Numerical solution of n'th order fuzzy initial value problems by six stages

Ali Jameel[a,*], N. R. Anakira[b], A. K. Alomari[c], Ishak Hashim[d], M. A. Shakhatreh[c]

[a]*School of Quantitative Sciences, Universiti Utara Malaysia (UUM), Kedah, Sintok, 06010 Malaysia.*

[b]*Department of Mathematics, Faculty of Science and Technology, Irbid National University, 2600 Irbid, Jordan.*

[c]*Department of Mathematics, Faculty of Science, Yarmouk University, 211-63 Irbid, Jordan.*

[d]*School of Mathematical Sciences, Universiti Kebangsaan Malaysia, 43600 Bangi, Selangor, Malaysia.*

Communicated by Wasfi Shatanawi

Abstract

The purpose of this paper is to present a numerical approach to solve fuzzy initial value problems (FIVPs) involving n-th order ordinary differential equations. The idea is based on the formulation of the six stages Runge-Kutta method of order five (RKM56) from crisp environment to fuzzy environment followed by the stability definitions and the convergence proof. It is shown that the n-th order FIVP can be solved by RKM56 by transforming the original problem into a system of first-order FIVPs. The results indicate that the method is very effective and simple to apply. An efficient procedure is proposed of RKM56 on the basis of the principles and definitions of fuzzy sets theory and the capability of the method is illustrated by solving second-order linear FIVP involving a circuit model problem. ©2016 All rights reserved.

Keywords: Fuzzy numbers, fuzzy differential equations, circuit model problem, six stages Runge-Kutta method of order five.
2010 MSC: 34A07, 65K15, 03B52.

1. Introduction

Many dynamical real life problems may be formulated as a mathematical model. These problems can be formulated either as a system of ordinary or partial differential equations [6, 15, 24]. Fuzzy differential

---

*Corresponding author
Email addresses:* kakarotte79@gmail.com (Ali Jameel), alanaghreh_nedal@yahoo.com (N. R. Anakira), abdomari2008@yahoo.com (A. K. Alomari), ishak_h@ukm.edu.my (Ishak Hashim), Mali@yu.edu.jo (M. A. Shakhatreh)

equations (FDEs) are a useful tool to model a dynamical system when information about its behavior is inadequate. Fuzzy Initial value problem (FIVP) appears when the modeling of these problems was imperfect and its nature is under uncertainty. Fuzzy ordinary differential equations are suitable mathematical models to model dynamical systems in which there exist uncertainties or vagueness. These models are used in various applications including, population models [26, 30] mathematical physics [9], and medicine [3, 7] An ordinary differential equation involving linear FIVP can be considered to be the simplest case to test the effectiveness of proposed methods for solving FDE. The explicit Runge-Kutta methods have been widely used in the solution of FIVPs. Even in the early days of these methods, the conditions for order seemed to have been thoroughly understood and methods of order 4 were successfully used to solve both single FIVPs [27] and systems of FIVPs [2]. The efficiency of Runge-Kutta methods depends on increasing the number of function evaluations per step [19]. In the last ducat various types of Runge Kutte methods have been used to solve n^{th} order FIVPs including, second order Runge Kutta method [28], third order Runge Kutta methods [12, 17, 21], fourth order Runge Kutta methods [2, 14], also fifth order Runge Kutta method of five stages in [18] and other methods [22, 31].

Our main motivation is to develop and analyze the use of RKM56 in order to obtain the numerical solution of FIVPs involving fuzzy linear second-order differential equation. To the best of our knowledge, this is the first attempt at solving n-th order FIVP by using RKM56. The structure of this paper is as follows. We will begin in Section 2 with some preliminary concepts about fuzzy numbers. In Section 3 we define the defuzzification procedure of n-th order FIVP. In Section 4, we analyze and formulate RKM56 for solving n-th order FIVP. In Section 5, we present the convergence and error analysis of RKM56 in detail. In Section 6, we employ RKM56 on test example involving second order linear FIVP model and finally, in Section 7, we give the conclusion of this study.

2. Preliminaries

Definition 2.1. Fuzzy numbers are a subset of the real numbers set, and represent uncertain values. Fuzzy numbers are linked to degrees of membership which state how true it is to say if something belongs or not to a determined set. A fuzzy number is called a triangular fuzzy number [8] if defined by three numbers $\alpha < \beta < \gamma$ where the graph of $\mu(x)$ is a triangle with the base on the interval $[\alpha, \beta]$ and vertex at $x = \beta$ and its membership function has the following form:

$$\mu(x; \alpha, \beta, \gamma) = \begin{cases} 0 , & if \ x < \alpha, \\ \frac{x-\alpha}{\beta-\alpha}, & if \ \alpha \leq x \leq \beta, \\ \frac{\gamma-x}{\gamma-\beta}, & if \ \beta \leq x \leq y, \\ 0 , & if \ x > \gamma \end{cases}$$
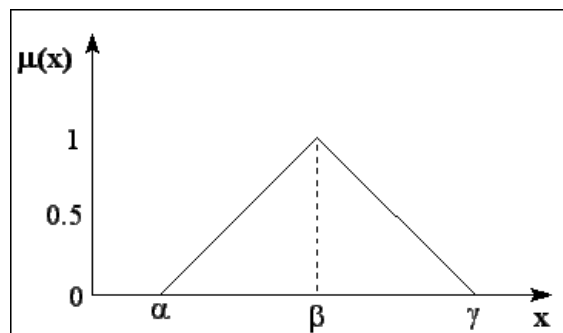

Figure 1: Triangular Fuzzy Number

and its r-level is: $[\mu]_r = [\alpha + r(\beta - \alpha), \gamma - r(\gamma - \beta)], r \in [0, 1].$

In this paper the class of all fuzzy subsets of $\Re$ will be denoted by $\tilde{E}$ and satisfy the following properties [8, 23]:

1. $\mu(t)$ is normal, i.e $\exists\, t_0 \in \Re$ with $\mu(t_0) = 1$.
2. $\mu(t)$ Is convex fuzzy set, i. e. $\mu(\lambda t + (1 - \lambda)s) \geq \min\{\mu(t), \mu(s)\} \quad \forall t, s \in \Re, \lambda[0, 1].$
3. μ upper semi-continuous on $\Re$.
4. $\overline{\{t \in \Re : \mu(t) > 0\}}$ is compact.

$\tilde{E}$ is called the space of fuzzy numbers and $\Re$ is a proper subset of $\tilde{E}$.

Define the r-level set $x \in \Re$, $[\mu]_r = \{x \setminus \mu(x) \geq r\}$, $0 \leq r \leq 1$, where $[\mu]_0 = \{x \setminus \mu(x) > 0\}$ is compact [30] which is a closed bounded interval and denoted by $[\mu]_r = (\underline{\mu}(t), \overline{\mu}(t))$. In the parametric form [20], a fuzzy number is represented by an ordered pair of functions $(\underline{\mu}(t), \overline{\mu}(t))$, $r \in [0, 1]$ which satisfies:

1. $\underline{\mu}(t)$ is a bounded left continuous non-decreasing function over $[0, 1]$.
2. $\overline{\mu}(t)$ is a bounded left continuous non-increasing function over $[0, 1]$.
3. $\underline{\mu}(t) \leq \overline{\mu}(t)$, $r \in [0, 1]$. A crisp number r is simply represented by $\underline{\mu}(r) = \overline{\mu}(r) = r$, $r \in [0, 1]$.

Definition 2.2 ([21]). If $\tilde{E}$ be the set of all fuzzy numbers, we say that $f(t)$ is a fuzzy function if $f : \Re \to \tilde{E}$.

Definition 2.3 ([13]). A mapping $f : T \to \tilde{E}$ for some interval $T \subseteq \tilde{E}$ is called a fuzzy function process and we denote r-level set by:

$$[\tilde{f}(t)]_r = \left[\underline{f}(t; r), \overline{f}(t; r)\right], t \in T, r \in [0, 1].$$

The r-level sets of a fuzzy number are much more effective as representation forms of fuzzy set than the above. Fuzzy sets can be defined by the families of their r-level sets based on the resolution identity theorem.

Definition 2.4 ([33, 34]). Each function $f : X \to Y$ induces another function $\tilde{f} : F(X) \to F(Y)$ defined for each fuzzy interval U in X by:

$$\tilde{f}(U)(y) = \left\{ \begin{array}{ll} \sup\limits_{x \in f^{-1}(y)} U(x), & if\, y \in range(f), \\ 0, & if\, y \notin range(f). \end{array} \right.$$

This is called the Zadeh extension principle.

