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Highlights

- An outranking method for assigning a score to considered actions is proposed
- Actions are compared by means of an outranking with reference sets of actions
- A deck of the cards method is used to assign a value to reference sets of actions
- A range for its score is assigned to each action
- Conditions ensuring desirable properties for the proposed method are discussed

Electre-Score: A first outranking based method for scoring actions

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Abstract

In this paper, we present (to the best of our knowledge) the first outranking method to assign a score to each alternative. This method is part of the Electre family, and we will call it Electre-Score. In contrast to the Multi-Attribute Value Theory methods, Electre-Score does not construct a value function for each criterion to proceed then to the aggregation into a single value. Instead, Electre-Score makes use of the outranking relations to make a comparison with sets of reference actions (to which a score is assigned) and assigns a score range to each action and a representative score in the range to each action. This is a more robust way of proceeding given the fragility of a single score. The reference actions have similar nature and characteristics of limiting profiles in Electre Tri-nB and the reference scores are assigned to them through the application of the deck of cards technique. Because this method uses outranking relations, it makes also possible to take into account the imperfect knowledge of data and to avoid systematic compensatory effects. Some fundamental theoretical results guaranteeing the consistency of the method, an illustrative example and a case study in the domain of healthcare system evaluation are also provided in this paper.

Keywords: Multiple criteria analysis, Electre methods, Scoring methods, Outranking relations, Decision support systems, Non-compensatory composite indicators, Healthcare system evaluation.

1. Introduction

Multiple criteria decision aiding (MCDA) is a discipline that comprises methods and techniques to produce information to enable decision-makers to take better and informed decisions. Over the last few years, there has been a tremendous growth in the development of new methods, strengthening the maturity of the existing ones and increasing their application to deal with real-world decision aiding situations of a crucial importance for organizations. One of the most promising features of these methods is their diversity (see Greco et al., 2016, for an example of the vast array of these methods).

According to Roy (1996) there are four main types of problem statements or problematics: the choice problem (where the objective is to choose the best or a small set of best actions or

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alternatives, being the others automatically rejected), ranking problems (where the objective is to rank the actions from the worst to the best), the sorting or ordinal classification problems (where the objective is to assign actions to predefined and ordered categories or classes) and the description problem (where the actions and their consequences are described in a systematic way). In approaching all these decision problems can be useful to define a score with a cardinal content permitting to measure the distance between the comprehensive evaluations of considered actions. This score permits the Decision Maker (DM) to be more comfortable in comparing the actions. In this perspective, MCDA is more and more adopted in the construction of composite indicators (Commission et al., 2008; Greco et al., 2019a), that are specific scores used for aggregating heterogeneous individual indicators into a synthetic index to describe an overall complex phenomenon such as the industrial competitiveness, the quality of life, the liveable cities and smart cities, the sustainable development, and the globalization and innovation. An important discussion in this domain has been related to the compensatory or non-compensatory nature of the aggregation procedure used to obtain the composite indicator (Munda and Nardo, 2009). Indeed, the basic approach to construct composite indicators is the linear aggregation among the elementary indicators formulated in terms of an arithmetic mean, very often non-weighted. This implies substitutability among the various indicators, in such a way that, bad performances on some elementary indices can be counterbalanced by good performances on other indices. These compensatory effects are not permitted in non-compensatory aggregation procedures that, in simple words, take only into account the ordinal nature of elementary indices. Consequently, they take into consideration that a is preferred to b with respect to index i, but they do not consider cardinal contents such as a is twice better than b (ratio scale), or the difference in evaluation between a and b is greater than the difference between c and d (interval scale); for a discussion on different types of cardinal scales and meaningfulness in their use see Roberts (1979). With respect to the output of the non-compensatory aggregation procedures, Munda and Nardo (2009) proposes to provide a complete ranking between alternatives obtained by the Kemeny rule (Kemeny, 1959; Young and Levenglick, 1978), which is in turn based on the Condorcet majority principle (Condorcet, 1785). In the same perspective Attardi et al. (2018) proposes to use an outranking method, in particular ELECTRE III (Roy and Bouyssou, 1993; Dias et al., 2006), to obtain a partial pre-order, that is a ranking admitting some incomparabilities. However, if the scoring problem should be handled in these terms, it would coincide with the ranking problem and there would not be any necessity to envisage some specific approach and methodology. Instead, in the common sense interpretation, a composite indicator must assign to each action a single value providing a meaningful synthesis of the information contained in the elementary indicators and, possibly, having a cardinal nature going beyond a mere ordinal comparison. Therefore, the challenge of a non-compensatory composite index is to aggregate elementary indices in terms of ordinal input to obtain a composite indicator in terms of cardinal output. This is the point of view taken by (Greco et al., 2021) that adopts PROMETHEE methods (Brans and Vincke, 1985) as a generalization of the Borda count (de Borda, 1781) to construct a non-compensatory composite indicator. With the same aim, we propose ELECTRE-SCORE, a new ELECTRE method providing a non-compensatory scoring procedure that assigns a cardinal evaluation on a ratio or an interval scale. Of course, ELECTRE-SCORE can be applied beyond the domain of composite indcators, in any decision problem in which a multicriteria score can be useful for the DM and the analyst.

Let us remember the main characteristics of ELECTRE methods (Roy, 1991) that suggests the proposal of a new method in the family to handle scoring decision problem. ELECTRE methods (Figueira et al., 2016) play a central role in the family of outranking approaches. Since their inception in the middle of the 1960s, they have been the object of several studies, extensions, generalizations, new developments, and many applications in real-world situations. For a comprehensive survey of these methods see Govindan and Jepsen (2016). The most relevant features of ELECTRE methods that make them adequate to deal with several situations are the following (Figueira et al., 2013): (1) they can deal with both the quantitative and the qualitative nature of the criteria scales; (2) the scales can be of very heterogeneous types (meters, noisy, delay, costs, return, etc); (3) the compensatory effects are not relevant in a systematic way (this is mainly due to the use of the concordance index and the existence of veto thresholds that avoid some compensability); (4) they are able to take into account the imperfect knowledge of data (uncertainty, imprecision, and ill-determination) and the arbitrariness when building the criteria; and, finally, (5) they are very adequate to take into account the reasons for and the reasons against an outranking.

Of course, as all the MCDA procedures, the methods from ELECTRE family are not perfect and they also suffer from some drawbacks, as follows: (1) the intransitivity phenomenon may occur (and even worse, it may be quite frequent); (2) the phenomenon of the dependence with respect to irrelevant alternatives may be present; (3) if all the criteria are of a quantitative nature and no imperfect knowledge and arbitrariness are present, and in addition the decision-maker allows for a systematic compensation, then we can use other methods, namely those of the Multi-Attribute Value Theory (MAVT) family (Keeney and Raiffa, 1976); and (4) if it is necessary to assign a score to considered actions, then these methods are not adequate.

In reality the compensation between criteria is not always allowed by decision-makers, and imperfect knowledge and arbitrariness are often present when dealing with practical decision aiding situations. Since it will be almost impossible to avoid the two other phenomena (i.e. intransitivities and dependence with respect to irrelevant alternatives), and since they have a clear relevance for a reliable score, the natural question is: is it be possible to build a scoring based ELECTRE method? The aim of this paper is just to propose such a scoring based ELECTRE method, and, moreover, we shall show that it does not suffer of any problem related to intransitivity or dependence with respect to irrelevant alternatives.

ELECTRE-SCORE, our new method, consists of following main steps:

1. Several sets of reference actions are built. We used the sets of limiting profiles as in ELECTRE-TRI-NB (Fernández et al., 2017) as our reference actions. These sets of reference actions must fulfill some important separability conditions. Note that whichever procedure is used to define *a priori* the set of reference actions, it must satisfy a certain number of conditions that should be as weak as possible for the method to be fruitfully applied in a vast generality of cases. To start, we consider rather restrictive conditions that must be relaxed as much as possible in the following but only if these less restrictive conditions would maintain the validity of the method.

2. With a deck of cards technique, we can assign a value to each set of limiting profiles after choosing two reference values. This is a similar technique to the one proposed in Bottero et al. (2018) or in Corrente et al. (2021) for building interval scales.

3. The last step consists of comparing each action to the reference sets and assigning a scoring interval to it: more precisely, each action a is assigned an interval $[s^l(a), s^u(a)]$ of admissible scores $s(a)$, and a representative score $s^\circ(a)$, $s^l(a) \leqslant s^\circ(a) \leqslant s^u(a)$, with

   – the lower bound $s^l(a)$ of admissible score equal to the value x_* assigned to the best sets of limiting profiles B_{x_*} for which there is a preference of a over B_{x_*}, and

- the upper bound $s^u(a)$ of admissible score equal to the value x^* assigned to the best sets of limiting profiles B_{x^*} for which there is a preference of B_{x^*} over a,
- the representative score $s^\circ(a) = \alpha \cdot s^u(a) + (1-\alpha) \cdot s^l(a)$, with $\alpha, 0 \leqslant \alpha \leqslant 1$, representing the degree of optimism of the decision-maker in the spirit of the Hurwicz decision rule (Hurwicz, 1951), so that, the greater the value of α the closer the representative score $s^\circ(a)$ to the upper bound $s^u(a)$.

Let us point out the following relevant points characterizing Electre-Score:

- The output of the method we are proposing is a score, which is of a cardinal nature, i.e., it is a cardinal numerical representation of the goodness of the considered actions. The evaluation supplied by Electre-Score is expressed on a ratio or on interval scale. Indeed, it is based on the assessment of a value to each set of limiting profiles that through the deck of the cards method can produce a ratio scale (Figueira and Roy, 2002) or an interval scale (Bottero et al., 2018) (for a recent improved version of this method see Corrente et al. 2021). Therefore, for example,
  - in case the deck of the cards method assigns values in terms of a ratio scale to the sets of limiting profiles, if an action a is assigned a score $s(a) = 20$ and an action b is assigned a score $s(b) = 10$, it is meaningful to say that a has an evaluation twice than the evaluation of b,
  - in case the deck of the cards method assigns values in terms of an interval scale to the sets of limiting profiles, if actions a, b, c and d are assigned scores $s(a) = 90, s(b) = 50, s(c) = 40, s(d) = 20$, respectively, it is meaningful to say that difference in evaluation between a and b is twice the difference in evaluation between c and d.
- The method was particularly designed to supply a range $[s^l(a), s^u(a)]$ of admissible scores $s(a)$ for each action a. Indeed, the obtained ranges are very robust because they consider the most pessimistic and the most optimistic evaluations $s^l(a)$ and $s^u(a)$ obtained by taking into account the outranking relation. Observe that the use of a range $[s^l(a), s^u(a)]$ is in line with the representation of preference in terms of semi-orders (Luce, 1956) or interval orders (Fishburn, 1973), that, to take into account not perfect discriminating utility, compare objects in terms of intervals of admissible evaluations. In the same perspective, in the ambit of composite indicators, punctual evaluations have been substituted by probability distributions of the score taking into account the different evaluations obtained changing the weights assigned to elementary indicators (Greco et al., 2018, 2019b). Observe also that in case the decision-maker needs to assign a single score to each action a, a representative score $s^\circ(a)$ can be provided with a rather flexible and intuitive procedure based on the decision-maker's degree of optimism.
- Let us observe that, differently from typical MCDA methods supplying a score to considered actions, such as Multiple Attribute Utility Theory (MAUT) methods (Keeney and Raiffa, 1976), Analytical Hierarchy Process (Saaty, 1990), MACBETH (Bana e Costa and Vansnick, 1994), Electre-Score does not consider any transformation of the scale for each criterion into partial value functions expressed in a common commensurable scale to be then aggregated into a single overall value function. Indeed, the score obtained with our procedure supplies a comprehensive evaluation without using a procedure for converting each criterion into a value function.

– In general, scoring decision problems take into consideration a rather large number of actions to be evaluated. This is the case of scoring procedures regarding countries all over the world or in some continents, universities, cities, companies and so on. Sometimes it is reasonable to assign a score to a small set of actions, so that the question arises if in these cases it is reasonable to apply a methodology relatively complex as ELECTRE-SCORE or if it is not the case to assign directly a score to the considered actions. We believe that also in case there are few actions can be reasonable to apply ELECTRE-SCORE because it permits to obtain a scoring which is coherent with the outranking relation, which makes the score more explainable and justifiable (Amgoud and Prade, 2009), and, consequently, in agreement with the principles of MCDA (Roy, 2010), more acceptable.

– Differently from other ELECTRE methods, ELECTRE-SCORE is not interested by the instransitivity of the outranking relation. In fact, ELECTRE-SCORE use outranking relation only to compare each actions with the sets of limiting profiles, and as we shall show in next Section 3, some reasonable dominance conditions among the set of reference profiles prevent intransitivity between sets of limiting profiles and actions to be evaluated. Observe also that for each action a the range $(s^l(a), s^u(a))$ of admissible scores $s(a)$ and the representative score $s^\circ(a)$ depends only on the preference comparison of a with the sets of reference profiles. Consequently, $s^l(a), s^u(a)$ and $s^\circ(a)$ do not depend on the comparison of a with other actions, which means that independence with respect to irrelevant alternatives holds for the scores provided by ELECTRE-SCORE.

The rest of this paper is organized as follows. Section 2 is devoted to some fundamental concepts, their definitions, and corresponding notation. Section 3 presents the new ELECTRE-SCORE method, including the conditions for the construction of the reference set, the assignment of a scoring range to each action, and the formal definitions of the lower and upper bounds of such a range. Section 4 is related to the conditions about the set of limiting profiles that allow the procedure to be in accordance with the objectives. Section 5 is devoted to the theoretical results proving desirable properties of the procedure. Section 6 presents the method by means of an illustrative example along with some practical aspects. In Section 7 ELECTRE-SCORE is applied to a real world case study in the domain of the healthcare system evaluation. Finally, the last section provides the main conclusions and some lines for future research.

2. Concepts, definitions, and notation

To start, we need to introduce a few notation. Let $A = \{a_1, \ldots, a_i, \ldots, a_m\}$ denote the set of actions to which an interval score must be assigned to each of them, $G = \{g_1, \ldots, g_j, \ldots, g_n\}$ denote the set of criteria used to assess the performance of such actions, and $g_j(a_i)$ denotes the performance of action a_i on criterion g_j (with all the performances we can build a performance table). Consider, without loss of generality, that the preference direction of the criteria is increasing. Consider also the collection of sets of reference actions $B = \{B_{x_1}, \ldots, B_{x_k}, \ldots, B_{x_\ell}\}$, for which a score is previously defined, and let $X = \{x_1, \ldots, x_k, \ldots, x_\ell\}$ the set of such scores. Each set is composed of at least one limiting profile as in ELECTRE TRI-NB (see Fernández et al., 2017), i.e., $B_{x_k} = \{b_{k1}, \ldots, b_{kp}, \ldots, b_{kp_k}\}$. As part of ELECTRE methods a credibility degree between all ordered pairs of actions, $\sigma(a, b)$, must be computed. This credibility measures on a scale $[0, 1]$ the degree in which action a outranks action b. To pass from a fuzzy relation to a crisp one, we need to define what is called the cutting level $\lambda \in]0.5, 1]$. A brief description of the way Electre methods compute the degree of credibility is available in the Appendix.

The next three definitions are fundamental.

