



THE UNIVERSITY *of* EDINBURGH

Edinburgh Research Explorer

## Piston-driven numerical wave tank based on WENO solver of well-balanced shallow water equations

### Citation for published version:

Jung, J, Hwang, JH & Borthwick, A 2020, 'Piston-driven numerical wave tank based on WENO solver of well-balanced shallow water equations', *KSCE Journal of Civil Engineering*, vol. 24, pp. 1959–1982. <https://doi.org/10.1007/s12205-020-1875-3>

### Digital Object Identifier (DOI):

[10.1007/s12205-020-1875-3](https://doi.org/10.1007/s12205-020-1875-3)

### Link:

[Link to publication record in Edinburgh Research Explorer](#)

### Document Version:

Peer reviewed version

### Published In:

KSCE Journal of Civil Engineering

### General rights

Copyright for the publications made accessible via the Edinburgh Research Explorer is retained by the author(s) and / or other copyright owners and it is a condition of accessing these publications that users recognise and abide by the legal requirements associated with these rights.

### Take down policy

The University of Edinburgh has made every reasonable effort to ensure that Edinburgh Research Explorer content complies with UK legislation. If you believe that the public display of this file breaches copyright please contact [openaccess@ed.ac.uk](mailto:openaccess@ed.ac.uk) providing details, and we will remove access to the work immediately and investigate your claim.



1  
2  
3  
4  
5  
6  
7  
8  
9  
10  
11  
12  
13  
14  
15  
16  
17  
18  
19  
20  
21  
22  
23  
24  
25  
26

# **Piston-Driven Numerical Wave Tank based on WENO Solver of Well-Balanced Shallow Water Equations**

**Jaeyoung Jung\*, Jin Hwan Hwang\*\*, Alistair G.L. Borthwick\*\*\***

\*PhD Student, Dept. of Civil and Environmental Engineering, Seoul National University, Seoul, 08826, Republic of Korea (E-mail: jlowc@snu.ac.kr)

\*\*Member, Associate Professor, Dept. of Civil and Environmental Engineering, Seoul National University, Seoul, 08826, Republic of Korea (Corresponding Author, E-mail: jinhwang@snu.ac.kr)

\*\*\*Professor, Institute for Energy Systems, The University of Edinburgh, EH9 3FB, United Kingdom (E-mail: Alistair.Borthwick@ed.ac.uk)

**Keywords:**

Shallow water equations, Well-balanced scheme, Piston type wave-maker, Exact C-property, WENO, Numerical wave tank

27 **ABSTRACT**

28 A numerical wave tank equipped with a piston type wave-maker is presented for long-duration  
29 simulations of long waves in shallow water. Both wave maker and tank are modelled using the nonlinear  
30 shallow water equations, with motions of the numerical piston paddle accomplished via a linear  
31 mapping technique. Three approaches are used to increase computational efficiency and accuracy. First,  
32 the model satisfies the exact conservation property (C-property), a stepping stone towards properly  
33 balancing each term in the governing equation. Second, a high-order WENO method is used to reduce  
34 accumulation of truncation error. Third, a cut-off algorithm is implemented to handle contaminated  
35 digits arising from round-off error. If not treated, such errors could prevent a numerical scheme from  
36 satisfying the exact C-property in long-duration simulations. Extensive numerical tests are performed  
37 to examine the well-balanced property, high order accuracy, and shock-capturing ability of the present  
38 scheme. Correct implementation of the wave paddle generator is verified by comparing numerical  
39 predictions against analytical solutions of sinusoidal, solitary, and cnoidal waves. In all cases, the model  
40 gives satisfactory results for small-amplitude, low frequency waves. Error analysis is used to investigate  
41 model limitations and derive a user criterion for long wave generation by the model.

42

43

44

45

46

## 47 **1. Introduction**

48 In wave tank tests, the wave-maker is usually installed at one end of an enclosed flume, with waves  
49 generated by a paddle whose movement is designed to mimic the Lagrangian motion of water particles  
50 within the waves. Various types of wave-makers have been devised including piston, flap, and wedge  
51 types (see e.g. [Hughes, 1993](#)). Of these, the piston type wave-maker is particularly well suited for the  
52 generation of waves in shallow water, where the wavelength  $L$  is much larger than the water depth  $h$ ,  
53 i.e.,  $h/L < 0.05$  ([Dean and Dalrymple, 1991](#)). Such piston type wave-makers have been widely used in  
54 experimental studies of coastal waves and long waves (see e.g., [Ursell \*et al.\*, 1960](#); [Madsen, 1970](#);  
55 [Zabusky and Galvin, 1971](#); [Goring, 1978](#); [Synolakis, 1987](#); [Synolakis, 1990](#); [Liu \*et al.\*, 1995](#); [Monaghan  
56 and Kos, 2000](#); [Guizien and Barthélemy, 2002](#); [Goseberg \*et al.\*, 2013](#); [Chen \*et al.\*, 2016](#); [Schimmels \*et\*  
57 \*al.\*, 2016](#)). To overcome error introduced by scale effects, several studies have been carried out at large-  
58 scale (e.g., [Streicher \*et al.\*, 2013](#); [Previsic \*et al.\*, 2014](#); [Schimmels \*et al.\*, 2014](#); [Yu \*et al.\*, 2015](#); [Wenneker  
59 \*et al.\*, 2016](#)). Alternatively, numerical wave tanks readily facilitate simulation of hydrodynamic  
60 phenomena at field scale and are advantageous in cases where field observations are unavailable or  
61 laboratory experiments are not at sufficiently large scale ([Hornsby, 2002](#); [Aly and Bitsuamlak, 2013](#)).

62 Many numerical studies have been carried out on long waves in the shallow water region (see e.g.  
63 [Walkley, 1999](#); [Toro, 2001](#); [Mader, 2004](#); [Vreugdenhil, 2013](#)) and on wave generation methods ([Table  
64 1](#); [Finnegan and Goggins, 2015](#)). Although detailed information can be gained from three-dimensional  
65 solutions of the viscous Navier-Stokes or inviscid Laplace equations, such approaches are  
66 computationally too demanding for long-duration simulations, and so depth-averaged models, such as  
67 those based on Boussinesq-type equations and shallow water equations, are widely employed. For  
68 example, [Orszaghova \*et al.\* \(2012\)](#) used a hybrid solver of the enhanced Boussinesq-type equations for  
69 pre-breaking waves and the nonlinear shallow water equations for broken waves to model a wave tank  
70 equipped with a piston type wave-maker. However, Boussinesq models can incur considerable  
71 computational overhead when applied to the long-term simulation of certain large-scale phenomena,  
72 such as bed morphological change and very long waves (tsunami, internal waves, storm surges, and  
73 planetary waves). In these cases, it is reasonable to solve the simpler shallow water equations, which

74 can resolve long waves where the pressure distribution is hydrostatic, provided limitations arising from  
75 the accumulation of numerical error due to long-duration integration and the requirement of a moving  
76 wave maker can be overcome. The present paper suggests three methods aimed at handling such issues  
77 affecting long-duration simulations with the shallow water equations.

78 First, we ensure that the momentum flux and source terms are well-balanced so that they satisfy  
79 the exact conservation property (C-property) (Bermúdez and Vázquez-Cendón, 1994) to prevent  
80 accumulation of error in the numerical wave tank which is essentially an isolated system. Table 2 lists  
81 a brief summary of pertinent literature, which can be divided into two categories. The exact C-property  
82 can be satisfied through either numerical methods (see e.g. Leveque, 1998; Vukovic and Sopta, 2002;  
83 and Xing and Shu, 2005) or by algebraic reformulation of the partial differential equations (see e.g.  
84 Rogers *et al.*, 2003 and Liang and Borthwick, 2009). More recently, Xing and Shu (2005)'s ideas have  
85 been further extended to more advanced approaches such as hybrid WENO (Zhu *et al.*, 2017) and  
86 weighted compacted nonlinear (WCN) schemes (Gao and Hu, 2017). Li *et al.* (2015) extended Xing  
87 and Shu (2005)'s well-balanced strategy to the 'pre-balanced' shallow water equations proposed by  
88 Rogers *et al.* (2003), and introduced a robust method that simultaneously combined both well-balanced  
89 strategies. Following a similar strategy, we construct a well-balanced scheme by applying Xing and Shu  
90 (2005)'s method to Liang and Borthwick (2009)'s shallow water equations.

91 Second, a high-order method is applied to obtain accurate simulations of long-duration unsteady  
92 flows, while reducing the magnitude of accumulated truncation error (Wang, 2007). Over the past  
93 twenty years, substantial research effort has been directed towards solving the nonlinear shallow water  
94 equations using high order schemes; examples include the discontinuous Galerkin method (DGM)  
95 (Giraldo *et al.*, 2002; Xing *et al.*, 2010; Bonev *et al.*, 2018; Li *et al.*, 2018), advective upwind splitting  
96 method (AUSM) (Ullrich *et al.*, 2010), the essentially non-oscillatory scheme (ENO) (Vukovic and  
97 Sopta, 2002), and the weighted ENO (WENO) (Xing and Shu, 2005; Noelle *et al.*, 2007; Li *et al.*, 2012).  
98 Herein, we solve the nonlinear shallow water equation using the fifth-order WENO method in space  
99 and the third-order Runge-Kutta scheme in time. Given that round off errors can accumulate at machine  
100 precision level and cause the scheme to fail to satisfy the exact C-property in long-duration simulations,

101 a cut-off algorithm is used to remove the effect of digits contaminated by machine error arising from  
102 floating-point arithmetic. This enables the present model to satisfy the exact C-property for long-  
103 duration simulations.

104 Third, we produce a numerical wave tank that mimics the behaviour of a tank with a piston-type  
105 wave-maker. Generally, there are two ways to generate waves numerically (Table 1). One involves  
106 directly imposing mathematical solutions obtained from wave theory on the boundary conditions. The  
107 other involves implementing the numerical piston paddle and operating it with wave-maker theory. The  
108 former is advantageous in implementing mathematically exact waves, but difficult to compare against  
109 corresponding laboratory generated waves. The latter method enables easier validation against  
110 laboratory measurements, and so is useful when simulating the behavior of an actual wave tank.  
111 Therefore, we use a linearly mapped time-varying domain in the region of the paddle domain following  
112 [Orszaghova \*et al.\* \(2012\)](#) whereby the physical grid contracts and expands as the paddle advances and  
113 retreats. This enables paddle displacement signals to be incorporated directly in the numerical model as  
114 a driving boundary condition.

115 In conclusion, the key contribution of this study comprises three aspects in implementing the  
116 piston-driven numerical wave tank. First, a well-balanced WENO method is formulated rigorously,  
117 combining ideas by [Xing and Shu \(2005\)](#) and [Liang and Borthwick \(2009\)](#). Second, the formulation is  
118 extended to linearly mapped shallow water equations which describe the movement of a piston paddle  
119 in the paddle sub-domain. Finally, by introducing a cut-off algorithm, we construct a model that satisfies  
120 the exact C-property for long-duration simulations.

121 The paper is organized as follows. Section 2 describes the governing equations, the conditions  
122 necessary to satisfy the exact C-property, and the construction of a well-balanced scheme. Section 3  
123 presents the 5th order WENO method that satisfies the exact C-property without loss of accuracy, the  
124 cut-off algorithm used to prevent the scheme from losing its well-balanced property when applied to  
125 long-duration simulation, and the implementation of the piston type wave-maker. Section 4 discusses  
126 results of benchmark tests conducted to verify the numerical model and devises a user criterion for the  
127 piston paddle. Section 5 summarizes the main conclusions.

129 **Table 1.** Example studies on numerical methods of wave generation.

Previous work	Numerical method	Governing equation	Wave generation method
Boo, 2002	BEM	Laplace	Boundary condition
Turnbull <i>et al.</i> , 2003a	FEM	Laplace	Boundary condition
Koo and Kim, 2004	BEM	Laplace	Boundary condition
Park <i>et al.</i> , 2004	FVM	Navier-Stokes	Boundary condition
Ning and Teng, 2007	BEM	Laplace	Boundary condition
Ning <i>et al.</i> , 2008	BEM	Laplace	Boundary condition
Yan and Lui, 2011	BEM	Laplace	Boundary condition
Yu and Li, 2013	FVM	Navier-Stokes	Boundary condition
Finnegan & Goggins, 2015	FVM	Navier-Stokes	Boundary condition
Turnbull <i>et al.</i> , 2003b	FEM	Laplace	Piston type wave-maker
Wu and Hu, 2004	FEM	Laplace	Piston type wave-maker
Sriram <i>et al.</i> , 2006	FEM	Laplace	Piston type wave-maker
Khayyer <i>et al.</i> , 2007	SPH	Navier-Stokes	Piston type wave-maker
Agamloh <i>et al.</i> , 2008	FVM	Navier-Stokes	Piston type wave-maker
Liang <i>et al.</i> , 2010	FVM	Navier-Stokes	Piston type wave-maker
Orszaghova <i>et al.</i> , 2012	FDM	Boussinesq	Piston type wave-maker
Wen and Ren, 2018	SPH	Navier-Stokes	Piston type wave-maker

130 BEM: boundary element method FEM: finite element method, FVM: finite volume method, FDM: finite  
 131 difference method, and SPH: smoothed particle hydrodynamics.

132

133 **Table 2.** Previous studies on well-balanced schemes for the shallow water equations

Paper	Content
Bermúdez and Vázquez-Cendón, 1994	Definition of exact conservation (C-) property
Greenberg and Leroux, 1996	Introduce concept of well-balanced scheme
Leveque, 1998	Propose quasi-steady wave propagation algorithm using C-property
Vukovic and Sopta, 2002	Combine the C-property with ENO/WENO
Rogers <i>et al.</i> , 2003	Propose algebraic balancing scheme for shallow water equations
Xing and Shu, 2005	Apply WENO scheme to shallow water equation using C-property
Liang and Borthwick, 2009	Algebraic balancing scheme for multiple wet/dry boundaries

134

## 135 2. Well-Balanced Model

136 After depth-integration invoking the hydrostatic assumption, the Reynolds-averaged continuity  
 137 and Navier-Stokes equations are simplified to give the shallow water equations over a non-erodible bed.

138 In one spatial dimension, the shallow water equation may be expressed in vector notation as:

139 
$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}}{\partial x} = \mathbf{s}, \quad (1)$$

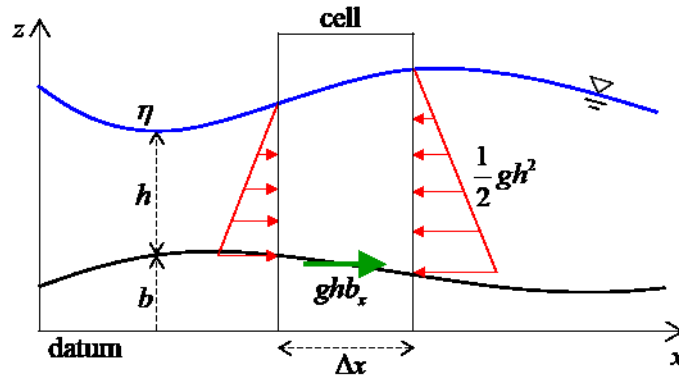
140 
$$\mathbf{u} = \begin{pmatrix} h \\ hU \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} hU \\ hUU \end{pmatrix}, \quad \text{and} \quad \mathbf{s} = \begin{pmatrix} 0 \\ -gh\eta_x \end{pmatrix}, \quad (2)$$

141 where  $\mathbf{u}$  is the vector of conserved dependent variables,  $\mathbf{f}$  is the vector of  $x$ -direction fluxes,  $\mathbf{s}$  is  
 142 the vector of source terms,  $t$  is time,  $x$  is stream-wise distance,  $g$  is gravitational acceleration,  $h$   
 143 is local water depth,  $U$  is depth-averaged velocity in the  $x$ -direction,  $\eta$  is the surface elevation  
 144 above a horizontal datum, and subscript  $x$  refers to the partial derivative in  $x$ . In practice (see e.g.  
 145 [Toro, 2001](#)), it is usual to split the source term using the fact that  $\eta = h + b$  where  $b$  is the bed  
 146 elevation above a fixed horizontal datum, giving:

147 
$$\mathbf{u} = \begin{pmatrix} h \\ hU \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} hU \\ hUU + \frac{1}{2}gh^2 \end{pmatrix}, \quad \text{and} \quad \mathbf{s} = \begin{pmatrix} 0 \\ -ghb_x \end{pmatrix}. \quad (3)$$

148 In Eq. (3),  $\frac{1}{2}gh^2$  expresses the hydrostatic pressure thrust acting on each side of the water column and  
 149  $-ghb_x$  is the  $x$ -direction component of the pressure thrust acting on the bed ([Fig. 1](#)).

150



151

152 **Fig. 1.** Schematic of water column used in derivation of the shallow water equations

153

154 A well-balanced numerical scheme for the shallow water equation should preserve the horizontal  
 155 free surface elevation of still water in a basin even when the bed has non-uniform elevation. By  
 156 definition, such a model must satisfy the exact C-property ([Bermúdez and Vázquez-Cendón, 1994](#))



157 when maintaining stationary conditions given by

$$158 \quad \eta = h + b = \text{const.} \quad \text{and} \quad hU = 0. \quad (4)$$

159 However, substituting (4) into (3), we obtain,

$$160 \quad \frac{\partial}{\partial x} \left( \frac{1}{2} gh^2 \right) = -ghb_x, \quad (5)$$

161 which is a function neither of  $\eta$  nor  $hU$ , but instead of  $h$ . It is necessary for both sides of Eq. (5)

162 to give exactly matching results in order for the hydrostatic force gradient to remain in balance.

