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Some Asymptotic Properties of Conditional Density Function for Functional Data Under Random Censorship

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Abstract

In this work, we investigate the asymptotic properties of a nonparametric mode of a conditional density when the real response variable is censored and the explanatory variable is valued in a semi-metric space under ergodic data. First of all, we establish asymptotic properties for a conditional density estimator from which we derive an central limit theorem (CLT) of the conditional mode estimator. Simulation study is also presented to illustrate the validity and finite sample performance of the considered estimator.

Keywords: Asymptotic normality; Censored data; Conditional mode; Ergodic processes functional data; Strong consistency

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1. Introduction

Survival analysis methods have been used in a number of applied fields (medicine, biology, epidemiology, engineering, econometrics, finance, social sciences, demography, etc.). The analysis of failure time data usually means addressing one of three problems: the estimation of survival functions, the comparison of treatments or survival functions, and the assessment of covariate effects or the dependence of failure time on explanatory variables. There are many reasons that make it difficult to get complete data in studies involving survival times. A study is often finished before the death of all patients, and we may keep only the information that some patients are still alive at the end of the study, not observing when they really die. In the presence of censored data, the time to event is unknown, and all we know is that the survival time has occurred before, between or after certain time points, this obviates the need for inference methods for censored data. When the failure time is observed completely, there are numerous methods to make non parametric inference on its conditional distribution. For instance, Nadaraya (1964) and Watson (1964) proposed a nonparametric estimator to estimate the conditional expectation as a locally weighted average using a kernel function. Beran (1981) extended the Kaplan-Meier estimator and proposed a method for non-parametric estimation (generalized Kaplan-Meier) of the conditional survival function for right-censored data.

Results regarding the estimation of the conditional models from right censored data can be found for instance in Dabrowska (1992), where author gave the nonparametric regression with censored survival time data. In Li and Doss (1995) an approach to nonparametric regression for life history data using local linear fitting was given. Dehghan and Duchesne (2016) established the estimation of the conditional survival function of a failure time given a time-varying covariate with interval-censored observations. Many works in the statistical literature deal with nonparametric estimation when the variable of interest is either complete or singly censored. However, in reliability and survival time studies, one can encounter a more complicated random censorship situation. An example of such a model, given in Patilea and Rolin (2006), is to consider a reliability system consisting of three components with two components in series and one component in parallel with the series system, the authors defined the product-limit estimators of the survival function with twice censored data.

On the other hand, the problem of nonparametric conditional models for censored data where the observations can be censored from either left or right are very limited in the literature. This gap can partially be explained by the difficulties arising in the estimation of the conditional distribution and/or density function with two-sided censored data. The problem of estimating the (unconditional) distribution function for data that may be censored from above and below has been considered by several authors.

Despite the regression function is of interest, other statistics such as quantile and mode regression might be important from a theoretical and a practical point of view. Quantile and/or mode regression is a common way to describe the dependence structure between a response variable X and some covariate Z. Unlike the regression function that relies only on the central tendency of the data, the conditional quantile function allows the analyst to estimate the functional dependence between variables for all portions of the conditional distribution of the response variable.

Mode regression is a common way to describe the dependence structure between a response variable X and some covariate Z. Unlike the regression function (which is defined as the conditional mean) that relies only on the central tendency of the data, the conditional mode function allows the analysts to estimate the functional dependence between variables for all portions of the conditional distribution of the response variable. On the other hand, compared with the standard approach based on functional conditional mean prediction that is sensitive to outliers, functional condition mode prediction could be seen as a reasonable alternative to conditional mean because of its robustness. Moreover, quantiles are well known for their robustness to heavy-tailed error distributions and outliers which allow to consider them as a useful alternative to the regression function see Chaouch and Khardani (2015). Conditional model are used in finance and/or insurance to model the risks of extreme values. The regression quantile function provide a well description of the data, specifically the conditional median function (see Chaudhuri et al. (1997)). Estimation of the conditional mode of a scalar response given a functional covariate has attracted the attention of many researchers.

In the censored case, Ould-Saïd and Cai (2005) stated the uniform strong consistency with rates of the kernel estimator of the conditional mode function, in this context, we refer to Ling et al. (2016) for the estimation of conditional mode for functional stationary ergodic data with missing at random. The ergodic theory has appeared in statistical mechanics, notably in Maxwell's and Gibbs's theories. It is necessary to make a sort of logical transition between the average behavior of the set of dynamic systems and the temporal average of the behaviors of a single dynamic system. It is derived from an ingenious hypothesis used for a long time without justifying it, and in various forms. In the context of the ergodic functional case with censored observations the literature is very restricted.

So, in the present work, we investigate the asymptotic properties of the conditional mode function of a randomly censored scalar response given a functional covariate when the data are sampled from a stationary and ergodic process. Our results can be used to construct prediction intervals, for instance in electricity when one wants to construct a maximum interval of demand (or needs) of electricity in the presence of censored data. In practice, this study has great importance because it permits us to construct a prediction method based on the conditional mode estimator. Here, we consider a model in which the response variable is censored but not the covariate. Besides the infinite dimensional character of the data, we avoid here the widely used strong mixing condition and its variants to measure the dependency and the very involved probabilistic calculations that it implies. Therefore, we consider, in our setting, the ergodic property to allow the maximum possible generality with regard to the dependence setting. Further motivations to consider ergodic data are discussed in Laib and Louani (2010, 2011) where details defining the ergodic property of processes are also given.

The layout of the paper is as follows. In the next section, our model is described. Section 3 is dedicated to fixing notations and hypotheses. We state our main result of strong consistency rate as well as the asymptotic normality, where the technical proofs are given with some auxiliary results. In Section 4. Lastly, an application of the proposed estimator is illustrated in Section 5.

Consider a random pair (Z, T) which is valued in $E \times \mathbb{R}$, where E is some semi-metric abstract space equipped with semi-metric $d(\cdot, \cdot)$, and T takes values in $\mathbb{R}$. Let $(T_i, Z_i)_{1 \le i \le n}$ be the statistical sample of pairs which are identically distributed as (Z, T) and supposed to be stationary and ergodic. Henceforward, Z is called functional random variable *f.r.v.*

For $z \in E$, we denote by $\varphi(\cdot|z)$ the conditional density function of T given $Z = z$ and we assume that $\varphi(\cdot|z)$ has an unique conditional mode $\theta(z)$ defined as

$$\theta(z) = \arg \sup_{t \in \mathcal{S}_{\mathbb{R}}} \varphi(t|z), \tag{1}$$

where $\mathcal{S}_{\mathbb{R}}$ is a fixed compact subset of $\mathbb{R}$.

2. The Model

Consider a randomly censored model given by two nonnegative stationary sequences of independent and identically distributed (i.i.d) random variables $T_1, \cdots, T_n$ (survival times) and $C_1, \cdots, C_n$ (censoring times) with the distribution functions F and G, respectively. In pratice, particularly, in medical applications, it is not possible to observe the lifetimes T of all patients under study in the presence of censoring. We only observe the triples (X_i, δ_i, Z_i), where $X_i = \min\{T_i, C_i\}$ and $\delta_i = \mathbf{1}_{\{T_i \le C_i\}}$, $1 \le i \le n$ with $\mathbf{1}_A$ denotes the indicator function of the set A, where both of T_i and C_i are expected to exhibit some kind of dependence which ensures the identifiability of the model.

In biomedical case studies, it is assumed that C_i and (T_i, Z_i) are independent, this condition is plausible whenever the censoring is independent of the patient's modality.

In this kind of model, it is well known that the empirical distribution is not a consistent estimator for the distribution function G. Therefore, Kaplan and Meier (1958) proposed a consistent estimator, for the survival function $\bar{G}(\cdot) = 1 - G(\cdot)$ which is constructed by

$$\bar{G}_n(t) = \begin{cases} \displaystyle\prod_{i=1}^{n} \left(1 - \frac{1 - \delta_{(i)}}{n - i + 1}\right)^{\mathbf{1}_{\{X_{(i)} \le t\}}}, & \text{if } t < X_{(n)}, \\ 0, & \text{otherwise}, \end{cases}$$

where $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ are the order statistics of $(X_i)_{1 \le i \le n}$ and $\delta_{(i)}$ is concomitant with $X_{(i)}$.

Because of the relation between the conditional mode and the conditional density given in statement (1), an estimator of $\theta(z)$ follows straightforwardly from an estimator of $\varphi(t|z)$. Now, we represent the kernel estimator of the conditional density function in the case of complete data, set

$$\varphi_n(t|z) = \frac{h_{n,H}^{-1} \sum_{i=1}^{n} K(h_{n,K}^{-1} d(z, Z_i)) H'(h_{n,H}^{-1}(t - T_i))}{\sum_{i=1}^{n} K(h_{n,K}^{-1} d(z, Z_i))}, \tag{2}$$

where K is a probability density function (so-called kernel function), $h_K = h_{n,K}$(resp. $h_H = h_{n,H}$) is a sequence of positive real numbers (so-called bandwidth) which goes to zero as n tends to infinity, $H'(\cdot)$ is the first derivative of a given distribution function $H(\cdot)$. An analogous estimator to Equation 2 was already given in Ferraty and Vieu (2006) in the general setting.

Firstly, we must know that our kernel type estimator of the conditional density $\varphi(t|z)$ adapted for censored samples is based on "a pseudo-estimator" of $\varphi(t|z)$ that is defined as

$$\widetilde{\varphi}_n(t|z) = \frac{\sum_{i=1}^{n} \delta_i \bar{G}^{-1}(X_i) K(h_K^{-1} d(z, Z_i)) H'(h_H^{-1}(t - X_i))}{h_H \sum_{i=1}^{n} K(h_K^{-1} d(z, Z_i))} = \frac{\widetilde{\varphi}_n(z,t)}{\psi_n(z)},$$

where

$$\widetilde{\varphi}_n(z,t) = \frac{1}{n h_H \mathbb{E}(\Delta_1(z))} \sum_{i=1}^{n} \delta_i \bar{G}^{-1}(X_i) H'(h_H^{-1}(t - X_i)) \Delta_i(z),$$

and

$$\psi_n(z) = \frac{1}{n\mathbb{E}(\Delta_1(z))} \sum_{i=1}^{n} \Delta_i(z), \quad \text{with} \;\; \Delta_i(z) = K(d(z, Z_i)/h_K).$$

In fact, this pseudo-estimator is not efficient since $\bar{G}(\cdot)$ is unknown in practice. So, we should replace $\bar{G}(\cdot)$ by its Kaplan and Meier's estimator $\bar{G}_n(\cdot)$ previously defined.

