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Numerical Solution of Fuzzy Fractional Differential Equation By Haar Wavelet

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Abstract

In this paper, we deal with a wavelet operational method based on Haar wavelet to solve the fuzzy fractional differential equation in the Caputo derivative sense. To this end, we derive the Haar wavelet operational matrix of the fractional order integration. The given approach provides an efficient method to find the solution and its upper bond error. To complete the discussion, the convergence theorem is subsequently expressed in detail. So far, no paper has used the Harr wavelet method using generalized difference and fuzzy derivatives, and this is the first time we have done so. Finally, the presented examples reflect the accuracy and efficiency of the proposed method.

Keywords: Generalized Hukuhara differentiability; Hilbert space; Fractional calculus; Haar wavelet; Fuzzy fractional differential equation; Caputo derivative; Convergence analysis

MSC 2010 No.: 34A07, 34A08

1. Introduction

In this paper, we deal with a wavelet operational method based on Haar. In the real world, to model and analyze a huge amount of problems, we need to apply fractional differential equations. Fractional calculus is used in many fields of mathematical and engineering sciences, including power grids, fluid mechanics, control theory, electromagnetism, biology, chemistry, propagation, and viscoelasticity (Diethelm et al. (2004); Kiryakova (1994); Podlubny (1999); Samko et al. (1993); Arikoglu et al. (2009)). In recent years, there has been many consideration to solve the ordinary fractional differential equations, integral equations, and differential equations with fractional partial derivatives.

Since there is no accurate analytical answer for fractional differential equations, we widely use numerical and approximation methods, such as Laplace transforms (Podlubny (1999)), Fourier transforms (Gaul et al. (1991)), Adomian decomposition method (Momani (2007)), fractional differential transformation method (Arikoglu et al. (2009); Erturk et al. (2008)), Haar wavelet operational matrix the fractional order differential equations (Wu (2009); Yi et al. (2014)) and fractional difference method (Meerschaert et al. (2006)), to solve them. Also, the fuzzy differential equations and fuzzy fractional differential equations have many applications in other sciences (Allahviranloo et al. (2016)).

The authors (Allahviranloo et al. (2012); Salahshour et al. (2012b)) considered the generalization of H-differentiability for the fractional case. A lot of research has been devoted to find the accurate and efficient methods for solving fuzzy fractional differential equations $(\mathbf{FFDEs})$. It is well known that the exact solutions of most of the $\mathbf{FFDEs}$ cannot be found easily; therefore, in the recent years, attempts have been made to address this problem (Mazandarani et al. (2013); Salahshour et al. (2012a)). So far, no paper has used the Harr wavelet method using generalized difference and fuzzy derivatives. It is with this motivation that we introduce in this paper Haar wavelet method for solving $\mathbf{FFDEs}$.

The paper is organized as follows. In Section 2, some necessary definitions, fuzzy integration and the fuzzy Caputo differentiability are brought, respectively. In Section 3, we explain Haar wavelet and function approximation and also how to solve fuzzy fractional differential equations using Haar wavelet. In Section 4, convergence analysis and error bound of the solution are expressed. In Section 5, several examples are solved for more illustration of the method.

2. Preliminaries

In this section, we present some definitions and introduce the necessary notation, which will be used throughout the paper.

We denote by $\mathbb{R}_{\mathcal{F}}$, the set of fuzzy numbers, that is normal, fuzzy convex, upper semi-continuous and compactly supported fuzzy sets which defined over the real line. For $0 < r \leq 1$, set $[u]_r$ of a fuzzy number u is the subset of points $t \in R$ with membership grade $u(t)$ of the least amount of r.

That is, $[u]_r = \{t \in R \mid u(t) \geq r\} = [u^-(r), u^+(r)]$ and $[u]_0 = cl\{t \in R \mid u(t) > 0\}$. If $u \in \mathbb{R}_\mathcal{F}$, the set $[u]_r$ is a bounded closed interval for all $r \in [0,1]$. For arbitrary $u, v \in \mathbb{R}_\mathcal{F}$ and $k \in R$, the addition and scalar multiplication are defined by $[u \oplus v]_r = [u]_r + [v]_r$ and if $k \geq 0$, then $k \odot u = (ku^-(r), ku^+(r))$, if $k < 0$ then $k \odot u = (ku^+(r), ku^-(r))$.

Remark 2.1.

Throughout the rest of this paper, we assume that the $gH-$difference always exists.

Remark 2.2.

If $f : [a,b] \to \mathbb{R}_\mathcal{F}$ be a fuzzy function with no switching point in interval $[a,b]$, then

i. If $f(t)$ is $[i-gH]-$differentiable, then $\int_a^b f'_{i.gH}(t)dt = f(b) \ominus f(a)$.
ii. If $f(t)$ is $[ii-gH]-$differentiable, then $\int_a^b f'_{ii.gH}(t)dt = (-1)f(a) \ominus (-1)f(b)$.

Lemma 2.1.

Suppose that $f(t) : [a,b] \to \mathbb{R}_\mathcal{F}$ be a fuzzy-valued function, then

(i) If $f(t)$ is $[i-gH]-$differentiable, then $\int_a^b f'_{i.gH}(t)dt = \ominus \int_b^a f'_{i.gH}(t)dt$.
(ii) If $f(t)$ is $[ii-gH]-$differentiable, then $\int_a^b f'_{ii.gH}(t)dt = \ominus \int_b^a f'_{ii.gH}(t)dt$.

Proof:

According to Remark 2.2 (i), we get

$$\ominus \int_b^a f'_{i.gH}(t)dt = \ominus\Big(f(a) \ominus f(b)\Big) = f(b) \ominus f(a) = \int_a^b f'_{i.gH}(t)dt.$$

For prove the second part by using Lemma 2.2 (ii), we have

$$\ominus \int_b^a f'_{ii.gH}(t)dt = \ominus\Big((-1)f(b) \ominus (-1)f(a)\Big) = (-1)f(a) \ominus (-1)f(b) = \int_a^b f'_{ii.gH}(t)dt. \quad \blacksquare$$

Lemma 2.2.

Let $f(t) : [a,b] \to \mathbb{R}_\mathcal{F}$ is $gH-$differentiable and $f'_{gH}(t)$ continues on $[a,b]$. Then

(i) $\ominus(-1) \int_a^b f'_{ii.gH}(t)dt = \int_a^b f'_{i.gH}(t)dt$.
(ii) $\int_a^b f'_{ii.gH}(t)dt = (-1) \int_b^a f'_{i.gH}(t)dt$.

Proof:

According to Remark 2.2 (ii), we get

$$\begin{aligned}0\ominus(-1)\int_a^b f'_{ii.gH}(t,r)dt &= 0\ominus(-1)[\int_a^b (f')^+(t,r)dt, \int_a^b (f')^-(t,r)dt]\\ &= 0\ominus[-f^-(b,r)+f^-(a,r), -f^+(b,r)+f^+(a,r)]\\ &= [f^-(b,r)-f^-(a,r), f^+(b,r)-f^+(a,r)]\\ &= [f^-(b,r), f^+(b,r)]\ominus[f^-(a,r), f^+(a,r)]\\ &= \int_a^b f'_{i.gH}(t,r)dt.\end{aligned}$$

The proof of part (ii) according to Remark 2.2 (i), is easily obtained:

$$\begin{aligned}0\oplus(-1)\int_b^a f'_{i.gH}(t,r)dt &= 0\oplus(-1)[\int_b^a (f')^-(t,r)dt, \int_b^a (f')^+(t,r)dt]\\ &= (-1)[f^-(a,r), f^+(a,r)]\ominus(-1)[f^-(b,r), f^+(b,r)]\\ &= \int_a^b f'_{ii.gH}(t,r)dt.\end{aligned}$$

■

Proposition 2.1.