163 Otherwise an unphysical flux arises from the truncation error, which increasingly contaminates the

164 results. To remove such error, the present study follows ideas expressed by [Xing and Shu \(2005\)](#) and

165 [Liang and Borthwick \(2009\)](#) in constructing a well-balanced model. [Liang and Borthwick \(2009\)](#),

166 reformulated the Eqs. (1) and (3) as functions of  $\eta$  and  $hU$ , and derived the following deviatoric

167 form of the shallow water equations that satisfies the exact C-property algebraically.

$$168 \quad \mathbf{u} = \begin{pmatrix} \eta \\ hU \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} hU \\ hUU + \frac{1}{2}g(\eta^2 - 2\eta b) \end{pmatrix}, \quad \text{and} \quad \mathbf{s} = \begin{pmatrix} 0 \\ -g\eta b_x \end{pmatrix}. \quad (6)$$

169 [Liang and Borthwick](#) solved Eqs. (1) and (6) using a second-order accurate, MUSCL-Hancock, HLLC

170 finite volume scheme, and demonstrated that the Eqs. (1) and (6) are well-balanced for HLLC. It

171 should also be noted that Eqs. (1) and (6) satisfy the exact C-property for more general cases. In other

172 words, if all  $x$ -derivative terms in Eqs. (1) and (6) are approximated using the same linear scheme

173 satisfying consistency, then the model satisfies the exact C-property (see Appendix for proof;

174 **Proposition 1**) where the consistency condition means that the  $x$ -derivative of constant functions is

175 zero. It is therefore important to maintain linearity and consistency of the spatial derivatives in the

176 numerical differentiation.

177 [Xing and Shu \(2005\)](#) also proposed a well-balanced scheme using the WENO method. To achieve

178 linearity, [Xing and Shu](#) split the source terms and rewrote the shallow water equations as follows:

$$179 \quad \frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}}{\partial x} = \frac{\partial \mathbf{s}_1}{\partial x} - g(h+b) \frac{\partial \mathbf{s}_2}{\partial x}, \quad (7)$$

$$180 \quad \mathbf{u} = \begin{pmatrix} h \\ hU \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} hU \\ hUU + \frac{1}{2}gh^2 \end{pmatrix}, \quad \mathbf{s}_1 = \begin{pmatrix} 0 \\ \frac{1}{2}gb^2 \end{pmatrix}, \quad \text{and} \quad \mathbf{s}_2 = \begin{pmatrix} 0 \\ b \end{pmatrix}, \quad (8)$$

181 where  $\mathbf{s}_1$  and  $\mathbf{s}_2$  are source terms. [Xing and Shu](#) approximated each  $x$ -derivative term in Eq. (7)  
 182 using the same WENO operator in order to satisfy consistency of spatial derivative. To guarantee  
 183 linearity of the WENO algorithm in stationary conditions, [Xing and Shu](#) formulated the flux splitting  
 184 method using the C-property ( $\eta$ ) instead of conservative variables ( $h$ ) as follows:

$$185 \quad \mathbf{f}^\pm = \frac{1}{2} \left[ \begin{pmatrix} hU \\ hUU + \frac{1}{2}gh^2 \end{pmatrix} \pm \max |\lambda| \begin{pmatrix} h+b \\ hU \end{pmatrix} \right], \quad (9)$$

$$186 \quad \mathbf{s}_1^\pm = \frac{1}{2} \begin{pmatrix} 0 \\ \frac{1}{2}gb^2 \end{pmatrix}, \quad \text{and} \quad \mathbf{s}_2^\pm = \frac{1}{2} \begin{pmatrix} 0 \\ b \end{pmatrix}, \quad (10)$$

187 where  $\lambda$  is an eigenvalue of the Jacobian matrix of flux terms;  $\mathbf{f}^\pm$  is the vector of split fluxes; and  
 188  $\mathbf{s}_1^\pm$  and  $\mathbf{s}_2^\pm$  are vectors of split source terms. [Xing and Shu \(2005\)](#)'s method therefore gives the same  
 189 results as if all  $x$ -derivative terms were treated as a single term using a single algorithm for the stationary  
 190 flow case. Consequently, [Xing and Shu](#)'s method satisfies the exact C-property up to machine level  
 191 without losing high order accuracy, and so meets all of the conditions for **Proposition 1**. Furthermore,  
 192 the flux splitting method in Eq. (9) is perfectly suitable for Eq. (6) because the dependent variables  
 193 in Eq. (6) comprise the C-property. Note that this deviatoric flux splitting method corresponds to a  
 194 Lax-Friedrichs flux splitting of Eq. (6). To implement a well-balanced scheme satisfying the exact C-  
 195 property, we therefore apply the WENO algorithm of [Xing and Shu \(2005\)](#) to the deviatoric shallow  
 196 water equations (i.e., Eqs. (1) and (6)) derived by [Liang and Borthwick \(2009\)](#).

197 Despite implementation of the foregoing approaches to achieve well-balanced shallow water  
 198 equations, numerical models can still suffer imbalance at machine level. Although the well-balanced  
 199 property has been numerically demonstrated in short-duration simulations of still water conditions (e.g.,  
 200 [Leveque, 1998](#); [Vukovic and Sopta, 2002](#); [Rogers et al. 2003](#); [Xing and Shu, 2005](#); [Castro et al., 2006](#);  
 201 [Lukáčová-Medvid'ová et al. 2007](#); [Liang and Borthwick 2009](#)), the accumulation of round-off error in

202 an isolated system such as a wave tank could cause serious deterioration in accuracy of long-duration  
 203 simulations. In other words, growth in round-off error could prevent the model from satisfying the exact  
 204 C-property, causing long-term deterioration in conservation of mass and momentum. Therefore, it is  
 205 desirable that the numerical scheme should eliminate the accumulation of round-off error in an isolated  
 206 system.

207

### 208 **3. Numerical Method**

#### 209 **3.1. The WENO Method Satisfying the Exact C-Property**

210 The governing equation is a conservative form of the one-dimensional shallow water equations  
 211 given by:

$$212 \quad \frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}}{\partial x} = -g\eta \frac{\partial \mathbf{s}}{\partial x}, \quad (11)$$

$$213 \quad \text{where } \mathbf{u} = \begin{pmatrix} \eta \\ hU \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} hU \\ hUU + \frac{1}{2}g(\eta^2 - 2\eta b) \end{pmatrix}, \quad \text{and } \mathbf{s} = \begin{pmatrix} 0 \\ b \end{pmatrix}. \quad (12)$$

214 Approximate weak solutions of the above governing equations at the  $i$ -th cell  $\mathbf{u}_i$  can be calculated  
 215 from

$$216 \quad \frac{d\mathbf{u}_i}{dt} + \frac{\hat{\mathbf{f}}_{i+1/2} - \hat{\mathbf{f}}_{i-1/2}}{\Delta x} = -g\eta_i \frac{\mathbf{s}_{i+1/2} - \mathbf{s}_{i-1/2}}{\Delta x}, \quad (13)$$

217

218 where the integer subscript  $i$  represents a given cell,  $\Delta x$  is the cell size, the subscript 1/2 refers to the  
 219 cell interface, and  $\hat{\mathbf{f}}_{i+1/2}$  is the cell interface flux. Here, the 5<sup>th</sup> order WENO method reconstructs  $\hat{\mathbf{f}}_{i+1/2}$   
 220 and  $\mathbf{s}_{i+1/2}$  in space, and the 3<sup>rd</sup> order Runge-Kutta method is used to integrate Eq. (13) in time. In the

221 first step of WENO reconstruction, the vector of flux terms is divided into positive and negative parts,

$$222 \quad \mathbf{f} = \mathbf{f}^+ + \mathbf{f}^-, \quad (14)$$

223 which satisfy  $\partial \mathbf{f}^+(\mathbf{u}) / \partial \mathbf{u} \geq 0$  and  $\partial \mathbf{f}^-(\mathbf{u}) / \partial \mathbf{u} \leq 0$ , and the source term vector is split such that

$$224 \quad \mathbf{s} = \mathbf{s}^+ + \mathbf{s}^-. \quad (15)$$

225 Following [Xing and Shu](#), in order to satisfy the exact C-property,

226 
$$\mathbf{f}^\pm = \frac{1}{2} \left[ \left( \begin{array}{c} hU \\ hUU + \frac{1}{2}g(\eta^2 - 2\eta b) \end{array} \right) \pm \max |\lambda| \left( \begin{array}{c} \eta \\ hU \end{array} \right) \right] \quad (16)$$

227 and

228 
$$\mathbf{s}^\pm = \frac{1}{2} \begin{pmatrix} 0 \\ b \end{pmatrix}. \quad (17)$$

229 When  $\partial \mathbf{f}^+ / \partial \mathbf{u} \geq 0$ ,  $\hat{\mathbf{f}}_{i+1/2}^+$  is obtained from

230 
$$\hat{\mathbf{f}}_{i+1/2}^+ = \chi_0 \hat{\mathbf{f}}_{i+1/2}^{(0)} + \chi_1 \hat{\mathbf{f}}_{i+1/2}^{(1)} + \chi_2 \hat{\mathbf{f}}_{i+1/2}^{(2)}, \quad (18)$$

231 where  $\chi_r$  is the nonlinear weight of the  $r$ -th sub-stencil calculated from:

232 
$$\chi_r = \frac{\tilde{\chi}_r}{\tilde{\chi}_0 + \tilde{\chi}_1 + \tilde{\chi}_2} \quad \text{and} \quad \tilde{\chi}_r = \frac{\beta_r}{(\sigma + \beta_r)^2}, \quad (19)$$

233 in which the linear weights  $\gamma_0 = 0.3$ ,  $\gamma_1 = 0.6$ , and  $\gamma_2 = 0.1$ , and  $\sigma = 10^{-6}$  is a parameter preventing

234 the denominator from becoming zero.  $\beta_r$  are smoothness indices given by

$$\begin{aligned} \beta_0 &= \frac{13}{12}(\mathbf{f}_i - 2\mathbf{f}_{i+1} + \mathbf{f}_{i+2})^2 + \frac{1}{4}(3\mathbf{f}_j - 4\mathbf{f}_{j+1} + \mathbf{f}_{j+2})^2, \\ \beta_1 &= \frac{13}{12}(\mathbf{f}_{i-1} - 2\mathbf{f}_i + \mathbf{f}_{i+1})^2 + \frac{1}{4}(\mathbf{f}_{i-1} - \mathbf{f}_{i+1})^2, \\ \beta_2 &= \frac{13}{12}(\mathbf{f}_{i-2} - 2\mathbf{f}_{i-1} + \mathbf{f}_i)^2 + \frac{1}{4}(\mathbf{f}_{i-2} - 4\mathbf{f}_{i-1} + 3\mathbf{f}_i)^2. \end{aligned} \quad (20)$$

236 The numerical fluxes  $\hat{\mathbf{f}}_{i+1/2}^{(r)}$  are represented by an affine combination of  $\mathbf{f}_i$  belonging to the  $r$ -th

237 candidate stencil,  $S_{r,i} = \{x_{i-r}, \dots, x_{i+r}\}$ ; where  $r \in \{0, 1, 2\}$ . The three numerical fluxes  $\hat{\mathbf{f}}_{i+1/2}^{(r)}$  are

238 obtained as follows;

239 
$$\begin{pmatrix} \hat{\mathbf{f}}_{i+1/2}^{(0)} \\ \hat{\mathbf{f}}_{i+1/2}^{(1)} \\ \hat{\mathbf{f}}_{i+1/2}^{(2)} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{1}{3} & \frac{5}{6} & -\frac{1}{6} \\ 0 & -\frac{1}{6} & \frac{5}{6} & \frac{1}{3} & 0 \\ \frac{1}{3} & -\frac{7}{6} & \frac{11}{6} & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{f}_{i-2} \\ \mathbf{f}_{i-1} \\ \mathbf{f}_i \\ \mathbf{f}_{i+1} \\ \mathbf{f}_{i+2} \end{pmatrix}. \quad (21)$$

240 For  $\partial \mathbf{f}^- / \partial \mathbf{u} \leq 0$ ,  $\hat{\mathbf{f}}_{i+1/2}^-$  can be obtained in a similar way to that outlined above. The flux vector at the

241 cell-interface  $\hat{\mathbf{f}}_{i+1/2}$  is then calculated using Eq. (14); i.e.,  $\hat{\mathbf{f}}_{i+1/2} = \hat{\mathbf{f}}_{i+1/2}^+ + \hat{\mathbf{f}}_{i+1/2}^-$ . For the source term, a  
 242 similar procedure is again used, except that  $\mathbf{s}_{i+1/2}$  is reconstructed using nonlinear weights obtained  
 243 from the flux reconstruction.

244 A 3<sup>rd</sup> order Runge-Kutta scheme is used to integrate the resulting ODE system (i.e., Eq. (13)) in  
 245 time, and satisfies the total variation diminishing (TVD) property for unity CFL number (as previously  
 246 shown by e.g., [Shu and Osher, 1988](#); [Jiang and Shu, 1996](#); [Gottlieb and Shu, 1998](#)). The time-marching  
 247 steps are

$$\begin{aligned}
 \mathbf{u}^{(1)} &= \mathbf{u}_i^n - \frac{\Delta t}{\Delta x} \left( \hat{\mathbf{f}}_{i+1/2}(\mathbf{u}^n) - \hat{\mathbf{f}}_{i-1/2}(\mathbf{u}^n) \right) - \frac{\Delta t}{\Delta x} g \eta_i (\mathbf{s}_{i+1/2} - \mathbf{s}_{i-1/2}), \\
 \mathbf{u}^{(2)} &= \frac{3}{4} \mathbf{u}_i^n + \frac{1}{4} \left( \mathbf{u}^{(1)} - \frac{\Delta t}{\Delta x} \left( \hat{\mathbf{f}}_{i+1/2}(\mathbf{u}^{(1)}) - \hat{\mathbf{f}}_{i-1/2}(\mathbf{u}^{(1)}) \right) - \frac{\Delta t}{\Delta x} g \eta_i (\mathbf{s}_{i+1/2} - \mathbf{s}_{i-1/2}) \right), \\
 \mathbf{u}_i^{n+1} &= \frac{1}{3} \mathbf{u}_i^n + \frac{2}{3} \left( \mathbf{u}^{(2)} - \frac{\Delta t}{\Delta x} \left( \hat{\mathbf{f}}_{i+1/2}(\mathbf{u}^{(2)}) - \hat{\mathbf{f}}_{i-1/2}(\mathbf{u}^{(2)}) \right) - \frac{\Delta t}{\Delta x} g \eta_i (\mathbf{s}_{i+1/2} - \mathbf{s}_{i-1/2}) \right),
 \end{aligned} \tag{22}$$

249 where superscript  $n$  refers to time level and  $\Delta t$  is the time step.

### 250 3.2. Cut-Off Algorithm

252 In long-duration simulations, error can accumulate owing to mismatches between different round-  
 253 off errors arising from both sides of the simplified shallow water momentum equation for stationary  
 254 flow,

$$\frac{\partial}{\partial x} \left( \frac{1}{2} g (\eta^2 - 2\eta b) \right) = -g \eta \frac{\partial b}{\partial x}. \tag{23}$$

256 Both sides of Eq. (23) are computed through different arithmetic procedures, and so the calculated  
 257 results can differ by several units in the last place (ULP) ([Goldberg, 1991](#)) due to round-off errors. Such  
 258 errors cause a tiny imbalance in momentum balance which in turn drives a very small flux that alters  
 259 the water elevation  $\eta$ . Round-off errors can also inherently affect the water elevation values calculated  
 260 from the mass conservation equation, again generating a spurious numerical flux. In this fashion, such  
 261 errors accumulate significantly as time progresses, causing the numerical model to fail to satisfy the  
 262 exact C-property. From an empirical perspective, such mismatches obviously occur more frequently  
 263 when more arithmetic operations are performed on finer meshes involving larger numbers of cells. It is

264 therefore desirable to reduce or remove cumulative round-off errors in high-order schemes used for  
 265 long-duration simulations. To reduce such errors, we propose a special rounding technique as follows.  
 266 First, the maximum machine error is determined by finding the  $k$  ULP, when computing each term in  
 267 Eq. (11). The maximum error is then reduced to 0.5 ULP by using the extended format in IEEE standard.  
 268 Second, the largest value is selected as reference. Third, the values of each term in Eq. (11) are rounded  
 269 at any digit larger than the reference. In short, this technique rounds off at the first digits that are not  
 270 affected by round-off error and cuts off lower digits possibly contaminated by round-off error. This  
 271 rounding method aims to use only true values that are unaffected by machine error. From now, this  
 272 method is termed the cut-off algorithm.

