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Stability Conditions for an Alternated Grid in Space and Time

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ABSTRAK

Keadaan kestabilan kekisi bertindan di dalam ruang dan masa telah dirumuskan. Grid yang sedang dirujuk ini, dikenali sebagai grid Eliassen (Eliassen 1956). Telah ditunjukkan bahawa keadaan kestabilan di dalam persamaan gelombang perairan cetek untuk kekisi jenis ini adalah keadaan kestabilan yang serupa untuk grid tidak bertindak kekisi Arakawa's B dan C. Bila melaksanakan skim 'leapfrog' di dalam grid bertindan mengikut ruang dan masa, didapati tiada mempunyai mod pengiraan. Tiada pengiraan purata diperlukan untuk mengira istilah Coriolis (gelombang graviti) sebagaimana yang diperlukan di dalam grid Arakawa's B dan C. Di samping itu, penggunaan grid Eliassen menjimatkan separuh masa pengiraan yang diperlukan dalam grid Arakawa's B atau C (Mesinger and Arakawa 1976). Oleh itu, adalah lebih menguntungkan dengan mengguna grid silih berganti di dalam ruang dan masa.

ABSTRACT

The stability conditions of a staggered lattice in space and time are derived. The grid used is known as the Eliassen grid (Eliassen 1956). It is shown that the stability conditions of the shallow water wave equations, for this type of lattice, have essentially the same stability condition as the unstaggered grid and Arakawa's B and C lattice. Upon implementation of a leapfrog scheme in a staggered grid in space and time, there will be no computational modes. No smoothing is needed to compute the Coriolis (gravity wave) terms as required in Arakawa's C (B) grid. Furthermore, the usage of an Eliassen grid halves the computation time required in Arawaka's B or C grid (Mesinger and Arakawa 1976). Therefore, there are fundamental advantages for the usage of an alternated grid in space and time.

Keywords: numerical stability, lattice, inertia - gravity waves, leapfrog

INTRODUCTION

Camerlengo and O' Brien (1980), using a staggered grid in space and time (*Fig. 1*), tested different sets of open boundary conditions for rotating (geophysical) fluids. In computing (with this type of grid) the Coriolis terms, quite an amount of averaging is avoided compared to the computation of these same terms by the widely-used Arakawa's C lattice.

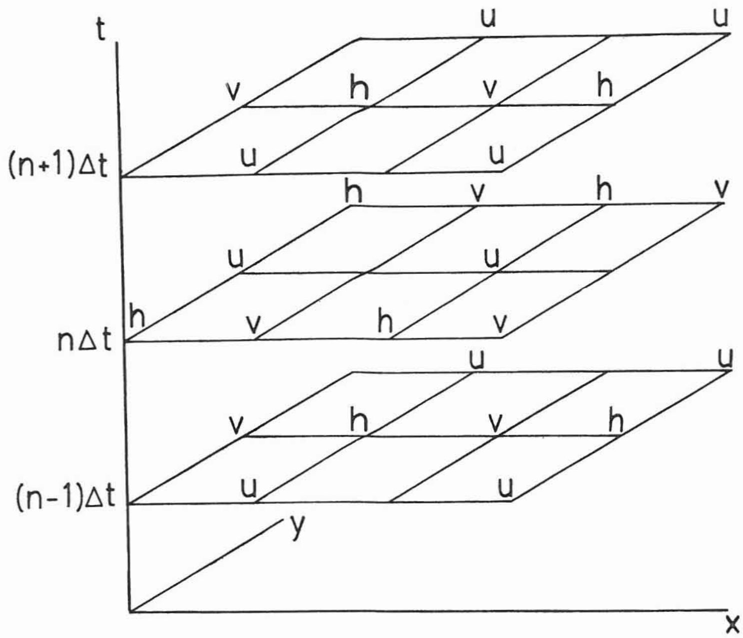

Fig. 1. The analysed staggered lattice in space and time

Upon using a leapfrog scheme, two solutions per time step are obtained. The first one (the physical mode) resembles the true (real) solution. The second one (the computational mode) is a spurious one. It travels in an opposite direction to the physical mode. Furthermore, its amplitude changes sign at every time level (Haltiner and Williams 1980).

