UNIVERSITI PUTRA MALAYSIA

ENERGY EIGENEQUATION EXPANSION FOR A PARTICLE ON SINGLY PUNCTURED TWO-TORUS AND TRIPLY PUNCTURED TWO-SPHERE SYSTEMS

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By

NURISYA MOHD SHAH

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Master of Science

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TABLE OF CONTENTS

		Page
DEDICATION		**ii**
ABSTRACT		**iii**
ABSTRAK		**vi**
ACKNOWLEDGEMENTS		**ix**
APPROVAL		**x**
DECLARATION		**xii**
LIST OF FIGURES		**xv**
LIST OF SYMBOLS AND ABBREVIATIONS		**xvii**

CHAPTER

1	**INTRODUCTION**	1
	1.1 Introduction	1
	1.2 Objectives of Research	3
	1.3 Outline of The Thesis	4
2	**LITERATURE REVIEW**	6
	2.1 Introduction	6
	2.2 Previous Work on Punctured Surfaces	6
	2.3 Automorphic Forms, Maass Waveforms and Maass Cusp Forms	10
	2.4 Relation of Scattering States and Eisensteins Series	12
	2.5 Conclusion	13
3	**THEORY**	14
	3.1 Preliminary Concepts on Hyperbolic Geometry	14
	3.2 Upper Half-Plane Model	14
	3.3 Concept of Isometries	16
	3.4 Definition of the Modular Group Γ	16
	3.4.1 Group-Theoretic Properties of Modular Group	18
	3.4.2 The Canonical Generators Relation	20
	3.5 Definition of the Principal Congruence Subgroup $\Gamma(N)$	22
	3.6 Fundamental Domain	23
	3.7 Concept of Tessellation	27
	3.8 The Quotient Space H/Γ	29
	3.9 Automorphic Forms	30

3.10 Modular Forms 32
3.11 The Fourier Series 33
3.11.1 Finite Fourier Transform 35
3.12 Computing Eisenstein Series 35

4 **METHODOLOGY** 38
4.1 Introduction 38
4.2 Identifying the Configuration Space 39
4.2.1 The Singly Punctured Two-Torus 39
4.2.2 The Triply Punctured Two-Sphere 43
4.3 Solving the Laplacian 45
4.4 The *K*-Bessel Function 47
4.5 The Automorphic Relation 49

5 **RESULTS AND DISCUSSION** 50
5.1 Introduction 50
5.2 Construction of Singly Punctured Two-Torus 50
5.3 Construction of Triply Punctured Two-Sphere 51
5.4 Fundamental Region of Singly Punctured Two-Torus 52
5.5 Fundamental Region of Triply Punctured Two-Sphere 56
5.6 The Algorithm 54
5.6.1 Maass Cusp Forms of Singly Punctured Two-Torus 62
5.6.2 Maass Cusp Forms of Triply Punctured Two-Sphere 67
5.7 Eisenstein Series for Singly Punctured Two-Torus 78

CONCLUSION 83
6.1 Conclusion 83
6.2 Suggestions 84

REFERENCES/BIBLIOGRAPHY 86
APPENDICES 89
BIODATA OF STUDENT 113

DEDICATION

To *Faizal* and *Ain*

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Master of Science

ENERGY EIGENEQUATION EXPANSION FOR A PARTICLE ON SINGLY PUNCTURED TWO-TORUS AND TRIPLY PUNCTURED TWO-SPHERE SYSTEMS

By

NURISYA MOHD SHAH

May 2008

Chairman : Associate Professor Hishamuddin Zainuddin, PhD

Institute : Institute of Advanced Technology

Ideas from topology have played a major role in physics especially to describe and explain exotic quantum phenomena. There has been a considerable interest among physicists who are working on string theory and quantum gravity to use ideas and results from topology to explain their work. However, often one is limited to the choice of spaces with relatively simpler topologies e.g. sphere, torus etc because more complex spaces are difficult to be characterized or even distinguished. It is our particular interest to consider singularities (i.e. having one or several punctures on it) as a tool to generate a family of complex two-dimensional configuration surfaces. These surfaces may find applications in to mathematical models of quantum chaos, cosmology, particle physics, condensed matter, quantum gravity and string theory.

