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On the Higher Order Edge-Connectivity of Complete Multipartite Graphs

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ABSTRAK

Biarkan G sebagai suatu graf terhubung yang mempunyai $p \geq 2$ titik. Untuk $k = 1, 2, ..., p - 1$, kehubungan-garis peringkat k yang diberi lambang $\lambda^{(k)}(G)$, ditakrifkan sebagai bilangan terkecil garis-garis yang apabila dikeluarkan daripada G akan meninggalkan suatu graf yang terdiri daripada $k + 1$ komponen. Dalam artikel ini kita akan menentukan kuantiti $\lambda^{(k)}(G_n)$ bagi sebarang graf multipartit lengkap G_n. Sebagai akibatnya kita perolehi syarat perlu dan cukup supaya graf G_n dapat difaktorkan menjadi pohon-pohon janaan.

ABSTRACT

Let G be a connected graph with $p \geq 2$ vertices. For $k = 1, 2, ..., p - 1$, the k^{th} order edge-connectivity of G, denoted by $\lambda^{(k)}(G)$, is defined to be the smallest number of edges whose removal from G leaves a graph with $k + 1$ connected components. In this note we determine $\lambda^{(k)}(G_n)$ for any complete multipartite graph G_n. As a consequence, we give a necessary and sufficient condition for the graph G_n to be factored into spanning trees.

1. INTRODUCTION

Let G be a connected simple graph of order p and size q. Denote by $V(G)$ and $E(G)$ the vertex set and edge set of G respectively. The *edge-connectivity* $\lambda = \lambda(G)$ of G is defined to be the smallest number of edges whose removal from G results in a disconnected or trivial graph. This notion has a natural generalization. Following Goldsmith *et al.* (1980), for each $k = 0, 1, ..., p - 1$, the kth order *edge-connectivity* of G, denoted by $\lambda^{(k)}(G)$, is defined as the minimum number of edges of G whose removal increases the number of components of G by k. Note that $\lambda^{(0)}(G) = 0$, $\lambda^{(1)}(G) = \lambda(G)$ and $\lambda^{(p-1)}(G) = q$. The properties of $\lambda^{(k)}(G)$ were studied previously in Boesch and Chen (1978), Goldsmith (1980 and 1981), Goldsmith *et al.* (1980) and Sampathkumar (1984).

It is easy to see that for any tree T, $\lambda^{(k)}(T) = k$. Furthermore, since any connected graph G contains a spanning tree, $\lambda^{(k)}(G) \geq k$. It was proved in Peng *et al.* (1988) that $\lambda^{(k)}(K_p) = \frac{1}{2}k(2_p - k - 1)$ for each $k = 0, 1, ..., p - 1$. In this note we shall determine the kth order edge-connectivity of a complete n-partite graph and then use the result to derive a necessary and sufficient condition for a complete n-partite graph to be factored into spanning trees.

Throughout this article, we write $G_n = K_n(m_1, m_2, ..., m_n)$, $n \geq 2$, to denote a complete n-partite graph with n partite sets $V_1, V_2, ..., V_n$ such that $|V_i| = m_i \geq 1$ for each $i = 1, 2, ..., n$. For the sake of convenience, we always assume

$$m_1 \leq m_2 \leq ... \leq m_n.$$

A graph G is called a *complete multipartite* graph if $G \cong G_n$ for some integer $n \geq 2$.

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For those graph-theoretic terms used but not defined here we refer to Behzad *et al.* (1979).

2. EFFICIENT SEPARATION

Let G be a connected graph of order p, and k be an integer such that $1 \leq k \leq p-1$. Following Goldsmith *et al.* (1980) again, by an *efficient k-separation* of G, we mean a removal of $\lambda^{(\kappa)}(G)$ edges from G so that G is separated into $k+1$ components. Call a component of a graph *trivial* if it is a singleton, and *non-trivial* otherwise.

It was pointed out in Peng *et al.* (1988) that every efficient *k-separation* of $K_p(1 \leq k \leq p-1)$ always results in at least k trivial components. In this section we shall study the possible situations after performing an efficient separation on G_n.

