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On the Stability of the Basic East-West Velocity Flow

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ABSTRAK

Ketidakstabilan menurut perbedaan dari kecepatan keadaan dasar U dijelaskan. Itu dijelaskan bahawa ketidakstabilan terjadi dimana $\partial^2 U/\partial y^2$ mengalikan dengan U secara negattif.

ABSTRACT

Instability due to the horizontal shear of the basic state velocity, U, is addressed. It is corroborated that instability happens whenever $\partial^2 U/\partial y^2$ covariates negatively with U.

Keywords: stabiliting effect, barotropic, instability, basic state velocity, perturbation

INTRODUCTION

Assume a westerly jet as shown in *Fig. 1*. U_o may be viewed as a reservoir in such a way that a small loss of U_o is enough to start the perturbation. It is an infinitesimal perturbation (*Fig. 2*).

Therefore, if there is a divergence of flux of momentum, energy is removed from the basic flow to the eddies, which will tend to grow. This may be done by displacing the parcel from the faster part of the flow to the slower part of the flow (*Fig. 3*).

The actual observed case is the opposite. Energy is removed from the eddies to the basic flow. This may be thought of as taking the faster particles of the slow part and interacting them with the slower particles of the quicker flow (*Fig. 4*).

If the eddies grow, there is an unstable situation. If they decay, it is stable. Otherwise, it is neutral.

THE PROBLEM

Let us consider a homogeneous, incompressible fluid with $U_o = U_o$ (y) only, where U represents the basic state east-west (zonal) velocity flow. The two-dimensional perturbation equations are:

$$\partial u/\partial t + U_o \partial u/\partial x + v \partial U_o/\partial y - f v + (1/\rho) \partial p/\partial x = 0 \quad (1)$$

$$\partial v/\partial t + U_o \partial v/\partial x + f u + (1/\rho) \partial p/\partial y = 0 \quad (2)$$

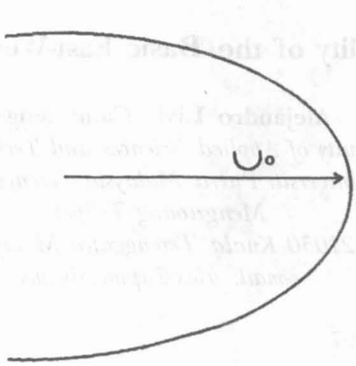

Fig 1. Idealized westerly jet showing the basic east-west velocity

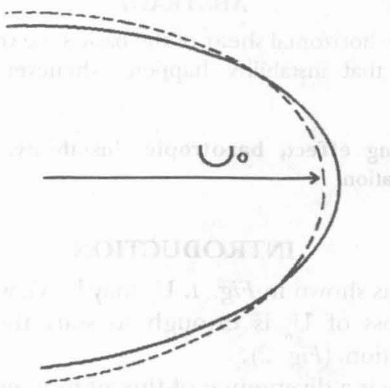

Fig 2. Infinitesimal perturbation of the basic state east-west velocity

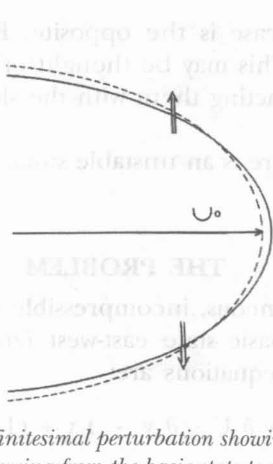

Fig 3. Infinitesimal perturbation showing energy flowing from the basic state to the eddies

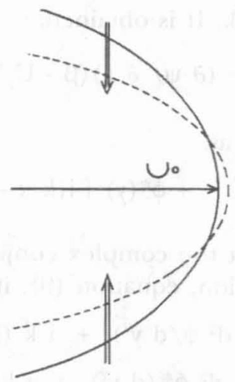

Fig 4. Infinitesimal perturbation showing energy flowing from the eddies to the basic state

where:

$$U_o = -(\rho_o f)^{-1} \partial P/\partial y \quad (3)$$

and u and v represent the perturbation equations.

Upon multiplication of equations (1) and (2) by u and v, respectively, the energy equation is obtained:

$$\partial (u^2/2 + v^2/2)/\partial t + U_o \partial (u^2/2 + v^2/2)\partial x + u v \partial U_o/\partial y + u \partial\pi/\partial x + v\partial\pi/\partial y = 0 \quad (4)$$

where $\pi = p/\rho$.