Therefore, feasible estimator of the conditional density function $\varphi(t|z)$ is denoted by

$$\widehat{\varphi}_n(t|z) = \frac{\sum_{i=1}^{n} \delta_i \bar{G}_n^{-1}(X_i) K(h_K^{-1} d(z, Z_i)) H'(h_H^{-1}(t - X_i))}{h_H \sum_{i=1}^{n} K(h_K^{-1} d(z, Z_i))} = \frac{\widehat{\varphi}_n(x,t)}{\psi_n(z)}, \tag{3}$$

where

$$\widehat{\varphi}_n(z,t) = \frac{1}{n h_H \mathbb{E}(\Delta_1(z))} \sum_{i=1}^{n} \delta_i \bar{G}_n^{-1}(X_i) H'(h_H^{-1}(t - X_i)) \Delta_i(z).$$

Then, a natural kernel estimator of $\theta(z)$ which maximizes the kernel estimator $\widehat{\varphi}_n(\cdot|z)$ of $\varphi(\cdot|z)$ is given by

$$\widehat{\theta}(z) = \arg\sup_{t \in \mathcal{S}_{\mathbb{R}}} \widehat{\varphi}_n(t|z). \tag{4}$$

Note that the estimate $\widehat{\theta}(z)$ is not necessarily unique and our results are valid for any chosen value satisfying (4). We point out that we can specify our choice by taking

$$\widehat{\theta}(z) = \inf \left\{ x \in \mathbb{R} \quad \text{such that}\; \widehat{\varphi}_n(x|z) = \sup_{t \in \mathcal{S}_{\mathbb{R}}} \widehat{\varphi}_n(t|z) \right\}.$$

3. Notations and Hypotheses

To formulate our assumptions, some additional notations are required. For $i = 1, \ldots, n$, we represent $\mathcal{F}_i$ as the σ-field generated by $((Z_1, T_1), \ldots, (Z_i, T_i))$ and $\mathcal{G}_i$ the one generated by $((Z_1, T_1), \ldots, (Z_i, T_i), Z_{i+1})$.

Let $\mathcal{N}_z$ be a fixed neighborhood of z, and let $\mathcal{B}(z, h)$ the ball of center z and radius h, denote $D_i(z) = d(z, Z_i)$ a nonnegative random variable such that its cumulative distribution function is determined by $F_z(u) = \mathbb{P}(D_i(z) \leq u) = \mathbb{P}(Z_i \in \mathcal{B}(z, u))$.

Furthermore, we define $F_z^{\mathcal{F}_{i-1}}(u) = \mathbb{P}(D_i(z) \leq u | \mathcal{F}_{i-1}) = \mathbb{P}(Z_i \in \mathcal{B}(z, u) | \mathcal{F}_{i-1})$ the conditional distribution function given the σ-field $\mathcal{F}_{i-1}$ of $(D_i(z))_{i \geq 1}$.

Our nonparametric model will be quite general in the sense that we will just need the following hypotheses:

(H0) For $x \in E$, there exists a sequence of nonnegative random functionals $(f_{i,1})_{i \geq 1}$ almost surely bounded by a sequence of deterministic quantities $(b_i(z))_{i \geq 1}$ accordingly, a sequence of random functions $(g_{i,z})_{i \geq 1}$, a deterministic nonnegative bounded functional f_1 and a nonnegative real function ϕ tending to zero, as its argument tends to 0, such that if $n \to \infty$ and $h \to 0$,

- **(a)** $F_z(h) = \phi(h) f_1(x) + o(\phi(h))$.
- **(b)** For any $i \in \mathbb{N}$, $F_z^{\mathcal{F}_{i-1}}(h) = \phi(h) f_{i,1}(z) + g_{i,z}(h)$ with $g_{i,z}(h) = o_{a.s.}(\phi(h))$ as $\frac{g_{i,z}(h)}{\phi(h)}$ almost surely bounded and $n^{-1} \sum_{i=1}^{n} g_{i,z}^{j}(h) = o_{a.s.}(\phi^j(h))$ for $j = 1, 2$.
- **(c)** $n^{-1} \sum_{i=1}^{n} f_{i,1}^{j}(z) \to f_1^j(z)$, almost surely, for $j = 1, 2$.
- **(d)** There exists a nondecreasing bounded function ς_0 such that, uniformly in $s \in [0, 1]$, $\phi(hs)/\phi(h) = \varsigma_0(s) + o(1)$, and, for $j \geq 1$, $\int_0^1 (K^j(t))' \varsigma_0(t)\, dt < \infty$.
- **(e)** $n^{-1} \sum_{i=1}^{n} b_i(z) \longrightarrow D(z) < \infty$.

(H1) The conditional density function $\varphi(t|z)$ satisfies

- **(a)** $\displaystyle\int_{\mathbb{R}} |t| \varphi(t|z)\, dt < \infty$, for all $z \in E$.
- **(b)** The Hölder condition, that is,
  $\forall (t_1, t_2) \in \mathcal{S}_{\mathbb{R}}^2$, $\forall (x_1, x_2) \in \mathcal{N}_x^2$, for some $\alpha_1 > 0$ and $\alpha_2 > 0$,
  $$|\varphi^{z_1}(t_1) - \varphi^{z_2}(t_2)| \leq C_z (d(z_1, z_2))^{\alpha_1} + |t_1 - t_2|^{\alpha_2}),$$
  with C_z is a positive constant depending on z.

(H2) $\varphi(\cdot|z)$ is twice continuously differentiable in a neighbourhood of $\theta(z)$ with

$$\begin{cases} \varphi^{(1)}(\theta(z)|z) = 0, \\ |\varphi^{(2)}(\theta(z)|z)| \neq 0. \end{cases}$$

(H3) The cumulative kernel H is derivable such that

$$\begin{cases} \exists C < \infty,\ \forall (v_1, v_2) \in \mathbb{R}^2, \quad |H'(v_1) - H'(v_2)| \leq C|v_1 - v_2|, \\ \displaystyle\int |v|^{\alpha_2} H'(v)\, dv < \infty, \quad \text{and} \quad \int H'(v)\, dv = 1. \end{cases}$$

(H4) For any $m \geq 1$, $\mathbb{E}[(H'(h_H^{-1}(t-T_i)))^m|\mathcal{G}_{i-1}] = \mathbb{E}[(H'(h_H^{-1}(t-T_i)))^m|Z_i]$.

(H5) For any $z' \in E$ and $m \geq 2$, $\sup_{t\in\mathcal{S}_\mathbb{R}} |g_m(z',t)| = \sup_{t\in\mathcal{S}_\mathbb{R}} |\mathbb{E}[H^m(h_H^{-1}(t-T_1))|Z_1 = x']| < \infty$ and $g_m(z',t)$ is continuous in $\mathcal{N}_z$ uniformly in t:

$$\sup_{t\in\mathcal{S}_\mathbb{R}} \sup_{z'\in B(z,h)} |g_m(z',t) - g_m(z,t)| = o(1).$$

(H6) K is a differentiable positive bounded function supported on $[0,1]$ of class $\mathcal{C}^1(0,1)$: $\exists\, C', C'',\ -\infty < C' < K'(t) < C'' < 0$ for $0 < t < 1$, $|\int_0^1 (K^j)'(t)\,dt| < \infty$ for $j = 1, 2$.

(H7) The bandwidth h_K and h_H, satisfying $\lim_{n\to\infty} h_K = 0$, $\lim_{n\to\infty} h_H = 0$ and $\dfrac{\log n}{n h_H \phi(h_K)} \underset{n\to\infty}{\longrightarrow} 0$.

(H8) $(C_n)_{n\geq 1}$ and $(Z_n, T_n)_{n\geq 1}$ are independent.

Remark 3.1.

Our assumptions are very standard for this kind of model. Assumption (H0) plays an important role in our methodology. It is known as the “concentration property.” (H1) is a regularity condition which characterize the functional space of our model and is needed to evaluate the bias term of our asymptotic results, while hypotheses (H3) and (H7) are technical conditions and are similar to those done in Ferraty and Vieu (2006). As for (H6), it is classical in nonparametric estimation.

4. Main resullts

In this part, we formulate the main results of strong consistency (with rate) as well as the asymptotic density and confidence interval of the conditional mode estimator are established.

4.1. Pointwise almost sure rate of convergence

We establish in Proposition 4.1 the rates of convergence of the kernel density estimator $\widehat{\varphi}_n(t|z)$, when couples of variables $(Z_n, T_n)_{n\geq 1}$ are independents. An immediate consequence is the almost sure convergence with a rate of the kernel mode estimator, as stated in Theorem 4.1.

Proposition 4.1.

Suppose that assumptions (H6)-(H7) and (H8) hold true, we get

$$\sup_{t\in\mathcal{S}_\mathbb{R}} |\widehat{\varphi_n}(t|z) - \varphi(t|z)| = O_{a.s.}(h_K^{\alpha_1} + h_H^{\alpha_2}) + O_{a.s.}\left(\sqrt{\frac{\log n}{n h_H \phi(h_K)}}\right).$$

Proof:

First of all, denote

$$\bar{\bar{\varphi}}_n(z,t) = \frac{1}{n h_H \mathbb{E}(\Delta_1(z))} \sum_{i=1}^n \mathbb{E}[\delta_i \bar{G}^{-1}(X_i) H'(h_H^{-1}(t - X_i))\Delta_i(z)|\mathcal{F}_{i-1}],$$

and

$$\bar{\psi}_n(z) = \frac{1}{n\mathbb{E}(\Delta_1(z))}\sum_{i=1}^{n}\mathbb{E}[\Delta_i(z)|\mathcal{F}_{i-1}],$$

the conditional bias which is given by

$$B_n(z,t) = \frac{\bar{\widetilde{\varphi}}_n(z,t)}{\bar{\psi}_n(z)} - \varphi(t|z). \tag{5}$$

In addition, there are quantities,

$$R_n(z,t) = -B_n(z,t)(\psi_n(z) - \bar{\psi}_n(z)),$$

and

$$Q_n(z,t) = (\widetilde{\varphi}_n(z,t) - \bar{\widetilde{\varphi}}_n(z,t)) - \varphi(t|z)(\psi_n(z) - \bar{\psi}_n(z)).$$

Lets's now introduce the following decomposition which is important to prove Proposition 4.1. For all $z \in E$, we state

$$\widehat{\varphi}_n(t|z) - \varphi(t|z) = \widehat{\varphi}_n(t|z) - \widetilde{\varphi}_n(t|z) + \widetilde{\varphi}_n(t|z) - \varphi(t|z). \tag{6}$$

Finally, the proof of this proposition is a direct consequence of the following intermediate results. It suffices to combine lemmas Lemma 4.1, Lemma 4.2 and decomposition 6.

Lemma 4.1.

Using (H6)-(H7) and (H8), we can show that

$$\sup_{t\in\mathcal{S}_{\mathbb{R}}}|\widehat{\varphi}_n(t|z) - \widetilde{\varphi}(t|z)| = O_{a.s.}\left(\sqrt{\frac{\log\log n}{n}}\right).$$

Proof:

By following the same steps as for the proof of Lemma 5.2 in Khardani et al. (2010), we can also prove our Lemma. ■

Lemma 4.2.

Because of the conditions (H6)-(H7) and (H8), we have as $n \to \infty$

$$\sup_{t\in\mathcal{S}_{\mathbb{R}}}|\widetilde{\varphi}_n(t|z) - \varphi(t|z)| = O_{a.s.}(h_K^{\alpha_1} + h_H^{\alpha_2}) + O_{a.s.}\left(\sqrt{\frac{\log n}{nh_H\phi(h_K)}}\right).$$

Proof:

We just need to prove that $\widetilde{\varphi}_n(t|z) - \varphi(t|z) = B_n(z,t) + \dfrac{R_n(z,t) + Q_n(z,t)}{\psi_n(z)}$ is negligible as $n \to \infty$, where $B_n(z,t)$ and $R_n(z,t)$ converge almost surely to zero by Lemma 4.4. Note that $\psi_n(z)$ has been studied in Lemma 4.3, where it converges to 1. Now, we deal only with $Q_n(z,t)$ in the following Lemma.

Lemma 4.3.

