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Newton Polyhedral Method of Determining p-adic Orders of Zeros Common to Two Polynomials in $Q_p[x, y]$

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ABSTRAK

Penelitian dibuat terhadap gabungan gambarajah-gambarajah penunjuk yang disekutukan dengan dua polinomial dalam $Q_p[x,y]$, *untuk mendapatkan peringkat p-adic pensifar-pensifar sepunya kedua-dua polinomial. Dibuktikan bahawa peringkat p-adic pensifar-pensifar sepunya dua polinomial tersebut adalah koordinat titik-titik persilangan tertentu tembereng gambarajah penunjuk, yang disekutukan dengan kedua-dua polinomial. Satu konjektur dibuat, bahawa jika* (λ, μ) *adalah titik persilangan tembereng-tembereng yang tidak bertindih, dalam gabungan gambarajah penunjuk yang disekutukan dengan dua polinomial dalam* $\bar{Q}_p[x,y]$, *maka wujud pensifar* (ξ, η) *sepunya kedua-dua polinomial sedemikian, sedemikian hingga* $\mathrm{ord}_p\ \xi = \lambda$ *dan* $\mathrm{ord}_p\ \eta = \mu$. *Satu kes khusus konjektur ini di bawah syarat-syarat tertentu dibuktikan.*

ABSTRACT

To obtain p-adic orders of zeros common to two polynomials in $\bar{Q}_p[x,y]$, *the combination of Indicator diagrams associated with both polynomials are examined. It is proved that the p-adic orders of zeros common to both polynomials give the coordinates of certain intersection points of segments of the Indicator diagrams associated with both polynomials. We make a conjecture that if* (λ, μ) *is a point of intersection of non-coincident segments in the combination of Indicator diagrams associated with two polynomials in* $\bar{Q}_p[x,y]$ *then there exists a zero* (ξ, η) *common to both polynomials such that* $\mathrm{ord}_p\ \xi = \lambda$, $\mathrm{ord}_p\ \eta = \mu$. *A special case of this conjecture is proved.*

1. INTRODUCTION

The classical Newton Polygon is a device for obtaining fractional power series expansions of algebraic functions. In his "Method Of Fluxions" (Whiteside, 1969) Newton gave a number of examples of this process. In ascertaining the properties of roots of polynomials in one variable, the Newton polygon plays an important role. For example, in the proof of Puiseux's theorem the Newton polygon can be usefully applied (Walker, 1962 and Lefschetz, 1953).

Koblitz (1977) discusses the Newton polygon method for polynomials and power series in $\Omega_p[x]$ where Ω_p denotes the completion of the algebraic closure of the field of p-adic numbers Q_p. As an extension of this idea to polynomials in $\Omega_p[x,y]$, Mohd Atan (1986) defines the analogue of the Newton Polygon associated with a polynomial $f(x, y) = \Sigma a_{ij} x^i y^j$ with coefficients in Ω_p as the lower convex hull of the set of points $(i, j, \mathrm{ord}_p\ a_{ij})$ with $\mathrm{ord}_p\ a_{rs} = \infty$ if $a_{rs} = 0$. This analogue, called the Newton polyhedron, consists of faces and edges on and above which lie the points (i, j, $\mathrm{ord}_p a_{ij}$). *Fig. 1* shows an example of the Newton polyhedron associated with the given polynomial. He further investigates the relationships between roots of a polynomial in $\Omega_p[x, y]$ and its Newton polyhedron. This is done by using a device called the Indicator diagram, which is defined as the set of line segments in R^2 joining pairs of points (x_i, y_j) and (x'_i, y'_j) that correspond to normals $(x_i, y_j, 1)$ and $(x'_i, y'_j, 1)$ respectively to adjacent faces in the associated Newton polyhedron. *Fig. 2* illustrates the Indicator diagram associated with the Newton polyhedron in *Fig. 1*.