Let A and B be two subsets of $V(G)$. We denote by $E_G(A,B)$ the set of edges of G each joining a vertex of A to a vertex of B, and by $e_G(A,B)$ the number of edges in $E_G(A,B)$. In particular, we write $e_G(A)$ for $e_G(A,A)$, and $e_G(v,B)$ for $e_G(\{v\},B)$ where $v \in V(G)$. The minimum degree of G is denoted by $\delta(G)$, i.e. $\delta(G) = \min\{\deg_G(v) \mid v \in V(G)\}$.

First of all, we have

LEMMA 1. *The number of edges of the graph G_n needed to be removed to separate G_n into two non-trivial components is greater than $\delta(G_n)$, except when $G_n = K_2(2,2)$, in which case, the number is equal to $\delta(G_n)$.*

Proof. We proceed by induction on n. For the case $n = 2$, let G_2, $G_2 \neq K_2(2,2)$, be separated into two non-trivial components, and let e^* denote the number of edges removed in this separation. We may assume that both partite sets V_1 and V_2 of G_2 are divided into two sets. Let V_1 be divided into a and b vertices, and V_2 be divided into c and d vertices. (*Figure 1(a)*) Then a,b,c and d are positive. Since $G_2 \neq K_2(2,2)$, not all of them are equal to 1. Thus $e^* = ad + bc \geq a + b$. If $ad + bc = a + b$, then $c = d = 1$ since a, b, c, d are positive integers. This implies $m_2 = c + d = 2$. Since $m_2 \geq m_1 \geq 2$, $m_1 = 2$. But this contradicts our assumption that $G_2 \neq K_2(2,2)$. Thus, we have $e^* > a + b = \delta(G_2)$.

Now, suppose that the statement holds for any graph $G_{n-1}(n \geq 3)$. We shall show that the statement is also true for any G_n. Assume that $G_n \neq K_2(2,2)$, and let G_n be separated into two non-trivial components Q_1 and Q_2. Except for the two cases of separation shown in *Figures 1(b)* and *(c)* for $n = 3$ and $n = 4$ respectively, it can be checked that there is always a partite set V_r of G_n such that $(Q_1 \cup Q_2) - V_r$ still consists of two non-trivial components $Q_1' = Q_1 - V_r'$ and $Q_2' = Q_2 - V_r''$ where V_r is separated into two sets V_r' and V_r'' in that separation. (*Figure 1(d)*) Note that V_r' or V_r'' may be empty.

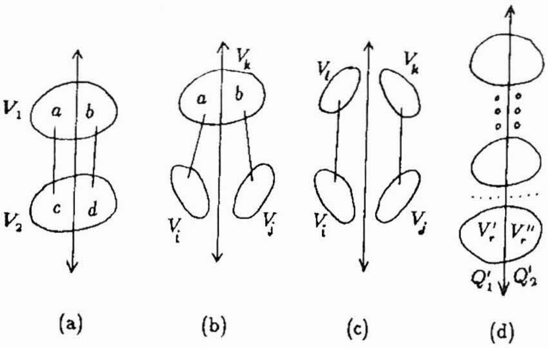

Figure 1.

So, the complete $(n-1)$-partite graph $G' = G_n - V_r$ is separated into two non-trivial components Q_1' and Q_2'. Let e' denote the number of edges removed in this separation of G', and e^* denote the number of edges deleted to separate G_n into Q_1 and Q_2. Then

$$e^* = e' + e_{G_n}(V_r', V(Q_2')) + e_{G_n}(V_r'', V(Q_1')).$$

By induction hypothesis,

$$e' > \delta(G').$$

But

$$\delta(G') = \begin{cases} \delta(G_n) - m_r & \text{if } r \neq n \\ \delta(G_n) - m_{n-1} & \text{if } r = n, \end{cases}$$

and

$$e_{G_n}(V_r', V(Q_2')) + e_{G_n}(V_r'', V(Q_1')) > \begin{cases} m_r & \text{if } r \neq n \\ m_{n-1} & \text{if } r = n. \end{cases}$$

Therefore $e^* > \delta(G_n)$, as required.

It remains to consider the two exceptional cases.

