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Adaptive Hybrid Projective Synchronization Of Hyper-chaotic Systems

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Abstract

In this paper, we design a procedure to investigate the hybrid projective synchronization (HPS) technique among two identical hyper-chaotic systems. An adaptive control method (ACM) is proposed which is based on Lyapunov stability theory (LST). The considered technique globally determines the asymptotical stability and establishes identification of parameter simultaneously via HPS approach. Additionally, numerical simulations are carried out for visualizing the effectiveness and feasibility of discussed scheme by using MATLAB.

Keywords: Adaptive control; hybrid projective synchronization; hyper-chaotic system; Lyapunov stability theory; MATLAB

MSC 2010 No.: 34K23, 34K35, 37B25, 37N35

1. Introduction

Undoubtedly, chaos theory has now become one of the most influential field of applicable mathematics having a broad ranging of applications in numerous areas of applied sciences, engineering and technology such as secure communication (Naderi and Kheiri (2016)), robotics (Patle et al. (2018)), finance models (Tong et al. (2015)), neural networks (Bouallegue (2017)), weather models (Russell et al. (2017)), ecological models (Sahoo and Poria (2014)), jerk systems (Wang et al.

(2017)), encryption (Wu et al. (2016)), oscillations (Ghosh et al. (2018)), etc. As a result, chaos theory has acquired a vital consideration in several research fields.

Fundamentally, chaotic system acquires the unique characteristic of exhibiting extreme high sensitivity for initial conditions and parameter data. Remarkably, Pecora and Carroll (1990) investigated chaos synchronization phenomenon in chaotic systems using a master-slave framework, which was unprecedented for more than two decades. Furthermore, Shinbrot et al. (1990) initiated a technique known as OGY methodology for controlling chaotic systems. Until now, different kinds of synchronization schemes and control techniques in chaotic systems have been proposed (Zhou and Zhu (2011); Ma et al. (2017); Singh et al. (2017); Li and Liao (2004); Khan and Chaudhary (2020b); Sudheer and Sabir (2009); Li (2007); Khan and Chaudhary (2020a); Khan and Chaudhary (2019); Ding and Shen (2016); Khan and Chaudhary (2019); Delavari and Mohadeszadeh (2018); Rasappan and Vaidyanathan (2012); Chen and Han (2003); Li and Zhang (2016); Jahanzaib et al. (2020); Sanjay et al. (2020)), etc.

In the available literature, a hyper-chaotic system has been identified as a chaotic system that has more than one +ve Lyapunov exponents. Most importantly, (Rossler (1979)) proposed the first classic hyper-chaotic system. Since then, many classic hyper-chaotic systems are emerged such as Lorenz system, Chen model, Nikolov system, Liu system, Vaidyanathan system, Pehlvian system and many more. Some hyper-chaotic systems may be generated directly by adding one or more variables to the original 3D-chaotic systems such as Lorenz system, Qi system, Chen model, Lu model and so on. The investigation for hyper-chaoticity of nonlinear models is still in its inception and dynamics of hyper-chaotic systems is not completely understood by the researchers. Consequently, hyper-chaos has drawn the attention from various engineering and scientific communities.

Synchronization phenomenon between chaotic systems via adaptive control method (ACM) was firstly initiated by Hubler (1989). Mainieri and Rehacek (1999) proposed the concept of projective synchronization while synchronizing the chaotic models. In Liao and Tsai (2000), synchronization of two chaotic systems has been studied separately via adaptive control method and also it is exhibited through numerical results that it has applications in secured communications. In Yassen et al. (2003), synchronization of a modified Chua's circuit system using adaptive control method has been discussed. Furthermore, projective synchronization and chaos in secure communication are studied in Li and Xu (2004). Additionally, in Li et al. (2012), adaptive backstepping scheme in synchronizing chaotic systems is discussed. Moreover, by Wu et al. (2012), complex projective synchronized technique is studied in complex chaotic systems. In Vaidyanathan (2015), ACM is discussed to synchronize the generalized three-species Lotka-Volterra biological systems. In Khan and Chaudhary (2020b), Khan and Chaudhary (2020a), Khan and Tyagi (2017c), and Khan and Tyagi (2017a), enormous control approaches have been analyzed in depth for newly formulated hyper-chaotic systems.

Keeping in view the aforesaid discussions, this manuscript focuses on proposing hybrid projective synchronization (HPS) among two identical hyper-chaotic systems by ACM. ACM is very significant in estimating the parameters used in master and slave systems. Thus, by applying this approach, a smaller amount of information is needed to synchronize the considered master and slave systems. Further, we discuss in much detail a desired adaptive control law and an estimating parameter update law, which is based on LST.

The present paper is described as follows. Section 2 deals with the essential preliminaries comprising of some notations and terminology that are used within this article. In Section 3, a methodology of ACM has been described comprehensively. Section 4 consists of basic structured features of the discussed systems. In addition, this section investigates the proposed ACM along with a parameter estimation update laws to stabilize asymptotically the given hyper-chaotic systems. Moreover, numerical simulations demonstrating the effectiveness as well as the feasibility of considered HPS approach are performed using MATLAB software. Section 5 deals with a comparative study. Finally, Section 6 concludes the paper with a precise list of references given at the end of the paper.

2. Preliminaries

In this section, we recall essential terminology and few notations and state some elementary results that are used throughout the article. Consider the master/drive system and the corresponding slave/response system as:

$$\dot{x_1} = f(x_1), \tag{1}$$

$$\dot{y_1} = g(y_1) + u, \tag{2}$$

where $x_1 = (x_{11}, x_{12}, \ldots, x_{1n})^T$, $y_1 = (y_{11}, y_{12}, \ldots, y_{1n})^T$ are the state vectors of (1) and (2) respectively, $f, g : R^n \to R^n$ are two nonlinear continuous vector functions and $u = (u_{11}, u_{12}, \ldots, u_{1n}) \in R^n$ is the suitable controller to be constructed.

Definition 2.1.

The master system (1) and the slave system (2) are said to be in hybrid projective synchronization (HPS) if

$$\lim_{t\to\infty} \|e(t)\| = \lim_{t\to\infty} \|y_1(t) - \alpha x_1(t)\| = 0, \tag{3}$$

for some $\alpha = \mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\|\cdot\|$ represents vector norm.

Remark 2.1.

Complete synchronized state of (1) and (2) is achieved if $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 1$.

Remark 2.2.

Anti-synchronized state of (1) and (2) is attained if $\alpha_1 = \alpha_2 = \ldots = \alpha_n = -1$.

3. Synchronization Phenomena

Despite several available synchronization techniques, in this paper, we study the adaptive control method as it is applicable in case of entirely unknown parameters.

Suppose the chaotic (or hyper-chaotic) master system and corresponding chaotic (hyper-chaotic) slave system are considered as:

$$\dot{x} = f(x) + g(x)\eta, \tag{4}$$

$$\dot{y} = f(y) + g(y)\eta + u, \tag{5}$$

where $x = (x_1, x_2, \ldots, x_n)^T$, $y = (y_1, y_2, \ldots, y_n)^T$ are state vectors, $u = (u_i, i = 1, 2, \ldots, n) \in R^n$ is the required controller, $f : R^n \to R^n$, $g : R^n \to R^{n\times p}$ are two nonlinear continuous vector functions. $\eta = (\eta_i; i = 1, 2, \ldots, p)^T$ is the known parameter vector and $(\cdot)^T$ describes transpose.

Error is defined by

$$e(t) = y(t) - x(t).$$

The systems (4) and (5) are in a synchronized state if

$$\lim_{t\to\infty} \|e(t)\| = 0,$$

where $e(t) = (e_i; i = 1, 2, \ldots, n)^T$ denotes error function and $\|\cdot\|$ represents vector norm.

From (4) and (5), the error dynamics becomes:

$$\dot{e}(t) = f(y) + g(y)\eta + u - f(x) - g(x)\eta.$$

We next design the control function u appropriately and parameter updating law to ensure that systems (4) and (5) get synchronized with unknown parameters.

Let us define the controller

$$u = -f(y) - g(y)\hat{\eta} + f(x) + g(x)\hat{\eta} - Ke,$$

where $\hat{\eta}(t) = (\hat{\eta}_i; i = 1, 2, \ldots, p)$ is uncertain parameter and K is any positive number chosen arbitrarily known as gain constant.

Define parameter update law by

$$\dot{\hat{\eta}} = -[g(x)]^T + K_\eta\tilde{\eta},$$

where $K = \mathrm{diag}(K_i; i = 1, 2, \ldots, n)$, $K_\eta = \mathrm{diag}(K_i; i = 1, 2, \ldots, p)$ and $\tilde{\eta} = \eta - \hat{\eta}$.

Choose the classic Lyapunov function as

$$V(t) = \frac{1}{2}(e^T e + \tilde{\eta}^T\tilde{\eta}),$$

which guarantees that V is positive definite.

