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Row Reduced Echelon Form for Solving Fully Fuzzy System with Unknown Coefficients

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Abstract

This study proposes a new method for finding a feasible fuzzy solution in positive Fully Fuzzy Linear System (FFLS), where the coefficients are unknown. The fully fuzzy system is transferred to linear system in order to obtain the solution using row reduced echelon form, thereafter; the crisp solution is restricted in obtaining the positive fuzzy solution.

The fuzzy solution of FFLS is included crisp intervals, to assign alternative values of unknown entries of fuzzy numbers. To illustrate the proposed method, numerical examples are solved, where the entries of coefficients are unknown in right or left hand side, to demonstrate the contributions in this study.

Keywords: Fully Fuzzy linear system, Row Reduced Echelon Form, Unknown Coefficients.

1 Introduction

Linear System of equations is considered the simplest model in solving mathematical problems. However, the coefficients of these systems are not completely obtainable. Therefore, the linear system is replaced by fuzzy systems, to substitute the crisp numbers by fuzzy numbers. The linear system of equations are called fuzzy linear system (FLS) if the elements of the matrix in left hand side are crisp numbers and the element for vector in right hand side are represented by fuzzy numbers. On the other hand, the linear system of equations is called fully fuzzy linear system (FFLS) where all the elements in both sides are fuzzy numbers.

The first achievable approach of solving FLS was carried out in [15] where they proposed a generic model for solving an FLS by transferring the FLS to linear system. In [10], [11] Dehghan and his colleagues obtained the solution for FFLS when the coefficient and parameters are positive. Furthermore for the same senario, scholars in [31-34],[16], proposed new methods for solving FFLS in a similar way to Dehghan

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and his colleagues. Malkawi and his colleagues in [24] proposed new matrix methods for solving a positive FFLS, the necessary and sufficient condition to have a positive solution was discussed, this method also capable for solving Left-Right fuzzy linear system (LR-FLS) and FLS.
In this study, we propose a method that can provide a positive solution for unknown coefficients in FFLS using row reduced echelon form. This method can deal with any system regardless of its size.
The structure of this study is organized as follows: In Section 2, the basic definitions of the fuzzy set theory will be reviewed. In Section 3, we introduce the proposed model. Section 4 contains some numerical examples to illustrate our work. The summary of our method is given in section 5.

2 Preliminaries and notations

In this section, basic definitions and notions of fuzzy set theory are reviewed ([7], [17]).

Definition 2.1. (*LR fuzzy number) A fuzzy number $\widetilde{m}$ is called LR fuzzy number where its membership function satisfy*

$$\mu_{\widetilde{m}}(x) = \begin{cases} L\left(\frac{m-x}{\alpha}\right), & for \quad x \leq m \ , \ \alpha > 0, \\ R\left(\frac{x-m}{\beta}\right), & for \quad m \leq x, \ \beta > 0, \end{cases} \tag{2.1}$$

where $m, \alpha, \beta \in \mathbb{R}$.
The function $L(.)$ is called a left shape function if the following hold:
1- $L(x) = L(-x)$,
2- $L(0) = 1, L(1) = 0$,
3- L is non-increasing on $[0, \infty]$.

Also, the definition of function $R(.)$ which is called right shape, is similar to that of $L(.)$. The LR fuzzy number can be symbolized as $\widetilde{m} = (m, \alpha, \beta)_{LR}$, where m denotes the mean value, while α and β are left and right spreads, respectively. We denote the set of LR fuzzy numbers as $F(\Re)$, Clearly, $\widetilde{m} = (m, \alpha, \beta)_{LR}$ is positive, if and only if $m - \alpha > 0$.

Remark 2.1. *Among the several shapes of fuzzy number, the most common one used is triangular fuzzy number (TFN), where $L = R = max(0, 1 - x)$. It is symbolically written as $\widetilde{m} = (m, \alpha, \beta)$.*

Definition 2.2. *Two fuzzy numbers $\tilde{n} = (n, \gamma, \delta)_{LR}$ and $\widetilde{m} = (m, \alpha, \beta)_{LR}$ are called equal iff $n = m, \gamma = \alpha, \delta = \beta$.*

Definition 2.3. (*Arithmetic operations on LR fuzzy numbers*) *We will represent arithmetic operations for two LR Fuzzy numbers $\widetilde{m} = (m, \alpha, \beta)_{LR}$ and $\tilde{n} = (n, \gamma, \delta)_{LR}$ as follows:*

- *Addition:*
  $$\widetilde{m} \oplus \tilde{n} = (m + n, \alpha + \gamma, \beta + \delta)_{LR}. \tag{2.2}$$
- *Opposite:*
  $$-\widetilde{m} = (-m, \alpha, \beta)_{RL}. \tag{2.3}$$
- *Subtraction:*
  $$\widetilde{m} \ominus \tilde{n} = (m - n, \alpha + \delta, \beta + \gamma)_{LR}. \tag{2.4}$$
- *Approximated multiplication operation of two fuzzy numbers:*

i- *If $\widetilde{m} > 0$ and $\tilde{n} > 0$ then*
$$\widetilde{m} \otimes \tilde{n} \cong (mn, m\gamma + n\alpha, m\delta + n\beta)_{LR}. \tag{2.5}$$

ii- *If* $\tilde{m} < 0$ *and* $\tilde{n} > 0$ *then*

$\tilde{m} \otimes \tilde{n} \cong (mn, n\alpha - m\delta, n\beta - m\gamma)_{RL}$. (2.6)

iii- $\tilde{m} < 0$ *and* $\tilde{n} < 0$ *then*

$\tilde{m} \otimes \tilde{n} \cong (mn, -n\beta - m\delta, -n\alpha - m\gamma)_{RL}$. (2.7)

- *Scalar multiplication:*

  Let $\lambda \in R$. *Then*

$$\lambda \otimes \tilde{m} = \begin{cases} (\lambda m, \lambda\alpha, \lambda\beta)_{LR} & \lambda \geq 0, \\ (\lambda m, -\lambda\beta, -\lambda\alpha)_{RL} & \lambda < 0. \end{cases} \quad (2.8)$$

Definition 2.4. *A crisp matrix A is called inverse-nonnegative if* $A > 0$ *and* $A^{-1} > 0$.