To eliminate the computational mode, a forward (or backward) time scheme is usually implemented (in the numerical integration) at every N (odd) time step. The time differencing scheme most commonly used in meteorology and physical oceanography modelling is the Matsuno scheme (Matsuno 1966). However, (due to the fact that the computational mode will reappear as soon as the leapfrog scheme is used) the implementation of the first order time differencing scheme does not seem to be the ideal solution. A more radical approach needs to be taken.

In using a staggered grid in time and space, the computational mode is avoided, as the variables at alternate time levels are missing.

The aim of this study is to gain some understanding of the numerical stability of the Eliassen grid. Following O'Brien (1986), stability analysis of a sequence of problems, for the Eliassen grid, leading to the linear shallow water wave equations, is considered. A series of analytical studies is conducted. The linear stability technique developed by von Neumann is used (Charney *et al.* 1950).

NUMERICAL STABILITY PROBLEM

The linear stability condition of the finite difference schemes is determined by the phase speed of the gravity waves, C, i.e., by the velocity of the fastest travelling waves. The general stability condition is :

$$C\frac{\Delta t}{\Delta x} \leq 0 \quad (1)$$

Stability Analysis for One-dimensional Gravity Wave

Let us consider the following set of partial differential equations:

$$\frac{\partial u}{\partial t} = -g\frac{\partial h}{\partial x}$$
$$\frac{\partial h}{\partial t} = -H\frac{\partial u}{\partial x} \qquad (1)$$

where g represents the earth's gravity; H, the mean sea-level depth; u and v, the velocity components in the x (east-west) and y (north-south) directions, respectively; and h, the free surface elevation.

A second order, centred, in space and time finite difference scheme, is considered (*Fig. 2*). Primed quantities represent variables at odd time levels. We obtain:

$$u'^{n+1}_{m,l} = u'^{n-1}_{m,l} - g\gamma_x\left(h^n_{m=1,l} - h^n_{m-1,l}\right)$$
$$h'^{n+1}_{m,l+1} = h'^{n-1}_{m,l+1} - \gamma_x H\left(u^n_{m+1,l=1} - u^n_{m-1,1+1}\right) \qquad (2)$$

where γ_x is equal to $\Delta t/\Delta x$; the superscript n, denotes the time level; the subscripts (m,l), the mesh of discrete points in the x and y directions, respectively; Δx and Δy, the grid size between grid points in the x and y directions, respectively; and Δt, the time step increment.

We define:

$$C^2 = gH,$$
$$\theta = \mu\Delta x,$$
$$\sigma = v\Delta y,$$

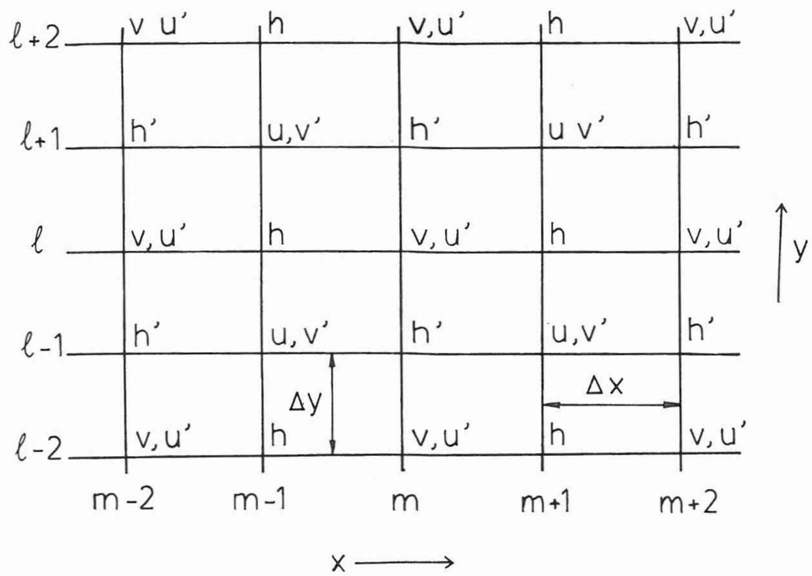

Fig. 2. Space representation of the alternated grid in space and time. Primed quantities represent variables at odd time levels