Differentiability of Lyapunov function implies that

$$\begin{aligned}
\dot{V}(t) &= e^T\dot{e} + \tilde{\eta}^T(-\dot{\hat{\eta}}) \\
&= e^T[-g(x)\tilde{\eta} - Ke] - \tilde{\eta}^T[-[g(x)]^T e + K_\eta\tilde{\eta}] \\
&= -e^T g(x)\tilde{\eta} - e^T Ke + \tilde{\eta}^T[g(x)]^T e - \tilde{\eta}^T K_\eta\tilde{\eta} \\
&= -[\tilde{\eta}^T[g(x)]^T e]^T + \tilde{\eta}^T[g(x)]^T e - e^T Ke - \tilde{\eta}^T K_\eta\tilde{\eta}
\end{aligned}$$

$$= -e^T K e - \tilde{\eta}^T K_\eta \tilde{\eta}$$
$$< 0,$$

which exhibits that $\dot{V}$ is negative definite.

In view of Lyapunov stability theory (Shevitz and Paden (1914); Perko (2013)), the error dynamics acquires global asymptotic stability in neighbourhood of considered equilibrium points.

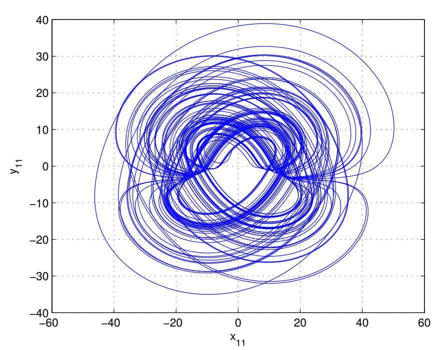
The next section presents the HPS technique to control hyper-chaos of (6) using adaptive control method.

4. Illustrative Example 1

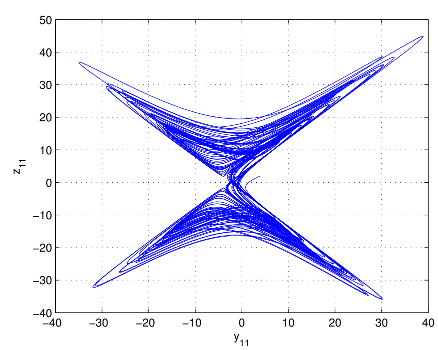
Introduced by Dong et al. (2016), the considered hyper-chaotic system is described as:

$$\begin{cases} \dot{x}_{11} = a_1 x_{11} - b_1 y_{11} z_{11}, \\ \dot{y}_{11} = -c_1 y_{11} + x_{11} z_{11}, \\ \dot{z}_{11} = k_1 x_{11} - d_1 z_{11} + x_{11} y_{11}, \\ \dot{w}_{11} = h_1 w_{11} + x_{11} y_{11}, \end{cases} \tag{6}$$

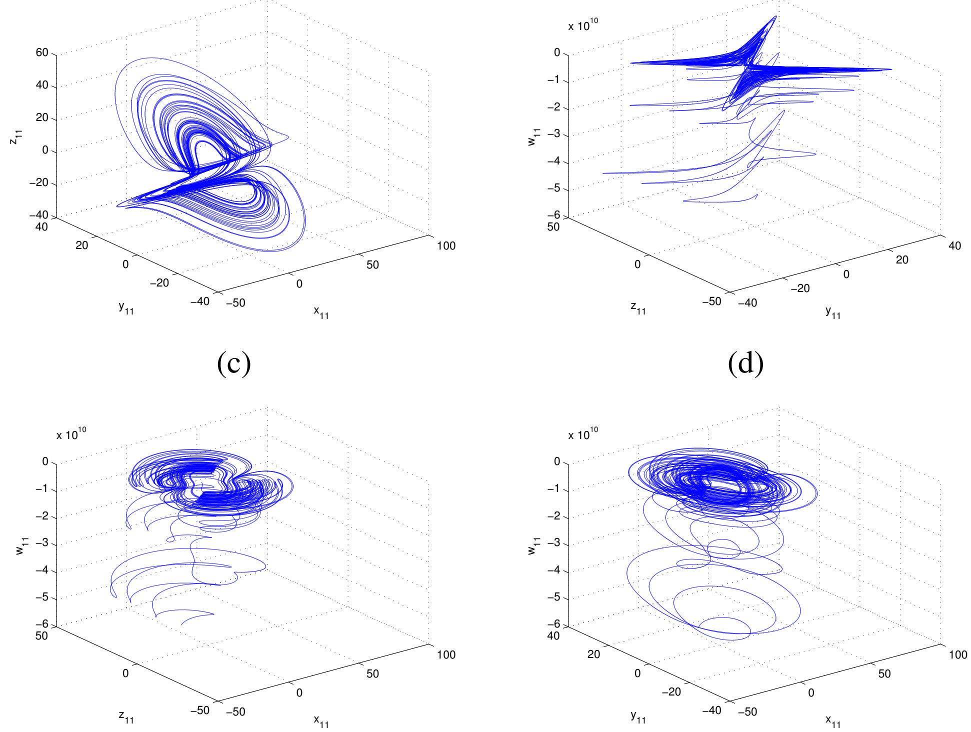
where $(x_{11}, y_{11}, z_{11}, w_{11})^T \in R^4$ is state vector and a_1,b_1,c_1,d_1,k_1 and h_1 are positive parameters. When $a_1 = 4.55$, $b_1 = 1.532$, $c_1 = 10.1$, $d_1 = 5.5$, $k_1 = 3.5$ and $h_1 = 0.04$, the system (6) exhibits hyper-chaos. Furthermore, Figure 1(a-f) depict phase diagrams of (6). In addition, Lyapunov exponents of system (6) are determined as $L_1 = 1.5278$, $L_2 = 0.041041$, $L_3 = 0.0023108$ and $L_4 = -12.5454$ which show the hyper-chaotic behaviour of (6). However, the detailed analytical study and numerical results for the system (6) can be found in Khan and Tyagi (2017a).



(a)



(b)



(c) (d)

(e) (f)

Figure 1. Phase diagrams of hyper-chaotic system in (a) $x_{11} - y_{11}$ plane, (b) $y_{11} - z_{11}$ plane, (c) $x_{11} - y_{11} - z_{11}$ space, (d) $y_{11} - z_{11} - w_{11}$ space, (e) $x_{11} - z_{11} - w_{11}$ space, (f) $x_{11} - y_{11} - w_{11}$ space

The next section presents the HPS technique to control hyper-chaos of (6) using adaptive control method.

For convenience, the system (6) is considered as the master system and the slave system can be defined by the following expression:

$$\begin{cases} \dot{x}_{21} = a_1 x_{21} - b_1 y_{21} z_{21} + u_{11}, \\ \dot{y}_{21} = -c_1 y_{21} + x_{21} z_{21} + u_{12}, \\ \dot{z}_{21} = k_1 x_{21} - d_1 z_{21} + x_{21} y_{21} + u_{13}, \\ \dot{w}_{21} = h_1 w_{21} + x_{21} y_{21} + u_{14}, \end{cases} \tag{7}$$

where u_{11}, u_{12}, u_{13} and u_{14} are adaptive controllers to be designed in such a manner that HPS between two identical hyper-chaotic models will be achieved.

Define the state errors by

$$\begin{cases} e_{11} = x_{21} - \alpha_1 x_{11}, \\ e_{12} = y_{21} - \alpha_2 y_{11}, \\ e_{13} = z_{21} - \alpha_3 z_{11}, \\ e_{14} = w_{21} - \alpha_4 w_{11}. \end{cases} \tag{8}$$

The main objective in this paper is the introduction of appropriate controllers $u_{1i}, (i = 1, 2, 3, 4)$ so that the considered state errors must satisfy

$$\lim_{t\to\infty} e_{1i}(t) = 0 \quad \text{for} \ (i = 1, 2, 3, 4).$$

Subsequent error dynamics is simplified as:

$$\begin{cases} \dot{e_{11}} = a_1 e_{11} - b_1(y_{21}z_{21} - \alpha_1 y_{11}z_{11}) + u_{11}, \\ \dot{e_{12}} = -c_1 e_{12} + x_{21}z_{21} - \alpha_2 x_{11}z_{11} + u_{12}, \\ \dot{e_{13}} = k_1 e_{11} + k_1(\alpha_1 - \alpha_3)x_{11} - d_1 e_{13} + x_{21}y_{21} - \alpha_3 x_{11}y_{11} + u_{13}, \\ \dot{e_{14}} = h_1 e_{14} + x_{21}y_{21} - \alpha_4 x_{11}y_{11} + u_{14}. \end{cases} \tag{9}$$

Now, we define the adaptive controllers as:

$$\begin{cases} u_{11} = -\hat{a_1} e_{11} + \hat{b_1}(y_{21}z_{21} - \alpha_1 y_{11}z_{11}) - K_1 e_{11}, \\ u_{12} = \hat{c_1} e_{12} - x_{21}z_{21} + \alpha_2 x_{11}z_{11} - K_2 e_{12}, \\ u_{13} = -\hat{k_1} e_{11} - \hat{k_1} x_{11}(\alpha_1 - \alpha_3) + \hat{d_1} e_{13} - (x_{21}y_{21} - \alpha_3 x_{11}y_{11}) - K_3 e_{13}, \\ u_{14} = -\hat{h_1} e_{14} - x_{21}y_{21} + \alpha_4 x_{11}y_{11} - K_4 e_{14}, \end{cases} \tag{10}$$

where $K_i > 0$, $i = 1, 2, 3, 4$ and are called gain constants.