In this paper we will prove a converse of an assertion in Mohd Atan (1986) and obtain the p-adic orders of zeros common to two polynomials in $\bar{Q}_p[x, y]$ by examining the combination of Indicator diagrams associated with both polynomials.

With p denoting a prime, the valuation $| \, |_p$ on Q_p is as usually defined:

$$|x|_p = \begin{cases} p^{-\mathrm{ord}_p x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

where $\mathrm{ord}_p x$ denotes the highest power of p dividing x and $\mathrm{ord}_p x = \infty$ if $x = 0$. $| \, |_p$ extends uniquely from Q_p to $\bar{Q}_p$ the algebraic closure of Q_p and to Ω_p which is complete and algebraically closed.

Our main assertion is as follows:

Theorem 1.1

Let f and g be polynomials in $\bar{Q}_p[x, y]$ with degrees at most d_f and d_g respectively and suppose $p > 2d_f d_g$. Let (λ, μ) be a point of intersection of the Indicator diagrams associated with f and g which is not a vertex of either diagram and suppose that the edges through (λ, μ) do not coincide. Then there are ξ and η in $\bar{Q}_p$ satisfying $f(\xi, \eta) = g(\xi, \eta) = 0$ and $\mathrm{ord}_p \xi = \lambda$, $\mathrm{ord}_p \eta = \mu$.

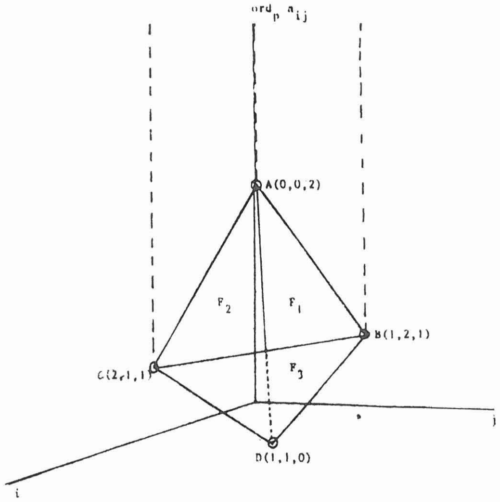

Fig. 1: The Newton polyhedron of polynomial $f(x, y) = 3x^2y + xy + 3xy^2 + 9$ with $p = 3$. AB, AC, AD, BC, BD and CD are the edges and F_1, F_2 and F_3 the non-vertical faces of the polyhedron. The vertical broken lines indicate the edges of its vertical faces passing through the outer edges.

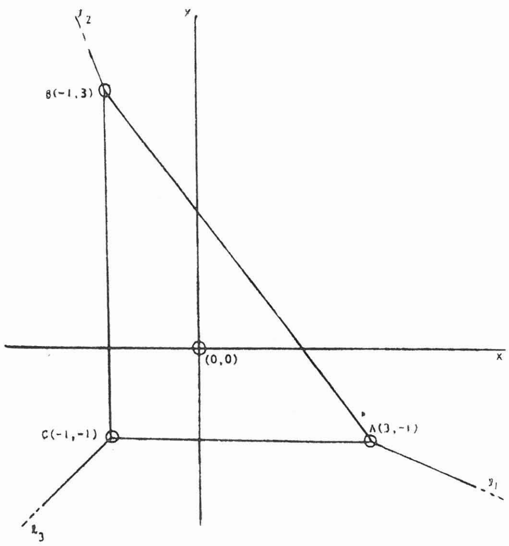

Fig. 2: Indicator diagram associated with the Newton polyhedron of $f(x, y) = 3x^2y + xy + 3xy^2 + 9$ with $p = 3$. Points A, B, C correspond to the upward-pointing normals of the faces F_1, F_2, F_3 of the Newton polyhedron as in Fig. 1. The broken lines l_i, $i = 1, 2, 3$ join A, B and C respectively to points at infinity corresponding to normals of the vertical faces of the Newton polyhedron.