Suppose that $f:[a,b]\to\mathbb{R}_{\mathcal{F}}$, such that $t\in[a,b]$ and there is Hukuhara difference. Then, $(-1)f(t)\ominus(-1)f(t)=0$.

Proof:

$$\begin{aligned}(-1)f(t,r)\ominus(-1)f(t,r) &= (-1)[f^-(t,r), f^+(t,r)]\ominus(-1)[f^-(t,r), f^+(t,r)]\\ &= [-f^+(t,r)+f^+(t,r), -f^-(t,r)+f^-(t,r)]\\ &= 0.\end{aligned}$$

■

Lemma 2.3.

Let $f:[a,b]\to\mathbb{R}_{\mathcal{F}}$ is both continuous and $gH-$differentiable over $[a,b]$ such that type of differentiability f in $[a,b]$ does not change. Then, if $c\in[a,b]$, hence

(i) If $f(t)$ is $[(i)-gH]$-differentiable, then $f'_{i.gH}(t)$ is (FR)-integrable over $[a,b]$ and

$$\int_a^c f'_{i.gH}(t)dt = \int_a^b f'_{i.gH}(t)dt\ominus\int_c^b f'_{i.gH}(t)dt.$$

(ii) If $f(t)$ is $[(ii)-gH]$-differentiable, then $f'_{ii.gH}(t)$ is (FR)-integrable over $[a,b]$ and

$$\int_a^c f'_{ii.gH}(t)dt = \int_a^b f'_{ii.gH}(t)dt\ominus\int_c^b f'_{ii.gH}(t)dt.$$

Proof:

Case (i): Consider $f(t)$ is $[(i)-gH]$-differentiable and Fredholm's integrating:

$$\begin{aligned}\int_a^b f'_{i.gH}(t)dt \ominus \int_c^b f'_{i.gH}(t)dt &= \Big(f(b)\ominus f(a)\Big)\ominus\Big(f(b)\ominus f(c)\Big)\\ &= f(b)\ominus f(a)\ominus f(b)\oplus f(c)\\ &= f(c)\ominus f(a)\\ &= \int_a^c f'_{i.gH}(t)dt.\end{aligned}$$

Case (ii): Consider $f(t)$ is $[(ii)-gH]$-differentiable by using Remark 2.2 (ii) and Proposition 2.1 we have

$$\begin{aligned}\int_a^b f'_{ii.gH}(t)dt \ominus \int_c^b f'_{ii.gH}(t)dt &= \Big((-1)f(a)\ominus(-1)f(b)\Big)\ominus\Big((-1)f(c)\ominus(-1)f(b)\Big)\\ &= (-1)f(a)\ominus(-1)f(b)\ominus(-1)f(c)\oplus(-1)f(b)\\ &= (-1)f(a)\ominus(-1)f(c)\\ &= \int_a^c f'_{ii.gH}(t)dt,\end{aligned}$$

which proves the lemma. ∎

Theorem 2.1.

Let f is continuous, bounded and $gH-$differentiable fuzzy function on $[a,b]$, such that the type of $gH-$differentiability does not change in $[a,b]$. Then, there is a constant number, $c\in[a,b]$, and by considering the type of $gH-$differentiability, we have

(i) If $f(t)$ is $[(i)-gH]$-differentiable, then $D\Big(f(b),f(a)\Big)\leq D\Big((b-a)\odot f'_{i.gH}(c),0\Big)$,

(ii) If $f(t)$ is $[(ii)-gH]$-differentiable, then $D\Big(f(b),f(a)\Big)\leq D\Big((a-b)\odot f'_{i.gH}(c),0\Big)$.

Proof:

Case (i): First, we prove the theorem for $[(i)-gH]$-differentiability. By using Remark 2.2 (i), we obtain $f(a)=f(c)\ominus\int_a^c f'_{i.gH}(t)dt$, and $f(b)=f(c)\oplus\int_c^b f'_{i.gH}(t)dt$. Then, by applying Lemma 2.3 (i), the definition of Hausdorff distance (Molliq et al. (2009)) and Lemma 2.2 (i) we get

$$\begin{aligned}D\Big(f(b),f(a)\Big) &= D\Big(f(c)\oplus\int_c^b f'_{i.gH}(t)dt, f(c)\ominus\int_a^c f'_{i.gH}(t)dt\Big)\\ &\leq D\Big(\int_c^b f'_{i.gH}(t)dt, \ominus\int_a^c f'_{i.gH}(t)dt\Big).\end{aligned}$$

According to the first part of Lemma 2.3 we obtain

$$D\Big(\ominus\int_a^c f'_{i.gH}(t)dt \oplus \int_a^b f'_{i.gH}(t)dt, \ominus\int_a^c f'_{i.gH}(t)dt\Big)$$
$$\leq D\Big(\ominus\int_a^c f'_{i.gH}(t)dt \ominus \int_a^c f'_{i.gH}(t)dt\Big) \oplus D\Big(\int_a^b f'_{i.gH}(t)dt, 0\Big) = D\Big(\int_a^b f'_{i.gH}(t)dt, 0\Big).$$

Therefore, according to Fuzzy Mean Value Theorem for Integrals (Armand et al. (2018)), we have $D\Big(\int_a^b f'_{i.gH}(t)dt, 0\Big) = (b-a)D\Big(f'_{i.gH}(c), 0\Big)$. So, the first part of proving is completed.

Case $(\mathbf{ii})$: To prove the second part of theorem, consider function to be $[(ii)-gH]$ -differentiable and also based on Lemma 2.3 (ii), the definition of Hausdorff distance (Molliq et al. (2009)) and Lemma 2.2 (ii), we get

$$\begin{aligned}
D\Big(f(b), f(a)\Big) &= D\Big(f(c)\ominus(-1)\int_a^b f'_{ii.gH}(t)dt, f(c)\oplus(-1)\int_a^c f'_{ii.gH}(t)dt\Big)\\
&\leq D\Big(\ominus(-1)\int_a^b f'_{ii.gH}(t)dt, (-1)\int_a^c f'_{ii.gH}(t)dt\Big)\\
&= D\Big((-1)\int_a^c f'_{ii.gH}(t)dt \ominus (-1)\int_a^b f'_{ii.gH}(t)dt, (-1)\int_a^c f'_{ii.gH}(t)dt\Big)\\
&= D\Big((-1)\int_a^c f'_{ii.gH}(t)dt, (-1)\int_a^c f'_{ii.gH}(t)dt\Big) \oplus D\Big((-1)\int_a^b f'_{ii.gH}(t)dt, 0\Big)\\
&= D\Big((-1)\int_a^b f'_{ii.gH}(t)dt, 0\Big) = D\Big(\int_b^a f'_{i-gH}(t)dt, 0\Big)\\
&\leq \int_b^a D\Big(f'_{i.gH}(t), 0\Big)dt.
\end{aligned}$$

Therefore, according to Fuzzy Mean Value Theorem for Integrals (Armand et al. (2018)) we obtain $\int_b^a D\Big(f'_{i.gH}(t), 0\Big)dt = (a-b)D\Big(f'_{i.gH}(c), 0\Big)$. The proof is complete. ■

Lemma 2.4.