273

### 274 **3.3 Well-Balanced Model for the Paddle Sub-Domain**

275 The numerical wave tank occupies paddle and main sub-domains (Fig. 2). Whereas the main sub-  
 276 domain comprises a fixed grid, the paddle sub-domain utilizes a moving grid that expands and contracts  
 277 in accordance with the motion of the piston. This paddle sub-domain is formulated as follows.

$$278 \quad l(t) = l_o - x_p(t), \quad (24)$$

$$279 \quad x \in [x_p(t), l_o], \quad (25)$$

280 where  $l(t)$  is the time-dependent length of the paddle sub-domain,  $l_o$  is the location of the fixed, end  
 281 point of the paddle sub-domain, and  $x_p(t)$  is the time-dependent displacement of the paddle. For  
 282 computational convenience, the moving domain is transformed to a fixed domain, using a similar  
 283 approach to Orszaghova *et al.* (2012). Here, the moving coordinate system  $x$  is transformed to a fixed  
 284 coordinate system,  $\bar{x}$ , using the linear mapping,  $V$ , defined as:

$$285 \quad x = V(\bar{x}) = \frac{l_o}{l(t)}(x - x_p(t)); \quad x \in [0, l_o]. \quad (26)$$

286 Using the chain rule, Eq. (11) defined in the moving coordinate system is mapped to the fixed new  
 287 coordinate system,  $(\bar{x}, t)$ , as follows. First, the derivatives in  $t$  and  $x$  are transformed as

$$288 \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial t} + \frac{\partial x}{\partial t} \frac{\partial}{\partial x} = \frac{\partial}{\partial t} + \frac{dl}{dt} \frac{(l_o - x)}{l} \frac{\partial}{\partial x} \quad (27)$$

289 and

$$290 \quad \frac{\partial}{\partial x} = \frac{\partial t}{\partial x} \frac{\partial}{\partial t} + \frac{\partial x}{\partial x} \frac{\partial}{\partial x} = \frac{l_o}{l} \frac{\partial}{\partial x}. \quad (28)$$

291 Then, applying Eqs. (27) and (28) to Eq. (11) in the paddle sub-domain,

$$292 \quad \frac{\partial \mathbf{u}}{\partial t} + \frac{dl}{dt} \frac{(l_o - x)}{l} \frac{\partial \mathbf{u}}{\partial x} + \frac{l_o}{l} \frac{\partial \mathbf{f}}{\partial x} = -\frac{l_o}{l} g\eta \frac{\partial \mathbf{s}}{\partial x}. \quad (29)$$

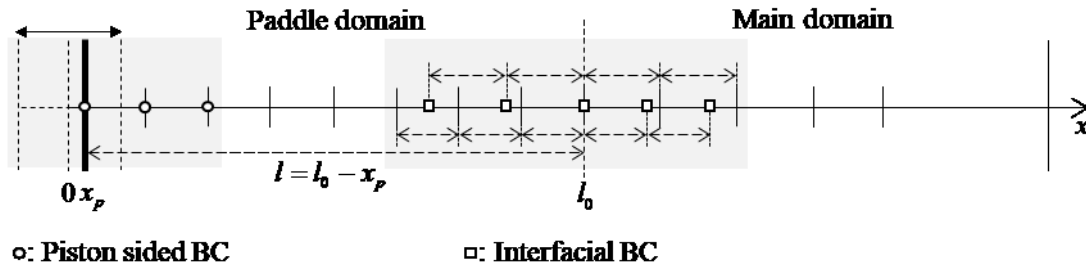
293 Numerical solutions of Eq. (29) are transformed to the original coordinate system  $x$  using the inverse  
294 mapping:

$$295 \quad V^{-1}(x) = \frac{l}{l_o} x + x_p = x. \quad (30)$$

296 To resolve flow discontinuities that may arise during the simulation, we use the 5th order WENO  
297 method which is a Riemann solver, based on [Xing and Shu \(2005\)](#)'s well-balanced scheme, noting its  
298 consistency, scalability to high-order accuracy, and exact C-property (see Appendix for more details;

299 **Proposition 2).**

300



301

302

**Fig. 2.** Schematic of numerical wave tank

303

### 304 **3.4 Interface and Boundary Conditions.**

305 At the interface between paddle and main sub-domains, the physical dimension of the cell on the  
306 paddle side of the interface changes with time, whereas that on the main side is invariant in time. This  
307 temporal inconsistency in mesh structure at the interface could inhibit the direct passage of information  
308 across the interface, and so a special method has been devised to handle this problem. Given that the 5<sup>th</sup>  
309 order WENO relies on 6 equally spaced neighboring cells for reconstruction, we use several ghost cells

310 to formulate the interface between the sub-domains (Fig. 2). Once the uniform ghost cells have been  
311 constructed, a smoothness indicator detects the presence or otherwise of any discontinuity. If a given  
312 ghost cell has a discontinuity, first-order interpolation is applied. Otherwise, a fifth-order upwind  
313 method is used for interpolation. This method is akin to an open boundary condition that exchanges  
314 information between two domains. For the hyperbolic shallow water equations considered herein, the  
315 Riemann invariants are selected as interpolation objects, with incoming and outgoing information  
316 treated using the method of characteristics. Assuming that the interface of the two sub-domains is flat-  
317 bottomed, the Riemann invariants are given by

$$318 \quad J = (U - 2\sqrt{gh}, U + 2\sqrt{gh})^T, \quad (31)$$

319 where superscript  $T$  represents the vector transpose.

320 At the piston boundary, the WENO method is no longer directly applicable because only three  
321 interior values can be assigned and it is physically inappropriate to place ghost cells behind the piston,  
322 outside the wave tank. Hence, we construct a piston-sided boundary condition for wave generation  
323 using fifth-order forward differences (consistent with the accuracy of the main scheme):

$$324 \quad \mathbf{f}_x(x_1) \approx \frac{1}{60\Delta x} (-137\mathbf{f}_1 + 300\mathbf{f}_2 - 300\mathbf{f}_3 + 200\mathbf{f}_4 - 75\mathbf{f}_5 + 12\mathbf{f}_6), \quad (32)$$

$$325 \quad \mathbf{f}_x(x_2) \approx \frac{1}{60\Delta x} (-12\mathbf{f}_1 - 65\mathbf{f}_2 + 120\mathbf{f}_3 - 60\mathbf{f}_4 + 20\mathbf{f}_5 - 3\mathbf{f}_6), \quad (33)$$

$$326 \quad \mathbf{f}_x(x_3) \approx \frac{1}{60\Delta x} (3\mathbf{f}_1 - 30\mathbf{f}_2 - 20\mathbf{f}_3 + 60\mathbf{f}_4 - 15\mathbf{f}_5 + 2\mathbf{f}_6). \quad (34)$$

327 The six consecutive cells closest to the piston paddle are assumed to be smooth without discontinuity.

328 The primitive boundary condition at the first cell is:

$$329 \quad U_1 = \frac{dx_p}{dt}, \quad (35)$$

330 where  $U_1$  is the depth-averaged velocity of the piston-sided first cell. This ensures that the paddle  
331 velocity is equal to the depth-averaged particle velocity (Hughes, 1993). In other words, the Lagrangian  
332 motions of the paddle match the depth-averaged water particle velocity at the first cell in keeping with  
333 the kinematic free surface boundary condition. Eq. (35) represents a clamped boundary, and so, when



334 the paddle starts to move, discontinuous values may be input into the first cell. To prevent spurious  
335 oscillations created by a discontinuity causing the solution to become unstable, the amplitude and  
336 frequency of the input waves are ramped up gradually to the desired values. During the spin-up period,  
337 wave information is exported across the interface using a Flather-type open boundary condition (OBC)  
338 (Blayo and Debreu, 2005). Once the wave attains its target amplitude, the OBC is no longer used and  
339 all information is passed to the main sub-domain.

340

## 341 **4. Numerical Results.**

342 Several numerical experiments are used to validate the numerical model, with the results compared  
343 to analytical or fine-grid solutions. Three types of numerical experiments are undertaken: the first to  
344 confirm whether the exact C-property is satisfied; the second to estimate the accuracy and stability of  
345 the numerical scheme; and the third to investigate the accuracy of the piston boundary condition.

346

### 347 **4.1. Tests for Exact C-Property**

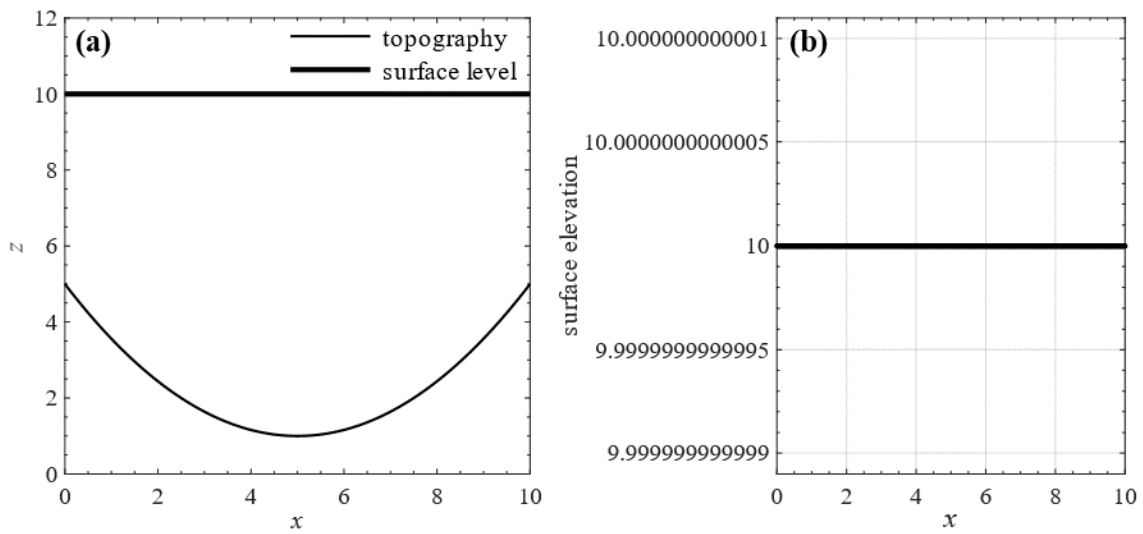
348 A case with non-flat bottom topology specified in the range of  $x \in [0, 10]$  is tested for still water  
349 conditions using three different models satisfying the exact C-property, formulated according to Xing  
350 and Shu (2005), Liang and Borthwick (2009) and a combination of these (the present scheme). All  
351 computations are undertaken in double precision on a computational mesh comprising 1000 cells, with  
352 CFL number set to 0.4. The tank has a smoothly varying parabolic bed elevation (Fig. 3) given by

$$353 \quad b(x) = 1 + 4 \left( \frac{x-5}{5} \right)^2. \quad (36)$$

354 Initial conditions are

$$355 \quad \eta(x, 0) = 10 \text{ and } U(x, 0) = 0. \quad (37)$$

356



357

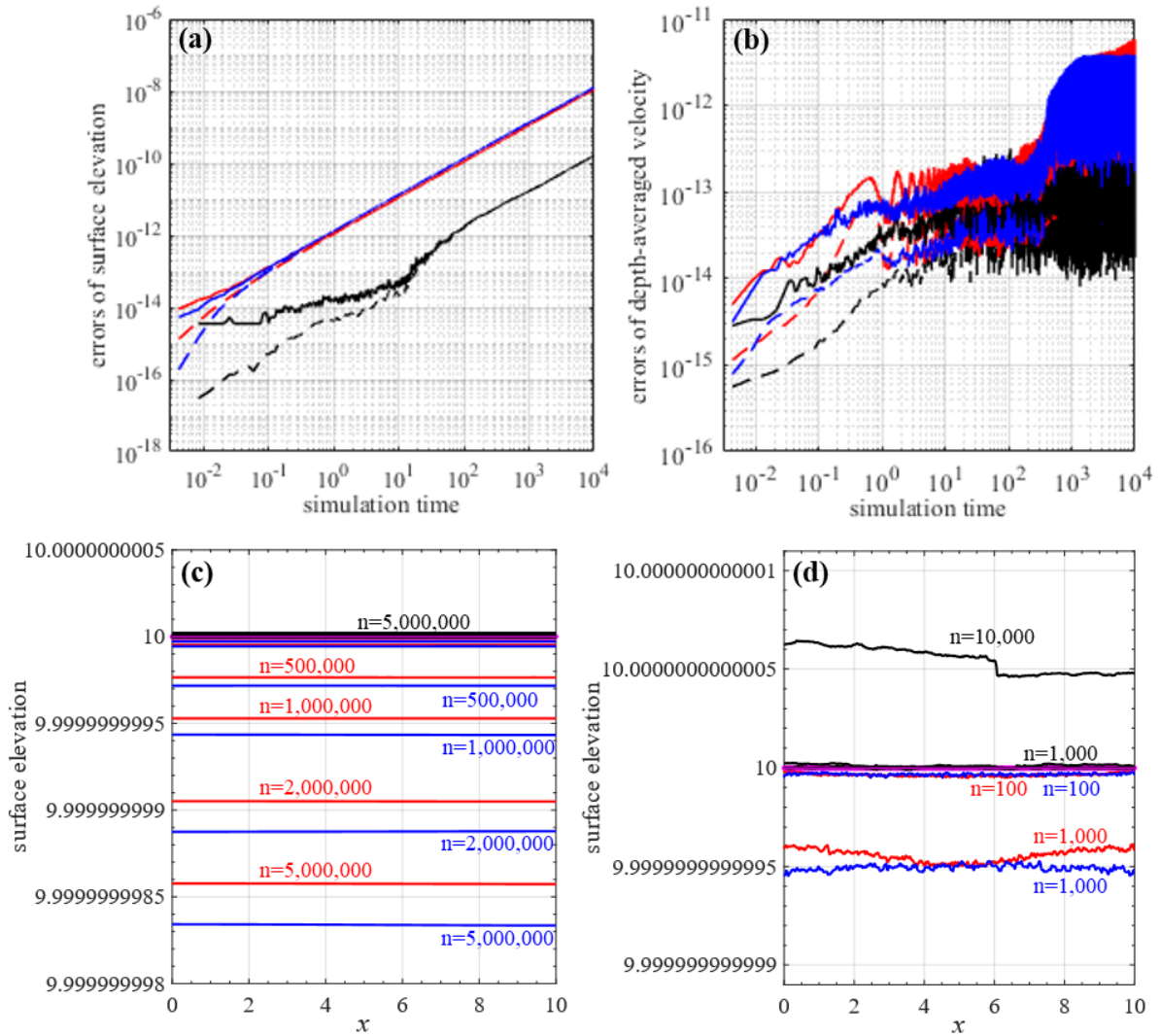
358 **Fig. 3.** Still water tests for exact C-property: (a) bed topography and initial water level; (b) zoomed-in  
 359 initial water level.

360

361 [Fig. 4\(a\)](#) shows the logarithmic growth in accumulated round-off error with simulation time. For a short  
 362 1 s simulation, the three models all produced small  $L_1$  and  $L_\infty$  errors ( $<10^{-12}$ ). For a longer 10,000  
 363 s simulation, the accumulated errors become noticeably amplified. It should be noted that even though  
 364 [Liang and Borthwick \(2009\)](#) used a second-order finite volume method, their scheme required fewer  
 365 arithmetic operations, leading to much smaller errors than [Xing and Shu \(2005\)](#)'s fifth-order finite  
 366 difference method ([Figs. 4\(a\)-4\(c\)](#)). From an empirical perspective, such mismatches obviously occur  
 367 more frequently when more arithmetic operations are performed on finer meshes involving larger  
 368 numbers of cells. It can be seen that the round-off errors of the high-order scheme initially accumulate  
 369 faster than the lower-order scheme ([Fig. 4\(a\)](#)) because of the increased number of computational steps.

370 Most previous studies (e.g., [Xing and Shu, 2005](#); [Castro et al., 2006](#); [Lukáčová-Medvid'ová et al.](#)  
 371 [2007](#)) tested the exact C-property for a finite short time. However, the accumulation of round-off errors  
 372 in long-term still water simulations is too significant to be neglected, providing justification for the use  
 373 of the cut-off algorithm in the present work which provided results that are perfectly well-balanced  
 374 (with errors remaining even below the round-off level as in [Fig. 4](#)).

375



376

377 **Fig. 4.** Still water tests for exact C-property: (a) and (b) accumulation of round-off  $L_\infty$  error (solid lines)  
 378 and  $L_1$  error (dotted lines); (c) and (d) water surface elevation at different time levels,  $n$ . Red, black,  
 379 blue, and purple lines refer to [Xing and Shu \(2005\)](#), [Liang and Borthwick \(2009\)](#), the present combined  
 380 scheme without cut-off algorithm, and the present scheme with the cut-off algorithm.

381

## 382 4.2. Main Solver Validation

383 Five numerical experiments with different initial and boundary values are now carried out to  
 384 examine the accuracy and stability of the present shallow water solver. The CFL number is set to 0.6  
 385 for all tests in this section.

386

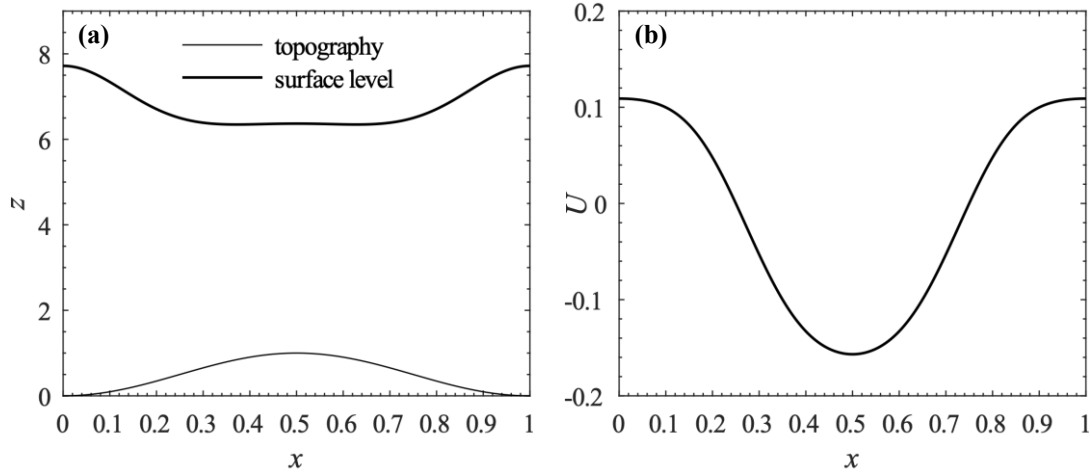
### 387 4.2.1. Accuracy of Smooth Solution

388 This test checks the fifth-order accuracy of the present scheme in computing a smooth solution to

389 the shallow water equations. Fig. 5 shows the bed topography and initial free surface and velocity  
 390 profiles proposed by Xing and Shu (2005), given by

391 
$$b(x) = \sin^2(\pi x), \quad h(x, 0) = 5 + \exp[\cos(2\pi x)], \quad \text{and} \quad U(x, 0) = \frac{\sin[\cos(2\pi x)]}{5 + \exp[\cos(2\pi x)]}. \quad (38)$$

392



393

394 **Fig. 5.** Smooth solution case devised by Xing and Shu (2005): (a) bed topography and initial water  
 395 surface level; and (b) depth-averaged velocity profiles.

396

397 This case is tested on a fine resolution reference mesh of  $N = 25,600$  cells, following Xing and Shu.

398 Table 3 lists the  $L_1$  and  $L_\infty$  errors and numerical order of accuracy obtained with respect to the

399 reference solution at time  $t = 0.1$  s; as the number of cells increases, the order of the accuracy converges

400 to fifth order, confirming that the scheme has been properly implemented.

401

402 **Table 3.**  $L_1$  and  $L_\infty$  errors and order of accuracy for Xing and Shu's (2005) smooth solution

N	$h$				$U$			
	$L_1$	order	$L_\infty$	order	$L_1$	order	$L_\infty$	order
25	1.69E-02		7.64E-02		1.41E-02		7.64E-02	
50	2.09E-03	3.014	1.70E-02	2.169	2.63E-03	2.426	2.13E-02	1.843
100	3.03E-04	2.787	4.14E-03	2.036	3.75E-04	2.810	4.96E-03	2.104
200	2.14E-05	3.821	5.01E-04	3.049	2.61E-05	3.845	6.13E-04	3.016
400	8.77E-07	4.609	2.69E-05	4.219	1.07E-06	4.615	3.26E-05	4.232
800	2.98E-08	4.878	9.53E-07	4.818	3.62E-08	4.881	1.15E-06	4.823

N is the number of cells,  $h$  is water depth, and  $U$  is depth-averaged velocity.

