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Convergence of the Variable Order and Variable Stepsize Direct Integration Methods for the Solution of the Higher Order Ordinary Differential Equations

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ABSTRAK

Kaedah Pengkamilan Terus bagi penyelesaian terus satu sistem persamaan pembezaan biasa peringkat tinggi dibincangkan. Penumpuan bagi kaedah Pengkamilan Terus dengan langkah dan peringkat yang tetap dibuktikan dahulu, sebelum penumpuan bagi kaedah tersebut dengan langkah dan peringkat yang berubah dibuktikan.

ABSTRACT

The DI methods for directly solving a system of a general higher order ODEs are discussed. The convergence of the constant stepsize and constant order formulation of the DI methods is proven first before the convergence for the variable order and stepsize case.

1. INTRODUCTION

The general system of higher order initial values ODEs is given by

$$y_i^{(d_i)} = f_i\,(x, Y),\ i = 1, 2, ..., s \qquad (1)$$

with initial conditions

$$Y(a) = \eta$$

in the interval $a \leqslant x \leqslant b$, where the i-th equation is of order d_i and

$$Y^T(x) = (y_1, ..., y_1^{(d_1-1)}, ..., y_s, ..., y_s^{(d_s-1)}),$$

$$\eta^T(x) = (\eta_{1,o}, ..., \eta_{1,d_1-1}, ..., \eta_{s,o}, ..., \eta_{s,d_s-1}).$$

Many problems in engineering and science can be formulated in terms of such a system. Some of the higher order ODEs problems found in the literature are on the symmetrical bending of a laterally loaded circular plate, the bending of a thin beam clamped at both ends. Both problems are given in Russell and Shampine (1972). Others are on the steady flow of a viscoelastic fluid parallel to an infinite plane surface with uniform sunction; given in Serth (1975), control theory in Enright *et al.* (1974), coplanar circular two body motion in Shampine and Gordon (1975), gravitational n-body problem in Dol *et al.* (1972).

The current technique of solving the problem in (1) is by reducing it to a system of first

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order equations. Let

$$z_{11} = y_1, z_{12} = y_1', ..., z_{1d_1} = y_1^{(d_1-1)},$$

$$z_{21} = y_2, z_{22} = y_2', ..., z_{2d_2} = y_2^{(d_2-1)},$$

$$\vdots$$

$$z_{s1} = y_s, z_{s2} = y_s', ..., z_{sd_s} = y_s^{(d_s-1)};$$

and

$$z^T = [z_{11}, z_{12}, ..., z_{1d_1}, ..., z_{s1}, z_{s2}, ..., z_{sd_s}]$$

and

$$F^T(x, z) = [z_{12}, z_{13}, ..., f_1, ..., z_{s2}, z_{s3}, ..., f_s]$$

then (1) reduces to the equivalent first order problem,

$$z' = F(x, z), \tag{2}$$

with

$$z(a) = \eta$$

The system in (2) is then solved using either the Adams or the Runge-Kutta class of methods. But the work of Krogh (1969) and Suleiman (1979) have shown that for many problems, it is more efficient to solve (1) by integrating directly, rather than reducing it to (2). This DI method is in fact a generalization of the Adams-type of methods to higher order ODEs. Hall and Suleiman (1981) and Gear (1978) have studied the stability of the method for solving second order ODEs. Gear (1971) proves the convergence for certain methods which solved (1), those which can be transformed into the Nordsieck's arrays, while Shampine and Gordon (1975) prove the case for the first order problem.

2 THE DI METHODS IN THE PREDICTOR-CORRECTOR MODE

For simplicity of description and without loss of generality, the discussion which follows will be for the single equation

$$y^{(d)} = f(x, Y), Y(a) = A \tag{3}$$

where $Y^T = (y, y', ..., y^{(d-1)})$

and $A^T = (\eta, \eta', ..., \eta^{(d-1)})$.