Definition 2.5 ([29]). Consider $\tilde{x}, \tilde{y} \in \tilde{E}$. If there exists $\tilde{z} \in \tilde{E}$ such that $\tilde{x} = \tilde{y} + \tilde{z}$, then z is called the H-difference (Hukuhara difference) of x and y and is denoted by $\tilde{z} = \tilde{x} \ominus \tilde{y}$.

Definition 2.6 ([29]). If $\tilde{f} : I \to \tilde{E}$ and $y_0 \in I$, where $I \in [t_0, T]$. We say that $\tilde{f}$ Hukuhara Differentiable at y_0, if there exists an element $\left[\widetilde{f'}\right]_r \in \tilde{E}$ such that for all $h > 0$ sufficiently small (near to 0), exists $\tilde{f}(y_0 + h; r) \ominus \tilde{f}(y_0; r)$, $\tilde{f}(y_0; r) \ominus \tilde{f}(y_0 - h; r)$ and the limits are taken in the metric $(\tilde{E}, D)$

$$\lim_{h \to 0+} \frac{\tilde{f}(y_0 + h; r) \ominus \tilde{f}(y_0; r)}{h} = \lim_{h \to 0+} \frac{\tilde{f}(y_0; r) \ominus \tilde{f}(y_0 - h; r)}{h}.$$

The fuzzy set $\left[\widetilde{f'}(y_0)\right]_r$ is called the Hukuhara derivative of $\left[\widetilde{f'}\right]_r$ at y_0.

These limits are taken in the space $(\tilde{E}, D)$ if t_0 or T, then we consider the corresponding one-side derivation. Recall that $\tilde{x} \ominus \tilde{y} = \tilde{z} \in \tilde{E}$ are defined on r-level set, where $[\tilde{x}]_r \ominus [\tilde{y}]_r = [\tilde{z}]_r$, $\forall\, r \in [0, 1]$ By consideration of definition of the metric D all the r-level set $\left[\tilde{f}(0)\right]_r$ are Hukuhara differentiable at y_0, with

Hukuhara derivatives $\left[\widetilde{f}'(y_0)\right]_r$, when $\tilde{f} : I \to \tilde{E}$ is Hukuhara differentiable at y_0 with Hukuhara derivative $\left[\widetilde{f}'(y_0)\right]_r$ it' lead to that $\tilde{f}$ is Hukuhara differentiable for all $r \in [0,1]$ which satisfies the above limits i.e. if f is differentiable at $t_0 \in [t_0 + \alpha, T]$ then all its r-levels $\left[\widetilde{f}'(t)\right]_r$ are Hukuhara differentiable at t_0.

Definition 2.7 ([23])**.** Define the mapping $\widetilde{f}' : I \to \tilde{E}$ and $y_0 \in$ I, where $I \in [t_0, T]$. We say that $\widetilde{f}'$ Hukuhara differentiable $t \in \tilde{E}$, if there exists an element $\left[\widetilde{f}^{(n)}\right]_r \in \tilde{E}$ such that for all $h > 0$ sufficiently small (near to 0), exists such that $\widetilde{f}^{(n-1)}(y_0 + h; r) \ominus \tilde{f}^{(n-1)}(y_0; r)$, $\tilde{f}^{(n-1)}(y_0; r) \ominus \tilde{f}^{(n-1)}(y_0 - h; r)$ and the limits are taken in the metric $(\tilde{E}, D)$

$$\lim_{h \to 0+} \frac{\tilde{f}^{(n-1)}(y_0 + h; r) \ominus \tilde{f}^{(n-1)}(y_0; r)}{h} = \lim_{h \to 0+} \frac{\tilde{f}^{(n-1)}(y_0; r) \ominus \tilde{f}^{(n-1)}(y_0 - h; r)}{h}$$

exists and equal to $\tilde{f}^{(n)}$ and for $n = 2$ we have second order Hukuhara derivative.

Theorem 2.8 ([29])**.** *Let $\tilde{f} : [t_0 + \alpha, T] \to \tilde{E}$ be Hukuhara differentiable and denote by*

$$\left[\widetilde{f}'(t)\right]_r = \left[\underline{f}'(t), \overline{f}'(t)\right]_r = \left[\underline{f}'(t;r), \overline{f}'(t;r)\right].$$

Then the boundary functions $\underline{f}'(t;r), \overline{f}'(t;r)$ are differentiable

$$\left[\widetilde{f}'(t)\right]_r = \left[\left(\underline{f}(t;r)\right)', \left(\overline{f}(t;r)\right)'\right], \forall r \in [0,1].$$

Theorem 2.9 ([23])**.** *Let $\tilde{f} : [t_0 + \alpha, T] \to \tilde{E}$ be Hukuhara differentiable and denote*

$$\left[\widetilde{f}'(t)\right]_r = \left[\underline{f}'(t), \overline{f}'(t)\right]_r = \left[\underline{f}'(t;r), \overline{f}'(t;r)\right].$$

Then the boundary functions

$$\underline{f}'(t;r), \overline{f}'(t;r)$$

are differentiable we can write for n^{th} order fuzzy derivative

$$\left[\tilde{f}^{(n)}(t)\right]_r = \left[\left(\underline{f}^{(n)}(t;r)\right)', \left(\overline{f}^{(n)}(t;r)\right)'\right], \forall r \in [0,1].$$

3.   Defuzzification of n-th order FIVP

Consider the n-th order FIVP

$$\tilde{y}^{(n)}(t) = f\left(t, \tilde{y}(t), \tilde{y}'(t), \tilde{y}''(t), \ldots \tilde{y}^{(n-1)}(t)\right) + \tilde{w}(t), \ t \in [t_0, T]. \tag{3.1}$$

Subject to the initial conditions

$$\tilde{y}(t_0) = \tilde{y}_0, y'(t_0) = \tilde{y}_0', \ldots, y^{(n-1)}(t_0) = \tilde{y}_0^{(n-1)}, \tag{3.2}$$

where $\tilde{y}$ is a fuzzy function of the crisp variable t with $\tilde{f}$ being a fuzzy function of the crisp variable t, the fuzzy variable $\tilde{y}$ and the fuzzy Hukuhara-derivatives $\tilde{y}'(t), \tilde{y}''(t), \ldots, \tilde{y}^{(n-1)}(t)$. Here $y^{(n)}$ is the fuzzy n^{th} order Hukuhara-derivative and $\tilde{y}(t_0), \tilde{y}'(t_0), \ldots \tilde{y}^{(n-1)}(t_0)$ are triangular fuzzy numbers. We denote the fuzzy function y by $\tilde{y} = [\underline{y}, \overline{y}]$ for $t \in [t_0, T]$ and $r \in [0,1]$. It means that the r-level sets of $\tilde{y}(t)$ can be defined as:

$$[\tilde{y}(t)]_r = \left[\underline{y}(t;r), \overline{y}(t;r)\right],$$

$$\left[\tilde{y}'\left(t\right)\right]_r = \left[\underline{y}'\left(t;r\right), \overline{y}'\left(t_0;r\right)\right], \dots \left[\tilde{y}^{(n-1)}\left(t\right)\right]_r = \left[\underline{y}^{(n-1)}\left(t;r\right), \overline{y}^{(n-1)}\left(t;r\right)\right],$$

$$\left[\tilde{y}\left(t_0\right)\right]_r = \left[\underline{y}\left(t_0;r\right), \overline{y}\left(t_0;r\right)\right], \left[\tilde{y}'\left(t_0\right)\right]_r = \left[\underline{y}'\left(t_0;r\right), \overline{y}'\left(t_0;r\right)\right], \dots, \left[\tilde{y}^{(n-1)}\left(t_0\right)\right]_r$$

$$= \left[\underline{y}^{(n-1)}\left(t_0;r\right), \overline{y}^{(n-1)}\left(t_0;r\right)\right],$$

where $w\left(t\right)$ is crisp or fuzzy inhomogeneous term such that $\left[\tilde{w}\left(t\right)\right]_r = \left[\underline{w}\left(t;r\right), \overline{w}\left(t;r\right)\right]$. Since

$$y^{(n)}\left(t\right) = f\left(t, y\left(t\right), y'\left(t\right), y''\left(t\right), \dots y^{(n-1)}\left(t\right)\right) + w(t).$$