2. p-adic ORDERS OF ZEROS COMMON TO TWO POLYNOMIALS IN $\bar{Q}_p(x, y)$

To determine the p-adic orders of common zeros of two polynomials in $\bar{Q}_p[x,y]$ we naturally examine the combination of the Indicator diagrams associated with both polynomials. First, however, we give a proof to an assertion by Mohd Atan (1986) for a polynomial in $\bar{Q}_p[x,y]$ and its converse in the following theorem, which will be useful in our discussion.

Theorem 2.1

Let (λ, μ) be in R^2 and let f be a polynomial in $\bar{Q}_p[x,y]$. Then there are ξ, η in $\bar{Q}_p$ with $\text{ord}_p\, \xi = \lambda$, $\text{ord}_p\, \eta = \mu$ and $f(\xi, \eta) = 0$ if and only if (λ, μ) is a point lying on a segment of the Indicator diagram of f.

Proof. Suppose (λ, μ) lies on a segment of the Indicator diagram of f. Then there are points (λ_1, μ_1) and (λ_2, μ_2) the vertices of this segment between which (λ, μ) is located. That is there is a γ with $0 \leqslant \gamma \leqslant 1$ such that:

$$\lambda = \gamma\lambda_1 + (1-\gamma)\lambda_2 \qquad (1)$$

$$\mu = \gamma\mu_1 + (1-\gamma)\mu_2$$

By the definition of Indicator diagram, $(\lambda_1, \mu_1, 1)$ and $(\lambda_2, \mu_2, 1)$ are upward-pointing normals to adjacent faces sharing a common edge in the Newton polyhedron of f. Hence by (1), $(\lambda, \mu, 1)$ is a normal lying in the same plane as and in between the normals $(\lambda_1, \mu_1, 1)$ and $(\lambda_2, \mu_2, 1)$. Therefore by Theorem 2.2 of Mohd Atan (1986) there are ξ, η in $\bar{Q}_p$ such that $\text{ord}_p\, \xi = \lambda$, $\text{ord}_p\, \eta = \mu$ and $f(\xi, \eta) = 0$ as required.

Conversely, suppose that there are ξ, η in $\bar{Q}_p$ with $\text{ord}_p\, \xi = \lambda$ and $\text{ord}_p\, \eta = \mu$ and $f(\xi, \eta) = 0$. Then by Theorem 2.1 of Mohd Atan (1986), $(\lambda, \mu, 1)$ is a normal to an edge E in the Newton polyhedron of f. Then $(\lambda, \mu, 1)$ lies in the same plane as and in between two upward-pointing normals $(\lambda_1, \mu_1, 1)$ and $(\lambda_2, \mu_2, 1)$ say, to two adjacent faces sharing the edge E in the Newton polyhedron of f. Therefore there exists γ with $0 \leqslant \gamma \leqslant 1$ such that

$$\lambda = \gamma\lambda_1 + (1-\gamma)\lambda_2$$
$$\mu = \gamma\mu_1 + (1-\gamma)\mu_2.$$

which implies that (λ, μ) is a point lying in between (λ_1, μ_1) and (λ_2, μ_2) the vertices of a segment in the Indicator diagram associated with f as asserted.

The following theorem gives the shape of the combination of the Indicator diagrams associated with two polynomials in $\bar{Q}_p[x,y]$ having common zeroes.

Theorem 2.2

Let f and g be polynomials in $\bar{Q}_p[x,y]$. If f and g have common zeros, then the Indicator diagrams associated with f and g intersect in at least one point.

Proof. Let (ξ, η) be a common zero of f and g. Then by Theorem 2.1 $(\text{ord}_p\, \xi, \text{ord}_p\, \eta)$ is a point on the Indicator diagrams of f and g. Clearly $(\text{ord}_p\, \xi, \text{ord}_p\, \eta)$ is a point of intersection of both Indicator diagrams.

An illustration of the assertion of Theorem 2.2 is given by the following example.

Example 2.1

Consider the polynomials

$$f(x, y) = ax^2 + by^2 + c$$
$$g(x, y) = dxy + e$$

in $\bar{Q}_p[x,y]$ with $p > 2$.

