UNIVERSITI PUTRA MALAYSIA

GENERALIZATIONS AND SOME APPLICATIONS OF KRONECKER AND HADAMARD PRODUCTS OF MATRICES

ZEYAD ABDEL AZIZ MAH'D AL ZHOUR

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By

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Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirement for the Degree of Doctor of Philosophy

November 2006

DEDICATION

Especially Dedicated

To My Beloved Parent, Brothers & Sisters

To My Beloved Wife & Children

To My Friends

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Doctor of Philosophy

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Chairman: **Associate Professor Adem Kiliçman, PhD**

Faculty: **Science**

In this thesis, generalizations of Kronecker, Hadamard and usual products (sums) that depend on the partitioned of matrices are studied and defined. Namely: Tracy-Singh, Khatri-Rao, box, strong Kronecker, block Kronecker, block Hadamard, restricted Khatri-Rao products (sums) which are extended the meaning of Kronecker, Hadamard and usual products (sums). The matrix convolution products, namely: matrix convolution, Kronecker convolution and Hadamard convolution products of matrices with entries in set of functions are also considered. The connections among them are derived and most useful properties are studied in order to find new applications of Tracy-Singh and Khatri-Rao products (sums). These applications are: a family of generalized inverses, a family of coupled singular matrix problems, a family of matrix inequalities and a family of geometric means.

In the theory of generalized inverses of matrices and their applications, the five generalized inverses, namely Moore-Penrose, weighted Moore-Penrose, Drazin, weighted Drazin and group inverses and their expressions and properties are studied. Moreover, some new numerous matrix expressions involving these generalized inverses and weighted matrix norms of the Tracy-Singh products matrices are also derived. In addition, we establish some necessary and sufficient conditions for the reverse order law of Drazin and weighted Drazin inverses. These results play a central role in our applications and many other applications.

In the field of system identification and matrix products work, we propose several algorithms for computing the solutions of the coupled matrix differential equations, coupled matrix convolution differential, coupled matrix equations, restricted coupled singular matrix equations, coupled matrix least-squares problems and weighted Least -squares problems based on idea of Kronecker (Hadamard) and Tracy-Singh (Khatri-Rao) products (sums) of matrices. The way exists which transform the coupled matrix problems and coupled matrix differential equations into forms for which solutions may be readily computed. The common vector exact solutions of these coupled are presented and, subsequently, construct a computationally - efficient solution of coupled matrix linear least-squares problems and non-homogeneous coupled matrix differential equations. We give new applications for the representations of weighted Drazin, Drazin and Moore-Penrose inverses of Kronecker products to the solutions of restricted singular matrix and coupled matrix equations. The analysis indicates that the Kronecker (Hadamard) structure method can achieve good efficient while the Hadamard structure method achieve more efficient when the unknown matrices are diagonal. Several special cases of these systems are also considered and solved, and then we prove the existence and

uniqueness of the solution of each case, which includes the well-known coupled Sylvester matrix equations. We show also that the solutions of non-homogeneous matrix differential equations can be written in convolution forms. The analysis indicates also that the algorithms can be easily to find the common exact solutions to the coupled matrix and matrix differential equations for partitioned matrices by using the connections between Tracy-Singh, Block Kronecker and Khatri -Rao products and partitioned vector row (column) and our definition which is the so-called partitioned diagonal extraction operators.

Unlike Matrix algebra, which is based on matrices, analysis must deal with estimates. In other words, Inequalities lie at the core of analysis. For this reason, it's of great importance to give bounds and inequalities involving matrices. In this situation, the results are organized in the following five ways: First, we find some extensions and generalizations of the inequalities involving Khatri-Rao products of positive (semi) definite matrices. We turn to results relating Khatri-Rao and Tracy-Singh powers and usual powers, extending and generalizing work of previous authors. Second, we derive some new attractive inequalities involving Khatri-Rao products of positive (semi) definite matrices. We remark that some known inequalities and many other new interesting inequalities can easily be found by using our approaches. Third, we study some sufficient and necessary conditions under which inequalities below become equalities. Fourth, some counter examples are considered to show that some inequalities do not hold in general case. Fifth, we find Hölder-type inequalities for Tracy-Singh and Khatri-Rao products of positive (semi) definite matrices. The results lead to inequalities involving Hadamard and Kronecker products, as a special case, which includes the well-known inequalities involving Hadamard product of matrices, for instance, Kantorovich-type inequalities and generalization of Styan's inequality. We utilize the commutativity of the Hadamard product (sum) for possible to develop and improve some interesting inequalities which do not follow simply from the work of researchers, for example, Visick's inequality.