Let

$$\mathcal{Y}\left(t\right) = y\left(t\right), y'\left(t\right), y''\left(t\right), \dots y^{(n-1)}\left(t\right),$$

such that

$$\widetilde{\mathcal{Y}}\left(t;r\right) = \left[\underline{\mathcal{Y}}\left(t;r\right), \overline{\mathcal{Y}}\left(t;r\right)\right]$$
$$= \left[\underline{y}\left(t;r\right), \underline{y}'\left(t;r\right), \dots, \underline{y}^{(n-1)}\left(t;r\right), \overline{y}\left(t;r\right), \overline{y}'\left(t;r\right), \dots, \overline{y}^{(n-1)}\left(t;r\right)\right].$$

Also, we can write

$$\left[\tilde{f}\left(t, \widetilde{\mathcal{Y}}\right)\right]_r = \left[\underline{f}\left(t, \widetilde{\mathcal{Y}};r\right), \overline{f}\left(t, \widetilde{\mathcal{Y}};r\right)\right].$$

By using Zadeh extension principles, we have

$$\tilde{f}\left(t, \widetilde{\mathcal{Y}}\left(t;r\right)\right) = \left[\underline{f}\left(t, \widetilde{\mathcal{Y}}\left(t;r\right)\right), \underline{f}\left(t, \widetilde{\mathcal{Y}}\left(t;r\right)\right),\right],$$

such that

$$\underline{f}\left(t, \widetilde{\mathcal{Y}}\left(t;r\right)\right) = \mathcal{F}\left(t, \underline{\mathcal{Y}}\left(t;r\right), \overline{\mathcal{Y}}\left(t;r\right)\right) = \mathcal{F}\left(t, \widetilde{\mathcal{Y}}\left(t;r\right)\right)$$
$$\overline{f}\left(t, \widetilde{\mathcal{Y}}\left(t;r\right)\right) = \mathcal{G}\left(t, \underline{\mathcal{Y}}\left(t;r\right), \overline{\mathcal{Y}}\left(t;r\right)\right) = \mathcal{G}\left(t, \widetilde{\mathcal{Y}}\left(t;r\right)\right).$$

Then we have

$$\underline{y}^{(n)}\left(t;r\right) = \mathcal{F}\left(t, \widetilde{\mathcal{Y}}\left(t;r\right)\right) + \underline{w}\left(t;r\right) \tag{3.3}$$

$$\overline{y}^{(n)}\left(t;r\right) = \mathcal{G}\left(t, \widetilde{\mathcal{Y}}\left(t;r\right)\right) + \overline{w}\left(t;r\right), \tag{3.4}$$

where the membership function of

$$\mathcal{F}\left(t, \widetilde{\mathcal{Y}}\left(t;r\right)\right) + \underline{w}\left(t;r\right)$$

and

$$\mathcal{G}\left(t, \widetilde{\mathcal{Y}}\left(t;r\right)\right) + \overline{w}\left(t;r\right)$$

can be defined as follows:

$$\mathcal{F}\left(t, \widetilde{\mathcal{Y}}\left(t;r\right)\right) + \underline{w}\left(t;r\right) = min\{\tilde{y}^{(n)}\left(t, \widetilde{\mu}(r)\right) : \mu | \mu \in [\widetilde{\mathcal{Y}}\left(t;r\right)]_r\},$$

$$\mathcal{G}\left(t, \widetilde{\mathcal{Y}}\left(t;r\right)\right) + \overline{w}\left(t;r\right) = max\{\tilde{y}^{(n)}\left(t, \widetilde{\mu}(r)\right) : \mu | \mu \in [\widetilde{\mathcal{Y}}\left(t;r\right)]_r\}.$$

4. Analysis of RKM56 for n-th order FIVP

In this section we consider the six order Runge-Kutta method of order five as our proposed method to obtain a numerical solution of n'th order FIVP. The structure of this method for solving crisp first order initial value problem have been introduced in [10, 11, 25]. Also Jameel *et al.* [16] proposed RK56 to solve

first order linear FIVPs In order to solve n-th order FIVPs, we need to fuzzify RKM56 then defuzzify it. We consider the n-th order FIVP (3.1)–(3.2) in Section 3 and since RKM56 is numerical method we need to reduce Eq. (3.1) in to system of first order FIVPs for all $r \in [0, 1]$ as follows

$$
\begin{cases}
\tilde{y}_1' \left(t;r \right) = \tilde{f}_1 \left(\tilde{y}_1 \left(t;r \right), \ldots, \tilde{y}_n \left(t;r \right) \right) \\
\tilde{y}_2' \left(t;r \right) = \tilde{f}_2 \left(\tilde{y}_1 \left(t;r \right), \ldots, \tilde{y}_n \left(t;r \right) \right) \\
\vdots \\
\tilde{y}_3' \left(t;r \right) = \tilde{f}_n \left(\tilde{y}_1 \left(t;r \right), \ldots, \tilde{y}_n \left(t;r \right) \right) \\
\tilde{y}_1 \left(t_0;r \right) = \tilde{y}_{1,0} \left(r \right) = \tilde{a}_1, \\
\tilde{y}_2 \left(t_0;r \right) = \tilde{y}_{2,0} \left(r \right) = \tilde{a}_2, \ldots, \tilde{y}_n \left(t_0;r \right) = \tilde{y}_{n,0} \left(r \right) = \tilde{a}_n,
\end{cases} \tag{4.1}
$$

where $t \in [t_0, T]$ and $a_1, \ldots, a_n$ are fuzzy numbers that defined Section 2 with r-level intervals

$$
\tilde{y}_{i,0} \left(r \right) = \left[\underline{y}_{i,0} \left(r \right), \overline{y}_{i,0} \left(r \right) \right] = [\underline{a}_i, \overline{a}_i]
$$

for $i = 1, 2, \ldots, n$ and $r \in [0, 1]$. We let

$$
\tilde{y} \left(t;r \right) = \left[\tilde{y}_1 \left(t;r \right), \tilde{y}_2 \left(t;r \right), \ldots, \tilde{y}_n \left(t_0;r \right) \right]^t
$$

be the fuzzy solution of Eq. (3.1) on $[t_0, T]$ if

$$
\underline{y}_i{}' \left(t;r \right) = min \left\{ \tilde{f}_i \left(\tilde{y}_1 \left(t;r \right), \ldots, \tilde{y}_n \left(t;r \right) \right); \tilde{y}_i \left(t;r \right) \in \left[\underline{y}_i \left(t;r \right), \overline{y}_i \left(t;r \right) \right] = \underline{f}_i \left(t; \tilde{y} \left(t;r \right) \right) \right\} \tag{4.2}
$$

$$
\overline{y}_i{}' \left(t;r \right) = max \left\{ \tilde{f}_i \left(\tilde{y}_1 \left(t;r \right), \ldots, \tilde{y}_n \left(t;r \right) \right); \tilde{y}_i \left(t;r \right) \in \left[\underline{y}_i \left(t;r \right), \overline{y}_i \left(t;r \right) \right] = \overline{f}_i \left(t; \tilde{y} \left(t;r \right) \right) \right\} \tag{4.3}
$$

and

$$
\begin{cases}
\left[\underline{f}_i \left(t; \tilde{y} \left(t;r \right) \right) \right]^t = \mathcal{F} \left(t, \underline{y} \left(t;r \right), \overline{y} \left(t;r \right) \right) = \mathcal{F} \left(t, \tilde{y} \left(t;r \right) \right) \\
\left[\overline{f}_i \left(t; \tilde{y} \left(t;r \right) \right) \right]^t = \mathcal{G} \left(t, \underline{y} \left(t;r \right), \overline{y} \left(t;r \right) \right) = \mathcal{G} \left(t, \tilde{y} \left(t;r \right) \right).
\end{cases} \tag{4.4}
$$

With respect to the above mentioned indicators, system (4.1) can be written as with assumptions for the

$$
\begin{cases}
\underline{y}' \left(t;r \right) = \mathcal{F} \left(t, \tilde{y} \left(t;r \right) \right) \\
\underline{y} \left(t_0;r \right) = \underline{y}_0 \in [t_0, T],
\end{cases} \tag{4.5}
$$

where

$$
\underline{y} \left(t_0;r \right) = \left[\underline{y}_{i,0} \left(r \right) \right]^t
$$

$$
\underline{y}' \left(t;r \right) = \left[\underline{y}'_i \left(t;r \right) \right]^t
$$

and

$$
\begin{cases}
\overline{y}' \left(t;r \right) = \mathcal{G} \left(t, \tilde{y} \left(t;r \right) \right) \\
\overline{y} \left(t_0;r \right) = \overline{y}_0 \in [t_0, T],
\end{cases} \tag{4.6}
$$

where

$$
\overline{y} \left(t_0;r \right) = \left[\underline{y}_{i,0} \left(r \right) \right]^t
$$

$$
\overline{y}' \left(t;r \right) = \left[\overline{y}'_i \left(t;r \right) \right]^t.
$$

