Sines, Cosines, and Conjugates

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by

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Abstract:

This thesis is an investigation of angles whose sine and cosine are algebraic conjugates over the field of rational numbers. That is to say, $\sin(\theta)$ and $\cos(\theta)$ are roots of the same irreducible polynomial with integer coefficients. These interesting families are explored. First, it is shown that for $n \geq 2$, the angles $\frac{\pi}{2^n}$ have this property. Second, all angles which are conjugate in this sense and which have a quadratic minimum polynomial are identified. The relationship between these two families is explored, and a family of conjugate angles with 4^{th} degree minimum polynomials is explored as well.

Questions for further investigation are proposed, including an intriguing connection to chaos theory.

Table of Contents

Introduction

It has been known since the late 19^{th} century that among real numbers most are transcendental while only countably many are algebraic. That is, only countably many are roots of integer polynomials. Thus algebraic numbers are "rare." One focus of this thesis will be an examination of angles θ that are "algebraic",in the sense that the trigonometric functions of θ are algebraic numbers.

For example, $\frac{\pi}{6}$ is algebraic since its sine, cosine, tangent,

cotangent, secant, and cosecant are $\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{3}, \sqrt{3}, \frac{2\sqrt{3}}{3}$, and 2 respectively. Further refining the search I will focus especially on angles whose sine and cosine have the same minimum polynomial, a very special condition indeed, as will be seen.

Details of definitions and discussion of theorems from abstract algebra can be found in [2]. Similarly, references to ideas of chaos theory can be found in [1], and the trigonometric formulas used here are available in [3].

Some Definitions

Definition 1: A real number α is said to be algebraic if there is a polynomial $f(x)=a_0+a_1x+...+a_nx^n$ with a_i an integer for every i, $a_n \neq 0$, and $f(\alpha)=0$.

Definition 2: If α is algebraic then the minimum polynomial of α is an integer polynomial $f_\alpha(x)$ of smallest degree with:

i.) $f_\alpha(\alpha)=0$

ii.) a positive leading coefficient

iii.) there exists no common divisor bigger than 1 between the coefficients.

It can be shown that the minimum polynomial of an algebraic number is unique and cannot be factored.

For example, $1+\sqrt{2}$ is algebraic and its minimum polynomial is x^2-2x-1. The definition of algebraic number is standard but the following, to the best of my knowledge, is my own.

Definition 3: An angle θ is said to be <u>algebraic</u> if the six trigonometric functions all exist at θ and are all algebraic in the sense defined above.

For example, $\frac{\pi}{6}$ is an algebraic angle but it is not an algebraic number. The search for algebraic angles is somewhat facilitated by the following theorem.

Theorem 4: If $\sin(\theta)$ is an algebraic number other than 0 or 1 then θ is an algebraic angle.

Proof: The set of algebraic numbers is closed under the arithmetic operations and the taking of roots. Hence $\pm\sqrt{1-\sin^2(\theta)}=\cos(\theta)$ is algebraic if $\sin(\theta)$ is algebraic. Since the other 4 trigonometric functions are quotients involving $1, \sin(\theta), \cos(\theta),$ they too are algebraic. ■

It is now an easy matter to see that as θ varies from say $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ (or some comparable interval) $\sin(\theta)$ varies from -1 to 1, taking on every value in between. Countably many of these values are algebraic, so countably many of the angles in $[-\frac{\pi}{2},\frac{\pi}{2}]$ are algebraic angles.

Of more compelling interest is the following strengthening of the idea of an algebraic angle.

Definition 5: If θ is an algebraic angle and $\sin(\theta)$ and $\cos(\theta)$ have the same minimum polynomial, we say that θ is a conjugate angle.

The terminology "conjugate" is based on the algebra notion of conjugate numbers: algebraic numbers with the same minimum polynomial.

As an easy example of a conjugate angle consider $\theta=\frac{\pi}{4}$.

$\sin(\theta)=\frac{\sqrt{2}}{2}=\cos(\theta)$, and these algebraic numbers certainly have the same minimum polynomial since they are the same number. In fact, the minimum polynomial is $2x^2-1$. A more substantial example is algebraic, so countably many of the angles in $[-\frac{\pi}{2},\frac{\pi}{2}]$ are algebraic angles.

Of more compelling interest is the following strengthening of the idea of an algebraic angle.

<u>Definition 5:</u> If θ is an algebraic angle and $\sin(\theta)$ and $\cos(\theta)$ have the same minimum polynomial, we say that θ is a <u>conjugate angle</u>.

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$\sin(\theta)=\frac{\sqrt{2}}{2}=\cos(\theta)$, and these algebraic numbers certainly have the same minimum polynomial since they are the same number. In fact, the minimum polynomial is $2x^2-1$. A more substantial example is

$\theta=\frac{\pi}{8}$. $\sin(\theta)=\frac{\sqrt{2-\sqrt{2}}}{2}$ and $\cos(\theta)=\frac{\sqrt{2+\sqrt{2}}}{2}$. These have the common minimum polynomial $8x^4-8x^2+1$, as may be verified by hand.

Definition 6: We say that a nonconstant polynomial f(x) in $\mathbb{Q}[x]$ is irreducible over $\mathbb{Q}$ if whenever f(x) is factored $f(x)=a(x)b(x)$ with $a(x),b(x)\in\mathbb{Q}[x]$ then either a(x) or b(x) is constant.

Theorem 7: If α is an algebraic number and $f_\alpha(x)$ is its minimum polynomial, then $f_\alpha(x)$ is irreducible over $\mathbb{Q}$.