Suppose that assumptions (H0)-(H6) and (H7) hold true. Then, for any $z \in E$, set

(i) $\psi_n(z) - \bar{\psi}_n(z) = O_{a.s.}\left(\sqrt{\log n / n\phi(h_K)}\right)$.
(ii) $\lim_{n\to\infty} \psi_n(z) = \lim_{n\to\infty} \bar{\psi}_n(z) = 1, \quad$ a.s.

Proof:

The proof of this Lemma is the same as of Lemma 3 and Lemma 5 in Laib and Louani (2011). ■

Lemma 4.4.

Under the hypotheses (H3)-(H6) and (H7) together with (H8), we have as n goes to infinity

$$\sup_{t\in\mathcal{S}_\mathbb{R}} |B_n(z,t)| = O_{a.s.}(h_K^{\alpha_1} + h_H^{\alpha_2}), \tag{7}$$

$$\sup_{t\in\mathcal{S}_\mathbb{R}} |R_n(z,t)| = O_{a.s.}\left((h_K^{\alpha_1} + h_H^{\alpha_2})\left(\frac{\log n}{n\phi(h_K)}\right)^{1/2}\right). \tag{8}$$

Proof:

In the beginning, we rewrite the statement 5 as

$$B_n(z,t) = \frac{\bar{\widetilde{\varphi}}_n(z,t) - \bar{\psi}_n(z)\varphi(t|z)}{\bar{\psi}_n(z)}.$$

If $(H4)$ is verified, and in addition if $\mathbf{1}_{\{T_i\le C_i\}}\chi(X_i) = \mathbf{1}_{\{T_i\le C_i\}}\chi(T_i)$, we obtain

$$\begin{aligned}
\bar{\widetilde{\varphi}}_n(z,t) &= \frac{1}{nh_H\mathbb{E}(\Delta_1(z))}\sum_{i=1}^n \mathbb{E}\{\Delta_i(z)\mathbb{E}[\delta_i\bar{G}^{-1}(X_i)H'(h_H^{-1}(t-X_i))|\mathcal{G}_{i-1},T_i]|\mathcal{F}_{i-1}\} \\
&= \frac{1}{nh_H\mathbb{E}(\Delta_1(z))}\sum_{i=1}^n \mathbb{E}\{\Delta_i(z)\mathbb{E}[\delta_i\bar{G}^{-1}(X_i)H'(h_H^{-1}(t-X_i))|Z_i,T_i]|\mathcal{F}_{i-1}\} \\
&= \frac{1}{nh_H\mathbb{E}(\Delta_1(z))}\sum_{i=1}^n \mathbb{E}\{\bar{G}^{-1}(X_i)H'(h_H^{-1}(t-T_i))\Delta_i(x)\mathbb{E}[\mathbf{1}_{\{T_i\le C_i\}}|Z_i,T_i]|\mathcal{F}_{i-1}\} \\
&= \frac{1}{nh_H\mathbb{E}(\Delta_1(z))}\sum_{i=1}^n \mathbb{E}\{\Delta_i(z)H'(h_H^{-1}(t-T_i))|\mathcal{F}_{i-1}\}.
\end{aligned}$$

Furthermore, simple calculations by using always a double conditioning with respect to $\mathcal{G}_{i-1}$ leads to

$$\bar{\widetilde{\varphi}}_n(z,t) - \bar{\psi}_n(z)\varphi(t|z) = \frac{1}{nh_H\mathbb{E}(\Delta_1(z))}\sum_{i=1}^n \mathbb{E}\left\{\Delta_i(z)\left[\mathbb{E}\left(H'\left(\frac{(t-T_i)}{h_H}\right)\Big|Z_i\right) - h_H\varphi(t|z)\right]\Big|\mathcal{F}_{i-1}\right\}.$$

In view of conditions (H1) and (H3), it follows that

$$|\mathbb{E}(H'(h_H^{-1}(t-T_i)|Z_i) - h_H\varphi(t|z)| \leq C_z h_H \int_{\mathbb{R}} H'(u)(h_K^{\alpha_1} + |u|^{\alpha_2} h_H^{\alpha_2})\,du. \tag{9}$$

Hence, we get

$$\begin{aligned}\bar{\bar{\varphi}}_n(z,t) - \bar{\psi}_n(z)\varphi(t|z) &= O_{a.s.}(h_K^{\alpha_1} + h_H^{\alpha_2}) \times \frac{1}{n\mathbb{E}(\Delta_1(z))}\sum_{i=1}^{n}\mathbb{E}\{\Delta_i(z)|\mathcal{F}_{i-1}\}.\\ &= O_{a.s.}(h_K^{\alpha_1} + h_H^{\alpha_2}) \times \bar{\psi}_n(x).\end{aligned}$$

■

As a last step, we combine the above result with Lemma 4.3(ii) to obtain the following:

$$\frac{\bar{\bar{\varphi}}_n(z,t) - \bar{\psi}_n(z)\varphi(t|z)}{\bar{\psi}_n(z)} = O_{a.s.}(h_K^{\alpha_1} + h_H^{\alpha_2}).$$

Now, the second part of Lemma 4.4 will be easily deduced from the definition of $R_n(z,t)$, together with Lemma 4.3 and Equation 7. ■

Lemma 4.5.

Assume that (H0)-(H4) and (H6)-(H8) are satisfied. Then, for any $z \in E$, set

$$\sup_{t\in S_{\mathbb{R}}} |\widetilde{\varphi}_n(z,t) - \bar{\widetilde{\varphi}}_n(z,t)| = O_{a.s.}\left(\left(\frac{\log n}{nh_H\phi(h_K)}\right)^{1/2}\right).$$

Proof:

To prove our result, we need the decomposition below,

$$\sup_{t\in\mathcal{S}_{\mathbb{R}}} |\widetilde{\varphi}_n(z,t) - \bar{\widetilde{\varphi}}_n(z,t)| \leq \mathcal{J}_{1,n} + \mathcal{J}_{2,n} + \mathcal{J}_{3,n},$$

where

$$\mathcal{J}_{1,n} = \max_{1\leq k\leq\gamma_n}\sup_{t\in\mathcal{B}_k} |\widetilde{\varphi}_n(z,t) - \widetilde{\varphi}_n(z,t_k)|, \quad \mathcal{J}_{2,n} = \max_{1\leq k\leq\gamma_n} |\widetilde{\varphi}_n(z,t_k) - \bar{\widetilde{\varphi}}_n(z,t_k)|,$$

$$\mathcal{J}_{3,n} = \max_{1\leq k\leq\gamma_n}\sup_{t\in\mathcal{B}_k} |\widetilde{\varphi}_n(z,t_k) - \bar{\widetilde{\varphi}}_n(z,t)|.$$

Indeed, $\mathcal{S}_{\mathbb{R}}$ may be written as: $\mathcal{S}_{\mathbb{R}} \subset \cup_{k=1}^{\gamma_n}\mathcal{B}_k = \cup_{k=1}^{\gamma_n}\mathcal{B}_k(t_k,\Re_n)$, with $t_k(1\leq k\leq\gamma_n)$ are the ball's centers. Let's now study our three terms.

On the one hand, by a standard analytical argument and by using hypothesis (H3) and the result of

Lemma 4.3, we can evaluate the first term in the following way:

$$\begin{aligned}\mathcal{J}_{1,n} &\leq \frac{1}{nh_H\mathbb{E}(\Delta_1(z))}\max_{1\leq k\leq\gamma_n}\sup_{t\in\mathcal{B}_k}\sum_{i=1}^{n}\left|\delta_i\bar{G}^{-1}(X_i)[H'(h_H^{-1}(t-X_i))-H'(h_H^{-1}(t_k-X_i))]\Delta_i(z)\right|\\ &\leq \frac{C}{nh_H\mathbb{E}(\Delta_1(z))}\max_{1\leq k\leq\gamma_n}\sup_{t\in\mathcal{B}_k}\frac{|t-t_k|}{h_H}\sum_{i=1}^{n}\delta_i\bar{G}^{-1}(X_i)\Delta_i(z)\\ &\leq \frac{\gamma_n}{nh_H^2\mathbb{E}(\Delta_1(z))}\sum_{i=1}^{n}\delta_i\bar{G}^{-1}(X_i)\Delta_i(z),\end{aligned}$$

more precisely, by the fact that $\lim_{n\to\infty} n^{\vartheta}h_H^2=\infty$, we obtain,

$$\mathcal{J}_{1,n}\longrightarrow 0 \qquad \text{a.s.} \quad \text{as } n\to\infty.$$

As the first and the third terms can be treated in the same manner, so $\mathcal{J}_{3,n}$ is also negligible almost surely,

$$\mathcal{J}_{3,n}\longrightarrow 0 \;\; a.s. \;\; \text{as} \;\; n\to\infty.$$

On the other hand, to examine the rest term, we start by showing that

$$\widetilde{\varphi}_n(z,t_k)-\bar{\widetilde{\varphi}}_n(z,t_k)=\frac{1}{nh_H\mathbb{E}(\Delta_1(z))}\sum_{i=1}^{n}\Psi_{i,n}(z,t_k),$$

where

$$\Psi_{i,n}(z,t_k)=\delta_i\bar{G}^{-1}(X_i)H'(h_H^{-1}(t_k-X_i))\Delta_i(z)-\mathbb{E}(\delta_i\bar{G}^{-1}(X_i)H'(h_H^{-1}(t_k-X_i))\Delta_i(z)|\mathcal{F}_{i-1}),$$

represents a triangular array of stationary martingale differences with respect to the σ-field $\mathcal{F}_{i-1}$. Based on the proof of Lemma 5 in Laib and Louani (2011) and the assumptions (H0)-(H4) and (H5), the quantity $\mathbb{E}(\Psi_{i,n}^p(z,t_k)|\mathcal{F}_{i-1})$ can be developed as

$$|\mathbb{E}(\Psi_{i,n}^p(z,t)|\mathcal{F}_{i-1})|=p!C^{p-2}[C_2\phi(h_K)f_{i,1}(z)+O_{a.s}(g_{i,z}(h_K))]\leq p!C^{p-2}\phi(h_K)[Mb_i(z)+1],$$

where $C=2\max(1,a_1^2)$ and $M=(C_2C)^2$.

Choosing $D_n=\sum_{i=1}^n d_i^2$ with $d_i^2=\phi(h_K)[Mb_i(z)+1]$. By using hypotheses (H0)(b) and (H0)(e), it yields $n^{-1}D_n=\phi(h_K)[MD(z)+o_{a.s}(1)]$ as $n\to\infty$.

Thus, we apply the exponential inequality given in Lemma 1 in Laib and Louani (2011) with taking $D_n=O_{a.s}(n\phi(h_K))$, $S_n=\sum_{i=1}^n\Psi_{i,n}(z,t)$, and for any $\epsilon_0>0$ and C_1 is a positive constant, the following calculations is valid

$$\begin{aligned}
\mathbb{P}\left(|\mathcal{J}_{2,n}| > \epsilon_0\sqrt{\frac{\log n}{nh_H\phi(h_K)}}\right) &\leq \mathbb{P}\left(\max_{k\in 1\cdots\gamma_n}|\widetilde{\varphi}_n(z,t_k) - \bar{\widetilde{\varphi}}_n(z,t_k)| > \epsilon_0\sqrt{\frac{\log n}{nh_H\phi(h_K)}}\right)\\
&\leq \max_{k\in 1\cdots\gamma_n}\mathbb{P}\left(|\sum_{i=1}^{n}\Psi_{i,n}(z,t_k)| > nh_H\mathbb{E}(\Delta_1(z))\epsilon_0\sqrt{\frac{\log n}{nh_H\phi(h_K)}}\right)\\
&\leq 2\gamma_n\exp\left(\frac{-\Big(nh_H\epsilon_0\mathbb{E}(\Delta_1(z))\Big)^2\frac{\log n}{nh_H\phi(h_K)}}{2D_n + 2Cnh_H\mathbb{E}(\Delta_1(x))\epsilon_0\sqrt{\frac{\log n}{nh_H\phi(h_K)}}}\right)\\
&\leq 2\gamma_n\exp\{-C_1\epsilon_0^2\log n\}\\
&\leq \frac{2}{n^{C_1\epsilon_0^2}}.
\end{aligned}$$

Lastly, to achieve the proof we need only to take ϵ_0 large enough and to use the Borel-Cantelli Lemma. ■

Lemma 4.6.