Finally, a family of geometric means for positive two definite matrices is studied; we discuss possible definitions of the geometric means of positive definite matrices. We study the geometric means of two positive definite matrices to arrive the definitions of the weighted operator means of k positive definite matrices. By means of several examples, we show that there is no known definition which is completely satisfactory. We have succeeded to find many new desirable properties and connections for geometric means related to Tracy-Singh products in order to obtain new unusual estimates for the Khatri-Rao (Tracy-Singh) products of several positive definite matrices.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia
sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

PENGITLAKAN DAN BEBERAPA APLIKASI HASIL DARAB KRONECKER DAN HADAMARD BAGI MATRIKS

Oleh

ZEYAD ABDEL AZIZ MAH'D AL ZHOUR

November 2006

Pengerusi: Profesor Madya Adem Kiliçman, PhD

Fakulti: Sains

Dalam tesis ini pengitlakan hasil darab (tambah) Kronecker, Hadamard yang berkaitan dengan matriks yang dipartisi dikaji dan ditak riftan. Perkara tersebut melihatkan: hasil darab (tambah) Tracy-Singh, Khatri-Rao, Kotak, Kronecker Kuat, Blok Kronecker, hasil darab terhadap Khatri-Rao yang diitlakan maksudnya. Hasildarab konvolusi matriks yakni Konvolusi matriks, Konvolusi Kronecker dan Konvolusi hadamard, ditertibkan dan sifat utamanya dikaji untuk memperlihatkan penggunaan hasil darab Tracy-Singh dan Khatri-Rao (hasil tambah) . Penggunaan ini termasuklah songsangan satu keluarga pengitlakan, keluarga masalah matriks singularan, keluarga ketaksamaan matriks dan keluarga min geometri.

Dalam teori songsangan matriks teritlak dan aplikasinya, lima songsangan teritlak, iaitu Moore-Penrose, Moore-Penrose berpemberat, Drazin, Drazin berpemberat dan songsangan kumpulan serta ungkapannya dan sifat-sifat dikaji. Lebih-lebih lagi, beberapa ungkapan matriks baru termasuk lima jenis songsangan teritlak dan norma

matriks berpemberat bagi matriks hasil darab Tracy-Singh juga diterbitkan. Tambahan pula, kita membina beberapa syarat perlu dan cukup untuk hukum tertib simpanan bagi Drazin dan songsangan Drazin berpemberat. Hasil-hasil ini memainkan suatu peranan pusat dalam aplikasi kita dan banyak aplikasi lain.

Dalam bidang pengecaman sistem dan kerja hasil darab matriks, kita syorkan beberapa algoritma untuk mengira penyelesaian bagi persamaan pembezaan matriks terganding, pembezaan konvolusi matriks terganding, persamaan matriks terganding, persamaan matriks singular gandingan tersekat, masalah gandingan matriks kuasa-dua terkecil dan masalah kuasa-dua terkecil berpemberat berasaskan ide hasil darab (hasil tambah) Kronecker (Hadamard) dan Tracy-Singh (Khatri-Rao) bagi matriks. Cara ini wujud untuk menjelmakan masalah matriks terganding dan persamaan pembezaan matriks terganding ke dalam bentuk yang penyelesaiannya mungkin dikira. Penyelesaian vektor tepat yang biasa bagi gandingan ini dikemukakan dan selanjutnya bina suatu penyelesaian pengiraan cekap bagi masalah matriks linear kuasa-dua terkecil terganding dan persamaan pembezaan matriks terganding yang tak homogen. Kita beri aplikasi baru untuk perwakilan bagi Drazin berpemberat, Drazin dan songsangan Moore-Penrose bagi hasil darab Kronecker kepada penyelesaian bagi matriks singular tersekat dan persamaan matriks terganding. Analisis menunjukkan bahawa kaedah struktur Kronecker (Hadamard) boleh mencapai kecekapan baik manakala kaedah struktur Hadamard mencapai lebih kecekapan apabila matriks anu adalah terpepenjuru. Beberapa kes khusus bagi sistem ini juga dipertimbangkan dan diselesaikan dan kita buktikan pula kewujudan dan keunikan bagi penyelesaian untuk setiap kes, termasuk persamaan matriks Sylvester terganding yang masyhur. Kita tunjukkan juga bahawa penyelesaian bagi persamaan pembezaan matriks tak homogen boleh ditulis dalam bentuk konvolusi. Analisis juga menunjukkan bahawa algoritmanya boleh mancari penyelesaian tepat biasa bagi persamaan pembezaan matriks dan matriks terganding untuk matriks berpetak dengan menggunakan kaitan antara hasil darab Tracy-Singh, Blok Kronecker dan Khatri-Rao dan vektor baris (lajur) berpetak dan takrifan kita yang dipanggil pengoperasi ekstraksi pepenjuru berpetak.