where μ and ν are the latitudinal and longitudinal wave numbers, respectively. Let us assume that:

$$P^n_{m,l} = P_n \exp\left(i\,\mu\, m\Delta x\right) \exp\left(i\,\nu\, 1\Delta y\right) \tag{3}$$

where P = (u, h). It is convenient to drop the primes. If we substitute equation (3) in (2), we obtain:

$$\begin{aligned} u_{n+1} &= u_{n-1} - g\,\gamma_x \left(2i \sin\theta\right) h_n \\ h_{n+1} &= h_{n-1} - H\,\gamma_x \left(2i \sin\theta\right) u_n \end{aligned} \tag{4}$$

If an amplification factor, Z, exists such that

$$P_{n+2} = Z \quad P_n \tag{5}$$

equation (4) can be rewritten as:

$$\begin{aligned} L_1\, u_n + L_2\, h_n &= 0 \\ L_1\, h_n + L_3\, u_n &= 0 \end{aligned} \tag{6}$$

The operators L_1, L_2, and L_3 are defined as:

$$L_1 = Z^{1/2} - Z^{-1/2}$$
$$L_2 = 2\ i\ g\ \gamma_x\ \sin\ \theta$$
$$L_3 = 2\ i\ H\ \gamma_x\ \sin\ \theta$$

For the system (6) to have a determinate solution:

$$\begin{vmatrix} L_1 & L_2 \\ L_3 & L_1 \end{vmatrix} = 0 \qquad (7)$$

The second order equation for Z is:

$$Z^2 - 2\left(1 - 2\,C^2\,\gamma_x^2\,\sin^2\theta\right)Z + 1 = 0 \qquad (8)$$

If the term $\left[1 - \left(1 - 2C^2\,\gamma_x^2\sin^2\theta\right)^2\right]$ is positive, then the amplification factor will be equal to one. Thus, the stability analysis shows that the chosen finite difference scheme is neutral. Namely that $|Z| = 1$. This instance will hold true if, and only if:

$$C^2\ \gamma_x^2\ \sin^2\theta \leq 1$$

That is:

$$\frac{C\Delta t}{\Delta x} \leq 1$$

which is the classical Courant-Friedrichs-Levy (CF) condition for computational stability.

Stability Conditions for the Inertial-gravity Waves

We consider the following system of partial differential equations:

$$\frac{\partial u}{\partial t} = f\ v - g\ \frac{\partial h}{\partial x}$$

$$\frac{\partial v}{\partial t} = - f\ u \tag{9}$$

$$\frac{\partial h}{\partial t} = - H\ \frac{\partial u}{\partial x}$$

We assume that:

$$Q^{n}_{m},l = Q_n\ \exp\left(i\ \mu\ m\Delta x\right)\ \exp\left(i\ \nu\ 1\Delta y\right) \tag{10}$$

where $Q = (u, v, h)$ and an amplification factor, Z, exists such that:

$$Q_{n+2} = ZQ_n$$

A leapfrog in time and second order space difference scheme is considered. We obtain:

$$\begin{aligned} u_{n+1} &= u_{n-1} + 2\,\Delta t\ f\ v_n - 2\ i\ g\ \gamma_x\ \sin\theta\ h_n \\ v_{n+1} &= v_{n-1} - 2\ \Delta t\ f\ u_n \\ h_{n+1} &= h_{n-1} - 2\ i\ H\ \gamma_x\ \sin\theta\ u_n \end{aligned} \tag{11}$$

Equation (11) may be rewritten as:

$$\begin{aligned} L_1\ u_n - L_4\ v_n + L_2\ h_n &= 0 \\ L_1\ v_n + L_4\ u_n &= 0 \\ L_1\ h_n + L_3\ u_n &= 0 \end{aligned}$$

where $L_4 = 2\ f\ \Delta t$.