Suppose that $t \in [0, 1]$. Haar wavelet functions are defined as in Chen et al. (2010). Then

$$h_i(t)h_i(t) \leq \int_0^1 h_i(t)h_i(t)dt.$$

Proof:

Suppose that $j = 0, k = 1$, so

$$h_i(t) = \frac{1}{\sqrt{m}}\begin{cases}1, & 0 \leq t < \frac{1}{2},\\ -1, & \frac{1}{2} \leq t < 1,\\ 0, & \text{elsewhere},\end{cases} \quad h_{i_1}(t)h_{i_2}(t) = \frac{1}{m}\begin{cases}1, & 0 \leq t < \frac{1}{2},\\ -1, & \frac{1}{2} \leq t < 1,\\ 0, & \text{elsewhere}.\end{cases}$$

So, we have

$$h_i(t)h_i(t) = \begin{cases} \frac{1}{m}, & h_i \text{ are both in the first and second intervals,} \\ -\frac{1}{m}, & h_i \text{ are one in the first interval and other in the second one,} \\ 0, & \text{elsewhere.} \end{cases}$$

Since the orthogonality of the sequence $\{h_i(t)\}$ on $[0,1]$ implies that

$$\int_0^1 h_{n_1}(t)h_{n_2}(t)dt = \begin{cases} \frac{1}{m}, & n_1 = n_2, \\ 0, & n_1 \neq n_2, \end{cases}$$

therefore, the proof is complete. ■

Lemma 2.5.

If $h(t) : [a,b] \to R$, then $(-1)h(t) = \ominus_{gH} h(t)$.

Proof:

Since $h(t) : [a,b] \to R$, then $h(t,r) = [h(t,r), h(t,r)] = [h(t), h(t)]$, so

$$\begin{aligned} 0 \ominus_{gH} h(t) &= 0 \ominus_{gH} [h(t), h(t)] \\ &= [\min\{0 - h(t), 0 - h(t)\}, \max\{0 - h(t), 0 - h(t)\}] \\ &= (-1)[h(t), h(t)] \\ &= (-1)h(t). \end{aligned}$$

■

Many properties of fuzzy gH-differentiability has been published very recently in Shahsavari et al. (2020).

3. Solving Fuzzy fractional differential equations

In this section, we intend to approximate the fuzzy function $y(t)$ using the Haar function. The space of all continuous fuzzy-valued function on $[a,b]$ denotes by $C^F[a,b]$. Also, the space of all Lebesque integrable fuzzy-valued functions over bounded interval $[a,b] \subset R$ shows by $L^F[a,b]$.

3.1. Haar Wavelet and Function approximation

For $t \in [0,1]$, the Haar wavelet functions are defined (Chen et al. (2010); Ezzati et al. (2016)) as follows:

$$h_0(t) = \frac{1}{\sqrt{m}}, \quad h_i(t) = \frac{1}{\sqrt{m}} \begin{cases} 2^{j/2}, & \frac{k-1}{2^j} \leq t < \frac{k-(1/2)}{2^j}, \\ -2^{j/2}, & \frac{k-(1/2)}{2^j} \leq t < \frac{k}{2^j}, \\ 0, & \text{elsewhere,} \end{cases} \tag{1}$$

where $i = 0, 1, 2, \cdots, m-1;\ m = 2^{p+1},\ p = 0, 1, 2, \cdots, j$. j and k represent integer decomposition of the index i, i.e., $i = 2^j + k - 1$.

According to Ziari et al. (2012), Mohamed et al. (2019), and Ezzati et al. (2016), let $y(t) \in L^2[0,1)$ and Haar wavelet function $h(t)$ be a real valued bounded function with support of $h(t) \subset [0,1]$. The fuzzy wavelet function it can be expanded into Haar series by

$$y(t) = \sum_{i=0}^{\infty} a_i \odot h_i(t). \tag{2}$$

Let $< .,. >$ denotes the inner product form, and $a_i =< y(t), h_i(t) >= \int_0^1 y(t) \odot h_i(t)dt$ are wavelet coefficients. In practice, only the first m terms of Equation (2) are considered, where m is a power of 2. So we have

$$y(t) \cong \sum_{i=0}^{m-1} a_i \odot h_i(t). \tag{3}$$

The matrix form of Equation (3) is

$$\mathbf{Y} = \mathbf{A^T} \odot \mathbf{H},$$

where $\mathbf{A} = \left[a_0, a_1, \cdots, a_{m-1}\right]^{\mathbf{T}}$. The row vector $\mathbf{Y}$, is the discrete form of the function $y(t)$ and $\mathbf{H}$ is the Haar wavelet matrix of order $m = 2^{p+1},\ p = 0, 1, 2, \cdots,\ j,$ i.e.,

$$\mathbf{H} = \begin{bmatrix} \mathbf{h_0(t_0)} & \mathbf{h_0(t_1)} & \cdots & \mathbf{h_0(t_{m-1})} \\ \mathbf{h_1(t_0)} & \mathbf{h_1(t_1)} & \cdots & \mathbf{h_1(t_{m-1})} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{h_{m-1}(t_0)} & \mathbf{h_{m-1}(t_1)} & \cdots & \mathbf{h_{m-1}(t_{m-1})} \end{bmatrix}.$$

According to Equation (1), we may know easily that $\mathbf{H}$ is an orthogonal matrix. Then, we have

$$\mathbf{A}^T = \mathbf{Y} \odot \mathbf{H}^{-1}.$$

In this paper, we use wavelet collocation method to determine the coefficients a_i. These collocation points are shown as follows:

$$t_l = (l - 0.5)/m,\ l = 1, 2, \cdots, m.$$

3.2. Solving Fuzzy fractional differential equations using Haar Wavelet

Consider to the following fuzzy fractional differential equation ($\mathbf{FFDE}$) of order $\alpha,\ \ 0 < \alpha \leq 1$ with the initial condition:

$$\begin{cases} {}^{C}_{gH}\mathfrak{D}^{\alpha} y(t) = f(t, y(t)), \\ y^{(i)}(t_0) = \delta_i \in \mathbb{R}_{\mathcal{F}}, \end{cases} \tag{4}$$

where $y \in C^F[a,b] \cap L^F[a,b]$ is a continuous fuzzy-valued function and $t_0 \in [a,b]$. We are going to solve Equation (4), so we consider approximate the highest derivative with Haar wavelet. Then,

$${}^{C}_{gH}\mathfrak{D}^{\alpha} y(t) = \sum_{i=0}^{\infty} a_i \odot h_i(t) = A^T \odot H_m(t),$$

in which A is coefficients vector and $H_m(x)$ is Haar vector. According to the sign $h_i(t)$, also, $y(t,r) = [y^-(t,r), y^+(t,r)]$ and $[A]_r = [A^-(r), A^+(r)]$, it is clear we have:

$$^{C}\mathfrak{D}^{\alpha} y^-(t,r) = \sum_{h_i(t)\geq 0} (a^-(r))_i h_i(t) + \sum_{h_i(t)\leq 0} (a^+(r))_i h_i(t), \tag{5}$$

$$^{C}\mathfrak{D}^{\alpha} y^+(t,r) = \sum_{h_i(t)\geq 0} (a^+(r))_i h_i(t) + \sum_{h_i(t)\leq 0} (a^-(r))_i h_i(t). \tag{6}$$

Equations (5) and (6) in matrix form are as follows:

$$^{C}\mathfrak{D}^{\alpha} y^-(t,r) = (A^-(r))^T H_m^+(t) + (A^+(r))^T H_m^-(t), \tag{7}$$

$$^{C}\mathfrak{D}^{\alpha} y^+(t,r) = (A^+(r))^T H_m^+(t) + (A^-(r))^T H_m^-(t). \tag{8}$$