Proof: Suppose $f_\alpha(x)=a(x)b(x)$ with $a(x),b(x)\in\mathbb{Q}$. Then $f_\alpha(\alpha)=0=a(\alpha)b(\alpha)$. So, either $a(\alpha)=0$ or $b(\alpha)=0$. Say $a(\alpha)=0$. But, since a(x) has rational coefficients it may be multiplied by an integer k to get $c(x)=k\alpha(x)\in\mathbb{Z}[x]$. But now $c(\alpha)=0$ and, since $f_\alpha(x)$ has the smallest degree among polynomials with α as a root, it follows that degree c(x) = degree a(x) = degree $f_\alpha(x)$. But. Since a(x)b(x) = $f_\alpha(x)$, it follows that b(x) has degree 0, so b(x) is constant. Hence $f_\alpha(x)$ is irreducible. ■

We now know that minimum polynomials are irreducible. The following is a short converse theorem to this.

Theorem 8: Let α be an algebraic number. Let f(x) be a polynomial in $\mathbb{Z}[x]$ such that $f(\alpha)=0$. Suppose the coefficients of f have no common divisor greater than 1 and that the leading coefficient of f is positive. If f(x) is irreducible over $\mathbb{Q}$, then f(x) is the minimum polynomial of α.

Proof: Let $f_\alpha(x)$ be the minimum polynomial of α. We wish to show that $f(x)=f_\alpha(x)$. Use the division algorithm in $\mathbb{Q}[x]$ to get

$f(x)=q(x)f_\alpha(x)+r(x)$ where either r(x) = 0 or

$\text{degree}(r(x))<\text{degree}(f_\alpha(x))$. Hence $f(\alpha)=q(\alpha)f_\alpha(\alpha)+r(\alpha)$, so

$r(\alpha)=0$.

If r(x) has degree smaller than $\text{degree}(f_\alpha(x))$ we have a contradiction. Hence r(x)=0, so $f(x)=q(x)f_\alpha(x)$. But f(x) is irreducible so either q(x) is constant or $f_\alpha(x)$ is constant. But $f_\alpha(x)$ is not constant, so q(x)=c for some c. Hence $f(x)=cf_\alpha(x)$. Since both polynomials have positive leading coefficients and since the coefficients of f(x) have no common divisor, c=1 and $f(x)=f_\alpha(x)$. ■

The following theorem is a convenient way, when it applies, to show polynomials in $\mathbb{Q}[x]$ to be irreducible.

Theorem 9: (Eisenstein's Criteria)

Let $c_n x^n + c_{n-1} x^{n-1} + \ldots + c_1 x + c_0 = f(x)$ be a polynomial in $\mathbb{Z}[x]$. If p is a prime so that

1.) $p \nmid c_n$;

2.) $p | c_i$ for $0 \leq i \leq n-1$;

3.) $p^2 \nmid c_0$;

Then f(x) is irreducible over $\mathbb{Q}$.

A Family of Irreducible Polynomials

The polynomial $2x^2-1$ is irreducible over $\mathbb{Q}$ because, aside from shifting constants, its only factorization in $\mathbb{R}[x]$ is $(\sqrt{2}x-1)(\sqrt{2}x+1)$, which is not a valid factorization in $\mathbb{Q}$. I define a family of polynomials as follows:

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$$f_2(x) = 2x^2 - 1$$

$$\vdots$$

$$f_{n+1}(x)=2(f_n(x))^2+1$$

$$\vdots$$

The first few polynomials $f_k(x)$ are displayed in table 1.

$f_2(x)$	$2x^2-1=0$
$f_3(x)$	$8x^4+8x^2+1=0$
$f_4(x)$	$64x^8+128z^6+80x^4+16x^2+1=0$
$f_5(x)$	

My next purpose is to show that these iterated compositions of $2x^2-1$ are irreducible over $\mathbb{Q}$.

<u>**Definition 10:**</u> For $n\geq 2$, using the above definitions for $f_n(x)$ I define $g_n(x)$:as follows:

$$g_2(x)=x^2-2$$

$$\vdots$$

$$g_{n+1}(x)=g(g_n(x)) \quad \text{or} \quad g_{n+1}(x)=(g_n(x))^2-2$$

It will be important to investigate the irreducibility of the $f_n(x)$ polynomials. For this purpose we will relate the $f_n(x)$ polynomials to the $g_n(x)$ polynomials.

Theorem 11: For $n \geq 2$, $g_n(x) = 2f_n(\frac{1}{2}x)$.

Proof: For $n=2$, $2(2(\frac{1}{2}x)^2 - 1)$

$$= 2(\frac{x^2}{2} - 1)$$

$$= x^2 - 2$$

$$= g_2(x).$$

Suppose $g_k(x) = 2f_k(\frac{1}{2}x)$.

Then
$$\begin{aligned} g_{k+1}(x) &= (g_k(x))^2 - 2 \\ &= (2f_k(\tfrac{1}{2}x))^2 - 2 \\ &= 4(f_k(\tfrac{1}{2}x))^2 - 2 \\ &= 2[2(f_k(\tfrac{1}{2}x))^2 - 1] \\ &= 2f_{k+1}(\tfrac{x}{2}). \end{aligned}$$

The result follows by the principle of mathematical induction. ■

Theorem 12: For $n \geq 2$, $g_n(x)$ is irreducible over $\mathbb{Q}$.