By substituting the controllers (10) in the error dynamics (9), we obtain

$$\begin{cases} \dot{e}_{11} = (a_1 - \hat{a_1})e_{11} - (b_1 - \hat{b_1})(y_{21}z_{21} - \alpha_1 y_{11}z_{11}) - K_1 e_{11}, \\ \dot{e}_{12} = -(c_1 - \hat{c_1})e_{12} - K_2 e_{12}, \\ \dot{e}_{13} = (k_1 - \hat{k_1})e_{11} + (k_1 - \hat{k_1})x_{11}(\alpha_1 - \alpha_3) - (d_1 - \hat{d_1})e_{13} - K_3 e_{13}, \\ \dot{e}_{14} = (h_1 - \hat{h_1})e_{14} - K_4 e_{14}, \end{cases} \tag{11}$$

where $\hat{a}_1$, $\hat{b}_1$, $\hat{c}_1$, $\hat{d}_1$, $\hat{h}_1$, $\hat{k}_1$ are estimated values for unknown parameters a_1, b_1, c_1, d_1, h_1, k_1, respectively.

We define parameter estimation error as follows:

$$\tilde{a_1} = a_1 - \hat{a_1}, \tilde{b_1} = b_1 - \hat{b_1}, \tilde{c_1} = c_1 - \hat{c_1}, \tilde{d_1} = d_1 - \hat{d_1}, \tilde{h_1} = h_1 - \hat{h_1}, \tilde{k_1} = k_1 - \hat{k_1}. \tag{12}$$

Using (12), the error dynamics (11) is written as:

$$\begin{cases} \dot{e}_{11} = \tilde{a}_1 e_{11} - \tilde{b}_1(y_{21}z_{21} - \alpha_1 y_{11}z_{11}) - K_1 e_{11}, \\ \dot{e}_{12} = -\tilde{c}_1 e_{12} - K_2 e_{12}, \\ \dot{e}_{13} = \tilde{k}_1 e_{11} + \tilde{k}_1(\alpha_1 - \alpha_3)x_{11} - \tilde{d}_1 e_{13} - K_3 e_{13}, \\ \dot{e}_{14} = \tilde{h}_1 e_{14} - K_4 e_{14}. \end{cases} \tag{13}$$

By differentiating the parameter estimation error (12), we obtain

$$\dot{\tilde{a_1}} = -\dot{\hat{a_1}}, \dot{\tilde{b_1}} = -\dot{\hat{b_1}}, \dot{\tilde{c_1}} = -\dot{\hat{c_1}}, \dot{\tilde{d_1}} = -\dot{\hat{d_1}}, \dot{\tilde{h_1}} = -\dot{\hat{h_1}}, \dot{\tilde{k_1}} = -\dot{\hat{k_1}}. \tag{14}$$

Considering the Lyapunov function as

$$V = \frac{1}{2}[e_{11}^2 + e_{12}^2 + e_{13}^2 + e_{14}^2 + \tilde{a_1}^2 + \tilde{b_1}^2 + \tilde{c_1}^2 + \tilde{d_1}^2 + \tilde{h_1}^2 + \tilde{k_1}^2], \tag{15}$$

which implies that V is positive definite.

Derivative of Lyapunov function V, using (14), is given by

$$\dot{V} = e_{11}\dot{e}_{11} + e_{12}\dot{e}_{12} + e_{13}\dot{e}_{13} + e_{14}\dot{e}_{14} - \tilde{a_1}\dot{\hat{a_1}} - \tilde{b_1}\dot{\hat{b_1}} - \tilde{c_1}\dot{\hat{c_1}} - \tilde{d_1}\dot{\hat{d_1}} - \tilde{h_1}\dot{\hat{h_1}} - \tilde{k_1}\dot{\hat{k_1}}. \tag{16}$$

In view of (16), parameter estimates law is defined as:

$$\begin{cases} \dot{\hat{a_1}} = (x_{21} - \alpha_1 x_{11})e_{11} + K_5(a_1 - \hat{a}_1), \\ \dot{\hat{b_1}} = -(y_{21}z_{21} - \alpha_1 y_{11} z_{11})e_{11} + K_6(b_1 - \hat{b}_1), \\ \dot{\hat{c_1}} = -(y_{21} - \alpha_2 y_{11})e_{12} + K_7(c_1 - \hat{c}_1), \\ \dot{\hat{d_1}} = -(z_{21} - \alpha_3 z_{11})e_{13} + K_8(d_1 - \hat{d}_1), \\ \dot{\hat{h_1}} = (w_{21} - \alpha_4 w_{11})e_{14} + K_9(h_1 - \hat{h}_1), \\ \dot{\hat{k_1}} = (x_{21} - \alpha_1 x_{11})e_{13} + x_{11}(\alpha_1 - \alpha_3)e_{13} + K_{10}(k_1 - \hat{k}_1), \end{cases} \tag{17}$$

where K_5, K_6, K_7, K_8, K_9 and K_{10} are positive gaining constants.

Theorem 4.1.

The hyper-chaotic system (6)-(7) are asymptotically hybrid projective synchronized globally for all initial states $(x_{11}(0), y_{11}(0), z_{11}(0), w_{11}(0)) \in R^4$ by appropriately designed adaptive controller (10) and the parameter updating law (17).

Proof:

The Lyapunov function V as mentioned in (15) is positive definite function. On simplifying Equations (16) and (17), we have

$$\dot{V} = -K_1 e_{11}^2 - K_2 e_{12}^2 - K_3 e_{13}^2 - K_4 e_{14}^2 - K_5\tilde{a_1}^2 - K_6\tilde{b_1}^2 - K_7\tilde{c_1}^2 - K_8\tilde{d_1}^2 - K_9\tilde{h_1}^2 - K_{10}\tilde{k_1}^2 < 0,$$

which implies that $\dot{V}$ is negative definite. Thus, by using Lyapunov stability theory (Shevitz and Paden (1914); Perko (2013)), we deduce that HPS error $e(t) \to 0$ exponentially for $t \to \infty$ for every initial conditions $e(0) \in R^4$. This completes the proof. ∎

4.1. Numerical Simulation

This section presents essential simulation results for the illustration of the effectiveness of proposed HPS scheme via ACM. Parameters of the given system are chosen as $a_1 = 4.55$, $b_1 = 1.532$, $c_1 = 10.1$, $d_1 = 5.5$, $k_1 = 3.5$ and $h_1 = 0.04$ to ensure that the system behaves chaotically without control inputs. The initial states of master (6) and slave system (7) are $(-2, 4, 2, -3)$ and $(-3, 5, 3, -4)$, respectively. We achieve HPS scheme between master system (6) and slave system

(7) by choosing the scaling matrix α with $\alpha_1 = 2$, $\alpha_2 = -2$, $\alpha_3 = 3$, $\alpha_4 = -3$. Here, control gains are taken as $K_i = 4$ for $i = 1, 2, \ldots, 10$. Numerical simulations are depicted in Figure 2(a-d) which exhibit the state trajectories of systems (6) and (7). The synchronization errors $(e_{11}, e_{12}, e_{13}, e_{14}) = (1, 13, -3, -13)$ as shown in Figure 2(e) tend to zero as t tends to infinity. Moreover, Figure 2(f) displays that estimated values $(\hat{a}_1, \hat{b}_1, \hat{c}_1, \hat{d}_1, \hat{h}_1, \hat{k}_1)$ of unknown parameter converge to their real values asymptotically as t tends to infinity. Hence, the proposed HPS technique between master and slave system is verified computationally. Furthermore, Figure 6(a-e) and Figure 7(a-e) exhibit some particular cases of HPS scheme, namely, complete synchronization and anti-synchronization among the systems (6) and (7) respectively and both are displayed in Appendix.