Extensive mathematical studies have been carried out for punctured surfaces but their literature in physics are scarce. Most have tackled the case of quantum mechanical

systems on punctured torus with respect to its scattering and chaotic behavior. Of particular interest in the present work are the quantum mechanical systems of singly punctured two-torus and triply punctured two-sphere. They both have two generators and three possible non-contractible loops. Both surfaces can be generated from the same parent generators of the modular group Γ, which is a discrete subgroup of linear fractional transformations of the upper half complex plane H (the universal cover of the punctured surfaces). In this dissertation, we construct both surfaces of singly punctured two-torus and triply punctured two-sphere stepwise using these generators.

The main aim however is to construct the energy eigenequation for particle on surfaces of singly punctured two-torus and triply punctured two-sphere. For that purpose, we first identify the configuration space explicitly by considering the tessellation of the upper half-plane and the required surfaces are determined. Next, by using the Fourier expansions, finite Fourier transform of the energy eigenequation is performed to give rise to a sought standard relation for generating the eigenfunction.

It is known that the eigenfunction on a punctured system exhibit both discrete and continuous energy spectra. The discrete energy spectrum will correspond to the computation of a countable number of Maass cusp forms while for the continuous spectrum, it is spanned by the Eisenstein series. In this work, we present the expressions for the Maass cusp forms of the singly punctured two-torus and triply punctured two-sphere and the expression of the Eisenstein series for the singly punctured two-torus.

At the end of this thesis a unified treatment of the Maass cusp forms and the Eisensteins series for the singly punctured two-torus and the triply punctured two-sphere are presented. The importance of each technique used on the formation of the energy eigenequation are explained in a more physical approach.

Abstrak tesis yand dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Master Sains

PENGEMBANGAN PERSAMAAN EIGEN TENAGA UNTUK SATU ZARAH ATAS SISTEM TORUS SATU JURING DAN SFERA TIGA JURING

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Idea daripada topologi memainkan peranan yang besar dalam bidang fizik terutamanya untuk memerihal dan menerangkan fenomena aneh kuantum. Terdapat sejumlah besar kecenderungan di kalangan ahli fizik yang berminat dalam kajian ke atas teori tetali dan graviti kuantum yang menggunakan idea-idea dan keputusan-keputusan daripada topologi untuk menerangkan kajian mereka. Walaubagaimanapun, kebiasaan pilihan ruang-ruang adalah terhad kepada contoh topologi mudah seperti sfera, torus dan sebagainya kerana ruang-ruang yang lebih kompleks adalah sukar untuk dikategorikan mahupun untuk dipastikan. Penjurusan minat kami adalah mengambilkira keadaan singulariti (iaitu mempunyai satu atau beberapa juring di atasnya) sebagai satu cara untuk menghasilkan sekeluarga permukaan konfigurasi dua dimensi yang kompleks. Permukaan-permukaan ini mampu mempunyai aplikasi-aplikasi dalam model matematik kelang-kabutan kuantum, kosmologi, fizik zarah, bahan terkondensasi, graviti kuantum dan teori tetali.

Banyak penyelidikan matematik yang dihasilkan untuk permukaan-permukaan juring tetapi bilangan literatur dalam fizik amatlah kurang. Kebanyakannya telah mengkaji kes sistem mekanikal kuantum ke atas torus berjuring yang merujuk kepada sifat serakan dan kelang-kabutan. Menjadi tumpuan penyelidikan di sini adalah sistem mekanik kuantum atas torus satu juring dan sfera tiga juring. Kedua-duanya mempunyai dua penjana kumpulan dan tiga kemungkinan gelungan yang tidak mengecut. Kedua permukaan ini boleh dijana daripada penjana kumpulan modular Γ yang sama, yang juga merupakan subkumpulan diskrit kepada transformasi linear pecahan bagi separuh satah kompleks atas H (litupan universal permukaan berjuring). Dalam penulisan ini, kami membina permukaan torus satu juring dan sfera tiga juring dengan menggunakan penjana-penjana ini, langkah demi langkah.