Following the same procedure as in the previous section, yields:

$$\begin{aligned} Z &= [1 - 2\ (f\Delta t)^2 - 2\ C^2\ \gamma_x^2\ \sin^2\theta] \pm \\ & i\ [1 - \left(1 - 2\ (f\Delta t)^2 - 2\ C^2\ \gamma_x^2\ \sin^2\theta\right)^2]^{1/2} \end{aligned} \tag{12}$$

If the term under the radical sign is positive, the stability analysis shows that the absolute value of the amplification factor is equal to one. We will have a neutral stability condition. This will require that:

$$(f\Delta t)^2 + C^2 \gamma_x^2 \sin^2 \theta \leq 1 \tag{13}$$

That is $$C \frac{\Delta t}{\Delta x} \leq [1 - (f\Delta t)^2]^{1/2}$$

Adamec and O'Brien (1978), in their reduced-gravity model, used a Δt of the order 10^4 sec. At mid=latitudes $\left(f \approx 10^{-4} \text{ sec}^{-1}\right)$. the product $f \Delta t$ could be of order one. Thus, the stability condition (for reduced-gravity models) could be easily violated, if caution is not taken.

However, for barotropic (vertically integrated) models, Δt varies from 100 to 600 sec. The product $(f \Delta t)^2$ could be of the order 10^{-3}. Therefore, the term $(f \Delta t)^2$ does not represent a serious problem, since it is very small.

Stability Conditions for 2D Flow

Consider the linear shallow water wave equations at a constant latitude, i.e., f = constant. We will have:

$$\begin{aligned}
\frac{\partial u}{\partial t} &= f v - g \frac{\partial h}{\partial x} \\
\frac{\partial v}{\partial t} &= -f u - g \frac{\partial h}{\partial y} \\
\frac{\partial h}{\partial t} &= - H \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)
\end{aligned} \tag{14}$$

Upon using, as in the two previous sections, a centred in space and time difference scheme yields:

$$\begin{aligned}
u_{n+1} &= u_{n-1} + 2 \Delta t f v_n - 2 i g \gamma_x \sin \theta h_n \\
v_{n+1} &= v_{n-1} - 2 \Delta t f u_n - 2 i g \gamma_y \sin \sigma h_n \\
h_{n+1} &= h_{n+1} - 2 H i [\gamma_x \sin \theta u_n + \gamma_y \sin \sigma v_n]
\end{aligned} \tag{15}$$

where γ_y is equal to $\Delta t / \Delta y$.

The above system of equations, (15), may be rewritten as:

$$\begin{aligned} L_1\ u_n - L_4\ v_n + L_2\ h_n &= 0 \\ L_1\ v_n + L_4\ u_n + L_5\ h_n &= 0 \\ L_1\ h_n + L_3\ u_n + L_6\ v_n &= 0 \end{aligned} \tag{16}$$

where

$$L_5 = 2\ i\ g\ \gamma_y\ \sin\sigma$$
$$L_6 = 2\ i\ H\ \gamma_y\ \sin\sigma$$

For the set of equations (16) to have a unique solution, it follows that:

$$L_1\left(L_1^2 - L_2\ L_3 - L_5\ L_6 + L_4^2\right) = 0 \tag{17}$$

Following the same procedure as in the previous cases, yields:

$$Z^2 - 2\ \chi\ Z + 1 = 0 \tag{18}$$

where

$$\chi = 1 - [2\ (f\Delta t)^2 + 2C^2\left(\gamma_x^2\ \sin^2\theta + \gamma_y^2\ \sin^2\sigma\right)]$$

Neutral stability conditions will be obtained if:

$$(f\Delta t)^2 + C^2\left(\gamma_x^2\ \sin^2\theta + \gamma_y^2\ \sin^2\sigma\right) \leq 1 \tag{19}$$