Definition 1 (Dominance)**.** *Consider two actions a and b. Action a* dominates *action b, whenever $g_j(a) \geqslant g_j(b)$, for all $j = 1, \ldots, n$, with at least one strict inequality. Let $a\Delta b$ denote such a binary dominance relation.*

Definition 2 (Fundamental outranking binary relation)**.** *Consider two actions a and b from set A. Once we have fixed the cutting level λ, we say that action a outranks (or is at least as good as) action b, denoted by $a \succsim^{\lambda} b$ iff $\sigma(a,b) \geqslant \lambda$. It is easy to see that $\succsim^{\lambda}$ is a reflexive, but not necessarily symmetric and transitive binary relation. In what follows and with some abuse of the mathematical language, we will use simply $a \succsim b$ instead of $a \succsim^{\lambda} b$ to denote this $\lambda-$outranking binary relation (the same applies for the binary relations introduced in the next definition).*

Definition 3 (Derived binary relations)**.** *From the fundamental outranking binary relation $\succsim$, we can derive, for two actions $a, b \in A$, the following three binary relations (which correspond to all possible combinations of the presence and non-presence of an outranking relation between a and b, and b and a, respectively).*

i) $a \succ b$ *(*preference *in favor of a, which means that a is preferred to b) iff $a \succsim b$ and not$(b \succsim a)$;*

ii) $b \succ a$ *(*preference *in favor of b, which means that b is preferred to a) iff $b \succsim a$ and not$(a \succsim b)$;*

iii) $a \sim b$ *(*indifference*, which means that actions, a and b, are indifferent) iff $a \succsim b$ and $b \succsim a$;*

iv) $a \parallel b$ *(*incomparability*, which means that actions, a and b, are incomparable) iff not$(a \succsim b)$ and not$(b \succsim a)$.*

($\succ$ is irreflexive and asymmetric; $\sim$ is reflexive and symmetric; and, $\parallel$ is irreflexive and symmetric.)

Remark 1. *From Definitions 2 and 3, it is easy to see that $a \succsim b$ implies, either $a \succ b$ or $a \sim b$. Taking into account the dominance relation of Definition 1, the following properties hold.*

$$a\Delta b \Rightarrow a \succsim b \tag{1.1}$$

$$a \succsim b \text{ and } b\Delta c \Rightarrow a \succsim c \tag{1.2}$$

$$a\Delta b \text{ and } b \succsim c \Rightarrow a \succsim c \tag{1.3}$$

$$a \succ b \text{ and } b\Delta c \Rightarrow a \succ c \tag{1.4}$$

$$a\Delta b \text{ and } b \succ c \Rightarrow a \succ c \tag{1.5}$$

3. Electre-Score

This section provides the basic foundations of the Electre-Score method; that is, the necessary elements for the construction of a reference set, the conditions needed for assigning a score range to each action, and the formal definition of the lower and upper bounds of such a range. Most of the material presented in this section is closely related to the Electre Tri-nB method (see Fernández et al., 2017).

3.1. Constructing a reference set

The definition of the reference set and the basic assumption with respect to such reference set are presented next. This subsection also provided some more results in the same line as in Fernández et al. (2017).

Definition 4 (Set of reference actions). *Let $X = \{x_1, \ldots, x_k, \ldots, x_\ell\}$ denote the set of values considered as references scores, and $B_{x_k} = \{b_{x_k1}, \ldots, b_{x_kp}, \ldots, b_{x_kp_k}\}$ denote the set of reference actions used to characterize score x_k. As a result $B = \bigcup_k^\ell B_{x_k}$ denotes a set containing all the reference actions.*

Condition 1 (Basic assumptions). *The score x_k is characterized by a set of reference actions, $B_{x_k} = \{b_{x_k1}, \ldots, b_{x_kp}, \ldots, b_{x_kp_k}\}$, for $k = 1, \ldots, \ell$, such that:*

i) For all $b_{kp}, b_{kq} \in B_{x_k}$ there is no preference between b_{kp} and b_{kq} (this implies, there is only the possibility to have either $b_{kp} \sim b_{kq}$ or $b_{kp} \parallel b_{kq}$);

ii) For all $b_{kp} \in B_{x_k}$ and $b_{hq} \in B_{x_h}$ ($x_k > x_h$), it is not possible to have $b_{hq} \succ b_{kp}$.

Definition 5 (Relations between an action and a reference set). *Consider the following relations between an action a, and a set of reference actions, B_{x_k} (see Fernández et al., 2017).*

i) $a \succsim B_{x_k}$ iff, for all $b_{kq} \in B_{x_k}$, either $a \parallel b_{kq}$ or $a \succsim b_{kq}$, the latter relation being fulfilled by at least one $b_{kq} \in B_{x_k}$ (note that, for all $b_{kq} \in B_{x_k}$, it is not possible to have $b_{kq} \succ a$);

ii) $B_{x_k} \succsim a$ iff, for all $b_{kq} \in B_{x_k}$, either $b_{kq} \parallel a$ or $b_{kq} \succsim a$, the latter relation being fulfilled by at least one $b_{kq} \in B_{x_k}$ (note that, for all $b_{kq} \in B_{x_k}$, it is not possible to have $a \succ b_{kq}$);

iii) $a \succ B_{x_k}$ iff, for all $b_{kq} \in B_{x_k}$, either $a \parallel b_{kq}$ or $a \sim b_{kq}$, or $a \succ b_{kq}$, the latter relation being fulfilled by at least one $b_{kq} \in B_{x_k}$ (note that, for all $b_{kq} \in B_{x_k}$, it is not possible to have $b_{kq} \succ a$);

iv) $B_{x_k} \succ a$ iff, for all $b_{kq} \in B_{x_k}$, either $b_{kq} \parallel a$ or $b_{kq} \sim a$, or $b_{kq} \succ a$, the latter relation being fulfilled by at least one $b_{kq} \in B_{x_k}$ (note that, for all $b_{kq} \in B_{x_k}$, it is not possible to have $a \succ b_{kq}$);

v) $a \sim B_{x_k}$ iff, for all $b_{kq} \in B_{x_k}$, either $a \parallel b_{kq}$ or $a \sim b_{kq}$, the latter relation being fulfilled by at least one $b_{kq} \in B_{x_k}$ (note that, because $\sim$ is symmetric, for all $b_{kq} \in B_{x_k}$, it is not possible to have $b_{kq} \succ a$ or $a \succ b_{kq}$);

vi) $a \parallel B_{x_k}$ iff, for all $b_{kq} \in B_{x_k}$, either $a \parallel b_{kq}$ or, when $a \succ b_{kq}$, for some $b_{kq} \in B_{x_k}$, $b_{kp} \succ a$, for some $b_{kp} \in B_{x_k}$, with $b_{kq} \neq b_{kp}$ (note that, because $\sim$ is symmetric, it is not possible to have $B_{x_k} \succsim a$ or $a \succsim B_{x_k}$).

Remark 2. *The following implications can be derived from Definitions 1 and 5 (see Fernández et al., 2017).*

i) $a \succ B_{x_k}$ implies $a \succsim B_{x_k}$;

ii) $a \succ B_{x_k}$ implies not($B_{x_k} \succsim a$), and consequently $a \succ B_{x_k}$ also implies not($B_{x_k} \succ a$);

iii) $B_{x_k} \succ a$ implies not($a \succsim B_{x_k}$), and consequently $B_{x_k} \succ a$ also implies not($a \succ B_{x_k}$);

iv) $B_{x_k} \succ a$ *implies* $B_{x_k} \succsim a$*;*

v) $B_{x_k} \succ a$ *and* $a\Delta b$ *implies* $B_{x_k} \succ b$*;*

vi) $a\Delta b$ *and* $b \succsim B_{x_k}$ *implies* $a \succsim B_{x_k}$*;*

vii) $a\Delta b$ *and* $b \succ B_{x_k}$ *implies* $a \succ B_{x_k}$*.*

3.2. Conditions that guarantee the existence of a score range to each alternative

The aim of our ELECTRE score method is to identify a range, $]s^l(a), s^u(a)[$ for the score $s(a)$ to be assigned to actions $a \in A$. In an informal way the lower bound $s^l(a)$ can be defined as the highest value, say x, for which action a is strictly preferred to the representative set B_x and there is no other value, say x', such that the representative set $B_{x'}$ is strictly preferred to a. For an informal definition of the upper bound $s^u(a)$ the reasoning is similar. Let us first discuss some conditions that guarantee the existence of such a range $]s^l(a), s^u(a)[$. Since we only know the relations between a and the elements of set B (see Definition 5), the lower bound, $s^l(a)$, and the upper bound, $s^u(a)$, of the range cannot be fixed *a priori*. However, it is easy to see that we cannot have $B_x \succ a$, for any score x lower than or equal to the lower bound, $s^l(a)$. Otherwise, the lower bound was not properly defined and we could move down (decrease) its value, which makes no sense. Analogously, it is easy to see that we cannot have $a \succ B_x$ for any score x greater than or equal to the upper bound, $s^u(a)$. Otherwise, the upper bound was not properly defined and we could move up (increase) its value, which makes no sense. This reasoning led us to establish the following two necessary conditions for the existence of the range $]s^l(a), s^u(a)[$.

Condition 2 (Lower bound necessary condition)**.** *If* $x \leqslant s^l(a)$*, then not*$(B_x \succ a)$*, for all* $a \in A$*.*

Condition 3 (Upper bound necessary condition)**.** *If* $x \geqslant s^u(a)$*, then not*$(a \succ B_x)$*, for all* $a \in A$*.*

These two conditions are necessary for the existence of the range, but they are not sufficient because we know nothing about the relations between B_x and a, for an x value strictly comprised within the range $]s^l(a), s^u(a)[$. However, it is easy to see that we cannot have $B_x \succ a$, for any score x strictly greater than the lower bound, $s^l(a)$. Otherwise, the lower bound was not properly defined and we could move up (increase) its value, which makes no sense. Analogously, it is easy to see that we cannot have $a \succ B_x$, for any score x strictly lower than the upper bound, $s^u(a)$. Otherwise, the upper bound was not properly defined and we could move down (decrease) its value, which makes no sense. This reasoning led us to establish the following two sufficient conditions for the existence of the range $]s^l(a), s^u(a)[$.

Condition 4 (No active preference condition of reference sets in the score range)**.** *If* $s^l(a) < x < s^u(a)$*, then not*$(B_x \succ a)$*, for all* $a \in A$*.*

If this condition was violated, then no score $s(a)$, such that $s^l(a) \leqslant s(a) < s^u(a)$, could be justified.

Condition 5 (No passive preference condition of reference sets in the score range)**.** *If* $s^l(a) < x < s^u(a)$*, then not*$(a \succ B_x)$*, for all* $a \in A$*.*

Analogously, if this condition was violated, then no score $s(a)$, such that $s^l(a) < s(a) \leqslant s^u(a)$, could be justified.

It is easy to see that Conditions 4 and 5 are fulfilled if and only if the next (equivalent) condition holds.

Condition 6 (No preference condition of reference sets in the interval)**.** *For all possible scores, x, such that $s^l(a) < x < s^u(a)$, not$(B_x \succ a)$ and not$(a \succ B_x)$, for all $a \in A$.*

This condition means that assigning to action a, any feasible score, i.e, any score x such that $s^l(a) < x < s^u(a)$, is only possible when, either $a \sim B_x$ or $a \parallel B_x$.

Observe that Conditions 2 and 4 can be replaced by the following condition.

Condition 7 (Lower bound general condition)**.** *If $x > s^l(a)$, then not$(a \succ B_x)$, for all $a \in A$.*

Analogously, Conditions 3 and 5 can be replaced by the following condition.

Condition 8 (Upper bound general condition)**.** *If $x < s^u(a)$, then not$(B_x \succ a)$, for all $a \in A$.*

Remark 3. *By no means do Conditions 2 to 5 imply that the next condition is automatically fulfilled.*

Condition 9 (Indifference/incomparability)**.** *If $a \sim B_x$ or $a \parallel B_x$, then $s^l(a) < x < s^u(a)$, for all $a \in A$.*

Figure 1 illustrates the above conditions. Notation [**Ck**] is used instead of Condition k, for $k = 3, 4, 5, 6, 7, 8, 9$.

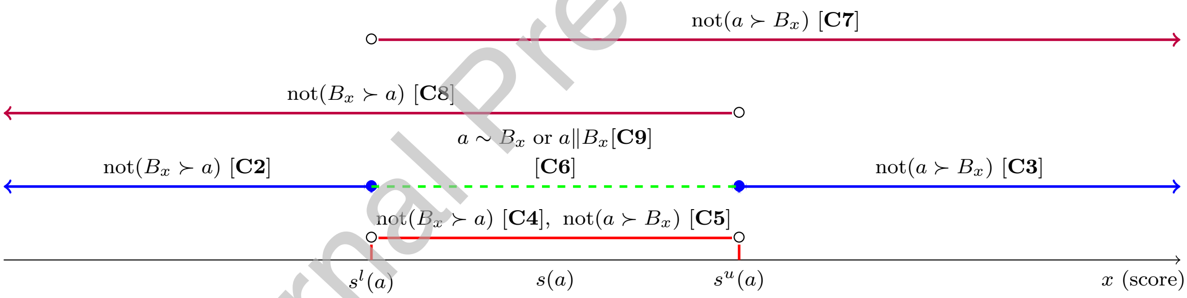

Figure 1: Illustration of the necessary and non active and passive preference conditions for the existence of a range for $s(a)$, $]s^l(a), s^u(a)[$

3.3. Definitions of the lower and upper bounds

The formal definitions of the lower and upper bound are as follows.

Definition 6 (Lower bound of the score range)**.** *The $s^l(a)$ value is the highest value $x \in X$, such that*

- *$a \succ B_x$ and*
- *not$(B_{x'} \succ a)$ for all $x' \in X$ such that $x' < x$.*

Definition 7 (Upper bound of the score range)**.** *The $s^u(a)$ value is the lowest value $x \in X$, such that*

- *$B_x \succ a$ and*
- *not$(a \succ B_{x'})$, for all $x' > x$.*

Remark 4. *It is obvious that Definitions 6 and 7 ensure that Conditions 2 and 3 are automatically fulfilled.*

Definition 8 (Representative score)**.**

$$s^{\circ}(a) = \alpha \cdot s^{u}(a) + (1-\alpha) \cdot s^{l}(a)$$

with $\alpha, 0 \leqslant \alpha \leqslant 1$, *representing the degree of optimism of the decision-maker.*

3.4. Basic notation

The following basic notation should be considered in the rest of this paper, especially in the two next sections.

- $A = \{a_1, \ldots, a_i, \ldots, a_m\}$ is the set of actions.
- $G = \{g_1, \ldots, g_j, \ldots, g_n\}$ is the set of criteria.
- $g_j(a_i)$ is the performance of action a_i on criterion g_j.
- $X = \{x_1, \ldots, x_k, \ldots, x_\ell\}$ is the set of scores, where x_k is a real value.
- $B = \{B_{x_1}, \ldots, B_{x_k}, \ldots, B_{x_\ell}\}$ is the set of sets of reference actions.
- $B_{x_k} = \{b_{k1}, \ldots, b_{kp}, \ldots, b_{kp_k}\}$ is the set of references actions with score x_k.
- $a\Delta b$ is the dominance relation of a over b.
- $a \succsim b$ is the outranking relation in favor of a.
- $b \succsim a$ is the outranking relation in favor of b.
- $a \succ b$ is the strict preference relation in favor o a.
- $b \succ a$ is the strict preference relation in favor o b.
- $a \sim b$ is the indifference relation between a and b.
- $a \parallel b$ is the incomparability relation between a and b.
- $a \succsim B_{x_k}$ is the outranking relation in favor of action a.
- $B_{x_k} \succsim a$ is the outranking relation in favor of set B_{x_k}.
- $a \succ B_{x_k}$ is the strict preference relation in favor of action a.
- $B_{x_k} \succ a$ is the strict preference relation in favor of set B_{x_k}.
- $a \sim B_{x_k}$ is the indifference relation between a and B_{x_k}.
- $a \parallel B_{x_k}$ is the incomparability relation between a and B_{x_k}.
- $s(a)$ is the score of action a.
- $s^{l}(a)$ is the lower bound for the score of action a.
- $s^{u}(a)$ is the upper bound for the score of action a.
- $s^{\circ}(a)$ is the representative score of action a.