Tujuan utama walaubagaimanapun adalah untuk menghasilkan persamaan eigen tenaga bagi zarah atas permukaan torus satu juring dan sfera tiga juring. Bagi tujuan tersebut, pertamanya kami mengenalpasti konfigurasi ruang secara eksplisit dengan mengambilkira penjubinan separuh satah atas dan permukaan yang ditentukan. Kemudian, dengan menggunakan pengembangan Fourier, jelmaan Fourier terhingga bagi persamaan eigen tenaga diguna untuk mendapatkan hubungan piawai bagi menghasilkan fungsi eigen.

Telah diketahui bahawa fungsi eigen atas sistem berjuring menghasilkan kedua-dua bentuk spektra tenaga yang diskrit dan selanjar. Spektrum tenaga diskrit akan berpadanan bilangan bentuk juring Maass manakala bagi spektrum selanjar, ia dijana oleh siri Eisenstein. Dalam hasil kerja ini, kami beri ungkapan untuk bentuk juring Maass bagi torus satu juring dan sfera tiga juring dan juga ungkapan siri Eisenstein untuk torus satu juring.

Di akhir tesis ini, satu kaedah tergabung untuk bentuk juring Maass dan siri Eisenstein bagi torus satu juring dan sfera tiga juring diperihalkan. Kepentingan setiap teknik yang digunakan dalam pembentukan persamaan eigen tenaga berkaitan diterangkan dengan pendekatan fizik.

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I certify that an Examination Committee met on 6th May 2008 to conduct the final examination of Nurisya Mohd Shah on her Master of Science thesis entitled “Energy Eigenequation Expansion for a Particle on Singly Punctured Two-Torus and Triply Punctured Two-Sphere Systems” in accordance with Universiti Putra Malaysia (Higher Degree) Act 1980 and Universiti Putra Malaysia (Higher Degree) Regulations 1981. The committee recommends that the candidate be awarded the degree of Master of Science.

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Date: 10 July 2008

DECLARATION

I declare that the thesis is my original work except for quotations and citations which have been duly acknowledged. I also declare that it has not been previously, and is not concurrently, submitted for any other degree at Universiti Putra Malaysia or at any other institution.

NURISYA MOHD SHAH

Date: 14 May 2008

LIST OF FIGURES

Figure		Page
3.1	The upper half-plane H. Free particle on H moves along the geodesic of straight line and semicircle perpendicular to x-axis	15
3.2	Transformation of T and S (both are elements of modular group Γ)	19
3.3	The gray region is the canonical fundamental domain satisfying equation (3.8), (3.9), (3.10) and (3.11) i.e. for the space H/Γ	26
3.4	A typical fundamental domain for the action of Γ on the upper half-plane (Cunning, 1962)	27
4.1	(a) Rectangular of local geometry of $\mathbf{R}^2$ and (b) The two dimensional torus	39
4.2	An arbitrary parallelogram by translation of t_1 and t_2	40
4.3	Region of $R \subset H$ bounded by two vertical of $\mathrm{Re}(z) = \pm 1$	41
4.4	Tranformation of A and B give the geometry of torus with one punctures	41
4.5	Transformation of A and B give the spherical geometry with punctures	44
5.1	The fundamental domain of singly punctured two-torus at point $z = i\infty$ which cover six adjacent copies of fundamental domain	53
5.2	Fundamental domain $\boldsymbol{F_{dom}}$ of triply punctured two-sphere. Punctures are at $z = i\infty$ (sub-domain $\boldsymbol{F_{i\infty}}$), at point $z = 0$ (sub-domain $\boldsymbol{F_0}$) and at point $z = 1$ (sub-domain $\boldsymbol{F_1}$)	55
5.3	A sub-domain fro the whole fundamental domain which considered the mapping of $z = 0$ to $z = i\infty$	57
5.4	Sub-domain which considered the mapping $z = 1$ to $z = i\infty$	58