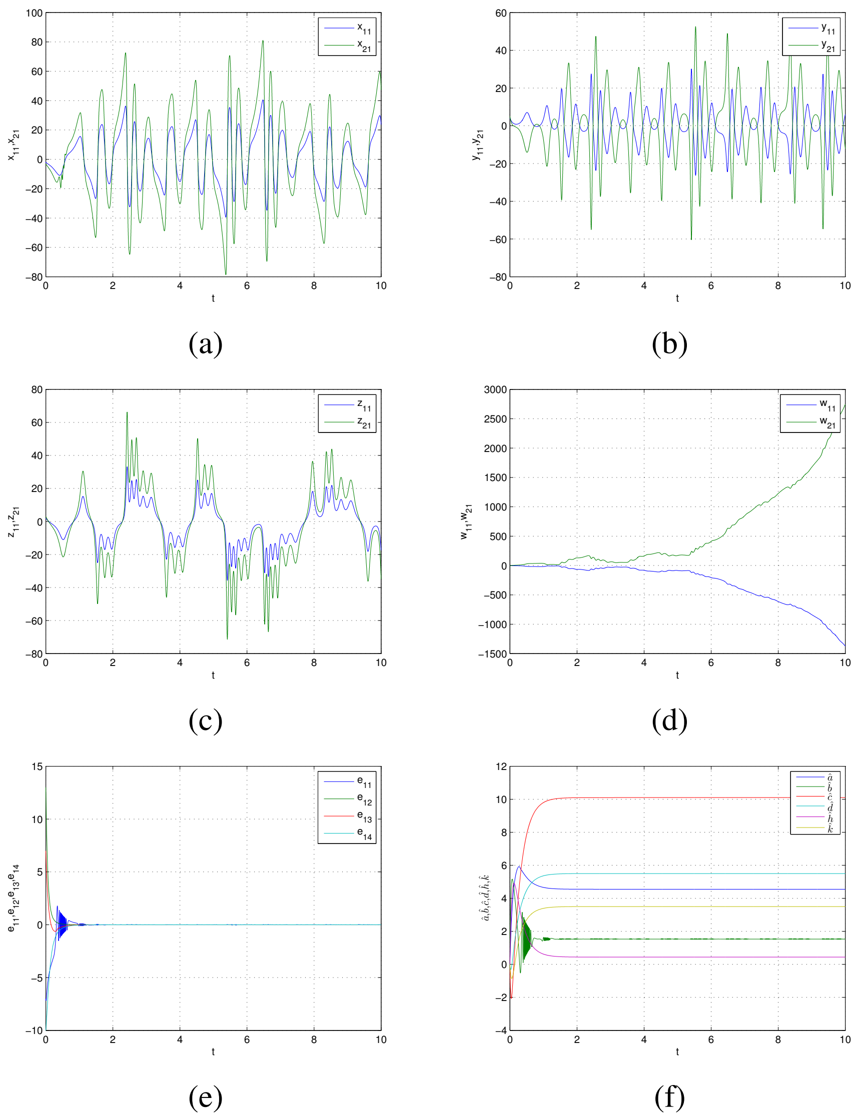


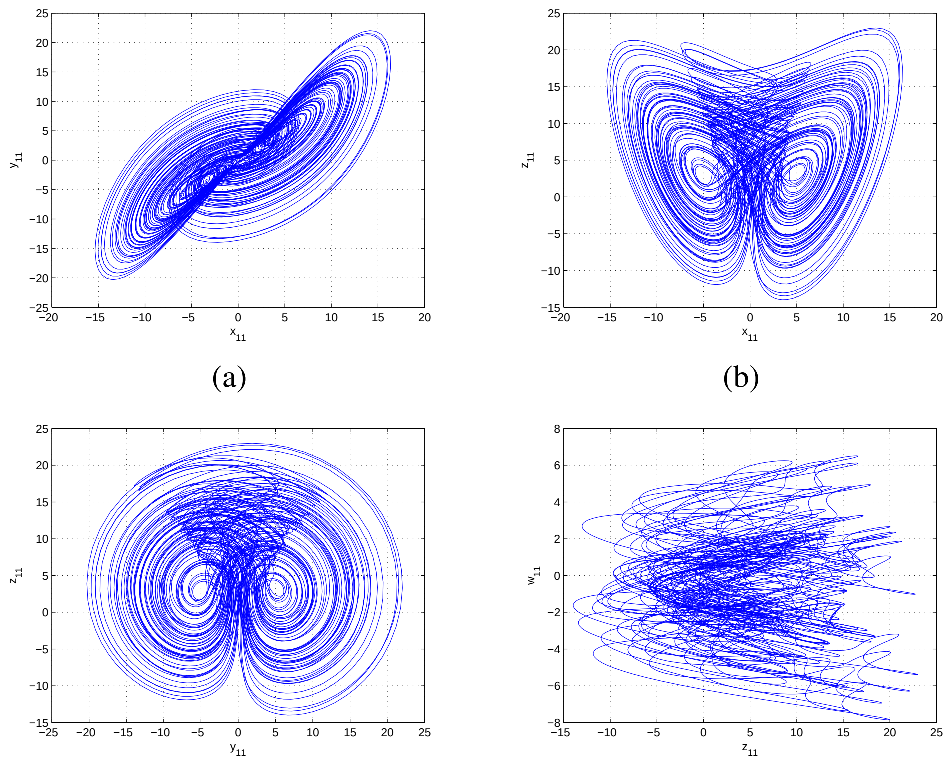
Figure 2. Hybrid projective synchronization for 4-D hyper-chaotic system (a) between $x_{11}(t) - x_{21}(t)$, (b) between $y_{11}(t) - y_{21}(t)$, (c) between $z_{11}(t) - z_{21}(t)$, (d) between $w_{11}(t) - w_{21}(t)$, (e) synchronization errors of the system, (f) parameter estimation

4.2. Illustrative example 2

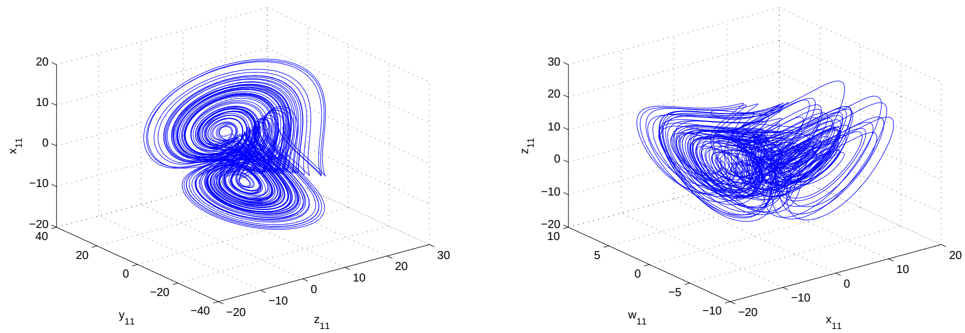
Proposed by Wei et al. (2014), the discussed hyper-chaotic system is defined as:

$$\begin{cases} \dot{x}_{11} = l_1(y_{11} - x_{11}), \\ \dot{y}_{11} = -x_{11}z_{11} - l_3 y_{11} + l_4 w_{11}, \\ \dot{z}_{11} = -l_2 + x_{11}y_{11}, \\ \dot{w}_{11} = -l_5 y_{11}, \end{cases} \tag{18}$$

where $(x_{11}, y_{11}, z_{11}, w_{11})^T \in R^4$ is state vector and l_1,l_2,l_3,l_4,k_1 and l_5 are positive parameters. When $l_1 = 10$, $l_2 = 25$, $l_3 = -2.5$, $l_4 = 1$ and $l_5 = 1$, the system (18) exhibits hyperchaos. Furthermore, Figure 3(a-f) depicts phase diagrams of (18) which show the hyperchaotic behaviour of (18). However, a thorough analytic study and numerical results for the system (18) may be found in Wei et al. (2014).



(a) (b) (c) (d)



(e) (f)

Figure 3. Phase diagrams of hyper-chaotic system in (a) $x_{11} - y_{11}$ plane, (b)$y_{11} - z_{11}$ plane, (c) $x_{11} - y_{11} - z_{11}$ space, (d) $y_{11} - z_{11} - w_{11}$ space, (e) $x_{11} - z_{11} - w_{11}$ space, (f) $x_{11} - y_{11} - w_{11}$ space

The system (18) is considered as the master system and the corresponding slave system can be described by the following expression:

$$\begin{cases} \dot{x}_{21} = l_1(y_{21} - x_{21}) + v_{11}, \\ \dot{y}_{21} = -x_{21}z_{21} - l_3y_{21} + l_4w_{21} + v_{12}, \\ \dot{z}_{21} = -l_2 + x_{21}y_{21} + v_{13}, \\ \dot{w}_{21} = -l_5y_{21} + v_{14}, \end{cases} \tag{19}$$

where v_{11}, v_{12}, v_{13} and v_{14} are adaptive controllers to be constructed in such a way that HPS among two identical hyper-chaotic models will be achieved.

Define the state errors by

$$\begin{cases} e_{11} = x_{21} - \beta_1 x_{11}, \\ e_{12} = y_{21} - \beta_2 y_{11}, \\ e_{13} = z_{21} - \beta_3 z_{11}, \\ e_{14} = w_{21} - \beta_4 w_{11}. \end{cases} \tag{20}$$

The main goal here is to introduce the appropriate controllers $v_{1i}, (i = 1, 2, 3, 4)$ so that the considered state errors must satisfy

$$\lim_{t\to\infty} e_{1i}(t) = 0 \quad \text{for} \;\; (i = 1, 2, 3, 4).$$

Subsequent error dynamics turns out to be

$$\begin{cases} \dot{e}_{11} = -l_1e_{11} + l_1(y_{21} - \beta_1 y_{11}) + v_{11}, \\ \dot{e}_{12} = -l_3e_{12} - (x_{21}z_{21} - \beta_2 x_{11}z_{11}) + l_4(w_{21} - \beta_1 w_{11}) + v_{12}, \\ \dot{e}_{13} = -l_2(1 - \beta 3) + x_{21}y_{21} - \beta_3 x_{11}y_{11} + v_{13}, \\ \dot{e}_{14} = -l_5(y_{21} - \beta_4 y_{11}) + v_{14}. \end{cases} \tag{21}$$

We now describe the adaptive controllers by the rule:

$$\begin{cases} v_{11} = \hat{l}_1 e_{11} - \hat{l}_1(y_{21} - \beta_1 y_{11}) - K_1 e_{11}, \\ v_{12} = \hat{l}_3 e_{12} + (x_{21}z_{21} - \beta_2 x_{11}z_{11}) - \hat{l}_4(w_{21} - \beta_1 w_{11}) - K_2 e_{12}, \\ v_{13} = \hat{(}l_2)(1 - \beta 3) - (x_{21}y_{21} - \beta_3 x_{11}y_{11}) - K_3 e_{13}, \\ v_{14} = \hat{(}l_5)(y_{21} - \beta_4 y_{11}) - K_4 e_{14}, \end{cases} \tag{22}$$

where $K_i > 0$, $i = 1, 2, 3, 4$ and are called gain constants.