Now, we explain the proposed method to solve Equation (7) by using Caputo derivative properties. By integrating to Equation (7), we achieve lower derivatives as follows:

$$^{C}\mathfrak{D}^{\alpha-1} y^-(t,r) = J^{\alpha-\alpha+1}\left({}^{C}\mathfrak{D}^{\alpha} y^-(t,r)\right) = (A^-(r))^T F H_m^+(t) + (A^+(r))^T F H_m^-(t).$$

But if $\alpha = q \in Z \geq 0$, so to calculate $\mathfrak{D}^q y^-(t,r), q = 0, 1, \cdots, n-1$, we have

$$\begin{aligned}
{}^{C}\mathfrak{D}^{n-1} y^-(t,r) &= J^{\alpha-n+1}\left({}^{C}\mathfrak{D}^{\alpha} y^-(t,r)\right) \\
&= (A^-(r))^T F^{\alpha-n+1} H_m^+(t) + (A^+(r))^T F^{\alpha-n+1} H_m^-(t) + \left(y^-(t_0,r)\right)^{(n-1)} + \left(y^+(t_0,r)\right)^{(n-1)} \\
&= (A^-(r))^T F^{\alpha-n+1} H_m^+(t) + (A^+(r))^T F^{\alpha-n+1} H_m^-(t) + \delta_{n-1}^- e^T + \delta_{n-1}^+ e^T, \\
{}^{C}\mathfrak{D}^{n-2} y^-(t,r) &= J^{\alpha-n+2}\left({}^{C}\mathfrak{D}^{\alpha} y^-(t,r)\right) \\
&= (A^-(r))^T F^{\alpha-n+2} H_m^+(t) + (A^+(r))^T F^{\alpha-n+2} H_m^-(t) + \delta_{n-1}^- e^T \Phi_{m\times m}^{-1} F^1 H_m^+(t) \\
&+ \delta_{n-2}^- + \delta_{n-1}^+ e^T \Phi_{m\times m}^{-1} F^1 H_m^-(t) + \delta_{n-2}^+ e^T, \\
&\vdots \\
y^-(t,r) &= (A^-(r))^T F^{\alpha} H_m^+(t) + (A^+(r))^T F^{\alpha} H_m^-(t) + \delta_{n-1}^- e^T \Phi_{m\times m}^{-1} (F^1)^{n-1} H_m^+(t) \\
&+ \delta_{n-2}^- e^T \Phi_{m\times m}^{-1} (F_{m\times m}^1)^{n-2} H_m^+(t) + \delta_1^- e^T \Phi_{m\times m}^{-1} F_{m\times m}^1 H_m^+(t) + \delta_0^- e^T + \delta_{n-1}^+ e^T \Phi_{m\times m}^{-1} (F^1)^{n-1} H_m^-(t) \\
&+ \delta_{n-2}^+ e^T \Phi_{m\times m}^{-1} (F_{m\times m}^1)^{n-2} H_m^-(t) + \delta_1^+ e^T \Phi_{m\times m}^{-1} F_{m\times m}^1 H_m^-(t) + \delta_0^+ e^T.
\end{aligned}$$

Also, we can use the mentioned above steps for Equation (8). By substituting the above relations in the original Equation (4), it will convert to a fully fuzzy system of equations.

4. Convergence Analysis

In this section, several theorems are presented for fuzzy function convergence analysis.

Theorem 4.1.

Assume that y is a fuzzy-value function on $L^2([0,1])$, ${}_{gH}^{C}\mathfrak{D}^{\alpha}y(t)$ is a fuzzy continuous, $a_i \in \mathbb{R}_{\mathcal{F}}$ and $h_i \in R$ (h_i are Haar bases). Then, the Haar wavelets expansion of $y(t)$ converges uniformly to

$$D(a_i^2, 0) \leq \frac{2^{-3j-2}}{m} M^2.$$

Proof:

By using the definition (Ziari et al. (2012); Mohamed et al. (2019)), we have $y(t) = \sum_{i=0}^{\infty} a_i \odot h_i(t)$. With regard to Definition Haar wavelet function (Chen et al. (2010)), we break the integral boundaries into the intervals of the Haar bases:

$$a_i = \int_0^1 y(t) \odot h_i(t)dt = \int_{\frac{k-1}{2^j}}^{\frac{k-(1/2)}{2^j}} y(t) \odot h_i(t)dt \oplus \int_{\frac{k-(1/2)}{2^j}}^{\frac{k}{2^j}} y(t) \odot h_i(t)dt.$$

By applying Equation (1) and Lemma 2.5 in the above equation, we have

$$a_i = \frac{2^{\frac{j}{2}}}{\sqrt{m}} \Big(\int_{\frac{k-1}{2^j}}^{\frac{k-(1/2)}{2^j}} y(t)dt \ominus_{gH} \int_{\frac{k-(1/2)}{2^j}}^{\frac{k}{2^j}} y(t)dt \Big).$$

Since there exist $t_1,\ t_2$ such that $\frac{k-1}{2^j} \leq t_1 < \frac{k-(1/2)}{2^j}$, $\frac{k-(1/2)}{2^j} \leq t_2 < \frac{k}{2^j}$, then by using fuzzy mean value theorem for integrals (Armand et al. (2018)), we have

$$a_i = \frac{2^{\frac{j}{2}}}{\sqrt{m}} \Big\{ (\frac{k-(1/2)}{2^j} - \frac{k-1}{2^j}) y(t_1) \ominus_{gH} (\frac{k}{2^j} - \frac{k-(1/2)}{2^j}) y(t_2) \Big\} = \frac{2^{\frac{-j}{2}-1}}{\sqrt{m}} \Big(y(t_1) \oplus (-1) y(t_2) \Big).$$

Therefore, $a_i^2 = \frac{2^{-j-2}}{m} \Big(y(t_1) \ominus_{gH} y(t_2) \Big)^2$. By using the definition of Hausdorff distance (Molliq et al. (2009)), we obtain

$$D(a_i^2, 0) = \frac{2^{-j-2}}{m} D^2 \Big(y(t_1) \ominus_{gH} y(t_2), 0 \Big).$$

And also by using the fuzzy mean value theorem of derivative under generalized differentiability (Theorem 2.1), we have:

$$D(a_i^2, 0) = \frac{2^{-j-2}}{m} D^2 \Big(y(t_1) \ominus_{gH} y(t_2), 0 \Big) \leq \frac{2^{-j-2}}{m} D^2 \Big((t_2 - t_1) \odot {}_{gH}^{C}\mathfrak{D}^{\alpha}(c), 0 \Big).$$

Since ${}_{gH}^{C}\mathfrak{D}^{\alpha}y(t)$ is continuous on the interval $[0,1]$, hence by applying the definition of continuous fuzzy function (Anastassion (2010)), we get: $\forall t \in [0,1],\ \exists M > 0,\ \ D\Big({}_{gH}^{C}\mathfrak{D}^{\alpha}y(t), 0\Big) \leq M$. So

$$\begin{aligned} D(a_i^2, 0) &\leq \frac{2^{-j-2}}{m} D^2 \Big((t_2 - t_1) \odot {}_{gH}^{C}\mathfrak{D}^{\alpha}(c), 0 \Big) \\ &\leq \frac{2^{-j-2}}{m} (t_2 - t_1)^2 M^2 \leq \frac{2^{-j-2}}{m} 2^{-2j} M^2 = \frac{2^{-3j-2}}{m} M^2. \end{aligned}$$

The proof is complete. ■

Corollary 4.1.