Integrating (3) d − r times, r = 0, 1, ..., d − 1 leads to the identity

$$y^{(r)}(x_{n+1}) = \sum_{i=0}^{d-1-r} \frac{h^i}{i!} y^{(r+i)}(x_n) + \tag{4}$$

$$\underbrace{\int_{x_n}^{x_{n+1}} \int_{x_n}^{x} \cdots \int_{x_n}^{x}}_{\leftarrow\, d-r \text{ times } \rightarrow} f(x, Y(x))\, dx \ldots dx .$$

f is then approximated with an interpolation polynomial $P_{k,n}(x)$ of degree k − 1 which interpolates f at the k preceding points $(f_n, x_n), (f_{n-1}, x_{n-1}), ..., (f_{n-k+1}, x_{n-k+1})$. Thus expressed in terms of divided differences,

$$P_{k,n}(x) = f_{[n]} + (x - x_n) f_{[n,n-1]} + ... + (x - x_n) ... (x - x_{n-k+2}) f_{[n,n-1,...,n-k+1]}$$

where

$$f_{[n,n-1,...,n-i]} = \frac{f_{[n,n-1,...,n-i+1]} - f_{[n-1,n-2,...,n-i]}}{x_n - x_{n-i}}$$

Hence a discrete replacement for (4) is

$$p_{n+1}^{(r)} = \sum_{i=0}^{d-1-r} \frac{h^i}{i!} y_n^{(r+i)} + \int_{x_n}^{x_{n+1}} \int_{x_n}^{x} \cdots \int_{x_n}^{x} P_{k,n}(x)\, dx \ldots dx \tag{5}$$

where $p_{n+1}^{(r)}$ are predicted values associated with totally know back values.

Define, the integration coefficients g_{it} to be the t fold integral

$$g_{it} = \underbrace{\int_{x_n}^{x_{n+1}} \int_{x_n}^{x} \cdots \int_{x_n}^{x}}_{\leftarrow\, t \text{ times } \rightarrow} (x - x_n) \ldots (x - x_{n-i+1})\, dx\, dx \ldots dx \tag{6}$$

Then the predicted values in (5) can be written as

$$p_{n+1}^{(r)} = \sum_{i=0}^{d-1-r} \frac{h^i}{i!} y_n^{(r+i)} + \sum_{i=0}^{k-1} g_{i,d-r} f[n, n-1, ..., n-i]. \quad (7a)$$

The corrector formula is obtained by approximating f in (3) by the k degree polynomial $P_{k+1,\ n+1}(x)$ interpolating the (k + 1) points $(f^*_{n+1}, x_{n+1}), (f_n, x_n), ..., (f_{n-k+1}, x_{n-k+1})$, where $f^*_{n+1} \equiv f(x_{n+1}, P^*_{n+1}), P^*_n = (p_n, p_n', ..., p_n^{(d-1)})$.

In terms of $P_{k,n}(x)$,

$$P_{k+1,n+1}(x) = P_{k,n}(x) + (x - x_n)(x - x_{n-1}) \dots (x - x_{n-k+1})$$

$f^*_{[n+1,n,...,n-k+1]}$, where the last divided difference is obtained using the f evaluated by the predicted values P^*_{n+1}. Replacing $P_{k+1,\ n+1}(x)$ in (4) and integrating, we have the corrected values

$$y_{n+1}^{(d-r)} = p_{n+1}^{(d-r)} + g_{k,r} f^*_{[n+1,\ n,\ ...,\ n-k+1]},$$

$$r = 1, 2, ..., d \quad (7b)$$

Formulae (7a, b) represent the k-step predictor-corrector mode of the DI methods. In the variable order implementation of the method, k, if changed, is usually either decreasing or increasing by one from the value at the previous step.

From the definition of g_{it} in (6) it can be shown by integrating by parts that

$$g_{i,t} = (x_{n+1} - x_{n-i+1})g_{i-1,\ t} - tg_{i-1,\ t+1}. \quad (8)$$

For the case i = 0,

$$g_{0t} = \int_{x_n}^{x_{n+1}} \int_{x_n}^{x} \dots \int_{x_n}^{x} dx\ dx \dots dx = \frac{h^t_{n+1}}{t!}$$

where $h_{n+1} = (x_{n+1} - x_n)$. (9)

These integration coefficients are generated in a triangular array of the form shown below starting with the first row and from left to right with k + d + 2 coefficients along the first row and (k + 2) along the first column.

i \ t	0	1	2	. . .	k + d + 1
0	g_{00}	g_{01}	g_{02}	. . .	$g_{0,k+d+1}$
1	g_{10}	g_{11}	g_{12}	. . .	$g_{1,k+d}$
2	g_{20}	g_{21}	g_{22}	. . .	$g_{2,k+d-1}$
.	.			.	
.	.				
.	.				
k + 1	$g_{k+1,\ 0}$				