Figure		Page

5.5	Fundamental domain is divided into three sub-domains with respect to the punctures $z = i\infty$, $z = 0$ (blue coloured line) and $z = 1$ (red coloured line) respectively	59
5.6	Fundamental domain is divided into three sub-domains after the mapping of $z = 0$ to $z = i\infty$	59
5.7	Fundamental domain is divided into three sub-domains after the mapping of $z = 1$ to $z = i\infty$	60
5.8	Few choices of $\Gamma-$pullback points $x + iy$ that may end up either in domain $z = i\infty$, $z = 1$ or $z = 0$	61
5.9	Fundamental domain is divided into three sub-domains ***F***, ***SF*** (blue coloured line) and $\boldsymbol{ST^{-1}F}$ (red coloured line).	71
B.1	Mapping for the construction of singly punctured two-torus	90
G.1	Transformation for the construction of triply punctured two-sphere (Phillips, 1998)	101
G.2	Transformation for the construction of triply punctured two-sphere (Stillwell, 1992)	102

LIST OF SYMBOLS AND ABBREVIATIONS

$\mathbf{Z}$	Integers
$\mathbf{R}$	Real numbers
Q	Rational numbers
$\mathbb{C}$	Complex numbers
∞	Infinity
H	The upper half-plane
Γ	The modular group
$\Gamma(2)$	Principal congruence subgroup of level 2
$\Gamma(N)$	Principal congruence subgroup of level N
FFT	Finite Fourier Transform
K_{ir}	K-Bessel function with imaginary index
$SL(2,\mathbf{Z})$	Special Linear Two-Dimensioanl of integers
$SL(2,\mathbf{R})$	Special Linear Two-Dimensioanl of reals
$PSL(2,\mathbf{Z})$	Projective Special Linear Two-Dimensioanl of integers
H/Γ	Quotient space of H over Γ
$<$	Smaller than
$>$	Bigger than
$\leq$	Less than or equal
$\geq$	Bigger than or equal
$\gamma \in \Gamma$	γ element of group Γ
$\gamma \notin \Gamma$	γ is not an element of Γ

z	Imaginary numbers
$\mapsto$	Mappings
$\Leftrightarrow$	If and only if
$\neq$	Not equal
$\gamma.z$	Group actions
F_{dom}	Fundamental Domain
$F_{i\infty}$	Puncture at point $z = i\infty$
F_0	Puncture at point $z = 0$
F_1	Puncture at point $z = 1$
$\lvert a \rvert$	Modulus of a
V	Finite-dimensional vector space

CHAPTER 1

INTRODUCTION

1.1 Introduction

Topology is the branch of mathematics that studies the qualitative properties of spaces, as opposed to the more delicate and refined geometric or analytic properties. The term comes from the Greek word *topos* for place and *logos* for study (Chinn and Steenrod, 1966). Topology begins with a consideration of the nature of space, investigating both its fine structure and its global structure. The ideas and results of topology have placed it a central role in mathematics, connecting to almost all other areas of mathematics. While there are earlier results on topology, the beginning of the subject as a separate branch of mathematics dates to the work of H. Poincare (Balachandran, 1993) during 1895-1904.

Recently, topological methods have played increasingly important roles not only in wide area of mathematics but also in various studies of physics (Nakahara, 1990). Particle physicists for example are among the first to witness an increasing inclusion of topological ideas into their discipline. The initial development is in soliton and monopole physics (Balachandran, 1993) and in investigations on the role of topology in quantum physics. Particle theorists have come to appreciate the importance of topology in both classical and in quantum domain over the years (Balachandran, 1993). Presently, the role of topology in physics has been uncovered in other areas of physics as well. The subject is highly important in studies like mathematical models of quantum chaos (Then, 2004A; Then, 2004B; Gutzwiller, 1983; Guztwiller 1990 and Gutzwiller, 1993), cosmology (Then, 2004A; Aurich *et al.*, 2004), general relativity (Giulini, 1993 and Geroch, 1967), detecting defects in condensed matter physics (Mermin, 1979), quantum field theory (Kim, 1999), quantum gravity (Klosch and Strobl, 1997) and string theory (Rey, 1999).

In research, it is common to begin with spaces with simpler topologies such as a sphere and torus, whose topological properties are well-known for quantum theory on such surfaces and is available in the literature. Lesser known however are those with punctures (in which our case are removed points, also known as cusps or leaks) so that their topological properties become more complex. These properties are often encoded in the symmetries associated to motion on the spaces.