3.5. Flowchart of the method

In this subsection we present a flowchart of our method, see Figure 2. We shall consider the case in which the score is expressed on an interval scale. The four phases of the method can be summarized as follows.

1. The first phase consists of the basic input data. This includes the set of criteria, the set of actions or alternatives (it contains those to be scored as well as the ones serving as dummy or reference actions), and the performance table.

2. The second phase consists of the preference elicitation phase and it contains also input for the next phase. It includes the following preference parameters:

   (a) The weights of criteria that can be obtained through the application of the deck of card method (see for example, Corrente et al. 2021).

   (b) Two anchoring actions and scores, which will be used to obtain all the remaining scores. These two scores are important to construct an interval scale.

   (c) A set of reference actions, other than the anchoring ones, well separated through the insertion of blank cards as in the deck of cards method.

   (d) From the two previous preference parameters we compute, by using the deck of cards method, the scores of the limiting profiles B_x. The decision-maker should be confronted to the scores and adjusting them, if necessary.

   (e) The veto thresholds, the cutting-off level λ and the degree of optimism α constitute the last pieces of preference information from the decision-makers.

3. The third phase is related with the scoring procedure, which makes use of the information provided in the performance table of Phase 1 and all the preference information from Phase 2. The objective of this phase is to build an outranking relation and making an exploration of such a relation, in order to provide a score range and a representative score for each action.

4. The fourth phase provide a score range $]s^l(a), s^u(a)[$ and a representative score $s^\circ(a)$ for each alternative $a \in A$, which is the output of the model.

The participation of decision-maker and analyst in the above procedure for co-constructing the preference parameters can be performed in the line of what is done to assess the weights of criteria as in SRF method (see Figueira and Roy 2002) and to assess the veto thresholds, which can be done in a similar way as the discriminating thresholds (see Roy et al. 2014). The only difference now, is related with the additional preference information needed, which comes from the definition of the reference actions corresponding to the limiting profiles and their reference scores. The process is done as in the deck of cards method (see, Corrente et al. 2021), where the interaction between analyst and decision-maker is even more intense to construct the most adequate reference scores; inconsistent judgments can lead to revise the preference information and progress or make evolve the decision aiding process. Even if the previous method was described in a linear way, any feedback resulting in any phase of the procedure can have consequences that have to be taken into consideration also in the other phases. In fact, a co-construction process between analyst and decision-maker is based on the dialog between these two actors to construct the preference parameters. This is quite different from a pure machine learning elicitation process, which can make sense and be useful in other contexts.

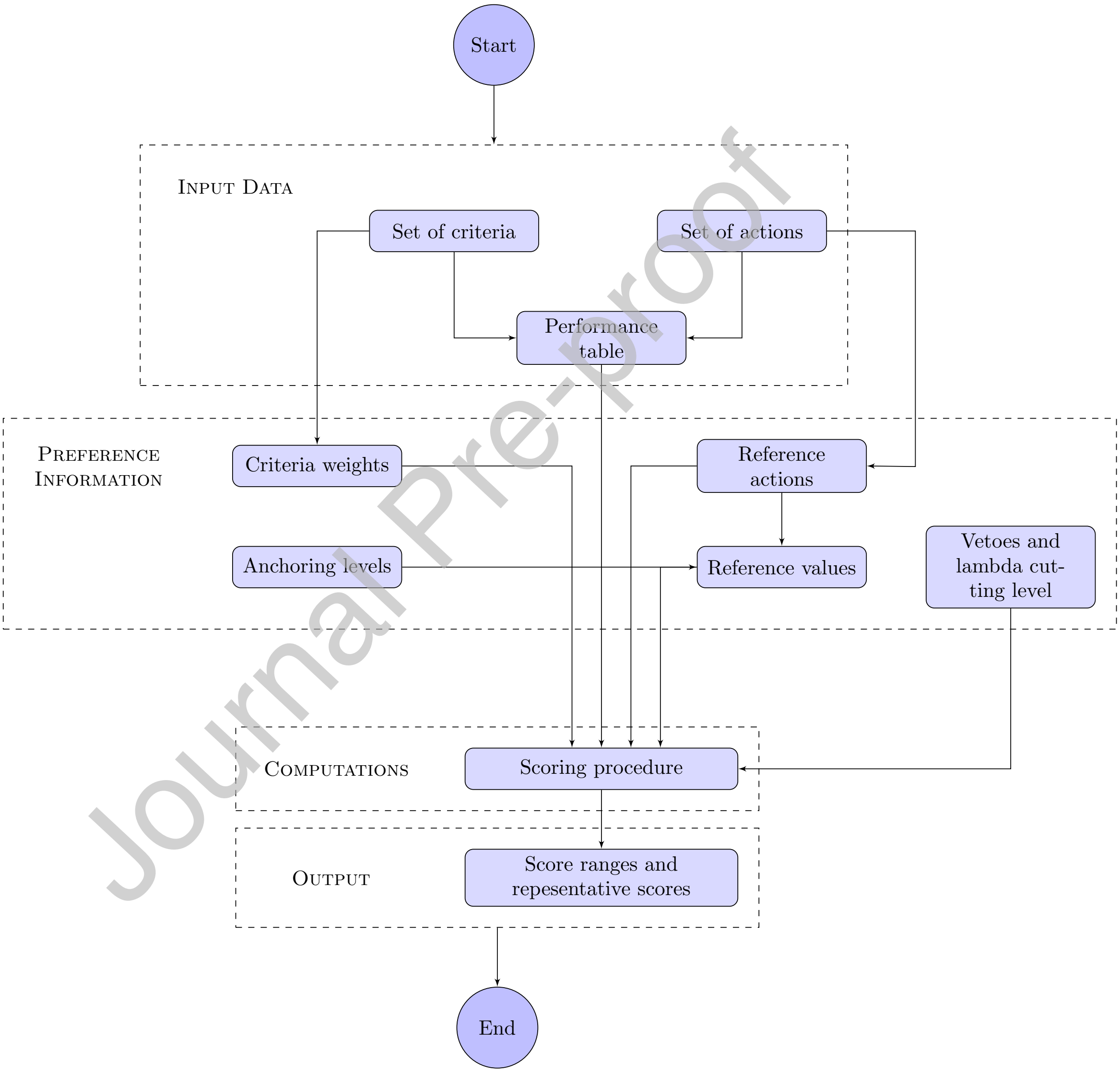

Figure 2: Flowchart of ELECTRE-SCORE

4. Conditions on B, which guarantee the procedure is in accordance with the objectives

This section presents the conditions of the set B that render the procedure or the new method coherent; that is, in accordance with the objectives the method have been designed for. From these conditions, a set of new results needed to be proven.

Condition 10 (Separability conditions)**.** *The following are the required separability conditions on set B.*

a Dominance based separability conditions*.*

- *a.1* Strong dominance*. Consider $x_h > x_k$. For any two reference actions, $b_{hq} \in B_{x_h}$ and $b_{kp} \in B_{x_k}$, the relation $b_{hq} \Delta b_{kp}$ holds.*
- *a.2* Soft dominance*.*
  - *a.2.p (*primal*): Consider $x_h > x_k$. For all $b_{kp} \in B_{x_k}$, there is at least one $b_{hq} \in B_{x_h}$ such that $b_{hq} \Delta b_{kp}$.*
  - *a.2.d (*dual*): Consider $x_h > x_k$. For all $b_{hq} \in B_{x_h}$, there is at least one $b_{kp} \in B_{x_k}$ such that $b_{hq} \Delta b_{kp}$.*

b Preference based separability conditions*.*

- *b.1* Strong preference*. Consider $x_h > x_k$. For any two reference actions, $b_{hq} \in B_{x_h}$ and $b_{kp} \in B_{x_k}$, the relation $b_{hq} \succ b_{kp}$ holds.*
- *b.2* Soft preference*.*
  - *b.2.p (*primal*): Consider $x_h > x_k$. For all $b_{kp} \in B_{x_k}$, there is at least one $b_{hq} \in B_{x_h}$ such that $b_{hq} \succ b_{kp}$.*
  - *b.2.d (*dual*): Consider $x_h > x_k$. For all $b_{hq} \in B_{x_h}$, there is at least one $b_{kp} \in B_{x_k}$ such that $b_{hq} \succ b_{kp}$.*

Proposition 1 (Comparisons of the actions against reference sets)**.**

i) If the primal soft dominance separability condition holds for B, then for all $a \in A$, $a \succsim B_{x_h}$ implies not$(B_{x_k} \succ a)$, for all $x_h > x_k$;

ii) If the dual soft dominance separability condition holds for B, then for all $a \in A$, $B_{x_k} \succsim a$ implies not$(a \succ B_{x_h})$, for all $x_h > x_k$;

iii) If the dual soft dominance separability condition holds for B, then for all $a \in A$, $a \succ B_{x_h}$ implies not$(B_{x_k} \succsim a)$, for all $x_h > x_k$;

iv) If the primal soft dominance separability condition holds for B, then for all $a \in A$, $B_{x_k} \succ a$ implies not$(a \succsim B_{x_h})$, for all $x_h > x_k$;

v) If both the primal and the dual soft dominance separability condition hold for B, then for all $a \in A$, $a \succsim B_{x_h}$ implies $a \succsim B_{x_k}$, for all $x_h > x_k$;

vi) If both the primal and the dual soft dominance separability condition hold for B, then for all $a \in A$, $B_{x_k} \succsim a$ implies $B_{x_h} \succsim a$, for all $x_h > x_k$;

vii) If both the primal and the dual soft dominance separability condition hold for B, then for all $a \in A$, $a \succ B_{x_h}$ implies $a \succ B_{x_k}$, for all $x_h > x_k$;

viii) If both the primal and the dual soft dominance separability condition hold for B, then for all $a \in A$, $B_{x_k} \succ a$ implies $B_{x_h} \succ a$, for all $x_h > x_k$.

Proof. We shall prove *i*), *iii*), *v*) and *vii*) because their proofs are analogous to *ii*), *iv*), *vi*) and *viii*), respectively. Analogous results have been obtained with respect to sorting procedures in (Fernández et al., 2017).

i) Suppose that the primal soft dominance separability condition holds for B, $a \succsim B_{x_h}$ for $a \in A$, and, by contradiction, $B_{x_k} \succ a$ with $x_h > x_k$. $B_{x_k} \succ a$ implies that there exists $b_{kp} \in B_{x_k}$ such that $b_{kp} \succ a$. By primal soft dominance condition there exists $b_{hq} \in B_{x_k}$ such that $b_{hq} \Delta b_{kp}$, so that $b_{hq} \succ a$, and, consequently, not$(a \succsim B_{x_h})$. But this is absurd because contradicts the hypothesis $a \succsim B_{x_h}$.

iii) Suppose that the dual soft dominance separability condition holds for B and $a \succ B_{x_h}$ for $a \in A$. $a \succ B_{x_h}$ implies that there exists $b_{hp} \in B_{x_h}$ such that $a \succ b_{hp}$. For $x_h > x_k$, by dual soft dominance condition, there exists $b_{kq} \in B_{x_k}$, such that $b_{hp} \Delta b_{kq}$, so that $a \succ b_{kq}$, and, consequently, *not*$(B_{x_k} \succsim a)$.

v) Suppose that the primal and the dual soft dominance separability condition hold for B and $a \succsim B_{x_h}$ for $a \in A$. $a \succsim B_{x_h}$ implies that there exists $b_{hp} \in B_{x_h}$ such that $a \succsim b_{hp}$. For $x_h > x_k$, by dual soft dominance condition there exists $b_{kq} \in B_{x_k}$ such that $b_{hp} \Delta b_{kq}$, so that $a \succsim b_{hp}$ implies $a \succsim b_{kq}$. By contradiction, suppose that there is $b_{kr} \in B_{x_k}$ such that $b_{kr} \succ a$. By primal soft dominance there exists $b_{hs} \Delta b_{kr}$, so that $b_{hs} \succ a$. But this is absurd because contradicts the hypothesis $a \succsim B_{x_h}$. Thus we get that there is $b_{kq} \in B_{x_k}$ for which $a \succsim b_{kq}$ and there is no $b_{kr} \in B_{x_k}$ for which $b_{kr} \succ a$, which imply $a \succsim B_{x_k}$.

vii) Suppose that the primal and the dual soft dominance separability condition hold for B and $a \succ B_{x_h}$ for $a \in A$. With arguments analogous to those used in the proof of *v*), we get that there is $b_{kq} \in B_{x_k}$ for which $a \succ b_{kq}$ and there is no $b_{kr} \in B_{x_k}$ for which $b_{kr} \succ a$, which imply $a \succ B_{x_k}$.

□

Proposition 2 (Upper and lower bound sufficiency). *If B fulfills both the primal and the dual soft dominance separability conditions (see Condition 10, points a.2.p and a.2.d), then Conditions 4 and 5 are also fulfilled.*

Proof. By contradiction, suppose that there exist $x_h \in X$ and $a \in A$ such that $x_h < s^u(a)$ and $B_{x_h} \succ a$. By Definition 7, this would imply that there should exist $x_k > x_h$ such that $a \succ B_{x_k}$. Thus, there would exist $b_{kp} \in B_{x_k}$ such that $a \succ b_{kp}$. By Condition *a.2.p* (primal), there should exist $b_{hq} \in B_{x_h}$ such that $b_{kp} \Delta b_{hq}$. Consequently, $a \succ b_{hq}$, which would imply not$(B_{x_h} \succ a)$, contradicting thus the hypothesis $B_{x_h} \succ a$. Therefore, Condition 4 holds.

Analogously, by contradiction, suppose that there exists $x_h \in X$ and $a \in A$ such that $x_h > s^l(a)$ and $a \succ B_{x_h}$. By Definition 6, this would imply that there should exist $x_k < x_h$ such that $B_{x_k} \succ a$. Thus, there would exist $b_{kp} \in B_{x_k}$ such that $b_{kp} \succ a$. By Condition *a.2.d* (dual), there should exist $b_{hq} \in B_{x_h}$ such that $b_{hq} \Delta b_{kp}$. Consequently, $b_{hq} \succ a$ which would imply not$(a \succ B_{x_h})$, contradicting thus the hypothesis $a \succ B_{x_h}$. Therefore, Condition 5 holds. □

Proposition 3 (Implications of soft dominance in a preference of an action w.r.t. a set)**.** *If B fulfills both the primal and the dual soft dominance separability conditions (see Points a.2.p and a.2.d of Condition 10), then, for any $a \in A$, $a \succ B_x$ for all $x \leqslant s^l(a)$.*

Proof. By Definition 6, we have $a \succ B_{s^l(a)}$, and, because Points $a.2.p$ and $a.2.d$ (primal and dual) of Condition 10 hold, by Proposition 1 vii), we obtain $a \succ B_x$, for all $x < s^l(a)$. □

Proposition 4 (Implications of soft dominance in a preference of a set w.r.t. an action)**.** *If B fulfills both the primal and the dual soft dominance separability conditions (see Points a.2.p and a.2.d of condition 10), then, for any $a \in A$, $B_x \succ a$ for all $x \geqslant s^u(a)$.*

Proof. By Definition 7, we have $B_{s^u(a)} \succ a$ and, because Points $a.2.p$ and $a.2.d$ (primal and dual) of Condition 10 hold, by Proposition 1 $viii$), we obtain $B_x \succ a$, for all $x > s^u(a)$. □

Proposition 5 (Implications of soft dominance in the indifference/incomparability region)**.** *If B fulfills both the primal and the dual soft dominance separability conditions (see Condition 10, points a.2.p and a.2.d), then Conditions 9 is fulfilled. Moreover, in such a case, the value $s^u(a)$ is simply the highest value, such that $a \succ B_x$, for $x < s^u(a)$, and $s^l(a)$ is the lowest value, such that $B_x \succ a$, for $x > s^l(a)$, for all $a \in A$ (i.e., the second parts of Definitions 6 and 7 can be neglected because they are automatically fulfilled by themselves)*

Proof. We have already noted that Conditions 4 and 5 are equivalent to Condition 6, in which, for all $a \in A$ and, for all values x such that $s^l(a) < x < s^u(a)$, not$(B_x \succ a)$ and not$(a \succ B_x)$. Therefore, we have to prove only that if Points $a.2.p$ and $a.2.d$ of Condition 10 are satisfied, then for all values x such that not$(B_x \succ a)$ and not$(a \succ B_x)$, we have $s^l(a) < x < s^u(a)$. This is true because

- by Proposition 3, if $x \leqslant s^l(a)$, then $a \succ B_x$; and,
- by Proposition 4, if $x \geqslant s^u(a)$, then $B_x \succ a$.