From (8) and (9) it is observed that $g_{0t} = g_{1,t-1}$ giving rise to the relation

$$g_{1t} = g_{1,\ t-1} \frac{h_{n+1}}{t+1},\ t = 1, ..., k + d,$$

and hence the computation of the first row is avoided. Constant stepsize will further account for the saving of computation, since if the stepsize remains constant for R steps, and the coefficients of the R-th row depend only on the current stepsize, therefore the coefficients the first R rows need not be recomputed and computation need only start at row R + 1. A change in order corresponds to adding or dropping an interpolation point. If at the current point to be computed, the step is not changed for R + 1 steps and the order is reduced, no special action is required. If however, the order is raised, one additional term needs to be added to the end table for each of the first R rows of the intergration coefficients, before commencing with computation of the row R + 1.

Despite the simplicity and advantages in speed, storage and accuracy of the DI methods, they are yet to be popularised as a general purpose code for the solution of the general order ODEs.

3. CONVERGENCE OF THE PREDICTOR-CORRECTOR MODE OF THE DI METHODS

Before the proof of convergence of the variable stepsize and order method in (7a, b) is given, the proof for the constant stepsize h and order k formulation of the method is first given.

Definition

A linear k-step method is said to be convergent if for any function $f(x, Y)$, for which the solution of the problem in (3) exists, and for all constants $\eta_o^{(m)}$,

$$m = 0, 1, ..., d - 1, \lim_{\substack{h\to 0\\ nh=x-a}} y_n^{(s)} = y^{(s)}(x),$$

$$s = 0, 1, ..., d - 1;$$

where $n = (x - a)/h$ for any $x \in [a, b]$, with starting values $y_\mu^{(m)} = \eta_\mu^{(m)}(h)$ satisfying the condition $\lim_{h\to 0} \eta_\mu^{(m)} = \eta_0^{(m)}$, $\mu = 0, 1, ..., k-1;$

$$m = 0, 1, ..., d - 1.$$

Let e_n be the global error between the true solution $Y(x_n)$ and the computed solution Y_n, i.e.,

$$e_n^T = Y^T(x_n) - Y_n^T$$
$$= (y(x_n) - y_n, y'(x_n) - y_n', ..., y^{(d-1)}(x_n) - y_n^{(d-1)})$$

We now quote the following theorem.

Theorem 1 If $f(x, Y)$ is a function such that the solution to (3) exists and the starting errors are such that $E_o = \max_i \| e_i \| \leqslant ch$, $c > 0$, then the constant stepsize h and constant order k formulation of 7 (a, b) converges.

Proof The Lagrangian form of the polynomial $P_{k,n}(x)$ which interpolates the points, $\{ (x_{n+1-i}, f_{n+1-i})\ i = 1, ..., k \}$ associated with the predictor is

$$P_{k,n}(x) = \sum_{i=1}^{k} L_i(x, 1) f_{n+1-i}$$

where $L_i(x, r) = \prod_{\substack{j=r\\ j\neq i}}^{k} \dfrac{(x - x_{n+1-j})}{(x_{n+1-i} - x_{n+1-j})}$ is the Lagrangian polynomial interpolating the set of values $\{ (x_{n+1-r}, y_{n+1-r}), ..., (x_{n-k+1}, y_{n-k+1}) \}$. This form of the interpolation polynomial is used here for convenience of proving the convergence of our theorem. The constant stepsize and order formulation of the predictor in (7a) is

$$p_{n+1}^{(d-r)} = y_n^{(d-r)} + hy_n^{(d-r+1)} + ... + \frac{h^{r-1}}{(r-1)!} y_n^{(d-1)}$$
$$+ h^r \sum_{i=1}^{k} \gamma_{i,\ d-r} f_{n+1-i},\ r = 1, ..., d \qquad (10)$$

where the coefficients

$$\gamma_{i,\ d-r} = \frac{1}{h^r} \underbrace{\int_{x_n}^{x_{n+1}} \int_{x_n}^{x} ... \int_{x_n}^{x}}_{\leftarrow r \text{ times} \rightarrow} L_i(x, 1)\, dx ... dx$$

are constants independent of h. To see this, put $x = sh + x_n$ then

$$\gamma_{i,d-r} = \int_0^1 \int_0^s ... \int_0^s c_i \prod_{\substack{j=0\\ j\neq i-1}}^{k-1} (s + j)\, ds...ds$$

and c_i are constants.

