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Quadrature Method for an Autonomous Ordinary Differential Equation

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RINGKASAN

Kaedah kuadratur digunakan bagi mendapatkan keputusan keunikan bagi persamaan pembezaan biasa tak linear autonomi. Kaedah ini membolehkan kami memberi syarat perlu dan cukup untuk penyelesaian unik bagi ketaklinearan cembung. Teknik ini kemudian digunakan untuk menentukan supremum bagi spektrum masalah dengan ketaklinearan cembung.

SUMMARY

A quadrature method is used to obtain uniqueness results for an autonomous nonlinear ordinary differential equation. This method enables us to give a necessary and sufficient condition for a unique solution when the nonlinearity is convex. The technique is then used to determine supremum of the spectrum for the problem with convex nonlinearity.

1. INTRODUCTION

Consider the autonomous ordinary differential equation

$$-u''(x) = \lambda f(u(x)), \quad x \in (0, 1)$$
$$u(0) = 0 = u(1). \tag{1.1}$$

We make the following assumptions on f:

(f1) $f \in C^2[0, \infty)$,

(f2) $f > 0$ on $[0, \beta]$, $0 \leqslant \beta < \infty$.

In this paper we are interested in positive solutions of (1.1). Such a problem has been extensively studied, among others by (Gelfand, 1963), (Joseph, 1965) , (Keller and Cohen, 1967), (Laetsch, 1970), and (Brown and Harun, 1977). The main result in Section 3 (Theorem 3.3) gives a necessary and sufficient condition for (1.1) to have a unique solution when f is convex. Our result is in contrast with what has been conjectured by Keller and Cohen that such a problem always has nonunique solutions.

2. QUADRATURE METHOD

In this paper we are interested in positive solutions of (1.1). We can extend f to a continuous nonnegative function on $(-\infty, \beta)$. By the maximum principle we may assume $\lambda > 0$ since if $\lambda \leqslant 0$ then no solution of (1.1) has a positive maximum and hence all solution are nonpositive. On the other hand, all solutions of (1.1) for $\lambda > 0$ are strictly positive and have precisely one maximum on (0, 1).

We shall first reduce (1.1) to a quadrature. For details of the procedure refer to Laetsch (1970).

Theorem 2.1 For any number $\rho \in (0,\beta)$, there is exactly one number $\lambda(\rho)$ and one nonnegative function $u(\rho)$ satisfying (1.1) such that $\| u \| = \sup \{ |u(x)| : x \in [0,1] \} = \rho$. We also have

$$\sqrt{\lambda(\rho)} = \sqrt{2} \int_0^\rho [F(\rho) - F(v)]^{-1/2} \cdot dv$$
$$= \sqrt{2}\, \rho \int_0^1 [F(\rho) - F(\rho s)]^{-1/2} ds \tag{2.1}$$

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and u is given by

$$\sqrt{2\lambda}\, x = \int_0^{u(x)} [\, F(\rho) - F(v)\,]^{-1/2}\, dv,$$

where $F(u) = \int_0^u f(s)\, ds.$

Furthermore, λ is a continuous function.

Proof The proof is straightforward and is a special case of the result in Brown and Harun (1977).

Clearly (2.1) has a solution for all λ in the range of λ (.) and if λ (.) is one-one then for a fixed λ, (2.1) has at most one solution.

We shall use equation (2.1) for our analysis. First we shall derive some preliminary results.

Lemma 2.2 The function

$$\int_0^1 [\, F(\rho) - F(\rho s)\,]^{-1/2}\, ds$$

is differentiable in ρ and

$$\frac{d}{d\rho}\int_0^1 [F(\rho)-F(\rho s)]^{-1/2}\, ds = -\tfrac{1}{2}\int_0^1 \frac{f(\rho)-sf(\rho s)}{[F(\rho)-F(\rho s)\,]^{3/2}}\, ds.$$

Proof The result can be obtained by applying simple calculus and the Lebesgue dominated convergence theorem.

Theorem 2.3 Let

$$\sqrt{\lambda(\rho)} = \sqrt{2\rho}\int_0^1 [F(\rho) - F(\rho v)\,]^{-1/2}\, dv.$$

Then

$$\frac{d\lambda}{d\rho}(\rho) > 0 \qquad (< 0) \quad \text{if}$$

$$J(\rho,v) = [F(\rho)-F(\rho v)] - \frac{1}{2}\rho^2 [f(\rho)-v f(\rho v)], \; v \in [0,1]$$

is positive (negative).

Proof. Direct differentiation.

