UNIVERSITI PUTRA MALAYSIA

A GENERALIZATION OF THE BERNOULLI NUMBERS

HASMIAH BT BAHARI

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A GENERALIZATION OF THE BERNOULLI NUMBERS

By

HASMIAH BT BAHARI

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfillment of the Requirement for the Degree of Master of Science

October 2006

Dedication

To

My Parents

Bahari Bin Pasahari

Hasmah Bt Aber

For Their Great Patience, Pray and Support

My fiancee

Lahammad Saleng

My Sisters and Brothers

Junina Bahari

Hasriani Bahari

Basri Bahari

Hasmilah Bahari

Azroy Bahari

Hastiah Bahari

Hasneera Bahari

Hasbiah Bahari

Hasnita Bahari

And

My Auntie

Muliaty

For Their Understanding and Encouragement

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in Fulfilment of the requirement for the degree of Master of Science

A GENERALIZATION OF THE BERNOULLI NUMBERS

By

HASMIAH BINTI BAHARI

October 2006

Chairman : **Associate Professor Bekbaev Ural, PhD**

Institute: **Mathematical Research**

The Bernoulli numbers are among the most interesting and important number sequences in mathematics. It plays an important and quite mysterious role in various places like number theory, analysis and etc. In general, many existing generalizations of Bernoulli numbers $\{B_n\}$ for example $[20, 21]$ are based on consideration of more general forms for the left side of the following equality

$$\frac{t}{\exp(t)-1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}$$

or for some related functions.

In this study, a generalization of Bernoulli numbers is offered by the use of their relations with Pascal's triangle. The thesis begins with the generalization of Bernoulli

numbers $\{B_n\}_{n=1}^{\infty}$ then a representation of B_n is presented, followed by the proof of the main result for odd n case (even case of n was considered in $[2]$). Then special cases of Bernoulli numbers, namely when the initial sequence is an geometric or arithmetic sequence, are considered. In these special cases more detailed representations of B_n are obtained. Then irreducibility problem over Z of polynomials closely related to B_n is considered followed by solution of this problem for some values of n. At the end some unsolved problems, with which we have come across in doing this thesis, over the field Z are formulated.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Master Sains

PENGITLAKAN NOMBOR BERNOULLI

Oleh

HASMIAH BINTI BAHARI

Oktober 2006

Pengerusi : **Profesor Madya Bekbaev Ural, PhD**

Institut: **Penyelidikan Matematik**

Nombor Bernoulli merupakan jujukan nombor yang menarik dan penting dalam matematik. Ia selalu diaplikasikan di dalam beberapa bidang matematik seperti teori nombor, analisis dan sebagainya. Secara amnya, pengitlakan nombor Bernoulli $\{B_n\}$ yang wujud pada masa kini, contohnya $[20, 21]$, adalah berdasarkan bentuk umum ungkapan sebelah kiri persamaan di bawah

$$\frac{t}{\exp(t)-1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}$$

atau bagi beberapa fungsi yang berkaitan.

Dalam kajian ini, pengitlakan nombor Bernoulli dilakukan dengan mengambilkira hubungannya dengan segitiga Pascal. Kajian ini dimulakan dengan melakukan pengitlakan terhadap nombor Bernoulli $\{B_n\}_{n=1}^{\infty}$ bagi menghasilkan perwakilan B_n dan diikuti dengan pembuktian hasil utama bagi n yang ganjil (n yang genap telah dikaji dalam [2]). Seterusnya, kes khusus bagi nombor Bernoulli dipertimbangkan iaitu apabila jujukan awal merupakan jujukan geometri dan aritmetik. Dalam kes-kes khusus ini, lebih banyak perwakilan yang terperinci bagi B_n telah diperoleh. Masalah ketidakbolehfaktoran dalam gelanggang Z bagi suatu polinomial yang berkait rapat dengan B_n juga dipertimbangkan dan disusuli dengan penyelesaian masalah tersebut bagi beberapa nilai n. Akhirnya, beberapa masalah yang tidak dapat diselesaikan yang timbul semasa kajian ini diatas medan Z dapat difomulasi.