By the same hypotheses of Lemma 4.5, it yields

$$\sup_{t\in\mathcal{S}_{\mathbb{R}}}|Q_n(z,t)| = O_{a.s.}\left(\sqrt{\frac{\log n}{nh_H\phi(h_K}}\right).$$

Proof:

Lemmas 4.3 and 4.5 lead directly to the proof. ■

Finally, the proof of Proposition 4.1 is a direct consequence of the intermediate results announced above, and consequently the proof is completed. ■

Theorem 4.1.

Again by (H6)-(H7) and (H8) in conjunction with (H2), we obtain

$$|\widehat{\theta}(z) - \theta(z)| = O_{a.s.}(h_K^{\alpha_1} + h_H^{\alpha_2}) + O_{a.s.}\left(\sqrt{\frac{\log n}{nh_H\phi(h_K)}}\right).$$

Proof:

The proof of Theorem 4.1 can be completed by the following lemma.

Lemma 4.7.

Under the assumptions of Proposition 4.1, we obtain

$$\lim_{n\to\infty} |\widehat{\theta}(z) - \theta(z)| = 0, \quad \text{a.s.}$$

Proof:

By the continuity of the function $f(t|x)$, it follows that

$$\forall \epsilon > 0,\ \exists \zeta(\epsilon) > 0, \quad |\varphi(t|z) - \varphi(\theta(z)|z)| \leq \zeta(\epsilon) \Rightarrow |t - \theta(z)| \leq \epsilon.$$

This allows us to write

$$\forall \epsilon > 0,\ \exists \zeta(\epsilon) > 0, \quad \mathbb{P}\Big(|\widehat{\theta}(z) - \theta(z)| > \epsilon\Big) \leq \mathbb{P}\Big(|\varphi(\widehat{\theta}(z)|z) - \varphi(\theta(z)|z)| > \zeta(\epsilon)\Big). \tag{10}$$

Next, by simple algebra, we also have

$$|\varphi(\widehat{\theta}(z)|z) - \varphi(\theta(z)|z)| \leq 2 \sup_{t\in\mathcal{S}_{\mathbb{R}}} |\widehat{\varphi}_n(t|z) - \varphi(t|z)|. \tag{11}$$

Lastly, the convergence of $\widehat{\theta}(z)$ to $\theta(z)$ almost surely will be easily deduced from the latter together with 10 and Proposition 4.1. ■

Finally the proof of Theorem 4.1 is based on the Taylor expansion of order two of $\varphi(\widehat{\theta}(z)|z)$ at the point $\theta(z)$, on the use of the first part of (H2). Let

$$\varphi(\widehat{\theta}(z)|z) - \varphi(\theta(z)|z) = \frac{1}{2}\varphi^{(2)}(\theta^*(z)|z)(\widehat{\theta}(z) - \theta(z))^2,$$

where $\min(\theta(z), \widehat{\theta}(z)) < \theta^*(z) < \max(\theta(z), \widehat{\theta}(z))$.
Consequently, by considering the last equality with the statement 11, we derive

$$|(\widehat{\theta}(z) - \theta(z))|^2 \leq \frac{1}{\varphi^{(2)}(\theta^*(z)|z)} \sup_{t\in\mathcal{S}_{\mathbb{R}}} |\widehat{\varphi}_n(t|z) - \varphi(t|z)|.$$

Now, because of $\varphi^{(2)}(\theta^*(z)|z) \longrightarrow \varphi^{(2)}(\theta(z)|z)$, and on the use of the second part of (H2), we directly obtain

$$|(\widehat{\theta}(z) - \theta(z))|^2 = O_{a.s.}\Big(\sup_{t\in\mathcal{S}_{\mathbb{R}}} |\widehat{\varphi}_n(t|z) - \varphi(t|z)|\Big).$$

Thus, Proposition 4.1 allow us to get the claimed result. ■

4.2. Asymptotique normality

The aim of this section is to establish the asymptotic normality which induces a confidence interval of the conditional mode estimator. For this purpose, we shall list some basic conditions.

(A0) The smoothing parameter h_H satisfies: $nh_H^3\phi(h_K) \longrightarrow 0$, as $n \to \infty$.
(A1) The distribution function of the censored random variable, G has a bounded first derivative $G^{(1)}$.
(A2) The *cdf* $\varphi(t|z)$ verifies the Hölder condition, $\forall (t_1, t_2) \in \mathcal{S}_{\mathbb{R}}^2$, $\forall j = 1, 2$, for some $\alpha_0 > 0$,

$$|\varphi^{(j)}(t_1|z) - \varphi^{(j)}(t_2|z)| \leq C(|t_1 - t_2|^{\alpha_0}).$$

(A3) The kernel H is twice differentiable such that

$$\int |t|^{\alpha_0}(H^{(j)}(v))^2\, dv < \infty, \text{ for } j = 1, 2, \quad \text{and} \quad \int (H'(v))^2\, dv < \infty.$$

Theorem 4.2.

Using the conditions (H0)-(H6)-(H7) and (A1)-(A3), it results in

$$\sqrt{nh_H\phi(h_K)}(\widehat{\varphi}_n(t|z) - \varphi(t|z)) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2(z,t)),$$

where

$$\sigma^2(z,t) = \frac{M_2}{M_1^2}\frac{\varphi(t|z)}{\bar{G}(t)f_1(z)}\int_{\mathbb{R}} (H'(v))^2\, dv,$$

with $M_j = K^j(1) - \int_0^1 (K^j)'\varsigma_0(u)\, du \quad \text{for} \quad j = 1, 2.$

Note that " $\xrightarrow{\mathcal{D}}$ " symbolyzes the convergence in distribution.

Proof:

Initially, we suggest the following decomposition:

$$\begin{aligned}\widehat{\varphi}_n(t|z) - \varphi(t|z) &= [\widehat{\varphi}_n(t|z) - \widetilde{\varphi}_n(t|z)] + [\widetilde{\varphi}_n(t|z) - \bar{\widetilde{\varphi}}_n(t|z)] + [\bar{\widetilde{\varphi}}_n(t|z) - \varphi(t|z)] \\ &= \mathcal{U}_{1,n} + \mathcal{U}_{2,n} + \mathcal{U}_{3,n}.\end{aligned}$$

According to Lemma 4.1, the term $\mathcal{U}_{1,n}$ converges almost surely to zero when n goes to infinity, where

$$\mathcal{U}_{1,n} = O_{a.s.}\left(\sqrt{\frac{\log\log n}{n}}\right). \tag{12}$$

Moreover, it is simple to show that $\mathcal{U}_{3,n}$ is also negligible, where we readily get

$$\mathcal{U}_{3,n} = \bar{\widetilde{\varphi}}_n(t|z) - \varphi(t|z) = B_n(t|z).$$

Therefore, from Lemma 4.4, we obtain

$$\mathcal{U}_{3,n} = O_{a.s.}(h_K^{\alpha_1} + h_H^{\alpha_2}). \tag{13}$$

Now, it suffices to prove the asymptotic normality of $\mathcal{U}_{2,n} = \dfrac{Q_n(z,t) + R_n(z,t))}{\psi_n(z)}$, where $R_n(z,t)$ is negligible as $n \to \infty$, and $\psi_n(z)$ converges almost surely towards 1, where

$$R_n(z,t) = -B_n(z,t)(\psi_n(z) - \bar{\psi}_n(z)),$$

with

$$B_n(z,t)=\frac{\bar{\widetilde{\varphi}}_n(z,t)}{\bar{\psi}_n(z)}-\varphi(t|z).$$

Thus, the asymptotic normality will be provided by the term $Q_n(z,t)=[\widetilde{\varphi}_n(z,t)-\bar{\widetilde{\varphi}}_n(z,t)]-\varphi(t|z)(\psi_n(z)-\bar{\psi}_n(z))$ which is treated in the lemmas Lemma 4.8 and Lemma 4.9 below.

Lemma 4.8.

Assume that conditions (H0)(a), (H0)(b) and (H0)(d) as well as (H6) are satisfied. Then, for any real numbers $1\leq j\leq 2+\delta$ and $1\leq k\leq 2+\delta$ with $\delta>0$, as $n\to\infty$, one has

(i) $\displaystyle \frac{1}{\phi(h_K)}\mathbb{E}\Big[\Delta_i^j(z)|\mathcal{F}_{i-1}\Big]=M_jf_{i,1}(z)+O_{a.s.}\left(\frac{g_{i,z(h_K)}}{\phi(h_K)}\right).$

(ii) $\displaystyle \frac{1}{\phi(h_K)}\mathbb{E}\Big[\Delta_i^j(z)\Big]=M_jf_1(z)+o(1).$

(iii) $\displaystyle \frac{1}{\phi^k(h_K)}(\mathbb{E}(\Delta_1(z)))^k=M_1^kf_1^k(z)+o(1).$

Proof:

The proof is given in Lemma 1 by Laib and Louani (2010). ■

Lemma 4.9.

By the same hypotheses of Theorem 4.2, one writes as $n\to\infty$,

$$\sqrt{nh_H\phi(h_K)}Q_n(z,t)\xrightarrow{\mathcal{D}}\mathcal{N}(0,\sigma^2(z,t)).$$

Recall that $\sigma^2(z,t)$ is defined in Theorem 4.2.

Proof:

Easily, we get

$$\sqrt{nh_H\phi(h_K)}Q_n(z,t)=\sum_{i=1}^n\mu_{ni}, \tag{14}$$

where

$$\mu_{ni}=\Xi_{ni}-\mathbb{E}[\Xi_{ni}|\mathcal{F}_{i-1}],$$

with

$$\Xi_{ni}=\left(\frac{\phi(h_K)}{nh_H}\right)^{1/2}\left(\frac{\delta_i}{\bar{G}(X_i)}H'(h_H^{-1}(t-X_i)-h_H\varphi(t|z)\right)\frac{\Delta_i(z)}{\mathbb{E}(\Delta_1(z))}.$$

Obviously, based on the central limit theorem for discrete-time arrays of real-valued martingales (see Hall and Heyde (1980)), the asymptotic normality of $Q_n(z,t)$ can be obtained if we demonstrate these two statements:

I. $\sum_{i=1}^{n} \mathbb{E}[\mu_{ni}^2|\mathcal{F}_{i-1}] \xrightarrow{\mathbb{P}} \sigma^2(z,t)$.
II. $n\mathbb{E}[\mu^2 \mathbf{1}_{[|\mu_{ni}|>\epsilon]}] = o(1)$ holds for any $\epsilon > 0$ (Linderberg condition).

