = n._h(\Box A)$.

$$(ii)\ \Diamond(n._h A) = \left[\left\langle \sqrt[3]{1-\left(\sqrt[3]{\frac{\delta^3_{a_{ij}}}{n-(n-1)\delta^3_{a_{ij}}}}\right)^3}, \sqrt[3]{\frac{\delta^3_{a_{ij}}}{n-(n-1)\delta^3_{a_{ij}}}} \right\rangle\right]$$

$$= \left[\left\langle \sqrt[3]{1-\frac{\delta^3_{a_{ij}}}{n-(n-1)\delta^3_{a_{ij}}}}, \sqrt[3]{\frac{\delta^3_{a_{ij}}}{n-(n-1)\delta^3_{a_{ij}}}} \right\rangle\right]$$

$$= \left[\left\langle \sqrt[3]{\frac{n-(n-1)\delta^3_{a_{ij}}-\delta^3_{a_{ij}}}{n-(n-1)\delta^3_{a_{ij}}}}, \sqrt[3]{\frac{\delta^3_{a_{ij}}}{n-(n-1)\delta^3_{a_{ij}}}} \right\rangle\right]$$

$$= \left[\left\langle \sqrt[3]{\frac{n - n\delta_{a_{ij}}^3 + \delta_{a_{ij}}^3 - \delta_{a_{ij}}^3}{n-(n-1)\delta_{a_{ij}}^3}}, \sqrt[3]{\frac{\delta_{a_{ij}}^3}{n-(n-1)\delta_{a_{ij}}^3}} \right\rangle \right],$$

$$\Diamond(n._h A) = \left[\left\langle \sqrt[3]{\frac{n(1-\delta_{a_{ij}}^3)}{n-(n-1)\delta_{a_{ij}}^3}}, \sqrt[3]{\frac{\delta_{a_{ij}}^3}{n-(n-1)\delta_{a_{ij}}^3}} \right\rangle \right].$$

$$n._h(\Diamond A) = \left[\left\langle \sqrt[3]{\frac{n(\sqrt[3]{1-\delta_{a_{ij}}^3})^3}{1+(n-1)(\sqrt[3]{1-\delta_{a_{ij}}^3})^3}}, \sqrt[3]{\frac{\delta_{a_{ij}}^3}{n-(n-1)\delta_{a_{ij}}^3}} \right\rangle \right]$$

$$= \left[\left\langle \sqrt[3]{\frac{n(1-\delta_{a_{ij}}^3)}{1+(n-1)(1-\delta_{a_{ij}}^3)}}, \sqrt[3]{\frac{\delta_{a_{ij}}^3}{n-(n-1)\delta_{a_{ij}}^3}} \right\rangle \right],$$

$$n._h(\Diamond A) = \left[\left\langle \sqrt[3]{\frac{n(1-\delta_{a_{ij}}^3)}{n-(n-1)\delta_{a_{ij}}^3}}, \sqrt[3]{\frac{\delta_{a_{ij}}^3}{n-(n-1)\delta_{a_{ij}}^3}} \right\rangle \right].$$

Hence, $\Diamond(n._h A) = n._h(\Diamond A)$.

$$(iii)\ \ \Box A^{\wedge_h n} = \left[\left\langle \sqrt[3]{\frac{\zeta_{a_{ij}}^3}{n-(n-1)\zeta_{a_{ij}}^3}}, \sqrt[3]{1 - \left(\sqrt[3]{\frac{\zeta_{a_{ij}}^3}{n-(n-1)\zeta_{a_{ij}}^3}}\right)^3} \right\rangle \right]$$

$$= \left[\left\langle \sqrt[3]{\frac{\zeta_{a_{ij}}^3}{n-(n-1)\zeta_{a_{ij}}^3}}, \sqrt[3]{1 - \frac{\zeta_{a_{ij}}^3}{n-(n-1)\zeta_{a_{ij}}^3}} \right\rangle \right]$$

$$= \left[\left\langle \sqrt[3]{\frac{\zeta_{a_{ij}}^3}{n-(n-1)\zeta_{a_{ij}}^3}}, \sqrt[3]{\frac{n-(n-1)\zeta_{a_{ij}}^3 - \zeta_{a_{ij}}^3}{n-(n-1)\zeta_{a_{ij}}^3}} \right\rangle \right]$$

$$= \left[\left\langle \sqrt[3]{\frac{\zeta_{a_{ij}}^3}{n-(n-1)\zeta_{a_{ij}}^3}}, \sqrt[3]{\frac{n - n\zeta_{a_{ij}}^3 + \zeta_{a_{ij}}^3 - \zeta_{a_{ij}}^3}{n-(n-1)\zeta_{a_{ij}}^3}} \right\rangle \right]$$

$$= \left[\left\langle \sqrt[3]{\frac{\zeta_{a_{ij}}^3}{n-(n-1)\zeta_{a_{ij}}^3}}, \sqrt[3]{\frac{n(1-\zeta_{a_{ij}}^3)}{n-(n-1)\zeta_{a_{ij}}^3}} \right\rangle \right],$$