Consequently, we obtain that, under the same conditions, $s^l(a)$ is simply the highest value x such that $a \succ B_x$, and, $s^u(a)$ is simply the lowest value x such that $B_x \succ a$. □

5. Theoretical results

This section will show the consistency of the method with respect to some fundamental and quite natural requirements, namely the uniqueness, independence, monotonicity, conformity, homogeneity, and stability.

Definition 9 (Inserting and deleting operations)**.** *The following two operations are considered:*

i) Inserting operation:

i.a) a new set B_x is inserted in B;

i.b) a new action b is inserted in B_x.

ii) Deleting operation.

ii.a) a set B_x is removed from B;

ii.b) an action b is removed from B_x.

Definition 10 (Structural requirements)**.** *The following structural requirements of the method are considered:*

i) Uniqueness*. For each action $a \in A$, there is a single value $s^l(a)$ and a single value $s^u(a)$.*

ii) Independence*. The definition of the values $s^l(a)$ and $s^u(a)$, for action $a \in A$, does not depend on the other actions in $A \setminus \{a\}$.*

iii) Monotonicity*. If $a\Delta a'$, then $s^l(a) \geqslant s^l(a')$ and $s^u(a) \geqslant s^u(a')$.*

iv) Conformity*. If $a \in B_{x_k}$, then $s^l(a) = x_{k-1}$ and $s^u(a) = x_{k+1}$.*

v) Homogeneity*. If two actions, a and a', compare the same way with respect to the reference actions in B, then $s^l(a) = s^l(a')$ and $s^u(a) = s^u(a')$.*

vi) Stability*. For $a \in A$ with $x_r = s^l(a) < s^u(a) = x_s$, let $s^{l*}(a)$ and $s^{u*}(a)$ be the new bounds in consequence of one of the operations of Definition 9. Then*

$$x_{r-1} \leqslant s^{l*}(a) \leqslant x_{r+1}$$

and

$$x_{s-1} \leqslant s^{u*}(a) \leqslant x_{s+1}.$$

The latter requirement means that single inserting or deleting operations according to Definition 9 imply a minimal perturbation on the score range of each action.

Note that uniqueness, independence, monotonicity and homogeneity are clearly satisfied. We only focus our attention on conformity and stability.

Theorem 1 (Conformity)**.** *If B fulfills both the primal and the dual soft dominance and preference separability conditions (see Points a.2.p, a.2.d, b.2.p, and b.2.d of Condition 10), then the conformity property of Definition 10 is fulfilled.*

Proof. By Points $b.2.p$ and $b.2.d$ of Condition 10, for each $a \in B_{x_k}$ there exist $b_{k-1p} \in B_{x_{k-1}}$, such that

$$a \succ b_{k-1p} \tag{2}$$

and $b_{k+1q} \in B_{x_{k+1}}$, such that

$$b_{k+1q} \succ a. \tag{3}$$

By contradiction, let us suppose that there exists $b_{k-1r} \in B_{x_{k-1}}$, such that

$$b_{k-1r} \succ a. \tag{4}$$

By Point $a.2.p$ of Condition 10, there would exist $b_{ks} \in B_{x_k}$ such that $b_{ks}\Delta b_{k-1r}$ that, together with (4), would imply $b_{ks} \succ a$, which is impossible because we cannot have a limiting profile of a class preferred to another limiting profile of the same class. Consequently, there cannot be any profile from $B_{x_{k-1}}$ preferred to b_{ks}, which, together with (2), implies that

$$a \succ B_{x_{k-1}}. \tag{5}$$

Again, by contradiction, let us suppose that there exists $b_{k+1t} \in B_{x_{k+1}}$, such that

$$a \succ b_{k+1t}. \tag{6}$$

By Point $b.2.d$ of Condition 10, there would exist $b_{ku} \in B_{x_k}$ such that $b_{k+1t} \Delta b_{ku}$ that, together with (6) would imply $a \succ b_{ku}$, which is impossible because we cannot have a limiting profile of a class preferred to another limiting profile of the same class. Consequently, a cannot be preferred to any profile from $B_{x_{k+1}}$, which, together with (3), implies that

$$B_{x_{k+1}} \succ a. \tag{7}$$

Based on these considerations and taking any $a \in B_{x_k}$, we can say that

– due to Proposition 1 vii) and (5) for any $x_h < x_k$,

$$a \succ B_{x_h}, \tag{8}$$

– from Point i) of Condition 1,

$$not(a \succ B_{x_k}) \text{ and } not(B_{x_k} \succ a), \tag{9}$$

– due to Proposition 1) $viii$) and (7) for any $x_h > x_k$,

$$B_{x_h} \succ a. \tag{10}$$

Therefore,

◦ $k-1$ is the maximum value of h, for which $a \succ B_{x_h}$ and, consequently, by Proposition 5, $s^l(a) = x_{k-1}$,

◦ $k+1$ is the minimum value of h, for which, $B_{x_h} \succ a$ and, consequently, by Proposition 5, $s^u(a) = x_{k+1}$.

□

Lemma 1 (Inserting a reference set: consequences w.r.t. lower bound)**.** *If B fulfills both the primal and the dual soft dominance separability conditions (see Points a.2.p and a.2.d of Condition 10) and if a set of limiting profiles B_x is inserted in B in such a way that these conditions are still fulfilled, and $s^l(a) = x_s$ we have either*

$$x_s < s^{l*}(a) = x < x_{s+1}$$

or

$$s^{l*}(a) = s^l(a).$$

Moreover, $s^{l*}(a) = x$ if and only if

– $x_s < x < x_{s+1}$*; and.*

– $a \succ B_x$*.*

Proof. If a set B_x is added we can have the following cases:

1. $x < s^l(a)$: in this case, by Proposition 3, we obtain $a \succ B_x$. Consequently, the highest value z from $X \cup \{x\}$ such that $a \succ B_z$ is again $s^l(a)$, which, by Proposition 5, implies $s^{l*}(a) = s^l(a)$;

2. $s^l(a) = x_s < x < x_{s+1}$ and not$(a \succ B_x)$: also in this case the maximal value z from $X \cup \{x\}$ such that $a \succ B_z$ is $s^l(a)$, which, again by Proposition 5, implies $s^{l*}(a) = s^l(a)$;

3. $s^l(a) = x_s < x < x_{s+1}$ and $a \succ B_x$: in this case the highest value z from $X \cup \{x\}$, such that, $a \succ B_z$ is x, which, again by Proposition 5, implies $s^{l*}(a) = x$;

4. $s^l(a) = x_s < x_{s+1} < x$: in this case we cannot have $a \succ B_x$ because from Condition 7 (that is, by the joint consideration of Conditions 2 and 4) not$(a \succ B_{x_{s+1}})$, but, by Proposition 1 *vii*), $a \succ B_x$ would imply $a \succ B_{x_{s+1}}$.

□

Lemma 2 (Inserting a reference set: consequences for the upper bound)**.** *If B fulfills both the primal and the dual soft dominance separability conditions (see Points a.2.p and a.2.d of Condition 10), and if a set of limiting profiles B_x is inserted in B in such a way that these conditions are still fulfilled, and $s^u(a) = x_t$ we have either*

$$x_{t-1} < s^{u*}(a) = x < x_t$$

or

$$s^{u*}(a) = s^u(a).$$

Moreover, $s^{u*}(a) = x$ if and only if

- *$x_{t-1} < x < x_t$; and,*
- *$B_x \succ a$.*

Proof. If a set B_x is added, then we can have the following cases:

1. $x > s^u(a)$: in this case, by Proposition 3, we obtain $B_x \succ a$. Consequently, the lowest value z from $X \cup \{x\}$ such that $B_z \succ a$ is again $s^u(a)$, which, from Proposition 5, implies $s^{u*}(a) = s^u(a)$;

2. $s^u(a) = x_t > x > x_{t-1}$ and not$(B_x \succ a)$: also in this case the lowest value z from $X \cup \{x\}$ such that $B_z \succ a$ is $s^u(a)$, which, again by Proposition 5, implies $s^{u*}(a) = s^u(a)$;

3. $s^u(a) = x_t > x > x_{t-1}$ and $B_x \succ a$: in this case the lowest value z from $X \cup \{x\}$, such that, $B_z \succ a$ is x, which, again by Proposition 5, implies $s^{u*}(a) = x$;

4. $s^u(a) = x_t > x_{t-1} > x$: in this case one cannot have $B_x \succ a$ because by Condition 8 (that is, by the joint consideration of Conditions 3 and 5) not$(B_{x_{t-1}} \succ a)$, but, by Proposition 1 *viii*), $B_x \succ a$ would imply $B_{x_{t-1}} \succ a$.

□

Remark 5 (Deleting a reference set)**.** *If both the primal and the dual soft dominance separability conditions (see Points a.2.p and a.2.d of Condition 10) hold for B and a set of limiting profiles B_x is deleted from B, clearly the primal and the dual soft dominance separability conditions continue to be fulfilled.*

Lemma 3 (Deleting a reference set: consequences w.r.t. lower bound)**.** *If B fulfills both the primal and the dual soft dominance separability conditions (see points a.2.p and a.2.d of Condition 10), a set of limiting profiles B_x is removed from B, and $s^l(a) = x_s$, then*

- $s^l(a) = x_s = x$ *and* $s^{l*}(a) = x_{s-1}$*; or*
- $s^l(a) = x_s \neq x$ *and* $s^{l*}(a) = s^l(a)$.

Proof. According to Remark 5, Points $a.2.p$ and $a.2.d$ of Condition 10 hold also after B_x is removed from B. Consequently, from Proposition 5, $s^{l*}(a)$ continues to be the highest value $z \in X \setminus \{x\}$ such that $a \succ B_z$. We can have two cases

$s^l(a) = x_s = x$: from Proposition 3 we get that the highest value $z \in X \setminus \{x\}$ such that $a \succ B_z$ becomes x_{s-1} and, consequently, $s^{l*}(a) = x_{s-1}$;

$s^l(a) = x_s \neq x$: the highest value $z \in X \setminus \{x\}$ such that $a \succ B_z$ continues to be x_s and, consequently, $s^{l*}(a) = s^l(a) = x_s$.

□

Lemma 4 (Deleting a reference set: consequences w.r.t. upper bound)**.** *If B fulfills both the primal and the dual soft dominance separability conditions (see Points a.2.p and a.2.d of Condition 10), a set of limiting profiles B_x is removed from B, and $s^u(a) = x_t$, then*

- $s^u(a) = x_t = x$ *and* $s^{u*}(a) = x_{t+1}$*; or,*
- $s^u(a) = x_t \neq x$ *and* $s^{u*}(a) = s^u(a)$.

Proof. Remember that from Remark 5, Points $a.2.p$ and $a.2.d$ of Condition 10 hold also after B_x is removed from B, and, taking into account Proposition 5, $s^{u*}(a)$ continues to be the lowest value $z \in X \setminus \{x\}$ such that $B_z \succ a$. We can have two cases:

1. $s^u(a) = x_t = x$: from Proposition 4 we get the lowest value $z \in X \setminus \{x\}$ such that $B_z \succ a$ becomes x_{t+1} and, consequently, $s^{u*}(a) = x_{t+1}$;
2. $s^u(a) = x_t \neq x$: the lowest value $z \in X \setminus \{x\}$ such that $B_z \succ a$ continues to be x_t and, consequently, $s^{u*}(a) = s^u(a) = x_t$.

□

Lemma 5 (Inserting a reference action: consequences w.r.t. lower bound)**.** *If B fulfills both the primal and the dual soft dominance separability conditions (see Points a.2.p and a.2.d of Condition 10), and if a limiting profile b_{kp} is added to B_{x_k} in such a way that these conditions are still fulfilled, and $s^l(a) = x_s$, then*

- $s^{l*}(a) = x_{s-1}$ *if and only if* $b_{kp} \succ a$ *and* $s^l(a) = x_s = x_k$*;*
- $s^{l*}(a) = x_{s+1}$*, if and only if* $a \succ b_{kp}$*,* $x_k = x_{s+1}$ *and there is no* $b_{s+1q} \in B_{x_{s+1}}$ *such that* $b_{s+1q} \succ a$*;*
- $s^{l*}(a) = s^l(a)$*, otherwise.*

Proof. If $b_{kp} \succ a$ and $s^l(a) = x_s = x_k$, after adding b_{kp},we have not$(a \succ B_{x_s})$. Consequently, taking into consideration Proposition 3, the highest value z such that $a \succ B_z$ becomes x_{s-1}, so that $s^{l*}(a) = x_{s-1}$.

If $a \succ b_{kp}$, $x_k = x_{s+1}$ and there is no other $b_{kq} \in B_{x_{s+1}}$ such that $b_{kq} \succ a$, after adding b_{kp}, we have $a \succ B_{x_{s+1}}$. In this case, the highest value z such that $a \succ B_z$ is no more x_s and it becomes x_{s+1}, in such a way that, according to Proposition 5, we obtain $s^{l*}(a) = x_{s+1}$.

All of the other possible cases are as follows:

1. $x_k < s^l(a)$: because $a \succ B_{s^l(a)}$ and after adding b_{kp}, Points $a.2.p$ and $a.2.d$ of Condition 10 are still satisfied, then the highest z such that $a \succ B_z$ remains $s^l(a)$, which, from Proposition 5, leads to $s^{l*}(a) = s^l(a)$;

2. $x_k = s^l(a)$, but not$(b_{kp} \succ a)$: in this case, after adding the limiting profile b_{kp}, we continue to have $a \succ B_{x_k}$, in such a way that the highest z such that $a \succ B_z$ remains $s^l(a)$, which, from Proposition 5, leads to $s^{l*}(a) = s^l(a)$;

3. $x_k = x_{s+1}$, but not$(a \succ b_{kp})$ or there is at least one $b_{kq} \in B_{x_{s+1}}$, such that, $b_{kq} \succ a$: in this case, after adding the reference action b_{kp}, we continue to have not$(a \succ B_{x_{s+1}})$, in such a way that the highest z, such that, $a \succ B_z$ remains $s^l(a)$, which, from Proposition 5, leads to $s^{l*}(a) = s^l(a)$;

4. $x_k > x_{s+1}$: because not$(a \succ B_{x_{s+1}})$ and because after adding b_{kp}, Points $a.2.p$ and $a.2.d$ of Condition 10 are still satisfied, then, from Proposition 1 vii), we obtain not$(a \succ B_{x_k})$, in such a way that the highest z such that $a \succ B_z$ remains $s^l(a)$ which, by Proposition 5, leads to $s^{l*}(a) = s^l(a)$.