Proof: Since $g_2(x)=x^2-2$ has integer coefficients it is clear that all $g_n(x)$ have integer coefficients. Now, $g_2(x)$ is irreducible by Eisenstein with p=2. I claim that every $g_n(x)$ is irreducible by Eisenstein with p=2. For the induction step assume $g_k(x)$ is irreducible by Eisenstein with p=2 and that, more specifically, $g_k(x)$ is monic and the constant coefficient of $g_k(x)$ is ± 2. I claim all the same properties for $g_{k+1}(x)$. $g_k(x)=x^n+2(h(x))\pm 2$ where $h(0)=0$.

Now, $g_{k+1}(x)=(g_n(x))^2-2$

$$= x^{2n}+4(h(x))^2+4x^n h(x)\pm 4x^n\pm 8h(x)+2$$

$$= x^{2n}+2[2(h(x))^2+2x^n h(x)\pm 2x^n\pm 4h(x)]+2$$

and we see that:

1.) $g_{k+1}(x)$ is monic;

2.) the constant coefficient is ± 2;

3.) $g_{k+1}(x)$ is irreducible by Eisenstein with p=2. ■

Now it will be shown that for $2\geqslant n$ $\sin(\frac{\pi}{2^n})$ and $\cos(\frac{\pi}{2^n})$ are roots of $f_n(x)$. For this we use mathematical induction and a small knowledge of trigonometric identities.

Theorem 13: For $n \geqslant 2$ and for $\alpha_n \in (\sin(\frac{\pi}{2^n}), \cos(\frac{\pi}{2^n}))$ we have $f_n(\alpha_n) = 0$.

Proof: If n=2, $\cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2} = \sin(\frac{\pi}{4}) = \alpha_2$ and $f_\alpha(x) = 2x^2 - 1$. Since $2(\frac{\sqrt{2}}{2})^2 - 1 = 0$, the statement holds for n=2. Now suppose that $f_k(\alpha_k) = 0$ and consider $f_{k+1}(\alpha_{k+1}) = f_k(f_2(\alpha_k))$. Now, since α_{k+1} is either $\cos(\frac{\pi}{2^{k+1}})$ or $\sin(\frac{\pi}{2^{k+1}})$, we must consider both cases.

We first examine the case when $\alpha_{k+1} = \cos(\frac{\pi}{2^{k+1}})$. Since $f_{k+1}(\alpha_{k+1}) = f_k(f_2(\alpha_k))$, we can write $f_{k+1}(\alpha_{k+1}) = f_k(2\alpha_k^2 - 1)$ since $f_2(\alpha_{k+1}) = 2\alpha_k^2 - 1$. But this is the same as $f_k(\alpha_k)$ which, by our induction hypothesis, is 0.

If $\alpha_{k+1} = \sin(\frac{\pi}{2^{k+1}})$, then $f_{k+1}(\alpha_{k+1}) = f_{k+1}(\sin(\frac{\pi}{2^{k+1}}))$. Using the double-angle formula for sin, we get:

$$f_{k+1}(\sin(\frac{\pi}{2^{k+1}})) = f_{k+1}(\sqrt{\frac{1-\cos\frac{\pi}{2^k}}{2}}) = f_k(f(\sqrt{\frac{1-\cos(\frac{\pi}{2^k})}{2}})) = f_k(-\cos(\frac{\pi}{2^k}))$$

since $f\left(\sqrt{\frac{1-\cos(\frac{\pi}{2^k})}{2}}\right)=2\left(\sqrt{\frac{1-\cos(\frac{\pi}{2^k})}{2}}\right)^2-1=2\left(\frac{1-\cos(\frac{\pi}{2^k})}{2}\right)-1$

$=1-\cos(\frac{\pi}{2^k})-1=-\cos(\frac{\pi}{2^k})$. So $f_{k+1}(\alpha_{k+1})=f_k(-\cos(\frac{\pi}{2^k}))=f_k(\cos(\frac{\pi}{2^k}))$

since the f_k's involve only even powers. But $f_k(\cos(\frac{\pi}{2^k}))$ is the same as $f_k(\alpha_k)$, which is 0.

Thus, by the principle of mathematical induction, $f_{k+1}(\alpha_{k+1})=0$. ■

So we have established that $f_n(x)$ is the minimum polynomial of $\cos(\frac{\pi}{2^n})$ and $\sin(\frac{\pi}{2^n})$. The values of these sines and cosines will now be determined. First note that for $n=2$ the sines and cosines of odd multiples of $\frac{\pi}{2^n}$ are $\pm\frac{\sqrt{2}}{2}$. Using the identity $\cos(\theta)=2\cos^2(\frac{\theta}{2})-1$, it follows that the cosines of odd multiples of $\frac{\pi}{2^3}$ are roots of the equation

$r^2=1\pm\frac{\sqrt{2}}{2}=\frac{2\pm\sqrt{2}}{2}$. Hence $r=\pm\frac{\sqrt{2\pm\sqrt{2}}}{2}$.

As expected, the fourth degree polynomial $f_3(x)$ has four distinct roots. For convenience, I focus on the positive roots.

$$\cos(\frac{\pi}{8})=\frac{\sqrt{2+\sqrt{2}}}{2}=\sin(\frac{3\pi}{8})$$

$$\cos(\frac{3\pi}{8})=\frac{\sqrt{2-\sqrt{2}}}{2}=\sin(\frac{\pi}{8})$$

It is easy to match the angles with the sines and cosines since the cosine function is decreasing and the sine function is increasing on the range $[0,\frac{\pi}{2}]$. Repeated use of the identity $\cos(\theta)=2\cos^2(\frac{\theta}{2})-1$ yields the values $\pm\frac{\sqrt{2\pm\sqrt{2\pm\sqrt{2\pm\ldots}}}}{2}$ for the cosines (and hence the sines as well) of odd multiples of $\frac{\pi}{2^n}$. There are n-1 occurrences of the " $\pm$ " sign, so there are 2^{n-1} distinct roots of the polynomial $f_n(x)$, as expected. Each of these occurs twice as the sine and twice as the cosine of an odd multiple of $\frac{\pi}{2^n}$.