Definition 2.5. *A matrix* $\tilde{A} = (\tilde{a}_{ij})$ *is called a fuzzy number matrix, or shortly fuzzy matrix, if each element of* $\tilde{A}$ *is a fuzzy number. A matrix* $\tilde{A}$ *will be positive fuzzy matrix (denoted by* $\tilde{A} \geq 0$*), if each element of* $\tilde{A}$ *is positive. We may represent such* $\tilde{a}_{ij} = (\tilde{\gamma}_{ij}, \tilde{\alpha}_{ij}, \tilde{\beta}_{ij})$ *with the new notation* $\tilde{A} = (A, M, N)$, *where* $A = (m_{ij})$, $M = (\alpha_{ij})$, $N = (\beta_{ij})$ *are three* $n \times n$ *crisp matrices. Clearly, if* $\tilde{A} = (\tilde{a}_{ij})_{n\times 1}$ *it is called fuzzy vector.*

Definition 2.6. *Let* $\tilde{A} = (\tilde{a}_{ij})$ *and* $\tilde{B} = (\tilde{b}_{ij})$ *be two* $m \times n$ *and* $n \times p$ *fuzzy matrices respectively. We define* $\tilde{A} \otimes \tilde{B} = \tilde{C} = (\tilde{c}_{ij})$ *which is the* $m \times p$ *matrix where*

$$\tilde{c}_{ij} = \sum\nolimits_{k=1,\dots,n}^{\oplus} \tilde{a}_{ik} \otimes \tilde{b}_{kj}. \quad (2.9)$$

Definition 2.7. *(Fully fuzzy linear system) Consider the* $n \times n$ *linear system of equations:*

$$\begin{cases} \tilde{a}_{11}\tilde{x}_1 + \tilde{a}_{12}\tilde{x}_2 + \ldots\ldots + \tilde{a}_{1n}\tilde{x}_n = \tilde{b}_1, \\ \tilde{a}_{21}\tilde{x}_1 + a_{22}\tilde{x}_2 + \ldots\ldots + \tilde{a}_{2n}\tilde{x}_n = \tilde{b}_2 \\ \qquad\qquad\vdots \\ \qquad\qquad\vdots \\ \tilde{a}_{n1}\tilde{x}_1 + \tilde{a}_{n2}\tilde{x}_2 + \ldots\ldots + \tilde{a}_{nn}\tilde{x}_n = \tilde{b}_n. \end{cases} \quad (2.10)$$

Where $\forall \tilde{a}_{ij}, \tilde{b}_j \in F(\mathfrak{R})$, *this system is called a fully fuzzy linear system (FFLS). The matrix* $\tilde{A} = \left(\tilde{a}_{ij}\right)_{i,j=1}^{n}$ *and the vector* $\tilde{B} = \left(\tilde{b}_j\right)_{j=1}^{n}$ *may be represented as*

$$\tilde{A} \otimes \tilde{X} = \tilde{B} \quad (2.11)$$

- *If the all entries of* $\tilde{A}$, $\tilde{B} \geq 0$ *Positives It is called Positive FFLS.*
- *If the vector* $\tilde{X} = \left(\tilde{x}_j\right)_{j=1}^{n}$ *satisfies* (2.3), *and all entries of* $\tilde{X} = (\tilde{x}_j)$ *are positives, where* $\forall \tilde{x}_j \in F(\mathfrak{R}), j = 1, 2, \dots, n$, *it is called positive fuzzy solution. If* $\exists \tilde{x}_j \notin F(\mathfrak{R}), j = 1, 2, \dots, n$, *it is called non fuzzy solution.*
- *If the vector* $\tilde{X}' = \left(\tilde{x}_j{}'\right)_{j=1}^{n}$ *satisfies*

$$\tilde{A} \otimes \tilde{X}' \approx \tilde{B} \quad (2.12)$$

where $\forall \tilde{x}_j{}' \in F(\mathfrak{R})$, $j = 1, 2, \dots, n$, *and all entries of* $\tilde{X}' = \left(\tilde{x}_j{}'\right)_{j=1}^{n}$ *are positives is called Positive approximated fuzzy solution. In the rest of this paper, we will find the positive solution.*

3 The Proposed Method

In this section a novel method will be constructed to obtain positive solution where the coefficients are unknown in FFLS.

Consider the following positive FFLS,

$\tilde{A} \otimes \tilde{X} = \tilde{B}$,
where,
$\tilde{A} = (\tilde{a}_{ij})_{n\times n} = (A, M, N), \tilde{B} = (\tilde{b}_i)_{n\times 1} = (m^b, \alpha^b, \beta^b)$,
And, $\widetilde{X} = (\tilde{x}_j)_{n\times 1} = (m^x, \alpha^x, \beta^x) \geq 0$,

$$(A, M, N) \otimes (m^x, \alpha^x, \beta^x) = (m^b, \alpha^b, \beta^b). \tag{3.13}$$

Let
$\tilde{a}_{ij} = (m^a_{i,j}, \alpha^a_{i,j}, \beta^a_{i,j}), \tilde{x}_j = (m^x_j, \alpha^x_j, \beta^x_j)$ and $\tilde{b}_i = (m^b_i, \alpha^b_i, \beta^b_i)$.
Define,

$$S = \begin{pmatrix} A & 0 & 0 \\ N & A & 0 \\ M & 0 & A \end{pmatrix},$$

where,

$$A = \begin{pmatrix} m^a_{11} & m^a_{12} & \dots & m^a_{1n} \\ m^a_{21} & m^a_{22} & \dots & m^a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m^a_{n1} & m^a_{n2} & \cdots & m^a_{nn} \end{pmatrix},$$

$$M = \begin{pmatrix} \alpha^a_{11} & \alpha^a_{12} & \dots & \alpha^a_{1n} \\ \alpha^a_{21} & \alpha^a_{22} & \dots & \alpha^a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha^a_{n1} & \alpha^a_{n2} & \cdots & \alpha^a_{nn} \end{pmatrix}, N = \begin{pmatrix} \beta^a_{11} & \beta^a_{12} & \dots & \beta^a_{1n} \\ \beta^a_{21} & \beta^a_{22} & \dots & \beta^a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \beta^a_{n1} & \beta^a_{n2} & \cdots & \beta^a_{nn} \end{pmatrix},$$

then,

$$S = \begin{pmatrix} \begin{pmatrix} m^a_{11} & m^a_{12} & \dots & m^a_{1n} \\ m^a_{21} & m^a_{22} & \dots & m^a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m^a_{n1} & m^a_{n2} & \cdots & m^a_{nn} \end{pmatrix} & \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \\ \begin{pmatrix} \alpha^a_{11} & \alpha^a_{12} & \dots & \alpha^a_{1n} \\ \alpha^a_{21} & \alpha^a_{22} & \dots & \alpha^a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha^a_{n1} & \alpha^a_{n2} & \cdots & \alpha^a_{nn} \end{pmatrix} & \begin{pmatrix} m^a_{11} & m^a_{12} & \dots & m^a_{1n} \\ m^a_{21} & m^a_{22} & \dots & m^a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m^a_{n1} & m^a_{n2} & \cdots & m^a_{nn} \end{pmatrix} & \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \\ \begin{pmatrix} \beta^a_{11} & \beta^a_{12} & \dots & \beta^a_{1n} \\ \beta^a_{21} & \beta^a_{22} & \dots & \beta^a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \beta^a_{n1} & \beta^a_{n2} & \cdots & \beta^a_{nn} \end{pmatrix} & \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} & \begin{pmatrix} m^a_{11} & m^a_{12} & \dots & m^a_{1n} \\ m^a_{21} & m^a_{22} & \dots & m^a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m^a_{n1} & m^a_{n2} & \cdots & m^a_{nn} \end{pmatrix} \end{pmatrix}.$$