□

Lemma 6 (Inserting a reference action: consequences w.r.t. upper bound)**.** *If B fulfills both the primal and the dual soft dominance separability conditions (see Points a.2.p and a.2.d of Condition 10), and if a limiting profile b_{kp} is added to B_{x_k} in such a way that these conditions are still fulfilled, and $s^u(a) = x_t$, then*

- $s^{u*}(a) = x_{t+1}$*, if* $a \succ b_{kp}$ *and* $s^u(a) = x_t = x_k$*;*
- $s^{u*}(a) = x_{t-1}$*, if* $b_{kp} \succ a$*,* $x_k = x_{t-1}$ *and there is no* $b_{kq} \in B_{x_{t-1}}$ *such that* $a \succ b_{kq}$*;*
- $s^{u*}(a) = s^u(a)$*, otherwise.*

Proof. If $a \succ b_{kp}$ and $s^u(a) = x_t = x_k$, then, after adding b_{kp}, we have not$(B_{x_k} \succ a)$. Consequently, taking into consideration Proposition 3, the lowest value z such that $B_z \succ a$ becomes x_{t+1}, in such a way that $s^{u*}(a) = x_{t+1}$.

If $b_{kp} \succ a$, $x_k = x_{t-1}$ and there is no other $b_{kq} \in B_{x_{t-1}}$ such that $a \succ b_{kq}$, after adding b_{kp}, we have $B_{x_{t-1}} \succ a$. In this case, the lowest value z such that $B_z \succ a$ is no more x_t and it becomes x_{t-1}, in such a way that, according to Proposition 5, we obtain $s^{u*}(a) = x_{t-1}$.

All of the other possible cases are as follows:

1. $x_k > s^u(a)$: because $B_{s^u(a)} \succ a$ and after adding b_{kp}, Points $a.2.p$ and $a.2.d$ of Condition 10 are still satisfied, the lowest z such that $B_z \succ a$ remains $s^u(a)$, which, by Proposition 5, leads to $s^{u*}(a) = s^u(a)$;

2. $x_k = s^u(a)$, but not$(a \succ b_{kp})$: in this case, after adding the reference action b_{kp}, we continue to have $B_{x_k} \succ a$, in such a way that the lowest z such that $B_z \succ a$ remains $s^u(a)$, which, by Proposition 5, leads to $s^{u*}(a) = s^u(a)$;

3. $x_k = x_{t-1}$, but not$(b_{kp} \succ a)$ or there is at least one $b_{kq} \in B_{x_{t-1}}$ such that $a \succ b_{kq}$: in this case, after adding the reference action b_{kp}, we continue to have not$(B_{x_{t-1}} \succ a)$, in such a way that the lowest z such that $B_z \succ a$ remains $s^u(a)$, which, by Proposition 5, leads to $s^{u*}(a) = s^u(a)$;

4. $x_k < x_{t-1}$: because not$(B_{x_{t-1}} \succ a)$ and because after adding b_{kp} Points $a.2.p$ and $a.2.d$ of Condition 10 are still satisfied, then, by Proposition 1 *viii)*, we obtain not$(B_{x_k} \succ a)$, in such a way that the lowest z such that $B_z \succ a$ remains $s^u(a)$, which, by Proposition 5, leads to $s^{u*}(a) = s^u(a)$.

□

Lemma 7 (Deleting a reference action: consequences w.r.t. lower bound)**.** *If B fulfills both the primal and the dual soft dominance separability conditions (see Points a.2.p and a.2.d of Condition 10), and if a limiting profile b_{kp} is removed from B_{x_k}, in such a way that, these conditions are still fulfilled, and $s^l(a) = x_s$, then*

- $s^{l*}(a) = x_{s-1}$, *if $s^l(a) = x_k$, $a \succ b_{kp}$ and there is no other $b_{kq} \in B_{x_k}$ such that $a \succ b_{kq}$,*
- $s^{l*}(a) = x_{s+1}$, *if $b_{kp} \succ a$, $x_k = x_{s+1}$, there is no other $b_{kq} \in B_{x_{s+1}}$ such that $b_{kq} \succ a$ and there exists at least one reference action $b_{kr} \in B_{x_{s+1}}$ such that $a \succ b_{kr}$;*
- $s^{l*}(a) = s^l(a)$, *otherwise.*

Proof. If $s^l(a) = x_k$, $a \succ b_{kp}$ and there is no other $b_{kq} \in B_{x_k}$ such that $a \succ b_{kq}$, then after b_{kp} is removed we have not$(a \succ B_{x_k})$. Consequently, taking into consideration Proposition 3, the highest value z such that $a \succ B_z$ becomes x_{s-1}, in such a way that $s^{l*}(a) = x_{s-1}$.

If $b_{kp} \succ a$, $x_k = x_{s+1}$, there is no other $b_{kq} \in B_{x_{s+1}}$ such that $b_{kq} \succ a$ and there exists at least one limiting profile $b_{kr} \in B_{x_{s+1}}$ such that $a \succ b_{kr}$, after removing b_{kp} we have $a \succ B_{x_{s+1}}$, in such a way that the highest z such that $a \succ B_z$ becomes x_{s+1} and, consequently, from Proposition 5, $s^{l*}(a) = x_{s+1}$.

All the other possible cases are as follows:

1. $x_k < s^l(a)$: because after removing b_{kp} the highest z such that $a \succ B_z$ remains $s^l(a)$ and Points $a.2.p$ and $a.2.d$ of Condition 10 are still satisfied, by Proposition 5, we obtain $s^{l*}(a) = s^l(a)$;

2. $x_k = s^l(a)$, and not$(a \succ b_{kp})$ or there is at least another $b_{kq} \in B_{x_k}$ such that $a \succ b_{kq}$: in this case, after removing the limiting profile b_{kp}, we continue to have $a \succ B_{s^l(a)}$, and, therefore, $s^{l*}(a) = s^l(a)$;

3. $x_k = x_{s+1}$, but not$(b_{kp} \succ a)$ or there is at least another $b_{kq} \in B_{x_s+1}$ such that $b_{kq} \succ a$ or there does not exist any reference action $b_{kr} \in B_{x_{s+1}}$ such that $a \succ b_{kr}$: in this case, after removing the limiting profile b_{kp}, we continue to have not$(a \succ B_{x_{s+1}})$, in such a way that the highest z such that $B_z \succ a$ remains $s^l(a)$, which gives $s^{l*}(a) = s^l(a)$;

4. $x_k > x_{s+1}$: because not$(a \succ B_{x_{s+1}})$ and because after removing b_{kp} Points $a.2.p$ and $a.2.d$ of Condition 10 are still satisfied, then, by Proposition 1 *vii)* we obtain not$(a \succ B_{x_k})$, in such a way that the highest z such that $a \succ B_z$ remains $s^l(a)$, which leads to $s^{l*}(a) = s^l(a)$.

□

Lemma 8 (Deleting a reference action: consequences w.r.t. upper bound)**.** *If B fulfills both the primal and the dual soft dominance separability conditions (see Points a.2.p and a.2.d of Condition 10) and if a limiting profile b_{kp} is removed from B_{x_k} in such a way that these conditions are still fulfilled, and $s^u(a) = x_t$, then*

- $s^{u*}(a) = x_{t+1}$, *if* $s^u(a) = x_k$, $b_{kp} \succ a$ *and there is no other* $b_{kq} \in B_{x_k}$ *such that* $b_{kq} \succ a$,
- $s^{u*}(a) = x_{t-1}$, *if* $a \succ b_{kp}$, $x_k = x_{t-1}$, *there is no* $b_{kq} \in B_{x_{t-1}}$ *such that* $a \succ b_{kq}$ *and there is at least one reference action* $b_{kr} \in B_{x_{t-1}}$ *such that* $b_{kr} \succ a$;
- $s^{l*}(a) = s^u(a)$, *otherwise.*

Proof. If $s^u(a) = x_k$, $b_{kp} \succ a$ and there is no other $b_{kq} \in B_{x_k}$ such that $b_{kq} \succ a$, then after b_{kp} is removed one has not$(B_{x_k} \succ a)$. Consequently, taking into consideration Proposition 3, the lowest value z such that $B_z \succ a$ becomes x_{t+1}, in such a way that $s^{u*}(a) = x_{t+1}$.

If $a \succ b_{kp}$, $x_k = x_{t-1}$, there is no other $b_{kq} \in B_{x_{t-1}}$ such that $a \succ b_{kq}$ and there is at least one limiting profile $b_{kr} \in B_{x_{t-1}}$ such that $b_{kr} \succ a$, after removing b_{kp} one has $B_{x_{t-1}} \succ a$, in such a way that the lowest z such that $a \succ B_z$ becomes x_{t-1} and, consequently, from Proposition 5, $s^{u*}(a) = x_{t-1}$.

All the other possible cases are as follows:

1. $x_k > s^u(a)$: because after removing b_{kp} the lowest z such that $B_z \succ a$ remains $s^u(a)$ and Points $a.2.p$ and $a.2.d$ of Condition 10 are still satisfied, by Proposition 5, we obtain $s^{u*}(a) = s^u(a)$;
2. $x_k = s^u(a)$, and not$(b_{kp} \succ a)$ or there is at least another $b_{kq} \in B_{x_k}$ such that $b_{kq} \succ a$: in this case, after removing the reference action b_{kp}, we continue to have $B_{s^u(a)} \succ a$, and, therefore, $s^{u*}(a) = s^u(a)$;
3. $x_k = x_{t-1}$, but not$(a \succ b_{kp})$ or there is at least another $b_{kq} \in B_{x_{t-1}}$ such that $a \succ b_{kq}$, or there does not exist any reference action $b_{kr} \in B_{x_{t-1}}$ such that $b_{kr} \succ a$: in this case, after removing the reference action b_{kp}, we continue to have not$(B_{x_{t-1}} \succ a)$, in such a way that the lowest z such that $B_z \succ a$ remains $s^u(a)$, which leads to $s^{u*}(a) = s^u(a)$;
4. $x_k < x_{t-1}$: because not$(B_{x_{t-1}} \succ a)$ and because after removing b_{kp} Points $a.2.p$ and $a.2.d$ of Condition 10 are still satisfied, then, by Proposition 1 *viii)*, we obtain not$(B_{x_k} \succ a)$, in such a way that the lowest z, such, that $B_z \succ a$ remains $s^u(a)$, which leads to $s^{u*}(a) = s^u(a)$.

□

Putting together the results of Lemmas 1-8 we get the following general result.

Theorem 2 (Stability)**.** *If B fulfills both the primal and the dual soft dominance separability conditions (see Points a.2.p and a.2.d of Condition 10) then the stability condition holds.*

With respect to the representative score $s^\circ(a)$ the following requirements seem interesting.

Definition 11 (Structural requirements for the representative score)**.** *The following structural requirements are considered:*

i) Uniqueness*. For each action $a \in A$, there is a single score $s^\circ(a)$.*

ii) Independence*. The definition of the score $s^\circ(a)$ for action $a \in A$ does not depend on the other actions in $A \setminus \{a\}$.*

iii) Monotonicity*. If $a\Delta a'$, then $s^\circ(a) \geqslant s^\circ(a')$.*

iv) Homogeneity*. If two actions, a and a', compare the same way with respect to the reference actions in B, then $s^\circ(a) = s^\circ(a')$.*

Clearly all the above requirements are satisfied by the representative score $s^\circ(a)$.

6. Practical issues and an illustrative example

This section presents some practical aspects of the method and an illustrative example

6.1. Building the set of reference actions

Let us start by observing that in defining the collection of sets of references actions

$$B = \{B_{x_1}, \ldots, B_{x_k}, \ldots, B_{x_\ell}\}$$

it is necessary to take care that all actions a from A the following condition is satisfied

$$B_{x_\ell} \succ a \succ B_{x_1}. \tag{11}$$

Condition (11) permits us to compare all actions a from A with sets B_x of reference actions, so that a range $]s^l(a), s^u(a)[$ for the score $s(a)$ is assigned to each $a \in A$.
There are several different ways of building the set of reference actions. Next we present two possible procedures.

Direct method. We ask the decision-maker to propose a set of reference actions that are used to characterize some scoring levels, for example, 20, 40, 60, and 80. Several reference actions can be proposed *a priori* to characterize the same scoring level. We assume that the scales are bounded from below and from above in such a way that we can consider reference the actions b_0 and b_{100}, characterizing the scores of 0 and 100 with actions having, on each considered criterion, the lower bound performances and the upper bound performances, respectively.

A deck of cards based technique. We can also ask the decision-maker to propose a set of actions to be taken as reference actions and use the deck of cads method in a similar way as in Figueira and Roy (2002). These actions must be ordered from the worst to the best, with the possibility of some ties. The decision-maker is required to put some blank cards between these equivalent sets of reference actions to assign a score to each equivalence class, on the basis of the extreme scores 0 and 100. The scores thus obtained are not necessarily equally spaced. It should be noticed that other values different from 0 and 100 can be used (this is also true when using the previous method).

6.2. The method

The method is now presented in a very simple way. Remember that to assign a score $s(a)$ to the action $a \in A$, we should firstly choose a λ cutting level to transform the fuzzy outranking relation into a crispy relation.

1. Find the set of reference actions, B_x, with the highest score x, such that $a \succ B_x$ and, for all $x' < x$, we have either $a \succ B_{x'}$ or $a \parallel B_{x'}$. It is then natural to consider a score $s(a)$ such that $s(a) > x = s^l(a)$, and indeed, according Definition 6, we have $s^l(a) = x$. Let us remember also that by Proposition 5, if both the primal and the dual soft dominance separability conditions (see Condition 10, points $a.2.p$ and $a.2.d$) hold, then $s^l(a)$ is simply the highest value, such that $a \succ B_x$.

2. Find the set of reference actions, B_x, with the lowest score x, such that $B_x \succ a$ and, for all $x' > x$, we have either $B_{x'} \succ a$ or $a \parallel B_{x'}$. It is then natural to consider a score $s(a)$ such that $s(a) < x = s^u(a)$, and indeed, according Definition 7, we have $s^u(a) = x$. Let us remember also that by Proposition 5, if both the primal and the dual soft dominance separability conditions (see Condition 10, points $a.2.p$ and $a.2.d$) hold, then $s^u(a)$ is simply the lowest value, such that $B_x \succ a$.

6.3. Illustrative example

We consider the example in Figueira et al. (2009) regarding the evaluation of some sites for the location of a new hotel. The set of criteria is as follows:

1. *Investment costs* (Scale unit: K€; Code: `ICOST`; notation: g_1; preference direction: minimization). This criterion comprises the land purchasing costs, as well as the costs for building the new hotel.

2. *Annual costs* (Scale unit: K€; Code: `ACOST`; notation: g_2; preference direction: minimization). This criterion comprises the hotel operating costs.

3. *Recruitment* (Scale unit: verbal levels (seven); Code: `RECRU`; notation: g_3; preference direction: maximization). This criterion models the possibility of recruiting workers.

4. *Image* (Scale unit: verbal levels (seven); Code: `IMAGE`; notation: g_4; preference direction: maximization). This criterion models the perceptions of the clients about the district where the new hotel will be located.

5. *Access* (Scale unit: verbal levels (seven); Code: `ACCES`; notation: g_5; preference direction: maximization). This criterion models the possibility of recruiting workers.

The verbal scale used for the last three criteria comprises the following levels (in between parenthesis we used a numerical code for each level): `very bad`[1]; `bad`[2]; `rather bad`[3]; `average`[4]; `rather good`[5]; `good`[6]; `very good`[7].

There are five potential sites for the location of the new hotel. The performance table can be presented as follows (Table 1).

The weights, discriminating (indifference and preference) thresholds used in the method are the following (see Table 2). Let us consider an ordered pair of actions $(a, b) \in A \times A$. The performance of b is assumed to be worse than the performance of a. This means that the variable thresholds

a	ICOST (g_1)	ACOST (g_2)	RECRU (g_3)	IMAGE (g_4)	ACCES (g_5)
a_1	13 000	3 000	4	4	4
a_2	15 000	2 500	6	2	7
a_3	10 900	3 400	6	6	1
a_4	15 500	3 500	6	6	6
a_5	15 000	2 600	6	1	2

Table 1: Performance table

presented in the next table with respect to criteria g_1 and g_2 are direct variable thresholds (see Roy et al., 2014). For the sake of simplicity, no veto thresholds are considered in this example. For the remaining criteria, the thresholds are constant (the numbers represent the differences of levels, not the scale levels. Thus, the indifference threshold is a difference of one performance level, while the preference threshold corresponds to a difference of two performance levels).