Now we show that under the assumption for the fuzzy functions $\tilde{f}_i$, for $i = 1, 2, \ldots, n$ we can obtain a unique fuzzy solution for system (4.1) for each r-level set (see [2]). For finding an approximate solution for (4.1) with RKM56, we first define the approximate numerical solution of systems (4.1) by

$$
\tilde{y} \left(t;r \right) = \left[\tilde{y}_1 \left(t;r \right), \tilde{y}_2 \left(t;r \right), \ldots, \tilde{y}_n \left(t_0;r \right) \right]^t
$$

such that $\tilde{s}_1 = \tilde{y}_1(t;r), \tilde{s}_2 = \tilde{y}_2(t;r), \ldots, \tilde{s}_n = \tilde{y}_n(t;r)$, where $\tilde{s}_j \in [\underline{s}_j, \overline{s}_j]$ for $1 \le i, j \le n$

$$\underline{K}_{i1}(t;\underline{y}(t;r)) = \min\left\{\tilde{f}_i(t, \tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n); \tilde{s}_j \in \left[\underline{y}_j(t;r), \overline{y}_j(t;r)\right]\right\}$$

$$\overline{K}_{i1}(t;\overline{y}(t;r)) = \max\left\{\tilde{f}_i(t, \tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n); \tilde{s}_j \in \left[\underline{y}_j(t;r), \overline{y}_j(t;r)\right]\right\}$$

$$\underline{K}_{i2}(t;\underline{y}(t;r)) = \min\left\{\tilde{f}_i\left(t+\frac{h}{2}, \tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n\right); \tilde{s}_j \in \left[\underline{z}_{j1}(t, \underline{y}(t;r), h), \overline{z}_{j1}(t, \overline{y}(t;r), h)\right]\right\}$$

$$\overline{K}_{i2}(t;\overline{y}(t;r)) = \max\left\{\tilde{f}_i\left(t+\frac{h}{2}, \tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n\right); \tilde{s}_j \in \left[\underline{z}_{j1}(t, \tilde{y}(t;r), h), \overline{z}_{j1}(t, \overline{y}(t;r), h)\right]\right\}$$

$$\underline{K}_{i3}(t;\underline{y}(t;r)) = \min\left\{\tilde{f}_i\left(t+\frac{h}{4}, \tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n\right); \tilde{s}_j \in \left[\underline{z}_{j2}(t, \overline{y}(t;r), h), \overline{z}_{j2}(t, \overline{y}(t;r), h)\right]\right\}$$

$$\overline{K}_{i3}(t;\overline{y}(t;r)) = \max\left\{\tilde{f}_i\left(t+\frac{h}{4}, \tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n\right); \tilde{s}_j \in \left[\underline{z}_{j2}(t, \overline{y}(t;r), h), \overline{z}_{j2}(t, \overline{y}(t;r), h)\right]\right\}$$

$$\underline{K}_{i4}(t;\underline{y}(t;r)) = \min\left\{\tilde{f}_i\left(t+\frac{h}{2}, \tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n\right); \tilde{s}_j \in \left[\underline{z}_{j3}(t, \overline{y}(t;r), h), \overline{z}_{j3}(t, \overline{y}(t;r), h)\right]\right\}$$

$$\overline{K}_{i4}(t;\overline{y}(t;r)) = \max\left\{\tilde{f}_i\left(t+\frac{h}{2}, \tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n\right); \tilde{s}_j \in \left[\underline{z}_{j3}(t, \overline{y}(t;r), h), \overline{z}_{j3}(t, \overline{y}(t;r), h)\right]\right\}$$

$$\underline{K}_{i5}(t;\underline{y}(t;r)) = \min\left\{\tilde{f}_i\left(t+\frac{3h}{4}, \tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n\right); \tilde{s}_j \in \left[\underline{z}_{j4}(t, \overline{y}(t;r), h), \overline{z}_{j4}(t, \overline{y}(t;r), h)\right]\right\}$$

$$\overline{K}_{i5}(t;\overline{y}(t;r)) = \max\left\{\tilde{f}_i\left(t+\frac{3h}{4}, \tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n\right); \tilde{s}_j \in \left[\underline{z}_{j4}(t, \overline{y}(t;r), h), \overline{z}_{j4}(t, \overline{y}(t;r), h)\right]\right\}$$

$$\underline{K}_{i6}(t;\underline{y}(t;r)) = \min\left\{\tilde{f}_i(t+h, \tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n); \tilde{s}_j \in \left[\underline{z}_{j5}(t, \overline{y}(t;r), h), \overline{z}_{j5}(t, \overline{y}(t;r), h)\right]\right\}$$

$$\overline{K}_{i6}(t;\overline{y}(t;r)) = \max\left\{\tilde{f}_i(t+h, \tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n); \tilde{s}_j \in \left[\underline{z}_{j5}(t, \overline{y}(t;r), h), \overline{z}_{j5}(t, \overline{y}(t;r), h)\right]\right\},$$

where

$$\underline{z}_{j1}(t, \underline{y}(t;r), h) = \underline{y}_j(t;r) + \frac{h}{2}\underline{K}_{i1}(r), \overline{z}_{j1}(t, \overline{y}(t;r), h) = \overline{y}_j(t;r) + \frac{h}{2}\overline{K}_{i1}(r)$$

$$\underline{z}_{j2}(t, \underline{y}(t;r), h) = \underline{y}_j(t;r) + \frac{h}{16}(3\underline{K}_{i1}(r) + \underline{K}_{i2}(r))$$

$$\overline{z}_{j2}(t, \overline{y}(t;r), h) = \overline{y}_j(t;r) + \frac{h}{16}(3\overline{K}_{i2}(r) + \overline{K}_{i2}(r))$$

$$\underline{z}_{j3}(t, \underline{y}(t;r), h) = \underline{y}_j(t;r) + \frac{h}{2}\underline{K}_{i3}(r), \overline{z}_{j3}(t, \overline{y}(t;r), h) = \overline{y}_j(t;r) + \frac{h}{2}\overline{K}_{i3}(r)$$

$$\underline{z}_{j4}(t, \underline{y}(t;r), h) = \underline{y}_j(t;r) + \frac{h}{16}(-3\underline{K}_{i2}(r) + 6\underline{K}_{i3}(r) + 9\underline{K}_{i4}(r))$$

$$\overline{z}_{j4}(t, \overline{y}(t;r), h) = \overline{y}_j(t;r) + \frac{h}{16}(-3\overline{K}_{i2}(r) + 6\overline{K}_{i3}(r) + 9\overline{K}_{i4}(r))$$

$$\underline{z}_{j5}(t, \underline{y}(t;r), h) = \underline{y}_j(t;r) + \frac{h}{7}(\underline{K}_{i1}(r) + 4\underline{K}_{i2}(r) + 6\overline{K}_{i3}(r) - 12\underline{K}_{i4}(r) + 8\underline{K}_{i5}(r))$$

$$\overline{z}_{j5}(t, \overline{y}(t;r), h) = \overline{y}_j(t;r) + \frac{h}{7}(\overline{K}_{i1}(r) + 4\overline{K}_{i2}(r) + 6\overline{K}_{i3}(r) - 12\overline{K}_{i4}(r) + 8\overline{K}_{i5}(r)),$$

where

$$\underline{K}_{ij}(t;\underline{y}(t;r), h) = \underline{K}_{ij}(r)$$

and

$$\overline{K}_{ij}(t;\overline{y}(t;r), h) = \overline{K}_{ij}(r).$$

Now if we can consider the following

$$F_i\left(t, \underline{y}\left(t; r\right), h\right) = 7\underline{K}_{i1}\left(r\right) + 32\underline{K}_{i3}\left(r\right) + 12\underline{K}_{i4}\left(r\right) + 32\underline{K}_{i5}\left(r\right) + 7\underline{K}_{i6}\left(r\right)$$

