Implementation of a Reducing Algorithm for Differential-Algebraic Systems in Maple

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Abstract: This paper discusses the implementation of a reduction algorithm for differential-algebraic systems in Maple. Using the proposed Maple package, the given system of differential-algebraic equations can be transformed into another simpler system having the same properties. Sample computations are presented to illustrate the proposed algorithm. This algorithm will be helpful to implement this in commercial software packages such as Mathematica, Matlab, Singular, SCIlab etc.

Keywords: Differential-algebraic systems, Reduction algorithms, Elementary row-column operations.

1 Introduction

A first-order matrix differential system can be represented as

$$\mathscr{A}(z)Du(z) + \mathscr{B}(z)u(z) = f(z), \qquad (1)$$

where z is a complex variable, $\mathscr{A}(z), \mathscr{B}(z)$ are $m \times n$ matrices of analytic functions, $f(z)$ is an m-dimensional vector of analytic functions, $u(z)$ is an n-dimensional unknown vector to be determined and $D = \frac{d}{dz}$ is a differential operator. In operator notations, the first-order matrix differential system (1) can be represented by an equation of the form

$$Lu = f,$$

where $L = \mathscr{A}D + \mathscr{B}$ is a matrix differential operator. If $m = n$ and $\mathscr{A}$ is regular (i.e., $\det(\mathscr{A}) \neq 0$), then the system (1) is called a *system of linear ordinary differential equations* or linear differential system (LDS) [5–10, 12, 14–16]. If $\mathscr{A} \equiv 0$, then the system (1) becomes a *purely algebraic system* and there are several methods available in the literature to find all possible solutions. If $m \neq n$ or $\mathscr{A}$ is singular matrix, then the system (1) turns out to be a *system of differential-algebraic equations* or simply, differential-algebraic system (DAS). Differential-algebraic system is a composed system of ordinary differential equations coupled with purely

algebraic equations, hence DAS differ from LDS in many aspects [1–4, 11]. This paper mainly focused on DAS with some necessary conditions. In the literature, there are many algorithms available for solving DAS as well as BVPs with maple implementations, for example see, [1, 2, 4–10, 12–18].

The aim of this paper is to discuss about the implementation of a reduction algorithm. More details about the proposed reduction algorithm for DAS are available in [12] and a part of this work has been presented in [19, 20]. In the proposed algorithm, the given DAS reduces into another simpler and equivalent system where we can easily apply the classical theory of differential equations.

2 Maple Package, `DAS_Reduction`, for System of DAEs

Now the Maple implementation of the proposed algorithm is presented by creating different data types. The implemented Maple package, `DAS_Reduction`, of the proposed algorithm is provided with Maple worksheet at www.srinivasaraothota.webs.com/research. Using the Maple package, one can obtain the two unimodular matrices S, T and the reduced DAS of the given system. In Maple implementation, x is complex variable and $D = \frac{d}{dz}$ is the differential operator.

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2.1 Pseudo-Code [12]

- Input: Coefficient Matrices A, B of a given DAS.
- Output: Two unimodular Matrices S, T and the reduced DAS.

1: $A, B \leftarrow$ coefficient matricies
2: $n \leftarrow$ size of A
3: $LNS\_A \leftarrow$ left null space of A
4: $Sa \leftarrow$ identity matrix with LNS_A
5: $Aa \leftarrow Sa.A$, $Ba \leftarrow Sa.B$
6: $LNS\_Ba \leftarrow$ left null space of Ba
7: $Sb \leftarrow$ identity matrix with LNS_Ba
8: $A1 \leftarrow Sb.Aa$, $B1 \leftarrow Sb.Ba$, $S1 \leftarrow Sb.Sa$
9: $RNS\_A1 \leftarrow$ right null space of $A1$
10: $Ta \leftarrow$ identity matrix with RNS_A1
11: $A1a \leftarrow A1.Ta$, $B1a \leftarrow B1.Ta$
12: $RNS\_B1a \leftarrow$ right null space of $B1a$
13: $Tb \leftarrow$ identity matrix with LNS_B1a
14: $A2 \leftarrow A1a.Tb$, $B2 \leftarrow B1a.Tb$, $T1 \leftarrow Ta.Tb$
15: $P \leftarrow$ left elementary matrix
16: $Q \leftarrow$ right elementary matrix
17: $S \leftarrow P.S1$, $T \leftarrow T1.Q$