An Algorithm

Now that we know so much about odd multiples of $\frac{\pi}{2^n}$, what if we wanted to know the exact value of $\cos(\frac{k\pi}{2^n})$ or $\sin(\frac{k\pi}{2^n})$? For simplicity I will consider the cosine function first and then move to the sine function. For example, consider $\cos(\frac{\pi}{8})$. A tedious computation will show that $\cos(\frac{\pi}{8})=\frac{\sqrt{2+\sqrt{2}}}{2}$. However, a simple algorithm will make this an easy transformation.

Algorithm: For $\cos(\frac{k\pi}{2^n})=\mu$ where k is an odd integer and $0\leq k\leq 2^{n-1}$, μ will be of the form $\frac{\sqrt{2\pm\sqrt{2\pm\sqrt{\ldots\pm\sqrt{2}}}}}{2}$ where there will be $(n-1)$ radicals and the $\pm$ signs will be determined by the binary representation of k. Since the numerator must be odd its binary equivalent will end in a one. Remove this last 1. Starting from the right in the radical sequence, that is, starting with the sign adjacent to the last $\sqrt{2}$, and starting from the right of the binary equivalent, each 0 will denote a plus sign and each 1 will denote a minus sign.

Example: Consider $\cos(\frac{3\pi}{8})$. Since $8=2^3$, we know the solution will have 3-1=2 radicals. In binary, 3 = 11, so erasing the last 1 will leave 1, denoting a minus sign. So, $\cos(\frac{3\pi}{8})=\frac{\sqrt{2-\sqrt{2}}}{2}$.

In the same manner we can transform a solution back to its angle by working the algorithm in reverse. ■

Example: Consider the value $\frac{\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2}}}}}{2}$. Then the denominator of the angle will be $2^{4+1}=2^5$ while the numerator will be - + - =101. Attaching a 1 to the right-hand side we get 1011, or 11.

So $\cos(\frac{11\pi}{32})=\frac{\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2}}}}}{2}$.

Using a slightly different process we can move between angles and values for $\sin(\frac{k\pi}{2^n})$.

Algorithm: For $\sin(\frac{k\pi}{2^n})=\mu$ where k is an odd integer, μ will be of the form $\frac{\sqrt{2\pm\sqrt{2\pm\sqrt{\ldots\pm\sqrt{2}}}}}{2}$ where there will be n-1 radicals and the numerator will be defined by converting k into binary and then changing each 0 to a 1 and vice versa. Adding a 1 from the end will give us the number we want. The plus and minus signs of the numerator are then defined by reading the number, left to right, where each 0 stands for a plus sign and each 1 stands for a minus sign. ■

Example: Again consider the value $\frac{\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2}}}}}{2}$. Applying the algorithm for sine we find the denominator to be $2^{4+1}=2^5$. k then will be the equivalent of 0101 = 5. (after changing 1's to 0's and vice versa and then attaching a 1 to the right-hand side) So

$$\sin(5\frac{\pi}{32})=\frac{\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2}}}}}{2}.$$

It is a simple process to go in the other direction, that is, obtain a value from an angle.

Example: Consider $\sin(\frac{7\pi}{32})$. Our value will have 5-1=4 radicals while the + and – signs will be defined by 7 = 0111. Removing a 1 and swapping 0's and 1's we get 100, or $\frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2}$.

Thus, for $\sin(\frac{k\pi}{2^n})$ and $\cos(\frac{k\pi}{2^n})$ we can easily move between the angles and the numeric values.

A Second Infinite set?

So far we have discovered an easily identifiable and predictable set of conjugate angles, namely those of the form $\frac{k\pi}{2^n}$ where k is an odd integer and $n \in \mathbb{Z}$. Surely these cannot be the only instances of conjugate angles. Above, $\frac{\pi}{4}$ posed a logical starting point since it is itself a conjugate angle. However, no other commonly known angles are readily recognizable as conjugates, and certainly $\frac{\pi}{4}$ is the only first quadrant angle whose sine equals its cosine. So the question turns from, "Do conjugate angles exist?" to "Do any angles $\theta \neq \{\frac{k\pi}{2^n} \mid k \text{ is an odd integer}, n \in \mathbb{Z}\}$ exist where θ is a conjugate?"

Surprisingly, the question can be answered by first examining a familiar non-conjugate angle, $\frac{\pi}{6}$. This common angle has

$\cos=\frac{\sqrt{3}}{2}$ and $\sin=\frac{1}{2}$, clearly marking this as an algebraic, non-conjugate angle. However, a little closer inspection is required. As before, the double angle formulas for $\sin\theta$ and $\cos\theta$ will be used. As

seen above, $\cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}$. So,

$$\cos^2\frac{\pi}{12}=\frac{1+\cos\frac{\pi}{6}}{2}=\frac{1+\frac{\sqrt{3}}{2}}{2}=\frac{2+\sqrt{3}}{4},\ \text{hence}\ \cos\frac{\pi}{12}=\frac{\sqrt{2+\sqrt{3}}}{2}.$$

Renaming $\cos(\theta)=x$, we find that, with a little work

$x=\frac{\sqrt{2+\sqrt{3}}}{2}\rightarrow 16x^4-16x^2+1=0$. So this is a polynomial for the cosine.