The constant stepsize, Lagrangian form of the corrector polynomial is given by

$$P_{k+1,n+1}(x) = \sum_{i=1}^{k} L_i(x, 0) f_{n+1-i} + L_0(x, 0) f_{n+1}^*$$

and the corrected values satisfy

$$y_{n+1}^{(d-r)} = \sum_{i=0}^{r-1} \frac{h^i y_n^{(d-r+i)}}{i!} + h^r \sum_{i=1}^{k} \gamma_{i,d-r}^* f_{n+1-i}$$
$$+ h^r \gamma_{0,d-r}^* f_{n+1}^*,\ r = 1, 2, ..., d. \qquad (11)$$

where

$$\gamma^*_{i,d-r} = \underbrace{\int_0^1 \int_0^s \cdots \int_0^s}_{\leftarrow\ r\ \text{times}\ \rightarrow} L_i\,(x_n + sh, 0)\, ds \ldots ds.$$

Let $H^T = (h^d, h^{d-1}, \ldots, h)$, α_i a $d \times d$ diagonal matrix, $=$ diagonal $(\gamma_{i,0}, \gamma_{i,1}, \ldots, \gamma_{i,d-1})$, α^*_i a $d \times d$ diagonal matrix, $\alpha^*_i =$ diagonal $(\gamma^*_{i,0}, \gamma^*_{i,1}, \ldots, \gamma^*_{i,d-1})$,

A a $d \times d$ matrix, A =

$$\begin{bmatrix} 1 & h & \frac{h^2}{2!} & \cdots & \frac{h^{d-1}}{(d-1)!} \\ 0 & 1 & h & \cdots & \frac{h^{d-2}}{(d-2)!} \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & & \cdot & \cdot & \\ 0 & \cdot & \cdot & 0 & 1 \end{bmatrix}$$

Then using the above notation, (10) and (11) can be written in the matrix form as

$$P_{n+1} = AY_n + \sum_{i=1}^{k} \alpha_i \; Hf_{n+1-i} \tag{12}$$

$$Y_{n+1} = AY_n + \sum_{i=1}^{k} \alpha^*_i \; Hf_{n+1-i} + \alpha^*_0 \; Hf^*_{n+1} \tag{13}$$

Since $Y(x_{n+1})$ is the true solution at x_{n+1}, then

$$Y(x_{n+1}) = \tag{14}$$

$$AY(x_n) + \sum_{i=1}^{k} \alpha_i \; Hf(Y(x_{n+1-i})) + \tau^*_{n+1}$$

$$Y(x_{n+1}) = AY(x_n) + \sum_{i=1}^{k} \alpha^*_i Hf(Y(x_{n+1-i}))$$

$$+ \alpha^*_0 \; Hf(Y(x_{n+1})) + \tau_{n+1} \tag{15}$$

where τ^*_{n+1} and τ_{n+1} are vectors for the local truncation errors of the explicit and implicit methods respectively.

Subtracting (15) from (13), the global error e_{n+1} satisfied

$$e_{n+1} = Ae_n + \sum_{i=1}^{k} \alpha^*_i \; H \frac{\partial f}{\partial Y}(\xi_{n+1-i})\,(Y(x_{n+1-i})$$

$$- Y_{n+1-i}) + \alpha^*_0 \; H \frac{\partial f}{\partial Y}(\xi_{n+1-i})\,(Y(x_{n+1}) - P_{n+1})$$

$$+ \tau_{n+1}$$

i.e.

$$e_{n+1} = Ae_n + \sum_{i=1}^{k} \alpha^*_i \, HG^T_{n+1-i} \; e_{n+1-i}$$

$$+ \alpha^*_0 HG^T{}_{n+1} \,(Ae_n + \sum_{i=1}^{k} \alpha_i \; HG^T_{n+1-i} \; e_{n+1-i})$$

$$+ \delta_{n+1}$$

where $\delta_{n+1} = (\alpha^*_0 \; HG^T_{n+1} \quad \tau^*_{n+1} + \tau_{n+1})$ is the local truncation error for the combined predictor-corrector method and