Also, $X = \begin{pmatrix} m^x \\ \alpha^x \\ \beta^x \end{pmatrix}$ where $m^x = \begin{pmatrix} m_1^x \\ m_2^x \\ \vdots \\ m_n^x \end{pmatrix}, \alpha^x = \begin{pmatrix} \alpha_1^x \\ \alpha_2^x \\ \vdots \\ \alpha_n^x \end{pmatrix}, \beta^x = \begin{pmatrix} \beta_1^x \\ \beta_2^x \\ \vdots \\ \beta_n^x \end{pmatrix}$, then $X = \begin{pmatrix} \begin{pmatrix} m_1^x \\ m_2^x \\ \vdots \\ m_n^x \end{pmatrix} \\ \begin{pmatrix} \alpha_1^x \\ \alpha_2^x \\ \vdots \\ \alpha_n^x \end{pmatrix} \\ \begin{pmatrix} \beta_1^x \\ \beta_2^x \\ \vdots \\ \beta_n^x \end{pmatrix} \end{pmatrix}$,

$B = \begin{pmatrix} m^b \\ \alpha^b \\ \beta^b \end{pmatrix}$ where $m^b = \begin{pmatrix} m_1^b \\ m_2^b \\ \vdots \\ m_n^b \end{pmatrix}, \alpha^b = \begin{pmatrix} \alpha_1^b \\ \alpha_2^b \\ \vdots \\ \alpha_n^b \end{pmatrix}, \beta^b = \begin{pmatrix} \beta_1^b \\ \beta_2^b \\ \vdots \\ \beta_n^b \end{pmatrix}$, then $B = \begin{pmatrix} \begin{pmatrix} m_1^b \\ m_2^b \\ \vdots \\ m_n^b \end{pmatrix} \\ \begin{pmatrix} \alpha_1^b \\ \alpha_2^b \\ \vdots \\ \alpha_n^b \end{pmatrix} \\ \begin{pmatrix} \beta_1^b \\ \beta_2^b \\ \vdots \\ \beta_n^b \end{pmatrix} \end{pmatrix}$.

Hence, the new linear system is $SX = B$:

$$\begin{pmatrix} A & 0 & 0 \\ N & A & 0 \\ M & 0 & A \end{pmatrix} \begin{pmatrix} m^x \\ \alpha^x \\ \beta^x \end{pmatrix} = \begin{pmatrix} m^b \\ \alpha^b \\ \beta^b \end{pmatrix}, \tag{3.14}$$

or,

$$\begin{pmatrix} \begin{pmatrix} m_{11}^a & m_{12}^a & \dots & m_{1n}^a \\ m_{21}^a & m_{22}^a & \dots & m_{2n}^a \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1}^a & m_{n2}^a & \cdots & m_{nn}^a \end{pmatrix} & \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \\ \begin{pmatrix} \alpha_{11}^a & \alpha_{12}^a & \dots & \alpha_{1n}^a \\ \alpha_{21}^a & \alpha_{22}^a & \dots & \alpha_{2n}^a \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1}^a & \alpha_{n2}^a & \cdots & \alpha_{nn}^a \end{pmatrix} & \begin{pmatrix} m_{11}^a & m_{12}^a & \dots & m_{1n}^a \\ m_{21}^a & m_{22}^a & \dots & m_{2n}^a \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1}^a & m_{n2}^a & \cdots & m_{nn}^a \end{pmatrix} & \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \\ \begin{pmatrix} \beta_{11}^a & \beta_{12}^a & \dots & \beta_{1n}^a \\ \beta_{21}^a & \beta_{22}^a & \dots & \beta_{2n}^a \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{n1}^a & \beta_{n2}^a & \cdots & \beta_{nn}^a \end{pmatrix} & \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} & \begin{pmatrix} m_{11}^a & m_{12}^a & \dots & m_{1n}^a \\ m_{21}^a & m_{22}^a & \dots & m_{2n}^a \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1}^a & m_{n2}^a & \cdots & m_{nn}^a \end{pmatrix} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} m_1^x \\ m_2^x \\ \vdots \\ m_n^x \end{pmatrix} \\ \begin{pmatrix} \alpha_1^x \\ \alpha_2^x \\ \vdots \\ \alpha_n^x \end{pmatrix} \\ \begin{pmatrix} \beta_1^x \\ \beta_2^x \\ \vdots \\ \beta_n^x \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} m_1^b \\ m_2^b \\ \vdots \\ m_n^b \end{pmatrix} \\ \begin{pmatrix} \alpha_1^b \\ \alpha_2^b \\ \vdots \\ \alpha_n^b \end{pmatrix} \\ \begin{pmatrix} \beta_1^b \\ \beta_2^b \\ \vdots \\ \beta_n^b \end{pmatrix} \end{pmatrix}. \tag{3.15}$$

The above linear system is equivalent to the following fuzzy system [24].

$\sum_{j=1}^{\oplus} \tilde{a}_{ij} \otimes \tilde{x}_j = \tilde{b}_i \ ,\forall i = 1, 2, \dots, n,$

$$\sum_{j=1}^{\oplus} \left(m_{i,j}^a, \alpha_{i,j}^a, \beta_{i,j}^a\right) \otimes \left(m_j^x, \alpha_j^x, \beta_j^x\right) = \left(m_i^b, \alpha_i^b, \beta_i^b\right), \forall i = 1, 2, \dots, n. \tag{3.16}$$

Then (3.14) is a linear system, it can be solved using a reduced row-echelon form, then the obtained crisp solution is equivalent to exact solution for (2.11),
Using the following inequalities,

$$\begin{cases} \alpha_i^x \geq 0, \\ \beta_i^x \geq 0, \quad \forall i = 1, 2, \dots, n, \end{cases} \tag{3.17}$$

The crisp solution in (3.14) should be examined using the inequalities

$$m_i^x \geq \alpha_i^x, \; i = 1, \dots n, \tag{3.18}$$

whether it is positive or non-positive.