2.2 Maple Code

```
with(MatrixPolynomialAlgebra):
with(LinearAlgebra):

DAS_Reduction:=proc(A,B)
local n, L, r, k, Id, transpose_A,
NS_A, Id_part1, NS_part1, Sa, La, Aa,
transpose_Ba, NS_Ba, Id_part2, NS_part2,
Sb, L1, A1, B1, S1, NS_A1,   Id_part3,
NS_part3, Ta, L1a, A1a, B1a, NS_B1a,
Id_part4, NS_part4, Tb, L2, A2, B2,
T1, P_A, Q_A, S, T, Ba, L_reduced;

n := LinearAlgebra:-RowDimension(A);
L := A*delta+B;
n := LinearAlgebra:-RowDimension(A);
r := MTM:-rank(A);
k := MTM:-rank(B);

Id := LinearAlgebra:-IdentityMatrix(n);
transpose_A:=LinearAlgebra:-
    Transpose(A);

NS_A:=LinearAlgebra:-
    NullSpace(transpose_A);
Id_part1:=LinearAlgebra:-Transpose
   (Matrix (LinearAlgebra: Transpose~
   (convert~([seq(Id[i],
   i=1..n-nops(NS_A))], Matrix)))));
NS_part1:=LinearAlgebra:-Transpose
   (Matrix(convert~([seq(NS_A[i],
   i=1..nops(NS_A))],Matrix)));
```

```
Sa:=convert(linalg:-blockmatrix
    (2,1,[Id_part1,NS_part1]), Matrix);
La := simplify~(Sa.L);
Aa:=seq(Coeff(La, delta,i),i=0..1)[2];
Ba:=seq(Coeff(La, delta,i),i=0..1)[1];
transpose_Ba:=LinearAlgebra:-
    Transpose(Ba);
NS_Ba:=LinearAlgebra:-NullSpace
    (transpose_Ba);
Id_part2:=LinearAlgebra:-Transpose
   (Matrix('~'[LinearAlgebra:
   Transpose]('~'[convert]
   ([seq(Id[i],i=nops(NS_Ba)..n-1)],
   Matrix)))));
NS_part2:=LinearAlgebra:-Transpose
   (Matrix('~'[convert]([seq
   (NS_Ba[i],i=1..nops(NS_Ba))],
   Matrix)));
Sb:=convert(linalg:-blockmatrix
    (2,1,[NS_part2,Id_part2]), Matrix);
L1 := simplify~(Sb.La);
A1:=seq(Coeff(L1,delta, i),i=0..1)[2];
B1:=seq(Coeff(L1, delta,i),i=0..1)[1];
S1 := simplify~(Sb.Sa);
NS_A1 := simplify~(LinearAlgebra:-
   NullSpace(A1));
Id_part3:=Matrix('~'[LinearAlgebra:-
   Transpose]('~'[convert]
   ([seq(Id[i],i=1..n-nops(NS_A1))],
   Matrix)));
NS_part3:=Matrix('~'[convert]
   ([seq(NS_A1[i],i=1..nops(NS_A1))],
   Matrix));
Ta:=convert(linalg:-blockmatrix
   (1,2,[Id_part3,NS_part3]), Matrix);
L1a := simplify~(L1.Ta);
A1a:=seq(Coeff(L1a,delta,i),i=0..1)[2];
B1a:=seq(Coeff(L1a,delta,i),i=0..1)[1];
NS_B1a := simplify~(LinearAlgebra:-
   NullSpace(B1a));
Id_part4:=Matrix('~'[LinearAlgebra:-
   Transpose]('~'[convert] ([seq(Id[i],
   i = nops(NS_B1a) .. n-1)], Matrix)));
NS_part4:=Matrix('~'[convert]
   ([seq(NS_B1a[i],i=1..nops(NS_B1a))],
   Matrix));
Tb:=convert(linalg:-blockmatrix
   (1,2,[NS_part4,Id_part4]), Matrix);

L2 := simplify~(L1a.Tb);
A2:=seq(Coeff(L2,delta,i),i=0..1)[2];
B2:=seq(Coeff(L2,delta,i),i=0..1)[1];
T1 := simplify~(Ta.Tb);

P_A:=convert(linalg:-submatrix
   (Student[LinearAlgebra]:-
   ReducedRowEchelonForm
   (<A2|Id>), 1 .. n, n+1 .. 2*n),
```