On solving $f(x, y) = 0$, $g(x, y) = 0$ the solutions are:

$$x^2 = \frac{-cd \pm (c^2d^2 - 4abe^2)^{1/2}}{2ad} \text{ with } y = \frac{-e}{dx}$$

or

$$y^2 = \frac{-cd \mp (c^2d^2 - 4abe^2)^{1/2}}{2bd} \text{ with } x \frac{-e}{dy}$$

There are three cases to consider.

Case 1: When $\text{ord}_p\, c^2d^2 > \text{ord}_p\, abe^2$. Then a solution (x, y) of f(x, y) = 0 and g(x, y) = 0 satisfies the following

$$\text{ord}_p\, x = \frac{1}{4}\,\text{ord}_p \frac{be^2}{ad^2} \text{ and } \text{ord}_p\, y = \frac{1}{4}\,\text{ord}_p \frac{ae^2}{bd^2}$$

All four zeros of f and g have the given orders.

Case 2: When $\text{ord}_p\, c^2d^2 < \text{ord}_p\, abe^2$. If (x, y) is a solution to f(x, y) = 0 and g(x, y) = 0, then

$$\text{ord}_p\, x = \frac{1}{2}\,\text{ord}_p \frac{c}{a},\ \text{ord}_p\, y = \frac{1}{2}\,\text{ord}_p \frac{ae^2}{cd^2}$$

or

$$\text{ord}_p\, y = \frac{1}{2}\,\text{ord}_p \frac{c}{b},\ \text{ord}_p\, x = \frac{1}{2}\,\text{ord}_p \frac{be^2}{cd^2}.$$

There is a pair of zeros of f and g for each order

Case 3: When $\text{ord}_p\, c^2d^2 = \text{ord}_p\, abe^2$. If (x, y) is a common zero of f and g then

$$\text{ord}_p\, x = \frac{1}{2}\,\text{ord}_p \frac{c}{a} \text{ and } \text{ord}_p\, y = \frac{1}{2}\,\text{ord}_p \frac{c}{b}$$

All four zeros of f and g have the given orders.

Fig. 3 shows all the possible intersection points of the Indicator diagrams associated with polynomials f and g in Example 3. The intersection points correspond to all the cases discussed in that example.

3. A CONJECTURE

We will now investigate the converse of the assertion in Theorem 2.2. On examining combinations of Indicator diagrams of pairs of polynomials, we make the following conjucture.