Similarly, for $\sin(\frac{\pi}{12})$ we have:

$\sin^2\frac{\pi}{12}=\frac{1-\cos\frac{\pi}{6}}{2}=\frac{1-\frac{\sqrt{3}}{2}}{2}=\frac{2-\sqrt{3}}{4}$, hence $\sin\frac{\pi}{12}=\frac{\sqrt{2-\sqrt{3}}}{2}$. Renaming

$\sin\frac{\pi}{12}=x$, we can see that $x=\frac{\sqrt{2\pm\sqrt{3}}}{2}\rightarrow 4x^2=2\pm\sqrt{3}\rightarrow$

$(4x^2-2)^2=3\rightarrow 16x^4-16x^2+4=3\rightarrow 16x^4-16x^2+1=0$. Thus

$\cos\frac{\pi}{12}$ and $\sin\frac{\pi}{12}$ are roots of the same polynomial. Setting

$x=(\frac{z+1}{2})$ and using Eisenstein, we can see that this is an irreducible polynomial. Thus $16x^4-16x^2+1=0$ is the minimum polynomial for

$\sin(\frac{\pi}{12})$ and $\cos(\frac{\pi}{12})$ so we can see that $\frac{\pi}{12}$ is a conjugate angle. A little work with the double angle formulas will show that,

surprisingly, $\cos\frac{\pi}{24}=\frac{\sqrt{2+\sqrt{2+\sqrt{3}}}}{2}$ and $\sin\frac{\pi}{24}=\frac{\sqrt{2-\sqrt{2-\sqrt{3}}}}{2}$, clearly

making $\frac{\pi}{24}$ a conjugate angle.

The Search for Others...

So it seems that we have discovered another set of conjugate angles that are easily identifiable and predictable. However, at this point we should note one important similarity that will lead to a startling revelation. Recall that the minimum polynomials for

$\frac{\pi}{8}$ and $\frac{\pi}{12}$ are $8x^4+8x^2+1$ and $16x^4+16x^2+1$, respectively. By rewriting $8x^4+8x^2+1$ as $16x^4+16x^2+2$ a striking similarity can be seen. It seems that fourth-degree polynomials may be a good place to start looking for other conjugate angles. But is there a way to discern exactly which angles are conjugates and which are non-conjugates?

Before we can begin an in-depth examination of these fourth-degree polynomials, we should remember that $\frac{\pi}{4}$ is a conjugate angle with a degree two polynomial, namely $2x^2-1=0$. It makes sense, then, to first examine second-degree polynomials to see if there are others that may be the minimum polynomials for conjugate angles.

So far the conjugate angles we have located have all arisen from $\frac{\pi}{4}$, repeated halving, and odd multiples. Are there other conjugate angles with quadratic minimum polynomials? In this search, it makes sense then to consider $x = \cos(\theta) = a+\sqrt{b}$ and $x = \sin(\theta) = a-\sqrt{b}$ where $a,b \in \mathbb{Q}$. Note that $\sin^2(\theta)+\cos^2(\theta) = 1 = (a+\sqrt{b})^2+(a-\sqrt{b})^2$, so $a^2+b = \frac{1}{2}$, or $b = \frac{1}{2}-a^2$. So we can rewrite

$$\cos(\theta) = a+\sqrt{\frac{1}{2}-a^2} \text{ and } \sin(\theta) = a-\sqrt{\frac{1}{2}-a^2}.$$

With all these specifications we are still looking for second-degree minimum polynomials for conjugate angles. With a little work we arrive at $x^2-2ax+(2a^2-\frac{1}{2})=0$. Since $\frac{1}{2}-a^2\geq 0$, we have $0\leq a<\frac{\sqrt{2}}{2}$

(since a has to be rational) as θ ranges from 0 to $\frac{\pi}{2}$. Interestingly,

when $a=0$, we have $x^2-\frac{1}{2}=0$, or $2x^2-1=0$, which is the minimum

polynomial for $\frac{\pi}{4}$.

The quadratic formula is a well known method for solving second-degree polynomials. We can apply it to our polynomial in order to find values for a.

For second-degree (or quadratic) polynomials $ax^2+bx+c=0$ we can find roots by the quadratic formula:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Applying this formula to $x^2-2ax+(2a^2-\frac{1}{2})=0$ we get

$$x=\frac{2a\pm\sqrt{4a^2-(4)(2a^2-\frac{1}{2})}}{2}=\frac{2a\pm\sqrt{2-4a^2}}{2}$$

Since $x=\cos(\theta)$ or $x=\sin(\theta)$ it seems reasonable to assign

$\frac{2a+\sqrt{2-4a^2}}{2}=\cos(\theta)$ and $\frac{2a-\sqrt{2-4a^2}}{2}=\sin(\theta)$. However, this should have been a more intuitive solution. By separating the two, we get $\frac{2a}{2}\pm\frac{\sqrt{2-4a^2}}{2}=a\pm\frac{2\sqrt{\frac{1}{2}-a^2}}{2}=a\pm\sqrt{\frac{1}{2}-a^2}$, which just happens to be $a\pm\sqrt{b}$, and we have returned to our starting point.