$$G_i\left(t, \overline{y}\left(t; r\right), h\right) = 7\overline{K}_{i1}\left(r\right) + 32\overline{K}_{i3}\left(r\right) + 12\overline{K}_{i4}\left(r\right) + 32\overline{K}_{i5}\left(r\right) + 7\overline{K}_{i6}\left(r\right),$$

where $t \in [t_0, T]$ and $r \in [0, 1]$. If the exact and approximate solution in the i-th r cut at t_m for $0 \le m \le N$ are denoted by

$$\tilde{Y}_i\left(t; r\right) = \left[\underline{Y}_i\left(t; r\right), \overline{Y}_i\left(t; r\right)\right]$$

$$\tilde{y}_i\left(t; r\right) = \left[\underline{y}_i\left(t; r\right), \overline{y}_i\left(t; r\right)\right]$$

respectively, then the numerical method for solution approximation in the i-th coordinate r level set, with RKM56 is

$$\underline{y}_i^{[m+1]}\left(t; r\right) = \underline{y}_i^{[m]}\left(t; r\right) + \frac{1}{90}\ F_i\left(t_m, \tilde{y}^{[m]}\left(t; r\right), h\right), \underline{y}_i^{[0]}\left(r\right) = \underline{Y}_i^{[0]}\left(r\right) \tag{4.7}$$

$$\overline{y}_i^{[m+1]}\left(t; r\right) = \overline{y}_i^{[m]}\left(t; r\right) + \frac{1}{90}\ G_i\left(t_m, \tilde{y}^{[m]}\left(t; r\right), h\right), \overline{y}_i^{[0]}\left(r\right) = \overline{Y}_i^{[0]}\left(r\right), \tag{4.8}$$

where

$$\tilde{y}^{[m]}\left(t; r\right) = \left[\underline{y}^{[m]}\left(t; r\right), \overline{y}^{[m]}\left(t; r\right)\right], \underline{y}^{[m]}\left(r\right) = \left[\underline{y}_1^{[m]}\left(r\right), \underline{y}_2^{[m]}\left(r\right), \ldots, \underline{y}_n^{[m]}\left(r\right)\right]^t$$

$$\overline{y}^{[m]}\left(r\right) = \left[\overline{y}_1^{[m]}\left(r\right), \overline{y}_2^{[m]}\left(r\right), \ldots, \overline{y}_n^{[m]}\left(r\right)\right]^t.$$

Now we let

$$\mathcal{F}\left(t, \tilde{y}^{[m]}\left(r\right), h\right) = \frac{1}{90}\left(F_1\left(t, \tilde{y}^{[m]}\left(r\right), h\right), \ldots, F_n\left(t, \tilde{y}^{[m]}\left(r\right), h\right)\right)^t \tag{4.9}$$

$$\mathcal{G}\left(t, \tilde{y}^{[m]}\left(r\right), h\right) = \frac{1}{90}\left(G_1\left(t, \tilde{y}^{[m]}\left(r\right), h\right), \ldots, G_n\left(t, \tilde{y}^{[m]}\left(r\right), h\right)\right)^t. \tag{4.10}$$

The RK5 for solutions approximation r-level sets of Eqs. (4.7)–(4.8) are as follows

$$\underline{y}^{[m+1]}\left(r\right) = \underline{y}^{[m]}\left(r\right) + h\ \mathcal{F}\left(t, \tilde{y}^{[m]}\left(r\right), h\right), \ \underline{y}^{[0]}\left(r\right) = \underline{Y}^{[0]}\left(r\right) \tag{4.11}$$

$$\overline{y}^{[m+1]}\left(r\right) = \overline{y}^{[m]}\left(r\right) + h\ \mathcal{G}\left(t, \tilde{y}^{[m]}\left(r\right), h\right), \ \overline{y}^{[0]}\left(r\right) = \overline{Y}^{[0]}\left(r\right) \tag{4.12}$$

and

$$\mathcal{F}\left(t, \tilde{y}^{[m]}\left(r\right), h\right) = \frac{1}{90}[7\underline{K}_{i1}\left(r\right) + 32\underline{K}_{i3}\left(r\right) + 12\underline{K}_{i4}\left(r\right) + 32\underline{K}_{i5}\left(r\right) + 7\underline{K}_{i6}\left(r\right)] \tag{4.13}$$

$$\mathcal{G}\left(t, \tilde{y}^{[m]}\left(r\right), h\right) = \frac{1}{90}[7\overline{K}_{i1}\left(r\right) + 32\overline{K}_{i3}\left(r\right) + 12\overline{K}_{i4}\left(r\right) + 32\overline{K}_{i5}\left(r\right) + 7\overline{K}_{i6}\left(r\right)], \tag{4.14}$$

where in Eqs. (4.13)–(4.14)

$$\tilde{K}_{ij}\left(r\right) = \tilde{K}_j\left(t_m, \tilde{y}^{[m]}\left(r\right), h\right) = \left(\tilde{K}_{1j}\left(t_m, \tilde{y}^{[m]}\left(r\right), h\right), \ldots, \tilde{K}_{1j}\left(t_m, \tilde{y}^{[m]}\left(r\right), h\right)\right)^t. \tag{4.15}$$

5. Stability, convergence and error analysis

According the definition given in [2] we can define the following:

Definition 5.1. A one-step method for approximating the solution of differential Eqs. (4.5) and (4.6) is a method which can be written in the form

$$\left[\underline{w}^{[n+1]}\right]_r = \left[\underline{w}^{[n]}\right]_r + h\underline{\phi}\left(t_n, \left[\tilde{w}^{[n]}\right]_r, h; r\right), \tag{5.1}$$

where the increment function $\underline{\phi}$ is determined by $\mathcal{F}$ in Eq. (4.5). Also from Eq. (4.6) we have

$$\left[\overline{w}^{[n+1]}\right]_r = \left[\overline{w}^{[n]}\right]_r + h\overline{\phi}\left(t_n, \left[\tilde{w}^{[n]}\right]_r, h; r\right), \tag{5.2}$$

where the increment function $\overline{\phi}$ is determined by $\mathcal{G}$ in Eq. (4.5).
According to Theorems 5.2 and 5.3 in [18] we can define the Theorems 5.2 and 5.3 as follows:

Theorem 5.2. *If* $\underline{\phi}\left(t_n, \left[\tilde{w}^{[n]}\right]_r, h; r\right)$ *and* $\overline{\phi}\left(t_n, \left[\tilde{w}^{[n]}\right]_r, h; r\right)$ *satisfies a Lipschitz condition in* $\tilde{y}(t; r)$ *then the method given by Eqs. (5.1)–(5.2) is stable.*

Theorem 5.3. *From Eqs. (10-11), if* $\mathcal{F}(t, \tilde{y}(t; r))$ *and* $\mathcal{G}(t, \tilde{y}(t; r))$ *satisfies a Lipschitz condition in* $\tilde{y}(t; r)$ *then the method given by Eqs. (4.7)–(4.8) is stable.*

Lemma 5.4 ([4])**.** *If the sequence of non-negative numbers* $\{\mathcal{W}_n\}_{n=0}^N$ *satisfy*

$$|\mathcal{W}_{n+1}| \le A|\mathcal{W}_n| + B \quad, \ 0 \le n \le N - 1,$$

for the given positive constants A and B, then

$$|\mathcal{W}_n| \le A^n|\mathcal{W}_0| + B\frac{A^n - 1}{A - 1} \quad, \ 0 \le n \le N.$$

Lemma 5.5 ([4])**.** *If the sequence of non-negative numbers* $\{\mathcal{W}_n\}_{n=0}^N$, *and* $\{\mathcal{V}_n\}_{n=0}^N$ *satisfy*

$$|\mathcal{W}_{n+1}| \le |\mathcal{W}_n| + A\max\{|\mathcal{W}_n|, |\mathcal{V}_n|\} + B$$

$$|\mathcal{V}_{n+1}| \le |\mathcal{V}_n| + A\max\{|\mathcal{W}_n|, |\mathcal{V}_n|\} + B$$

for the given positive constants A and B, then

$$|\mathcal{U}_n| = |\mathcal{W}_n| + |\mathcal{V}_n|, 0 \le n \le N,$$

then we have

$$\mathcal{U}_n \le \hat{A}^n\mathcal{U}_0 + \hat{B}\frac{\hat{A}^n - 1}{\hat{A} - 1}, 0 \le n \le N,$$

where $\hat{A} = 1 + 2A$ *and* $\hat{B} = 2B$.