• **Proof of the first part (I.)**:

Firstly, let us consider

$$\left|\sum_{i=1}^{n} \mathbb{E}[\Xi_{ni}^2|\mathcal{F}_{i-1}] - \sum_{i=1}^{n} \mathbb{E}[\mu_{ni}^2|\mathcal{F}_{i-1}]\right| \leq \sum_{i=1}^{n} (\mathbb{E}[\Xi_{ni}|\mathcal{F}_{i-1}])^2.$$

Applying Lemma 4.8 together with inequality 9, it yields

$$\begin{aligned}|\mathbb{E}[\Xi_{ni}|\mathcal{F}_{i-1}]| &= \frac{1}{\mathbb{E}(\Delta_1(z))}\left(\frac{\phi(h_K)}{nh_H}\right)^{1/2}\left|\mathbb{E}\left[\Delta_i(z)\left(\frac{\delta_i}{\bar{G}(X_i)}H'(h_H^{-1}(t-X_i)) - h_H\varphi(t|z)\right)|\mathcal{F}_{i-1}\right]\right| \\ &\leq C(h_K^{\alpha_1} + h_H^{\alpha_2})\left(\frac{\phi(h_K)h_H}{n}\right)^{1/2}\left(\frac{f_{i,1}(z)}{f_1(z)} + O_{a.s}\left(\frac{g_{i,z}(h_K)}{\phi(h_K)}\right)\right).\end{aligned}$$

Subsequently, by (H0)(b)-(c), it follows that

$$\sum_{i=1}^{n} (\mathbb{E}[\Xi_{ni}|\mathcal{F}_{i-1}])^2 = O_{a.s.}\left(h_H\phi(h_K)(h_K^{\alpha_1} + h_H^{\alpha_2})^2\right).$$

So, we just need to prove the following

$$\lim_{n\to\infty} \sum_{i=1}^{n} \mathbb{E}[\Xi_{ni}^2|\mathcal{F}_{i-1}] \xrightarrow{\mathbb{P}} \sigma^2(z,t). \tag{15}$$

For this, let's use (H4) to get

$$\begin{aligned}\sum_{i=1}^{n} \mathbb{E}[\Xi_{ni}^2|\mathcal{F}_{i-1}] &= \frac{\phi(h_K)}{nh_H(\mathbb{E}(\Delta_1(z)))^2}\sum_{i=1}^{n} \mathbb{E}\left\{\Delta_i^2(z)\mathbb{E}\left[\left(\frac{\delta_i}{\bar{G}(X_i)}H'(h_H^{-1}(t-X_i)) - h_H\varphi(t|z)\right)^2|Z_i\right]|\mathcal{F}_{i-1}\right\} \\ &= \frac{\phi(h_K)}{nh_H(\mathbb{E}(\Delta_1(z)))^2}\sum_{i=1}^{n} \mathbb{E}\left\{\Delta_i^2(z)\mathbb{E}\left[\left(\frac{\delta_i}{\bar{G}(X_i)}H'(h_H^{-1}(t-X_i)) - h_H\varphi(t|z)\right)^2|Z_i\right]|\mathcal{F}_{i-1}\right\}.\end{aligned}$$

Moreover, set

$$\begin{aligned}\mathbb{E}\left[\left(\frac{\delta_i}{\bar{G}(X_i)}H'(h_H^{-1}(t-X_i)) - h_H\varphi(t|z)\right)^2|Z_i\right] &= Var\left[\frac{\delta_i}{\bar{G}(X_i)}H'(h_H^{-1}(t-X_i))|Z_i\right] \\ &\quad + \left[\mathbb{E}\left(\frac{\delta_i}{\bar{G}(X_i)}H'(h_H^{-1}(t-X_i))|Z_i\right) - h_H\varphi(t|z)\right]^2 \\ &= \Gamma_{1,n} + \Gamma_{2,n}.\end{aligned}$$

It should be noted that the second term is negligible: $\Gamma_{2,n} \longrightarrow 0, \quad$ as $n \to \infty$, where we used inequality 9 and assumptions (H1), (H3) in order to get our result.

Now, all what is left to be study is $\Gamma_{1,n}$. Thus, we state

$$\Gamma_{1,n} = \underbrace{\mathbb{E}\Big[\frac{\delta_i}{\bar{G}^2(X_i)}\Big(H'\Big(\frac{t-X_i}{h_H}\Big)\Big)^2\Big|Z_i\Big]}_{\Lambda_1} - \underbrace{\Big[\mathbb{E}\Big(\frac{\delta_i}{\bar{G}(X_i)}H'\Big(\frac{t-X_i}{h_H}\Big)\Big|Z_i\Big)\Big]^2}_{\Lambda_2}. \quad (16)$$

• Concerning Λ_1, by simple calculations, we obtain

$$\begin{aligned}\Lambda_1 &= \mathbb{E}\Big[\mathbb{E}\Big(\frac{\delta_i}{\bar{G}^2(X_i)}H'^2\Big(\frac{t-X_i}{h_H}\Big)\Big|Z_i,T_i\Big)\Big]\\ &= \mathbb{E}\Big(\frac{1}{\bar{G}(T_i)}H'^2\Big(\frac{t-T_i}{h_H}\Big)\Big|Z_i\Big)\\ &= \int_{\mathbb{R}}\frac{1}{\bar{G}(\omega)}H'^2\Big(\frac{t-\omega}{h_H}\Big)f(\omega|Z_i)\,d\omega\\ &= \int_{\mathbb{R}}\frac{1}{\bar{G}(t-vh_H)}H'^2(v)dF(t-vh_H|Z_i).\end{aligned}$$

Writing a Taylor expansion of order one of the function $\bar{G}^{-1}(\cdot)$ around zero leads to the existence of some t^* between t and $(t-vh_H)$ such that

$$\begin{aligned}\Lambda_1 &= \int_{\mathbb{R}}\frac{1}{\bar{G}(t)}(H'(v))^2dF(t-vh_H|Z_i) + \frac{h_H^2}{\bar{G}^2(t)}\int_{\mathbb{R}}v(H'(v))^2\bar{G}^{(1)}(t^*)\varphi(t-vh_H|Z_i)\,dv + o(1)\\ &= \lambda_1+\lambda_2.\end{aligned}$$

If the hypotheses (H1), (A3) are verified, one has

$$\begin{aligned}\lambda_1 &= h_H\int_{\mathbb{R}}\frac{1}{\bar{G}(t)}(H'(v))^2\varphi(t-vh_H|Z_i)\,dv\\ &\le \frac{h_H}{\bar{G}(t)}\int_{\mathbb{R}}(H'(v))^2(\varphi(t-vh_H|Z_i)-\varphi(t|z))\,dv\\ &\quad+\frac{h_H}{\bar{G}(t)}\int_{\mathbb{R}}(H'(v))^2\varphi(t|z)\,dv\\ &\le \frac{h_H}{\bar{G}(t)}\Big(C_z\int_{\mathbb{R}}(H'(v))^2(h_K^{\alpha_1}+|v|^{\alpha_2}h_H^{\alpha_2})\,dv+\varphi(t|z)\int_{\mathbb{R}}(H'(v))^2\,dv\Big)\\ &= O\Big(h_K^{\alpha_1}+h_H^{\alpha_2}\Big)+\frac{h_H}{\bar{G}(t)}\varphi(t|z)\int_{\mathbb{R}}(H'(v))^2\,dv.\end{aligned}$$

On the other hand, by (A1), one can write

$$\lambda_2 \le h_H^2(\sup_{v\in\mathbb{R}}|\bar{G}^{(1)}(v)|/\bar{G}^2(t))\int_{\mathbb{R}}v\varphi(t-vh_H|Z_i)\,dv.$$

This means that as $n\to\infty$, $\lambda_2 = O(h_H^2)$.

• For the second term of (16), it suffices to evaluate its square root,

$$\begin{aligned}\Lambda_2' &= \mathbb{E}\left(\frac{\delta_i}{\bar{G}(X_i)}H'\left(\frac{t-X_i}{h_H}\right)\Big|Z_i\right)\\ &= \mathbb{E}\left(H'\left(\frac{t-T_i}{h_H}\right)\Big|Z_i\right)\\ &= \int_{\mathbb{R}} H'\left(\frac{t-\omega}{h_H}\right)f(\omega|Z_i)\,d\omega.\end{aligned}$$

By changing variables, we arrive at

$$\Lambda_2' = h_H\int_{\mathbb{R}} H'(v)(\varphi(t-vh_H|Z_i)-\varphi(t|z))\,dv + h_H\varphi(t|z)\int_{\mathbb{R}} H'(v)\,dv.$$

So, under (H1) and (H3) we would have

$$\Lambda_2' = O\left(h_K^{\alpha_1}+h_H^{\alpha_2}\right)+h_H\varphi(t|z),$$

which permit us to conclude that Λ_2 is negligible. By Lemma 4.8, all of the above results leads to

$$\begin{aligned}\frac{\phi(h_K)}{nh_H(\mathbb{E}(\Delta_1(z)))^2}\sum_{i=1}^n \mathbb{E}\{\Delta_i^2(z)\Gamma_{1,n}|\mathcal{F}_{i-1}\} &= \frac{h_H}{\bar{G}(t)}\varphi(t|z)\int_{\mathbb{R}}(H'(v))^2\,dv\\ &\quad\times\frac{\phi(h_K)}{nh_H(\mathbb{E}(\Delta_1(z)))^2}\sum_{i=1}^n \mathbb{E}(\Delta_i^2(z)|\mathcal{F}_{i-1}),\\ &\longrightarrow \frac{M_2}{M_1^2}\frac{\varphi(t|z)}{\bar{G}(t)f_1(z)}\int_{\mathbb{R}}(H'(v))^2\,dv.\end{aligned}$$

Lastly, we could establish that

$$\sum_{i=1}^n \mathbb{E}[\Xi_{ni}^2|\mathcal{F}_{i-1}] = \frac{M_2}{M_1^2}\frac{\varphi(t|z)}{\bar{G}(t)f_1(z)}\int_{\mathbb{R}}(H'(v))^2\,dv = \sigma^2(z,t),$$

which is enough to confirm part (I).

• **Proof of the second part (II.)**:

The definition of μ_{ni} allows us to write: $n\mathbb{E}[\mu_{ni}^2\mathbf{1}_{[|\mu_{ni}|>\epsilon]}] \le 4n\mathbb{E}[\Xi_{ni}^2\mathbf{1}_{[|\Xi_{ni}|>\epsilon/2]}]$.