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I certify that an Examination Committee met on 5^{th} October 2006 to conduct the final examination of Hasmiah Binti Bahari on her Master of Science thesis entitle "A Generalization of the Bernoulli Numbers" in accordance with Universiti Pertanian Malaysia (Higher Degree) Act 1980 and Universiti Pertanian Malaysia (Higher Degree) Regulation 1981. The Committee recommends that the candidate be awarded the relevant degree. Members of the Examination Committee are as follows:

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This thesis submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Master of Science. The members of the Supervisory Committee are as follows:

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DECLARATION

I hereby declare that the thesis is based on my original work except for quotations and citations which have been duly acknowledged. I also declare that it has not been previously or concurrently submitted for any other degree at UPM or other institutions.

HASMIAH BAHARI

Date: 21 NOVEMBER 2006

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CHAPTER 1

INTRODUCTION

1.1 Short history

Two thousand years ago, Greek mathematician Phytagoras first noted about triangle numbers which are $1+2+3+...+n.$ Archimedes found out

$$1^2+2^2+3^2+...+n^2=\frac{1}{6}n(n+1)(2n+1)$$

Later in the fifth century, Indian mathematician Aryabhata proposed

$$1^3+2^3+3^3+...+n^3=\left[\frac{1}{2}n(n+1)\right]^2$$

Which Jacobi gave the first vigorous proof in 1834. It is not until five hundred years later that Arabian mathematician Al-Khwarizm showed

$$1^4+2^4+3^4+...+n^4=\frac{1}{30}n(n+1)(2n+1)\left(3n^2+3n-1\right).$$

Studies of the more generalized formula for $\sum_{k=1}^{n-1} k^r$ for any natural number r was only carried out in the last few centuries. Among them, the investigation of Bernoulli numbers is much significant.

Swiss mathematician Jacob Bernoulli (1654-1705) once claimed that instead of laboring for hours to get a sum of powers, he only used several minutes to calculate sum of powers such as

$$1^{10} + 2^{10} + 3^{10} + ... + 1000^{10} = 91,409,924,241,424,243,424,241,924,242,500 .$$

Obviously, he had used a summation formula, knowing the first 10 Bernoulli numbers.

The Bernoulli numbers are among the most interesting and important number sequences in mathematics. Its play an important and quite mysterious role in various places like number theory, analysis, differential topology and etc. They first appeared in posthumous work "*Ars Conjectandi*" by Jacob Bernoulli which was published in 1713 after 8 years his death.

After Jacob Bernoulli, his brother Johann Bernoulli (1667-1748) continued to discover those Bernoulli numbers. These numbers were assisted in developing Fermat's Last Theorem. The modern Bernoulli numbers are a superset of the archaic version. The term Bernoulli numbers was used for the first time by Abraham De Moivre (1667-1754) and Leonard Euler (1707-1783) which found its recursion relation. In 1735, the solution of the *Basel problem,* the relation between zeta function and Bernoulli numbers was one of Euler's most sensational discoveries.

The famous Clausen-von Staudt's theorem regarding Bernoulli numbers fractional part was published by Karl von Staudt (1798-1867) and Thomas Clausen (1801-1885) independently in 1840. It allows computing easily the fractional part of Bernoulli numbers and thus also permits to compute the denominator of those numbers. It is very useful and significant in the sense that it permits to compute exactly Bernoulli numbers as soon as there is sufficiently good approximation of it.