By putting the controllers (22) in the error dynamics (21), we get

$$\begin{cases} \dot{e}_{11} = -(l_1 - \hat{l}_1)e_{11} + (l_1 - \hat{l}_1)(y_{21} - \beta_1 y_{11}) - K_1 e_{11}, \\ \dot{e}_{12} = -(l_3 - \hat{l}_3)e_{12} + (l_4 - \hat{l}_4)(w_{21} - \beta_2 w_{11}) - K_2 e_{12}, \\ \dot{e}_{13} = -(l_2 - \hat{l}_2)(1 - \beta 3) - K_3 e_{13}, \\ \dot{e}_{14} = -(l_5 - \hat{l}_5)(y_{21} - \beta_4 y_{11}) - K_4 e_{14}, \end{cases} \tag{23}$$

where $\hat{l}_1$, $\hat{l}_2$, $\hat{l}_3$, $\hat{l}_4$, $\hat{l}_5$ are estimated values for unknown system parameters l_1, l_2, l_3, l_4, l_5, respectively.

We represent parameter estimation error as follows:

$$\tilde{l_1} = l_1 - \hat{l_1}, \tilde{l_2} = l_2 - \hat{l_2}, \tilde{l_3} = l_3 - \hat{l_3}, \tilde{l_4} = l_4 - \hat{l_4}, \tilde{l_5} = l_5 - \hat{l_5}. \tag{24}$$

Using (24), the error dynamics (23) is written as:

$$\begin{cases} \dot{e}_{11} = -\tilde{l}_1 e_{11} + \tilde{l}_1(y_{21} - \beta_1 y_{11}) - K_1 e_{11}, \\ \dot{e}_{12} = -\tilde{l}_3 e_{12} + \tilde{l}_4(w_{21} - \beta_2 w_{11}) - K_2 e_{12}, \\ \dot{e}_{13} = -\tilde{l_2}(1 - \beta 3) - K_3 e_{13}, \\ \dot{e}_{14} = -\tilde{l_5})(y_{21} - \beta_4 y_{11}) - K_4 e_{14}. \end{cases} \tag{25}$$

On differentiation, the parameter estimation error (24) becomes

$$\dot{\tilde{l_1}} = -\dot{\hat{l_1}}, \dot{\tilde{l_2}} = -\dot{\hat{l_2}}, \dot{\tilde{l_3}} = -\dot{\hat{l_3}}, \dot{\tilde{l_4}} = -\dot{\hat{l_4}}, \dot{\tilde{l_5}} = -\dot{\hat{l_5}}. \tag{26}$$

Define the Lyapunov function by

$$V = \frac{1}{2}[e_{11}^2 + e_{12}^2 + e_{13}^2 + e_{14}^2 + \tilde{l_1}^2 + \tilde{l_2}^2 + \tilde{l_3}^2 + \tilde{l_4}^2 + \tilde{l_5}^2, \tag{27}$$

which shows that V is positive definite.

The derivative of Lyapunov function V, using (26), is given by

$$\dot{V} = e_{11}\dot{e}_{11} + e_{12}\dot{e}_{12} + e_{13}\dot{e}_{13} + e_{14}\dot{e}_{14} - \tilde{l_1}\dot{\hat{l_1}} - \tilde{l_2}\dot{\hat{l_2}} - \tilde{l_3}\dot{\hat{l_3}} - \tilde{l_4}\dot{\hat{l_4}} - \tilde{l_5}\dot{\hat{l_5}}. \tag{28}$$

Keeping (28) in view, the parameter estimates law is defined as:

$$\begin{cases} \dot{\hat{l}}_1 = (y_{21} - \beta_1 y_{11})e_{11} - e11^2 + K_5(l_1 - \hat{l}_1), \\ \dot{\hat{l}}_2 = (1 - \beta_3)e_{13}z + K_6(l_2 - \hat{l}_2), \\ \dot{\hat{l}}_3 = -e_{12}^2 + K_7(l_3 - \hat{l}_3), \\ \dot{\hat{l}}_4 = (w_{21} - \beta_2 w_{11})e_{12} + K_8(l_4 - \hat{l}_4), \\ \dot{\hat{l}}_5 = -(y_{21} - \beta_4 y_{11})e_{14} + K_9(l_5 - \hat{l}_5), \end{cases} \tag{29}$$

where K_5, K_6, K_7, K_8 and K_9 are positive gaining constants.

Theorem 4.2.

The hyper-chaotic system (18)-(19) are asymptotically hybrid projective synchronized globally for all initial states $(x_{11}(0), y_{11}(0), z_{11}(0), w_{11}(0)) \in R^4$ by properly designed adaptive controller (22) and the parameter updating law (29).

Proof:

The Lyapunov function V as defined in (27) is positive definite function. On solving Equations (28) and (29), we obtain

$$\begin{aligned} \dot{V} = &-K_1 e_{11}^2 - K_2 e_{12}^2 - K_3 e_{13}^2 - K_4 e_{14}^2 - K_5 \tilde{l_1}^2 - K_6 \tilde{2_1}^2 - K_7 \tilde{l_3}^2 - K_8 \tilde{l_4}^2 - K_9 \tilde{l_5}^2 \\ &< 0, \end{aligned}$$

which depicts that $\dot{V}$ is negative definite. Thus, using Lyapunov stability theory (Shevitz and Paden (1914); Perko (2013)), we conclude that HPS error $e(t) \to 0$ exponentially for $t \to \infty$ for every initial conditions $e(0) \in R^4$. The proof is completed. ■

4.3. Numerical Simulation

In this section, we present few simulation results for illustrating the effectiveness of proposed HPS scheme via ACM. Parameters of the given hyper-chaotic system (18) are chosen as $l_1 = 10$, $l_2 = 25$, $l_3 = -2.5$, $l_4 = 1$ and $l_5 = 1$ to make sure that the system behaves chaotic without control inputs. The initial states of master (18) and slave system (19) are $(0.2, 0.1, 0.75, -2)$ and $(0.35, 0.4, 0.6, -3)$, respectively. We attain HPS scheme among master system (18) and slave system (19) by selecting the scaling matrix β with $\beta_1 = 4$, $\beta_2 = -3$, $\beta_3 = 2$, $\beta_4 = -5$. Here, control gains are taken as $K_i = 4$ for $i = 1, 2, \ldots, 9$. Numerical simulations are shown in Figure 4(a-d) which depict the state trajectories of systems (18) and (19). The synchronization errors $(e_{11}, e_{12}, e_{13}, e_{14}) = (-0.45, 0.7, -0.9, -13)$ as displayed in Figure 4(e) tend to zero as t tends to infinity. Moreover, Figure 4(f) displays that estimated values $(\hat{l}_1, \hat{l}_2, \hat{l}_3, \hat{l}_4, \hat{l}_5)$ of unknown parameter converge to their real values asymptotically as t tends to infinity. Thus, the proposed HPS technique between master and slave system is verified computationally. In addition, Figure 8(a-e) and Figure 9(a-e) display some particular cases of HPS scheme, for example, complete synchronization and anti-synchronization among the systems (18) and (19), respectively, and both are shown in the Appendix.