Denote: $A>1$ and $B>1$ such that $1/A+1/A=1$. According to Hölder and Markov inequalities, we have for any $\epsilon>0$

$$\mathbb{E}[\Xi_{ni}^2\mathbf{1}_{[|\Xi_{ni}|>\epsilon/2]}] \le \frac{\mathbb{E}|\Xi_{ni}|^{2A}}{(\epsilon/2)^{2A/B}}.$$

Choosing C_0 a positive constant and $2A=2+\delta$ for all $\delta>0$, it follows that

$$
\begin{aligned}
4n\mathbb{E}[\Xi_{ni}^2\mathbf{1}_{[|\Xi_{ni}|>\epsilon/2]}] &\le C_0\left(\frac{\phi(h_K)}{nh_H}\right)^{(2+\delta)/2}\frac{n}{(\mathbb{E}(\Delta_1(z)))^{2+\delta}}\\
&\quad\times\mathbb{E}\left(\left[\Delta_i(z)\left|\frac{\delta_i}{\bar{G}(X_i)}H'(h_H^{-1}(t-X_i)-h_H\varphi(t|z)\right|\right]^{2+\delta}\right)\\
&\le C_0\left(\frac{\phi(h_K)}{nh_H}\right)^{(2+\delta)/2}\frac{n}{(\mathbb{E}(\Delta_1(z)))^{2+\delta}}\mathbb{E}((\Delta_i(z))^{2+\delta}\\
&\quad\times\mathbb{E}\left[\left|H'(h_H^{-1}(t-T_i)-h_H\varphi(t|z)\right|^{2+\delta}\Big|Z_i\right].
\end{aligned}
$$

Meanwhile,

$$
\begin{aligned}
\mathbb{E}\left[\left|H'(h_H^{-1}(t-T_i)-h_H\varphi(t|z)\right|^{2+\delta}\Big|Z_i\right] &= \int_{\mathbb{R}}\left(H'\left(\frac{t-\omega}{h_H}\right)-h_H\varphi(t|z)\right)^{2+\delta}\varphi(\omega|Z_i)\,d\omega\\
&\le C\int_{\mathbb{R}}H'^{2+\delta}\left(\frac{t-\omega}{h_H}\right)\varphi(\omega|Z_i)\,d\omega+h_H^{2+\delta}\varphi^{2+\delta}(t|z)\\
&= Ch_H\int_{\mathbb{R}}H'^{2+\delta}(v)\varphi(t-vh_H|Z_i)\,dv+h_H^{2+\delta}\varphi^{2+\delta}(t|z)\\
&= h_H\left[\int_{\mathbb{R}}H'^{2+\delta}(v)\varphi(t-vh_H|Z_i)\,dv+h_H^{1+\delta}\varphi^{2+\delta}(t|z)\right],
\end{aligned}
$$

which implies that

$$
\begin{aligned}
4n\mathbb{E}[\Xi_{ni}^2\mathbf{1}_{[|\Xi_{ni}|>\epsilon/2]}] &\le C_0\left(\frac{\phi(h_K)}{nh_H}\right)^{(2+\delta)/2}\frac{nh_H}{\mathbb{E}(\Delta_1(z)))^{2+\delta}}\\
&\quad\times\mathbb{E}\left((\Delta_i(z))^{2+\delta}\left[\int_{\mathbb{R}}\left(H'^{2+\delta}(v)\varphi(t-vh_H|z)\,dv+h_H^{1+\delta}\varphi^{2+\delta}(t|z)\right]\right)\\
&\le C_0\left(\frac{\phi(h_K)}{nh_H}\right)^{(2+\delta)/2}\frac{nh_H\mathbb{E}[(\Delta_i(z))^{2+\delta}]}{(\mathbb{E}(\Delta_1(z)))^{2+\delta}}.
\end{aligned}
$$

Making use of Lemma 4.8, then

$$
\begin{aligned}
4n\mathbb{E}[\Xi_{ni}^2\mathbf{1}_{[|\Xi_{ni}|>\epsilon/2]}] &\le C_0(nh_H\phi(h_K))^{-\delta/2}\frac{M_{2+\delta}f_1(z)+o(1)}{M_1^{2+\delta}f_1^{2+\delta}(z)+o(1)}\\
&= O((nh_H\phi(h_K))^{-\delta/2}).
\end{aligned}
$$

Ultimately, the proof of the second part is completed. Thus, Lemma 4.9 is proved. ■

From that, the Theorem 4.2 is valid by combining Equations (12), (13) and Lemma 4.9. ■

Theorem 4.3.

If the hypotheses (A0)-(A1)-(A3) as well as (H0)-(H2)-(H6) are satisfied, then we have

$$\sqrt{\frac{nh_H^3\phi(h_K)}{\varrho^2(z,\theta(z))}}(\widehat{\theta}(z)-\theta(z)) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1),$$

where

$$\varrho^2(z,\theta(z)) = \frac{M_2}{M_1^2}\frac{\varphi(\theta(z)|z)}{\bar{G}(t)f_1(z)(\varphi^{(2)}(\theta(z)|z))^2}\int_{\mathbb{R}}(H^{(2)}(v))^2\,dv.$$

Proof:

By the first order Taylor expansion of $\widehat{\varphi}_n^{(1)}(\cdot|z)$ in the neighborhood of $\widehat{\theta}(z)$, and since $\widehat{\varphi}_n^{(1)}(\widehat{\theta}(z)|z)=0$, one has

$$\sqrt{nh^3\phi(h_K)}|\widehat{\theta}(z)-\theta(z)| = \frac{-\sqrt{nh^3\phi(h_K)}\widehat{\varphi}_n^{(1)}(\theta(z)|z)}{\widehat{\varphi}_n^{(2)}(\theta^*(z)|z)},$$

where $\theta^*(z)$ is between $\theta(z)$ and $\widehat{\theta}(z)$.

In the verity, the proof of the statement below is analogous to that of Theorem 4.2. Let

$$-\sqrt{nh^3\phi(h_K)}\widehat{\varphi}_n^{(1)}(\theta(z)|z) \xrightarrow{\mathcal{D}} \mathcal{N}(0,\varrho_1^2(z,\theta(z)),$$

with $\varrho_1^2(z,\theta(z)) = \dfrac{M_2}{M_1^2}\dfrac{\varphi(\theta(z)|z)}{\bar{G}(t)f_1(z)}\displaystyle\int_{\mathbb{R}}(H^{(2)}(v))^2\,dv.$

Then, proceeding as in Ferraty and Vieu (2006), where $\widehat{\varphi}_n^{(2)}(\theta(z)|z) \longrightarrow \varphi^{(2)}(\theta(z)|z)$ as $n\to\infty$, and the fact that $\theta^*(z)$ is lying between $\theta(z)$ and $\widehat{\theta}(z)$, which gives

$$\widehat{\varphi}_n^{(2)}(\theta^*(z)|z) \longrightarrow \varphi^{(2)}(\theta(z)|z), \quad \text{as } n\to\infty.$$

■

4.3. Application and Confidence Bands

Observe that both the asymptotic variance $\sigma^2(z,t)$ and $\varrho^2(z,\theta(z))$ are not useful in practice since some of its related quantities ($\varphi(\cdot|z),\varphi^{(2)}(\cdot|z),\theta(z),\bar{G}(\cdot),M_j$ for $j=1,2$) and functions ($\phi(h_K),f_1(z)$) are unknown. To overcome this difficulty and to make it usable, we have to estimate it.

Hence, $\varphi(\cdot|z)$, $\varphi^{(2)}(\cdot|z)$, $\theta(z)$ and $\bar{G}(\cdot)$ must be changed respectively by the conditional density estimators $\widehat{\varphi}_n(\cdot|z)$ and $\widehat{\varphi}_n^{(2)}(\cdot|z)$, the conditional mode estimator $\widehat{\theta}(z)$ and the Kaplan-Meier's estimator $\bar{G}_n(\cdot)$. Furthermore, under the conditions (H0)-(a) and (H0)-(d), $\varsigma_0(\cdot)$ can be estimated by

$$\varsigma_n(\cdot) = \frac{F_{z,n}(uh)}{F_{z,n}(h)},$$

where

$$F_{z,n}(u) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{\{d(z,Z_i)\leq u\}}.$$

Finally, since ς_0 is replaced with ς_n, so we can directly estimate M_1 and M_2 by $M_{1,n}$ and $M_{2,n}$, respectively.

Now, we can simply obtain a confidence interval in practice since all quantities are known. For this purpose, let us introduce the following corollaries.

Corollary 4.1.

By the same assumptions of Theorem 4.2, one gets

$$\sqrt{\frac{nh_H^3 F_{z,n}(h_K)}{\widehat{\sigma}^2(z,t)}}(\widehat{\varphi}_n(t|z) - \varphi(t|z)) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1), \tag{17}$$

where

$$\widehat{\sigma}^2(z,t) = \frac{M_{2,n}}{M_{1,n}^2}\frac{\widehat{\varphi}_n(t|z)}{\bar{G}_n(t)}\int_{\mathbb{R}}(H'(v))^2\,dv.$$

Proof:

Note that

$$\sqrt{\frac{nh_H^3 F_{z,n}(h_K)}{\widehat{\varrho}^2(z,\widehat{\theta}(x))}}(\widehat{\theta}(z) - \theta(z)) = \frac{M_{1,n}}{M_1}\frac{\sqrt{M_2}}{\sqrt{M_{2,n}}}\frac{[\widehat{\varphi}_n^{(2)}(\widehat{\theta}(z)|z)]}{[\varphi^{(2)}(\theta(z)|z)]}\sqrt{\frac{F_{z,n}(h_K)\bar{G}_n(t)\varphi(\theta(z)|z)}{\phi(h_K)\bar{G}(t)\widehat{\varphi}_n(\widehat{\theta}(z)|z)f_1(z)}}$$
$$\times\sqrt{\frac{nh_H^3\phi(h_K)}{\varrho^2(z,\theta(z))}}(\widehat{\theta}(z) - \theta(z)).$$

By Theorem 4.3, it follows that

$$\sqrt{\frac{nh_H^3\phi(h_K)}{\varrho^2(z,\theta(z))}}(\widehat{\theta}(z) - \theta(z)) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1).$$

Making use of results given by Laib and Louani (2010), we obtain $M_{1,n} \xrightarrow{\mathbb{P}} M_1, M_{2,n} \xrightarrow{\mathbb{P}} M_2, F_{z,n}(h_K)/\phi(h_K)f_1(z) \xrightarrow{\mathbb{P}} 1$ as $n\to\infty$. On the other hand, we have $\bar{G}_n \longrightarrow \bar{G}$, according to Deheuvels and Einmahl (2000). In addition, we have $\widehat{\varphi}_n^{(2)}(\widehat{\theta}(z)|z) \longrightarrow \varphi^{(2)}(\theta(z)|z)$.

Finally, in conjunction with Lemma 4.7 and Proposition 4.1, one writes

$$\frac{M_{1,n}}{M_1}\frac{\sqrt{M_2}}{\sqrt{M_{2,n}}}\frac{[\widehat{\varphi}_n^{(2)}(\widehat{\theta}(z)|z)]}{[\varphi^{(2)}(\theta(z)|z)]}\sqrt{\frac{F_{z,n}(h_K)\bar{G}_n(t)\varphi(\theta(z)|z)}{\phi(h_K)\bar{G}(t)\widehat{\varphi}_n(\widehat{\theta}(z)|z)f_1(z)}} \xrightarrow{\mathbb{P}} 1, \quad \text{as } n\to\infty.$$

This yields the proof. ∎

Corollary 4.2.