The following results give us some elementary properties of the function J which we shall need later.

Lemma 2.4 The following relations hold

$$\frac{\partial J(\rho,v)}{\partial v} = \tfrac{1}{2}\rho[\,\rho v f'(\rho v) - f(\rho v)\,] \qquad (2.2)$$

$$\frac{\partial^2 J(\rho,v)}{\partial v^2} = \tfrac{1}{2}\rho^3\, v f''(\rho v). \qquad (2.3)$$

If $\quad H(\rho) = J(\rho,0) = F(\rho) - \frac{1}{2}\rho f(\rho),$ (2.4)

then $\quad H'(\rho) = \frac{1}{2}[f(\rho) - \rho f'(\rho)]$ (2.5)

and $\quad H''(\rho) = -\frac{1}{2}\rho f''(\rho)\,]$ (2.6)

Proof Direct differention.

3. UNIQUENESS RESULTS

The following results on uniqueness of solutions are obtained by considering the behaviour of the functions $J(\rho,.)$ and H. Similar results have been obtained by other authors using different techniques.

Theorem 3.1 There exists $r > 0$, sufficiently small such that (1.1) has a unique solution u with $\| u \| = \rho < r$.

Proof $H'(\rho) > 0$ for sufficiently small $\rho < r$ and since $H(0) = 0$ we get $H(\rho) > 0$ for $\rho < r$. Hence $J(\rho,0) > 0$ if $\rho < r$. Also from (2.2) $J(\rho,.)$ is a decreasing function if $\rho < r$ such that $J(\rho,1) = 0$. Hence $J(\rho,v) > 0$ for $\rho < r$ and $v \in [0,1]$. Hence λ is an increasing function and by Theorem 2.1 the solution of (1.1) is always unique for $\rho \in (0,r)$.

Theorem 3.2 If f satisfies (f1) and (f2) and in addition

either (i) $f' < 0$

or (ii) $f'' < 0$

then (1.1) possesses a unique solution.

Proof If $f' < 0$ then $H' > 0$. This together with $H(0) = 0$ implies that $H(\rho) > 0$ for all ρ. Also from (2.3) $J(\rho,v)$ decreases for $v \in [0,1]$. Then $J > 0$ on $(0,\infty) \times [0,1]$. Hence $\lambda(\rho)$ increases with ρ. Consequently (1.1) possesses a unique solution.

Suppose $f'' < 0$. Then H is convex with $H'(0) > 0$. Therefore $H(\rho) > 0$ for all ρ. Hence from (2.3) $J(\rho,.)$ is concave. Hence $J > 0$ on $(0, \infty) \times [0, 1]$. This implies that $\lambda(\rho)$ is increasing with ρ and hence (1.1) possesses a unique solution.

An interesting situation arises when the nonlinearity is convex. $J(\rho,.)$ and H are convex and concave respectively. H may change signs as ρ increases from zero, initially positive and eventually negative. However, if we impose the stronger condition that $f(\rho) - \rho f'(\rho) > 0$ for all $\rho > 0$, then $H > 0$.

Counter example

Consider the function $f(u) = 3u + (1+u)^{-2}$. Then f is positive for positive u, asymptotically linear, convex and increasing. Furthermore $f(\rho) - \rho f'(\rho) > 0$ for all $\rho > 0$ and tends to zero as $\rho \to \infty$. For this choice of nonlinearity $H > 0$, $\partial J(\rho,0)/\partial v < 0$ and $\partial J(\rho,1)/\partial v < 0$. The convexity of $J(\rho,.)$ implies $J(\rho,.) > 0$. Hence $\lambda(.)$ is an increasing function.

The above example shows that it is possible to obtain a unique solution for problem (1.1) when the nonlinearity f is convex. This is in contrast with what has been conjectured in (Keller and Cohen, 1967) that the solution of this class of nonlinearity is always nonunique.

The above example gives us a motivation for the following:

Theorem 3.3 Let f satisfy (f1), f2) and in addition $f'' > 0$. Then a necessary and sufficient condition that (1.1) possesses a unique solution is that $f(\rho) - \rho f'(\rho) > 0$ for all ρ.

Proof Assume $f(\rho) - \rho f'(\rho) > 0$ for all ρ. Then from (2.3) and (2.6) we have $J(\rho,.)$ convex and H concave. By the hypotheses of the theorem, H remains positive with a decreasing gradient. Hence $\partial J(\rho,1)/\partial v < 0$ and the convexity of $J(\rho,.)$ implies $J(\rho,.) > 0$. Hence $\lambda(.)$ increases and by Theorem 2.1 equation (1.1) possesses a unique solution.