In order to obtain the positive fuzzy solution for an unknown coefficients, the obtained crisp solution from (3.14) can be modified using

$$\begin{cases} m_i^x \geq \alpha_i^x, \\ \alpha_i^x \geq 0, \\ \beta_i^x \geq 0, \qquad \forall i = 1, 2, \dots, n. \end{cases} \tag{3.19}$$

4 Numerical examples

In this section, numerical examples are illustrated to show the easiness and efficiency of the proposed method to obtain the positive solution for arbitrary FFLS.
The following example is taken from previous study to compare the result, where the coefficients are fully known.

Example 4.1. [10] *Consider the following FFLS,*

$$\begin{cases} (6,1,4) \otimes (m_1^x, \alpha_1^x, \beta_1^x) \oplus (5,2,2) \otimes (m_2^x, \alpha_2^x, \beta_2^x) \oplus (3,2,1) \otimes (m_3^x, \alpha_3^x, \beta_3^x) = (58,30,60), \\ (12,8,20) \otimes (m_1^x, \alpha_1^x, \beta_1^x) \oplus (14,12,15) \otimes (m_2^x, \alpha_2^x, \beta_2^x) \oplus (8,8,10) \otimes (m_3^x, \alpha_3^x, \beta_3^x) = (142,139,257), \\ (24,10,34) \otimes (m_1^x, \alpha_1^x, \beta_1^x) \oplus (32,30,30) \otimes (m_2^x, \alpha_2^x, \beta_2^x) \oplus (20,19,24) \otimes (m_3^x, \alpha_3^x, \beta_3^x) = (316,297,514), \end{cases}$$

where $\tilde{x}_i = ({m_i}^x, {\alpha_i}^x, {\beta_i}^x) \geq 0, i = 1, 2, 3.$

Solution
The system may be written in matrix form, as following,

$$\begin{pmatrix} (6,1,4) & (5,2,2) & (3,2,1) \\ (12,8,20) & (14,12,15) & (8,8,10) \\ (24,10,34) & (32,30,30) & (20,19,24) \end{pmatrix} \otimes \begin{pmatrix} (m_1^x, \alpha_1^x, \beta_1^x) \\ (m_2^x, \alpha_2^x, \beta_2^x) \\ (m_3^x, \alpha_3^x, \beta_3^x) \end{pmatrix} = \begin{pmatrix} (58,30,60) \\ (142,139,257) \\ (316,297,514) \end{pmatrix},$$

where

$$A = \begin{pmatrix} 6 & 5 & 3 \\ 12 & 14 & 8 \\ 24 & 32 & 20 \end{pmatrix}, M = \begin{pmatrix} 1 & 2 & 2 \\ 8 & 12 & 8 \\ 10 & 30 & 19 \end{pmatrix}, N = \begin{pmatrix} 4 & 2 & 1 \\ 20 & 15 & 10 \\ 34 & 30 & 24 \end{pmatrix},$$

and

$$m^b = \begin{pmatrix} 58 \\ 142 \\ 316 \end{pmatrix}, \alpha^b = \begin{pmatrix} 30 \\ 139 \\ 297 \end{pmatrix}, \beta^b = \begin{pmatrix} 60 \\ 257 \\ 514 \end{pmatrix},$$

$$m^x = \begin{pmatrix} m_1^x \\ m_2^x \\ m_3^x \end{pmatrix}, \alpha^x = \begin{pmatrix} \alpha_1^x \\ \alpha_2^x \\ \alpha_3^x \end{pmatrix}, \beta^x = \begin{pmatrix} \beta_1^x \\ \beta_2^x \\ \beta_3^x \end{pmatrix}.$$

Then, S, B and X are appointed as following,

$$S=\begin{pmatrix} A & 0 & 0\\ M & A & 0\\ N & 0 & A\end{pmatrix}=\begin{pmatrix} \begin{pmatrix}6&5&3\\12&14&8\\24&32&20\end{pmatrix} & \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix} & \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\\ \begin{pmatrix}1&2&2\\8&12&8\\10&30&19\end{pmatrix} & \begin{pmatrix}6&5&3\\12&14&8\\24&32&20\end{pmatrix} & \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\\ \begin{pmatrix}4&2&1\\20&15&10\\34&30&24\end{pmatrix} & \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix} & \begin{pmatrix}6&5&3\\12&14&8\\24&32&20\end{pmatrix}\end{pmatrix}.$$

Also,

$$B=\begin{pmatrix}m^b\\ \alpha^b\\ \beta^b\end{pmatrix}=\begin{pmatrix}\begin{pmatrix}58\\142\\316\end{pmatrix}\\ \begin{pmatrix}30\\139\\297\end{pmatrix}\\ \begin{pmatrix}60\\257\\514\end{pmatrix}\end{pmatrix}=\begin{pmatrix}58\\142\\316\\30\\139\\297\\60\\257\\514\end{pmatrix} \text{ and } X=\begin{pmatrix}m^x\\ \alpha^x\\ \beta^x\end{pmatrix}=\begin{pmatrix}\begin{pmatrix}m_1^x\\m_2^x\\m_3^x\end{pmatrix}\\ \begin{pmatrix}\alpha_1^x\\\alpha_2^x\\\alpha_3^x\end{pmatrix}\\ \begin{pmatrix}\beta_1^x\\\beta_2^x\\\beta_3^x\end{pmatrix}\end{pmatrix}=\begin{pmatrix}m_1^x\\m_2^x\\m_3^x\\\alpha_1^x\\\alpha_2^x\\\alpha_3^x\\\beta_1^x\\\beta_2^x\\\beta_3^x\end{pmatrix}.$$

Then the associated linear system is given by $SX=B$:

$$\begin{pmatrix} 6&5&3&0&0&0&0&0&0\\ 12&14&8&0&0&0&0&0&0\\ 24&32&20&0&0&0&0&0&0\\ 1&2&2&6&5&3&0&0&0\\ 8&12&8&12&14&8&0&0&0\\ 10&30&19&24&32&20&0&0&0\\ 4&2&1&0&0&0&6&5&3\\ 20&15&10&0&0&0&12&14&8\\ 34&30&24&0&0&0&24&32&20 \end{pmatrix}\begin{pmatrix}m_1^x\\m_2^x\\m_3^x\\\alpha_1^x\\\alpha_2^x\\\alpha_3^x\\\beta_1^x\\\beta_2^x\\\beta_3^x\end{pmatrix}=\begin{pmatrix}58\\142\\316\\30\\139\\297\\60\\257\\514\end{pmatrix}.$$