Several cases are considered.

a) For $L_x = 2\Delta x$ and $L_y = 2\Delta y$, i.e. $\theta = \sigma = \pi$, where L_x and L_y are the wavelengths in the x and y directions, respectively, we obtain:

$$(f\,\Delta t)^2 \leq 1 \text{ for stability} \tag{20}$$

b) For $L_x = 4\Delta x$ and $L_y = 4\Delta y$, i.e., $\theta = \sigma = \pi/2$, yields:

$$(f\ \Delta t)^2 + C^2\left(\gamma_x^2 + \gamma_y^2\right) \leq 1 \tag{21}$$

If $\Delta x = \Delta y = \Delta$ the following relation holds:

$$C\frac{\Delta t}{\Delta} \leq \left[\frac{\left(1-(f\Delta t)^2\right)}{2}\right]^{1/2} \tag{22}$$

Close to the equator, $f \approx 0$, we obtain:

$$C\frac{\Delta t}{\Delta} \leq \frac{1}{\sqrt{2}} \tag{23}$$

The same stability condition is obtained for the two-dimensional unstaggered grid case.

c) For $L_x = 8\ \Delta x$ and $L_y = 8\ \Delta y$, we will have :

$$(f\Delta t)^2 + \frac{C^2}{2}\left(\gamma_x^2 + \gamma_y^2\right) \leq 1 \tag{24}$$

If $\Delta x = \Delta y = \Delta$, it may be obtained:

$$(f\Delta t)^2 + C^2\,\gamma_x^2 \leq 1 \tag{25}$$

Therefore,

$$C\frac{\Delta t}{\Delta} \leq \left[1 - (f\Delta t)^2\right]^{1/2} \tag{26}$$

Stability condition is larger than in the two previous cases. This is natural since this stability condition is for the 8 Δx wave.

Stability Condition for a 2D Gravity Wave with Advection.
We consider the following set of partial differential equations:

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} + B\frac{\partial u}{\partial y} = f\ v - g\ \frac{\partial h}{\partial y}$$
$$\frac{\partial v}{\partial t} + A\frac{\partial v}{\partial x} + B\frac{\partial v}{\partial y} = -\ f\ u - g\ \frac{\partial h}{\partial y} \qquad (27)$$
$$\frac{\partial h}{\partial t} + A\frac{\partial h}{\partial x} + B\frac{\partial h}{\partial y} = -H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

Use of a centred in space and time differencing scheme yields:

$$u_{n+1} = u_{n+1} + 2\,\Delta t\, f\, v_n - 2\, i\, g\, \gamma_x \sin\theta\, h_n$$
$$-2\, i\left[A\gamma_x \sin\theta\cos\sigma + B\gamma_y \cos\theta\sin\sigma\right] u_n$$
$$v_{n+1} = v_{n-1} - 2\,\Delta t\, f\, u_n - 2\, i\, g\, \gamma_y \sin\sigma\, h_n$$
$$-\,2\, i\left[A\gamma_x \sin\theta\cos\sigma + B\gamma_y \cos\theta\sin\sigma\right] v_n \qquad (28)$$
$$h_{n+1} = h_{n-1} - 2\, H\, i\left[\gamma_x \sin\theta\, u_n + \gamma_y \sin\sigma\, v_n\right]$$
$$-\,2\, i\left[A\, \gamma_y \sin\theta\cos\sigma + B\gamma_y \cos\theta\sin\sigma\right] h_n$$

It is convenient to redefine the amplification factor, Z, in the following manner:

$$Q_{n+1} = Z\, Q_n \qquad (29)$$

The operator L_1 will change due to the incorporation of the advective terms. Its new value is:

$$L_1\, Z - Z^{-1} + 2\, i\, [A\gamma_x \sin\theta\cos\sigma + B\,\gamma_y \cos\theta\sin\sigma] \qquad (30)$$

The value of the other operators will remain unchanged.