Let y be fuzzy-value function on $L^2([0,1])$ and ${}^{C}_{i-gH}\mathfrak{D}^{\alpha}y(t)$, ${}^{C}_{ii-gH}\mathfrak{D}^{\alpha}y(t)$ are fuzzy continuous, $a_i \in \mathbb{R}_{\mathcal{F}}$ and $h_i \in R$ (h_i are Haar bases). If $y(t)$ is expanded on Haar series $y(t) = \sum_{i=0}^{\infty} a_i \odot h_i(t)$, and by considering the type of ${}^{cf}[gH]-$differentiability, we have

(i) If $y(t)$ is ${}^{cf}[(i)-gH]$-differentiable, then $D(a_i^2, 0) \leq \frac{2^{-3j-2}}{m} M_1^2$,
(ii) If $y(t)$ is ${}^{cf}[(ii)-gH]$-differentiable, then $D(a_i^2, 0) \leq \frac{2^{j-2}}{m} M_1^2$.

Proof:

Similar to Theorem 4.1, according to the first part of Theorem 2.1, let $y(t)$ be ${}^{cf}[(i)-gH]$-differentiability. Then we get

$$D(a_i^2, 0) = \frac{2^{-j-2}}{m} D^2\Big(y(t_1) \ominus y(t_2), 0\Big) \leq \frac{2^{-j-2}}{m} D^2\Big((t_2 - t_1) \odot {}^{C}_{i-gH}\mathfrak{D}^{\alpha}y(c), 0\Big).$$

Since ${}^{C}_{i-gH}\mathfrak{D}^{\alpha}y(t)$ is continuous on the interval $[0,1]$, hence by using the definition of continuous fuzzy function (Anastassion (2010)), we have:

$$\forall t \in [0,1], \exists M_1 > 0, D\Big({}^{C}_{i-gH}\mathfrak{D}^{\alpha}y(t), 0\Big) \leq M_1.$$

So

$$\begin{aligned} D(a_i^2, 0) &\leq \frac{2^{-j-2}}{m} D^2\Big((t_2 - t_1) \odot {}^{C}_{i-gH}\mathfrak{D}^{\alpha}y(c), 0\Big) \\ &\leq \frac{2^{-j-2}}{m} (t_2 - t_1)^2 M_1^2 \\ &\leq \frac{2^{-j-2}}{m} 2^{-2j} M_1^2 \\ &= \frac{2^{-3j-2}}{m} M_1^2. \end{aligned}$$

So, the first part is complete. Using the second part of Theorem 2.1 (ii) and suppose that $y(t)$ is ${}^{cf}[(ii)-gH]$-differentiability. So we get

$$D(a_i^2, 0) = \frac{2^{-j-2}}{m} D^2\Big(y(t_1) \ominus y(t_2)\Big) \leq \frac{2^{-j-2}}{m} D^2\Big((t_1 - t_2) \odot {}^{C}_{i-gH}\mathfrak{D}^{\alpha}y(c), 0\Big).$$

According to the assumption, ${}^{C}_{i-gH}\mathfrak{D}^{\alpha}y(t)$ be continuous on the interval $[0,1]$, so: $\forall t \in [0,1], \quad \exists M_2 > 0, \quad D\Big({}^{C}_{i-gH}\mathfrak{D}^{\alpha}y(t), 0\Big) \leq M_1$. Then

$$\begin{aligned} D(a_i^2, 0) &\leq \frac{2^{-j-2}}{m} D^2\Big((t_1 - t_2) \odot {}^{C}_{i-gH}\mathfrak{D}^{\alpha}y(c), 0\Big) \\ &= \frac{2^{-j-2}}{m} (t_1 - t_2)^2 M_1^2 \\ &\leq \frac{2^{-j-2}}{m} 2^{2j} M_1^2 \\ &= \frac{2^{j-2}}{m} M_1^2. \end{aligned}$$

The proof is complete. ∎

Theorem 4.2.

Suppose that y is a fuzzy continuous function on the interval $[0,1]$. If the fuzzy functions $y_m(t)$ obtained by using Haar wavelet are the approximation of $y(t)$, then $y_m(t)$ converges to $y(t)$.

Proof:

According to the assumption, let $y_m(t)$ be the approximation of m-first sentence of $y(t)$ obtained by Haar wavelet. So if we prove the $D\Big(y(t), y_m(t)\Big) \to 0$ tends to zero, then $y_m(t)$ converges to $y(t)$.

For this purpose, by using the definition of Hausdorff distance (Molliq et al. (2009)), we have

$$\begin{aligned}
D\Big(y(t), y_m(t)\Big) &= D\Big(\sum_{i=0}^{\infty} a_i \odot h_i(t), \sum_{i=0}^{m} a_i \odot h_i(t)\Big) \\
&\leq D\Big(a_0 \odot h_0, a_0 \odot h_0\Big) \oplus \cdots \oplus D\Big(a_m \odot h_m, a_m \odot h_m\Big) \\
&\qquad \oplus D\Big(\sum_{i=m+1}^{\infty} a_i \odot h_i(t), 0\Big) \\
&= D\Big(\sum_{i=m+1}^{\infty} a_i \odot h_i(t), 0\Big) \\
&= D\Big(\sum_{n=2^{m+1}}^{\infty} a_n \odot h_n(t), 0\Big).
\end{aligned}$$

So by applying Lemma 2.4, we get

$$\begin{aligned}
D^2\Big(y(t), y_m(t)\Big) &\leq D\Big(\sum_{n=2^{m+1}}^{\infty} a_n \odot h_n(t), 0\Big)^2 \\
&= D\Big(\sum_{n_1=2^{m+1}}^{\infty} \sum_{n_1=2^{m+1}}^{\infty} a_{n_1} \odot a_{n_2} \odot h_{n_1} h_{n_2}, 0\Big) \\
&\leq D\Big(\sum_{n_1=2^{m+1}}^{\infty} \sum_{n_1=2^{m+1}}^{\infty} a_{n_1} \odot a_{n_2} \odot \int_0^1 h_{n_1} h_{n_2} dt, 0\Big) \\
&= \frac{1}{m} D\Big(\sum_{j=p+1}^{\infty} \sum_{n=2^j}^{2^{j+1}-1} a_n^2, 0\Big).
\end{aligned}$$

It is enough to prove $D\Big(\sum_{j=p+1}^{\infty} \sum_{n=2^j}^{2^{j+1}-1} a_n^2, 0\Big) \longrightarrow 0$. According to Theorem 4.1,

$$
\begin{aligned}
D^2\Big(y(t), y_m(t)\Big) &\leq \frac{1}{m} D\Big(\sum_{j=p+1}^{\infty} \sum_{n=2^j}^{2^{j+1}-1} a_n^2, 0\Big) \\
&= \frac{1}{m} \sum_{j=p+1}^{\infty} \sum_{n=2^j}^{2^{j+1}-1} D(a_n^2, 0) \\
&\leq \frac{1}{m} \sum_{j=p+1}^{\infty} \sum_{n_1=2^j}^{2^{j+1}-1} \frac{2^{-3j-2}}{m} M^2 \\
&= \frac{M^2}{m^2} \sum_{j=p+1}^{\infty} 2^{-3j-2}(2^{j+1} - 1 - 2^j + 1) \\
&= \frac{M^2}{3m^2} \frac{1}{m^2} \\
&= \frac{M^2}{3m^4}.
\end{aligned}
$$

So $D\Big(y(t), y_m(t)\Big) \leq \frac{M}{\sqrt{3m^2}}$. We can see that $D\Big(y(t), y_m(t)\Big) \longrightarrow 0$ when $m \longrightarrow \infty$. As a result $y(t) = y_m(t)$. ∎

5. Numerical experiments

In this section, the proposed method is utilized to study three examples of the fuzzy fractional differential equations. The computations associated with the examples are performed using Mathematica software.