Case (i). The separation of G_3 as shown in *Figure 1(b).*

Let the partite set V_k be divided into a and b vertices, and let e^* denote the number of edges removed in this separation. Then

$$e^* = m_i b + m_i m_j + m_j a$$

$$= m_i(b + m_j) + m_j a$$
$$> m_i + m_j \text{ (since } b + m_j \geq 2,\ a \geq 1)$$
$$\geq \delta(G_2),$$

as required.

Case (ii). The separation of G_4 as shown in *Figure 1(c)*.

Let e^* denote the number of edges deleted in this separation. Then

$$\begin{aligned} e^* &= m_i m_j + m_i m_k + m_j m_l + m_k m_i \\ &= m_i(m_j + m_k) + m_j m_l + m_k m_i \\ &> m_i + m_j + m_k \\ &> \delta\ (G_4). \end{aligned}$$

The proof is now complete. □

We are now ready to prove the following main result of this section.

THEOREM 1. *Let p be the order of the graph G_n, and k be any integer with $1 \leq k \leq p$-1. If G_n is separated into components by an efficient k-separation, then*

either (i) *at least k of the components are trivial,*

or (ii) *k - 1 of the components are trivial, and the other two are K_2.*

Proof. Suppose there are two non-trivial components Q_1 and Q_2 of G_n after the removal of $\lambda^{(\kappa)}(G_n)$ edges in an efficient *k-separation* of G_n. We shall show that the induced subgraph $H = \langle Q_2 \cup Q_3\rangle_{G_n}$ is $K_2(2,2)$.

We first note that H is a complete multipartite subgraph of G_n. If $H \neq K_2(2,2)$, then by Lemma 1, the number of edges removed to separate H into two components Q_1 and Q_2 is greater than $\delta(H)$. But $\delta(H)$ is equal to the number of edges removed to separate H into a trivial component $\{\upsilon\}$, and a component H - υ where $\upsilon \in V(H)$ such that $\deg_H(\upsilon) = \delta(H)$. Thus G_n can be separated into k+1 components by removing less than $\lambda^{(\kappa)}(G_n)$ edges. This contradicts the definition of $\lambda^{(\kappa)}(G_n)$. Therefore $Q_1 = Q_2 = K_2$ and $H = K_2(2,2)$.

Now, suppose that there is another non-trivial component Q_3 of G_n after the removal of $\lambda^{(\kappa)}(G_n)$ edges in an efficient *k-separation* of G_n. Then, by the argument above, we conclude that $H_1 = \langle Q_2 \cup Q_3\rangle_{G_n}$ and $H_2 = \langle Q_2 \cup Q_3\rangle_{Gn}$ are all isomorphic with $K_2(2,2)$. Thus, $Q_3 = K_2$ and the number of edges removed to separate $H^* = \langle Q_1 \cup Q_2 \cup Q_3\rangle$ into three components Q_1, Q_2 and Q_3 is six. However, if we delete all the five edges of H^* which are incident with the two vertices of Q_3, we also separate H^* into three components. But this contradicts the minimality of $\lambda^{(\kappa)}(G_n)$. The result thus follows.˙ □

Remark. We note that the result (ii) in Theorem 1 can occur only when $G_n = K_2(m_1, m_2)$, where $m_1,\ m_2 \geq 2$.

3. HIGHER ORDER EDGE-CONNECTIVITY

In this section we shall apply Theorem 1 to determine the k^{th} order edge-connectivity of any complete n-partite graph.

We begin with the following result.

LEMMA 2. *Let $T \supseteq V(G_n)$ such that $|T| = t$* ≥ 1 and $e_{G_n}(T) + e_{G_n}(T, G_n - T) = \lambda^{(t)}(G_n)$. Then (i) *there exists $\omega \in T$ such that* $\deg_{G_n}(\omega) = \delta(G_n)$, *and*

(ii) *if* $T' = T - \{\upsilon\}$ and $G' = G_n - \upsilon$, where $\upsilon \in T$, then $e_{G'}(T') + e_{G'}(T', G' - T') = \lambda^{(t-1)}(G')$.