#### 4.2.2. Solutions with Discontinuity

Three numerical experiments tested the shock capturing ability of the solver. The first involved the generation of an upstream-directed critical rarefaction and downstream-directed bore proposed by Toro (2001) with initial conditions,

$$h(x,0) = \begin{cases} 1 & \text{if } 0 \leq x \leq 25, \\ 0.1 & \text{otherwise,} \end{cases} \quad \text{and } U(x,0) = \begin{cases} 2.5 & \text{if } 0 \leq x \leq 25, \\ 0 & \text{otherwise.} \end{cases} \quad (39)$$

Fig. 6 shows the initial and final free surface and velocity profiles for a simulation at  $t = 0$  s and  $t = 7$  s on a mesh of 500 cells in a channel 50 m long. There is very close agreement between the model predictions and results obtained by Toro (2001) on a very fine mesh.

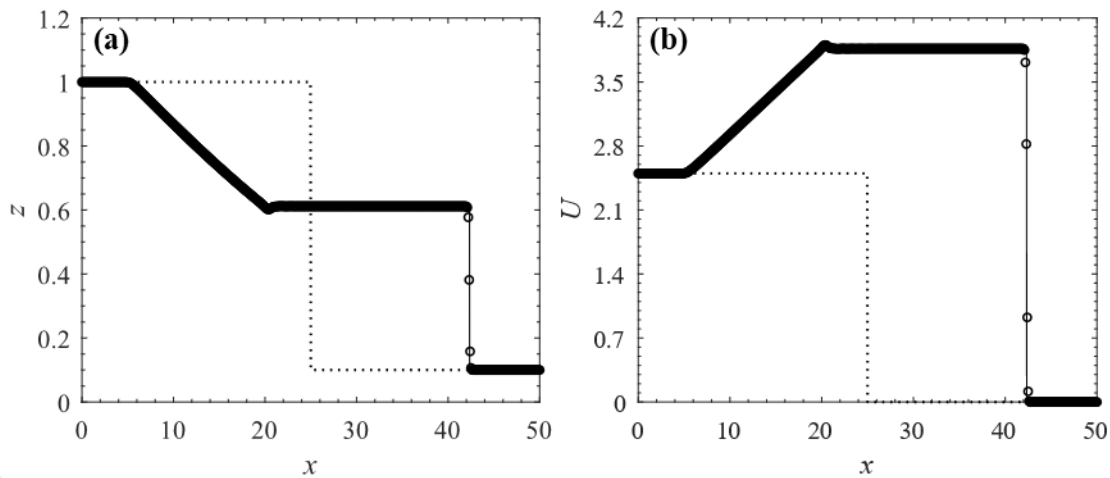
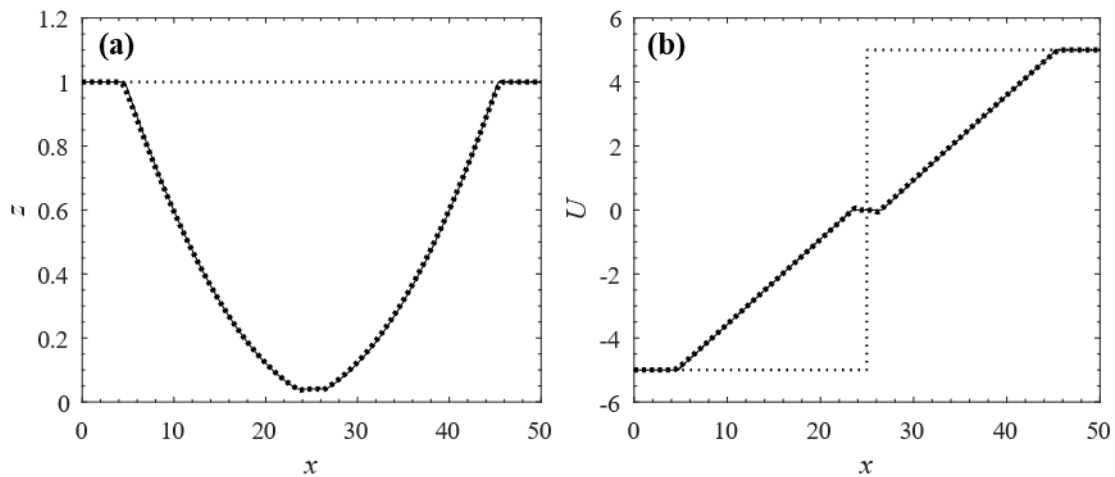


Fig. 6. Left critical rarefaction and right bore, where dotted lines represent the initial conditions at  $t = 0$  s, and black circular dots represent numerical predictions and solid lines represent Toro's (2001) quasi-analytical solution at  $t = 7$  s: (a) free surface elevation; and (b) depth-averaged velocity profiles.

The second test simulates two rarefaction waves propagating in opposite directions over a nearly dry bed in a channel of length 50 m. The equations of the initial conditions are given as

$$h(x,0) = 1 \quad \text{and } U(x,0) = \begin{cases} -5 & \text{if } 0 \leq x \leq 25, \\ 5 & \text{otherwise.} \end{cases} \quad (40)$$

422 Fig. 7 shows the initial flat free surface with oppositely directed flow at  $t = 0$  s. By  $t = 2.5$  s, forward  
 423 and backward propagating rarefaction waves can be seen, in satisfactory agreement with corresponding  
 424 fine mesh results presented by Toro (2001).  
 425



426  
 427 **Fig. 7.** Discontinuous solution with two rarefaction waves over a nearly dry bed, where dotted lines  
 428 represent the initial conditions at  $t = 0$  s, and black circular dots represent numerical predictions and  
 429 solid lines represent Toro's (2001) quasi-analytical solution at  $t = 2.5$  s: (a) free surface elevation; and  
 430 (b) depth-averaged velocity profiles.

431  
 432 The third test concerns a dam break over discontinuous topography, comprising a rectangular hump,  
 433 proposed by Bermúdez and Vázquez-Cendón (1994). This case examines the ability of the scheme to  
 434 handle shocks in the presence of a non-zero source term. Figs. 8(a) and 8(b) shows the bed topography  
 435 and initial conditions for  $x \in [0, 1500]$  given by,

$$436 \quad b(x) = \begin{cases} 8 & \text{if } |x - 750| \leq 1500/8, \\ 0 & \text{otherwise,} \end{cases} \quad (41)$$

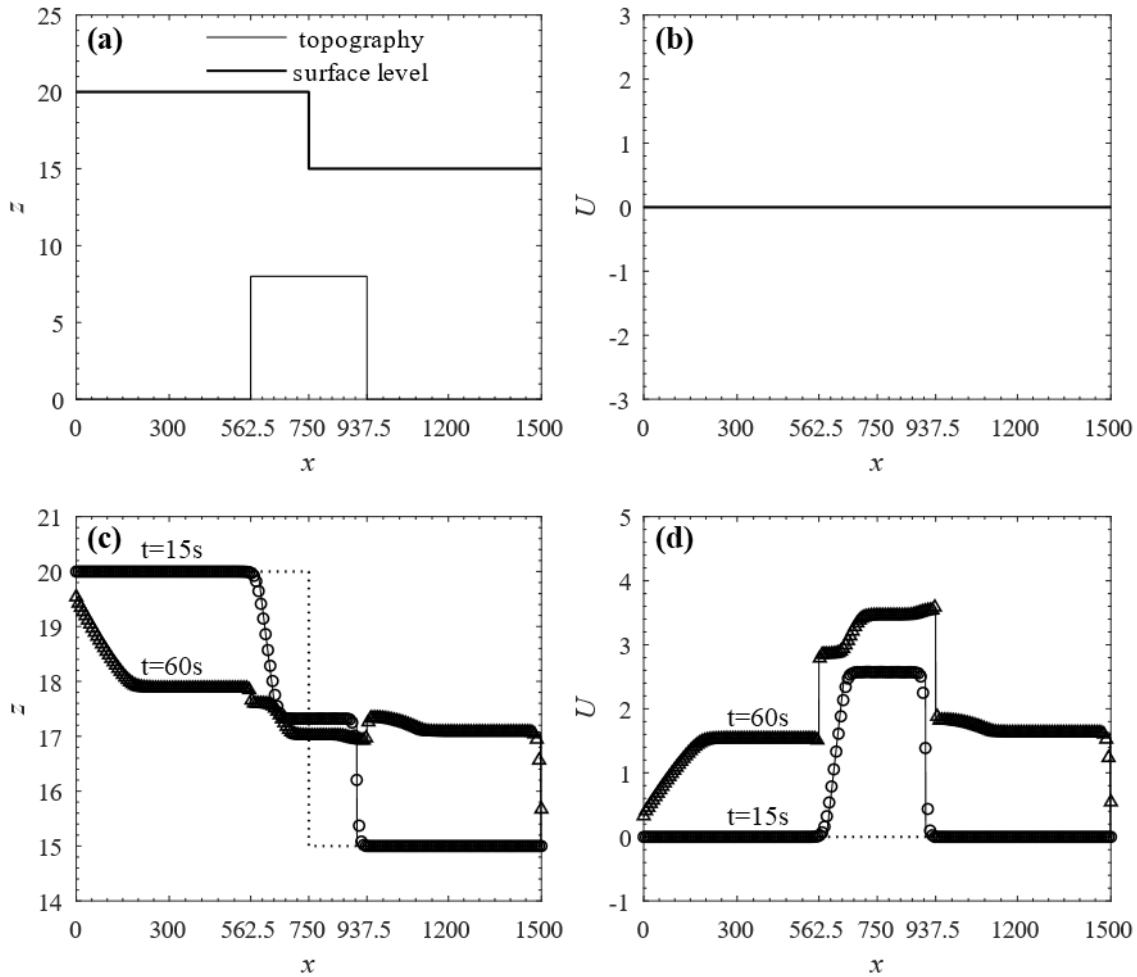
437 and

$$438 \quad h(x,0) = \begin{cases} 20 - b(x) & \text{if } 0 \leq x \leq 750, \\ 15 - b(x) & \text{otherwise} \end{cases} \quad \text{and } U(x,0) = 0. \quad (42)$$

439 Figs. 8(c) and 8(d) show the free surface elevation and depth-averaged velocity profiles at  $t = 15$  s and  
 440  $t = 60$  s on three grids of 200 and 3000 cells. The predictions on both the coarse and fine meshes match  
 441 those of Bermúdez and Vázquez-Cendón (1994) confirming the present solver correctly reproduces

442 discontinuous solutions, with correct wave speeds and amplitudes.

443



444

445 **Fig. 8.** Dam break over a box at  $t = 15$  s and  $t = 60$  s: (a) topography and initial free surface elevation  
 446 profiles; (b) initial depth-averaged velocity profile; (c) and (d) the numerical predictions of surface  
 447 elevation and depth-averaged velocity (dotted line is the initial condition, and solid lines, circular and  
 448 triangular dots are numerical solutions on meshes of 3,000, 200, and 200 grid points.

449

#### 450 4.2.3. A Small Perturbation Applied to Still Water

451 This case tests a quasi-stationary condition, similar to that suggested by [Leveque \(1998\)](#). The bed  
 452 topography and initial conditions for  $x \in [0, 2]$  ([Figs. 9\(a\) and 9\(b\)](#)) are

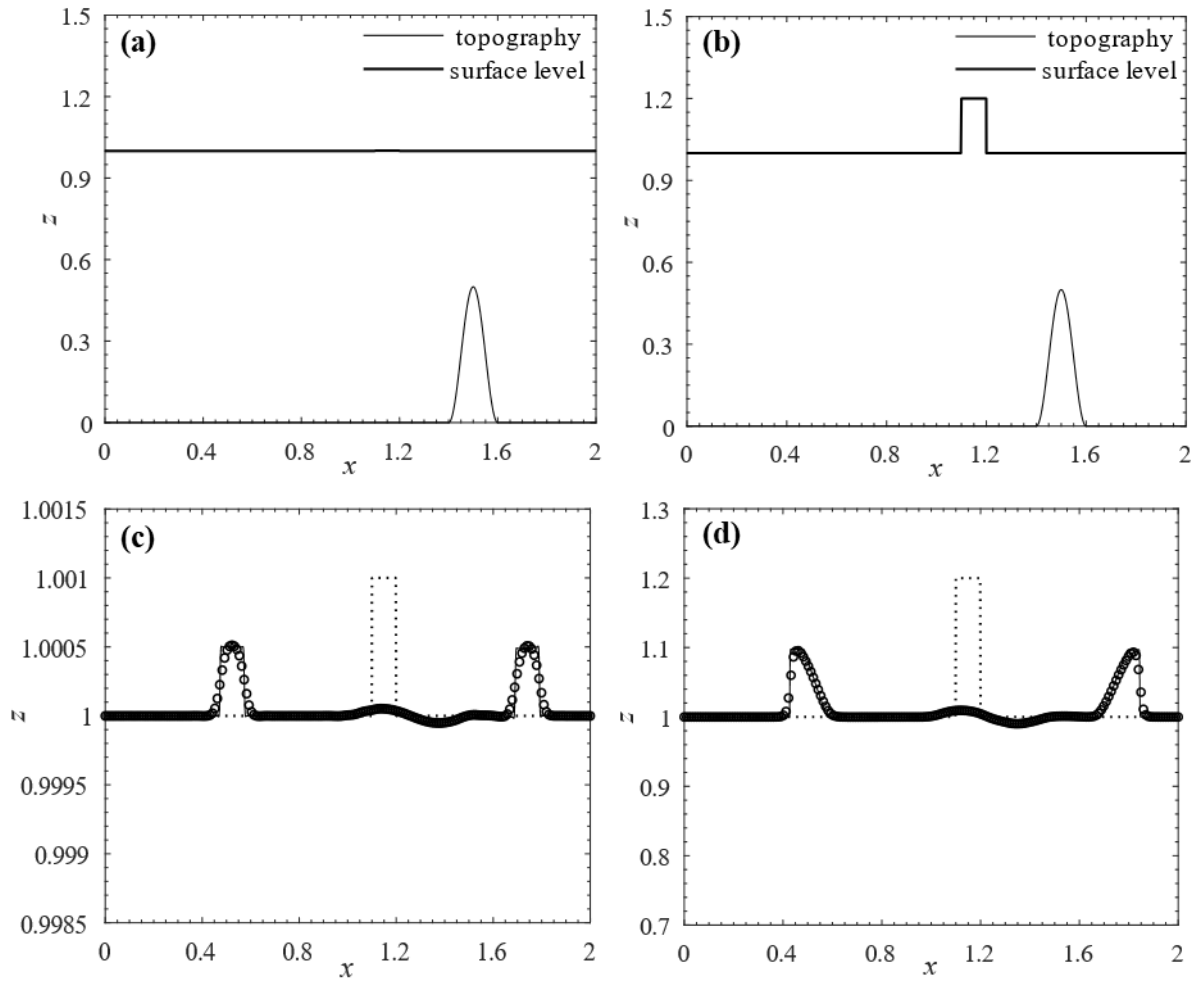
$$453 \quad b(x) = \begin{cases} 0.25 \{ \cos[10\pi(x-1.5)] + 1 \} & \text{if } 1.4 \leq x \leq 1.6, \\ 0 & \text{otherwise,} \end{cases} \quad (43)$$

454 and

455 
$$h(x,0) = \begin{cases} 1-b(x) + \xi & \text{if } 1.1 \leq x \leq 1.2, \\ 1-b(x) & \text{otherwise,} \end{cases} \text{ and } U(x,0) = 0. \quad (44)$$

456 The perturbation amplitude,  $\xi$ , is set to 0.001 m and 0.2 m in order to generate linear and nonlinear  
457 wave cases. In both cases, two waves of the same amplitude propagate in opposite directions. The  
458 numerical experiments were performed on meshes with  $N = 200$  and 3,000 cells. [Figs. 9\(c\) and 9\(d\)](#)  
459 show the results at  $t = 0.2$  s, where it can be seen that both linear and nonlinear waves have been  
460 generated stably and correctly. Almost identical results are obtained on the coarse and fine meshes,  
461 indicating that the higher order scheme works well in both cases of the linear and nonlinear waves. The  
462 linear small amplitude pulse is reproduced without contamination from the truncation error at low  
463 resolution ([Fig. 9\(c\)](#)). The front façade of the large amplitude, nonlinear waves steepens ([Fig. 9\(d\)](#)).  
464 When the wave propagating to the east passes over the hump, its amplitude slightly decreases, and small  
465 amplitude undulations are generated, propagating westward ([Figs. 9\(c\) and 9\(d\)](#)).  
466





467

468 **Fig. 9.** Small perturbation applied to still water. Topography and initial free surface elevation with: (a)  
 469 small amplitude,  $\xi = 0.001$  m; and (b) large amplitude,  $\xi = 0.2$  m perturbations. Free surface elevation  
 470 profiles at  $t = 0.2$  s for: (c)  $\xi = 0.001$  m; and (d)  $\xi = 0.2$  m. Circular dots and solid line are predictions  
 471 on meshes with 200 and 3000 cells respectively. Dotted line represents the initial condition.

472

473

#### 4.2.4. Tidal Flow

474

475

476

477

To check that the equations remain well-balanced in the present solver, we consider a benchmark test proposed by [Bermúdez and Vázquez-Cendón \(1994\)](#) whereby a long, small-amplitude tidal wave is simulated on variable bed topography. The domain lies in the range  $x \in [0, 14,000]$ , and the topography and initial conditions are

$$478 \quad b(x) = 10 + \frac{40x}{14,000} + 10 \sin\left(\frac{4\pi x}{14,000} - \frac{\pi}{2}\right), \quad (45)$$

$$479 \quad h(x, 0) = 60.5 - b(x) \quad \text{and} \quad U(x, 0) = 0. \quad (46)$$

480 The inflow depth and outflow velocity boundary conditions are

481  $h(x=0,t) = 64.5 - 4 \sin\left(\frac{4\pi t}{86,400} + \frac{\pi}{2}\right)$  and  $U(x=14,000,t) = 0$ . (47)

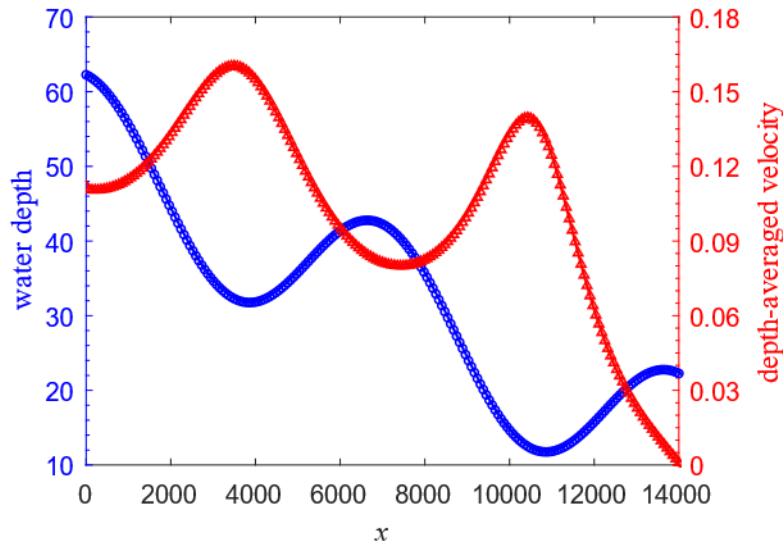
482 Here, numerical results on a mesh with cell size,  $\Delta x = 70$  m, are compared against the following very  
 483 accurate approximate solution, obtained by asymptotic analysis by [Bermúdez and Vázquez-Cendón](#)  
 484 (1994).