Example 5.1.

Consider the following fuzzy Bagley-Torvik equation (Diethelm et al. (2002)),

$$
\begin{cases}
{}_{gH}\mathfrak{D}^2 y(t) \oplus {}^{C}_{gH}\mathfrak{D}^{1.5} y(t) \oplus y(t) \ominus_{gH} (t+1) = 0, \\
y(0,r) = [r, r+1], \quad y'_{gH}(0,r) = [r, r-2].
\end{cases}
$$

The exact value of this equation is $y(t,r) = [r, r-2](t+1)$.

By attention to method describe in detail, for $m = 8$ we have the following approximation solutions for $y^-(t,r)$ and $y^+(t,r)$, respectively.

$$y^-(t,r) = \begin{cases} 0.2126t^2 + r(1 + t - 0.2555t^2), & 0 \le t < 0.125; \\ -0.00105 + 0.0168t + 0.1452t^2 + r(1.0003 + 0.9950t - 0.2356t^2), & 0.125 \le t < 0.250; \\ -0.0036 + 0.0373t + 0.1043t^2 + r(0.9992 + 1.0038t - 0.2532t^2), & 0.25 \le t < 0.375; \\ -0.0082 + 0.0621t + 0.0712t^2 + r(1.0003 + 0.9975t - 0.2449t^2), & 0.375 \le t < 0.5; \\ -0.0156 + 0.0916t + 0.0417t^2 + r(0.9898 + 1.0398t - 0.2871t^2), & 0.5 \le t < 0.625; \\ -0.0263 + 0.1259t + 0.0142t^2 + r(0.9956 + 1.0212t - 0.2723t^2), & 0.625 \le t < 0.75; \\ -0.0411 + 0.1653t - 0.0120t^2 + r(0.9888 + 1.0393t - 0.2843t^2), & 0.75 \le t < 0.875; \\ -0.0605 + 0.2097t - 0.0373t^2 + r(1.0013 + 1.0106t - 0.2679t^2), & True. \end{cases}$$

$$y^+(t,r) = \begin{cases} 2 + 2t + 0.3893t^2 + r(-1 - t - 0.2555t^2), & 0 \le t < 0.125; \\ 1.9989 + 2.0168t + 0.3220t^2 + r(-0.9996 - 1.0049t - 0.2356t^2), & 0.125 \le t < 0.250; \\ 1.9963 + 2.0373t + 0.2810t^2 + r(-1.0007 - 0.9961t - 0.2532t^2), & 0.25 \le t < 0.375; \\ 1.9917 + 2.0621t + 0.2480t^2 + r(-0.9996 - 1.0024t - 0.2449t^2), & 0.375 \le t < 0.5; \\ 1.9843 + 2.0916t + 0.2185t^2 + r(-1.0101 - 0.9601t - 0.2871t^2), & 0.5 \le t < 0.625; \\ 1.9736 + 2.1259t + 0.1910t^2 + r(-1.0043 - 0.9787t - 0.2723t^2), & 0.625 \le t < 0.75; \\ 1.9588 + 2.1653t + 0.1647t^2 + r(-1.0111 - 0.9606t - 0.2843t^2), & 0.75 \le t < 0.875; \\ 1.9394 + 2.2097t + 0.1394t^2 + r(-0.9986 - 0.9893t - 0.2679t^2), & True. \end{cases}$$

The exact and approximation solutions in Figure 1 and 2. Table 1 reports the computational results. The results show that the approximate solutions are in good agreement with the exact solutions.

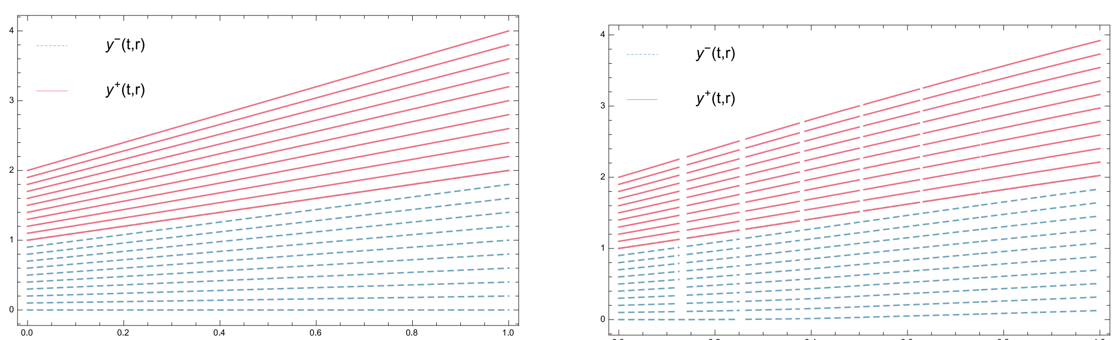

Figure 1. Graph of exact value (Left) and approximation value (Right) of Example 5.1 for $0 \le t \le 1$ and $0 \le r \le 1$

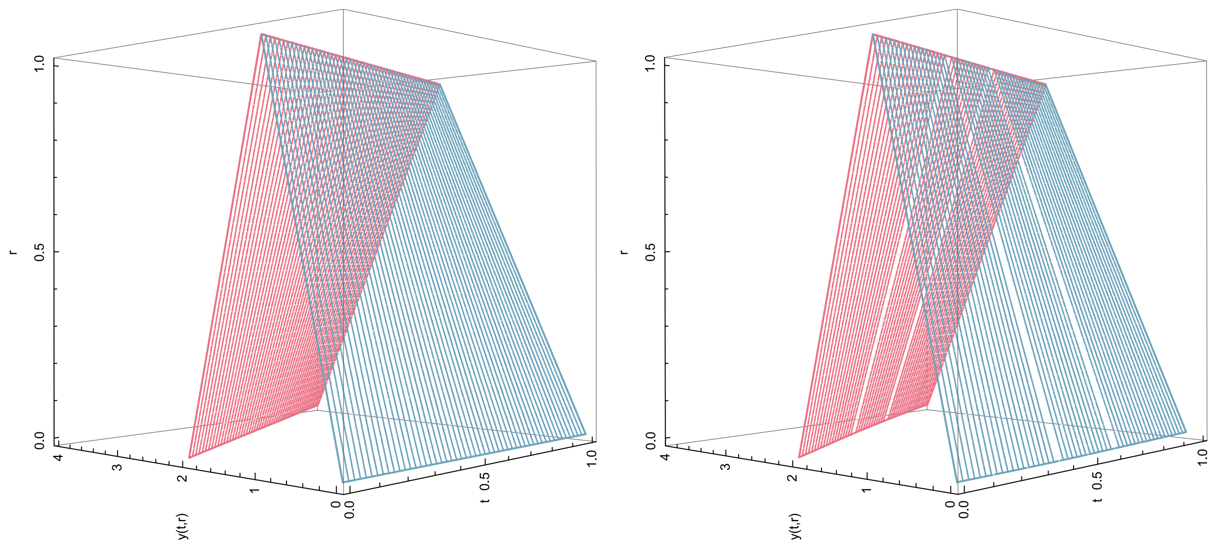

Figure 2. Graph of exact value (Left) and approximation value (Right) of Example 5.1 for $0 \leq t \leq 1$ and $0 \leq r \leq 1$ and $m = 32$

		$y^-(t,r)$	$y^+(t,r)$
r	t	$m = 8$	$m = 8$
0.3	0.3	3.81639×10^{-8}	6.38378×10^{-8}
	0.6	9.71445×10^{-7}	7.63278×10^{-7}
	0.9	8.18789×10^{-8}	1.31839×10^{-8}
0.6	0.3	1.66533×10^{-6}	9.99201×10^{-6}
	0.6	3.33067×10^{-6}	4.44089×10^{-6}
	0.9	5.55112×10^{-5}	4.99693×10^{-6}
0.9	0.3	2.22045×10^{-6}	1.77636×10^{-8}
	0.6	4.24624×10^{-7}	1.26745×10^{-7}
	0.9	2.22045×10^{-7}	2.22045×10^{-7}

Table 1. Numerical results for Example 5.1

Example 5.2.