Note. By the assumption of Lemma 2, we are, indeed, given an efficient t-separation of G_n which separates it into $t + 1$ components {x} (x $\in T$) and $G_n - T$. The subgraph $G_n - T$ must be connected as $\lambda^{(t)}(G_n) < \lambda^{(t+1)}(G_n)$.

Proof. (i) We suppose the contrary. Then no element of T is in V_n or in any other partite set V_i of G_n such that $|V_i| = |V_n|$ Let $\upsilon \in T$ and $T^* = T - \{\upsilon\}$. Consider the graph $G_n - T^*$. Note that $G_n - T^*$ is a complete multipartite subgraph of G_n. So V_n is one of its partite sets. Let $u \in V_n$ and $u \in V^*$ where V^* is also a partite set in the partition of $G_n - T^*$. Since $|V_n| > |V^*|$, we have

$$e_{G_n}(u,\ G_n - T^*) < e_{G_n}(u,\ G_n - T^*).$$

Therefore (Figure 2)

$$\begin{aligned} &e_{G_n}(T^* \cup \{u\}) + e_{G_n}(T^* \cup \{u\},\ G_n - (T^* \cup \{u\})) \\ &= e_{G_n}(T^*) + e_{G_n}(T^*,\ G_n - T^*) + e_{G_n}(u,\ G_n - T^*) \\ &< e_{G_n}(T^*) + e_{G_n}(T^*,\ G_n - T^*) + e_{G_n}(\upsilon,\ G_n - T^*) \\ &= e_{G_n}(T^* \cup \{\upsilon\}) + e_{G_n}(T^* \cup \{\upsilon\},\ G_n - (T^* \cup \{\upsilon\})) \\ &= e_{G_n}(T) + e_{G_n}(T,\ G_n - T) \\ &= \lambda^{(t)}(G), \end{aligned}$$

which contradicts the minimality of $\lambda^{(t)}(G_n)$. Thus (i) follows.

(ii) Since $|T'| = t - 1$, by the minimality of $\lambda^{(t-1)}(G')$, we have

$$e_{G'}(T') + e_{G'}(T',\ G' - T') \geq \lambda^{(t-1)}(G').$$

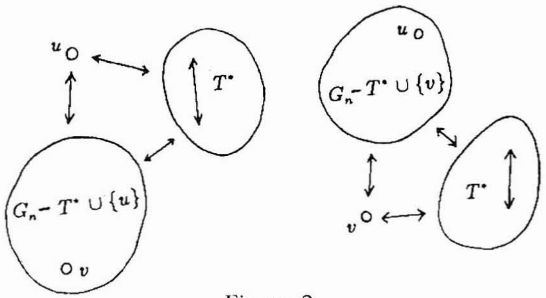

Figure 2.

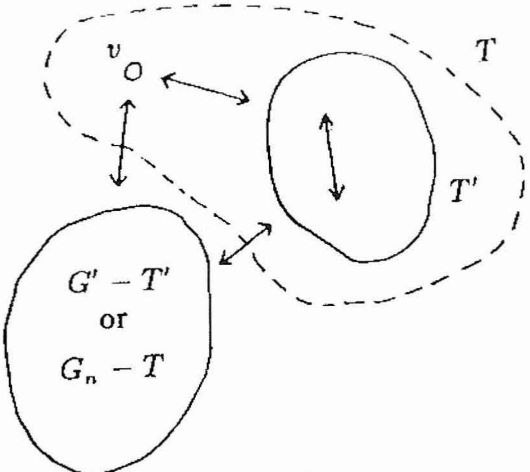

Figure 3.

If the equality does not hold, then (*Figure 3*)

$$\begin{aligned}\lambda^{(t)}(G_n) &= e_{G_n}(T) + e_{G_n}(T, G_n - T)\\ &= e_{G'}(T') + e_{G'}(T', G' - T') + \deg_{G_n}(u)\\ &> \lambda^{(t-1)}(G') + \deg_{G_n}(u)\\ &\geq \lambda^{(t)}(G_n),\end{aligned}$$

which is impossible. □

Let $H_k = \{H_0, H_1, ..., H_k\}$ be a family of subgraphs of G_n defined as follows: $H_0 = G_n$ and for i = 1,2,...,k, $H_i = H_{i-1} - \upsilon_i$ for some $\upsilon_i \in V(H_{i-1})$ such that $\deg_{H_{i-1}}(\upsilon_i) = \delta(H_{i-1})$.