Upon integration over a closed volume, it yields:

$$\partial/\partial t\int\left(u^2/2+v^2/2\right) dm = -\int(u v)\,\partial U_O/\partial y\, d m = -\int\left(V.\nabla_H\pi\right) d m \quad (5)$$

where:

$$\int\left(V.\nabla_H\pi\right) d m = \int V_H\cdot(\nabla\pi)\, d m - \int\pi V_H\cdot(V)\, d m = 0 \quad (6)$$

It is convenient to define a space average of the form:

$$\overline{(\ )} = \left[\int(\ )\, dx\, dy\right]/\left[\int dx\, dy\right] \quad (7)$$

Therefore,

$$\partial K_E/\partial t = \int\overline{u'\, v'}\,\partial U_0/\partial y\, d m = \int U_0\,\overline{\partial(u'\, v')}/\partial y\, d m \quad (8)$$

Without specifying the profile, the vorticity equation is:

$$\partial \varsigma/\partial t + U_o \partial \varsigma/\partial x - v\, \partial^2 U_o/\partial y^2 + \beta v = 0 \quad (9)$$

where $\varsigma = \nabla^2 \psi$ and $c = \sigma/k$. It is obtained:

$$\{\partial / \partial t + U_o \partial / \partial x \; \nabla^2 \psi + (\partial \psi / \partial x)(\beta - U_o'') = 0 \quad (10)$$

It is convenient to define ψ as:

$$\psi = 0.5 \{\phi(y) \exp[i(k x - \sigma t)] + \phi^*(y) [-i(k x - \sigma^* t)]\} \quad (11)$$

where the asteriscs represent the complex conjugate. Plugging equation (11) back into the vorticity equation, equation (9), it is obtained:

$$(- i \sigma + i U_o k) (- k^2 \phi + d^2 \phi/d y^2) + i k (\beta - U_o'') \phi = 0 \quad (12)$$

$$(- i \sigma^* - i U_o k) (- k^2 \phi^* + d^2 \phi^*/d y^2) + i k (\beta - U_o'') \phi^* = 0 \quad (13)$$

Upon division by k and multiplication by i, it is obtained:

$$(U_o - c) (d^2 /d y^2 - k^2) \phi + (\beta - U_o'') \phi = 0 \quad (14)$$

$$(U_o - c^*) (d^2 /d y^2 - k^2) \phi^* + (\beta - U_o'') \phi^* = 0 \quad (15)$$

Assuming c to be complex, and dividing by $(U_o - c) \neq 0$, it is obtained:

$$d^2 \phi/d y^2 - \{k^2 - [(\beta - U_o'')/(U_o - c)]\} \phi = 0 \quad (16)$$

$$d^2 \phi^*/d y^2 - \{k^2 - [(\beta - U_o'')/(U_o - c^*)]\} \phi^* = 0 \quad (17)$$

It is convenient to define:

$$u' = \partial \psi/\partial y \quad (18a)$$

$$v' = - \partial \psi/\partial x \quad (18b)$$

Then:

$$\overline{u' v'} = -(\partial \psi/\partial y)(\partial \psi/\partial x) \quad (19)$$

It is obtained:

$$\partial \psi/\partial x = (i k/2) \{\phi \exp[i(k x - \sigma t)] - \phi^* [-i(k x - \sigma^* t)]\} \quad (20)$$

$$\partial \psi/\partial y = (1/2) \{\phi' \exp[i(k x - \sigma t)] + (\phi^*)' \exp[- i (k x - \sigma^* t)]\} \quad (21)$$

Multiplication of equations (20) and (21) yields:

$$\overline{u' v'} = (i k/4) \{\phi^* \phi' - \phi (\phi^*)'\} \exp(2 i \sigma t) \quad (22)$$

It follows naturally that:

$$\partial \overline{u' v'}/\partial y = (i k/4) \exp(2 i \sigma t) \{\partial (\phi^* \phi' - \phi (\phi^*)')/\partial y\} \quad (23)$$

The value of $K_E = (1/2)\{(u')^2 + (v')^2\}$ is seeked. After some algebraic manipulations, it is obtained:

$K_E = (1/2)\{(1/4)\ |\phi'|(2\ + (1/4)\ k^2\ |\phi|^2\}\ \exp(2\ i\ \sigma\ t)$ (24)

On the other hand,

$\phi^*\ d^2\ \phi/d\ y^2 = \phi^*\ d\ (d\ \phi/d\ y)/d\ y = d\ (\phi^* d\ \phi/d\ y)/d\ y - (d\ \phi/d\ y)^2$ (25)