Generalization of Bernoulli numbers are defined starting from suitable generating function. The number sequences of Euler, Genocchi, Stirling and others, as well as the tangent numbers, secant numbers are closely related to the Bernoulli numbers. The same is true for the numerous generalizations and expansions of the Bernoulli numbers and the corresponding polynomials. Perhaps one of the most important results is *Euler-Maclaurin* summation formula, where Bernoulli numbers are contained and which allows accelerating the computation of slow converging series. They also appear in number theory (Fermat's theorem). Realized that the Bernoulli numbers are important, the Indian mathematician Srinivasa Ramanujan (1887-1920) rediscovered those Bernoulli numbers in 1904. He investigated the series and calculated Euler's constant to 15 decimal places. He began to study the Bernoulli numbers, although this was entirely his own independent discovery.

In the year 2001, Radoslav Jovanovic found that there is surprising connection with Bernoulli numbers and Pascal's Triangle. To illustrate the Bernoulli numbers, he considers the function $f(x) = \dfrac{x}{e^x - 1}$. Taking advantage of the familiar exponential expansion

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

hence,

$$f(x)=\frac{x}{\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots}=\frac{1}{1+\frac{x}{2!}+\frac{x^2}{3!}+\ldots}$$

The function $f(x)$ can be expanded in a power series about $x=0$; for the sake of convenience in subsequent computations, he represent this series as

$$\frac{x}{e^x-1}=\sum_{n=0}^{\infty}\frac{B_n}{n!}x^n$$

where $B_0 = f(0)=1$. In order to determine the other coefficient $B_n(n=1,2\ldots)$ of the expansion, which are called Bernoulli numbers, he make use of the identity

$$\sum_{n=0}^{\infty}\frac{x^n}{(n+1)!}\sum_{n=0}^{\infty}\frac{B_n}{n!}x^n=1$$

Multiplying together the power series and equating to zero the coefficients of the positive powers of the variable *x*, he obtain an infinite system of linear equations:

$$\frac{B_n}{n!}.\frac{1}{1!}+\frac{B_{n-1}}{(n-1)!}.\frac{1}{2!}+\ldots+\frac{B_0}{0!}.\frac{1}{(n+1)!}=0$$

or, multiplying by $(n+1)!$ and noting that

$$\frac{(n+1)!}{(n-k)!(k+1)!}=C_{n+1}^{n-k}$$

Then last formula he written in the following form:

$$(B+1)_{n+1}-B_{n+1}=0$$

or, replacing $(n+1)$ by n,

$$(B+1)_n-B_n=0;\ n=1,2,3,\ldots$$

he obtain an infinite system of equation :

$$
\begin{aligned}
&0 = 1B_0 + 2B_1 \\
&0 = 1B_0 + 3B_1 + 3B_2 \\
&0 = 1B_0 + 4B_1 + 6B_2 + 4B_3 \\
&0 = 1B_0 + 5B_1 + 10B_2 + 10B_3 + 5B_4 \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
$$

Hence, he successively find the connection with Bernoulli's numbers and Pascal's triangle.

Bernoulli himself calculated the numbers up to B_{10}. Later, Euler worked up to B_{30}, then Martin Ohm extended the calculation up to B_{62} in 1840. A few years later, in 1877, Adams made the computation of all Bernoulli numbers up to B_{124}. For instance, the numerator of B_{124} has 110 digits and the denominator is the number 30. In 1996, Simon Plouffe and Greg J. Fee computed $B_{200,000}$ and this huge number has about 800,000 digits. In July 2002, they improved the record to $B_{750,000}$ which has 3,391,993 digits by a 21 hours computation on their personal computer. The method is based on the relation between zeta function and Bernoulli numbers, which allow a direct computation of the target number without the need of calculating the previous numbers.

In this research generalization of Bernoulli numbers is offered by the use of their relation with Pascal's triangle.

1.2 Relation between Bernoulli numbers and Pascal's Triangle

Roughly speaking, all existing generalizations of Bernoulli numbers $\{B_n\}$, for example $[1,2]$, are based on consideration of more general forms for the left side of the following equality

$$\frac{t}{\exp(t)-1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}$$

or for some related functions.