We shall now apply Theorem 1 and Lemma 2 to prove the following result.

LEMMA 3. *Let p be the order of the graph G_n, and k be any integer satisfying $1 \leq k \leq p-1$. Then there exists a set of vertices S of G_n such that $|S| = k$ and $e_{G_n}(S) + e_{G_n}(S, G_n - S) = \lambda^{(k)}(G_n)$.*

Furthermore, $S = \{\upsilon_1, \upsilon_2, ..., \upsilon_k\} = V(G_n) - V(H_k)$ *where H_k is a member of some H_k of G_n.*

Proof. To prove the first part, we show that, for each $k = 1,2,..., p-1$, there is an efficient k-separation of G_n such that, after performing this separation, at least k of the components are trivial. By Theorem 1, there are at most two non-trivial components Q_1 and Q_2 in any efficient k-separation of G_n, and $H = \langle Q_1 \cup Q_2 \rangle = K_2(2,2)$. But the number of edges removed to separate H into Q_1 and Q_2 is equal to the number of edges whose removal separates H into a trivial component and a $K_2(1,2)$ component. This completes the proof of the first part.

By Lemma 2(i), there exists $\upsilon_1 \in S$ such that $\deg_{G_n}(\upsilon_1) = \delta(G_n)$. If k > 1, let us write $S_1 = S - \{\upsilon_1\}$ and $H_1 = G_n - \upsilon_1$. By Lemma 2(ii), we have

$$e_{H_1}(S_1) + e_{H_1}(S_1, H_1 - S_1) = \lambda^{(k-1)}(H_1).$$

Note that H_1 is also a complete multipartite graph, and $|S_1| = k - 1$. Thus by Lemma 2(i), there exists $\upsilon_2 \in S_1$ such that $\deg_{H_1}(\upsilon_2) = \delta(H_1)$. If $k > 2$, by using the same argument as above, we conclude that for i = 3, 4, ..., k, S_i-1 contains υ_i such that $\deg_{H_{i-1}}(\upsilon_i) = \delta(H_{i-1})$ where $H_i = H_{i-1} - \upsilon_i$. The proof is now complete. □

We are now in a position to establish the main result of this note.

THEOREM 2. *Let p be the order of the graph G_n. Then for $k = 1,2,...,p - 1$,*

$$\lambda^{(k)}(G_n) = \sum_{H_i \in H_{k-1}} \delta(H_i)$$

for some H_{k-1} of G_n.

Proof. By Lemma 3, there exists $S \subseteq V(G_n)$ such that $|S| = k$ and $\lambda^{(k)}(G_n) = e_{G_n}(S) + e_{G_n}(S, G_n - S)$. Since $S = \{\upsilon_1, \upsilon_2, ..., \upsilon_k\}$ is such that for i = 1,2,...,k, $\deg_{H_{i-1}}(\upsilon_i) = \delta(H_{i-1})$, we have

$$\begin{aligned}&E_{G_n}(S) \cup E_G(S, G_n - S)\\ &= \bigcup_{i=0}^{k-1} E_{G_n}(\upsilon_{i+1}, H_i).\end{aligned}$$

Thus,

$$\begin{aligned}\lambda^{(k)}(G_n) &= e_{G_n}(S) + e_{G_n}(S, G_n - S)\\ &= \sum_{i=0}^{k-1} e_{G_n}(\upsilon_{i+1}, H_i) = \sum_{i=0}^{k-1} \delta(H_i),\end{aligned}$$

as required. □

4. SPANNING TREE FACTORIZATION

It is known that a complete graph K_p can be factored into spanning trees (indeed spanning paths) if and only if p is even (see for instance, Behzad *et al.* (1979), p. 168). In the following theorem we give a necessary and sufficient condition for the graph G_n to be factored into spanning trees.