In this work, we consider two punctured spaces namely the singly punctured two-torus and the triply punctured two-sphere. Using their topological properties and group theoretic structures, we construct the corresponding energy eigenequation for a particle that move on such surfaces.

These introduced surfaces are known to have two generators and three possible non-contractible loops. However, they are considerably different geometrically with different genus and different number of punctures. Nevertheless, both surfaces do share similar initial construction condition according to the group structure. This is due to the fact that, they are generated from the same parent generators of the modular group. The properties of the modular group itself will give advantages in our current analysis and later it may be used to distinguish between such surfaces topologically.

1.2 Objectives of Research

Present research is meant to provide an explicit construction of energy eigenequation for the singly punctured two-torus and the triply punctured two-sphere giving due consideration of their topological and geometrical properties. Of particular interest is the group structure for both surfaces whose representations should classify the available quantum states of particles moving on the surfaces.

We start from constructing the configuration space explicitly by considering the tessellation of group plus the boundary conditions in representing the two surfaces, then determining the periodicity of function in the surfaces. Next, by using the Fourier expansion with finite Fourier transforms, a standard relation of energy eigenequation for both systems will be established. The equation obeys an automorphy condition which corresponds to the symmetry properties due to the surfaces' group structure. The real analytic solution of the energy eigenequation will then correspond to the so-called Maass cusp forms.

We will analytically compute the Maass cusp forms of singly punctured two-torus and triply punctured two-sphere. The results will correspond to the energy spectrum for both the discrete and continuous parts. For the continuous part, one should later consider to compute the Eisenstein series analytically. In this work however, the computation of Eisenstein series is with respect to the case of singly punctured two-torus only. Later, results for the analytic computation of Maass cusp forms and Eisenstein series of the singly punctured two-torus will be discussed.

In principle, the energy spectra can be worked out using the constructed eigenequations but this is beyond the score of the present work. Nevertheless, it is still hoped that this report will serve as a guideline for further research on other physical properties of the chosen topology and provide input on related topics like group-theoretic quantization for particle on more general surfaces in the future.

1.3 Outline of the Thesis

There are six chapters in this thesis. We started with the main introduction and objectives of the present research. Later, in Chapter Two we presented selected related literatures which inspired and rejuvenated our interest on this particular topic. This chapter is divided into three subtopics which include the configuration surfaces namely the singly punctured two-torus and the triply punctured two-sphere, the Maass waveforms, and Maass cusp forms. It will also include the scattering states which utilises the Eisenstein series.

Chapter Three explains on the elementary notations and definitions, mostly from the study of discrete groups (more specifically, principal congruence subgroups) and hyperbolic geometry. More of the theoretical background needed to understand the rich structure of the space of Maass waveforms, Maass cusp forms and the closed relationship to the Eisenstein series will then be introduced. We started from giving a basic understanding of the upper half-plane model and hyperbolic geometry, some definitions of the modular group Γ, principal congruence subgroup, the understanding of fundamental domain or *region*, the concept of tessellation, the subject of quotient spaces, definition on automorphic forms, modular forms and a review on the Fourier series with the finite Fourier transform.

The methodology of the present research is presented in Chapter Four. We begin from the singly punctured two-torus and triply punctured two-sphere. From there, we deal with the identification of the surfaces, and later compute its Maass cusp forms analytically using the standard Laplacian and automorphy conditions. The algorithm for determining the *K*-Bessel function will be presented for the computation of the Maass cusp forms together with the Eisenstein series.

In Chapter Five, every single detail of the results will be shown and explicitly explained. They included the construction of singly punctured two-torus and triply punctured two-sphere, their fundamental regions, the algorithm for computing the Maass cusp forms analytically and some remarks on the scattering states (i.e. the Eisenstein series of singly punctured two-torus). A unified treatment of both Maass cusp forms and the Eisenstein series of those surfaces will be emphasized in a more physical approach.

The final chapter provided the conclusion of this work and also discussion on further directions of the research that can be taken. The significant impact of the present findings are highlighted and suggestions are given with regards to present research.