Parameters	ICOST (g_1)	ACOST (g_2)	RECRU (g_3)	IMAGE (g_4)	ACCES (g_5)
k_j	5	4	3	3	3
$q_j(g_j(b))$	$250 + 0.03g_1(b)$	$50 + 0.05g_2(b)$	1	1	1
$p_j(g_j(b))$	$500 + 0.05g_1(b)$	$100 + 0.07g_2(b)$	2	2	2

Table 2: Parameters table

Our set of limiting profiles is composed of seven subsets; that is, $B = \{B_{x_1}, B_{x_2}, B_{x_3}, B_{x_4}, B_{x_5}, B_{x_6}, B_{x_7}\}$. This means that will be defined seven reference values $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$. These values were obtained with the deck of cards method proposed in Bottero et al. (2018) to build interval scales, by fixing two reference levels. In this work, and without loss of generality, we will put $x_1 = 0$ and $x_7 = 100$.

The B sets are characterized by at least a single limiting profile as follows.

$$B_{x_1=0} = \{b_{11} = (18\,00, 4\,000, 1, 1, 1)\};$$

$$B_{x_2} = \{b_{21} = (17\,000, 3\,500, 2, 2, 1),\ b_{22} = (16\,500, 3\,700, 1, 2, 1)\};$$

$$B_{x_3} = \{b_{31} = (15\,350, 3\,200, 3, 1, 2)\};$$

$$B_{x_4} = \{b_{41} = (14\,250, 2\,850, 3, 4, 3),\ b_{42} = (13\,750, 3\,150, 4, 3, 3)\};$$

$$B_{x_5} = \{b_{51} = (12\,650, 2\,650, 4, 4, 5)\};$$

$$B_{x_6} = \{b_{61} = (11\,500, 2\,100, 5, 6, 5),\ b_{62} = (11\,000, 2\,500, 6, 5, 7)\};$$

$$B_{x_7=100} = \{b_{71} = (10\,000, 2\,000, 7, 7, 7)\};$$

The deck of cards method applied to this problem works as follows.

1. Suppose the decision-maker provides the following strict order on the subsets of reference profiles ($\prec$ means "strictly less preferred than"):

$$B_{x_1} \prec B_{x_2} \prec B_{x_3} \prec B_{x_4} \prec B_{x_5} \prec B_{x_6} \prec B_{x_7}.$$

2. We then call the attention of the decision-maker to the fact that if the difference between two consecutive sets is bigger than the difference of other pair of consecutive subsets, she/he should add more blank cards in the first two consecutive ones than in the second ones (for more details see Bottero et al., 2018). We may obtain the following ranking of the subsets with the blank cards in between consecutive ones (in brackets):

$$B_{x_1}\ [1]\ B_{x_2}\ [2]\ B_{x_3}\ [0]\ B_{x_4}\ [1]\ B_{x_5}\ [0]\ B_{x_6}\ [2]\ B_{x_7}.$$

   Zero blank cards between two consecutive sets of limiting profiles does not mean that the limiting profiles of the two consecutive sets have the same value, but only that the difference is minimal. We know that the number of units between B_{x_1} and B_{x_7} is $h = (1+1)+(2+1)+(0+1)+(1+1)+(0+1)+(2+1) = 12$.

3. In this step we should define the two reference levels. As stated before, we considered that the value of all the limiting profiles in B_{x_1} is 0; that is, $x_1 = 0$ and that the value of all the limiting profiles in B_{x_7} is 100, i.e., $x_7 = 100$.

4. Consequently, we can compute the value of the unit as follows.

$$u = \frac{x_7 - x_1}{h} = \frac{100-0}{12} = 8.33.$$

5. The computations of the remaining values in X is easy because we know the number of units separating two consecutive sets of limiting profiles. We have thus,

$$x_1 = 0,\ x_2 = 16.67,\ x_3 = 41.67,\ x_4 = 50,\ x_5 = 66.67,\ x_6 = 75.00,\ x_7 = 100.$$

Let us consider now the comparison table of all of our five sites against the limiting profiles (Table 3).

b	$a_1 \succ b$	$a_2 \succ b$	$a_3 \succ b$	$a_4 \succ b$	$a_5 \succ b$	$b \succ a_1$	$b \succ a_2$	$b \succ a_3$	$b \succ a_4$	$b \succ a_5$
b_{11}	$\succ$	$\succ$	$\succ$	$\succ$	$\succ$					
b_{21}	$\succ$	$\succ$	$\succ$	$\succ$	$\succ$					
b_{22}	$\succ$	$\succ$	$\succ$	$\succ$	$\succ$					
b_{31}	$\succ$	$\succ$	$\succ$	$\succ$	$\succ$					
b_{41}		$\succ$		$\succ$	$\succ$					
b_{42}		$\succ$	$\succ$							
b_{51}									$\succ$	$\succ$
b_{61}						$\succ$	$\succ$	$\succ$	$\succ$	$\succ$
b_{62}						$\succ$	$\succ$	$\succ$	$\succ$	$\succ$
b_{71}						$\succ$	$\succ$	$\succ$	$\succ$	$\succ$

Table 3: Comparison table

Now, for the definition of the score range of each alternative, we can take advantage of this table. Let us consider action a_1 and try to identify $s^l(a_1) < s(a) < s^u(a_1)$. For the lower bound, it is provided by x_3 and the upper bound is provided by x_6. Thus, the range for score of a_1 becomes.

$$41.67 < s(a_1) < 75.$$

With the same procedure we can derive the range for all the five actions:

$41.67 < s(a_1) < 75;$

$50 < s(a_2) < 75;$

$50 < s(a_3) < 75;$

$50 < s(a_4) < 66.67;$

$50 < s(a_5) < 66.67.$

Adopting a degree of optimism $\alpha = 0.7$, we obtain the following representative scores:

$$s^{\circ}(a_1) = 65, s^{\circ}(a_2) = 67.5, s^{\circ}(a_3) = 67.5, s^{\circ}(a_4) = 61.67, s^{\circ}(a_5) = 61.67.$$

7. A case study: Measuring performances of Healthcare Access and Quality Index

Assessing the performances of healthcare systems are becoming more and more important; see e.g. (Organization, 2000; Nolte and McKee, 2004, 2008; Radley et al., 2015). Overall healthcare system assessment can benefit from a non-compensatory approach partially or completely preventing that bad performances on some criteria could be outweighed by a combination of relatively better performances on some other criteria. In this perspective, we applied ELECTRE-SCORE to the assessment of the Healthcare Access and Quality (HAQ) index (Fullman et al., 2018) build on the basis of Global Burden of Disease Study 2016 (Vos et al., 2017). HAQ uses 32 causes from which death should not occur in the presence of effective care. Each cause is transformed into a scale of 0100, with 0 as the first percentile (worst) observed between 1990 and 2016, and 100 as the 99th percentile (best). HAQ overall index is obtained applying principal components analysis using all scaled cause values. We applied our methodology on the first 60 best ranked countries in the HAQ 2016 index.

The list of the 32 causes considered is the following: g_1, Tuberculosis; g_2, Diarrhoeal diseases; g_3, LRIs; g_4, URIs; g_5, Diphtheria; g_6, Whooping cough; g_7, Tetanus; g_8, Measles; g_9, Maternal disorders; g_{10}, Neonatal disorders; g_{11}, NM skin cancer (SCC); g_{12}, Breast cancer; g_{13}, Cervical cancer; g_{14}, Uterine cancer; g_{15}, Colon cancer; g_{16}, Testicular cancer; g_{17}, Hodgkins lymphoma; g_{18}, Leukaemia; g_{19}, Rheumatic HD; g_{20}, Ischaemic HD; g_{21}, Stroke; g_{22}, Hypertensive HD; g_{23}, Chronic respiratory; g_{24}, Peptic ulcer; g_{25}, Appendicitis; g_{26}, Hernia; g_{27}, Gallbladder; g_{28}, Epilepsy; g_{29}, Diabetes; g_{30}, Chronic kidney; g_{31}, Congenital heart; g_{32}, Adverse med treat. These 32 causes constitutes the set of criteria we considered.

We fixed ten reference actions with the following procedure. For each criterion $g_j, j = 1, \ldots, 32$, we considered the worst (minimal) and the best (maximal) performances denoted b_j^1 and b_j^{10}, respectively, in the set of 60 considered countries. After we fixed the values of the performances of the other reference actions as follows:

$$b_j^r = int\left[b_j^1 + \frac{(b_j^1 - b_j^{10}) \cdot (r-1)}{9}\right], r = 1, \ldots, 32,$$

where $int(x)$ is the approximation of x to the closest integer value. The reference actions obtained with this procedure are presented in Table 4.

For all criteria $g_j, j = 1 \ldots, 32$, we considered indifference threshold $q_j = 2$, preference threshold $p_j = 5$ and veto threshold $v_j = 10$, respectively. To show the versatility of the proposed approach we considered two basic scenarios:

b_j^r	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}	g_{11}	g_{12}	g_{13}	g_{14}	g_{15}	g_{16}
b^1	51	73	41	100	100	67	100	83	78	46	20	55	33	53	43	39
b^2	56	76	47	100	100	70	100	84	80	52	28	60	40	58	49	45
b^3	61	79	54	100	100	74	100	86	82	58	37	65	47	63	55	52
b^4	67	82	60	100	100	78	100	88	85	64	46	70	55	68	62	59
b^5	72	85	67	100	100	81	100	90	87	70	55	75	62	73	68	66
b^6	78	88	73	100	100	85	100	92	90	76	64	80	70	79	74	72
b^7	83	91	80	100	100	89	100	94	92	82	73	85	77	84	81	79
b^8	89	94	86	100	100	92	100	96	95	88	82	90	85	89	87	86
b^9	94	97	93	100	100	96	100	98	97	94	91	95	92	94	93	93
b^{10}	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100

b_j^r	g_{17}	g_{18}	g_{19}	g_{20}	g_{21}	g_{22}	g_{23}	g_{24}	g_{25}	g_{26}	g_{27}	g_{28}	g_{29}	g_{30}	g_{31}	g_{32}
b^1	26	16	54	15	24	24	68	51	69	69	51	48	42	25	28	30
b^2	34	25	59	24	32	32	71	56	72	72	56	53	48	33	36	37
b^3	42	34	64	33	40	40	75	61	75	75	61	59	54	41	44	45
b^4	50	44	69	43	49	49	78	67	79	79	67	65	61	50	52	53
b^5	58	53	74	52	57	57	82	72	82	82	72	71	67	58	60	61
b^6	67	62	79	62	66	66	85	78	86	86	78	76	74	66	68	68
b^7	75	72	84	71	74	74	89	83	89	89	83	82	80	75	76	76
b^8	83	81	89	81	83	83	92	89	93	93	89	88	87	83	84	84
b^9	91	90	94	90	91	91	96	94	96	96	94	94	93	91	92	92
b^{10}	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100

Table 4: Reference actions

1. Scenario 1:
   - all criteria were assigned equal weights, that is $w_j = \frac{1}{32} = 0.03125$;
   - the cutting-off threshold was fixed at $\lambda = 0.75$;
   - the ten reference actions $b^1, \ldots, b^{10}$ were assigned a values $x_1, \ldots, x_{10}$ by using the deck of the cards method, considering no blank card between each two consecutive reference actions and by fixing $x_1 = 0$ and $x_{10} = 100$; consequently, we obtained the following values:
   $$x_1 = 0, \quad x_2 = 11.11, \quad x_3 = 22.22, \quad x_4 = 33.33, \quad x_5 = 44.44,$$
   $$x_6 = 55.56, \quad x_7 = 66.67, \quad x_8 = 77.78, \quad x_9 = 88.89, \quad x_{10} = 100;$$
   - the representative score $s_1^\circ(a)$ was computed by considering a degree of optimism $\alpha = 0.5$, so that, for each country a,
   $$s_1^\circ(a) = 0.5 \cdot s_1^l(a) + 0.5 \cdot s_1^u(a).$$

   .

2. Scenario 2:
   - the 32 criteria were assigned a weight increasing with the standard deviation of their performances with the deck of cards method (SRF version); more precisely the following procedure was applied:
     - the criteria were split in four classes:
       * class A: criteria with standard deviation equal to 0;
       * class B: criteria with standard deviation greater than 0, but not greater than 10;
       * class C: criteria with standard deviation greater than 10, but not greater than 20;
       * class D: criteria with standard deviation greater than 20;
     - the following blank cards were added to represent difference in importance between criteria of different classes: no blank cards between class A and class B, one blank card between class B and class C, two blank cards between class C and class D;
     - the ratio between the weights of the most important criteria (those ones in class D) and the least important criteria (those ones in class A) was given a value $z = 20$;

- ◦ the following weights were obtained: criteria of classes A, B, C and D obtained weights $w_A = 0,0032$, $w_B = 0,0134$, $w_C = 0,034$, $w_D = 0,0644$. The weights w_j together with its standard deviation σ_j of each criterion $g_j, j = 1, \ldots, 32$, are shown in Table 5.

– the cutting-off threshold was $\lambda = 0.65$;

– the ten reference actions $b^1, \ldots, b^{10}$ were assigned values $x_1, \ldots, x_{10}$ with the same procedure of scenario 1, but considering the following blank cards: no blank cards between b_1 and b_2, b_2 and b_3, b_3 and b_4; one blank card between b_4 and b_5, b_5 and b_6, b_6 and b_7; 2 blank cards between b_8 and b_9, and b_9 and b_10. Consequently, we obtained the following values:

$$x_1 = 0, \quad x_2 = 5.88, \quad x_3 = 11.76, \quad x_4 = 17.65, \quad x_5 = 29.41,$$

$$x_6 = 41.18, \quad x_7 = 52.94, \quad x_8 = 64.71, \quad x_9 = 82.35, \quad x_{10} = 100.$$

– the representative score $s_2^\circ(a)$ was computed by considering a degree of optimism $\alpha = 0.25$, so that, for each country a,

$$s_1^\circ(a) = 0.75 \cdot s_1^l(a) + 0.25 \cdot s_1^u(a).$$

	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}	g_{11}	g_{12}	g_{13}	g_{14}	g_{15}	g_{16}
σ_j	14.56	7.63	16.39	0.00	0.00	6.98	0.00	2.66	5.20	13.85	19.38	12.54	14.83	11.78	15.50	15.81
w_j	0.0338	0.0134	0.0338	0.0032	0.0032	0.0134	0.0032	0.0134	0.0134	0.0338	0.0338	0.0338	0.0338	0.0338	0.0338	0.0338

	g_{17}	g_{18}	g_{19}	g_{20}	g_{21}	g_{22}	g_{23}	g_{24}	g_{25}	g_{26}	g_{27}	g_{28}	g_{29}	g_{30}	g_{31}	g_{32}
σ_j	20.50	25.34	13.02	22.55	22.72	21.58	8.67	14.90	7.18	7.74	11.40	11.88	13.84	19.06	18.24	15.99
w_j	0.0644	0.0644	0.0338	0.0644	0.0644	0.0644	0.0134	0.0338	0.0134	0.0134	0.0338	0.0338	0.0338	0.0338	0.0338	0.0338

Table 5: Weights of criteria in scenario 2

Let us explain the rational supporting the weighting procedure in Scenario 2. The idea is that the standard deviation σ_j measures the variability of the performances on criterion $g_j, j = 1, \ldots, 32$. As, in general, the greater the variability the greater the difference between the performances of the best health systems and the others, it is reasonable to assign a greater importance to criteria with greater standard deviation.