485  $h(x,t) = 64.5 - b(x) - 4 \sin\left(\frac{4\pi t}{86,400} + \frac{\pi}{2}\right)$  (48)

486 and

487  $U(x,t) = \frac{\frac{(x-14,000)\pi}{5,400} \cos\left(\frac{4\pi t}{86,400} + \frac{\pi}{2}\right)}{64.5 - b(x) - 4 \sin\left(\frac{4\pi t}{86,400} + \frac{\pi}{2}\right)}$ . (49)

488 [Fig. 10](#) shows the excellent agreement between the semi-analytical and numerical results at  $t = 7552.13$   
 489 s, demonstrating that the present scheme is uncontaminated by any spurious numerical flux due to  
 490 imbalance between flux and source terms in the shallow water equations, and so satisfies the well-  
 491 balanced conditions for unsteady flow simulation.



492  
 493 **Fig. 10.** Well-balanced solutions for tidal flow over spatially varying topography at  $t = 7552.13$  s: solid  
 494 lines represent semi-analytical solutions from asymptotic analysis ([Bermúdez and Vázquez-Cendón](#),  
 495 1994), and circular and triangular dots represent numerical solutions of (blue) water depth and (red)  
 496 depth-averaged velocity, respectively, on a mesh with  $\Delta x = 70$  m.

497

#### 4.2.5. Steady Flow Over a Hump

Free surface flow over a bed hump is a well-established verification test for shallow water solvers of subcritical flow, trans-critical flow without a shock, and trans-critical flow with a shock (see e.g., LeVeque, 1998; Vázquez-Cendón, 1999; Xing and Shu, 2005; Liang and Borthwick, 2009). In this case, we consider a one-dimensional open channel of length 25 m, and bed elevation profile and initial conditions (Fig. 11(a)) given by

$$b(x) = \begin{cases} 0.2 - 0.05(x-10)^2 & \text{if } 8 \leq x \leq 12, \\ 0 & \text{otherwise,} \end{cases} \quad (50)$$

and

$$\eta(x,0) = 10, \text{ and } U(x,0) = 0. \quad (51)$$

The following case-dependent clamped boundary conditions are assigned at the upstream and downstream ends of the channel:

Case 1. Subcritical flow

- upstream:  $hU = 4.42 \text{ m}^2/\text{s}$ , downstream:  $h = 2 \text{ m}$ .

Case 2. Trans-critical flow with a shock

- upstream:  $hU = 0.18 \text{ m}^2/\text{s}$ , downstream:  $h = 0.33 \text{ m}$ .

Case 3. Trans-critical flow without a shock

- upstream:  $hU = 1.53 \text{ m}^2/\text{s}$ , downstream:  $h = \begin{cases} 0.66 \text{ m} & \text{if } Fr < 1 \\ \text{open boundary condition} & \text{otherwise} \end{cases}$ .

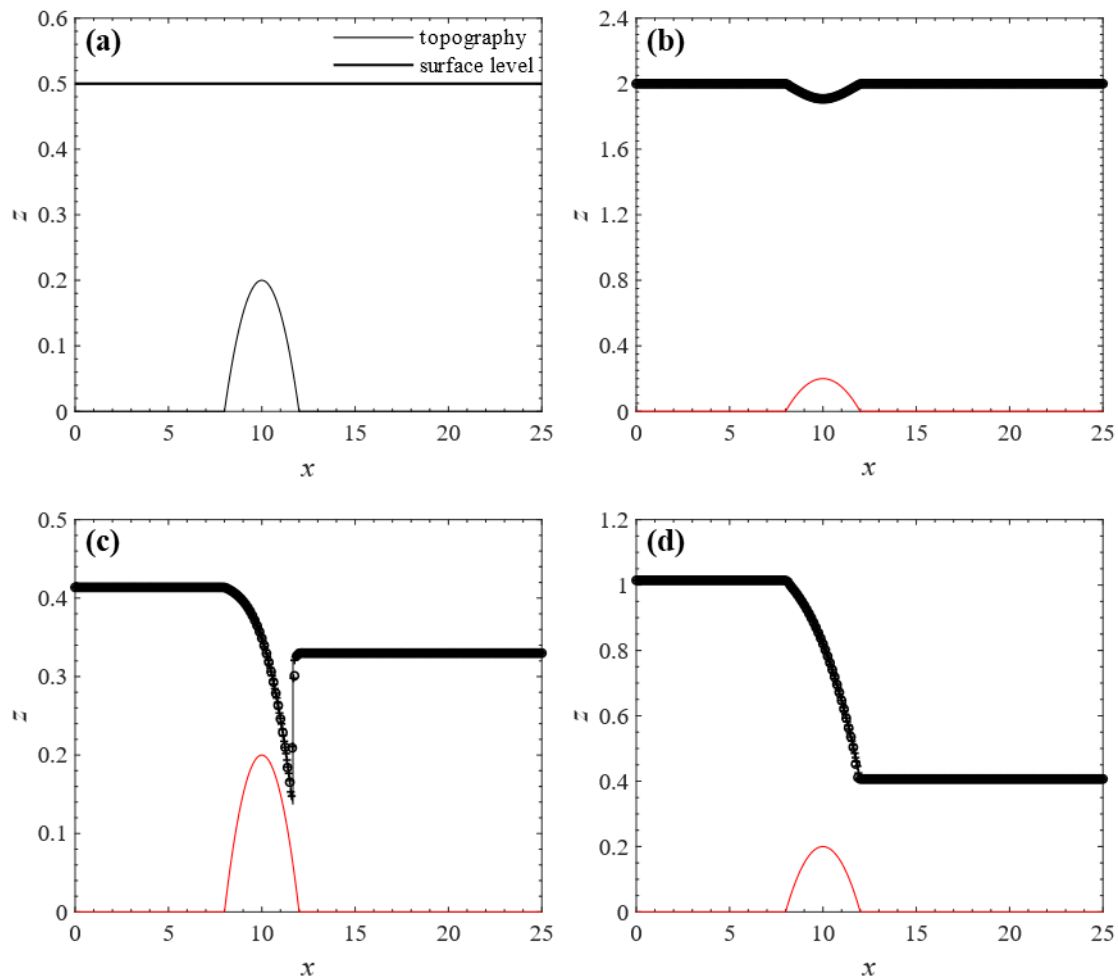
In Case 3, when the downstream flow is not subcritical (i.e.,  $Fr \geq 1$ ), zero-order open boundary conditions are used.

Figs. 11(b)-11(d) shows the excellent agreement achieved between the analytical and predicted steady state free surface elevation profiles obtained at  $t = 200 \text{ s}$ . The numerical predictions are carried out on coarse and fine meshes of 200 and 500 cells.

In all the foregoing tests, the present numerical model predictions converged properly to analytical or fine-grid solutions of the shallow water equation confirming the well-balanced, high-order, accurate nature of the scheme in the presence of flow discontinuities, flow transitions, and long-duration steady

523 and unsteady flows.

524



525

526 **Fig. 11.** Steady flow over a hump in a one-dimensional channel: (a) bed topography and initial surface  
527 elevation profile; and steady-state results at time  $t = 200$  s for (b) subcritical flow; (c) trans-critical flow  
528 with a shock; and (d) trans-critical flow without a shock. The black solid lines display the analytical  
529 solutions, and the circular and + symbols display the numerical predictions using the present scheme  
530 on meshes of 200 and 500 cells, respectively. The red solid line shows the bed profile.

531

### 532 4.3. Piston Boundary

533 We now consider generation of sinusoidal, solitary, and cnoidal waves using the piston boundary  
534 condition where the piston paddle velocity is set to be the same as the local depth-averaged particle  
535 velocity of the target wave. In the numerical model, the depth averaged velocity at the piston side of the  
536 first cell next to the piston,  $U_1$ , is prescribed as a clamped boundary condition, Eq. (35). Meanwhile,  
537 the water elevation at the first cell,  $\eta_1$ , is determined adaptively from Eqs. (32), (33), and (34) using

538 values from adjacent interior cells. The following tests are undertaken to check that long shallow water  
 539 waves are correctly generated by the piston-sided boundary. The tank is of length 500 m, such that  
 540  $x \in [0, 500]$ , the mesh size is  $\Delta x = 1$  m, and the time step is  $\Delta t = 0.01$  s.

541  
 542 **4.3.1. Sinusoidal Waves**

543 For simple sinusoidal wave generation, the paddle displacement time series is

544  $x_p = a \sin(\omega t),$  (52)

545 where  $a$  is the paddle displacement amplitude, and  $\omega$  is its frequency (Dean and Dalrymple, 1991).

546 For small-amplitude waves, the free surface elevation is given by linear wave theory equation as

547  $\eta_a(x, t) = A \sin(\kappa x - \omega t),$  (53)

548 where  $\eta_a$  is the analytic solution of water elevation,  $\kappa$  is the wave number, and  $A$  is the wave  
 549 amplitude, which in turn is given by Dean and Dalrymple (1991) as

550 
$$A = \frac{\omega \int_{-h}^0 a \omega \cosh[\kappa(h+z)] dz \cosh(\kappa h)}{g \kappa \int_{-h}^0 \cosh^2[\kappa(h+z)] dz} = \frac{\omega \frac{a\omega}{\kappa} \sinh(\kappa h) \cosh(\kappa h)}{g \frac{2\kappa h + \sinh(2\kappa h)}{4}} = \frac{4a\omega^2 \sinh(\kappa h) \cosh(\kappa h)}{\kappa g (2\kappa h + \sinh(2\kappa h))}. \quad (54)$$

551 A total of 22 cases were simulated using the present numerical model (Table 4). Fig. 12 shows the  
 552 analytical solution and free surface elevation time history at the piston cell for Case 1, involving high-  
 553 frequency relatively large amplitude waves of amplitude 1.139 m, period 3.5696 s, and mean water  
 554 depth 5 m. Fig. 13 shows the corresponding results for Case 15, involving lower frequency, small-  
 555 amplitude waves of amplitude 0.016 m, period 14.2784 s, and mean water depth 5 m. As would be  
 556 expected, the larger amplitude, higher frequency waves gave rise to greater error owing to their inherent  
 557 nonlinearity.

558

559 **Table 4.** Sinusoidal wave test parameters.

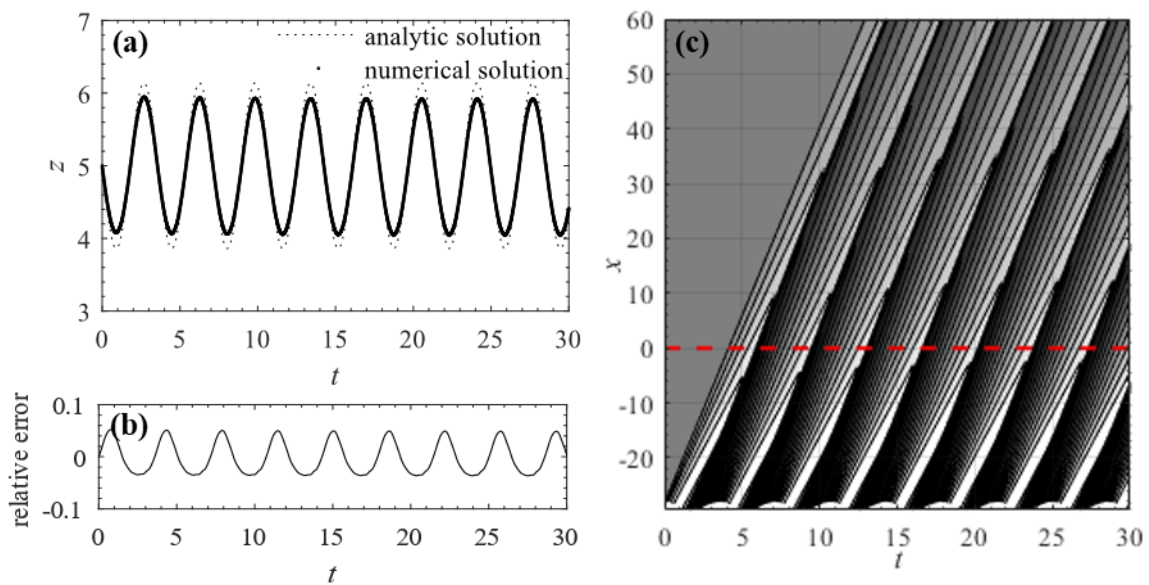
Case	Ur	H/L	h <sub>s</sub> /L	H/h <sub>s</sub>	L (m)	T (s)	A (m)	h <sub>s</sub> (m)
1	6.2946	0.1226	0.269	0.4556	18.5852	3.5696	1.139	5
2	4.1964	0.0817	0.269	0.3037	18.5852	3.5696	0.7593	5
3	0.8393	0.0163	0.269	0.0607	18.5852	3.5696	0.1519	5

4	0.4196	0.0082	0.269	0.0304	18.5852	3.5696	0.0759	5
5	0.0839	0.0016	0.269	0.0061	18.5852	3.5696	0.0152	5
6	23.3742	0.0287	0.1071	0.268	46.6962	7.1392	0.67	5
7	11.6871	0.0143	0.1071	0.134	46.6962	7.1392	0.335	5
8	2.3374	0.0029	0.1071	0.0268	46.6962	7.1392	0.067	5
9	1.1687	0.0014	0.1071	0.0134	46.6962	7.1392	0.0335	5
10	0.2337	0.0003	0.1071	0.0027	46.6962	7.1392	0.0067	5
11	98.8524	0.013	0.0508	0.2555	98.3526	14.2784	0.6387	5
12	49.4262	0.0065	0.0508	0.1277	98.3526	14.2784	0.3193	5
13	24.7131	0.0032	0.0508	0.0639	98.3526	14.2784	0.1597	5
14	4.9426	0.0006	0.0508	0.0128	98.3526	14.2784	0.0319	5
15	2.4713	0.0003	0.0508	0.0064	98.3526	14.2784	0.016	5
16	49.5879	0.0188	0.0723	0.2595	96.7672	12.0675	0.9082	7
17	23.3742	0.0287	0.1071	0.268	93.3923	10.0964	1.3399	10
18	9.2667	0.0506	0.1761	0.2874	85.1802	8.2437	2.1552	15
19	200.232	0.0032	0.0251	0.1262	199.1772	28.5569	0.3155	5
20	100.116	0.0016	0.0251	0.0631	199.1772	28.5569	0.1577	5
21	50.058	0.0008	0.0251	0.0315	199.1772	28.5569	0.0789	5
22	10.0116	0.0002	0.0251	0.0063	199.1772	28.5569	0.0158	5

560 Ur is Ursell number,  $H$  is wave height ( $=2A$ ),  $A$  is wave amplitude,  $L$  is wavelength,  $h_s$  is still-water depth, and  $T$   
561 is wave period.

562

563

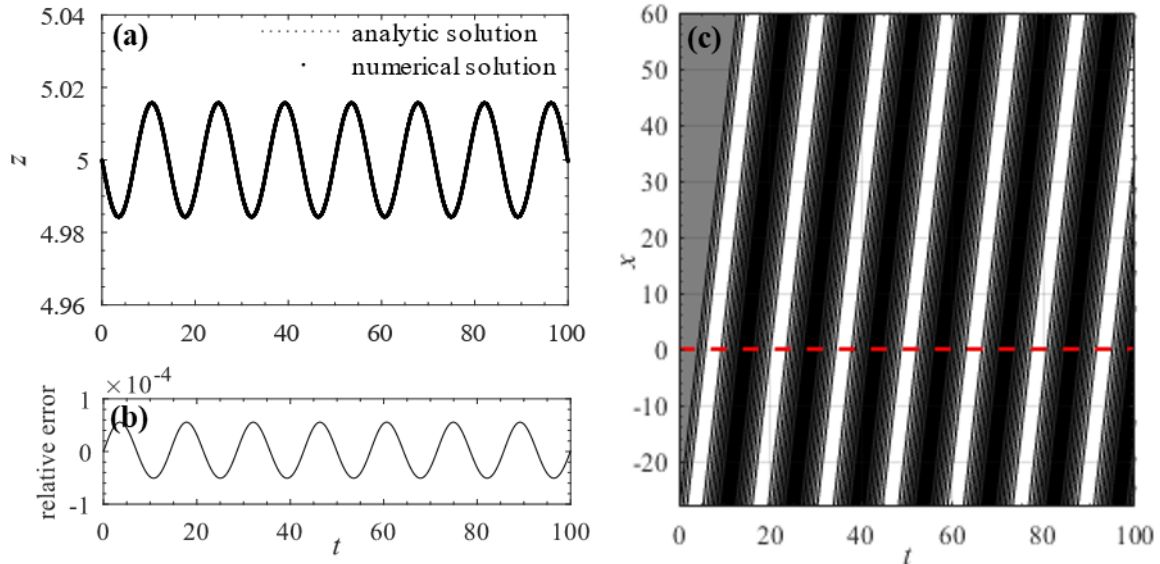


564

565 **Fig. 12.** Relatively large amplitude, high frequency sinusoidal waves (case 1,  $A=1.139$  m,  
566  $T=3.5696$  s,  $h_s=5$  m): (a) free surface elevation time series at piston-sided first cell; (b) relative error  
567 based on (a); and (c) wave generation and propagation in the phase plane. The red dashed line indicates

568 the interface between main (upper) and paddle (lower) sub-domains.