This process was not in vain, however. We see that if $x^2-2ax+(2a^2-\frac{1}{2})$ is to be the minimum polynomial for a conjugate angle, then a must be chosen so that $\sqrt{2-4a^2}$ is irrational. This ensures that the roots of $x^2-2ax+(2a^2-\frac{1}{2})$ are really quadratic. Hence $2-4a^2$ cannot be a "perfect square," for if it is, then the roots $\frac{2a\pm\sqrt{2-4a^2}}{2}$ would be rational. So $2-4a^2\neq r^2$ where $r\in\mathbb{Q}$. This means $a^2\neq\frac{1}{2}-\frac{r^2}{4}$. Using bounds on a we see that $0<r\leq\sqrt{2}$. Thus we know exactly what values of a lead to conjugate angles. Just compute $2-4a^2$ and check to see whether or not it is a perfect square. To illustrate:

If $a=.7$ we have $2-4a^2=2-4(.49)=(.04)=(.2)^2$ Hence $a=.7$ will yield a polynomial with roots that are rational, in particular $x^2-1.4x+.48$ with roots .8 and .6 . These <u>are</u> the sine and cosine of an algebraic angle since $(.6)^2+(.8)^2=1$, but the minimum polynomials of .8 is $5x-4$ while the minimum polynomial of .6 is $5x-3$. So this angle is not a conjugate.

Now try $a=.6$. $2-4a^2=.56=\frac{14}{25}$, which is not a "perfect square," so $a=.6$ should yield a conjugate angle. Indeed, the minimum polynomial is $x^2-1.2x+.22$, or, multiplying by 50 to get integer coefficients, $50x^2-60x+11$. Now $\cos(\theta)=.6+\frac{\sqrt{14}}{25}$ and $\sin(\theta)=.6-\frac{\sqrt{14}}{25}$, and θ is a conjugate angle with a quadratic minimum polynomial.

Can all Conjugates be Halved?

With the discovery of a countably infinite conjugate family of angles with degree-two polynomials its logical to wonder if these new angles can be halved repeatedly and still produce conjugate angles.

Above we saw that $\cos(\theta)=a+\sqrt{\frac{1}{2}-a^2}$. So, again using the double angle formulas, we can find a formula for $\cos(\frac{\theta}{2})$:

$$\cos^2\frac{\theta}{2}=\frac{1+a+\sqrt{\frac{1}{2}-a^2}}{2}=\frac{2+2a+2\sqrt{\frac{1}{2}-a^2}}{4},$$

so $\cos(\frac{\theta}{2})=\frac{\sqrt{2+2a+\sqrt{2-4a^2}}}{2}$

Similarly, $\sin(\frac{\theta}{2})=\frac{\sqrt{2-\sqrt{2a-\sqrt{2-4a^2}}}}{2}$. Setting

$x=\frac{\sqrt{2+2a+\sqrt{2-4a^2}}}{2}=\cos(\frac{\theta}{2})$ and doing the arithmetic we obtain

$4x^2=2+2a+\sqrt{2-4a^2}\rightarrow(4x^2-2-2a)^2=2-4a^2$, so

$16x^4-8(2-2a)x^2+(2-2a)^2=8-4a^2$, and

$16x^4-16(1+a)x^2+(8a^2+8a+2)=0.$

This polynomial, multiplied by the appropriate constant, is the minimum polynomial of $\cos(\frac{\theta}{2})$. For $\sin(\frac{\theta}{2})$, we proceed as above.

$x=\frac{\sqrt{2-2a-\sqrt{2-4a^2}}}{2}=\sin(\frac{\theta}{2})$

$4x^2=2-2a-\sqrt{2-4a^2}\rightarrow(4x^2-2+2a)^2=2-4a^2$, so

$16x^4-8(2-2a)x^2+(2-2a)^2=8-4a^2$, and

$16x^4-16(1+a)x^2+(8a^2+8a+2)=0.$

This polynomial, multiplied by an appropriate constant, is the minimum polynomial of $\sin(\frac{\theta}{2})$. But, if $\frac{\theta}{2}$ is to be a conjugate angle, the sine and cosine must have the same minimum polynomial. It is easy to see that this can happen only in the case where a=0, in which case we have the polynomial $16x^4-16x^2+2=0$, or $8x^4-8x^2+1=0$, and $\frac{\theta}{2}=\frac{\pi}{8}$, so $\theta=\frac{\pi}{4}$. The effort spent generating conjugate angles with repeated halving of $\frac{\pi}{4}$ was well spent, as this is the only instance of a <u>quadratic</u> angle for which this can be done.

We now turn our attention back to fourth-degree polynomials.

From our previous examples, $\frac{\pi}{8}$ with minimum polynomial $8x^4-8x^2+1=0$ and $\frac{\pi}{12}$ with minimum polynomial $16x^4-16x^2+1=0$, it seems reasonable to consider the form x^4+px^2+q where p and q are rational numbers. Solutions x then represent values for $\sin\theta$ and $\cos\theta$. So an equation involving sine would be $\sin^4\theta+p\sin^2\theta+q=0$ and an equation involving cosine would be $\cos^4\theta + p\cos^2\theta + q = 0$ By subtracting the two equations, we get:

$$\cos^4\theta + p\cos^2\theta + q - (\sin^4\theta + p\sin^2\theta + q) = 0$$

$$\cos^4\theta - \sin^4\theta + p\cos^2\theta - p\sin^2\theta = 0$$

$$(\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta) + p(\cos^2\theta - \sin^2\theta) = 0$$

Here we should note that $\sin^2\theta + \cos^2\theta = 1$ is a well known trigonometric identity, of which we will make use here.

$$(\cos^2\theta - \sin^2\theta)(1) + p(\cos^2\theta - \sin^2\theta) = 0$$

$$(\cos^2\theta - \sin^2\theta)(1 + p) = 0$$

Here we use another trigonometric identity, $\cos^2\theta - \sin^2\theta = \cos 2\theta$, to proceed:

$$(\cos 2\theta)(1 + p) = 0,$$

so either $\cos(2\theta) = 0$, in which case $\theta = \frac{\pi}{4}$, or $p = -1$. Our fourth degree polynomial has become $x^4 - x^2 + q = 0$.