The score assigned to each country is shown in Table 6 having the following content:

– the first column shows the countries;

– the second column presents the HAQ index (Fullman et al., 2018);

– the third, fourth and fifth column presents the lower bound $s_1^l(a)$, the upper bound $s_1^u(a)$ and the representative score $s_1^\circ(a)$ assigned to country a in Scenario 1 by Electre-Score;

– the sixth, seventh and eighth column present the lower bound $s_2^l(a)$, the upper bound $s_2^u(a)$ and the representative score $s_2^\circ(a)$ assigned to country a in Scenario 2 by Electre-Score;

– the nineth, tenth and eleventh column show the ranking position of each country according with HAQ index, Electre-Score in Scenario 1 and Electre-Score in Scenario 2, respectively.

To interpret the score provided by Electre-Score and compare it with the HAQ score, it is useful to remember that

– $s_r^l(a) = x_k, r = 1, 2$, means that $a \succ b_k$ and $not(a \succ b_h)$ for all $h = k+1, \ldots, 10$;

– $s_r^u(a) = x_k, r = 1, 2$, means that $b_k \succ a$ and $not(b_h \succ a)$ for all $h = k-1, \ldots, 1$.

Country	HAQ	$s_1^l(a)$	$s_1^u(a)$	$s_1^\circ(a)$	$s_2^l(a)$	$s_2^u(a)$	$s_2^\circ(a)$	HAQ	ES_1	ES_2
Iceland	97	66.67	100.00	83.33	52.94	100.00	64.71	1	4	5
Norway	97	66.67	100.00	83.33	52.94	100.00	64.71	1	4	5
Netherlands	96	77.78	100.00	88.89	64.71	100.00	73.53	3	1	1
Luxembourg	96	66.67	100.00	83.33	52.94	100.00	64.71	3	4	5
Australia	96	66.67	100.00	83.33	52.94	100.00	64.71	3	4	5
Finland	96	66.67	100.00	83.33	52.94	100.00	64.71	3	4	5
Switzerland	96	77.78	100.00	88.89	64.71	100.00	73.53	3	1	1
Sweden	95	77.78	100.00	88.89	64.71	82.35	69.12	8	1	3
Italy	95	66.67	100.00	83.33	52.94	100.00	64.71	8	4	5
Andorra	95	44.44	100.00	72.22	29.41	100.00	47.06	8	13	18
Ireland	95	77.78	88.89	83.33	64.71	82.35	69.12	8	4	3
Japan	94	11.11	100.00	55.56	5.88	100.00	29.41	12	29	28
Austria	94	33.33	100.00	66.67	17.65	82.35	33.82	12	21	25
Canada	94	55.56	100.00	77.78	41.18	82.35	51.47	12	11	12
Belgium	93	55.56	88.89	72.22	41.18	82.35	51.47	15	13	12
New Zealand	92	55.56	88.89	72.22	41.18	82.35	51.47	16	13	12
Denmark	92	44.44	88.89	66.67	29.41	82.35	42.65	16	21	21
Germany	92	33.33	88.89	61.11	17.65	82.35	33.82	16	27	25
Spain	92	44.44	100.00	72.22	29.41	100.00	47.06	16	13	18
France	92	55.56	100.00	77.78	41.18	100.00	55.88	16	11	11
Slovenia	91	33.33	100.00	66.67	17.65	82.35	33.82	21	21	25
Singapore	91	11.11	100.00	55.56	5.88	82.35	25.00	21	29	34
UK	90	55.56	88.89	72.22	41.18	82.35	51.47	23	13	12
Greece	90	55.56	88.89	72.22	41.18	64.71	47.06	23	13	18
South Korea	90	11.11	88.89	50.00	5.88	82.35	25.00	23	38	34
Cyprus	90	44.44	88.89	66.67	29.41	82.35	42.65	23	21	21
Malta	90	55.56	88.89	72.22	41.18	82.35	51.47	23	13	12
Czech Republic	89	55.56	88.89	72.22	41.18	82.35	51.47	28	13	12
USA	89	44.44	88.89	66.67	29.41	82.35	42.65	28	21	21
Croatia	87	33.33	88.89	61.11	17.65	64.71	29.41	30	27	28
Estonai	86	11.11	88.89	50.00	5.88	64.71	20.59	31	38	42
Portugal	86	22.22	88.89	55.56	11.76	64.71	25.00	31	29	34
Lebanon	86	33.33	77.78	55.56	17.65	64.71	29.41	31	29	28
Taiwan	85	11.11	88.89	50.00	5.88	64.71	20.59	34	38	42
Israel	85	22.22	88.89	55.56	11.76	82.35	29.41	34	29	28
Slovakia	83	33.33	77.78	55.56	17.65	64.71	29.41	36	29	28
Bermuda	83	44.44	88.89	66.67	29.41	82.35	42.65	36	21	21
Puerto Rico	83	11.11	77.78	44.44	5.88	64.71	20.59	36	43	42
Poland	82	11.11	66.67	38.89	5.88	41.18	14.71	39	53	50
Hungary	82	22.22	77.78	50.00	11.76	64.71	25.00	39	38	34
Qatar	82	22.22	88.89	55.56	11.76	64.71	25.00	39	29	34
Montenegro	81	22.22	88.89	55.56	11.76	52.94	22.06	42	29	40
Latvia	81	22.22	77.78	50.00	11.76	52.94	22.06	42	38	40
Kuwait	81	22.22	66.67	44.44	11.76	41.18	19.12	42	43	46
Lithuania	80	11.11	77.78	44.44	5.88	41.18	14.71	45	43	50
Belarus	79	11.11	77.78	44.44	5.88	41.18	14.71	46	43	50
Romania	78	22.22	66.67	44.44	11.76	41.18	19.12	47	43	46
China	78	11.11	66.67	38.89	5.88	41.18	14.71	47	53	50
Chile	78	33.33	55.56	44.44	17.65	41.18	23.53	47	43	39
Serbia	77	22.22	66.67	44.44	11.76	41.18	19.12	50	43	46
Bulgaria	77	11.11	55.56	33.33	5.88	41.18	14.71	50	57	50
Saudi Arabia	77	11.11	77.78	44.44	5.88	29.41	11.76	50	43	58
Brunei	76	11.11	66.67	38.89	5.88	52.94	17.65	53	53	49
Oman	76	11.11	77.78	44.44	5.88	41.18	14.71	53	43	50
Cuba	76	22.22	88.89	55.56	11.76	82.35	29.41	53	29	28
Albania	75	0.00	88.89	44.44	0.00	82.35	20.59	56	43	45
Macedonia	75	11.11	55.56	33.33	5.88	29.41	11.76	56	57	58
Russia	75	11.11	55.56	33.33	5.88	41.18	14.71	56	57	50
Ukraine	75	11.11	66.67	38.89	5.88	17.65	8.82	56	53	60
Turkey	74	11.11	55.56	33.33	5.88	41.18	14.71	60	57	50

Table 6: Health care score

This permits to give an answer to the questions: Why Iceland and Norway are the first in the HAQ ranking and only the fourth and the fifth in the ranking supplied by ELECTRE-SCORE in the Scenarios 1 and 2, respectively? And why, the Netherlands are the third in HAQ ranking and the first both in the Scenarios 1 and 2 of ELECTRE-SCORE? Since $s_r^u(Iceland) = s_r^u(Norway) = s_r^u(Netherlands) = 100, r = 1, 2$, we have to conclude that the different representative scores $s_1^\circ(a)$

depend on $s_r^l(a)$. In fact,

- $s_1^l(Iceland) = s_1^l(Norway) = 66.67 = x_7$ and consequently, in scenario 1, $Iceland \succ b_7$ and $Norway \succ b_7$, but neither $Iceland \succ b_h$ nor $Norway \succ b_h, h = 8, 9, 10$;
- $s_2^l(Iceland) = s_2^l(Norway) = 52.94 = x_7$ and consequently, in scenario 2, $Iceland \succ b_7$ and $Norway \succ b_7$, but neither $Iceland \succ b_h$ nor $Norway \succ b_h, h = 8, 9, 10$;
- $s_1^l(Netherlands) = 77.78 = x_8$ and consequently $Netherlands \succ b_8$ in Scenario 1;
- $s_2^l(Netherlands) = 64.71 = x_8$ and consequently $Netherlands \succ b_8$ in Scenario 2.

Therefore, Electre-Score ranks the Netherlands better than Iceland and Norway because the Netherlands are preferred to the reference action b_8 while this is not the case of Iceland and Norway. Similar arguments hold for the ranking assigned by Electre-Score to other countries. Since the preference relation provided by Electre-Score is based on an outranking relation having a noncompensatory content, this shows that we can consider the value assigned by Electre-Score a noncompensatory composite indicator. Observe that the explanation of the score obtained by Iceland and Norway has important policy implications. Indeed Iceland is not preferred to the reference action b_8 because a certain deficiency of its performances with respect to b_8 on LRIs and NM skin cancer having as a consequence a veto effect (more precisely, Iceland has scaled cause values 76 and 72 on LRIs and NM skin cancer, respectively, while b_8 has scaled cause values 86 and 82 on the same criteria). Analogously, Norway is not preferred to the reference action b_8 for a veto on Epilepsy (more precisely Norway has a scaled cause values of 78 on Epilepsy, while b_8 has a scaled cause values of 88 on the same criterion). This suggest adequate interventions on policies related to RIs and NM skin cancer for Iceland and Epilepsy for Norway.

Finally, taking into account robustness concerns, it can be interesting to investigate how the score changes when perturbing the parameters of the model such as weights, indifference, preference, and veto thresholds, value assigned to the reference actions, optimism coefficient α or cuting-off threshold λ. As an example of such type of analysis, let us observe the consequences on the score of the threshold λ for the Scenario 1 as shown in Table 7. We considered the values of $\lambda = 0.5, 0.55, 0.6, 0.65, 0.7, 0.75, 0.8, 0.85, 0.9, 0.95, 1$, and for each one of these values we computed the representative score $s_\lambda^\circ(a)$ and the ranking positions ES_λ. The last two columns of Table 7 show the best and worst ranking position for each country. We can observe that no country maintains the same ranking position for all the considered λ values. More precisely, the width of the interval of ranking positions taken by a country varies from a minimum of 3, being the case of the Netherlands, Switzerland and Sweden that stay between the first and the third ranking position, to a maximum of 25, being the case of Albania that stays between the 31st and 55th position. The large variability of the ranking with respect to the value of λ suggests to select with care this parameter. Consequently, in case it is possible to interact with one or more decision-makers, the cutting-off threshold should be discussed with the involved decision makers. Instead, in case the score is aimed to supply a more "neutral" evaluation, it is reasonable to fix some central value of λ in its value range [0.5,1]. In this perspective, to fix $\lambda = 0.75$ seems an acceptable option.

8. Conclusions

In this paper we have presented a new method to assign a score range $]s^l(a), s^u(a)[$ and a representative score $s^\circ(a)$ to each action $a \in A$. The theoretical soundness of the method has been proven;