Lain sekali dalam aljabar matriks yang menggunakan matriks, analisisnya mesti dikendalikan dengan anggaran. Dengan lain perkataan, ketaksamaannya terletak pada teras analisis. Oleh sebab itulah paling mustahak untuk memberi batas dan ketaksamaan yang melibatkan matriks. Dalam keadaan ini, keputusannya disusun dalam lima cara berikut. Pertama, kita cari beberapa pengembangan dan pengitlakan bagi ketaksamaan melibatkan hasil darab Khatri-Rao bagi matriks tentu (semi) positif. Kita balik pula kepada keputusan yang menghubungkan kuasa Khatri-Rao dan Tracy-Singh dan kuasa biasa, mengembangkan dan mengitlakkan kerja yang dibuat oleh penulis terdahulu. Kedua, kita terbitkan beberapa ketaksamaan baru yang menarik melibatkan hasil darab Khatri-Rao bagi matriks tentu (semi) positif. Kita perhatikan bahawa beberapa ketaksamaan ketahuan dan banyak ketaksamaan lain yang baru dan menarik boleh diperolehi dengan menggunakan pendekatan kita. Ketiga, kita kaji beberapa syarat cukup dan perlu di mana ketaksamaan menjadi kesamaan. Keempat, beberapa contoh penyangkal dipertimbangkan untuk menunjukkan beberapa ketaksamaan tidak dipatuhi dalam kes umum. Kelima, kita cari ketaksamaan jenis-Hölder untuk hasil darab Tracy-Singh dan Khatri-Rao bagi matriks tentu (semi) positif. Hasilnya menuju kepada ketaksamaan melibatkan hasil darab Hadamard dan Kronecker, sebagai satu kes khusus, termasuk ketaksamaan tersohor melibatkan hasildarab Hadamard bagi matriks, umpamanya, ketaksamaan jenis-Kantorovich dan pengitlakan ketaksamaan Styan. Kita gunakan kekalisan tukar tertib bagi hasil darab (hasil tambah) Hadamard kemungkinan untuk membangunkan dan menambahbaikkan beberapa ketaksamaan menarik yang tidak mengikuti kerja para penyelidik, misalnya, ketaksamaan Visick .

Akhir sekali, suatu famili min geometri untuk dua matriks tentu positif dikaji; kita bincangkan takrifan yang mungkin bagi min geometri untuk matriks tentu positif. Kita mengkaji min geometri bagi dua matriks tentu positif untuk tiba ke takrifan bagi min pengoperasi berpemberat bagi *k* matriks positif. Dengan cara contoh mudah, kita tunjukkan bahawa tiada takrif ketahuan yang memuaskan selengkapnya. Kita berjaya mencari banyak sifat baru yang diinginkan dan kaitan untuk min geometri berhubung dengan hasil darab Tracy-Singh untuk memperolehi anggaran tak biasa baru bagi hasil darab Khatri-Rao (Tracy-Singh) bagi beberapa matriks tentu positif.

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I certify that an Examination Committee met on 27 November 2006 to conduct the final examination of Zeyad Abdel Aziz Mah'd Al Zhour on his Doctor of Philosophy thesis entitled "Generalizations and Some Applications of Kronecker and Hadamard Products of Matrices" in accordance with Universiti Pertanian Malaysia (Higher Degree) Act 1980 and Universiti Pertanian Malaysia (Higher Degree) Regulations 1981. The Committee recommends that the candidate be awarded the relevant degree. Members of the Examination Committee are as follows:

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This thesis submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Doctor of Philosophy. The members of the Supervisory Committee are as follows:

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Date: 16 JANUARY 2007

DECLARATION

I hereby declare that the thesis is based on my original work except for quotations and citations which have been duly acknowledged. I also declare that it has not been previously or concurrently submitted for any other degree at UPM or other institutions.