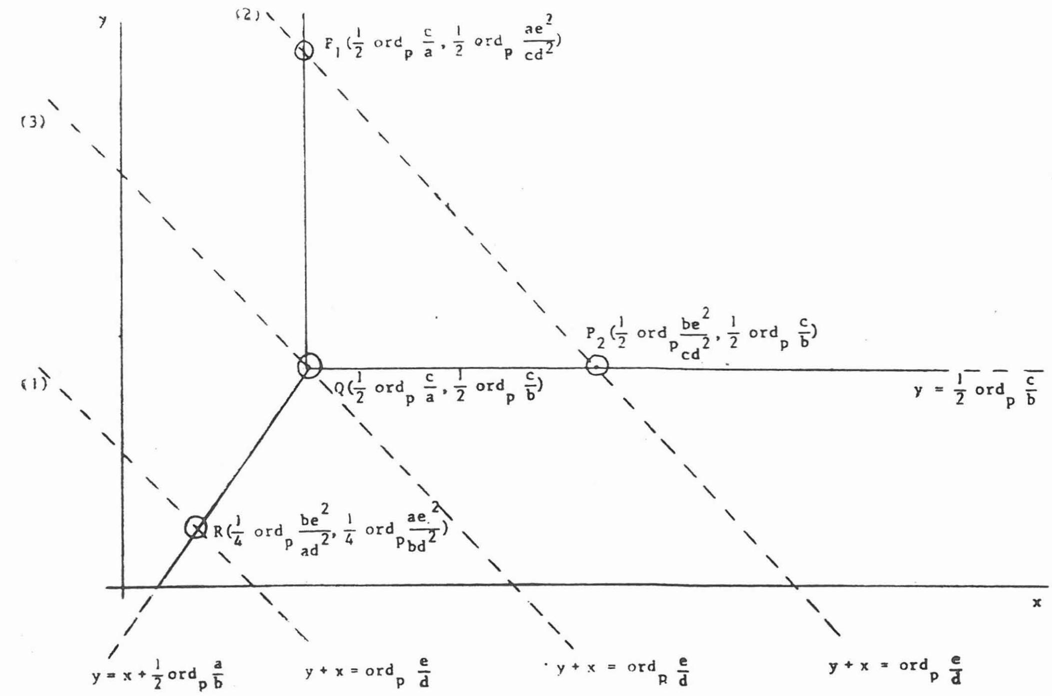

Fig. 3: Combinations of the Indicator diagrams associated with Newton polyhedrons of $f(x, y) = ax^2 + by^2 + c$ in bold and $g(x, y) = dxy + e$ in broken lines with their intersection points in the various cases possible.

Conjecture 3.1

Let f and g be polynomials in $\bar{Q}_p[x,y]$ and let (λ, μ) be a point of intersection of their Indicator diagrams and suppose that the edges through (λ, μ) do not coincide. Then there are ξ and η in $\bar{Q}_p$ satisfying $f(\xi, \eta) = g(\xi, \eta) = 0$ and $\mathrm{ord}_p\, \xi = \lambda$, $\mathrm{ord}_p\, \lambda = \mu$..

Theorem 1.1 is a special case of Conjecture 3.1 for polynomials of low degree. The proof of this theorem depends on the following assertion.

Lemma 3.1

Let S be a closed subset of Ω^2_p. Suppose that $f : S \to S$ is a map such that for some real number M with $0 < M < 1$ and for all a and b in S.

$$|f(b) - f(a)|_p \leqslant M\, |b - a|_p$$

Then there is a unique x in S with $f(x) = x$.

Proof We follow a standard argument of the standard fixed point theorem (Porteous, 1969), adapted however for the p-adic case.

Let x_o be any point of S and consider the sequence defined by $f(x_n) = x_{n+1}$ with initial value x_o. This is a Cauchy sequence since for any $n \geqslant 1$,

$$|x_{n+1} - x_n|_p = |f(x_n) - f(x_{n-1})|_p$$

$$\leqslant M\, |x_n - x_{n-1}|_p \leqslant M^n\, |x_1 - x_o|_p$$

whence for all $m \geqslant 0$

$$|x_{n+m+1} - x_n|_p \leqslant \max_{n \leqslant j \leqslant n+m} |x_{j+1} - x_j|_p$$

$$\leqslant M^n\, |x_1 - x_o|_p$$

Since Ω_p is complete and S is closed the sequence has a limit x say, in S. Moreover, since f is continuous, $f(x) = x$. Finally if $f(x') = x'$ then.

$$|x' - x|_p = |f(x') - f(x)|_p \leqslant M\, |x' - x|_p$$

which implies that x', $= x$.

We will now give the proof to Theorem 1.1.

Proof of Theorem 1.1 Since (λ, μ) is not a vertex on the Indicator diagram of f there are exactly two terms of f which dominate the other terms at any point $[x,y]$ in Ω_p^2 with $\mathrm{ord}_p\, x = \lambda$. and $\mathrm{ord}_p\, y = \mu$. We write $f = s_1 + s_2 + f_1$, where s_1 and s_2 are the dominant terms of f, that is $\mathrm{ord}_p\, s_1(x, y) = \mathrm{ord}_p\, s_2(x, y)$ and this order is less than the order of the remaining terms in $f_1(x, y)$ whenever $\mathrm{ord}_p\, x = \lambda$ and $\mathrm{ord}_p\, y = \mu$. Similarly, we write $g = t_1 + t_2 + g_1$, where t_1 and t_2 are the dominant terms of g. After replacing x by $p^\lambda x$ and y and $p^\mu . y$, we can suppose $\lambda = \mu = 0$. Also after multiplying f and g by suitable constants we can suppose $\mathrm{ord}_p\, s_i[x,y] = \mathrm{ord}_p\, t_i[x,y] = 0$ for x, y with $\mathrm{ord}_p\, x = \mathrm{ord}_p\, y = 0$ and $i = 1, 2$. Note that now all coefficients of the polynomials f_1 and g_1 have positive orders.