Country	$s^\circ_{0.5}(a)$	$s^\circ_{0.55}(a)$	$s^\circ_{0.6}(a)$	$s^\circ_{0.65}(a)$	$s^\circ_{0.7}(a)$	$s^\circ_{0.75}(a)$	$s^\circ_{0.8}(a)$	$s^\circ_{0.85}(a)$	$s^\circ_{0.9}(a)$	$s^\circ_{0.95}(a)$	$s^\circ_{1}(a)$	$ES_{0.5}(a)$	$ES_{0.55}(a)$	$ES_{0.6}(a)$	$ES_{0.65}(a)$	$ES_{0.7}(a)$	$ES_{0.75}(a)$	$ES_{0.8}(a)$	$ES_{0.85}(a)$	$ES_{0.9}(a)$	$ES_{0.95}(a)$	$ES_{1}(a)$	Best	Worst
Iceland	83.33	83.33	83.33	83.33	83.33	83.33	83.33	83.33	83.33	83.33	77.78	1	2	3	3	4	4	5	5	3	1	6	1	6
Norway	77.78	77.78	77.78	83.33	83.33	83.33	83.33	83.33	83.33	83.33	77.78	6	7	8	3	4	4	5	5	3	1	6	1	8
Netherlands	83.33	83.33	88.89	88.89	88.89	88.89	88.89	88.89	83.33	83.33	83.33	1	2	1	1	1	1	1	1	3	1	1	1	3
Luxembourg	77.78	83.33	83.33	83.33	83.33	83.33	83.33	83.33	83.33	83.33	77.78	6	2	3	3	4	4	5	5	3	1	6	1	6
Australia	77.78	77.78	83.33	83.33	83.33	83.33	83.33	83.33	83.33	83.33	77.78	6	7	3	3	4	4	5	5	3	1	6	1	7
Finland	77.78	77.78	77.78	83.33	83.33	83.33	83.33	83.33	83.33	83.33	83.33	6	7	8	3	4	4	5	5	3	1	1	1	8
Switzerland	83.33	88.89	88.89	88.89	88.89	88.89	88.89	88.89	83.33	83.33	83.33	1	1	1	1	1	1	1	1	3	1	1	1	3
Sweden	83.33	83.33	83.33	83.33	88.89	88.89	88.89	88.89	88.89	83.33	83.33	1	2	3	3	1	1	1	1	1	1	1	1	3
Italy	77.78	77.78	77.78	83.33	83.33	83.33	83.33	83.33	83.33	77.78	77.78	6	7	8	3	4	4	5	5	3	10	6	3	10
Andorra	66.67	66.67	66.67	66.67	66.67	72.22	72.22	72.22	72.22	72.22	61.11	18	18	18	19	20	13	15	17	19	16	23	13	23
Ireland	83.33	83.33	83.33	83.33	83.33	83.33	88.89	88.89	88.89	83.33	83.33	1	2	3	3	4	4	1	1	1	1	1	1	4
Japan	50.00	50.00	55.56	55.56	55.56	55.56	55.56	55.56	55.56	55.56	55.56	31	31	28	28	29	29	30	33	35	36	34	28	36
Austria	61.11	61.11	61.11	61.11	66.67	66.67	66.67	66.67	66.67	66.67	61.11	25	25	25	25	20	21	24	23	23	23	23	20	25
Canada	72.22	72.22	72.22	72.22	72.22	77.78	77.78	77.78	77.78	72.22	72.22	11	11	11	11	11	11	11	11	11	16	11	11	16
Belgium	72.22	72.22	72.22	72.22	72.22	72.22	72.22	77.78	77.78	77.78	72.22	11	11	11	11	11	13	15	11	11	10	11	10	15
New Zealand	72.22	72.22	72.22	72.22	72.22	72.22	77.78	77.78	77.78	77.78	72.22	11	11	11	11	11	13	11	11	11	10	11	10	13
Denmark	66.67	66.67	66.67	66.67	66.67	66.67	72.22	72.22	72.22	72.22	72.22	18	18	18	19	20	21	15	17	19	16	11	11	21
Germany	61.11	61.11	61.11	61.11	61.11	61.11	66.67	66.67	66.67	66.67	61.11	25	25	25	25	26	27	24	23	23	23	23	23	27
Spain	66.67	66.67	66.67	66.67	72.22	72.22	72.22	72.22	72.22	72.22	72.22	18	18	18	19	11	13	15	17	19	16	11	11	19
France	72.22	72.22	72.22	72.22	72.22	77.78	77.78	77.78	77.78	77.78	72.22	11	11	11	11	11	11	11	11	11	10	11	10	11
Slovenia	61.11	61.11	61.11	61.11	61.11	66.67	66.67	66.67	66.67	66.67	66.67	25	25	25	25	26	21	24	23	23	23	21	21	26
Singapore	50.00	50.00	50.00	50.00	50.00	55.56	55.56	55.56	55.56	55.56	50.00	31	31	33	34	35	29	30	33	35	36	40	29	40
UK	72.22	72.22	72.22	72.22	72.22	72.22	72.22	77.78	77.78	72.22	72.22	11	11	11	11	11	13	15	11	11	16	11	11	16
Greece	66.67	66.67	66.67	72.22	72.22	72.22	77.78	77.78	77.78	77.78	72.22	18	18	18	11	11	13	11	11	11	10	11	10	18
South Korea	50.00	50.00	50.00	50.00	50.00	50.00	50.00	55.56	55.56	55.56	50.00	31	31	33	34	35	38	40	33	35	36	40	31	40
Cyprus	66.67	66.67	66.67	66.67	66.67	66.67	72.22	72.22	72.22	72.22	66.67	18	18	18	19	20	21	15	17	19	16	21	15	21
Malta	72.22	72.22	72.22	72.22	72.22	72.22	72.22	72.22	77.78	72.22	72.22	11	11	11	11	11	13	15	17	11	16	11	11	17
Czech Republic	72.22	72.22	72.22	72.22	72.22	72.22	72.22	72.22	77.78	77.78	72.22	11	11	11	11	11	13	15	17	11	10	11	10	17
USA	66.67	66.67	66.67	66.67	66.67	66.67	72.22	66.67	66.67	66.67	61.11	18	18	18	19	20	21	15	23	23	23	23	15	23
Croatia	50.00	50.00	55.56	55.56	61.11	61.11	61.11	61.11	66.67	66.67	61.11	31	31	28	28	26	27	28	29	23	23	23	23	31
Estonai	38.89	38.89	44.44	50.00	50.00	50.00	50.00	50.00	55.56	55.56	50.00	41	43	39	34	35	38	40	43	35	36	40	34	43
Portugal	44.44	44.44	50.00	50.00	55.56	55.56	55.56	55.56	55.56	61.11	61.11	38	38	33	34	29	29	30	33	35	29	23	23	38
Lebanon	50.00	50.00	50.00	55.56	55.56	55.56	61.11	66.67	66.67	61.11	55.56	31	31	33	28	29	29	28	23	23	29	34	23	34
Taiwan	33.33	38.89	44.44	44.44	44.44	50.00	50.00	50.00	55.56	55.56	50.00	46	43	39	41	43	38	40	43	35	36	40	35	46
Israel	55.56	55.56	55.56	55.56	55.56	55.56	55.56	55.56	61.11	61.11	55.56	28	28	28	28	29	29	30	33	30	29	34	28	34
Slovakia	55.56	55.56	55.56	55.56	55.56	55.56	55.56	61.11	61.11	66.67	61.11	28	28	28	28	29	29	30	29	30	23	23	23	30
Bermuda	66.67	66.67	66.67	66.67	66.67	66.67	66.67	66.67	66.67	61.11	61.11	18	18	18	19	20	21	24	23	23	29	23	18	29
Puerto Rico	44.44	44.44	44.44	44.44	44.44	44.44	44.44	50.00	50.00	55.56	50.00	38	38	39	41	43	43	49	43	46	36	40	36	49
Poland	33.33	33.33	33.33	38.89	38.89	38.89	50.00	50.00	55.56	55.56	50.00	46	49	50	46	50	53	40	43	35	36	40	35	53
Hungary	50.00	50.00	50.00	50.00	50.00	50.00	50.00	55.56	55.56	55.56	55.56	31	31	33	34	35	38	40	33	35	36	34	31	40
Qatar	38.89	44.44	44.44	50.00	50.00	55.56	55.56	61.11	61.11	61.11	61.11	41	38	39	34	35	29	30	29	30	29	23	23	41
Montenegro	44.44	44.44	44.44	44.44	50.00	55.56	55.56	61.11	61.11	61.11	61.11	38	38	39	41	35	29	30	29	30	29	23	23	41
Latvia	33.33	38.89	44.44	44.44	50.00	50.00	55.56	55.56	55.56	61.11	61.11	46	43	39	41	35	38	30	33	35	29	23	23	46
Kuwait	38.89	38.89	38.89	38.89	44.44	44.44	55.56	55.56	61.11	55.56	55.56	41	43	46	46	43	43	30	33	30	36	34	30	46
Lithuania	27.78	33.33	33.33	38.89	44.44	44.44	44.44	50.00	50.00	55.56	50.00	53	49	50	46	43	43	49	43	46	36	40	36	53
Belarus	27.78	33.33	38.89	38.89	38.89	44.44	50.00	50.00	50.00	55.56	50.00	53	49	46	46	50	43	40	43	46	36	40	36	53
Romania	33.33	38.89	38.89	38.89	38.89	44.44	50.00	55.56	55.56	50.00	50.00	46	43	46	46	50	43	40	33	35	52	40	33	52
China	27.78	33.33	33.33	38.89	38.89	38.89	44.44	50.00	50.00	50.00	50.00	53	49	50	46	50	53	49	43	46	52	40	40	53
Chile	38.89	44.44	44.44	44.44	44.44	44.44	50.00	50.00	55.56	55.56	55.56	41	38	39	41	43	43	40	43	35	36	34	34	43
Serbia	38.89	38.89	38.89	38.89	38.89	44.44	44.44	50.00	50.00	55.56	50.00	41	43	46	46	50	43	49	43	46	36	40	36	50
Bulgaria	33.33	33.33	33.33	33.33	33.33	33.33	33.33	38.89	50.00	50.00	50.00	46	49	50	53	56	57	60	59	46	52	40	40	60
Saudi Arabia	27.78	27.78	27.78	33.33	44.44	44.44	44.44	50.00	50.00	50.00	50.00	53	56	57	53	43	43	49	43	46	52	40	40	57
Brunei	22.22	27.78	33.33	33.33	38.89	38.89	44.44	44.44	38.89	44.44	50.00	58	56	50	53	50	53	49	55	60	60	40	40	60
Oman	33.33	33.33	33.33	33.33	44.44	44.44	50.00	50.00	50.00	50.00	50.00	46	49	50	53	43	43	40	43	46	52	40	40	53
Cuba	55.56	55.56	55.56	55.56	55.56	55.56	55.56	55.56	50.00	55.56	50.00	28	28	28	28	29	29	30	33	46	36	40	28	46
Albania	50.00	50.00	50.00	50.00	50.00	44.44	44.44	44.44	50.00	50.00	50.00	31	31	33	34	35	43	49	55	46	52	40	31	55
Macedonia	27.78	27.78	27.78	33.33	33.33	33.33	44.44	50.00	50.00	55.56	50.00	53	56	57	53	56	57	49	43	46	36	40	36	57
Russia	22.22	27.78	27.78	33.33	33.33	33.33	38.89	38.89	50.00	50.00	50.00	58	56	57	53	56	57	57	59	46	52	40	40	59
Ukraine	22.22	22.22	27.78	33.33	33.33	38.89	38.89	44.44	50.00	50.00	50.00	58	60	57	53	56	53	57	55	46	52	40	40	60
Turkey	33.33	33.33	33.33	33.33	33.33	33.33	38.89	44.44	50.00	55.56	50.00	46	49	50	53	56	57	57	55	46	36	40	36	57

Table 7: Health care score variation with respect to cutting-off level λ

that is, the fundamental requirements of uniqueness, independence, monotonicity, conformity, homogeneity, and stability with respect to insertion and deletion operations, became properties of the method, which provide some consistency to the method.

All of the main strengths of the Electre methods are present in this new method: it deals with different types of scales without the need of converting them into a single unit; it is able to cope with the imperfect knowledge of data and arbitrariness when building the criteria; it takes into account the reasons for and against an outranking; and, it avoids the compensatory phenomenon in a systematic way. In addition, the method is able to provide a score for each action (more precisely score range and a representative score), which was previously considered a weak point of Electre methods.

We believe that future research could start from software development and real-world applications. Also several extensions are possible, such as,

- The design of a hierarchical Electre-Score method, as in Corrente et al. (2013, 2016) and Del Vasto-Terrientes et al. (2015), allowing in our case to assign scores to different macro-criteria in the hierarchical trees of criteria;
- The application of Monte-Carlo based pseudo-robustness analysis in the same line as in Corrente et al. (2017) to provide more accurate score ranges;
- The use of robust ordinal techniques and other robustness techniques in the same line of

Greco et al. (2011) and Kadziński and Ciomek (2016).

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A. Appendix

In this Appendix we will present the main concepts and steps that lead to the construction of a credibility degree, $\sigma(a, b)$ for the pair of actions (a, b).

State-of-the art versions of ELECTRE methods make use of the so-called *pseudo-criterion* model (Figueira et al., 2016; Roy, 1996; Roy and Bouyssou, 1993; Roy and Vincke, 1984) when comparing two actions, a and b, on criterion g_j, from their performances, $g_j(a)$ and $g_j(b)$, respectively. This model associates with each criterion function, $g_j(\cdot)$, two threshold functions: an *indifference threshold* function, denoted by $q_j(\cdot)$, and a *preference threshold* function, denoted by $p_j(\cdot)$. Assume that g_j is a criterion to be maximized and that the performance $g_j(a)$ is better than the performance $g_j(b)$. The threshold functions or simply thresholds can be constant or may vary in a direct way, i.e., with respect to the worst performance, $q_j(g_j(b))$ and $p_j(g_j(b))$, or in an inverse way, i.e., with respect to the best performance, $q_j(g_j(a))$ and $p_j(g_j(a))$. For the sake of simplicity and without loss of generality, we consider in the following that the thresholds are constant and use the simple notation q_j and p_j, for the indifference and preference thresholds, respectively.

It is very important to note that the main purpose of these thresholds is not to model the preferences, but rather the imperfect knowledge of data as it can be seen in Roy et al. (2014).

The choice of a pseudo-criterion model for the comparison of two actions, a and b, from their performances on criterion g_j, leads to the definition of three *per*-criterion binary relations, as follows.

- A per-criterion *indifference binary relation*, which is used to model a situation in which a is indifferent to b on criterion g_j, denoted by $a \sim_j b$; this occurs whenever $|(g_j(a) - g_j(b))| \leqslant q_j$. In other words, a situation where no one of the two actions, a and b, has a significant advantage over the other on the considered criterion. Let $C(a \sim b)$ denote the set or coalition of criteria for which a is indifferent to b.

- A per-criterion *strict preference binary relation*, which is used to model a situation in which a is strictly preferred to b on criterion g_j, denoted by $a \succ_j b$; this occurs whenever $(g_j(a) - g_j(b)) > p_j$. In other words, a situation where action a has a significant advantage over b on the considered criterion. Let $C(a \succ b)$ denote the set or coalition of criteria for which a is strictly preferred to b.

- A per-criterion *weak preference binary relation*, which is used to model hesitation situations of a with respect to b on criterion g_j, denoted by $a \succsim_j^? b$; this occurs whenever $q_j < (g_j(a) - g_j(b)) \leqslant p_j$. In other words, a situation where there is an ambiguity zone between indifference and strict preference of a over b on the considered criterion. Let $C(a \succsim^? b)$ denote the set or coalition of criteria for which a is weakly preferred to b. Note that the word "weak" has nothing to do with intensities of preferences, it models hesitation or ambiguity (due to the imperfect knowledge of data), not preferences.

Whenever a is indifferent, weak, or strict preferred to b, on criterion g_j, we say that "a outranks b" because a is at least as good as b in a *stricto sensu* on this criterion. This situation thus occurs, when $a \sim_j b$, $a \succ_j b$, or $a \succsim_j^? b$, and can be denoted by $a \succsim_j b$. In a more *lato sensu* we can also say that "a outranks b" when $b \succsim_j^? a$ because there is hesitation between $b \sim_j a$ and $b \succ_j a$ on criterion g_j.

As in all outranking-based methods, ELECTRE methods also make use of the *per*-criterion outranking relations to build one or several comprehensive outranking relations. In the method proposed in this paper only one comprehensive outranking relation is considered, which allows to conclude whether or not "a comprehensively outranks b", denoted by $a \succsim b$. More precisely, to conclude about the assertion "a outranks b", the strength of the coalition in its favor of should be powerful enough to overcome the opposition effect of the coalition against this assertion. How should the power of the coalition in favor (or concordant with the assertion) and the effects of the coalition against (or discordant with the assertion) be measured? To model the power of the concordant coalition is modeled and measured in Electre methods through what is called in these methods a *comprehensive concordance index*, while the opposition effect of each criterion is modeled and measured to what is called a *per-criterion discordance index*. Both will be combined to devise a *credibility* (outranking) *index* for each ordered pair of alternatives, $(a, b) \in A \times A$. Next, we will present the three main steps to obtain the credibility index.

1. Computing the *comprehensive concordance index* $c(a, b)$. Again, for the sake of simplicity, the formula we present next, for this index, is the classical one as in Roy and Bouyssou (1993). A more sophisticated and recent version of the concordance index, which takes into account the interaction between criteria can also be used in this context (see Figueira et al., 2009) with no additional changes in the method proposed in this paper. As stated previously, the concordance index is used to measure the power of the concordant coalition, where each criterion g_j contributes with its relative importance coefficient or weigh, w_j, for $j = 1, \ldots, n$ (we assume w.l.o.g. that $\sum_{j=1}^{n} w_j = 1$). The formula for the index can thus be stated as follows.
$$c(a, b) = \sum_{C(a\{\sim, \succsim^?, \succ\}b)} w_j + \sum_{C(b \succsim^? a)} \varphi_j w_j,$$
where
$$\varphi = \frac{(g_j(a) - g_j(b)) + p_j}{p_j - q_j} \in [0, 1].$$
This means that if a criterion g_j belongs to the concordant coalition *stricto sensu*, i.e., $g_j \in C(a\{\sim, \succsim^?, \succ\}b$ its contribution to the coalition power corresponds to its total weight, w_j, but if this criterion belongs to $C(b \succsim^? a)$ it only contributes with a fraction of its weight, $\varphi_j w_j$.

2. Computing the *per-criterion discordance indices* $d_j(a, b)$, $j = 1, \ldots, n$. To model the opposition effect of each criterion against the concordant coalition *lato sensu*, i.e., when $g_j \in C(b \succ a)$, it is necessary to introduce another concept and preference parameter, the *veto threshold* $v_j(\cdot)$. This threshold can also be constant or vary in a direct or indirect way as in case of indifference and preference thresholds. To render things simple and without loss of generality we will keep its value constant and will denote it simply by v_j. A criterion g_j is discordant with the assertion "a outranks b", when the difference of performances $(g_j(b) - g_j(a))$ is considered significantly large to validate such an assertion. The more or less degree of discordance of each criterion can be measured through a *per*-criterion discordance index of the form,
$$d_j(a, b) = \begin{cases} 1 & \text{if} \quad v_j > (g_j(a) - g_j(b)) \\ \dfrac{(g_j(a) - g_j(b)) + p_j}{p_j - v_j} & \text{if} \quad -v_j \leqslant (g_j(a) - g_j(b)) < -p_j \\ 0 & \text{if} \quad (g_j(a) - g_j(b)) \geqslant -p_j \end{cases}$$

3. Computing the *credibility index* $\sigma(a, b)$. This index measures the credibility degree of the outranking relation; that is, the degree in which a outranks b. This can be modeled though

the following formula.

$$\sigma(a,b) = c(a,b)\prod_{j=1}^{n} T_j(a,b),$$

where

$$T_j(a,b) = \begin{cases} \dfrac{1-d_j(a,b)}{1-c(a,b)} & \text{if } g_j(a,b) > c(a,b) \\ 1 & \text{otherwise} \end{cases}$$

It is thus a fuzzy measure. It can be converted into a crispy by making use of a cutting-off level, denote by λ as in Section 2.

For the main features, advantages, and drawbacks of Electre the reader can refer to Figueira et al. (2013).

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