$G(\xi_{n+1-i})$, where ${}_{n+1-i} = \dfrac{\partial f}{\partial Y}$

ξ_{n+1-i} is a vector between

$$\begin{cases} Y(x_{n+1-i}) \text{ and } Y_{n+1-i}, & i=1, 2, \ldots k \\ Y(x_{n+1}) \text{ and } P_{n+1}, & i = 0. \end{cases}$$

Therefore,

$$\| e_{n+1} \| \leqslant \| A \| \, \| e_n \| + h^*L \sum_{i=1}^{k} \| \alpha^*_i \| \, \| e_{n+1-i} \|$$

$$+ h^*L\,\alpha^* \, \| A \| \, \| e_n \| + h^{*2} \, L^2 \, \alpha^*$$

$$(\sum_{i=1}^{k} \| e_{n+1-i} \| \, \| \alpha_i \|) + \delta \tag{16}$$

where $h^* = \| H \|$, $L = \max_i \| G_{n+1-i} \|$,

$$\delta = \max_m \| \delta_m \|, \; \alpha^* = \sum_{i=0}^{k} \| \alpha_i{}^* \|,$$

$$\alpha = \sum_{i=1}^{k} \| \alpha_i \|.$$

Here, we introduce $\{E_i\}$ a sequence such that $\| e_j \| \leqslant E_i$ for $j = 0, 1, \ldots, i$ and for all $i = 0, 1, \ldots, n\ (n \geqslant k-1)$ and E_0 is given, $E_0 = \max \| y(x_r) - y_r \|$, $r = 0, 1, \ldots, k-1$. Noting that $\| A \| \leqslant 1 + c_1 h$ and $h^* \leqslant c_2 h$, for some constants $c_1 > 0$, $c_2 > 0$ and sufficiently small h, then from (17), we have

$$\| e_{n+1} \| \leqslant \{1 + c_1 h + 2hc_2 L\alpha^* + c_3 h^2\} E_n + \delta, \qquad (17)$$

for some constant $c_3 > 0$, where $c_3 > (c_1 c_2 L\alpha^* + c_2 L^2 \alpha^* \alpha)$).

Define

$$E_{n+1} = RE_n + \delta \qquad (18)$$

where $R = 1 + c_1 h + 2hc_2 L\alpha^* + c_3 h^2$

then $\| e_{n+1} \| \leqslant E_{n+1}$ and $\| e_j \| \leqslant E_n \leqslant E_{n+1}$, $j = 0, 1, \ldots, n.$ (19)

The equation in (18), for R and δ positive gives rise to the relationship

$$E_{n+1} = R^{n+1} E_0 + \frac{R^{n+1} - 1}{R - 1} \delta, \; n = 0, 1, \ldots \qquad (20)$$

Further, since $(1 + x)^n \leqslant \exp(nx)$, $x \geqslant 0$, using (18), (19) and (20) then (17) becomes

$$\| e_{n+1} \| \leqslant R^{n+1} E_0 + \frac{R^{n+1}}{R-1} \delta, \text{ since } (R^{n+1} \geqslant R^{n+1} - 1) \leqslant (E_0 + \frac{\delta}{R-1}) \exp((n+1)x),$$

where $x = (c_1 h + 2hc_2 L\alpha^* + c_3 h^2)$. Noting that $(n+1)h = x_{n+1} - a$, the above inequality becomes

$$\| e_{n+1} \| < \{E_0 + \frac{\delta}{h(c_1 + 2c_2 L\alpha^* + c_3 h)}\} \exp((x_{n+1} - a)(c_1 + 2c_2 L\alpha^* + c_3 h))$$

Since $\delta = 0(h^{k+2})$ and since $E_0 \leqslant ch$, therefore $\| e_{n+1} \| \to 0$ as $h \to 0$.

Therefore the constant stepsize and order formulation of (7a, b) converges. If E_0 is $0(h^{k+1})$, then $\| e_{n+1} \|$ converges with order $k + 1$.

4. CONVERGENCE FOR VARIABLE STEP, VARIABLE ORDER METHOD

The proof is very similar to that of constant stepsize and order. In this case the variable stepsize is first bounded and then a lemma is stated.

Let the variable stepsize h_j and order k_j, be used on the j – th step, x_{j-1} to x_j, $j = 1, 2, \ldots, N$, such that $b - a = \sum_{j=1}^{N} h_j$. Let $H^* = \max_j h_j$, D^*, K are positive constants such that $NH^* \leqslant D^*$ and $k_j \leqslant K$ for all j. Let the ratio of successive stepsize be bounded,

$$\mu \leqslant \frac{h_{j-1}}{h_j} \leqslant \eta \quad \text{for all j and } k_j > 1. \text{ For } k_j = 1,$$

the above restriction is not necessary.