569



570

571 **Fig. 13.** Small-amplitude, low frequency sinusoidal waves (case 15,  $A = 0.016$  m,  $T = 14.2784$  s,  
572  $h_s = 5$  m): (a) free surface elevation time series at piston-sided first cell; (b) relative error based on (a);  
573 and (c) wave generation and propagation in the phase plane. The red dashed line indicates the interface  
574 between main (upper) and paddle (lower) sub-domains.

575

### 576 4.3.2. Solitary Waves

577 We now consider the generation of a solitary wave, with free surface profile given by

$$578 \eta_a(x, t) = H \operatorname{sech}^2(\kappa(x - Ct)) + h_s, \quad (55)$$

579 where  $h_s$  is the still water depth,  $H$  is the wave height of the solitary wave, the wave number

$$580 \kappa = \sqrt{3H / (4h_s^3)}, \text{ and the wave celerity } C = \sqrt{g(H + h_s)} \quad (\text{Goring, 1978}). \text{ In the numerical model, the}$$

581 paddle displacement signal required to produce the foregoing solitary wave is obtained using the

582 Newton method by solving the implicit equation

$$583 x_p = \frac{H}{\kappa h} \tanh(\kappa(x_p - Ct)). \quad (56)$$

584 Table 5 lists five cases that were simulated. Fig. 14 and Fig. 15 show the results obtained for a large

585 wave height ( $H = 1$  m), short duration ( $T = 10.5729$  s) solitary wave and small wave height ( $H = 0.05$

586 m), long duration ( $T = 51.53936$  s) solitary wave, both propagating over water of still depth 5 m. In

587 both cases, reasonable agreement is achieved between the numerical predictions and analytical solution.  
 588 Again, as would be expected, the larger wave height and shorter the period of the wave, the greater the  
 589 error.

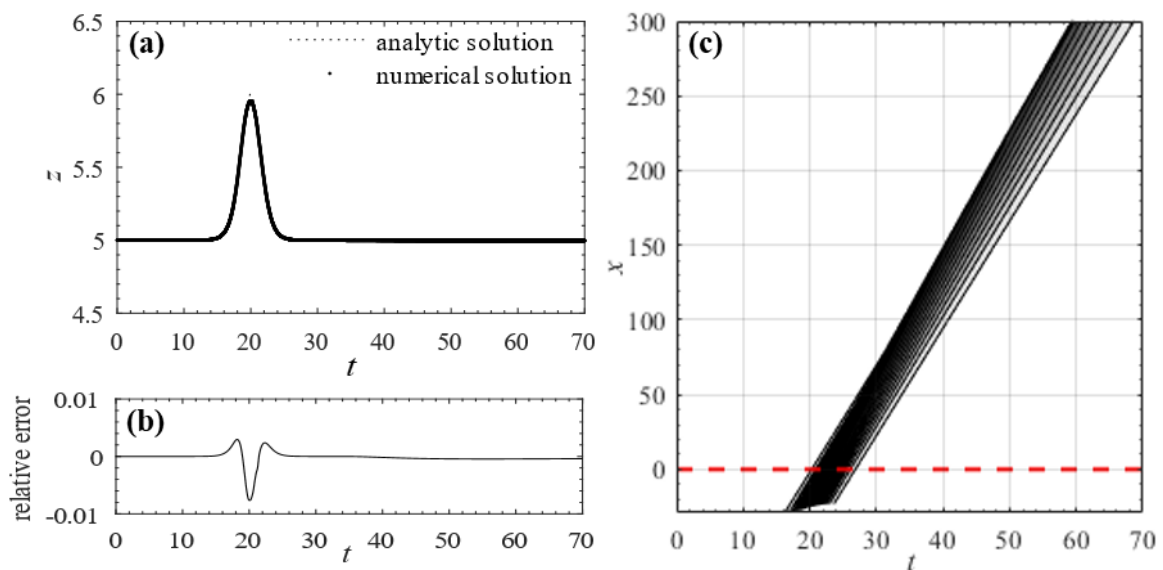
590

591 **Table 5.** Solitary wave test parameters.

Cases	$Ur$	$H/L$	$h_s/L$	$H/h_s$	$L$ (m)	$T$ (s)	$H$ (m)	$h_s$ (m)
1	52.63789	0.012328	0.06164	0.2	81.11557	10.5729	1	5
2	52.63789	0.004359	0.043586	0.1	114.7147	15.61721	0.5	5
3	52.63789	0.001103	0.027566	0.04	181.3799	25.39528	0.2	5
4	52.63789	0.00039	0.019492	0.02	256.51	36.26475	0.1	5
5	52.63789	0.000138	0.013783	0.01	362.7599	51.53936	0.05	5

592  $Ur$  is Ursell number,  $H$  is wave height,  $L$  is wavelength,  $h_s$  is still-water depth, and  $T$  is wave period.

593

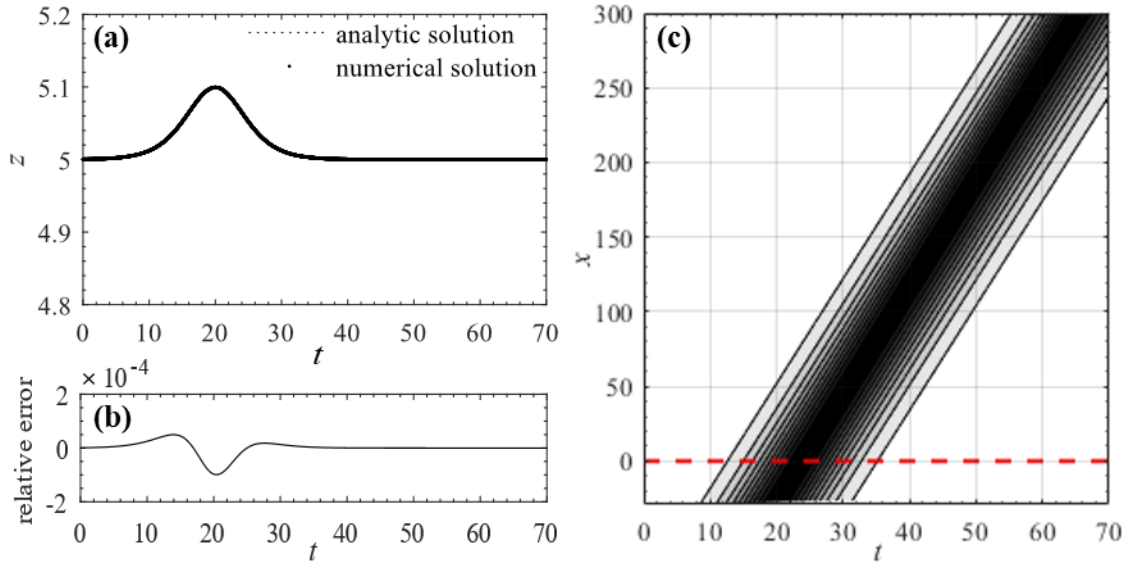


594

595 **Fig. 14.** Relatively large wave height, short duration solitary wave (case 1,  $H = 1$  m,  $T = 10.5729$  s,  
 596  $h_s = 5$  m): (a) free surface elevation time series at piston-sided first cell; (b) relative error based on (a);  
 597 and (c) wave generation and propagation in the phase plane. The red dashed line indicates the interface  
 598 between main (upper) and paddle (lower) sub-domains.

599





600

601 **Fig. 15.** Small wave height, long duration solitary wave (case 5,  $H = 0.05$  m,  $T = 51.53936$  s,  $h_s = 5$   
 602 m): (a) free surface elevation time series at piston-sided first cell; (b) relative error based on (a); and (c)  
 603 wave generation and propagation in the phase plane. The red dashed line indicates the interface between  
 604 main (upper) and paddle (lower) sub-domains.

605

### 606 4.3.3. Cnoidal Waves

607 As a periodic solution of the Korteweg–de Vries (KdV) equation, the time-dependent free surface  
 608 profile of a cnoidal wave may be written (Korteweg and de Vries, 1895; Svendsen, 1974):

$$609 \quad \eta_a(x, t) = y_t - h_s + H cn^2 \left\{ 2K \left( \frac{x}{L} - \frac{t}{T} \right) \right\}_m, \quad (57)$$

610 where  $\eta_a$  is the analytic solution of water elevation,  $y_t$  is the height of the wave trough above a datum,

611  $h_s$  is the still water depth,  $H$  is the wave height,  $cn$  is a Jacobian elliptic function,  $K = K(m)$  is a

612 complete elliptic integral of the first kind in which  $m$  is the elliptic parameter,  $L$  is the wave length

613 and  $T$  is the wave period. For given depth  $h_s$ , the cnoidal wave is determined knowing any two among

614  $L$  (or  $T$ ),  $H$ , and  $m$ , where the relationship between the variables is as follows:

$$615 \quad y_t = \frac{H}{Km} (K - E) + h_s - H, \quad (58)$$

$$616 \quad \frac{L}{T} = \sqrt{gh_s} \left\{ 1 + \frac{H}{h_s} \left[ \left( \frac{2}{m} \right) - \left( \frac{3E}{mK} \right) \right] \right\}, \quad (59)$$

617 and

$$618 \quad \frac{HL^2}{h_s^3} = \frac{16}{3} K^2 m, \quad (60)$$

619 where  $E$  is the complete elliptic integral of the second kind. The paddle displacement signal required  
620 to produce this cnoidal wave is obtained using the Newton method from the following implicit equation,

$$621 \quad x_p(t) = \frac{L}{2Kh_s} \left[ (y_t - h_s)\theta + \frac{H}{m} E(\theta|_m) - (1-m)\theta \right], \quad (61)$$

622 where  $\theta = 2K(t/T - x_p/L)$ , and  $E(\theta|_m)$  is the incomplete elliptic integral of the second kind  
623 (Goring, 1978). Table 6 lists the 23 cases tested. Fig. 16 and Fig. 17 display results for a large wave  
624 height, high frequency cnoidal waves (case 7:  $H = 1$  m,  $T = 10.9553$  s,  $h_s = 5$  m) and small wave  
625 height, low frequency cnoidal waves (case 9:  $H = 0.1$  m,  $T = 41.8746$  s,  $h_s = 5$  m). Very satisfactory  
626 agreement is achieved between the analytical solution and numerical predictions, with larger errors  
627 obtained for the higher wave height and frequency case.

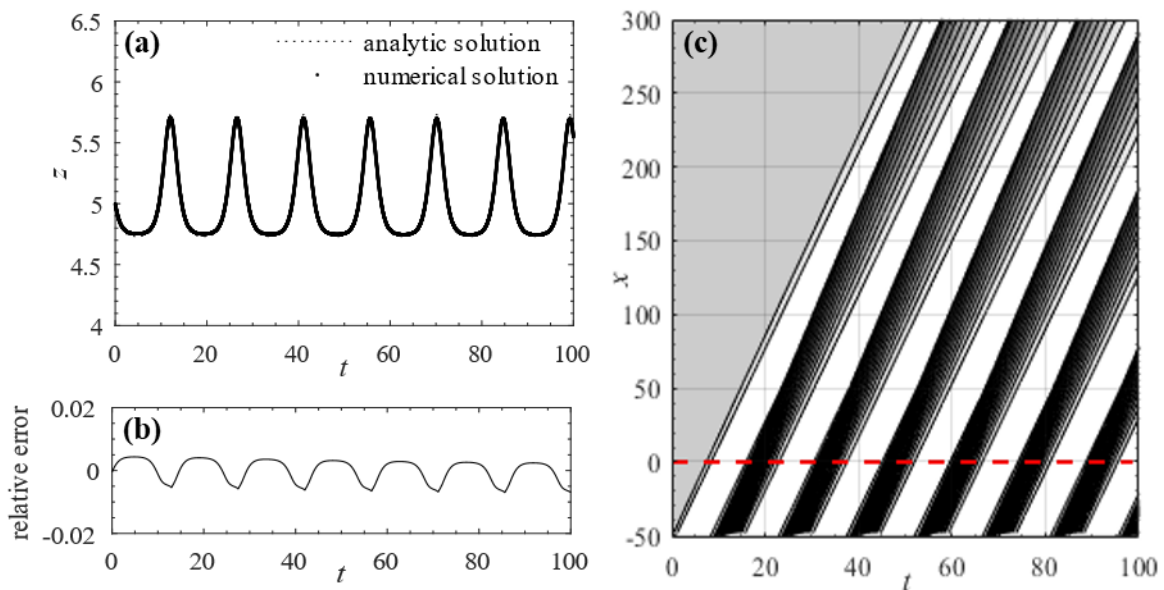
628

629 **Table 6.** Cnoidal wave test parameters.

Cases	Ur	H/L	h <sub>s</sub> /L	H/h <sub>s</sub>	L (m)	T (s)	H (m)	h <sub>s</sub> (m)	m
1	35.3351	0.0053	0.0532	0.1	93.9882	12.3665	0.5	5	0.92
2	35.3351	0.015	0.0752	0.2	66.4597	8.1079	1	5	0.92
3	42.856	0.0137	0.0683	0.2	73.1915	8.7764	1	5	0.95
4	42.856	0.008	0.0572	0.14	87.4806	11.0194	0.7	5	0.95
5	42.856	0.0012	0.0306	0.04	163.6613	22.5095	0.2	5	0.95
6	42.856	0.0002	0.0153	0.01	327.3226	46.295	0.05	5	0.95
7	72.1128	0.0105	0.0527	0.2	94.9426	10.9553	1	5	0.99
8	72.1128	0.0037	0.0372	0.1	134.2692	17.1371	0.5	5	0.99
9	72.1128	0.0003	0.0167	0.02	300.235	41.8746	0.1	5	0.99
10	72.1128	0.0001	0.0118	0.01	424.5964	59.9144	0.05	5	0.99
11	72.1128	0.0147	0.0589	0.25	33.9677	5.9136	0.5	2	0.99
12	72.1128	0.0037	0.0372	0.1	53.7077	10.8385	0.2	2	0.99
13	72.1128	0.0013	0.0263	0.05	75.9541	16.1868	0.1	2	0.99
14	72.1128	0.0005	0.0186	0.025	107.4153	23.5513	0.05	2	0.99
15	31.9035	0.002	0.0396	0.05	50.5201	10.9701	0.1	2	0.9
16	31.9035	0.0056	0.056	0.1	35.7231	7.4718	0.2	2	0.9
17	124.87	0.0112	0.0447	0.25	44.6981	7.5022	0.5	2	0.999

18	124.87	0.0028	0.0283	0.1	70.6739	14.0202	0.2	2	0.999
19	124.87	0.001	0.02	0.05	99.948	21.1076	0.1	2	0.999
20	191.4429	0.0008	0.0162	0.05	123.7555	25.9909	0.1	2	0.9999
21	191.4429	0.0023	0.0229	0.1	87.5084	17.1802	0.2	2	0.9999
22	272.0997	0.0007	0.0136	0.05	147.5397	30.8701	0.1	2	0.99999
23	2081.323	0.0007	0.0069	0.1	288.5358	54.9794	0.2	2	1

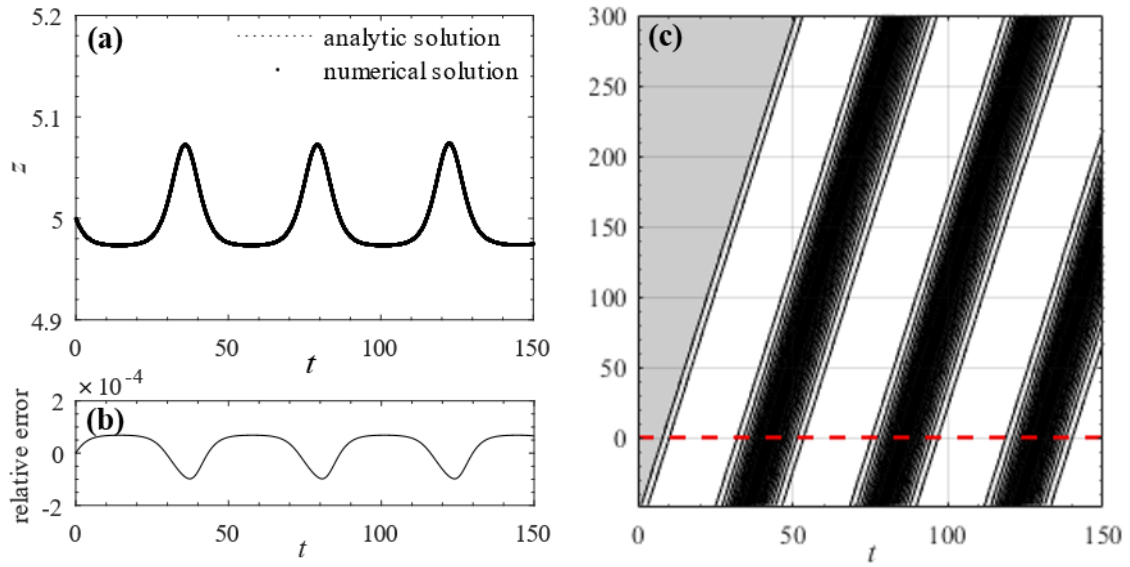
630  $Ur$  is Ursell number,  $H$  is wave height,  $L$  is wavelength,  $h_s$  is still-water depth,  $T$  is wave period, and  $m$  is a shape  
631 factor also called elliptic parameter. The larger the value of  $m$ , the sharper the wave profile.  
632



633

634 **Fig. 16.** Relatively large wave height, high frequency cnoidal waves (case 7,  $H = 1$  m,  $T = 10.9553$  s,  
635  $h_s = 5$  m): (a) free surface elevation time series at piston-sided first cell; (b) relative error based on (a);  
636 and (c) wave generation and propagation in the phase plane. The red dashed line indicates the interface  
637 between main (upper) and paddle (lower) sub-domains.

638



639

640 **Fig. 17.** Small wave height, low frequency cnoidal waves (case 9,  $H = 0.1$  m,  $T = 41.8746$  s,  $h_s = 5$   
641 m): (a) free surface elevation time series at piston-sided first cell; (b) relative error based on (a); and (c)  
642 wave generation and propagation in the phase plane. The red dashed line indicates the interface between  
643 main (upper) and paddle (lower) sub-domains.