Example 1.2.1

Consider infinitely smooth function $y = f(t) = \dfrac{t}{e^t - 1}$ defined on some neighbourhood of $t = t_0$. The Taylor expansion of it at $t = t_0$ is

$$f(t) = f(t_0) + \frac{f'(t_0)}{1!}(t - t_0) + \frac{f''(t_0)}{2!}(t - t_0)^2 + \ldots = \sum_{k=0}^{\infty} \frac{f^{(k)}(t_0)}{k!}(t - t_0)^k.$$

Therefore due to the ordinary definition of Bernoulli numbers $\sum_{n=0}^{\infty} B_n \frac{t^n}{n!}$ one has $B_n = f^{(n)}(0)$. Therefore $B_0 = 1$ as far as $\lim_{t\to 0} \dfrac{t}{e^t - 1} = 1$.

$$B_1 = \lim_{t\to 0}\left(\frac{t}{e^t - 1}\right)' = \lim_{t\to 0}\frac{e^t(1-t)-1}{(e^t - 1)^2} = \lim_{t\to 0}\left(\frac{e^t(1-t)-1}{(e^t - 1)^2}\right)' =$$

$$\lim_{t\to 0}\left(-\frac{t}{2(e^t - 1)}\right) = -\frac{1}{2}\lim_{t\to 0}\left(\frac{t}{(e^t - 1)}\right) = -\frac{1}{2} \quad \textit{and etc.}$$

Example 1.2.2

For $C_{n+1}^{i}=\binom{n+1}{i}$ and $\sum_{i=0}^{n}\binom{n+1}{i}B_i=0$. We know the first number is always $B_0=1$.

For $n=1; \binom{2}{0}B_o+\binom{2}{1}B_1=0,$ then $B_1=-\frac{1}{2}$. Recursively, for $n=2; B_2=\frac{1}{6}$ and so on.

May be the simplest definition of Bernoulli numbers $\{B_n\}$ is the following: $B_0=1$ and if you already have known $B_0, B_1, ..., B_{n-1}$, where $n\geq 1$, then find B_n by solving the equation

$$\sum_{i=0}^{n}\binom{n+1}{i}B_i=0$$

Now we give a generalization of Bernoulli numbers by their relations with Pascal's triangle. In future "sequence $\{a_n\}$" means $a_1, a_2, a_3, ...$ except for "Bernoulli sequence $\{B_n\}$", which stands for $B_0, B_1, B_2, B_3, ...$.

The ordinary Bernoulli numbers can be defined in the following way as well:

(1) Consider Pascal's triangle

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & \vdots & & \vdots & & \vdots & & \vdots & & &
\end{array}
$$

(2) Delete its "right side" consisting of ones and the rest write as the following matrix

$$
A = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
1 & 2 & 0 & 0 & 0 & 0 & \ldots \\
1 & 3 & 3 & 0 & 0 & 0 & \ldots \\
1 & 4 & 6 & 4 & 0 & 0 & \ldots \\
1 & 5 & 10 & 10 & 5 & 0 & \ldots \\
. & . & . & . & . & . & \ldots
\end{pmatrix}
$$

(3) Then evaluate A^{-1} and you will get $\{B_n\}$ as the first column of A^{-1}.

For a while consider the sequence $\{a_n\} = \{n\}$. If $1 \le i < j$ then

$$
\binom{j}{i} = \frac{j.(j-1)...(j-i+1)}{1.2...i} = \frac{a_j.a_{j-1}...a_{j-i+1}}{a_1.a_2...a_i}
$$

So one can consider any sequence $\{a_n\}$ of nonzero numbers, and its "Pascal's triangle"

(a)