265

```
    Matrix);
Q_A:=LinearAlgebra:-Transpose(convert
    (linalg:-submatrix(Student
    [LinearAlgebra]:-
    ReducedRowEchelonForm
    (<LinearAlgebra:-
    Transpose(A2)|Id>),
    1..n,n+1..2*n),Matrix));
S := simplify~(P_A.S1);
T := simplify~(T1.Q_A);
L_reduced:=simplify~(S.L.T);
return S,T,L_reduced;
end proc:
```

3 Sample Computations

Example 1 *Consider a matrix differential operator of DAS*

$$L = \mathscr{A}D + \mathscr{B}$$

$$= \begin{pmatrix} D+z & 0 & D-1 & (2-z)D+1 & D \\ 0 & zD+2 & D+1 & 0 & 1 \\ 2D+z & zD-2 & 3D-2 & (4-2z)D+1 & 2D-1 \\ D+\frac{z}{2} & -2zD+z & -D+\frac{1}{2} & (2-z)D+\frac{1}{2} & D+1 \\ 3D+2z & zD+2 & 4D-1 & (6-3z)D+2 & 3D+1 \end{pmatrix},$$

where $\mathscr{A} = \begin{pmatrix} 1 & 0 & 1 & 2-z & 1 \\ 0 & z & 1 & 0 & 0 \\ 2 & z & 3 & 4-2z & 2 \\ 1 & -2z & -1 & 2-z & 1 \\ 3 & z & 4 & 6-3z & 3 \end{pmatrix}$, *and*

$$\mathscr{B} = \begin{pmatrix} z & 0 & -1 & 1 & 0 \\ 0 & 2 & 1 & 0 & 1 \\ z & -2 & -2 & 1 & -1 \\ \frac{z}{2} & 2 & \frac{1}{2} & \frac{1}{2} & 1 \\ 2z & 2 & -1 & 2 & 1 \end{pmatrix}.$$

Applying the proposed algorithm, one can obtain the reduced matrix operator as follows

$$\widetilde{L}$$

$$= \begin{pmatrix} -\frac{1}{12}(3z-5+\frac{1}{z})D & -\frac{1}{6}(z-3-\frac{1}{z})D & 0 & 0 & 0 \\ \frac{1}{12}(1+3z)D & \frac{1}{6}(z-1)D & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z & -z^2+2z-1 \\ 0 & 0 & 0 & \frac{z}{2}+3 & -\frac{1}{2}z^2+z-\frac{1}{2} \end{pmatrix}.$$

Now we apply the Maple implementation to the given system of DAS (1).

```
> A := Matrix([[1,0,1,2-x,1],[0,x,1,0,0],
    [2,x,3,4-2*x,2],[1,-2*x,-1,2-x,1],
    [3,x,4,6-3*x,3]]);

> B := Matrix([[x,0,-1,1,0],[0,2,1,0,1],
    [x,-2,-2,1,-1],[(1/2)*x,2,1/2,1/2,1],
    [2*x,2,-1,2,1]]);
```

$$A := \begin{bmatrix} 1 & 0 & 1 & 2-x & 1 \\ 0 & x & 1 & 0 & 0 \\ 2 & x & 3 & 4-2x & 2 \\ 1 & -2x & -1 & 2-x & 1 \\ 3 & x & 4 & 6-3x & 3 \end{bmatrix}$$

$$B := \begin{bmatrix} x & 0 & -1 & 1 & 0 \\ 0 & 2 & 1 & 0 & 1 \\ x & -2 & -2 & 1 & -1 \\ \frac{x}{2} & 2 & \frac{1}{2} & \frac{1}{2} & 1 \\ 2x & 2 & -1 & 2 & 1 \end{bmatrix}$$

```
> S, T, Lreduced := DAS_Reduction(A,B);
```