Conversely suppose that $f(\rho_o) - \rho_o f'(\rho_o) \leqslant 0$ for some $\rho_o \geqslant 0$. Then $H'(\rho_o) \leqslant 0$ and by (2.6) $H'(\rho) \leqslant H'(\rho_o) \leqslant 0$ for $\rho \geqslant \rho_o$. Hence $H(\rho) < 0$ and so $J(\rho,0) < 0$ for sufficiently large ρ. Since $J(\rho,1) = 0$ and $J(\rho,.$ convex, it follows that $J(\rho,v) < 0$ for $v \in [0,1$ and so $\lambda(.)$ is strictly decreasing for sufficie tly large ρ. Since $\lambda(0) = 0$, $\lambda(\rho) > 0$ foı $\rho > 0$ and $\lambda(.)$ is eventually decreasing, λ cannot be one-one i.e. there exists λ for which (1.1) possesses more than one solution.

4. BOUND ON EIGENVALUES

We shall define the term 'spectrum' as the set of real values of λ for which positive solutions of (1.1) exist. The least upper bound of the spectrum is denoted by λ^*. Much interest has been focussed on evaluating λ^* since it plays a direct role in determining the multiplicity of solutions of (1.1), (Keller and Cohen, 1967) and (Laetsch, 1970). In this section we shall use the quadrature method developed in Section 1 to evaluate λ^* for the well-known problem

$$-u''(x) = \lambda \exp u(x), \quad x \in (0,1) \tag{4.1}$$

$$u(0) = 0 = u(1).$$

The value of λ^* has been determined by Gelfand (1963) using phase plane technique. Joseph (1965) has also obtained λ^* for (4.1).

For $f(u) = \exp(u)$, equation (2.1) takes the particular form

$$\sqrt{\lambda(\rho)} = 2^{3/2} e^{-\rho/2} \cosh^{-1}(e^{\rho/2}).$$

It is easy to show that λ has a maximum turning point at ρ, where ρ satisfies the transcendental equation

$$(e^{\rho} - 1)^{-1/2} - e^{-\rho/2} \cosh^{-1}(e^{\rho/2}) = 0.$$

Solving this, we obtain $\rho = 1.18683$ and the corresponding value of $\lambda^* = 3.51383$. We now compare this value with those obtained by Gelfand and Joseph.

Joseph studied the problem

$$u''(x) + \lambda \exp(u(x)) = 0, \quad x \in (-1,1)$$
$$u(-1) = 0 = u(1) \tag{J}$$

and obtained $\lambda^* = 0.893$. The transformation $t = (x+1)/2$ will transform (J) into

$$-u''(t) = 4\lambda \exp u(t), \quad t \in (0,1)$$
$$u(0) = 0 = u(1)$$

which is of the form (1.1). Hence $\lambda^* = 4(0.893) = 3.572$.

Gelfand's problem is

$$u''(\xi) + 2\exp u(\xi) = 0$$
$$u'(0) = 0, \; u(\lambda) = 0 \tag{G}$$

with $\lambda^* = 0.66$. The transformation $x = \xi/\lambda$ leads to the problem

$$u''(x) + 2\lambda^2 \exp u(x) = 0$$
$$u'(0) = 0, \; u(1) = 0,$$

which is equivalent to

$$u''(x) + 2\lambda^2 \exp u(x) = 0$$

$$u(-1) = 0, \; u(1) = 0.$$

Letting $t = (x+1)/2$ we get

$$-u''(t) = 8\lambda^2 \exp u(t)$$
$$u(0) = 0 = u(1).$$

Hence $\lambda^* = 8(0.66)^2 = 3.4846$.

We see that the value of λ^* obtained by the quadrature method falls between λ^* obtained by Gelfand and λ^* obtained by Joseph.

5. REMARKS ON GENERALIZATION

As a conclusion we remark that many of the results we have obtained for the autonomous ordinary differential equation hold true for more general cases of non-autonomous ordinary and partial differential equations by different methods. In this section we mention works that have been done on these more general cases. The general case for Theorem 3.1 can be be proved using contraction mapping principle and this can be found in Harun (1977). The result corresponding to Theorem 3.2(i) is found in Stuart (1975) and the result corresponding to Theorem 3.2(ii) is in Keller and Cohen (1967); the condition $f(\rho)-\rho f'(\rho)>0$ is equivalent to $u \rightarrow f(u)/u$ decreasing and this condition is shown to be sufficient for uniqueness in Cohen and Laetsch (1970).

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