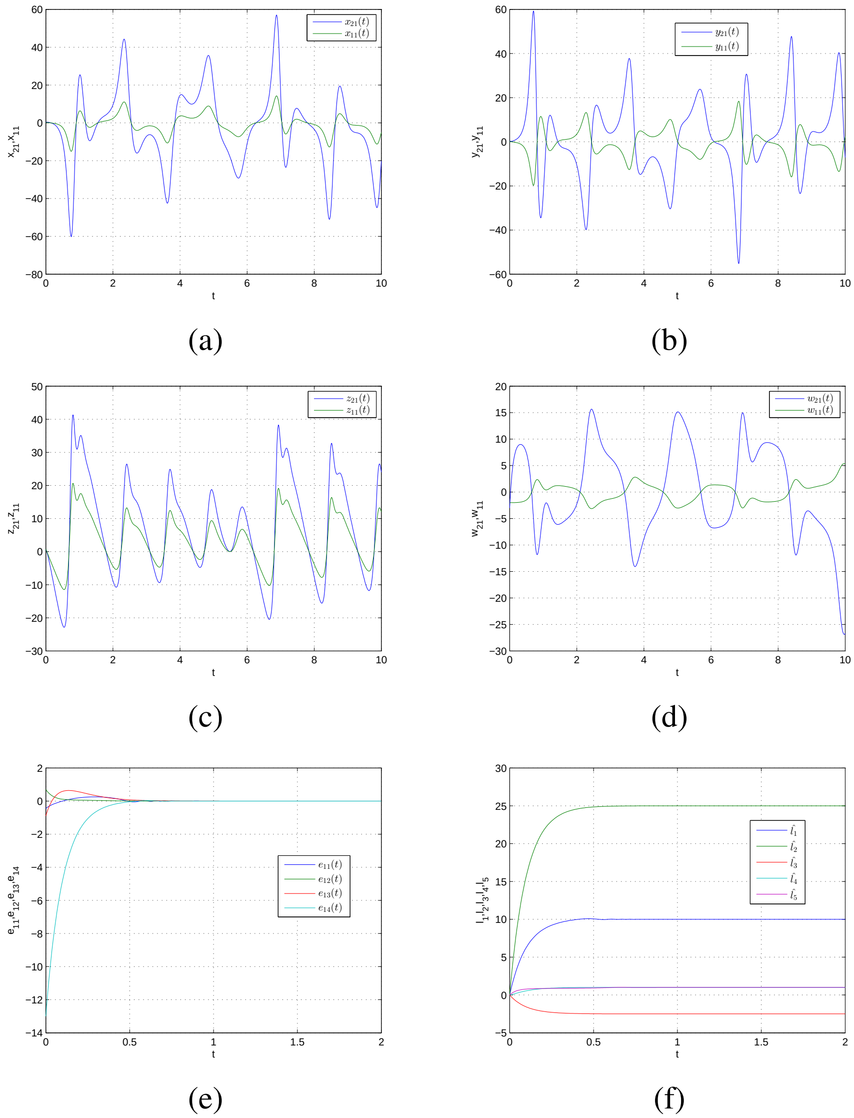
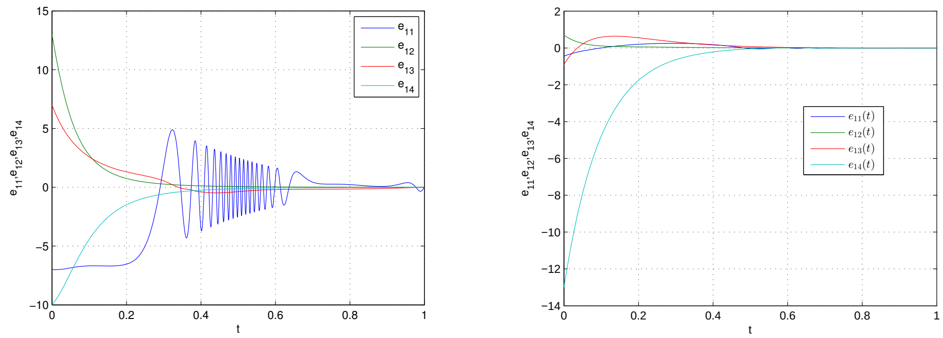


Figure 4. Hybrid projective synchronization for 4-D hyper-chaotic system (a) between $x_{11}(t) - x_{21}(t)$, (b) between $y_{11}(t) - y_{21}(t)$, (c) between $z_{11}(t) - z_{21}(t)$, (d) between $w_{11}(t) - w_{21}(t)$, (e) synchronization errors of the system, (f) parameter estimation

5. A Comparative Study

The HPS scheme is attained in Khan and Tyagi (2017c) via active control method while performing on the same hyper-chaotic system with similar parameters. It is noticed here that synchronization error converges to zero at $t = 5.1$ (approx) as shown in Khan and Tyagi (2017c), whereas in our study, HPS scheme is achieved via adaptive control method, in which it is observed that synchronization error is convergent with limit to zero at $t = 0.9$ (approx) as depicted in Figure 5(a). Also, the synchronization error of systems (18)-(19) converges to zero at $t = 0.5$ (approx) as shown in Figure 5(b). It shows that the proposed HPS approach via adaptive control method is more preferable over other published work.



(a) (b)

Figure 5. Hybrid projective synchronization for 4-D hyper-chaotic system using (a) active control method, (b) adaptive control method

6. Conclusion

In this paper, we have investigated our proposed HPS scheme between identical hyper-chaotic systems via adaptive control technique. By designing suitable controllers based on Lyapunov stability theory, the considered HPS technique has been achieved. The particular cases of anti-synchronization and complete synchronization are also discussed. The effectivity and feasibility of the theoretical results are verified in simulations by using MATLAB. Exceptionally, the theoretical analysis and numerical results both agree completely. In fact, the discussed HPS technique is very efficient as it has various applications in encryption, control theory and secure communication. In this research, time taken by the synchronization errors converging to zero is very less in contrasting with prior published work. Furthermore, we believe that the proposed HPS approach would be generalized by applying many more control techniques.

REFERENCES

Bouallegue, K. (2017). A new class of neural networks and its applications, Neurocomputing, Vol. 249, pg. 28–47.

Chen, M. and Han, Z. (2003). Controlling and synchronizing chaotic Genesio system via nonlinear feedback control, Chaos, Solitons & Fractals, Vol. 17, No. 4, pg. 709–716.

Delavari, H. and Mohadeszadeh, M. (2018). Hybrid complex projective synchronization of complex chaotic systems using active control technique with nonlinearity in the control input, Journal of Control Engineering and Applied Informatics, Vol. 20, No. 1, pg. 67–74.

Ding, D., Qian, X., Wang, N. and Liang, D. (2018). Synchronization and anti-synchronization of a fractional order delayed memristor-based chaotic system using active control, Modern Physics Letters B, Vol. 32, No. 14, 1850142.

Ding, Z. and Shen, Y. (2016). Projective synchronization of nonidentical fractional-order neural networks based on sliding mode controller, Neural Networks, Vol. 76, pg. 97–105.

Dong, E., Liang, Z., Du, S. and Chen, Z. (2016). Topological horseshoe analysis on a four-wing chaotic attractor and its FPGA implement, Nonlinear Dynamics, Vol. 83, No. 1-2, pg. 623–630.

Fedorovich, E. (1986). *Numerical Modelling of Atmospheric Boundary Layer Flow Over Topography Elements*, Voeikov Main Geophysical Observatory, Leningrad, USSR.

Filali, R. L., Benrejeb, M. and Borne, P. (2014). On observer-based secure communication design using discrete-time hyperchaotic systems, Communications in Nonlinear Science and Numerical Simulation, Vol. 19, No. 5, pg. 1424–1432.

Garratt, J. R. (1994). *The Atmospheric Boundary Layer*, Cambridge University Press, Cambridge, UK, 316.

Ghosh, D., Mukherjee, A., Das, N. R. and Biswas, B. N. (2018). Generation & control of chaos in a single loop optoelectronic oscillator, Optik, Vol. 165, pg. 275–287.

Hubler, A. W. (1989). Adaptive control of chaotic system, Helv. Phys. Acta., Vol. 62, pg. 343–346.

Jahanzaib, L.S., Trikha, P., Chaudhary, H., Haider, S.M. and others (2020). Compound synchronization using disturbance observer based adaptive sliding mode control technique, J. Math. Comput. Sci., Vol. 10, No. 5, pg. 1463–1480.

Khan, A. and Chaudhary, H. (2019). Adaptive control and hybrid projective combination synchronization of chaos generated by generalized Lotka-Volterra biological systems, *BLOOMSBURY INDIA*, 174.

Khan, A. and Chaudhary, H. (2020a). Hybrid projective synchronization in identical hyperchaotic systems using active control, AIP Conference Proceedings, Vol. 2253, No. 1, 020014, AIP Publishing LLC.

Khan, T. and Chaudhary, H. (2020b). Estimation and identifiability of parameters for generalized Lotka-Volterra biological systems using adaptive controlled combination difference anti-synchronization, Differential Equations and Dynamical Systems, Vol. 28, pg. 515–526.

Khan, A. and Tyagi, A. (2017a). Analysis and hyper-chaos control of a new 4-D hyper-chaotic system by using optimal and adaptive control design, International Journal of Dynamics and Control, Vol. 5, No. 4, pg. 1147–1155.

Khan, A. and Tyagi, A. (2017b). Fractional order disturbance observer based adaptive sliding mode synchronization of commensurate fractional order Genesio-Tesi system, AEU-International Journal of Electronics and Communications, Vol. 82, pg. 346–357.

Khan, A. and Tyagi, A. (2017c). Hybrid projective synchronization between two identical new 4-D hyper-chaotic systems via active control method, Int. J. Nonlinear. Sci, Vol. 23, No. 3, pg. 142–150.

Khan, A. and Tyagi, A. (2018). Fractional order disturbance observer based adaptive sliding mode hybrid projective synchronization of fractional order Newton–Leipnik chaotic system, International Journal of Dynamics and Control, Vol. 6, No. 3, pg. 1136–1149.

Kumar, S., Motouk, A. E., Chaudhary, H. and Kant, S. (2011). Control and synchronization of fractional-order chaotic satellite systems using feedback and adaptive control techniques, International Journal of Adaptive Control and Signal Processing.

Li, C. and Liao, X. (2004). Complete and lag synchronization of hyperchaotic systems using small impulses, Chaos, Solitons & Fractals, Vol. 22, No. 4, pg. 857–867.

Li, D. and Zhang, X. (2016). Impulsive synchronization of fractional order chaotic systems with time-delay, Neurocomputing, Vol. 216, pg. 39–44.

Li, G.-H. (2007). Modified projective synchronization of chaotic system, Chaos, Solitons & Fractals, Vol. 32, No. 5, pg. 1786–1790.

Li, G.-H. and Zhou, S.-P. (2007). Anti-synchronization in different chaotic systems, Chaos, Solitons & Fractals, Vol. 32, No. 2, pg. 516–520.

Li, S.-Y., Yang, C.-H., Lin, C.-T., Ko, L.-W. and Chiu, T.-T. (2012). Adaptive synchronization of chaotic systems with unknown parameters via new backstepping strategy, Nonlinear Dynamics, Vol. 70, No. 3, pg. 2129–2143.