644

#### 645 4.3.4. Error Analysis and the Appropriate Usage Criterion for the Piston Paddle

646 The trend in errors is examined for the fifty cases considered; i.e., 22 sinusoidal, 5 solitary, and 23  
647 cnoidal waves. The relative error is determined from:

$$648 \quad \varepsilon_n^* = \frac{\eta_1^n - \eta_a(x_p, t_n)}{\eta_a(x_p, t_n)}, \quad (62)$$

649 where  $n$  is the time level,  $t_n$  is time associated with  $n$ ,  $\varepsilon_n^*$  is relative error at time  $t_n$ ,  $\eta_1^n$  is the  
650 numerically predicted water elevation at the first cell at time  $t_n$ , and  $\eta_a(x_p, t_n)$  is the analytic solution  
651 of water elevation at the piston paddle; obtained from Eqs. (53), (55), and (57) for sinusoidal, solitary,  
652 and cnoidal waves. Table 7 lists correlations between  $L_1 / L_\infty$  errors and non-dimensional Ursell  
653 number, wave steepness, and nonlinearity parameters. Here, the relative errors ( $L_1$ ,  $L_\infty$ ) exhibit greatest  
654 correlation with wave steepness,  $H / L$  (Table 7, Figs. 19(a) and 19(b)). In Stokes perturbation theory,  
655 the velocity potential is expressed as

$$656 \quad \phi^* = \phi_1^* + \delta \phi_2^* + \delta^2 \phi_3^* + \dots \quad (63)$$

657 where the wave steepness,  $\delta = H/L$ , is the perturbation parameter and superscript \* represents  
 658 dimensionless form. The dynamic boundary condition (DBC) is thus given by (Dean and Dalrymple,  
 659 1991),

$$660 \quad p^* + \delta^2 \left( \frac{(\partial\phi^*/\partial x^*)^2 + (\partial\phi^*/\partial z^*)^2}{2} \right) - \delta \frac{\partial\phi^*}{\partial t^*} + z^* = 0, \quad (64)$$

661 which simplifies to hydrostatic pressure when the wave steepness is very small (in accordance with the  
 662 hydrostatic assumption in the shallow water equations). Table 7 indicates that  $h_s/L$  is less important  
 663 in determining accuracy than wave steepness, implying that the hydrostatic approximation ( $H/L \ll 1$ )  
 664 is a more important influence factor than the long wave approximation ( $h_s/L < 0.05$ ) in wave  
 665 generation in the region of the shallow water assumptions. In other words, the numerical piston paddle  
 666 generates accurate waveforms provided the wave steepness is sufficiently small that the hydrostatic  
 667 assumption is satisfied. Table 7 also indicates that the errors correlate with  $(H/L)(h_s/L)^2$ . This is  
 668 reasonable because the difference between the non-dimensional Boussinesq and shallow water  
 669 momentum equations can be expressed by the dispersion term  $(H/L)(h_s/L)^2 U_{x^*x^*t^*}^*$  (Goring, 1978).  
 670 Fig. 18 depicts the almost linear relationships between the relative errors and the two non-dimensional  
 671 parameters (wave steepness and  $(H/L)(h_s/L)^2$ ). The scatter plots indicate that stable, highly  
 672 accurate waves are generated by the numerical wave tank, provided the non-dimensional numbers are  
 673 suitably small. For example, to generate shallow water waves with a relative  $L_1$ -error  $< 1\%$ , it is  
 674 necessary to ensure  $H/h_s < 3.97 \times 10^{-2}$  and  $(H/L)(h_s/L)^2 < 1.42 \times 10^{-3}$  (Table 8 and Fig. 19).

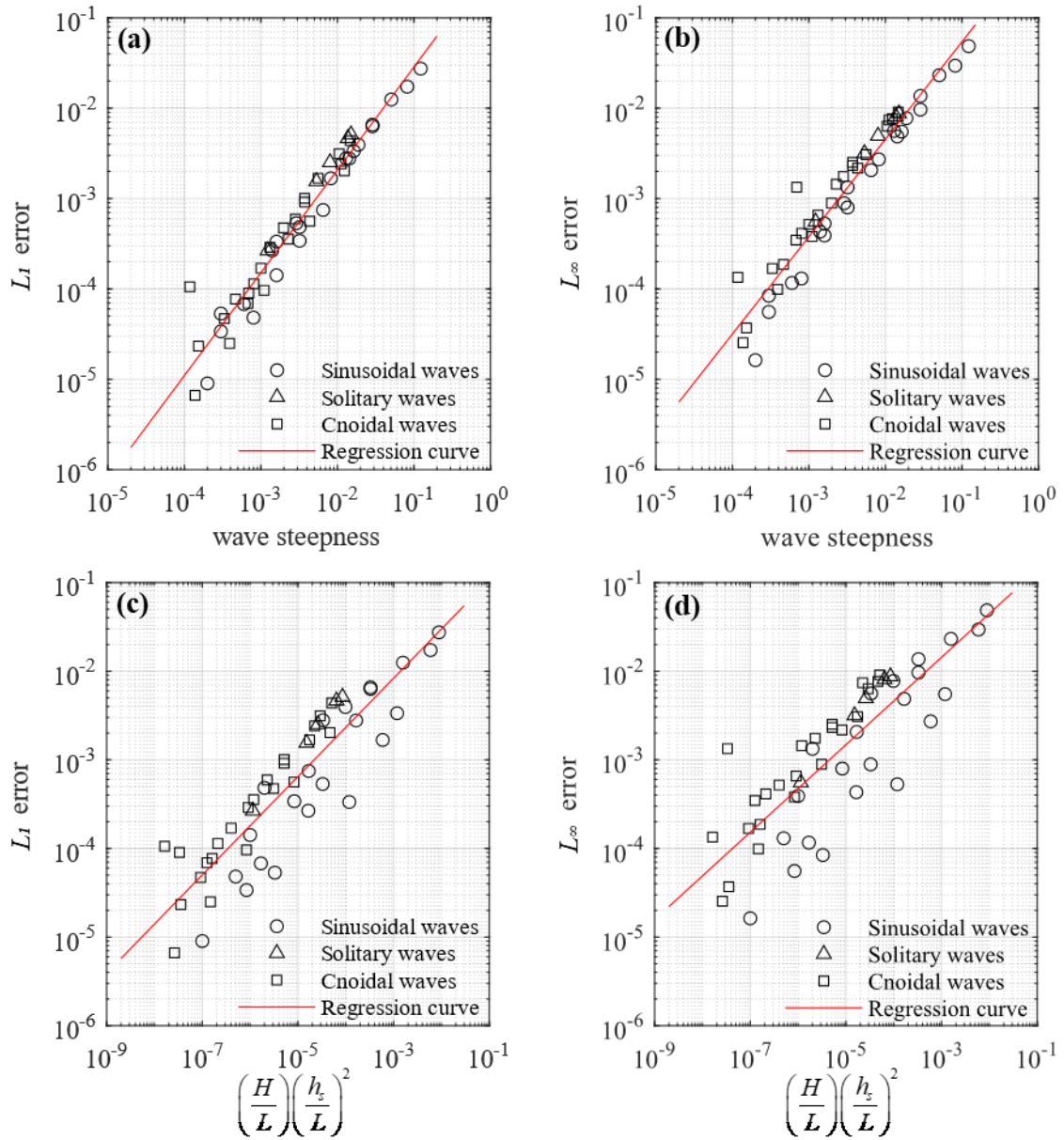
675

676 **Table 7.** Correlations between  $L_1$  and  $L_\infty$  errors and non-dimensional numbers

	Ur	$H/L$	$h_s/L$	$H/h_s$	$\left(\frac{H}{L}\right)\left(\frac{h_s}{L}\right)^2$
$L_1$	-0.121	0.995	0.634	0.812	0.925
$L_\infty$	-0.103	0.990	0.611	0.843	0.909

677 Ur is Ursell number,  $H$  is wave height,  $L$  is wavelength, and  $h_s$  is still-water depth.

678



679

680 **Fig. 18.** Relative errors plotted against non-dimensional numbers: (a)  $L_1$  error with respect to  $H/L$   
 681 ( $H/L$ ); (b)  $L_\infty$  error with respect to  $H/L$  ( $H/L$ ); (c)  $L_1$  error with respect to  
 682  $(H/L)(h_s/L)^2$ ; and (d)  $L_\infty$  error with respect to  $(H/L)(h_s/L)^2$ .

683

684

685

686

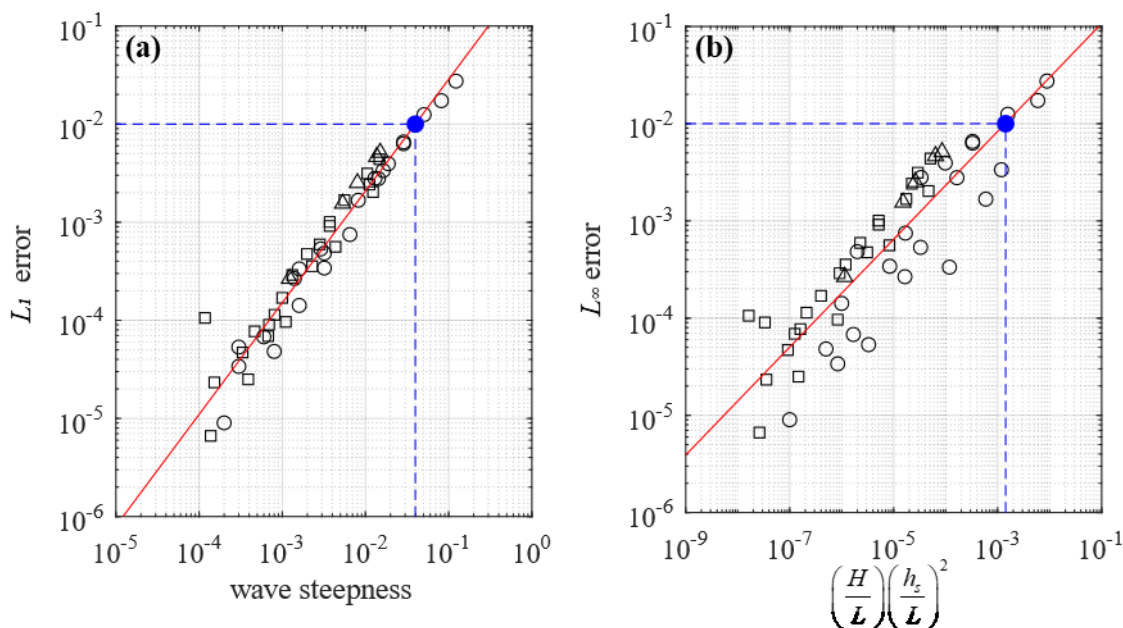
687

688

689 **Table 8.** Relative error in non-dimensional numbers,  $H/L$  and  $(H/L)(h_s/L)^2$ , from linear  
 690 regression

	Error	5%	1%	0.5%	0.1%
$H/L$	$L_1$	1.63E-01	3.97E-02	2.16E-02	5.26E-03
	$L_\infty$	9.27E-02	2.08E-02	1.09E-03	2.46E-03
$\left(\frac{H}{L}\right)\left(\frac{h_s}{L}\right)^2$	$L_1$	2.55E-02	1.42E-03	3.98E-04	2.20E-05
	$L_\infty$	1.29E-02	4.80E-04	1.20E-04	4.53E-06

691



692

693 **Fig. 19.** Example of user criteria for 1% threshold in relative  $L_1$  error with respect to (a)  $H/L$  (wave  
 694 steepness); and (b)  $(H/L)(h_s/L)^2$ . Circular, triangular and squared symbols represent sinusoidal,  
 695 solitary and cnoidal waves; the red solid lines are regression curves; and the blue dashed lines indicate  
 696 the threshold of 1% in relative  $L_1$  error.

697

698

## 699 5. Conclusions

700 A high-order numerical wave tank has been presented based on the shallow water equations to  
 701 simulate long wave phenomena that satisfy the hydrostatic pressure assumption. The governing  
 702 equations were formulated as a well-balanced hyperbolic system and solved using a fifth-order WENO  
 703 scheme in space and third-order Runge-Kutta method in time, with a specialized cut-off algorithm used  
 704 to prevent accumulation of round-off errors. The resulting high-order scheme was computationally

705 efficient, and produced accurate, stable, long-duration simulations. To model a piston-type paddle, the  
706 computational domain was divided into two sub-domains, one a moving sub-domain adjacent to the  
707 paddle, the other a fixed sub-domain representing the remainder of the tank. A mapped version of the  
708 shallow water equations was solved using a modified version of the fifth-order WENO method in the  
709 sub-domain adjacent to the paddle, and specialized interface boundary conditions implemented at the  
710 join between the two sub-domains. Error analysis suggested criteria for wave simulations by the present  
711 numerical model. The model has been verified extensively. Still water tests demonstrated the well-  
712 balanced C-property of the numerical scheme. Fifth-order accuracy for the smooth solution was  
713 demonstrated for a test case originally devised by [Xing and Shu \(2005\)](#). Discontinuous and trans-critical  
714 flow tests confirmed the shock-capturing ability of the present solver, and its correct reproduction of  
715 steady and time-dependent flows. The numerical wave tank was used to generate sinusoidal, solitary,  
716 and cnoidal waves; in each case the model predictions agreed well with analytical solutions, provided  
717 criteria that limited wave steepness and dispersion were met. Even so, in a few cases, the model tended  
718 to underestimate slightly the wave amplitude owing to the hydrostatic assumption, causing the wave  
719 steepness and dispersion criteria to be over-restrictive. Future development of the high-order numerical  
720 wave tank should therefore include extension to non-hydrostatic pressure.

721

## 722 **Appendix.**

### 723 **Appendix A. Propositions**

724 The following propositions relate to the properties required by numerical schemes to satisfy the C-  
725 property for [Liang and Borthwick \(2009\)](#)'s version of balanced shallow water equations (Eqs. (1) and  
726 (6)). Propositions 1 and 2 refer to well-balanced conditions on Cartesian and linear-mapped grids,  
727 respectively.

728

729 **Proposition 1.** If the same linear operator  $W$  for approximating an  $x$ -derivative,  $W(\cdot) \approx \frac{\partial}{\partial x}(\cdot)$ ,

730 satisfying  $W(\text{const.})=0$  is applied to the each  $x$ -derivative term in Eq. (6), the exact C-property is



731 satisfied. Here  $W(\text{const.})=0$  is a kind of consistency condition, noting that the  $x$ -derivative of a  
732 constant function is by definition zero.

733

734 *Proof.* Substituting the stationary condition Eq. (4) into Eqs. (1) and (6), we obtain

$$735 \quad \frac{\partial}{\partial x} \left( \frac{1}{2} g(\eta^2 - 2\eta b) \right) = -g\eta \frac{\partial b}{\partial x} \quad (65)$$

736 for stationary cases. Applying the linear operator  $W(\cdot)$  to each  $x$ -derivative term in Eq. (65),

$$737 \quad W \left( \frac{1}{2} g(\eta^2 - 2\eta b) \right) = -g\eta W(b) \quad (66)$$

738 By linearity of the operator  $W$ , this becomes

$$739 \quad \frac{1}{2} g\eta^2 W(\mathbf{1}) - g\eta W(b) = -g\eta W(b) \quad (67)$$

740 which reduces to  $W(\mathbf{1})=0$ . Thus, if an identical linear numerical scheme satisfying  $W(\text{const.})=0$  is  
741 applied to every  $x$ -differential term in Eqs. (1) and (6), the model satisfies the exact C-property.  $\square$

742

743 **Proposition 2.** If the same linear operator  $W$  satisfying consistency is applied to each  $x$ -differential  
744 term in Eq. (29), the equation satisfies the C-property.

745

746 *Proof.* When Eq. (4) is applied to Eq. (29), Eq. (66) is obtained. The subsequent procedure is identical  
747 to that of Proposition 1.  $\square$

748

749 Note that Eq. (4) is trivial in the mass conservation part of Eq. (29) because all the  $x$ -derivative terms  
750 in Eq. (29) satisfy the conservation property. This means that the linear numerical operator satisfying  
751 exact mass conservation implies consistency for the differential operators.

752

## 753 **Appendix B. $L_1$ and $L_\infty$ Errors**

754 The  $L_1$  and  $L_\infty$  errors are defined as:

$$755 \quad L_1 = \frac{1}{t_{total}} \sum_{n=1}^M |\varepsilon_n| \Delta t_n \quad (68)$$

756 And

$$757 \quad L_\infty = \max_n |\varepsilon_n| \quad (69)$$

758 where  $n$  is the time index,  $M$  is the number of time steps in the simulation,  $t_{total}$  is the total  
759 simulation time ( $t_{total} = \sum_{n=1}^M \Delta t_n$ ),  $\Delta t_n$  is the  $n$ -th time step, and  $\varepsilon_n$  is the error at time  $t_n$ .

760

## 761 **References**

- 762 Agamloh, E. B., Wallace, A. K., and Von Jouanne, A. (2008). "Application of fluid–structure interaction  
763 simulation of an ocean wave energy extraction device." *Renewable Energy*, Vol. 33, No. 4, pp. 748-757,  
764 DOI: 10.1016/j.renene.2007.04.010.
- 765 Aly, A. M., and Bitsuamlak, G. (2013). "Aerodynamics of ground-mounted solar panels: test model scale effects."  
766 *Journal of Wind Engineering and Industrial Aerodynamics*, Vol. 123, pp. 250-260, DOI:  
767 10.1016/j.jweia.2013.07.007.
- 768 Bermúdez, A. and Vázquez-Cendón, M. E. (1994). "Upwind methods for hyperbolic conservation laws with  
769 source terms." *Computers and Fluids*, Vol. 23, No. 8, pp. 1049-1071, DIO: 10.1016/0045-7930(94)90004-  
770 3.
- 771 Blayo, E., and Debreu, L. (2005). "Revisiting open boundary conditions from the point of view of characteristic  
772 variables." *Ocean modelling*, Vol. 9, No. 3, pp. 231-252, DOI: 10.1016/j.ocemod.2004.07.001.
- 773 Bonev, B., Hesthaven, J. S., Giraldo, F. X., and Kopera, M. A. (2018). "Discontinuous Galerkin scheme for the  
774 spherical shallow water equations with applications to tsunami modeling and prediction." *Journal of*  
775 *Computational Physics*, Vol. 362, pp. 425-448, DOI: 10.1016/j.jcp.2018.02.008.
- 776 Boo, S. Y. (2002). "Linear and nonlinear irregular waves and forces in a numerical wave tank." *Ocean*  
777 *Engineering*, Vol. 29, No. 5, pp. 475-493, DOI: 10.1016/s0029-8018(01)00055-5.
- 778 Castro, M., Gallardo, J., and Parés, C. (2006). "High order finite volume schemes based on reconstruction of states  
779 for solving hyperbolic systems with nonconservative products. Applications to shallow-water systems."  
780 *Mathematics of computation*, Vol. 75, No. 255, pp. 1103-1134, DOI: 10.1090/s0025-5718-06-01851-5.
- 781 Chen, J., Jiang, C., Yang, W., and Xiao, G. (2016). "Laboratory study on protection of tsunami-induced scour by  
782 offshore breakwaters." *Natural Hazards*, Vol. 81, No. 2, pp. 1229-1247, DOI: 10.1007/s11069-015-2131-x.
- 783 Dean, R. G., and Dalrymple, R. A. (1991). *Water wave mechanics for engineers and scientists*. Vol. 2. World

784 Scientific Publishing Company, DOI: 10.1142/1232.

785 Finnegan, W., and Goggins, J. (2015). "Linear irregular wave generation in a numerical wave tank." *Applied*  
786 *Ocean Research*, Vol. 52, pp. 188-200, DOI: 10.1016/j.apor.2015.06.006.