$\phi\ d^2\ \phi^*/d\ y^2 = \phi\ d\ (d\ \phi^*/d\ y)/d\ y = d\ (\phi d\ \phi^*/d\ y)/d\ y - (d\ \phi/d\ y)^2$ (26)

Upon subtraction of the product of equation (25) by ϕ^* minus the product of equation (26) by ϕ, it follows that:

$0.5\ d\ (\phi^*\ \phi' - \phi\ (\phi^*)'/\ dy + 0.5\ (\beta - U_o'')\{(\ U_o - c)^{-1} - (U_o - c^*)^{-1}\}|\phi|^2 = 0$ (27)

Upon addition of the product of equation (25) by ϕ^* plus the product of equation (26) by ϕ, it yields:

$0.5\ d\ (\phi^*\ \phi' + \phi\ (\phi^*)'/\ dy\ - |\phi'|^2 - k^2\ |\phi|^2 + 0.5\ (\beta - U_o'')\{(U_o - c)^{-1} +$
$(U_o - c^*)^{-1}\}|\phi|^2\ =\ 0$ (28)

Equation (27) may be written as:

$0.5\ d\ (\phi^*\ \phi' - \phi\ (\phi^*)'/\ dy + (1/2)\{\ [(U_o - c^*) - (U_o - c)]/(\ U_o - c)$
$(U_o - c^*)\}(\beta - U_o'')|\phi|^2 = 0$ (29)

whereas equation (28) may be written as:

$0.5\ d\ (\phi^*\ \phi' + \phi\ (\phi^*)')/\ dy\ - |\phi'|^2 - k^2\ |\phi|^2 +$
$(1/2)\{\ [(U_o - c^*) + (U_o - c)]/(U_o - c)\ \ (U_o - c^*)\}(\beta - U_o'')|\phi|^2 = 0$ (30)

After some algebraic manipulations, it yields: :

$(U_o - c^*) - (U_o - c) = 2\ i\ c_i$ (31)

$(U_o - c^*) + (U_o - c) = 2(U_o - c_r)$ (32)

$(U_o - c)\ (U_o - c^*) = |U_o - c|^2$ (33)

where $c = c_r +\ i\ c_i$

Therefore, equation (29) may be written as:

$0.5\ d\ (\phi^*\ \phi' - \phi\ (\phi^*)'/\ dy + i\ c_i\ (\beta - U_o'')\ |U_o - c|^{-2}\ |\phi|^2 = 0$ (34)

whereas equation (30) may be rewritten as:

$0.5\ d\ (\phi^*\ \phi' + \phi\ (\phi^*))'/\ dy\ - |\phi'|^2 - k^2\ |\phi|^2 +$
$0.5\ ((U_o - c_r)(\beta - U_o'')\ |U_o - c|^{-2}\ |\phi|^2 = 0$ (35)

The boundary conditions for this problem are:

$\phi^* =\ \phi\ = 0$, at the boundaries. (36)

DISCUSSION

Whenever the profiles of ϕ and ϕ^* look more or less than in *Fig. 5*, upon integration over the entire region, it is obtained:

$$\int d\left(\phi^* \phi' - \phi(\phi^*)' / dy = 0 \qquad (37)$$

Therefore, it is obtained:

$$i\, c_i \int (\beta - U_o") \, |U_o - c|^{-2} \, |\phi|^2 = 0 \qquad (38)$$

Because of the fact that both $|U_o - c|^{-2}$ and $|\phi|^2$ are positive, from equation (38) it may be inferred that either c_i is zero or the integral must vanish. If c_i is zero, we expect neutral solutions. The existence of the integral becomes questionable. If $c_i \neq 0$, equation (38) cannot be satisfied unless $(\beta - U_o")$ has at least one zero (Rayleigh, 1888). This is the *necessary condition for barotropic instability*. Namely that $(\beta - U_o")$ changes sign somewhere in the region.