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & \frac{a_2}{a_1} & & 1 & & & \\
 & & 1 & & \frac{a_3}{a_1} & & \frac{a_3}{a_1} & & 1 & & \\
 & 1 & & \frac{a_4}{a_1} & & \frac{a_4 a_3}{a_1 a_2} & & \frac{a_4}{a_1} & & 1 & \\
1 & & \frac{a_5}{a_1} & & \frac{a_5 a_4}{a_1 a_2} & & \frac{a_5 a_4}{a_1 a_2} & & \frac{a_5}{a_1} & & 1 \\
 & & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Remark 1.2.3

The property of Pascal's triangle that "An inside number of the $(n+1)th$ row (base) can be computed by going up to the $(n)th$ row(base) and adding two neighbouring numbers above it" is not inherited by the above "Pascal's triangle for $\{a_n\}$." But it inherits Pascal's triangle's following property related to its lateral sides: Consider its *nth* right lateral side. If you know its *kth* number then multiply it by $\frac{a_{n+k-1}}{a_k}$ to get its $(k+1)th$ number.

(b) Now delete its "right side" consisting of ones and the rest of ones and the rest write as the following matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & \frac{a_2}{a_1} & 0 & 0 & 0 & \ldots \\ 1 & \frac{a_3}{a_1} & \frac{a_3}{a_1} & 0 & 0 & \ldots \\ 1 & \frac{a_4}{a_1} & \frac{a_4 a_3}{a_1 a_2} & \frac{a_4}{a_1} & 0 & \ldots \\ 1 & \frac{a_5}{a_1} & \frac{a_5 a_4}{a_1 a_2} & \frac{a_5 a_4}{a_1 a_2} & \frac{a_5}{a_1} & \ldots \\ . & . & . & . & . & . \end{pmatrix}$$

(c) Evaluate A^{-1} and call the sequence of entries of the first column of A^{-1} the sequence of Bernoulli numbers for the given sequence $\{a_n\}$. We have

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ -\frac{a_1}{a_2} & \frac{a_1}{a_2} & 0 & 0 & \ldots \\ -\frac{a_1}{a_3}\frac{(a_2 - a_3)}{a_2} & -\frac{a_1}{a_2} & \frac{a_1}{a_3} & 0 & \ldots \\ -\frac{a_1}{a_4}\frac{(a_2^2 - 2a_2 a_4 + a_3 a_4)}{a_2^2} & \frac{a_1(a_3 - a_2)}{a_2^2} & -\frac{a_1}{a_2} & \frac{a_1}{a_4} & \ldots \\ . & . & . & . & \ldots \\ . & . & . & . & \ldots \end{pmatrix}$$

The first column of A^{-1} is the sequence of Bernoulli numbers.

Here is the beginning part of that sequence of Bernoulli numbers for the given sequence $\{a_n\}$.

$$1, \quad -\frac{a_1}{a_2}, \quad -\frac{a_1}{a_3}\frac{a_2 - a_3}{a_2}, \quad -\frac{a_1}{a_4}\frac{a_4 a_3 - 2a_4 a_2 + a_2^2}{a_2^2},$$

$$-\frac{a_1}{a_5}\frac{-a_5 a_4 a_3^2 + 3a_5 a_4 a_3 a_2 - 2a_5 a_3 a_2^2 - a_5 a_4 a_2^2 + a_3 a_2^3}{a_3 a_2^3},\ldots$$

Here we are not going to fix some $\{a_n\}$ and consider the corresponding Bernoulli numbers. In opposite, we will consider $a_1, a_2, a_3, \ldots$ as independent variables (i.e. there is no polynomial relation among $a_1, a_2, a_3, \ldots$) and deal with entries of the corresponding matrix A^{-1} as rational function in $a_1, a_2, a_3, \ldots$. Let $\{B_n\}$ stand for the sequence of entries of the first column of A^{-1} and $[r]$ stand for the integer part of a real number *r*.

1.3 Objective of Research

The following theorem was announced by Dr. Ural (2003)

Theorem 1.3.1 For any $n > 1$, the rational function B_{n-1} is of the following form

$$B_{n-1} = -\frac{a_1}{a_n}\frac{N_n}{D_n},$$