787 Gao, Z., and Hu, G. (2017). "High Order Well-Balanced Weighted Compact Nonlinear Schemes for Shallow  
788 Water Equations." *Communications in Computational Physics*, Vol. 22, No. 4, pp. 1049-1068, DOI:  
789 10.4208/cicp.OA-2016-0200.

790 Giraldo, F. X., Hesthaven, J. S., and Warburton, T. (2002). "Nodal high-order discontinuous Galerkin methods  
791 for the spherical shallow water equations." *Journal of Computational Physics*, Vol. 181, No. 2, pp. 499-525,  
792 DOI: 10.21236/ada633613.

793 Goldberg, D. (1991). "What every computer scientist should know about floating-point arithmetic." *ACM*  
794 *Computing Surveys (CSUR)*, Vol. 23, No. 1, pp. 5-48, DOI: 10.1145/103162.103163.

795 Goring, D. G. (1978). *Tsunamis--the propagation of long waves onto a shelf*. Report No KH-R-38, W. M. Keck  
796 Laboratory of Hydraulics and Water Resources Division of Engineering and Applied Science, California  
797 Institute of Technology, Pasadena, California.

798 Goseberg, N., Wurpts, A., and Schlurmann, T. (2013). "Laboratory-scale generation of tsunami and long waves."  
799 *Coastal Engineering*, Vol. 79, pp. 57-74, DOI: 10.1016/j.coastaleng.2013.04.006.

800 Gottlieb, S., and Shu, C. W. (1998). "Total variation diminishing Runge-Kutta schemes." *Mathematics of*  
801 *Computation*, Vol. 67, No. 221, pp. 73-85, DOI: 10.1090/s0025-5718-98-00913-2.

802 Greenberg, J. M., and LeRoux, A. Y. (1996). "A well-balanced scheme for the numerical processing of source  
803 terms in hyperbolic equations." *SIAM Journal on Numerical Analysis*, Vol. 33, No. 1, pp. 1-16, DOI:  
804 10.1137/0733001.

805 Guizien, K., and Barthélemy, E. (2002). "Accuracy of solitary wave generation by a piston wave maker." *Journal*  
806 *of Hydraulic Research*, Vol. 40, No. 3, pp. 321-331, DOI: 10.1080/00221680209499946.

807 Hornsby, C. (2002). "CFD-driving pump design forward." *World Pumps*, Vol. 2002, No. 431, pp. 18-22, DOI:  
808 10.1016/S0262-1762(02)80195-X.

809 Hughes, S. A. (1993). *Physical models and laboratory techniques in coastal engineering*, Advanced Series in  
810 Ocean Engineering Vol. 7, World Scientific, Singapore.

811 Jiang, G. S., and Shu, C. W. (1996). "Efficient implementation of weighted ENO schemes." *Journal of*  
812 *computational physics*, Vol. 126, No. 1, pp. 202-228, DOI: 10.1006/jcph.1996.0130.

813 Khayyer, A., Gotoh, H., and Shao, S. D. (2008). "Corrected incompressible SPH method for accurate water-  
814 surface tracking in breaking waves." *Coastal Engineering*, Vol. 55, No. 3, pp. 236-250, DOI:  
815 10.1016/j.coastaleng.2007.10.001.

816 Koo, W., and Kim, M. H. (2004). "Freely floating-body simulation by a 2D fully nonlinear numerical wave tank."  
817 *Ocean Engineering*, Vol. 31, No. 16, pp. 2011-2046, DOI: 10.1016/j.oceaneng.2004.05.003.

818 Korteweg, D. J., and de Vries, G. (1895). "XLI. On the change of form of long waves advancing in a rectangular  
819 canal, and on a new type of long stationary waves." *The London, Edinburgh, and Dublin Philosophical*  
820 *Magazine and Journal of Science*, Vol. 39, No. 240, pp. 422-443, DOI: 10.1080/14786449508620739.

821 LeVeque, R. J. (1998). "Balancing source terms and flux gradients in high-resolution Godunov methods: the  
822 quasi-steady wave-propagation algorithm." *Journal of Computational Physics*, Vol. 146, No. 1, pp. 346-365,  
823 DOI: 10.1006/jcph.1998.6058.

824 Li, G., Lu, C., and Qiu, J. (2012). "Hybrid well-balanced WENO schemes with different indicators for shallow  
825 water equations." *Journal of Scientific computing*, Vol. 51, No. 3, pp. 527-559, DOI: 10.1007/s10915-011-  
826 9520-4.

827 Li, G., Caleffi, V., and Qi, Z. (2015). "A well-balanced finite difference WENO scheme for shallow water flow  
828 model." *Applied Mathematics and Computation*, Vol. 265, pp. 1-16, DOI: 10.1016/j.amc.2015.04.054.

829 Li, G., Song, L., and Gao, J. (2018). "High order well-balanced discontinuous Galerkin methods based on  
830 hydrostatic reconstruction for shallow water equations." *Journal of Computational and Applied  
831 Mathematics*, Vol. 340, pp. 546-560, DOI: 10.1016/j.cam.2017.10.027.

832 Liang, Q., and Borthwick, A. G. L. (2009). "Adaptive quadtree simulation of shallow flows with wet-dry fronts  
833 over complex topography." *Computers and Fluids*, Vol. 38, No. 2, pp. 221-234, DOI:  
834 10.1016/j.compfluid.2008.02.008.

835 Liang, X. F., Yang, J. M., Jun, L. I., Xiao, L. F., and Xin, L. I. (2010). "Numerical simulation of irregular wave-  
836 simulating irregular wave train." *Journal of Hydrodynamics*, Ser. B, Vol. 22, No. 4, pp. 537-545, DOI:  
837 10.1016/s1001-6058(09)60086-x.

838 Liu, P. L.-F., Cho, Y.-S., Briggs, M. J., Kanoglu, U., and Synolakis, C. E. (1995). "Runup of solitary waves on a  
839 circular island." *Journal of Fluid Mechanics*, Vol. 302, pp. 259-285, DOI: 10.1017/s0022112095004095.

840 Lukáčová-Medvid'ová, M., Noelle, S., and Kraft, M. (2007). "Well-balanced finite volume evolution Galerkin  
841 methods for the shallow water equations." *Journal of Computational Physics*, Vol. 221, No. 1, pp. 122-147,  
842 DOI: 10.1016/j.jcp.2006.06.015.

843 Mader, C. L. (2004). *Numerical modeling of water waves*. 2nd Edition, CRC Press, Boca Raton.

844 Madsen, O. S. (1970). "Waves generated by a piston-type wavemaker." *Proc. of 12<sup>th</sup> Int. Conf. on Coastal  
845 Engineering.*, ASCE, Washington, D.C., USA, pp. 589-607, DOI: 10.1061/9780872620285.036.

846 Monaghan, J. J., and Kos, A. (2000). "Scott Russell's wave generator." *Physics of Fluids*, Vol. 12, No. 3, pp. 622-  
847 630, DOI: 10.1063/1.870269.

848 Ning, D. Z., and Teng, B. (2007). "Numerical simulation of fully nonlinear irregular wave tank in three dimension."  
849 *International Journal for numerical methods in fluids*, Vol. 53, No. 12, pp. 1847-1862, DOI:  
850 10.1002/flid.1385.

851 Ning, D. Z., Teng, B., Taylor, R. E., and Zang, J. (2008). "Numerical simulation of non-linear regular and focused  
852 waves in an infinite water-depth." *Ocean Engineering*, Vol. 35, No. 8-9, pp. 887-899, DOI:  
853 10.1016/j.oceaneng.2008.01.015.

854 Noelle, S., Xing, Y., and Shu, C. W. (2007). "High-order well-balanced finite volume WENO schemes for shallow  
855 water equation with moving water." *Journal of Computational Physics*, Vol. 226, No. 1, pp. 29-58, DOI:  
856 10.1016/j.jcp.2007.03.031.

857 Orszaghova, J., Borthwick, A. G. L., and Taylor, P. H. (2012). "From the paddle to the beach - A Boussinesq

858 shallow water numerical wave tank based on Madsen and Sørensen's equations." *Journal of Computational*  
859 *Physics*, Vol. 231, No. 2, pp. 328-344, DOI: 10.1016/j.jcp.2011.08.028.

860 Park, J. C., Uno, Y., Sato, T., Miyata, H., and Chun, H. H. (2004). "Numerical reproduction of fully nonlinear  
861 multi-directional waves by a viscous 3D numerical wave tank." *Ocean Engineering*, Vol. 31, No. 11-12, pp.  
862 1549-1565, DOI: 10.1016/j.oceaneng.2003.12.009.

863 Previsic, M., Shoele, K., and Epler, J. (2014). "Validation of theoretical performance results using wave tank  
864 testing of heaving point absorber wave energy conversion device working against a subsea reaction plate."  
865 *In 2nd Marine Energy Technology Symposium*, Seattle, Washington, USA, pp. 1-8.

866 Rogers, B. D., Borthwick, A. G. L., and Taylor, P. H. (2003). "Mathematical balancing of flux gradient and source  
867 terms prior to using Roe's approximate Riemann solver." *Journal of Computational Physics*, Vol. 192, No.  
868 2, pp. 422-451, DOI: 10.1016/j.jcp.2003.07.020.

869 Schimmels, S., Sriram, V., Didenkulova, I., and Fernández, H. (2014). "On the generation of tsunami in a large  
870 scale wave flume." *Proc. of 34<sup>th</sup> Int. Conf. on Coastal Engineering.*, Seoul, S. Korea, Vol. 1, No. 34, currents,  
871 14, DOI: 10.9753/icce.v34.currents.14.

872 Schimmels, S., Sriram, V., and Didenkulova, I. (2016). "Tsunami generation in a large scale experimental facility."  
873 *Coastal Engineering*, Vol. 110, pp. 32-41, DOI: 10.1016/j.coastaleng.2015.12.005.

874 Shu, C. W., and Osher, S. (1988). "Efficient implementation of essentially non-oscillatory shock-capturing  
875 schemes." *Journal of Computational Physics*, Vol. 77, No. 2, pp. 439-471, DOI: 10.1016/0021-  
876 9991(88)90177-5.

877 Sriram, V., Sannasiraj, S. A., and Sundar, V. (2006). "Simulation of 2-D nonlinear waves using finite element  
878 method with cubic spline approximation." *Journal of Fluids and Structures*, Vol. 22, No. 5, pp. 663-681,  
879 DOI: 10.1016/j.jfluidstructs.2006.02.007.

880 Streicher, M., Hofland, B., and Lindenbergh, R. C. (2013). "Laser ranging for monitoring water waves in the new  
881 Deltares Delta Flume." *In ISPRS Annals of the photogrammetry, remote sensing and spatial information*  
882 *sciences*, ISPRS, Antalya, Turkey, DOI: 10.5194/isprsannals-ii-5-w2-271-2013.

883 Svendsen, I. A. (1974). *Cnoidal waves over a gently sloping bottom*. Series Paper No. 6, Institute of  
884 Hydrodynamics and Hydraulic Engineering, Technical University of Denmark.

885 Synolakis, C. E. (1987). "The runup of solitary waves." *Journal of Fluid Mechanics*, Vol. 185, pp. 523-545, DOI:  
886 10.1017/S002211208700329X.

887 Synolakis, C. E. (1990). "Generation of long waves in laboratory." *Journal of Waterway, Port, Coastal, and ocean*  
888 *engineering*, Vol. 116, No. 2, pp. 252-266, DOI: 10.1061/(ASCE)0733-950X(1990)116:2(252).

889 Toro, E. F. (2001). *Shock-capturing methods for free-surface shallow flows*. John Wiley, Chichester, New York.

890 Turnbull, M. S., Borthwick, A. G. L., and Taylor, R. E. (2003). "Wave-structure interaction using coupled  
891 structured-unstructured finite element meshes." *Applied ocean research*, Vol. 25, No. 2, pp. 63-77, DOI:  
892 10.1016/s0141-1187(03)00032-4.

893 Turnbull, M. S., Borthwick, A. G. L., and Taylor, R. E. (2003). "Numerical wave tank based on a  $\sigma$ -transformed  
894 finite element inviscid flow solver." *International Journal for Numerical Methods in Fluids*, Vol. 42, No. 6,

895 pp. 641-663, DOI: doi.org/10.1002/fld.539.

896 Ullrich, P. A., Jablonowski, C., and van Leer, B. (2010). "High-order finite-volume methods for the shallow-water  
897 equations on the sphere." *Journal of Computational Physics*, Vol. 229, No. 17, pp. 6104-6134, DOI:  
898 10.1016/j.jcp.2010.04.044.

899 Ursell, F., Dean, R. G., and Yu, Y. S. (1960). "Forced small-amplitude water waves: a comparison of theory and  
900 experiment." *Journal of Fluid Mechanics*, Vol. 7, No. 1, pp. 33-52, DOI: 10.1017/s0022112060000037.

901 Vázquez-Cendón, M. E. (1999). "Improved treatment of source terms in upwind schemes for the shallow water  
902 equations in channels with irregular geometry." *Journal of Computational Physics*, Vol. 148, No. 2, pp. 497-  
903 526, DOI: 10.1006/jcph.1998.6127.

904 Vreugdenhil, C. B. (2013). *Numerical methods for shallow-water flow*. Vol. 13 of Water Science and Technology  
905 Library. Springer Science and Business Media.

906 Vukovic, S., and Sopta, L. (2002). "ENO and WENO schemes with the exact conservation property for one-  
907 dimensional shallow water equations." *Journal of Computational Physics*, Vol. 179, No. 2, pp. 593-621, DOI:  
908 10.1006/jcph.2002.7076.

909 Walkley, M. A. (1999). *A numerical method for extended Boussinesq shallow-water wave equations*. PhD  
910 Dissertation, The University of Leeds, UK.

911 Wang, Z. J. (2007). "High-order methods for the Euler and Navier-Stokes equations on unstructured grids."  
912 *Progress in Aerospace Sciences*, Vol. 43, No. 1-3, pp. 1-41, DOI: 10.1016/j.paerosci.2007.05.001.

913 Wenneker, I., Hoffmann, R., and Hofland, B. (2016). "Wave generation and wave measurements in the new Delta  
914 Flume." *Proc. of 6th Int. Conf. on the Application of Physical Modelling in Coastal and Port Engineering  
915 and Science*, Ottawa, Canada.

916 Wen, H., and Ren, B. (2018). "A non-reflective spectral wave maker for SPH modeling of nonlinear wave motion."  
917 *Wave Motion*, Vol. 79, pp. 112-128, DOI: 10.1016/j.wavemoti.2018.03.003.

918 Wu, G. X., and Hu, Z. Z. (2004). "Simulation of nonlinear interactions between waves and floating bodies through  
919 a finite-element-based numerical tank." *Proc. of the Royal Society of London. Series A: Mathematical,  
920 Physical and Engineering Sciences*, Vol. 460, No. 2050, pp. 2797-2817, DOI: 10.1098/rspa.2004.1302.

921 Xing, Y., and Shu, C. W. (2005). "High order finite difference WENO schemes with the exact conservation  
922 property for the shallow water equations." *Journal of Computational Physics*, Vol. 208, No. 1, pp. 206-227,  
923 DOI: 10.1016/j.jcp.2005.02.006.

924 Xing, Y., Zhang, X., and Shu, C.-W. (2010). "Positivity-preserving high order well-balanced discontinuous  
925 Galerkin methods for the shallow water equations." *Advances in Water Resources*, Vol. 33, No. 12, pp. 1476-  
926 1493, DOI: 10.1016/j.advwatres.2010.08.005.

927 Yan, H., and Liu, Y. (2011). "An efficient high-order boundary element method for nonlinear wave-wave and  
928 wave-body interactions." *Journal of Computational Physics*, Vol. 230, No. 2, pp. 402-424, DOI:  
929 10.1016/j.jcp.2010.09.029.

930 Yu, Y. H., and Li, Y. (2013). "Reynolds-Averaged Navier-Stokes simulation of the heave performance of a two-  
931 body floating-point absorber wave energy system." *Computers and Fluids*, Vol. 73, pp. 104-114, DOI:

932 10.1016/j.compfluid.2012.10.007.

933 Yu, Y. H., Lawson, M., Li, Y., Previsic, M., Epler, J., and Lou, J. (2015). *Experimental wave tank test for reference*  
934 *model 3 floating-point absorber wave energy converter project*. Technical Report No. NREL/TP-5000-  
935 62951, National Renewable Energy Laboratory, Golden, CO, US, DOI: 10.2172/1169792.

936 Zabusky, N. J., and Galvin, C. J. (1971). "Shallow-water waves, the Korteweg-deVries equation and solitons."  
937 *Journal of Fluid Mechanics*, Vol. 47, No. 4, pp. 811-824, DOI: 10.1017/s0022112071001393.

938 Zhu, Q., Gao, Z., Don, W. S., and Lv, X. (2017). "Well-balanced hybrid compact-WENO scheme for shallow  
939 water equations." *Applied Numerical Mathematics*, Vol. 112, pp. 65-78, DOI: 10.1016/j.apnum.2016.10.001.

940