By the same assumptions of Theorem 4.3, one gets

$$\sqrt{nh_H^3 F_{z,n}(h_K)}(\widehat{\theta}(z)-\theta(z)) \xrightarrow{\mathcal{D}} \mathcal{N}(0,\widehat{\varrho}^2(z,\widehat{\theta}(z))), \tag{18}$$

where

$$\widehat{\varrho}^2(z,\widehat{\theta}(z)) = \frac{M_{2,n}}{M_{1,n}^2}\frac{\widehat{\varphi}_n(\widehat{\theta}(z)|z)}{\bar{G}_n(t)(\widehat{\varphi}_n^{(2)}(\widehat{\theta}(z)|z))^2}\int_{\mathbb{R}}(H^{(2)}(v))^2\,dv.$$

Proof:

Similarly as in Theorem 4.1, by using Taylor's expansion of the function $\widehat{\varphi}_n(\widehat{\theta}(z)|z)$ around $\theta(z)$, and making use of Theorem 4.2, Theorem 4.3 and Lemma 4.7, we achieve the proof of the Corollary. ∎

- From Corollaries 4.1 and 4.2, it is possible to construct confidence bands. Exactly, we can obtain for each fixed $\eta \in (0,1)$ approximate $(1-\eta)\%$ confidence intervals for the conditional density and conditional mode, namely

$$\left[\widehat{\varphi}_n(t|z) - \frac{I_{\eta/2}\widehat{\sigma}(z,t)}{\sqrt{nh_H F_{z,n}(h_K)}}\,,\,\widehat{\varphi}_n(t|z) + \frac{I_{\eta/2}\widehat{\sigma}(z,t)}{\sqrt{nh_H F_{z,n}(h_K)}}\right],$$

and

$$\left[\widehat{\theta}(z) - \frac{I_{\eta/2}\widehat{\varrho}(z,\widehat{\theta}(z))}{\sqrt{nh_H^3 F_{z,n}(h_K)}}\,,\,\widehat{\theta}(z) + \frac{I_{\eta/2}\widehat{\varrho}(z,\widehat{\theta}(z))}{\sqrt{nh_H^3 F_{z,n}(h_K)}}\right],$$

where $I_{\eta/2}$ denotes the $\eta/2$ quantile of the standard normal distribution.

5. Simulation Study and Real Data Application

This section is proposed to illustrate our study for the conditional mode and to evaluate the effectiveness of the suggested estimator (i.e. in the censored nonparametric functional data analysis case) (CNPFDA) (3) in comparison with the one for complete data (NPFDA) (2).

First of all, note that all the routines for functional data used in this application (developed in R/S-Plus software) are available on the website *https://www.math.univ-toulouse.fr/staph/npfda/*

5.1. Simulation study

Now, we start by introducing the following stationary ergodic process defined on $[0,\pi/3]$, where the covariates are curves

$$Z_i(t) = -1-\cos(2W_i(t-\pi/3)), \quad i=1,\cdots,200;\; t\in[0,\pi/3], \tag{19}$$

where W_i is generated by the model constructed as: $W_i = \dfrac{1}{\sqrt{2}} W_{i-1} + \zeta_i$, with ζ_i are i.i.d. uniformly distributed on $(0, 1)$ and W_i is also simulated independently by $W_0 \sim \mathcal{U}(0, 1)$. For more clarification, some of these curves $(200$ samples) are simulated, and the corresponding graph is presented in Figure 1 below.

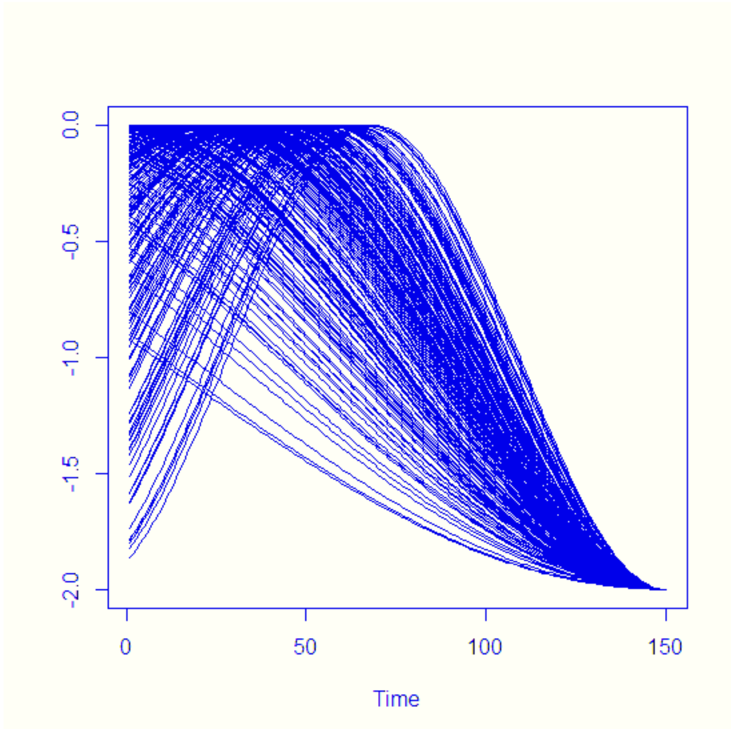

Figure 1. A sample of curves $\{Z_i(t), t \in [0, \pi/3]\}_{i=1,\cdots,200}$

The scalar response variable is defined by the following regression relation $T_i = r(Z_i) + \epsilon_i$, where $r(Z_i) = \left(\int_0^1 Z_i'(t)\, dt \right)^2$ and $\epsilon \sim \mathcal{N}(0, 0.075)$. Then, n i.i.d. random variables C_i, $i = 1, \ldots, n$ are drawn from an exponential distribution $\varepsilon(1.5)$.

Recall that the calculations of our estimator (for the incomplete data) are linked to the observed triplets $(Z_i, X_i, \delta_i)_{i=1,\ldots,n}$, where $X_i = \min(T_i, C_i)$ and $\delta_i = \mathbf{1}_{\{T_i \le C_i\}}$ denotes the censorship indicator.

Concerning the other parameters of our study: the regularity of the curves Z_i leads directly to choose the semi metric in E,

$$d(z_i, z_j) = \sqrt{\int_0^{\pi/3} (z_i'(t) - z_j'(t))^2\, dt} \quad z_i, z_j \in E.$$

For the kernels $K(\cdot)$ and $H(\cdot)$ were chosen to be of quadratic type as

$$K(u) = \frac{3}{2}(1 - u^2)\mathbf{1}_{(0,1)}(u), \quad H(u) = \frac{3}{4}(1 - u^2)\mathbf{1}_{(-1,1)}(u),$$

respectively.

Then, the smoothing parameter $h_H \sim h_K =: h$ is obtained by the cross-validation method on the k-nearest neighbors (Ferraty and Vieu (2006)).

In our experience, we consider a sample of 200 observations distributed on two parts A and B: the first one is a learning subsample $(Z_i, X_i)_{i\in A}$ with size$(A) = 150$, and the other is a testing subsample $(Z_j, X_j)_{j\in B}$ with size$(B) = 50$. We also compute the estimators $\widetilde{X}_j = \widetilde{\theta}(Z_j)$ and $\widehat{X}_j = \widehat{\theta}(Z_j)$ $j = \{151, \ldots, 200\}$ for complete data and censored data, respectively, through the learning sample. To evaluate the performance of both estimators (2) and (3), we propose the following mean square errors (MSE):

✠ **Under the complete data case:** $$NPFDA.MSE = \frac{1}{50}\sum_{j=151}^{200}(X_j - \widetilde{X}_j)^2.$$

✠ **Under the censored data case:** $$CNPFDA.MSE = \frac{1}{50}\sum_{j=151}^{200}(X_j - \widehat{X}_j)^2.$$

In order to simplifying the obtained results, Figure 2 and Figure 3 plot the predicted values as functions of the true ones for the MSE under the complete data and censored data cases, respectively.

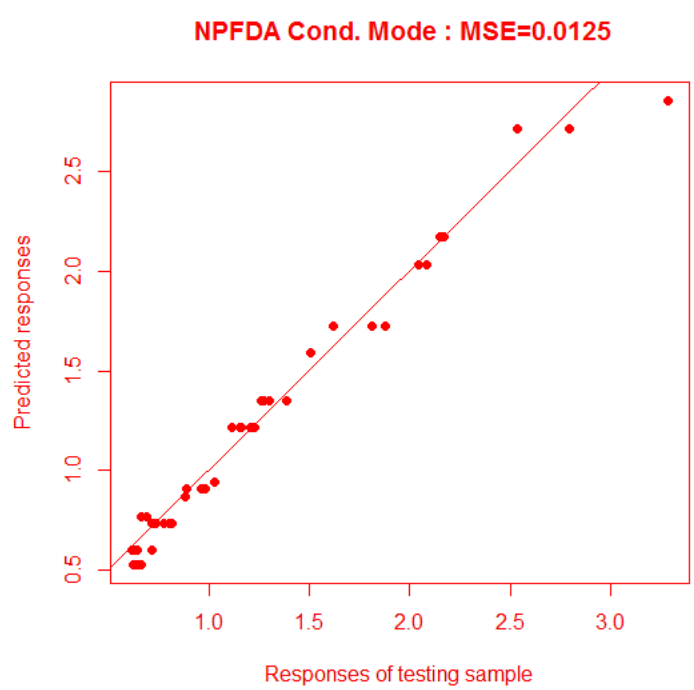

Figure 2. Prediction via the conditional mode for complete data case

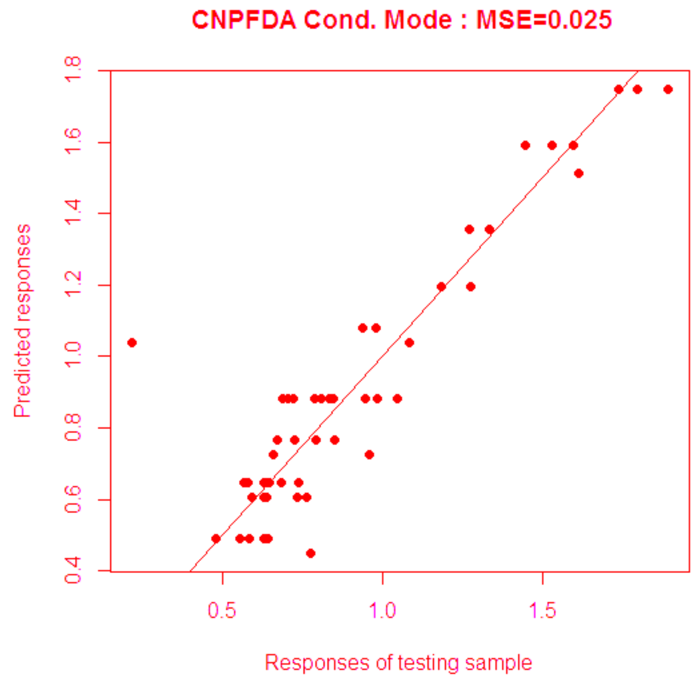

Figure 3. Prediction via the conditional mode for censored data case (CR$\sim 3\%$)

At the end of this section, we will explain the impact of the different censored rates (CRs) on the prediction results. For this purpose, we represent in Table 1 some CRs $(6\%, 17\%, 25\%, 50\%)$ and their corresponding MSE under censorship, where we fix the sample sizes by taking $n = 200$ then $n = 300$ and we vary the percentage of censure for each case.

The performance of the conditional mode estimator $\widehat{\theta}(z)$ is evaluated on $N = 300$ replications using different sample sizes $n = 50, 100, 150$, and 200. The mean square error (MSE) is considered here, such that, for a fixed z. Figure 2 displays the distribution of the obtained MSE given by the N replications. It can be observed that the proposed estimator performs well, especially when the sample size increases. This conclusion is confirmed by Table 1 which provides a numerical summary of the distribution of the MSE, with different censored rates (CR).