Consider the following fuzzy fractional differential equation:

$$\begin{cases} {}_{gH}\mathfrak{D}^2 y(t) \ominus_{gH} 2\,{}_{gH}\mathfrak{D} y(t) \oplus {}^{C}_{gH}\mathfrak{D}^{\frac{1}{2}} y(t) \ominus_{gH} (6t \ominus_{gH} 6t^2 \oplus \frac{16}{5\sqrt{\pi}} t^{\frac{5}{2}} + t^3) = 0, \\ y(0,r) = 0, \quad y'_{gH}(0,r) = 0. \end{cases}$$

This equation has the exact solution $y(t,r) = [0.5 + 1.5r, 6 - 4r]t^3$. By using the Haar wavelet method for $m = 8$, we have

$$y^-(t,r) = \begin{cases} (0.0994 + 0.2983r)t^2, & 0 \le t < 0.125; \\ 0.0029 - 0.0469t + 0.2874t^2 + r(0.00881 - 0.1409t + 0.8622t^2), & 0.125 \le t < 0.250; \\ 0.01468 - 0.1409t + 0.4753t^2 + r(0.0441 - 0.4229t + 1.4261t^2), & 0.25 \le t < 0.375; \\ 0.04110 - 0.2818t + 0.6632t^2 + r(0.1233 - 0.8456t + 1.9898t^2), & 0.375 \le t < 0.5; \\ 0.1613 - 0.7042t + 1.0387t^2 + r(0.4840 - 2.1127t + 3.1162t^2), & 0.625 \le t < 0.75; \\ 0.26681 - 0.9854t + 1.2262t^2 + r(0.8004 - 2.9564t + 3.6786t^2), & 0.75 \le t < 0.875; \\ 0.4101 - 1.3131t + 1.4134t^2 + r(1.2305 - 3.9395t + 4.2404t^2), & 0.75 \le t < 0.875; \\ 0.0880 - 0.4696t + 0.8510t^2 + r(0.2641 - 1.4090t + 2.5532t^2), & True. \end{cases}$$

$$y^+(t,r) = \begin{cases} (1.19354 - 0.795695r)t^2, & 0 \le t < 0.125; \\ 0.0352 - 0.5639t + 3.4491t^2 + r(-0.0234 + 0.3759t - 2.2994t^2), & 0.125 \le t < 0.250; \\ 0.1762 - 1.6916t + 5.7047t^2 + r(-0.1174 + 1.1277t - 3.8031t^2), & 0.25 \le t < 0.375; \\ 0.4932 - 3.3827t + 7.9595t^2 + r(-0.3288 + 2.2551t - 5.3063t^2), & 0.375 \le t < 0.5; \\ 1.9362 - 8.4510t + 12.4648t^2 + r(-1.2908 + 5.6340t - 8.3098t^2), & 0.625 \le t < 0.75; \\ 0.26681 - 0.9854t + 1.2262t^2 + r(0.8004 - 2.9564t + 3.6786t^2), & 0.75 \le t < 0.875; \\ 3.2017 - 11.8256t + 14.7145t^2 + r(-2.1345 + 7.8837t - 9.8096t^2), & 0.75 \le t < 0.875; \\ 4.9222 - 15.7582t + 16.9617t^2 + r(-3.2814 + 10.5054t - 11.3078t^2), & 0.875 \le t < 1; \\ 1.0566 - 5.6362t + 10.213t^2 + r(-0.7044 + 3.7575t - 6.8086t^2), & True. \end{cases}$$

The exact and approximation solutions in Figures 3 and 4 for $0 \le r \le 1$ and $0 \le t \le 1$. The computational results are reported in Table 2. The results show that the approximate solutions are in good agreement with the exact solutions.

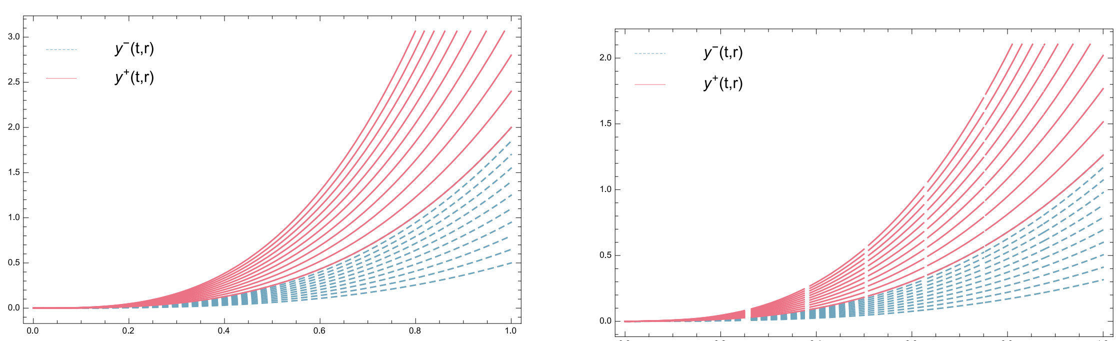

Figure 3. Graph of exact value (Left) and approximation value (Right) of Example 5.2 for $0 \le t \le 1$ and $0 \le r \le 1$

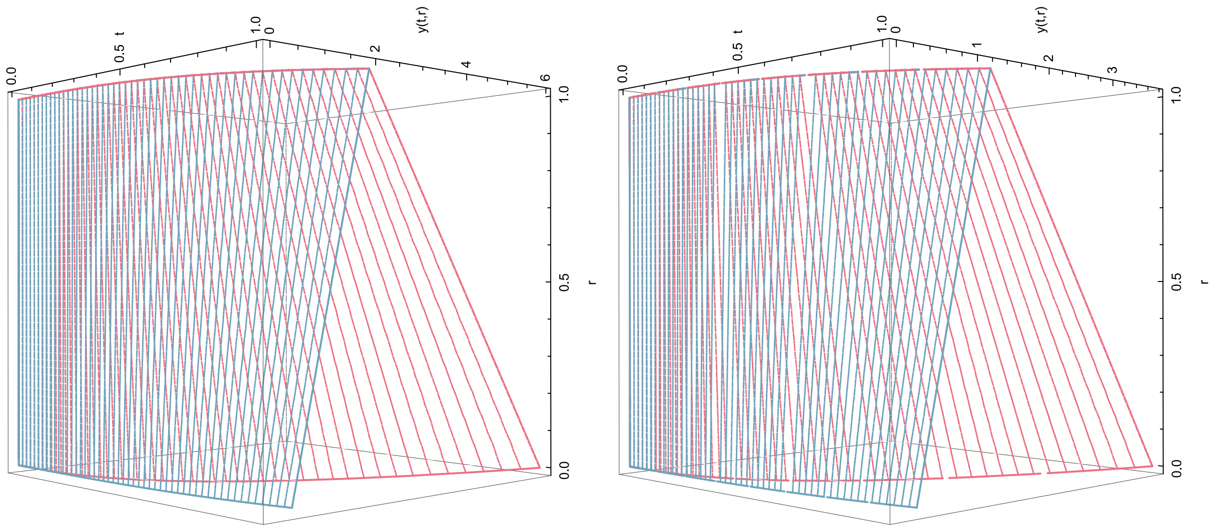

Figure 4. Graph of exact value (Left) and approximation value (Right) of Example 5.1 for $0 \leq t \leq 1$ and $0 \leq r \leq 1$ and $m = 32$