Now let

$$\tilde{y}(t; r) = \left[\underline{y}(t; r), \overline{y}(t; r)\right] = [\mathcal{U}, \mathcal{V}]$$

such that

$$\widehat{\mathcal{F}}\left(t, \tilde{Y}(t; r); h\right) = \widehat{\mathcal{F}}(t, \mathcal{U}, \mathcal{V}; h), \widehat{\mathcal{G}}(t, \tilde{y}(t; r); h) = \widehat{\mathcal{G}}(t, \mathcal{U}, \mathcal{V}; h),$$

then the domain of $\widehat{\mathcal{F}}(t, \mathcal{U}, \mathcal{V}; h)$ and $\widehat{\mathcal{G}}(t, \tilde{y}(t; r); h)$ is give by the set all nonempty fuzzy sets RF such that

$$\tilde{E}\{(t, \mathcal{U}, \mathcal{V}) \mid t \in [0, T], -\infty < V < \infty, -\infty < U \le \mathcal{V}\}.$$

According to the Theorem 5.4 that was mentioned in [2] convergence of fourth order Runge–Kutta methods followed by Theorem 4.3 in [1] for the convergence of p order Taylor's method, we can define the following theorem related to the convergence of RKM56 in Section 3.

Theorem 5.6. *Let* $\widehat{\mathcal{F}}\left(t, \tilde{Y}(t; r); h\right)$ *and* $\widehat{\mathcal{G}}\left(t, \tilde{Y}(t; r); h\right)$ *belong to* $C^5(\tilde{E})$ *and the partial derivatives of* $\widehat{\mathcal{F}}$ *and* $\widehat{\mathcal{G}}$ *are bounded on RF. Then, for arbitrarily fixed* $r \in [0, 1]$ *the numerical* $\underline{y}(t; r), \overline{y}(t; r)$ *solutions of Eqs. (4.5)–(4.6) converge to the exact solutions* $\underline{Y}(t; r), \overline{Y}(t; r)$ *of Eq. (3.1).*

Proof. It is sufficient to show

$$\lim_{h\to\infty} \underline{y}(T;r) = \underline{Y}(T;r) \ , \ \lim_{h\to\infty} \overline{y}(T;r) = \overline{Y}(T;r)$$

for $0 \le i \le N$. Using the Taylor theorem in formulas (4.7)–(4.8) with RKM56 formulas (4.13)–(4.14) we get

$$\begin{cases} \underline{Y}(t_{i+1};r) = \underline{Y}(t_i;r) + h \, \widehat{\mathcal{F}}\left(t_i, \tilde{Y}(t_i;r);h\right) + \frac{h^6}{6!} \underline{Y}^{(6)}(\theta_{i,1};r) \\ \overline{Y}(t_{i+1};r) = \overline{Y}(t_i;r) + h \, \widehat{\mathcal{G}}\left(t_i, \tilde{Y}(t_i;r);h\right) + \frac{h^6}{6!} \overline{Y}^{(6)}(\theta_{i,2};r) , \end{cases} \tag{5.3}$$

where $\theta_{i,1}, \theta_{i,2} \in (t_i, t_{i+1})$. Denoting

$$\mathcal{W}_i = \underline{Y}(t_i;r) - \underline{y}(t_i;r) , \mathcal{Z}_i = \overline{Y}(t_i;r) - \overline{y}(t_i;r) ,$$

from (4.7)–(4.8) and (5.3) we have

$$\mathcal{W}_{i+1} = \mathcal{W}_i + h\left\{ \widehat{\mathcal{F}}\left(t_i, \tilde{Y}(t_i;r);h\right) - \widehat{\mathcal{F}}\left(t_i, \tilde{y}(t_i;r);h\right) \right\} + \frac{h^6}{6!} \underline{Y}^{(6)}(\theta_{i,1};r) ,$$

$$\mathcal{Z}_{i+1} = \mathcal{Z}_i + h\left\{ \widehat{\mathcal{G}}\left(t_i, \tilde{Y}(t_i;r);h\right) - \widehat{\mathcal{G}}\left(t_i, \tilde{Y}(t_i;r);h\right) \right\} + \frac{h^6}{6!} \overline{Y}^{(6)}(\theta_{i,2};r) .$$

Hence

$$|\mathcal{W}_{i+1}| \le |\mathcal{W}_i| + 2Lh \max\{|\mathcal{W}_i|, |\mathcal{Z}_i|\} + \frac{h^6}{6!} M$$

$$|\mathcal{Z}_{i+1}| \le |\mathcal{Z}_i| + 2Lh \max\{|\mathcal{W}_i|, |\mathcal{Z}_i|\} + \frac{h^6}{6!} M,$$

where $M = \max\{\underline{M}, \overline{M}\}$ for $t \in [t_0, T]$ such that

$$\underline{M} = \min\left|\underline{Y}^{(6)}(t;r), \overline{Y}^{(6)}(t;r)\right|, \overline{M} = \max\left|\underline{Y}^{(6)}(t;r), \overline{Y}^{(6)}(t;r)\right|$$

and $L > 0$ is a bound for the partial derivatives of $\widehat{\mathcal{F}}$ and $\widehat{\mathcal{G}}$. Therefore, from Lemma 5.4–5.5 we obtain

$$|\mathcal{W}_i| \le (1 + 4Lh)^i |\mathcal{U}_0| + \frac{2h^6}{6!} M \frac{(1 + 4Lh)^i - 1}{4lh},$$

$$|\mathcal{Z}_i| \le (1 + 4Lh)^i |\mathcal{U}_0| + \frac{2h^6}{6!} M \frac{(1 + 4Lh)^i - 1}{4lh},$$

where $|\mathcal{U}_0| = |\mathcal{W}_0| + |\mathcal{Z}_0|$. In particular, according to Lemma 1.2 in [32] we have

$$|\mathcal{W}_N| \le (1 + 4Lh)^N |\mathcal{U}_0| + \frac{h^6}{6!} M \frac{(1 + 4Lh)^{\frac{T}{h}} - 1}{2lh},$$

$$|\mathcal{Z}_N| \le (1 + 4Lh)^N |\mathcal{U}_0| + \frac{h^6}{6!} M \frac{(1 + 4Lh)^{\frac{T}{h}} - 1}{2lh}.$$

Since $\mathcal{W}_0 = \mathcal{Z}_0 = 0$ and from Proposition 4 in [27] we have

$$|\mathcal{W}_N| \le \underline{M} \frac{e^{4LT} - 1}{1440L} h^6, |\mathcal{Z}_N| \le \overline{M} \frac{e^{4LT} - 1}{1440L} h^6.$$

Thus, if $h \to 0$, we get $\mathcal{W}_N \to 0$ and $\mathcal{Z}_N \to 0$. Thus absolute error bound of RKM56 for the numerical approximate solution of the FIVP (3.1)–(3.2) can be defined as:

$$[\underline{E}_{i+1}]_r = \left|\underline{Y}(t_{i+1};r) - \underline{y}(t_{i+1};r)\right| \le \underline{\epsilon}, \tag{5.4}$$

$$[\overline{E}_{i+1}]_r = \left|\underline{Y}(t_{i+1};r) - \underline{y}(t_{i+1};r)\right| \le \overline{\epsilon}, \tag{5.5}$$

where $\underline{\epsilon} \approx \underline{M} \frac{e^{4LT}-1}{1440L} h^6$, $\overline{\epsilon} \approx \overline{M} \frac{e^{4LT}-1}{1440L} h^6$ for $0 = i = N$ and $t \in [0, T]$, with

$$\underline{M} = \max\left|\underline{Y}^6(t;r), \overline{Y}^6(t;r)\right|, \overline{M} = \max\left|\underline{Y}^6(t;r), \overline{Y}^6(t;r)\right|,$$

which completes the proof. $\qquad\square$

6. Numerical example

Consider the circuit model problem [23] shown in Figure 2, where $L = 1h$, $R = 2\Omega$, $C = 0.25f$ and $E(t) = 50\cos t$. If $Q(t)$ is the charge on the capacitor at time $t > 0$.
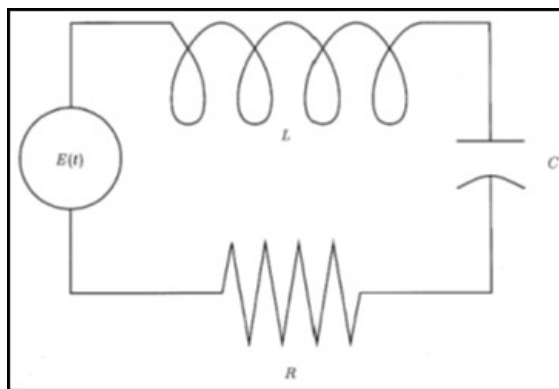