$$\begin{bmatrix} \frac{1}{9}\frac{6x^2-11x+1}{x} & \frac{1}{9}\frac{3x^2-7x-1}{x} & 0 & \frac{2}{9}\frac{x+1}{x} & -\frac{1}{9}\frac{3x^2-5x+1}{x} \\ -\frac{1}{9}-\frac{2}{3}x & \frac{1}{9}-\frac{1}{3}x & 0 & -\frac{2}{9} & \frac{1}{9}+\frac{1}{3}x \\ \frac{4}{3} & \frac{5}{3} & 1 & \frac{2}{3} & -\frac{4}{3} \\ -3 & -1 & 0 & 0 & 1 \\ -1 & 2 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{4}\frac{2x-3}{(x^2-3x+2)(x-1)} & \frac{1}{2}\frac{2x-5}{x^3-4x^2+5x-2} & -\frac{x^2-2x-2}{x^3-4x^2+5x-2} & -1 & x-2 \\ \frac{1}{8(x-2)} & -\frac{1}{4(x-2)} & \frac{1}{x-2} & 0 & 0 \\ -\frac{1}{4(x-2)} & \frac{1}{2(x-2)} & -\frac{x}{x-2} & 0 & 0 \\ -\frac{1}{4}\frac{3x^2-5x+1}{(x^2-3x+2)(x-1)} & -\frac{1}{2}\frac{x^2-3x-1}{x^3-4x^2+5x-2} & -\frac{3x}{x^3-4x^2+5x-2} & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{1}{12}(3x-5+\frac{1}{x})\delta & -\frac{1}{6}(x-3-\frac{1}{x})\delta & 0 & 0 & 0 \\ \frac{1}{12}(1+3x)\delta & \frac{1}{6}(x-1)\delta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & -x^2+2x-1 \\ 0 & 0 & 0 & \frac{1}{2}x+3 & -\frac{1}{2}x^2+x-\frac{1}{2} \end{bmatrix}$$

From Maple implementation, we have two unimodular matrices $S, T \in GL_n(\mathbb{K}[[z]])$,

$$S = \begin{pmatrix} \frac{1}{9}\frac{6z^2-11z+1}{z} & \frac{1}{9}\frac{3z^2-7z-1}{z} & 0 & \frac{2}{9}\frac{z+1}{z} & -\frac{1}{9}\frac{3z^2-5z+1}{z} \\ -\frac{1}{9}-\frac{2}{3}z & \frac{1}{9}-\frac{1}{3}z & 0 & -\frac{2}{9} & \frac{1}{9}+\frac{1}{3}x \\ \frac{4}{3} & \frac{5}{3} & 1 & \frac{2}{3} & -\frac{4}{3} \\ -3 & -1 & 0 & 0 & 1 \\ -1 & 2 & 0 & 1 & 0 \end{pmatrix}$$

$$T = \begin{pmatrix} \frac{1}{4}\frac{2z-3}{(z^2-3z+2)(z-1)} & \frac{1}{2}\frac{2z-5}{z^3-4z^2+5z-2} & -\frac{z^2-2z-2}{z^3-4z^2+5z-2} & -1 & z-2 \\ \frac{1}{8(z-2)} & -\frac{1}{4(z-2)} & \frac{1}{z-2} & 0 & 0 \\ -\frac{1}{4(z-2)} & \frac{1}{2(z-2)} & -\frac{z}{z-2} & 0 & 0 \\ -\frac{1}{4}\frac{3z^2-5z+1}{(z^2-3z+2)(z-1)} & -\frac{1}{2}\frac{z^2-3z-1}{z^3-4z^2+5z-2} & -\frac{3z}{z^3-4z^2+5z-2} & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

and the reduced matrix differential operator is

$$\widetilde{L} = \begin{pmatrix} -\frac{1}{12}(3z-5+\frac{1}{z})D & -\frac{1}{6}(z-3-\frac{1}{z})D & 0 & 0 & 0 \\ \frac{1}{12}(1+3z)D & \frac{1}{6}(z-1)D & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z & -z^2+2z-1 \\ 0 & 0 & 0 & \frac{1}{2}z+3 & -\frac{1}{2}z^2+z-\frac{1}{2} \end{pmatrix}$$

4 Conclusion

In this paper, we discussed the Maple implementation of a reduction algorithm for differential-algebraic systems, `DAS_Reduction`. Sample computations are presented to illustrate the proposed algorithm. This algorithm will be helpful to implement this in commercial software packages such as Mathematica, Matlab, Singular, SCIlab etc. Some numerical examples are presented to demonstrate the proposed algorithm. This package is provided at https://srinivasaraothota.webs.com/research with example worksheets.

Competing interests

The authors declare that they have no competing interests.

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