The crisp solution can be easily obtained only by computing,

$$\begin{pmatrix} 6m_1^x+5m_2^x+3m_3^x\\ 12m_1^x+14m_2^x+8m_3^x\\ 24m_1^x+32m_2^x+20m_3^x\\ m_1^x+2m_2^x+2m_3^x+6\alpha_1^x+5\alpha_2^x+3\alpha_3^x\\ 8m_1^x+12m_2^x+8m_3^x+12\alpha_1^x+14\alpha_2^x+8\alpha_3^x\\ 10m_1^x+30m_2^x+19m_3^x+24\alpha_1^x+32\alpha_2^x+20\alpha_3^x\\ 4m_1^x+2m_2^x+m_3^x+6\beta_1^x+5\beta_2^x+3\beta_3^x\\ 20m_1^x+15m_2^x+10m_3^x+12\beta_1^x+14\beta_2^x+8\beta_3^x\\ 34m_1^x+30m_2^x+24m_3^x+24\beta_1^x+32\beta_2^x+20\beta_3^x \end{pmatrix}=\begin{pmatrix}58\\142\\316\\30\\139\\297\\60\\257\\514\end{pmatrix},$$

$$\begin{cases} 6m_1^x + 5m_2^x + 3m_3^x = 58, \\ 12m_1^x + 14m_2^x + 8m_3^x = 142, \\ 24m_1^x + 32m_2^x + 20m_3^x = 316, \\ m_1^x + 2m_2^x + 2m_3^x + 6\alpha_1^x + 5\alpha_2^x + 3\alpha_3^x = 30, \\ 8m_1^x + 12m_2^x + 8m_3^x + 12\alpha_1^x + 14\alpha_2^x + 8\alpha_3^x = 139, \\ 10m_1^x + 30m_2^x + 19m_3^x + 24\alpha_1^x + 32\alpha_2^x + 20\alpha_3^x = 297, \\ 4m_1^x + 2m_2^x + m_3^x + 6\beta_1^x + 5\beta_2^x + 3\beta_3^x = 60, \\ 20m_1^x + 15m_2^x + 10m_3^x + 12\beta_1^x + 14\beta_2^x + 8\beta_3^x = 257, \\ 34m_1^x + 30m_2^x + 24m_3^x + 24\beta_1^x + 32\beta_2^x + 20\beta_3^x = 514. \end{cases}$$

The crisp solution can be easily obtained by solving the above linear system,

$$X = \begin{pmatrix} m_1^x \\ m_2^x \\ m_3^x \\ \alpha_1^x \\ \alpha_2^x \\ \alpha_3^x \\ \beta_1^x \\ \beta_2^x \\ \beta_3^x \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 3 \\ 1 \\ \frac{1}{2} \\ \frac{1}{2} \\ 3 \\ 2 \\ 1 \end{pmatrix} \text{ or } \begin{pmatrix} \begin{pmatrix} m_1^x \\ m_2^x \\ m_3^x \end{pmatrix} \\ \begin{pmatrix} \alpha_1^x \\ \alpha_2^x \\ \alpha_3^x \end{pmatrix} \\ \begin{pmatrix} \beta_1^x \\ \beta_2^x \\ \beta_3^x \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix} \\ \begin{pmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \\ \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \end{pmatrix}.$$

The inequalities are satisfied,

$$\begin{cases} m_i^x \geq \alpha_i^x, \\ \alpha_i^x \geq 0, \\ \beta_i^x \geq 0, \qquad \forall i = 1, 2, 3. \end{cases}$$

Then the crisp solution is positive fuzzy solution,

$$\tilde{X} = \begin{pmatrix} (m_1^x, \alpha_1^x, \beta_1^x) \\ (m_2^x, \alpha_2^x, \beta_2^x) \\ (m_3^x, \alpha_3^x, \beta_3^x) \end{pmatrix} = \begin{pmatrix} (4, 1, 3) \\ \left(5, \frac{1}{2}, 2\right) \\ \left(3, \frac{1}{2}, 1\right) \end{pmatrix}.$$

In the next Examples 4.2 and 4.3, we show that our proposed method is able to obtain the solution when the entry coefficient is unknown.

Examples 4.2. *Consider the following FFLS,*

$$\begin{cases}(\delta,2,1)\otimes(m_1^x,\alpha_1^x,\beta_1^x)\oplus(7,3,1)\otimes(m_2^x,\alpha_2^x,\beta_2^x)=(54,46,54),\\ \\ (6,1,2)\otimes(m_1^x,\alpha_1^x,\beta_1^x)\oplus(4,1,2)\otimes(m_2^x,\alpha_2^x,\beta_2^x)=(52,27,56),\end{cases}$$

where $\tilde{x}_i=({m_i}^x,{\alpha_i}^x,{\beta_i}^x)\geq 0, i=1,2,$ *and* δ *is an arbitrary real valued parameter not less than* 2, *determine when the system has a positive solution based on values of* δ.

Solution

The system may be written in matrix form, is as follows

$$\begin{pmatrix}(\delta,2,1) & (7,3,1)\\ \\ (6,1,2) & (4,1,2)\end{pmatrix}\otimes\begin{pmatrix}(m_1^x,\alpha_1^x,\beta_1^x)\\ \\ (m_2^x,\alpha_2^x,\beta_2^x)\end{pmatrix}=\begin{pmatrix}(54,46,54)\\ \\ (52,27,56)\end{pmatrix},$$

where,

$A=\begin{pmatrix}\delta & 7\\ 6 & 4\end{pmatrix}, M=\begin{pmatrix}2 & 3\\ 1 & 1\end{pmatrix}, N=\begin{pmatrix}1 & 1\\ 2 & 2\end{pmatrix},$

$$S=\begin{pmatrix}A & 0 & 0\\ \\ M & A & 0\\ \\ N & 0 & A\end{pmatrix}=\begin{pmatrix}\begin{pmatrix}\delta & 7\\ 6 & 4\end{pmatrix} & \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} & \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\\ \begin{pmatrix}2 & 3\\ 1 & 1\end{pmatrix} & \begin{pmatrix}\delta & 7\\ 6 & 4\end{pmatrix} & \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\\ \begin{pmatrix}1 & 1\\ 2 & 2\end{pmatrix} & \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} & \begin{pmatrix}\delta & 7\\ 6 & 4\end{pmatrix}\end{pmatrix}.$$