Using this new value for p, we solve for q by adding the equations together:

$$(\cos^4\theta - \cos^2\theta + q) + (\sin^4\theta - \sin^2\theta + q) = 0$$

$$\cos^4\theta + \sin^4\theta - (\cos^2\theta + \sin^2\theta) = -2q$$

Here it is important to note that $\cos^2\theta = \frac{1 + \cos 2\theta}{2}$ and

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2}.$$

$$(\cos^2\theta)^2+(\sin^2\theta)^2-(1)=-2q$$

$$(\frac{1+\cos 2\theta}{2})^2+(\frac{1-\cos 2\theta}{2})^2=1-2q$$

$$\frac{2+2\cos^2 2\theta}{4}+2q=1$$

$$2q=\frac{2-2\cos^2 2\theta}{4}, \text{ and } q=\frac{1-\cos^2 2\theta}{4}$$

Using another simple trigonometric substitution, we arrive at

$q=\frac{\sin^2 2\theta}{4}$. This result gives $0\le q\le\frac{1}{4}$ since $\sin^2 2\theta$ ranges from 0 to 1.

However, we can restrict this bound a little further. If q=0, then $x^4-x^2+q=x^4-x^2+0=x^4-x^2$. This clearly is reducible to $x^2(x^2-1)$ and so cannot be a minimum polynomial. Similarly, if $q=\frac{1}{4}$, then $x^4-x^2+\frac{1}{4}$ is reducible to $(x^2-\frac{1}{2})(x^2-\frac{1}{2})$ and so cannot be a minimum polynomial. So we have restricted our bound to $0<q<\frac{1}{4}$.

Using $q=\frac{\sin^2(\theta)}{4}$, we find that $\theta=\frac{1}{2}\arcsin(2\sqrt{q})$.

Could there be other values that create reducible polynomials and thus must be excluded? Lets consider some familiar values in our search for an answer. First we should consider values $\frac{y}{8}$ where y is any integer. What values of y should we examine? $q>0$, So we must consider $y>0$. $y=2$ gives $\frac{2}{8}=\frac{1}{4}$. But $q<\frac{1}{4}$, so we have $0<y<2$ for this case. Since y must be an integer, $\frac{1}{8}$ is the only value we should be concerned with. $q=\frac{1}{8}$ will lead us to a familiar polynomial:

$$x^4-x^2+\frac{1}{8}=8x^4-8x^2+1,$$

or the minimum polynomial for $\frac{\pi}{8}$. The next logical range of values to consider is $\frac{y}{16}$. In this instance we should consider $y=1, 3$ since $y=0$ gives 0, $y=4$ gives $\frac{1}{4}$, and $y=2$ gives $\frac{1}{8}$, which we have previously studied. So $y=1$ gives $\frac{1}{16}$, which clearly leads to the equation $16x^4-16x^2+1$, the minimum polynomial for $\frac{\pi}{12}$. For $y=3$ we have $q=\frac{3}{16}$. This value leads to the equation $x^4-x^2+\frac{3}{16}=(x^2-\frac{3}{4})(x^2-\frac{1}{4})$, which is reducible, so $q\neq\frac{3}{16}$.

Moving to $\frac{y}{64}$ we should examine $y=\{1,3,5,7\}$. $y=1,3,5$ will give us pleasant fourth-degree polynomials, but $y=7$ produces a reducible polynomial, namely $(x^2-\frac{7}{8})(x^2-\frac{1}{8})=0$. I conjecture that, for $q=\frac{r-1}{r^2}$, that is, one less than a number divided by the number squared will produce a reducible polynomial.

So we have again found a collection of values to exclude. It also seems that we may well be on our way to completely describing and classifying conjugate angles with fourth-degree minimum polynomials.

Conclusion

In this essay we found at least two infinite families of angles whose sine and cosine are roots are the same polynomial. We have also had an in-depth examination of second-degree polynomials.

However, many questions (with few answers) have been raised. The following is a sample of these questions:

Connections to the Complex Plane

So far we have restricted our search to values that lie of the real number plane. If instead we expanded our search to the complex plane would similar patterns of conjugacy emerge, and what, if anything, could they tell us about conjugates in the real number plane?

Connections to Chaos Theory

The polynomial $2x^2-1$ is an extremely important function in the context of this thesis. In particular, we saw that we could define the minimum polynomials of $\frac{k\pi}{2^n}$ by plugging $2x^2-1$ into itself a specific number of times. This is known as iterating a function. A branch of mathematics that takes an interest in functions and their iterations is chaos theory. A particular concern is with cycles.

Definition:

Cycles of 1 are known as fixed points. These are easy to find, namely by solving $f(x)=x$. In our case, $2x^2-1=x$. A little work will show that 1 and $-\frac{1}{2}$ are fixed points of $2x^2-1=0$.

Cycles of 2, while slightly more difficult to find, tell us a little about the chaotic properties of a function. We find these through solving the following system of equations: for $a, b \in \mathbb{R}$, $2a^2-1=b$ and $2b^2-1=0$. Again, a little work will show that $\frac{-1\pm\sqrt{5}}{4}$ is the only 2-cycle of $2x^2-1=0$.