Lemma

$$\gamma_{i,d-r} = \int_0^1 \int_0^s \cdots \int_0^s L_i(x_n + sh_{n+1}, 1)\, ds \ldots ds$$

is bounded, where $0 < s < 1$.

Proof $\quad |\gamma_{i,d-r}| \leqslant \int_0^1 \int_0^s \cdots \int_0^s \prod_{\substack{j=1 \\ j \neq i}}^{k} |\theta_{ji}(s)|\, ds$

where $\quad \theta_{ji}(s) = \dfrac{x_n + sh_{n+1} - x_{n+1-j}}{x_{n+1-i} - x_{n+1-j}}$

and $\quad L_i(x_n + sh_{n+1}, 1) = \prod_{\substack{j=1 \\ j \neq i}}^{k} \theta_{ji}(s)$;

We have only to show that each term $|\theta_{ji}(s)|$ is bounded in order for $\gamma_{i,d-r}$ to be bounded. This is done by showing

that $\quad \beta_j = \left| \dfrac{x_n + sh_{n+1} - x_{n+1-j}}{h_{n+1}} \right|$

and $\quad \beta_j^* = \left| \dfrac{x_{n+1-i} - x_{n+1-j}}{h_{n+1}} \right|$

are bounded.

$$\beta_j \leqslant \left| \frac{x_{n+1} - x_{n+1-j}}{h_{n+1}} \right| = \left| \frac{h_{n+1} + \dots + h_{n+2-j}}{h_{n+1}} \right|$$

$$\leqslant 1 + \eta + \eta^2 + \dots + \eta^{j-1} = (\eta^j - 1)/(\eta - 1)$$

for all j, since for all m = 0, 1, ...

$$\frac{h_{n-m}}{h_{n+1}} = \frac{h_{n-m}}{h_{n+1-m}} \cdot \frac{h_{n+1-m}}{h_{n+2-m}} \cdot \dots \cdot \frac{h_n}{h_{n+1}}$$

$$\leqslant \eta^{m+1} .$$

In a similar manner, the denominator of $|\theta_{ji}(s)|$ is bounded below, the bound depending on the relative sizes of i and j. For $j > i$,

$$\left| \frac{x_{n+1-i} - x_{n+1-j}}{h_{n+1}} \right| = \left| \frac{h_{n+1-i} + h_{n-i} + \dots + h_{n+2-j}}{h_{n+1}} \right|$$

$$\geqslant \mu^i + \mu^{i+1} + \dots + \mu^{j-1} = \frac{\mu^j - \mu^i}{\mu - 1} \cdot$$

Similar bounds exist for $j < i$.

Hence $|\alpha_{i,r}|$ is bounded for all r. Therefore $\|\alpha_i\|$ is bounded. Similarly $\|\alpha_i^*\|$ is bounded.

Theorem 2 If $f(x, y)$ is a function such that the solution to (3) exists and the starting errors are such that $E_0 = \max_i \|e_i\| \leqslant c^* H^*$, $i = 0, 1, \dots, k_1 - 1$, then the variable stepsize, variable order method (7a, b) converges.

Proof The remaining argument in proving convergence for the variable stepsize, variable order case is the same as in Theorem 1. Equation (17) remains true except that H is replaced by H_{n+1} where $H_{n+1}^T = (h_{n+1}^d, h_{n+1}^{d-1}, \dots, h_{n+1})$.

Let $\quad \alpha \geqslant \sum_{j=1}^{k} \|\alpha_j\|$ and $\alpha^* \geqslant \sum_{j=O}^{k} \|\alpha_j^*\|$,

then we find that

$$\|e_{n+1}\| \leqslant (1 + c_1 H^* + 2H^* c_2 L^* + c_3 H^{*2}) E_n + \delta ,$$

this leads to

$$\|e_{n+1}\| \leqslant \{E_0 + \frac{\delta}{H^*(c_1 + 2c_2 L \alpha^* + c_3 H^*} \}$$

$$\exp (D^* (c_1 + 2c_2 L \alpha^* + c_3 H^*))$$

Hence the convergence of the variable stepsize, variable order method (7a, b).

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