ZEYAD ABDEL AZIZ AL ZHOUR

Date: 16 JANUARY 2007

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LIST OF ABBREVIATIONS

$\mathbb{R}$	The Set of Real Numbers
$\mathbb{C}$	The Set of Complex Numbers
$\mathbb{C}^n$	The Set of n-Vectors Over $\mathbb{C}$
$\subseteq$	Subset
$\in$	Belong to
$M_{m,n}$	The Set of $m \times n$ Matrices Over the Complex (Real) Field $\mathbb{C}$ or $\mathbb{R}$ and when $m = n$, we write M_n instead of $M_{n,n}$
$M^I_{m,n}$	The Set of $m \times n$ Matrices for whose entries are integrable one some fixed finite interval $0 \le t < b$ and when $m = n$, we write M^I_n instead of $M^I_{n,n}$
$[A]_{\alpha,\beta}$	The (Sub) matrix of A resulting from laying in the intersection of rows α and columns β and when $\alpha = \beta$, we write $[A]_\alpha$ instead of $[A]_{\alpha,\alpha}$
I_n or I	The Identity Matrix of order $n \times n$
$\sum_{i=1}^{k} A_i$	The Usual Summation of matrices A_i $(1 \le i \le k)$ from 1 to k
$\prod_{i=1}^{k} A_i$	The Usual Product of matrices A_i $(1 \le i \le k)$ from 1 to k
$\lim_{k \to a} A_k$	The Limit of A_k as k goes to a
$\int_a^b A(t)dt$	The Integral of matrix function $A(t)$ from a to b
$null(A)$	The Null Space of matrix A

$rang(A)$	The Range of matrix A
$rank(A)$	The Rank of matrix A
$tr(A)$	The Trace of matrix A
e_i	The i-th Column of the identity matrix I_n or Unit vector
$E_{ij} = e_i e_j^T = e_i \otimes e_j^T$	The Elementary Matrix
P_{mn}	The Permutation Matrix of order $mn \times mn$
Q_{mn}	The Generalized Permutation Matrix of order $mn \times mn$
K_{mn} and G_{mn}	The Partitioned Permutation matrices of order $mn \times mn$
$\mathrm{Re}(A)$ and $\mathrm{Im}(A)$	The Real and Imaginary Parts of complex matrix A, respectively.
$\sigma(A)$	The Set of All Eigenvalues of a square matrix A (Spectrum)
$W(A)$	The Numerical Range of matrix A
$w(A)$	The Numerical Radius of matrix A
$\rho(A)$	The Spectral Radius of matrix A
$cond(A)$	The Condition Number of matrix A
$s(A)$	The Set of All Singular Values of matrix A
$S_{A_{11}}$	The Schur Complement of block A_{11} in partitioned matrix $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$
A^*	The Conjugate Transpose of matrix A
A^T	The Transpose of matrix A
$\overline{A}$	The Closure of matrix A
$\det A$	The Determinant of matrix A

$VecA$	The Vector Operator of matrix A
$Vecd(A)$	The Diagonal Extraction-Operator of matrix A
$\underset{r}{Vec} A$	The Row-Blockwise Vector Operator of matrix A
$\underset{c}{Vec} A$	The Column-Blockwise Vector Operator of matrix A
$Vecd_p(A)$	The Partitioned Diagonal Extraction Operator of matrix A
$A \geq 0$	The Hermitian Positive Semi Definite Matrix A
$A > 0$	The Hermitian Positive Definite Matrix A
$A \geq B$ or $A - B \geq 0$	The Hermitian Positive Semi Definite Matrix $A - B$
$A > B$ or $A - B > 0$	The Hermitian Positive Definite Matrix $A - B$
H_n	The Space of n-Square Hermitian Matrices
H_n^+	The Space of n-Square Positive Definite Matrices
$A^{\#}$	The Weighted Conjugate of matrix A
$Ind(A)$	The Index of matrix A
A^{-1}	The Inverse of a non-singular matrix A
A^+	The Moore-Penrose Inverse of matrix A
$A_{M,N}^+$	The weighted Moore-Penrose Inverse of matrix A
A_d	The Drazin Inverse of matrix A
A_g	The Group Inverse of matrix A
$A_{d,W}$	The weighted Drazin Inverse of matrix A
e^A or $\exp A$	The Exponential Matrix of A
$\sin A$	The Sine Matrix of A