Let $S = \{(x, y)$ in Ω^2_p: $\mathrm{ord}_p\, x = \mathrm{ord}_p\, y = 0\}$. If we write $s_1 s_2^{-1} = -a^{-1} x^{\alpha} y^{\beta}$ and $t_1 t_2^{-1} = -b^{-1} x^{\gamma} y^{\delta}$, then the equations $f = g = 0$ become:

$$x^{\alpha} y^{\beta} = a(1 + f_1 s_2^{-1}),\; x^{\gamma} y^{\delta} = b(1 + g_1 t_2^{-1}).$$

Since the edges of two Indicator diagrams actually cross at (0, 0), the two monomials on the left-hand sides of these equations are multiplicatively independent, that is $d = \alpha\delta - \beta\gamma$ is non-zero. Consequently, we can "solve" the equations by:

$$x = h_1(x, y) =$$

$$a^{\delta/d}\, b^{-\beta/d} (1 + f_1 s_2^{-1})^{\delta/d} (1 + g_1 t_2^{-1})^{-\beta/d},$$

$$y = h_2(x,y) =$$

$$a^{-\gamma/d}\, b^{\alpha/d} (1 + f_1 s_2^{-1})^{-\gamma/d} (1 + g_1 t_2^{-1})^{\alpha/d}$$

For (x, y) in S, the d-th roots occurring here can be defined by their bionomial expansions; these converge p-adically because $p > d$ and f_1 and g_1 have positive orders.

Define $h: S \to S$ by the rule $h(x, y) = (h_1(x, y), h_2(x, y))$. We assert that h has the contraction mapping property of Lemma 3.1. By the previous remarks, $h_1(x, y)$ has an absolutely convergent expansion on S, say:

$$h_1(x, y) = a_o + \sum_{(i, j) \neq (0, 0)} a_{ij} x^i y^j$$

with $|a_{ij}|_p \leqslant M < 1$ and $|a_{ij}|_p \to 0$ as $|i| + |j| \to \infty$.

For (x, y) in S,

$$|x'^i y'^j - x^i y^j|_p = |(x'^i - x^i) y'^j + x^i (y'^j - y^j)|_p$$

$$\leqslant \max \{|x' - x|_p, |y' - y|_p\}$$

and consequently

$$|h_1(x', y') - h_1(x, y)|_p$$

$$\leqslant M \max (|x' - x|_p, |y' - y|_p)$$

The same argument applies to h_2. The norm $\|(x, y)\| = \max \{|x_p|, |y|_p\}$ turns Q_p^2 into a complete metric space and S is a closed subset. So the hypotheses of Lemma 3.1 are satisfied. Hence h has a unique fixed point (ξ, η) in S. By the earlier construction (ξ, η) is a common zero of f and g and its components have the required orders.

CONCLUSION

Theorem 2.1 gives a proof of an assertion in Mohd Atan (1986) and its converse. Theorem 2.2 gives the shape of the combination of Indicator diagrams associated with the Newton polyhedrons of two polynomials in $\bar{Q}_p(x, y)$ having common zeros. It is shown that the Indicator diagrams of such polynomials will have a point of intersection corresponding to each common zero. Conversely we conjecture that a point of intersection in the combination of Indicator diagrams of two polynomials in $Q_p[x,y]$ gives the p-adic orders of a common zero of the polynomials. Theorem 1.1 gives a proof of a special case of this assertion.

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