Figure 2: Electric circuit model

Thus the FIVP of this model is given as follows:

$$\tilde{Q}''(t;r) + 2\tilde{Q}'(t;r) + 4\tilde{Q}(t;r) = 50\cos t, \ t > 0 \tag{6.1}$$

$$\tilde{Q}(0;r) = [4 + r, 6 - r]$$
$$\tilde{Q}'(0;r) = [r, 2 - r], \ r \in [0,1].$$

According to Section 2 Eq. (6.1) can be written into first order system as follows

$$\begin{cases} \tilde{Q}_1'(t;r) = \tilde{Q}_2(t;r) \\ \tilde{Q}_1(t;r) = [4 + r, 6 - r] \\ \tilde{Q}_2'(t;r) = 50\cos t - 4\tilde{Q}_1(t;r) - 2\tilde{Q}_2(t;r) \\ \tilde{Q}_2(t;r) = [r, 2 - r]. \end{cases}$$

The exact analytical solution of Eq. (6.1) was given in [5]. According to RK56 in Section 4, we have:

$$U_1\left(t, \tilde{Q}_1'(t;r), \tilde{Q}_2(t;r)\right) = U_1\left(t, \tilde{Q}_1, \tilde{Q}_2\right) = \tilde{Q}_2(t;r)$$
$$U_2\left(t, \tilde{Q}_1'(t;r), \tilde{Q}_2(t;r)\right) = U_2\left(t, \tilde{Q}_1, \tilde{Q}_2\right) = 50\cos t - 4\tilde{Q}_1(t;r) - 2\tilde{Q}_2(t;r)$$

then

$$\tilde{K}_1(r) = hU_1\left(t_i, \tilde{Q}_1, \tilde{Q}_2\right)$$

$$\tilde{D}_1(r) = hU_2\left(t_i, \tilde{Q}_1, \tilde{Q}_2\right)$$

$$\tilde{K}_2(r) = hU_1\left(t_i + \frac{h}{2}, \tilde{Q}_1 + \frac{\tilde{K}_1(r)}{2}, \tilde{Q}_2 + \frac{\tilde{D}_1(r)}{2}\right)$$

$$\tilde{D}_2(r) = hU_2\left(t_i + \frac{h}{2}, \tilde{Q}_1 + \frac{\tilde{K}_1(r)}{2}, \tilde{Q}_2 + \frac{\tilde{D}_1(r)}{2}\right)$$

$$\tilde{K}_3(r) = hU_1\left(t_i + \frac{h}{4}, \tilde{Q}_1 + \frac{3\tilde{K}_1(r) + \tilde{K}_2(r)}{2}, \tilde{Q}_2 + \frac{3\tilde{D}_1(r) + \tilde{D}_2(r)}{2}\right)$$

$$\tilde{D}_3(r) = hU_2\left(t_i + \frac{h}{4}, \tilde{Q}_1 + \frac{3\tilde{K}_1(r) + \tilde{K}_2(r)}{16}, \tilde{Q}_2 + \frac{3\tilde{D}_1(r) + \tilde{D}_2(r)}{16}\right)$$

$$\tilde{K}_4(r) = hU_1\left(t_i + \frac{h}{2}, \tilde{Q}_1 + \frac{\tilde{K}_4(r)}{2}, \tilde{Q}_2 + \frac{\tilde{D}_4(r)}{2}\right)$$

$$\tilde{D}_4(r) = hU_2\left(t_i + \frac{h}{2}, \tilde{Q}_1 + \frac{\tilde{K}_3(r)}{2}, \tilde{Q}_2 + \frac{\tilde{D}_3(r)}{2}\right)$$

$$\tilde{K}_5(r) = hU_1\left(t_i + \frac{3h}{4}, \tilde{Q}_1 + \frac{-3\tilde{K}_1(r) + 6\tilde{K}_3(r) + 9\tilde{K}_4(r)}{16}, \tilde{Q}_2 + \frac{-3\tilde{D}_1(r) + 6\tilde{D}_3(r) + 9\tilde{D}_4(r)}{16}\right)$$

$$\tilde{D}_5(r) = hU_2\left(t_i + \frac{3h}{4}, \tilde{Q}_1 + \frac{-3\tilde{K}_1(r) + 6\tilde{K}_3(r) + 9\tilde{K}_4(r)}{16}, \tilde{Q}_2 + \frac{-3\tilde{D}_1(r) + 6\tilde{D}_3(r) + 9\tilde{D}_4(r)}{16}\right)$$

$$\tilde{K}_6(r) = hU_1(t_i + h, A_2, A_3)$$

$$\tilde{D}_6(r) = hU_2(t_i + h, A_4, A_5),$$

where

$$A_2 = \tilde{Q}_1 + \frac{\tilde{K}_1(r) + 4\tilde{K}_2(r) + 6\tilde{K}_3(r) - 12\tilde{K}_4(r) + 9\tilde{K}_8(r)}{7}$$

$$A_3 = \tilde{Q}_2 + \frac{\tilde{D}_1(r) + 4\tilde{D}_2(r) + 6\tilde{D}_3(r) - 12\tilde{D}_4(r) + 9\tilde{D}_5(r)}{7}$$

$$A_4 = \tilde{Q}_1 + \frac{\tilde{K}_1(r) + 4\tilde{K}_2(r) + 6\tilde{K}_3(r) - 12\tilde{K}_4(r) + 9\tilde{K}_8(r)}{7}$$

$$A_5 = \tilde{Q}_2 + \frac{\tilde{D}_1(r) + 4\tilde{D}_2(r) + 6\tilde{D}_3(r) - 12\tilde{D}_4(r) + 9\tilde{D}_5(r)}{7}$$

then

$$\tilde{Q}_1(t_{i+1}; r) = \tilde{Q}_1(t_i; r) + \frac{7\tilde{K}_1(r) + 32\tilde{K}_3(r) + 12\tilde{K}_4(r) + 32\tilde{K}_5(r) + 7\tilde{K}_6(r)}{90} \qquad (6.2)$$

$$\tilde{Q}_2(t_{i+1}; r) = \tilde{Q}_2(t_i; r) + \frac{7\tilde{D}_1(r) + 32\tilde{D}_3(r) + 12\tilde{D}_4(r) + 32\tilde{D}_5(r) + 7\tilde{D}_6(r)}{90}, \qquad (6.3)$$

where

$$\tilde{Q}_1(t_i; r) = \left[\underline{Q}_1(t_i; r), \overline{Q}_1(t_i; r)\right],$$

$$U_1\left(t_i, \tilde{Q}_1, \tilde{Q}_2\right) = \left[U_1\left(t_i, \underline{Q}_1, \underline{Q}_2\right), U_1\left(t_i, \overline{Q}_1, \overline{Q}_2\right)\right],$$

$$U_2\left(t_i, \tilde{Q}_1, \tilde{Q}_2\right) = \left[U_2\left(t_i, \underline{Q}_1, \underline{Q}_2\right), U_2\left(t_i, \overline{Q}_1, \overline{Q}_2\right)\right],$$

$$\tilde{K}_j(r) = \left[\underline{K}_j(r), \overline{K}_j(r)\right], \ \tilde{D}_j(r) = \left[\underline{D}_j(r), \overline{D}_j(r)\right]$$

and $t_{i+1} = t_i + h$ for $j = 1, \ldots, 6$, $i = 1, \ldots, n$ and $r \in [0, 1]$. From Eqs. (6.2)–(6.3), the results obtained and represented in Table 1 and Figure 3. One can note that RKM56 approximate solution with 100 iterations at $t=1$ and for all $0=r=1$ satisfy the fuzzy numbers properties in Section 2 by taking the triangular shape.