Table 1. MSE under the case of censored data

size(n)	CR%	MSE(for CNPFDA)	size(n)	CR%	MSE(for CNPFDA)
	6%	0.0443		6%	0.0401
200	17%	0.1005	300	17%	0.0941
	25%	0.1330		25%	0.1306
	50%	0.2712		50%	0.2487

5.2. Confidence intervals

For a deeper analysis, a Monte-Carlo simulations based on 100 replications are performed to assess the accuracy of both predictors based on CNPFDA and NPFDA approaches. Table 2 gives the mean square error obtained for both estimators, across 100 replications, for different values of η, n and for Censorship Rate CR = 30%.

Table 2. Comparison of average MSEs of conditional quantile estimators for different sample sizes and η with (CR$\sim$ 30%)

	η	0.05	0.5	0.95
n=100	CNPFDA	0.0108	0.0100	0.0110
	NPFDA	0.0635	0.0616	0.0639
n=300	CNPFDA	0.0079	0.0074	0.0080
	NPFDA	0.0426	0.0402	0.0437
n=500	CESIM	0.0059	0.0053	0.0061
	NPFDA	0.0300	0.0297	0.0308

With the increasing of n, the MSE decreases for both estimators decrease. In addition, when $\eta = 0.5$, we obtain the smallest mean square error. Further, again, it is well remarked that estimator CNPFDA produces much more accurate estimation than NPFDA estimator.

Next, we verify if our estimator is remains the best (MSE smaller) by choosing other semi-metrics. Table 3 compares the MSE, across 100 replications, between the two estimators considering the semi-metric based on 2^{nd} derivatives $(deriv_2)$, and the semi-metric based on functional principal component analysis (pca), for different values of η and n and censored data case (CR$\sim$ 60%).

Table 3. Estimation accuracy of the conditional quantile function between the functional single index model and the nonparametric functional model with different choices of semi-metrics and different values of η and n with (CR$\sim$ 60%)

Error	Model	Semi-metric	$n = 200$			$n = 400$		
			$\eta = 0.05$	$\eta = 0.50$	$\eta = 0.95$	$\eta = 0.05$	$\eta = 0.50$	$\eta = 0.95$
MSE	CPFDA	$deriv_2$	0.08087	0.08080	0.08090	0.03052	0.03049	0.03055
		pca	0.08094	0.08091	0.08097	0.03062	0.03061	0.03066
	NPFDA	$deriv_2$	0.08438	0.08434	0.08489	0.03276	0.03271	0.03292
		pca	0.08487	0.08483	0.08495	0.03301	0.03299	0.03309

As intuitively expected, with the increasing of the sample size n, the MSEs of both models for different semi-metrics and different values of η monotonically decrease. In addition, it is well observed that the mean square errors of our estimator are smaller than that of NPFDA.

Here we emphasizes a significant advantage of our estimator regarding prediction bands. One can think then our estimator may has an advantage with regard to prediction intervals compared to that of NPFDA. To this end, we conduct a simulation in order to we examine the efficiency of our assumption. The conditional median curves of both estimators (CNPFDA and NPFDA) are plotted in Figure 4. The curve of NPFDA estimator is represented in red colour.

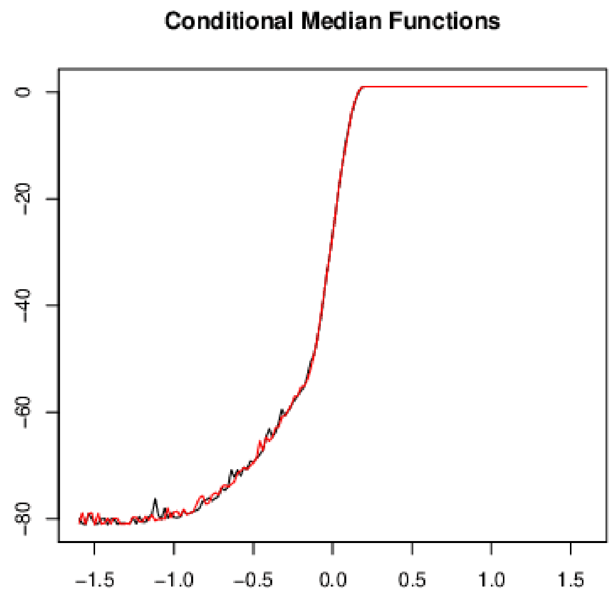

Figure 4. Conditional median ($\eta = 0.5$) curve

Next, we compare the 95% confidence intervals of both estimators for different values of n and for Censorship Rates CR = 60%. The results are arranged in Table 4.

Table 4. Confidence intervals for CNPFDA with (CR$\sim 48\%$)and NPFDA estimators for different values of n

n	100	200	300	400
CNPFDA	$[0.149, 0.194]$	$[0.171, 0.196]$	$[0.227, 0.257]$	$[0.226, 0.248]$
NPFDA	$[0.151, 0.190]$	$[0.171, 0.198]$	$[0.227, 0.258]$	$[0.226, 0.249]$

It is well seen that when n is large, our estimator is still better compared to NPFDA estimator.

5.3. Real data example: Peak electricity demand

This subsection applies our estimator to a real data. We evaluate and compare the finite sample performance between a nonparametric functional model from Chaouch and Khardani (2015) and our estimator (the functional single index model).

So, we apply our method to the data constituting hourly electricity demand for the Rocky Mountain region (WACM) of the United States. The data are daily electricity demands divided into 24 grids, where each hour of the day corresponds to a grid, from July 2015 to November 2018. The updated version of the data can be found on the site http://www.eia.gov/.

We construct our variables as follows. The observations of our covariate Z are the daily electricity demands from 2016 to 2018, $Z_i = (z_{i1}, \ldots, z_{i24})$. Our sample consists of $n = 1037$ observations. The observations of our response variable X are $X_i = min(max(Z_i), 1408),\ i = 1, \ldots, n$, where 1408 is the maximum peak of electricity demands in 2015.

In this part, we use Kaplan-Meier's estimator $\bar{G}_n(\cdot)$ as an estimator of $\bar{G}(\cdot)$ to construct our conditional distribution estimator, by taking the variables $(C_i)_i$ as deterministic (all equal to 1408, which is the maximum of the peak observed in 2015).

Since we are performing analysis on a time series spread over 4 years, considering the year 2015 as a base year, and in the simulation we are interested only in the years 2016-2018, we can consider 1408 as a maximum amplitude, that is, any value (or hourly observation) greater than 1408 can be considered as aberrant data. So, on this basis, we built our response variable.

Concerning the estimation of our parameters, we chose $deriv_1$ (the semi-metric based on the first derivatives of the curves) as semi-metric, the kernels $K(\cdot)$ and $H\,(u)$ are defined in the subsection 5.1. Then, as discussed previously, the optimal bandwidth $h = h_H = h_K$, are chosen using the cross-validation method on the k-nearest neighbors. Finally, the curves of the data are represented in Figure 5.

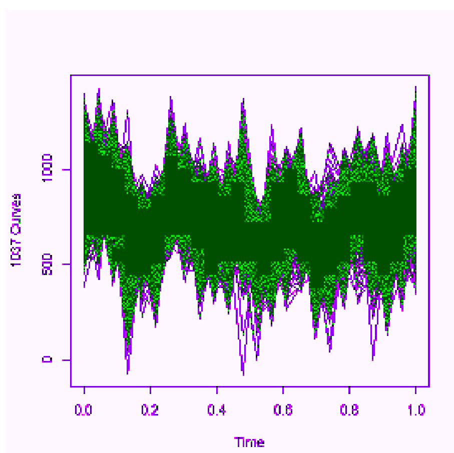

Figure 5. A sample of curves $\{Z_i\,(t)\,,\, t \in [0, 1]\}_{i=1,\ldots,1037}$

To assess the in-sample estimation accuracy and out-of sample prediction accuracy of the models, we split the original 1037 samples into two samples. The first one (learning set), from 1 to 960, used for the estimation, while the second sample (testing set), from 961 to 1037, is served for the prediction. To measure the estimation and prediction accuracies, we evaluate and compare the forecast accuracy using the testing sample, from which we predict responses in the testing sample.

To measure the performance of each functional prediction method, we consider the mean square errors (MSE).

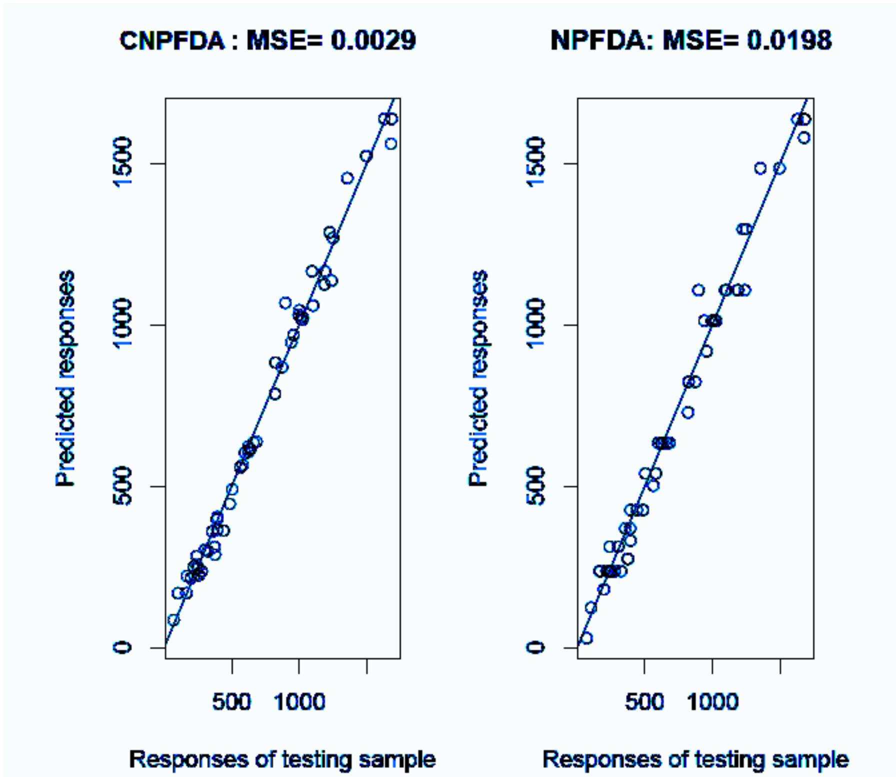

Figure 6. Prediction via the conditional mode by CNPFDA (CR$\sim$ 4.5%)with MSE =0.0029 against NPFDA with MSE =0.0198

Across 100 replications, the values of MSE of the two different models are given as $MSE = 0.0029$ (for our estimator) and $MSE = 0.0198$ (using NPFDA). The results by plotting the predicted values versus the true values are displayed in Figure 6. We therefore can conclude that there is an improvement in estimation for our model in comparison to the nonparametric functional model.

6. Conclusion

This paper focused on nonparametric estimation of conditional mode for dependant stationary ergodic data under random censorship and defined as an argument of the maximum of the conditional density. The resulting estimator has been shown to be asymptotically normally distributed under some regularity conditions. The main implication is to obtain the confidence bands which have been given in section 4.3. Of course, we use the plug-in rules to obtain an estimator of the asymptotic variance term.

Our prime aim was to improve the performance of this model for the conditional mode with censored response variable under the ergodic property. The simulations experiments in this paper show that our methodology can be easily implemented and work very well in both simulated and real data. It is well known that the kernel choice do not affect substantially the quality of the estimator. In addition, in order to explore the effectiveness of our method in real situations, we applied the CNPFDA estimator to data constituting hourly electricity demand for the Rocky Mountain region of the United States as well as spectrometric data.

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