$$(\Box A)^{\wedge_h n} = \left[\left\langle \sqrt[3]{\frac{\zeta_{a_{ij}}^3}{n-(n-1)\zeta_{a_{ij}}^3}}, \sqrt[3]{\frac{n(\sqrt[3]{1-\zeta_{a_{ij}}^3})^3}{n-(n-1)(\sqrt[3]{1-\zeta_{a_{ij}}^3})^3}} \right\rangle \right]$$

$$= \left[\left\langle \sqrt[3]{\frac{\zeta_{a_{ij}}^3}{n-(n-1)\zeta_{a_{ij}}^3}}, \sqrt[3]{\frac{n(1-\zeta_{a_{ij}}^3)}{n-(n-1)(1-\zeta_{a_{ij}}^3)}} \right\rangle \right],$$

$$(\Box A)^{\wedge_h n} = \left[\left\langle \sqrt[3]{\frac{\zeta_{a_{ij}}^3}{n-(n-1)\zeta_{a_{ij}}^3}}, \sqrt[3]{\frac{n(1-\zeta_{a_{ij}}^3)}{n-(n-1)\zeta_{a_{ij}}^3}} \right\rangle\right].$$

Hence, $\Box A^{\wedge_h n} = (\Box A)^{\wedge_h n}$.

$$(iv) \ \ \Diamond A^{\wedge_h n} = \left[\left\langle \sqrt[3]{1-\left(\sqrt[3]{\frac{n\delta_{a_{ij}}^3}{1+(n-1)\delta_{a_{ij}}^3}}\right)^3}, \sqrt[3]{\frac{n\delta_{a_{ij}}^3}{1+(n-1)\delta_{a_{ij}}^3}} \right\rangle\right]$$

$$= \left[\left\langle \sqrt[3]{1-\frac{n\delta_{a_{ij}}^3}{1+(n-1)\delta_{a_{ij}}^3}}, \sqrt[3]{\frac{n\delta_{a_{ij}}^3}{1+(n-1)\delta_{a_{ij}}^3}} \right\rangle\right]$$

$$= \left[\left\langle \sqrt[3]{\frac{1+(n-1)\delta_{a_{ij}}^3-n\delta_{a_{ij}}^3}{1+(n-1)\delta_{a_{ij}}^3}}, \sqrt[3]{\frac{n\delta_{a_{ij}}^3}{1+(n-1)\delta_{a_{ij}}^3}} \right\rangle\right],$$

$$\Diamond A^{\wedge_h n} = \left[\left\langle \sqrt[3]{\frac{1-\delta_{a_{ij}}^3}{1+(n-1)\delta_{a_{ij}}^3}}, \sqrt[3]{\frac{n\delta_{a_{ij}}^3}{1+(n-1)\delta_{a_{ij}}^3}} \right\rangle\right],$$

$$(\Diamond A)^{\wedge_h n} = \left[\left\langle \sqrt[3]{\frac{\left(\sqrt[3]{1-\delta_{a_{ij}}^3}\right)^3}{1+(n-1)\left(\sqrt[3]{1-\delta_{a_{ij}}^3}\right)^3}}, \sqrt[3]{\frac{n\delta_{a_{ij}}^3}{1+(n-1)\delta_{a_{ij}}^3}} \right\rangle\right]$$

$$= \left[\left\langle \sqrt[3]{\frac{\left(1-\delta_{a_{ij}}^3\right)}{1+(n-1)\left(1-\delta_{a_{ij}}^3\right)}}, \sqrt[3]{\frac{n\delta_{a_{ij}}^3}{1+(n-1)\delta_{a_{ij}}^3}} \right\rangle\right],$$

$$(\Diamond A)^{\wedge_h n} = \left[\left\langle \sqrt[3]{\frac{1-\delta_{a_{ij}}^3}{1+(n-1)\delta_{a_{ij}}^3}}, \sqrt[3]{\frac{n\delta_{a_{ij}}^3}{1+(n-1)\delta_{a_{ij}}^3}} \right\rangle\right].$$

Hence, $\Diamond A^{\wedge_h n} = (\Diamond A)^{\wedge_h n}$. ■

6. Conclusion

The work has extended the Hamacher operation results under Fermatean fuzzy environment. In this paper, the Hamacher operations of Fermatean fuzzy matrices are developed and investigated their algebraic properties. We also proved that the set of all FFMs with respect to Hamacher sum and Hamacher product forms a commutative monoid. A study of the algebraic structure of FFMs with respect to Hamacher operations gives us a deep insight into the applications. Then, De Morgan's laws are verified. Furthermore, the scalar multiplication and exponentiation operations on Fermatean fuzzy matrices are constructed and their algebraic properties are investigated. Finally, some properties of necessity and possibility operators on Fermatean fuzzy matrices are verified.

It is worth pointing out that the proposed Hamacher operations over FFMs will be applied to aggregating Fermatean fuzzy information in the future.

Acknowledgment:

The authors would like to thank the reviewers for their valuable suggestions and comments to improve the presentation of this paper.

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