For the set of equations (28) to have a unique solution, it must satisfy condition (17).

Upon substitution of the operators L_1,............, L_6 in equation (17), and after some mathematical manipulations, a second and a fourth order equation will be obtained. The second order equation yields:

$$Z^2 + 2\,i\,F\;Z - 1 = 0 \tag{31}$$

where:

$$F = A\,\gamma_x \sin\theta \cos\sigma + B\gamma_y \cos\theta \sin\sigma \tag{32}$$

Equation (31) represents the advection equation. Following the same procedures as in previous sections, a neutral stability solution is obtained if:

$$|F| \le 1$$

In other words:

$$\left|A\,\gamma_x \sin\theta \cos\sigma + B\,\gamma_y \cos\theta \sin\sigma\right| \le 1 \tag{33}$$

On the other hand, the fourth order equation is:

$$Z^4 + 4\,i\,F\,Z^3 - 2\,G\,Z^2 - 4\,i\,F\,Z + 1 = 0 \tag{34}$$

where

$$G = 2\,F^2 + 1 - 2\,(f\,\Delta t)^2 - 2g\,H\,(\gamma_x^2 \sin^2\theta + \gamma_y^2 \sin^2\sigma) \tag{35}$$

The reader can verify that equation (34) may be factorized in two parts. Namely, that:

$$\begin{aligned} &\left[Z^2 + \left(2\,i\,F + \left(2\,G - 4\,F^2 - 2\right)^{1/2}\right)Z - 1\right] * \\ &\left[Z^2 + \left(2\,i\,F - \left(2\,G - 4\,F^2 - 2\right)^{1/2}\right)Z - 1\right] = 0 \end{aligned} \tag{36}$$

Using either of the quadratic factors, after some algebraic manipulations, a general condition for stability is obtained:

$$[F + (g\, H(\gamma_x^2 \sin^2 \theta + \gamma_y^2 \sin^2 \sigma) + (f\ \Delta t)^2)^{1/2}]^2 \leq 1 \qquad (37)$$

For $L_x = 2\Delta x\,(4\,\Delta x)$ and $L_y = 2\Delta y\,(4\,\Delta y)$, results similar to those in sections 3.a and 3.b are obtained;

However, for $L_x = 8\Delta x$ and $L_y = 8\ \Delta y$; i.e., $\theta = \sigma = \pi / 4$; $\Delta x = \Delta y = \Delta$; $|A| = |B|$; $\gamma_x = \gamma_y = \gamma$; and $|U| = \left(A^2 + B^2\right)^{1/2}$ yields :

$$\left[\sqrt{2}\,|U|\ \gamma + 2\left(C\,\gamma^2 + (f\Delta t)^2\right)^{1/2}\right]^2 \leq 4$$

This can be written as:

$$\left[\frac{|U|\,\gamma}{\sqrt{2}} + C\,\gamma^2 + (f\Delta t)^2\right]^{1/2} \leq 1 \qquad (38)$$

Advective terms are computed over a $4\,\Delta$distance. Therefore, the stability condition, due to the advective terms, is twice the value for the unstaggered grid. As expected, the CFL condition for the Coriolis and gravity terms are in perfect agreement with stability condition obtained in section 3.c.

We have conducted a similar stability analysis for variables at even time levels. The results were identical.

CONCLUSION

In using a leapfrog scheme in an staggered grid in space and time, there are no computational modes. Furthermore, the excessive smoothing needed to compute the Coriolis (gravity wave) terms in the C (B) lattice is avoided.

It is shown that the stability conditions for the shallow water wave equations, using an Eliassen grid, are practically the same as for the unstaggered (O' Brien 1986). As expected, the truncation error remains unchanged.

The usage of an Eliassen grid saves by half the computation time required either on an Arawaka's B or C grid (Mesinger and Arakawa 1976: 53). Furthermore, the computational modes are nonexistent. Therefore, there are fundamental advantages in the usage of an alternated grid in space and time.

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