Also,

$$m^b=\begin{pmatrix}54\\ 52\end{pmatrix}, \alpha^b=\begin{pmatrix}46\\ 27\end{pmatrix}, \beta^b=\begin{pmatrix}54\\ 56\end{pmatrix}, \text{ then } B=\begin{pmatrix}m^b\\ \alpha^b\\ \beta^b\end{pmatrix}=\begin{pmatrix}\begin{pmatrix}54\\ 52\end{pmatrix}\\ \begin{pmatrix}46\\ 27\end{pmatrix}\\ \begin{pmatrix}54\\ 56\end{pmatrix}\end{pmatrix}=\begin{pmatrix}54\\ 52\\ 46\\ 27\\ 54\\ 56\end{pmatrix},$$

$$m^x=\begin{pmatrix}m_1^x\\ m_2^x\end{pmatrix}, \alpha^x=\begin{pmatrix}\alpha_1^x\\ \alpha_2^x\end{pmatrix}, \beta^x=\begin{pmatrix}\beta_1^x\\ \beta_2^x\end{pmatrix}, \text{ then } X\begin{pmatrix}m^x\\ \alpha^x\\ \beta^x\end{pmatrix}=\begin{pmatrix}\begin{pmatrix}m_1^x\\ m_2^x\end{pmatrix}\\ \begin{pmatrix}\alpha_1^x\\ \alpha_2^x\end{pmatrix}\\ \begin{pmatrix}\beta_1^x\\ \beta_2^x\end{pmatrix}\end{pmatrix}=\begin{pmatrix}m_1^x\\ m_2^x\\ \alpha_1^x\\ \alpha_2^x\\ \beta_1^x\\ \beta_2^x\end{pmatrix}.$$

Hence, the associated linear system is given by $SX=B$:

$$\begin{pmatrix}\delta & 7 & 0 & 0 & 0 & 0\\ 6 & 4 & 0 & 0 & 0 & 0\\ 2 & 3 & \delta & 7 & 0 & 0\\ 1 & 1 & 6 & 4 & 0 & 0\\ 1 & 1 & 0 & 0 & \delta & 7\\ 2 & 2 & 0 & 0 & 6 & 4\end{pmatrix}\begin{pmatrix}m_1^x\\ m_2^x\\ \alpha_1^x\\ \alpha_2^x\\ \beta_1^x\\ \beta_2^x\end{pmatrix}=\begin{pmatrix}54\\ 52\\ 46\\ 27\\ 54\\ 56\end{pmatrix},$$

$$\begin{pmatrix} \delta m_1^x + 7m_2^x \\ 6m_1^x + 4m_2^x \\ 2m_1^x + 3m_2^x + \delta\alpha_1^x + 7\alpha_2^x \\ m_1^x + m_2^x + 6\alpha_1^x + 4\alpha_2^x \\ m_1^x + m_2^x + \delta\beta_1^x + 7\beta_2^x \\ 2m_1^x + 2m_2^x + 6\beta_1^x + 4\beta_2^x \end{pmatrix} = \begin{pmatrix} 54 \\ 52 \\ 46 \\ 27 \\ 54 \\ 56 \end{pmatrix},$$

$$\begin{cases} \delta m_1^x + 7m_2^x = 54, \\ 6m_1^x + 4m_2^x = 52, \\ 2m_1^x + 3m_2^x + \delta\alpha_1^x + 7\alpha_2^x = 46, \\ m_1^x + m_2^x + 6\alpha_1^x + 4\alpha_2^x = 27, \\ m_1^x + m_2^x + \delta\beta_1^x + 7\beta_2^x = 54, \\ 2m_1^x + 2m_2^x + 6\beta_1^x + 4\beta_2^x = 56. \end{cases}$$

The crisp solution can be obtained by solving the above linear system

$$X = \begin{pmatrix} m_1^x \\ m_2^x \\ \alpha_1^x \\ \alpha_2^x \\ \beta_1^x \\ \beta_2^x \end{pmatrix} = \begin{pmatrix} \frac{74}{21-2\delta} \\ \frac{162-26\delta}{21-2\delta} \\ \frac{989-140\delta}{2(21-2\delta)^2} \\ \frac{1992-415\delta+28\delta^2}{2(21-2\delta)^2} \\ \frac{668-46\delta}{(21-2\delta)^2} \\ \frac{2694-598\delta+30\delta^2}{(21-2\delta)^2} \end{pmatrix}.$$

- $\tilde{x}_1$ is a positive fuzzy number if and only if $\begin{cases} \alpha_1^x \le m_1^x, \\ \alpha_1^x \ge 0, \\ \beta_1^x \ge 0, \end{cases}$

hence, $\begin{cases} \frac{74}{21-2\delta} \geq \frac{989-140\delta}{2(21-2\delta)^2}, \\ \frac{989-140\delta}{2(21-2\delta)^2} \geq 0, \\ \frac{668-46\delta}{(21-2\delta)^2} \geq 0, \end{cases}$ or $\begin{cases} \frac{163-12\delta}{(21-2\delta)^2} \geq 0, \\ \frac{989-140\delta}{(21-2\delta)^2} \geq 0, \\ \frac{334-23\delta}{(21-2\delta)^2} \geq 0, \end{cases}$ then, $\delta \leq \frac{989}{140}$,

- $\tilde{x}_2$ is a positive fuzzy number if and only if $\begin{cases} \alpha_2^x \leq m_1^x, \\ \alpha_2^x \geq 0, \\ \beta_2^x \geq 0, \end{cases}$

$\begin{cases} \frac{162-26\delta}{21-2\delta} \geq \frac{1992-415\delta+28\delta^2}{2(21-2\delta)^2}, \\ \frac{1992-415\delta+28\delta^2}{2(21-2\delta)^2} \geq 0, \\ \frac{2694-598\delta+30\delta^2}{(21-2\delta)^2} \geq 0, \end{cases}$ or $\begin{cases} \frac{4812-1325\delta+76\delta^2}{(21-2\delta)^2} \geq 0, \\ \frac{1992-415\delta+28\delta^2}{(21-2\delta)^2} \geq 0, \\ \frac{1347-299\delta+15\delta^2}{(21-2\delta)^2} \geq 0, \end{cases}$ then, $\begin{cases} \delta \leq \frac{1}{152}\left(1325 - \sqrt{292777}\right), \\ \delta \geq \frac{1}{30}(299 + \sqrt{8581}, \end{cases}$

hence, the positive solution for $\tilde{x}_1$, $\tilde{x}_2$ where

$\begin{cases} \delta \leq \frac{989}{140}, \\ \delta \leq \frac{1}{152}\left(1325 - \sqrt{292777}\right), \\ \text{and } \delta \geq \frac{1}{30}(299 + \sqrt{8581}, \end{cases}$ or $\begin{cases} \delta \leq 7.0642, \\ \delta \leq 5.157 \text{ and } \delta \geq 13.054, \end{cases}$

hence, $\delta \leq \frac{1}{152}(1325 - \sqrt{292777}$ or $\delta \leq 5.157$.