While more difficult yet, the existence of one 3-cycle will tell us much about our function with respect to its chaotic properties. Quite a bit of work (to solve a system of equations in 3 variables) will reveal $\{\frac{6\pi}{7}\rightarrow\frac{2\pi}{7}\rightarrow\frac{4\pi}{7}\}$ to be one of two 3-cycles of $2x^2-1=0$. This result tells us that $2x^2-1=0$ contains at least one k-cycle for every k that is an integer.

What do these cycles tell us, and how do these chaotic properties relate to conjugacy?

Connections to other Trigonometric Functions

From basic trigonometry we know that $\frac{\sin(\theta)}{\cos(\theta)}=\tan(\theta)$, but what can we say about the tangent of an angle whose sine and cosine are both roots of the same minimum polynomial? Lets consider our favorite conjugate angle, $\frac{\pi}{4}$. We immediately notice that

$\tan(\frac{\pi}{4})=\frac{\sin(\frac{\pi}{4})}{\cos(\frac{\pi}{4})}=1$. So the minimum polynomial for $\tan(\frac{\pi}{4})$ is

$x-1=0$. As before, this tells us very little, but perhaps moving to

half of $\frac{\pi}{4}$, namely $\frac{\pi}{8}$, and comparing the two will shed some light

on an answer. $\tan(\frac{\pi}{8})=\frac{\sqrt{2-\sqrt{2}}}{\sqrt{2+\sqrt{2}}}=\sqrt{2}-1$. So $x^2+2x-1=0$ is the

minimum polynomial for $\tan(\frac{\pi}{8})$. Halving the angle once more we

arrive at $\tan(\frac{\pi}{16})$ is $\frac{\sqrt{2-\sqrt{2-\sqrt{2}}}}{\sqrt{2+\sqrt{2+\sqrt{2}}}}$. Quite a bit of work will give us a

minimum polynomial for $\tan(\frac{\pi}{16})=x^4+4x^3-6x^2-4x+1$. We now

notice, surprisingly, that the absolute values of the coefficients of these minimum polynomials are exactly following Pascal's Triangle! A bit of algebra will give us the solution $\sqrt{4-2\sqrt{2}}-\sqrt{2}-1$ for $\tan(\frac{\pi}{16})$.

Is there a way to predict the polynomial for $\tan(\frac{\pi}{2^n})$? And can we predict the numeric values of $\tan(\frac{k\pi}{2^n})$ like we could with the sine and cosine?

For every tangent there is a cotangent. Do these behave anything like the tangents do? Lets again consider our favorite conjugate angle, $\frac{\pi}{4}$. Like $\tan(\frac{\pi}{4})$, $\cot(\frac{\pi}{4})=1$ and so has the same minimum polynomial, namely $x-1=0$. Moving to $\cot(\frac{\pi}{8})=\frac{\sqrt{2+\sqrt{2}}}{\sqrt{2-\sqrt{2}}}$ we find that $\cot(\frac{\pi}{8})=\sqrt{2}+1$ leading us to a minimum polynomial of $x^2-2x-1=0$. For $\frac{\pi}{16}$, $\cos(\frac{\pi}{16})=\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{\sqrt{2-\sqrt{2-\sqrt{2}}}}$, which gives us a minimum polynomial of $x^4-4x^3-5x^2+4x+1=0$. Some startling similarities between the tangent and cotangent can now be observed.

1.) Both follow Pascal's Triangle.

2.)For minimum polynomials of $\frac{\pi}{2^n}$ we can see that the + and – signs travel in opposite directions but always in groups of two.

3.) While θ is a conjugate because its sine and cosine are roots of the same minimum polynomial, its tangent and cotangent have different minimum polynomials. So a conjugate with respect to the sine and cosine is not necessarily a conjugate with respect to the tangent and cotangent.

With respect to the tangent and cotangent, I can show conjugate angles exist, but it would be interesting to characterize them. And what about the secant and cosecant? It is interesting to note that if θ is a conjugate angle, then $\sec(\theta)$ and $\csc(\theta)$ are algebraic conjugates. To see this suppose that the minimum polynomial of $\sin(\theta)$ and $\cos(\theta)$ is

$$a_n x^n + a_{n-1} x^{n+1} + \ldots a_1 x^1 + a_0$$

Then $a_n \sin^n(\theta) + a_{n-1} \sin^{n-1}(\theta) + \ldots + a_1 \sin(\theta) + a_0 = 0$.

Dividing by $\sin^n(\theta)$, or, equivalently, multiplying by $\csc^n(\theta)$ yields

$$a_0 \csc^n(\theta) + + a_1 \csc^{n-1}(\theta) + .. + a_{n-1} \csc^1(\theta) + a_n = 0$$

and the polynomial

$$a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n = 0$$

has $\csc(\theta)$ as a root. Similarly, it has $\sec(\theta)$ as a root.

Do angles exist for which the sine and cosine have the same minimum polynomial, the tangent and cotangent have the same minimum polynomial, and the secant and cosecant have the same minimum polynomial? Do very special angles exist for which all six trigonometric functions are roots of the same polynomial?

I consider this work to be the beginning of a larger enterprise. I have settled completely the classifying of conjugate angles with quadratic minimum polynomials and have learned that exactly one of these sits at the top of a huge family. I have begun the investigation of conjugate angles with 4^{th} degree minimum polynomials and I have noted a possible connection between conjugate angles and chaos theory. It is my hope that my graduate studies will leave time for pursuing this project.

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