On the other hand, the same reasoning as before applies for the integration over the entire region of equation (35). It follows that:

$$\int d\left(\phi^* \phi' + \phi(\phi^*)' / dy = 0 \qquad (39)$$

Therefore, the integration of equation (35) yields:

$$\int_{y_2}^{y_1} \left(\left(U_o - c_r\right)\left(\beta - U_o"\right) \left|U_o - c\right|^{-2} \left|\phi\right|^2 = \int_{y_2}^{y_1} \left\{\left|\phi'\right|^2 + k^2\left|\phi\right|^2\right\} dy \qquad (40)$$

As c_r is constant, it yields:

$$c_r \int_{y_2}^{y_1} \left(\beta - U_o"\right) \left|U_o - c\right|^{-2} \left|\phi\right|^2 dy = 0 \qquad (41)$$

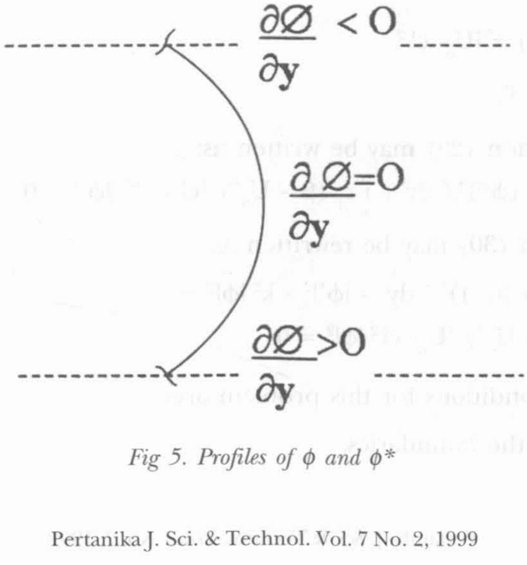

Fig 5. Profiles of ϕ and ϕ^*

due to equation (38). Therefore, equation (40) yields:

$$\partial K_E/\partial t = \int_{y_2}^{y_1} U_o \partial \overline{u' v'}/\partial y \, d m = \int_{y_2}^{y_1} \left\{ |\phi'|^2 - k^2 |\phi|^2 \right\} d y =$$

$$\int_{y_2}^{y_1} U_o \left(\beta - U_o''\right) | U_o - c|^{-2} |\phi|^2 \, d y \qquad (42)$$

For this to happen, the integral $\int_{y_2}^{y_1} U_o \left(\beta - U_o''\right) | U_o - c|^{-2} |\phi|^2 \, d y$ must be negative. This is certain only if U_o (b - U_o") is negative in the interval of integration (Hoilland 1953). Therefore, only certain shapes of velocity profiles of velocity are permissible (Necco 1980). It means that unstable oscillations can exist only if the mean velocity has an absolute maximum within the flow.

CONCLUSIONS

The barotropic instability depends on the basic flow. Whenever U" > 0 and U < 0, we would have instability. The same situation holds whenever U" < 0 and U > 0. Let us assume a westerly flow with $\beta = 0$ (*Fig. 6*). This represents an unstable case. The peak of the curve is also where U" < 0.

If the β-effect is added, we will have a stabilizing effect. *Fig. 7* shows an easterly flow profile U" > 0. When U < 0 we will have instability. This results as in perfect agreement with previous findings.

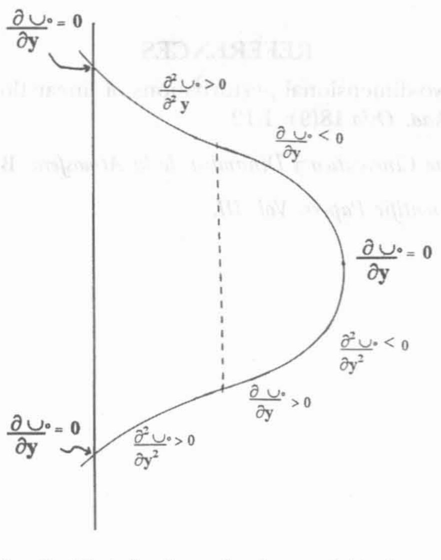

Fig 6. Westerly flow showing a typical case of barotropic instability

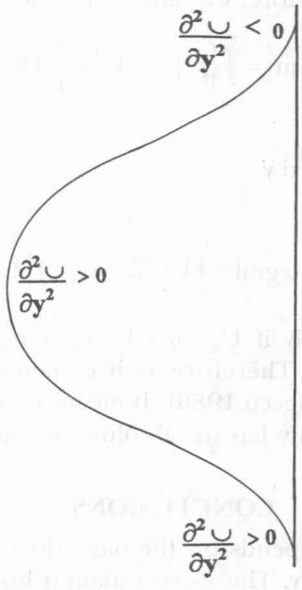

Fig 7. Easterly flow showing a typical case of barotropic instability

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