The positive fuzzy solution,

$$\tilde{X} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \begin{pmatrix} (m_1^x, \alpha_1^x, \beta_1^x) \\ (m_2^x, \alpha_2^x, \beta_2^x) \end{pmatrix} = \begin{pmatrix} \left(\frac{74}{21-2\delta}, \frac{989-140\delta}{2(21-2\delta)^2}, \frac{668-46\delta}{(21-2\delta)^2}\right) \\ \left(\frac{162-26\delta}{21-2\delta}, \frac{1992-415\delta+28\delta^2}{2(21-2\delta)^2}, \frac{2694-598\delta+30\delta^2}{(21-2\delta)^2}\right) \end{pmatrix},$$

where $2 \leq \delta \leq 5.157$.

Similar to the previous example, the system has a positive fuzzy solution if and only if $\delta \in [2, 5.157]$.

- For instance, when $\delta = 5.1$,

$$\tilde{X} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \begin{pmatrix} (m_1^x, \alpha_1^x, \beta_1^x) \\ (m_2^x, \alpha_2^x, \beta_2^x) \end{pmatrix} = \begin{pmatrix} (6.851, 1.178, 3.715) \\ (2.722, 2.588, 3.639) \end{pmatrix},$$

we get a nonnegative fuzzy solution.

Also, if we suppose $\delta \notin [2, 5.157]$, then the solution will be negative or non-fuzzy solution with the following values:

- When $\delta = 5.2142$,

$$\tilde{X} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \begin{pmatrix} (m_1^x, \alpha_1^x, \beta_1^x) \\ (m_2^x, \alpha_2^x, \beta_2^x) \end{pmatrix} = \begin{pmatrix} (7, 1.1587, 3.831) \\ (2.5, 2.636, 3.503) \end{pmatrix},$$

we get negative fuzzy solution, since $m_2^x - \alpha_2^x = -0.13682$.

- When $\delta = 6.9$,

$$\tilde{X}=\begin{pmatrix}\tilde{x}_1\\ \tilde{x}_2\end{pmatrix}=\begin{pmatrix}(m_1^x,\alpha_1^x,\beta_1^x)\\ (m_2^x,\alpha_2^x,\beta_2^x)\end{pmatrix}=\begin{pmatrix}(10.27,0.221,6.763)\\ (-2.41,4.451,-0.0752)\end{pmatrix},$$

we get non-fuzzy solution, since $\beta_2^x \ngeq 0$.

Examples 4.3. *Consider the following FFLS,*

$$\begin{cases}(5,2,1)\otimes(m_1^x,\alpha_1^x,\beta_1^x)\oplus(7,3,1)\otimes(m_2^x,\alpha_2^x,\beta_2^x)=(\delta,46,54),\\ (6,1,2)\otimes(m_1^x,\alpha_1^x,\beta_1^x)\oplus(4,1,2)\otimes(m_2^x,\alpha_2^x,\beta_2^x)=(52,27,56),\end{cases}$$

where $\tilde{x}_i=({m_i}^x,{\alpha_i}^x,{\beta_i}^x)\geq 0, i=1,2,$ *and* δ *is an arbitrary real valued parameter not less than* 46, *determine when the system has a positive solution based on values of* δ.

Solution

$$\begin{pmatrix}(5,2,1) & (7,3,1)\\ (6,1,2) & (4,1,2)\end{pmatrix}\otimes\begin{pmatrix}(m_1^x,\alpha_1^x,\beta_1^x)\\ (m_2^x,\alpha_2^x,\beta_2^x)\end{pmatrix}=\begin{pmatrix}(\delta,46,54)\\ (52,27,56)\end{pmatrix}.$$

Where,

$A=\begin{pmatrix}5 & 7\\ 6 & 4\end{pmatrix}$, $M=\begin{pmatrix}2 & 3\\ 1 & 1\end{pmatrix}$, $N=\begin{pmatrix}1 & 1\\ 2 & 2\end{pmatrix}$,

then,

$$\boldsymbol{S}=\begin{pmatrix}\boldsymbol{A} & \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{M} & \boldsymbol{A} & \boldsymbol{0}\\ \boldsymbol{N} & \boldsymbol{0} & \boldsymbol{A}\end{pmatrix}=\begin{pmatrix}\begin{pmatrix}\mathbf{5} & \mathbf{7}\\ \mathbf{6} & \mathbf{4}\end{pmatrix} & \begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix} & \begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\\ \begin{pmatrix}\mathbf{2} & \mathbf{3}\\ \mathbf{1} & \mathbf{1}\end{pmatrix} & \begin{pmatrix}\mathbf{5} & \mathbf{7}\\ \mathbf{6} & \mathbf{4}\end{pmatrix} & \begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\\ \begin{pmatrix}\mathbf{1} & \mathbf{1}\\ \mathbf{2} & \mathbf{2}\end{pmatrix} & \begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix} & \begin{pmatrix}\mathbf{5} & \mathbf{7}\\ \mathbf{6} & \mathbf{4}\end{pmatrix}\end{pmatrix}.$$

$m^b=\begin{pmatrix}\delta\\ 52\end{pmatrix}$, $\alpha^b=\begin{pmatrix}46\\ 27\end{pmatrix}$, $\beta^b=\begin{pmatrix}54\\ 56\end{pmatrix}$**,** then, $B=\begin{pmatrix}m^b\\ \alpha^b\\ \beta^b\end{pmatrix}=\begin{pmatrix}\begin{pmatrix}\delta\\ 52\end{pmatrix}\\ \begin{pmatrix}46\\ 27\end{pmatrix}\\ \begin{pmatrix}54\\ 56\end{pmatrix}\end{pmatrix}=\begin{pmatrix}\delta\\ 52\\ 46\\ 27\\ 54\\ 56\end{pmatrix}$,

$m^x=\begin{pmatrix}m_1^x\\ m_2^x\end{pmatrix}$, $\alpha^x=\begin{pmatrix}\alpha_1^x\\ \alpha_2^x\end{pmatrix}$, $\beta^x=\begin{pmatrix}\beta_1^x\\ \beta_2^x\end{pmatrix}$, then, $X=\begin{pmatrix}m^x\\ \alpha^x\\ \beta^x\end{pmatrix}=\begin{pmatrix}\begin{pmatrix}m_1^x\\ m_2^x\end{pmatrix}\\ \begin{pmatrix}\alpha_1^x\\ \alpha_2^x\end{pmatrix}\\ \begin{pmatrix}\beta_1^x\\ \beta_2^x\end{pmatrix}\end{pmatrix}=\begin{pmatrix}m_1^x\\ m_2^x\\ \alpha_1^x\\ \alpha_2^x\\ \beta_1^x\\ \beta_2^x\end{pmatrix}$.