Li, Z. and Xu, D. (2004). A secure communication scheme using projective chaos synchronization, Chaos, Solitons & Fractals, Vol. 22, No. 2, pg. 477–481.

Liao, T.-L. and Tsai, S.-H. (2000). Adaptive synchronization of chaotic systems and its application to secure communications, Chaos, Solitons & Fractals, Vol. 11, No. 9, pg. 1387–1396.

Ma, J., Mi, L., Zhou, P., Xu, Y. and Hayat, T. (2017). Phase synchronization between two neurons induced by coupling of electromagnetic field, Applied Mathematics and Computation, Vol. 307, pg. 321–328.

Mainieri, R. and Rehacek, J. (1999). Projective synchronization in three-dimensional chaotic systems, Physical Review Letters, Vol. 82, No. 15, pg. 3042.

Naderi, B. and Kheiri, H. (2016). Exponential synchronization of chaotic system and application in secure communication, Optik, Vol. 127, No. 5, pg. 2407–2412.

Newsom, R. K., Sivaraman, C., Shippert, T. and Riihimaki, L. D. (2015). Doppler lidar vertical velocity statistics value-added product, DOE ARM Climate Research Facility, No. DOE/SC-ARM/TR-149.

Ouannas, A. and Abu-Saris, R. (2016). On matrix projective synchronization and inverse matrix projective synchronization for different and identical dimensional discrete-time chaotic systems, Journal of Chaos, Vol. 2016.

Patle, B.K., Parhi, D.R.K., Jagadeesh, A. and Kashyap, S. K. (2018). Matrix-binary codes based genetic algorithm for path planning of mobile robot, Computers & Electrical Engineering, Vol. 67, pg. 708–728.

Pecora, L. M. and Carroll, T. L. (1990). Synchronization in chaotic systems, Physical Review Letters, Vol. 64, No. 8, 821.

Perko, L. (2013). *Differential Equations and Dynamical Systems*, Vol. 7.

Rasappan, S. and Vaidyanathan, S. (2012). Synchronization of hyperchaotic Liu system via backstepping control with recursive feedback, International Conference on Eco-friendly Computing and Communication Systems, pg. 212–221.

Rossler, O. E. (1979). An equation for hyperchaos, Physics Letters A, Vol. 71, No. 2-3, pg. 155–157.

Russell, F. P., Düben, P. D., Niu, X., Luk, W. and Palmer, T. N. (2017). Exploiting the chaotic behaviour of atmospheric models with reconfigurable architectures, Computer Physics Communications, Vol. 221, pg. 160–173.

Sahoo, B. and Poria, S. (2014). The chaos and control of a food chain model supplying additional food to top-predator, Chaos, Solitons & Fractals, Vol. 58, pg. 52–64.

Shevitz, D. and Paden, B. (1914). Lyapunov stability theory of nonsmooth systems, IEEE Transactions on automatic control, Vol. 39, No. 9, pg. 1910–1914.

Shinbrot, T., Ott, E., Grebogi, C. and Yorke, J. A. (1990). Using chaos to direct trajectories to targets, Physical Review Letters, Vol. 65, No. 26, 3215.

Singh, A. K., Yadav, V. K. and Das, S. (2017). Synchronization between fractional order complex chaotic systems, International Journal of Dynamics and Control, Vol. 5, No. 3, pg. 756–770.

Sudheer, K. S. and Sabir, M. (2009). Hybrid synchronization of hyperchaotic Lu system, Pramana, Vol. 73, No. 4, 781.

Tong, X.-J., Zhang, M., Wang, Z., Liu, Y. and Ma, J. (2015). An image encryption scheme based on a new hyperchaotic finance system, Optik, Vol. 126, No. 20, pg. 2445–2452.

Vaidyanathan, S. (2015). Adaptive biological control of generalized Lotka-Volterra three-species biological system, International Journal of PharmTech Research, Vol. 8, No. 4, pg. 622–631.

Wang, X., Vaidyanathan, S., Volos, C., Pham, V.-T. and Kapitaniak, T. (2017). Dynamics, circuit realization, control and synchronization of a hyperchaotic hyperjerk system with coexisting attractors, Nonlinear Dynamics, Vol. 89, No. 3, pg. 1673–1687.

Wei, Z., Wang, R. and Liu, A. (2014). A new finding of the existence of hidden hyperchaotic attractors with no equilibria, Mathematics and Computers in Simulation, Vol. 100, pg. 13–23.

Wu, G.-C., Baleanu, D. and Lin, Z.-X. (2016). Image encryption technique based on fractional chaotic time series, Journal of Vibration and Control, Vol. 22, No. 8, pg. 2092–2099.

Wu, Z., Duan, J. and Fu, X. (2012). Complex projective synchronization in coupled chaotic complex dynamical systems, Nonlinear Dynamics, Vol. 69, No. 3, pg. 771–779.

Yassen, M. T. (2003). Adaptive control and synchronization of a modified Chua's circuit system, Applied Mathematics and Computation, Vol. 135, No. 1, pg. 113–128.

Zhou, P. and Zhu, W. (2011). Function projective synchronization for fractional-order chaotic systems, Nonlinear Analysis: Real World Applications, Vol. 12, No. 2, pg. 811–816.

Appendix

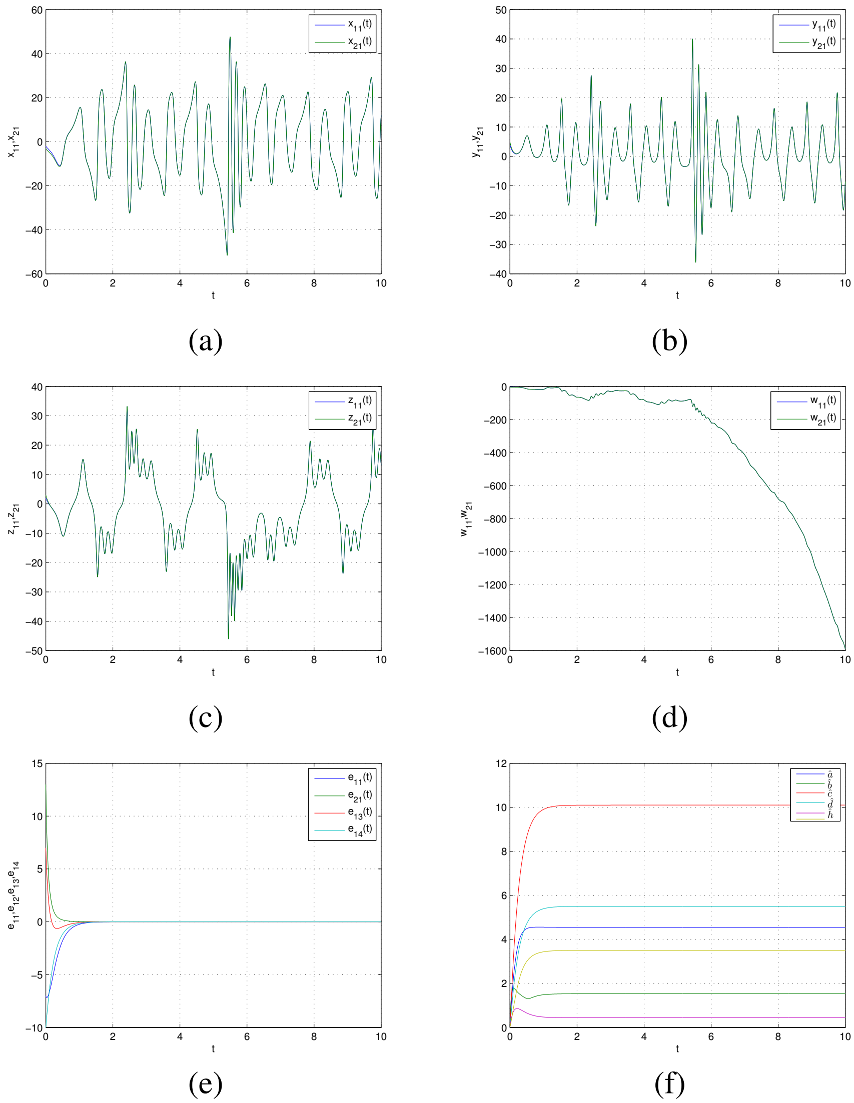


Figure 6. Complete synchronization of 4-D hyper-chaotic system (a) between $x_{11}(t) - x_{21}(t)$, (b) between $y_{11}(t) - y_{21}(t)$, (c) between $z_{11}(t) - z_{21}(t)$, (d) between $w_{11}(t) - w_{21}(t)$, (e) synchronization errors of the system, (f) parameter estimation