$\cos A$	The Cosine Matrix of A
$\sinh A$	The Hyperbolic Sine Matrix of A
$\cosh A$	The Hyperbolic Cosine Matrix of A
$\log A$	The Logarithmic Matrix of A
$\langle Ax, x\rangle = x^* Ax$	The Euclidean Inner Product on $\mathbb{C}^n$
$\lvert A\rvert = (A^* A)^{\frac{1}{2}}$	The Absolute Value of matrix A
$\lVert A\rVert$	The Matrix Norm of A
$\lVert A\rVert_p$	The Schatten p-Norms of matrix A
$\lVert A\rVert_\infty$	The $\infty -$Norm or Maximum Row Sum Norm of matrix A
$\lVert A\rVert_{\max}$	The Max-Norm or Maximum Column Sum Norm of matrix A
$\lVert A\rVert_{sp}$	The Spectral Norm of matrix A
$\lVert A\rVert_F$ or $\lVert A\rVert_2$	The Frobenius Norm of matrix A
$\lVert x\rVert_M$	The Weighted Vector Spectral Norm
$\lVert A\rVert_{M,N}$	The Weighted Matrix Spectral Norm
$\lVert A\rVert_{M,N}^{(F)}$	The Weighted Matrix Frobenius Norm
$\lVert A\rVert_{M,N}^{(p)}$	The Weighted Matrix Schatten p-Norms
$£\{A(t)\}$	The Laplace Transform of matrix function $A(t)$
$A'(t)$ or $\frac{d}{dt}A(t)$	The First-Order Derivative of matrix function $A(t)$
$\frac{\partial A}{\partial y}$	The First-Order partial Derivative of a matrix function A with

	respect to the vector y
$\delta(t)$	The Dirac Delta Function
$D_n(t)$	The Dirac Identity Matrix
$A^{\{-1\}}(t)$	The Inverse of matrix function $A(t)$ respect to the convolution
$A^{\{m\}}(t)$	The m-Power Convolution Product
AB	The Usual Product of matrices A and B
$A \otimes B$	The Kronecker Product of matrices A and B
$A \circ B$	TheHadamard Product of matrices A and B
$A \oslash B$	The Khatri-Rao of the First Kind Product of matrices A and B
$A \circledcirc B$	The Fan Product of matrices A and B
$A \odot B$	The Tracy-Singh Product of matrices A and B
$A * B$	The Khatri-Rao Product of matrices A and B
$A \boxtimes B$	The Block Kronecker Product of matrices A and B
$A \ltimes B$	The Block-Pair Kronecker Product of matrices A and B
$A \boxdot B$	The Box Product of matrices A and B
$A \circledS B$	The Strong Kronecker Product of matrices A and B
$A \bullet B$	The Block Hadamard Product of matrices A and B
$A \Diamond B$	The Restricted Khatri- Rao (Box) Product of matrices A and B
$A(t) \circledast B(t)$	The Convolution Product of matrix functions $A(t)$ and $B(t)$
$A(t) \amalg B(t)$	The Kronecker Convolution Product of matrix functions $A(t)$ and $B(t)$
$A(t) \sqcap B(t)$	The Hadamard Convolution Product of matrix functions $A(t)$

	and $B(t)$
$A \oplus B$	The Kronecker Sum of matrices A and B
$A \uplus B$	The Hadamard Sum of matrices A and B
$A \nabla B$	The Tracy-Singh Sum of matrices A and B
$A \bowtie B$	The Khatri-Rao Rao Sum of matrices A and B
$A \boxplus B$	The Block Kronecker Sum of matrices A and B
$A \sim B$	The Arithmetic Mean of matrices A and B
$A ! B$	The Harmonic Mean of positive definite matrices A and B
$A \# B$	The Ando's Geometric Mean of positive definite matrices A and B
$A \underset{\alpha}{\#} B$	The α -Power Mean of positive definite matrices A and B
$A \triangleright B$	The Fiedler's and Ptak Geometric Mean of positive definite matrices A and B
$A \Bumpeq B$	The Operator Mean of positive definite matrices A and B
$\prod_{i=1}^{k} \underset{(w)}{\#} A_i$	The Weighted Geometric Mean of positive definite matrices A_i
$\prod_{i=1}^{k} \underset{(w)}{\sim} A_i$	The Weighted Arithmetic Mean of positive definite matrices A_i
$\prod_{i=1}^{k} \underset{(w)}{!} A_i$	The Weighted Harmonic Mean of positive definite matrices A_i
LSP	Least-Squares Problems
WLSP	Weighted Least-Squares Problems
SVD	Singular Value Decomposition
WSVD	Weighted Singular Value Decomposition
MPI	Moore-Penrose Inverse

WMPI	Weighted Moore-Penrose Inverse
DI	Drazin Inverse
WDI	Weighted Drazin Inverse
GI	Group Inverse
GAS	General Algebraic Structure
WMSN	Weighted Matrix Spectral Norm
WMPN	Weighted Matrix Schatten p-Norms
WVSP	Weighted Vector Spectral Norm
AM	Harmonic Mean
GM	Geometric Mean
OM	Operator Means
AGM	Ando's Geometric Mean
$\alpha-$PM	α-Power Means
WAM	Weighted Arithmetic Mean
WHM	Weighted Harmonic Mean
WGM	Weighted Geometric Mean
WOM	Weighted Operator Means
Wα-PM	Weighted α-Power Means