THEOREM 3. *The complete n-partite graph* $K_n(m_1, m_2, ..., m_n)$ *can be factored into spanning trees if and only if*

$$\sum_{ij=1}^{n} m_i m_j = k\left(\sum_{i=1}^{n} m_i - 1\right)$$

for some positive integer k.

COROLLARY. *The graph* $K_2(m,n)$ *is spanning tree factorizable for the following integers m and n* $(n \geq m)$:

(i) $m = 1$, *and* $n \geq 1$;
(ii) $m \equiv 1 \pmod 2$, $m > 1$, *and* $n = m + 1$;
(iii) $m > 2$, *and* $n = (m - 1)^2$;
(iv) $m > 4$, *and* $n = (m - 1)(m - 2)/2$;
(v) $m = ab$ *and* $n = (ab - 1)(b - 1)$ *where a and b are integers* > 2.

COROLLARY 2. (i) *The graph* $K_n(m - 1, m, m, ..., m)$ *is spanning tree factorizable if and only if* $(n - 1)\ m \equiv 0 \pmod 2$.

(ii) *The graph* $K_n(1, m, m, ..., m)$ *is spanning tree factorizable if and only if* $nm \equiv 0 \pmod 2$.

(iii) *The graph* $K_n(m, m, ..., m)$ *is spanning tree factorizable if and only if* $m = 1$ *and n is even.*

Denote by $\omega(G)$ the number of components of G. A subset X of $E(G)$ is called an *edge-cutset* of G if $\omega(G - X) > 1$. Following Peng *et al.* (1988), the *edge-toughness* of G, denoted by $\tau_1(G)$, is defined as

$$\tau_1(G) = \min\left\{\frac{|X|}{\omega(G-X)-1} \middle| X \text{ is an edge-cutset of } G\right\}$$

The above definition of $\tau_1(G)$ is, as a matter of fact, motivated by the following result due to Nash-Williams (1961) and Tutte (1961) independently.

THEOREM A. *A connected graph G has s edge-disjoint spanning trees if and only if* $|X| \geq s(\omega(G - X) - 1)$ *for each* $X \subseteq E(G)$.

It follows from Theorem A that a connected graph G has k edge-disjoint spanning trees if and only if $\tau_1(G) \geq k$. Thus Theorem 3 is an immediate consequence of the following result.

THEOREM 4. $\tau_1(K_n(m_1, m_2, ..., m_n))$

$$= \frac{\sum_{ij=1}^{n} m_i m_j}{\sum_{i=1}^{n} m_i - 1}$$

To prove the above theorem we shall make use of the following result which was obtained in Peng *et al.* (1988) as a corollary of a more general theorem.

For each i = 1,2,..., $|V(G)| - 1$, we write

$$\Delta\lambda_i(G) = \lambda^{(i-1)}(G) - \lambda^{(i)}(G).$$

THEOREM B. *Let G be a connected graph of order p and size q. If the sequence* $(\Delta_i(G) \mid 1 \leq i \leq p-1)$ *is non-increasing, i.e.* $\Delta_i(G) \geq \Delta_{i+1}(G)$ *for each* i = 1,2,...,p - 2, *then* $\tau_1(G) = q/(p-1)$.

Proof of Theorem 4. By Theorem B, we only need to show that the sequence $(\Delta_i(G_n) \mid 1 \leq i \leq p - 1)$ is non-increasing. By Theorem 2, $\Delta_i(G_n) = \lambda^{(i)}(G_n) - \lambda^{(i-1)}(G_n) = \delta(H_{i-1})$. Note that for i = 1,2,...,p-1, $H_i = H_{i-1} - \upsilon_i$ where $\deg_{H_{i-1}}(\upsilon_i) = d(H_{i-1})$ and υ_i is adjacent to every vertex of H_{i-1}, except those in the partite set (of the partition of H_{i-1}) that υ_i belongs to. So, it is clear that for i = 1,2,...,p - 1, $\delta(H_i) \leq \delta(H_{i-1})$. Therefore, the sequence $(\Delta_i(G_n) \mid 1 \leq i \leq p - 1) = (\delta(H_i) \mid 0 \leq i \leq p - 2)$ is non-increasing. □

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