r	RK5 $\underline{y}(1;r)$	$\underline{E}$ RK5	RK5 $\overline{y}(1;r)$	RK5 $\overline{E}$
0	9.959398084113598	6.13553×10^{-12}	10.679826444068837	3.26849×10^{-12}
0.2	10.031440920109120	5.84776×10^{-12}	10.607783608073317	3.55981×10^{-12}
0.4	10.103483756104641	5.55822×10^{-12}	10.535740772077789	3.84225×10^{-12}
0.6	10.175526592100164	5.27222×10^{-12}	10.463697936082266	4.12825×10^{-12}
0.8	10.247569428095690	4.98623×10^{-12}	10.391655100086743	4.41779×10^{-12}
1	10.319612264091214	4.69846×10^{-12}	10.319612264091214	4.69840×10^{-12}

Table 1: RKM56 solution and accuracy of Eq. (6.1) at $t = 1$ with step size $h = 0.01$
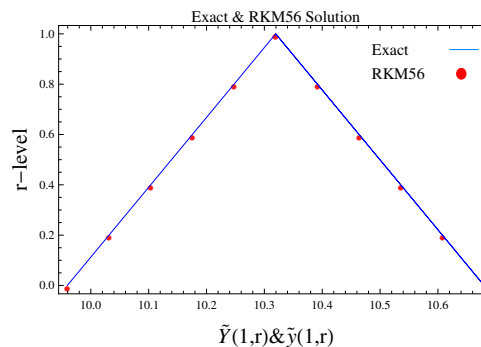
Figure 3: Exact and RKM56 solutions of Eq. (6.1) at $t = 1$ with step size $h = 0.01$

7. Conclusions

In this paper, we presented and developed a numerical method for solving fuzzy ordinary differential equations. The scheme is based on the six stages Runge-Kutta method of order five for solving n-th order FIVP. The stability, convergence and error analysis of RKM56 have been presented and proved. The numerical example involving second-order linear FIVP model showed that the RKM56 is a capable and accurate method for n-th order FIVP. Also the obtained results by RKM56 satisfy the properties of fuzzy numbers by taking triangular shape.

Acknowledgements:

References

[1] S. Abbasbandy, T. Allahvinloo, *Numerical solutions of fuzzy differential equations by Taylor method*, Comput. Methods Appl. Math., **2** (2002), 113–124. 5

[2] S. Abbasbandy, T. Allahvinloo, P. Darabi, *Numerical solution of n-order fuzzy differential equations by Runge-Kutta method*, Math. Comput. Appl., **16** (2011), 935–946. 1, 4, 5, 5

[3] M. F. Abbod, D. G. Von Keyserlingk, M. Mahfouf, *Survey of utilization of fuzzy technology in medicine and healthcare*, Fuzzy Sets and Systems, **120** (2001), 331–349. 1

[4] Z. Akbarzadeh, M. Mohseni, *Solving fuzzy differential equations by Runge-Kutta method*, J. Math. Computer Sci., **2** (2011), 208–221. 5.4, 5.5

[5] T. Allahviranloo, E. Ahmady, N. Ahmady, *nth-order fuzzy linear differential equations*, Inform. Sci., **178** (2008), 1309–1324. 6

[6] A. K. Alomari, N. R. Anakira, A. S. Bataineh, I. Hashim, *Approximate solution of nonlinear system of BVP arising in fluid flow problem*, Math. Probl. Eng., **2013** (2013), 7 pages. 1

[7] S. Barro, R. Marin, *Fuzzy logic in medicine*, Physica-Verlag Heidelberg, Berlin, (2002). 1

[8] D. Dubois, H. Prade, *Towards fuzzy differential calculus Part 3: Differentiation*, Fuzzy Sets and Systems, **8** (1982), 225–233. 2.1, 2

[9] M. S. El Naschie, *From experimental quantum pptics to quantum gravity via a fuzzy Kähler manifold*, Chaos, Solotons Fract., **25** (2005), 969–977. 1

[10] S. Emre, *Comparison of Runge-Kutta methods of order 4 and 5 on Lorenz equation*, Int. J. Art. Sci., (2004), 61–69. 4

[11] A. *Faéq*, A. Radwan, *Solution of initial value problem using fifth order Runge-Kutta method using excel spread-sheet*, Pakistan J. Appl. Sci., **2** (2002), 44–47. 4

[12] R. Faranak, I. Fudziah, A. Ali, S. Soheil, *Numerical solution of second-order fuzzy differential equation using improved Runge-Kutta Nystrom method*, Math. Probl. Eng., **2013** (2013), 10 pages. 1

[13] O. S. Fard, *An iterative scheme for the solution of generalized system of linear fuzzy differential equations*, World Appl. Sci. J., **7** (2009), 1597–1604. 2.3

[14] R. S. Gethsi, A E. Henry, *Numerical solution of Nth-order fuzzy initial value problems by fourth order Runge-Kutta method based on centroidal mean*, IOSR J. Math, **6** (2013), 47–63. 1

[15] A. F. Jameel, S. A. Altaie, *Numerical solution of linear Emden fowler boundary value problem in fuzzy environment*, J. Math. Com. Sci., **15** (2015), 179–194. 1

[16] A. Jameel, A. Izani, A. Sadeghi, *Numerical solution of fuzzy IVP with trapezoidal and triangular fuzzy numbers by using fifth order Runge-Kutta method*, World Appl. Sci. J., **17** (2012), 1667–1674. 4

[17] T. Jayakumar, K. Kanagarajan, S. Indrakumar, *Numerical solution of Nth-order fuzzy differential equation by Runge-Kutta Nystrom method*, Int. Journal Math. Eng. Sci., **1** (2012), 1–13. 1

[18] T. Jayakumar, K. Kanagarajan, S. Indrakumar, *Numerical solution of Nth-order fuzzy differential equation by Runge-Kutta method of order five*, Int. Journal of Math. Anal., **6** (2012), 2885–2896. 1, 5

[19] H. E. Julyan, P. Oreste, *The dynamics of Runge-Kutta methods*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., **2** (1992), 427–449. 1

[20] O. Kaleva, *Fuzzy differential equation*, Fuzzy Sets and Systems, **24** (1987), 301–317. 2

[21] K. Kanagarajan, M. Sambath, *Runge-Kutta Nystrom method of order three for solving fuzzy differential equations*, Comput. Methods Appl. Math., **10** (2010), 195–203. 1, 2.2

[22] M. T. Khodadad, M. M. Moghadam, *Linear k-step methods for solving Nth-order fuzzy differential equations*, Adv. Fuzzy Math., **1** (2006), 23–34. 1

[23] S. S. Mansouri, N. Ahmady, *A numerical method for solving Nth-order fuzzy differential equation by using characterization theorem*, Commun. Numer. Anal., **2012** (2012), 12 pages. 2, 2.7, 2.9, 6

[24] K. Nemati, M. Matinfar, *An implicit method for fuzzy parabolic partial differential equations*, J. Nonlinear Sci. Appl., **1** (2008), 61–71. 1

[25] S. C. Nikolaos, *An algorithm using Runge-Kutta methods of orders 4 and 5 for systems of ODEs*, Int. J. Numer. Methods Appl., **2** (2009), 47–57. 4

[26] A. Omer, O. Omer, *A pray and predator model with fuzzy initial values*, Hacet. J. Math. Stat., **41** (2012), 387–395. 1

[27] S. Palligkinis, G. Papageorgiou, T. Famelis, *Runge-Kutta methods for fuzzy differential equations*, Appl. Math. Comput., **209** (2009), 97–105. 1, 5

[28] N. Parandin, *Numerical solution of fuzzy differential equations of 2nd order by RungeKutta method*, J. Math. Ext., **7** (2013), 47–62. 1

[29] S. Salahshour, *Nth-order fuzzy differential equations under generalized differentiability*, J. Fuzzy Set Valued Anal., **2011** (2011), 14 pages. 2.5, 2.6, 2.8

[30] S. Seikkala, *On the fuzzy initial value problem*, Fuzzy Sets and Systems, **24** (1987), 319–330. 1, 2

[31] S. Sekar, S. Senthilkumar, *Numerical solution of Nth-order fuzzy differential equations by STHWS method*, Int. J. Sci. Eng. Res., **4** (2013), 1639–1644. 1

[32] F. Sunday, O. Bosede, O. Tayo, *Euler's method for solving initial value problems in ordinary differential equation*, Pac. J. Sci. Technol., **13** (2012), 152–158. 5

[33] L. A. Zadeh, *Fuzzy sets*, Inform. Cont., **8** (1965), 338–353. 2.4

[34] L. A. Zadeh, *Toward a generalized theory of uncertainty*, Inform. Sci., **172** (2005), 1–40. 2.4