		$y^-(t,r)$	$y^+(t,r)$
r	t	$m = 32$	$m = 32$
0.3	0.3	8.88178×10^{-10}	3.56218×10^{-10}
	0.6	1.77636×10^{-11}	8.88178×10^{-11}
	0.9	5.32907×10^{-10}	8.88175×10^{-10}
0.6	0.3	4.44089×10^{-9}	3.55271×10^{-10}
	0.6	2.66454×10^{-10}	3.55271×10^{-10}
	0.9	8.88178×10^{-11}	5.32907×10^{-11}
0.9	0.3	1.77636×10^{-9}	2.34636×10^{-9}
	0.6	1.95399×10^{-10}	1.06581×10^{-9}
	0.9	1.42109×10^{-10}	5.50671×10^{-10}

Table 2. Numerical results for Example 5.2

Example 5.3.

Consider the following fuzzy fractional differential equation:

$$\begin{cases} {}_{gH}\mathfrak{D}^{\frac{3}{2}} y(t) \oplus y(t) \ominus_{gH} (t^{\frac{5}{2}} + \frac{5.8905}{\sqrt{\pi}} t) = 0, \\ y(0,r) = 0, \quad y'_{gH}(0,r) = 0. \end{cases}$$

The exact fuzzy solution of the problem is $y(t,r) = [2+3r, 11-6r]t^{\frac{5}{2}}$. This exact solution and the approximation solution by the Haar wavelet method are shown in Figures 5 and 6. Table 3 shows the approximate solutions obtained by the Haar wavelet method. It is evident from Table 3 that the numerical solutions converge to the exact solution.

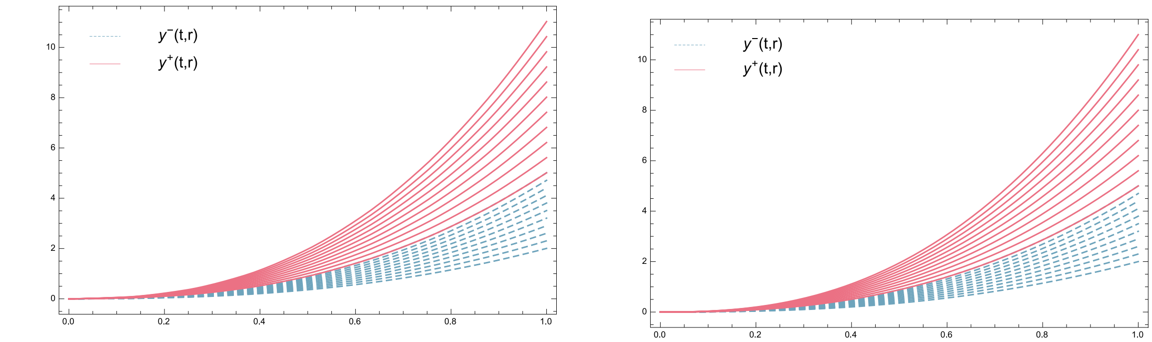

Figure 5. Graph of exact (Left) and approximation (Right) solution of Example 5.3 for $0 \leq t \leq 1$ and $0 \leq r \leq 1$

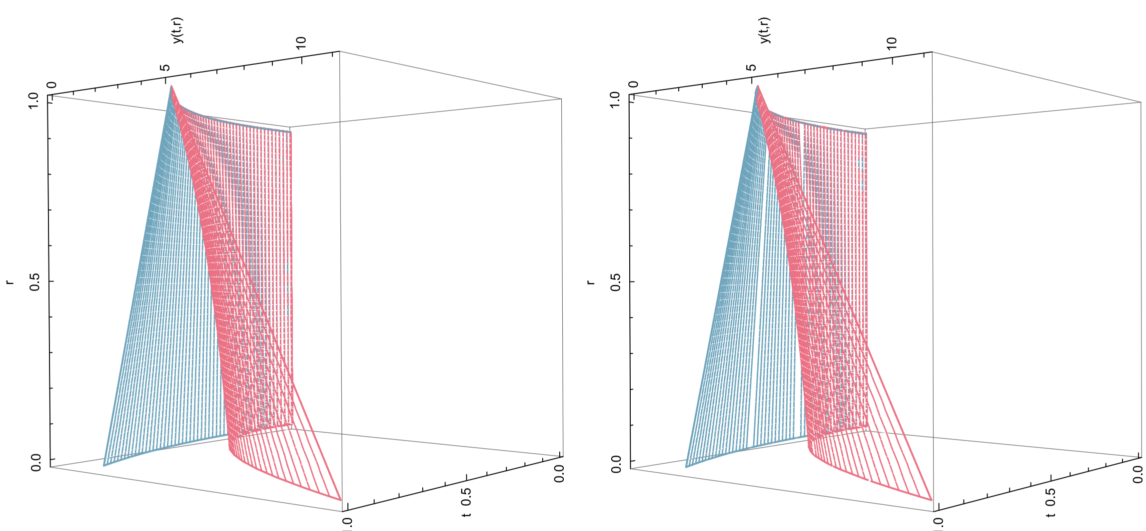

Figure 6. Graph of exact (Left) and approximation (Right) value of $y(t, r)$ of Example 5.3 for $0 \leq t \leq 1$ and $0 \leq r \leq 1$ and $m = 8$

		$y^{-}(t,r)$	$y^{+}(t,r)$
r	t	$m=32$	$m=32$
0.3	0.3	1.85211×10^{-10}	1.03556×10^{-10}
	0.6	1.54845×10^{-10}	1.08187×10^{-10}
	0.9	1.24485×10^{-10}	1.12842×10^{-10}
0.6	0.3	2.43887×10^{-10}	5.17273×10^{-11}
	0.6	1.89542×10^{-10}	7.97329×10^{-11}
	0.9	1.35185×10^{-10}	1.07727×10^{-10}
0.9	0.3	1.11655×10^{-10}	8.68369×10^{-10}
	0.6	6.46392×10^{-11}	6.24644×10^{-10}
	0.9	2.40913×10^{-10}	3.80909×10^{-10}

Table 3. Numerical results for Example 5.3

6. Conclusion

We derive a numerical method for fuzzy fractional differential equations based on Haar wavelet operational matrices of the fractional-order integration. Some examples were examined using the Haar wavelet and the results show remarkable performance. So far no paper has used the Haar wavelet method using generalized difference and fuzzy derivatives, and this is the first time we have done so. In future research, we will investigate the solution of fuzzy nonlinear Volterra-Fredholm integro-differential equations with arbitrary fuzzy and crisp kernels.

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