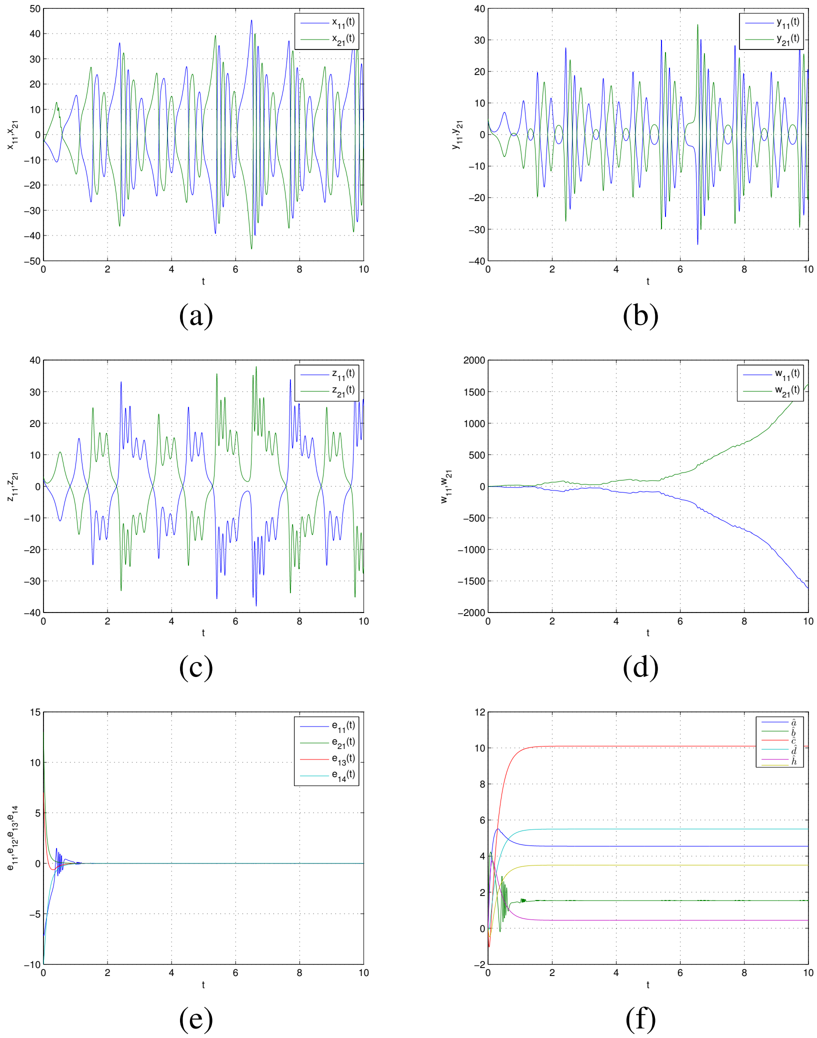


Figure 7. Anti-synchronization of 4-D hyper-chaotic system (a) between $x_{11}(t) - x_{21}(t)$, (b) between $y_{11}(t) - y_{21}(t)$, (c) between $z_{11}(t) - z_{21}(t)$, (d) between $w_{11}(t) - w_{21}(t)$, (e) synchronization errors of the system, (f) parameter estimation

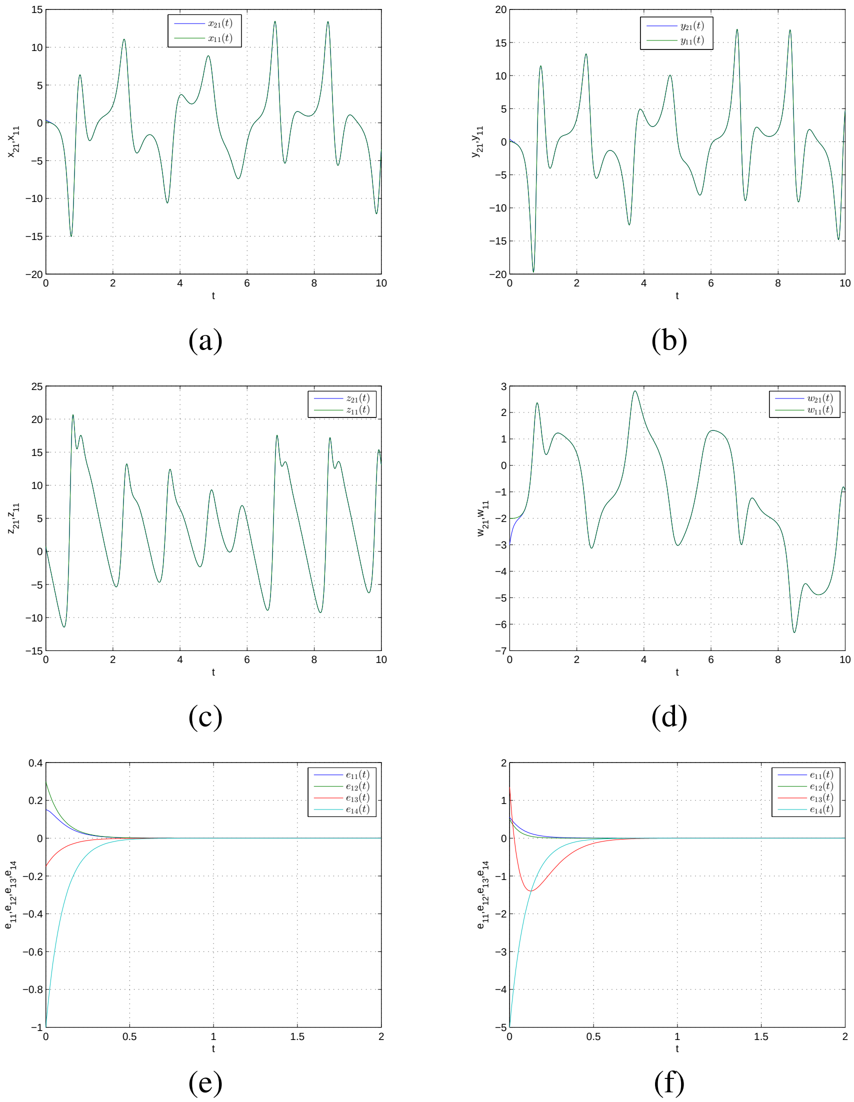


Figure 8. Complete synchronization of 4-D hyper-chaotic system (a) between $x_{11}(t) - x_{21}(t)$, (b) between $y_{11}(t) - y_{21}(t)$, (c) between $z_{11}(t) - z_{21}(t)$, (d) between $w_{11}(t) - w_{21}(t)$, (e) synchronization errors of the system, (f) parameter estimation

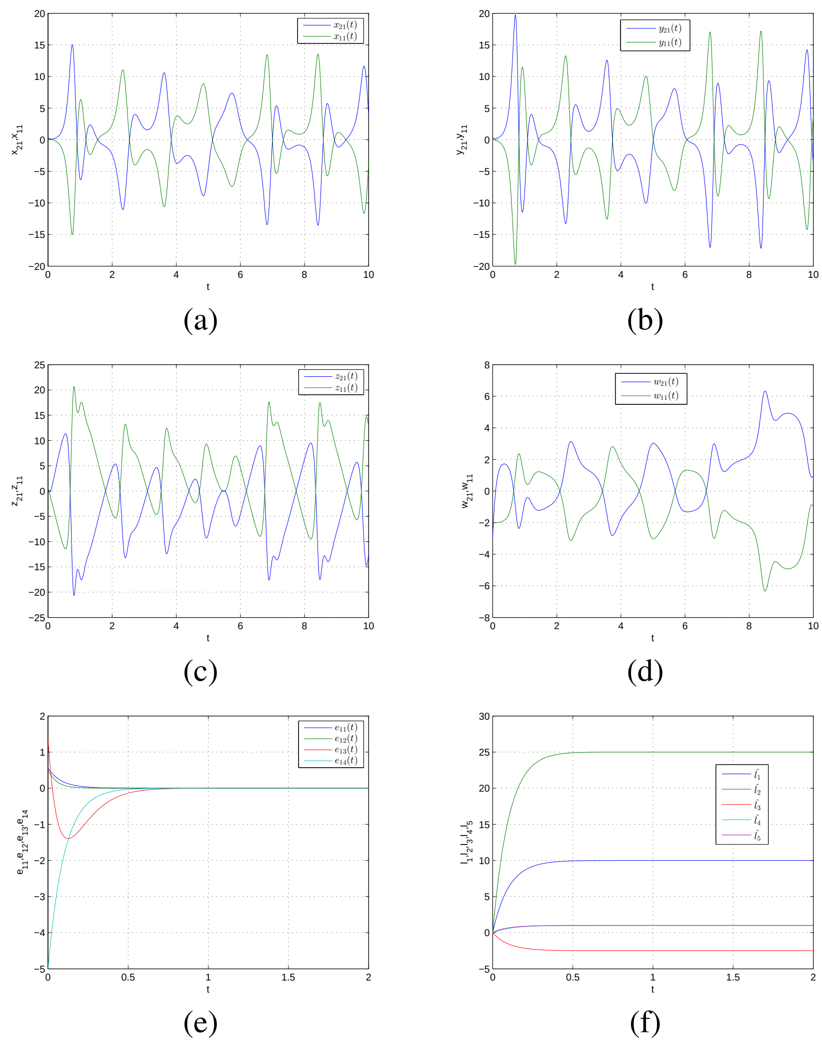


Figure 9. Anti-synchronization of 4-D hyper-chaotic system (a) between $x_{11}(t) - x_{21}(t)$, (b) between $y_{11}(t) - y_{21}(t)$, (c) between $z_{11}(t) - z_{21}(t)$, (d) between $w_{11}(t) - w_{21}(t)$, (e) synchronization errors of the system, (f) parameter estimation