Then the associated linear system is given by $SX=B$:

$$\begin{pmatrix}5 & 7 & 0 & 0 & 0 & 0\\ 6 & 4 & 0 & 0 & 0 & 0\\ 2 & 3 & 5 & 7 & 0 & 0\\ 1 & 1 & 6 & 4 & 0 & 0\\ 1 & 1 & 0 & 0 & 5 & 7\\ 2 & 2 & 0 & 0 & 6 & 4\end{pmatrix}\begin{pmatrix}m_1^x\\ m_2^x\\ \alpha_1^x\\ \alpha_2^x\\ \beta_1^x\\ \beta_2^x\end{pmatrix}=\begin{pmatrix}\delta\\ 52\\ 46\\ 27\\ 54\\ 56\end{pmatrix},$$

$$\begin{cases} 5m_1^x + 7m_2^x = \delta, \\ 6m_1^x + 4m_2^x = 52, \\ 2m_1^x + 3m_2^x + 5\alpha_1^x + 7\alpha_2^x = 46, \\ m_1^x + m_2^x + 6\alpha_1^x + 4\alpha_2^x = 27, \\ m_1^x + m_2^x + 5\beta_1^x + 7\beta_2^x = 54, \\ 2m_1^x + 2m_2^x + 6\beta_1^x + 4\beta_2^x = 56. \end{cases}$$

The crisp solution can be easily obtained by solving the above linear system,

$$X = \begin{pmatrix} m_1^x \\ m_2^x \\ \alpha_1^x \\ \alpha_2^x \\ \beta_1^x \\ \beta_2^x \end{pmatrix} = \begin{pmatrix} -\frac{2}{11}(-91+\delta) \\ \frac{1}{11}(-130+3\delta) \\ \frac{1}{242}(-413+13\delta) \\ \frac{1}{242}(1967-25\delta) \\ \frac{1}{121}(708-5\delta) \\ \frac{2(173+\delta)}{121} \end{pmatrix},$$

$$\tilde{X} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \begin{pmatrix} (m_1^x, \alpha_1^x, \beta_1^x) \\ (m_2^x, \alpha_2^x, \beta_2^x) \end{pmatrix} = \begin{pmatrix} \left(-\frac{2}{11}(-91+\delta), \frac{1}{242}(-413+13\delta), \frac{1}{121}(708-5\delta)\right) \\ \left(\frac{1}{11}(-130+3\delta), \frac{1}{242}(1967-25\delta), \frac{2(173+\delta)}{121}\right) \end{pmatrix}.$$

- $\tilde{x}_1$ is a positive fuzzy number if and only if $\begin{cases} \alpha_1^x \leq m_1^x, \\ \alpha_1^x \geq 0, \\ \beta_1^x \geq 0, \end{cases}$

hence $\begin{cases} -\frac{2}{11}(-91+\delta) \geq \frac{1}{242}(-413+13\delta), \\ \frac{1}{242}(-413+13\delta) \geq 0, \\ \frac{1}{121}(708-5\delta) \geq 0, \end{cases}$, or $\begin{cases} 57\delta \leq 4417, \\ 13\delta \geq 413, \\ 5\delta \leq 708, \end{cases}$ then, $\frac{413}{13} \leq \delta \leq \frac{4417}{57}$,

- $\tilde{x}_2$ is a positive fuzzy number if and only if $\begin{cases} \alpha_2^x \le m_2^x, \\ \alpha_2^x \ge 0, \\ \beta_2^x \ge 0, \end{cases}$

  hence $\begin{cases} \frac{1}{11}(-130+3\delta) \ge \frac{1}{242}(1967-25\delta), \\ \frac{1}{242}(1967-25\delta) \ge 0, \\ \frac{2(173+\delta)}{121} \ge 0, \end{cases}$, or $\begin{cases} 91\delta \ge 4827, \\ 25\delta \le 1967, \\ 173+\delta \ge 0, \end{cases}$ then, $\frac{4827}{91} \le \delta \le \frac{1967}{25}$,

  using $\tilde{x}_1, \tilde{x}_2$ are positive fuzzy numbers where

  $\begin{cases} \frac{413}{13} \le \delta \le \frac{4417}{57}, \\ \frac{4827}{91} \le \delta \le \frac{4417}{57}, \end{cases}$ or $\delta \in [53.0439, 77.491]$.

- For instance, when $\delta = 54$,

  $$\tilde{X} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \begin{pmatrix} (m_1^x, \alpha_1^x, \beta_1^x) \\ (m_2^x, \alpha_2^x, \beta_2^x) \end{pmatrix} = \begin{pmatrix} (6.72, 1.19, 3.61) \\ (2.90, 2.54, 3.75) \end{pmatrix}.$$

  we get nonnegative fuzzy solution.

  Also, if we suppose that $\delta \notin [53.0439, 77.491]$, then the solution will be negative or non-fuzzy solution with the following values:

- When $\delta = 53$,

  $$\tilde{X} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \begin{pmatrix} (m_1^x, \alpha_1^x, \beta_1^x) \\ (m_2^x, \alpha_2^x, \beta_2^x) \end{pmatrix} = \begin{pmatrix} (6.909, 1.140, 3.661) \\ (2.6363, 2.652, 3.735) \end{pmatrix},$$

  we get negative fuzzy solution, since $m_2^x, -\alpha_2^x = -0.0157$.

- When $\delta = 80$,

  $$\tilde{X} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \begin{pmatrix} (m_1^x, \alpha_1^x, \beta_1^x) \\ (m_2^x, \alpha_2^x, \beta_2^x) \end{pmatrix} = \begin{pmatrix} (2, 2.59, 2.545) \\ (10 - 0.136, 4.18) \end{pmatrix},$$

  we get non-fuzzy solution, since $\alpha_2^x \ngeq 0$.

5 Conclusion

Solution of fully fuzzy linear systems can be obtained by classical methods of linear algebra through Gauss elimination method, Cramer's rule, Cholesky method, decomposition method, and other iterative methods. These methods obtained the solutions where the coefficients are known. In this study, row reduced echelon form is employed for FFLS. As a result, the solutions are obtained where some coefficients are known. In addition to that, many steps coming from fuzzy operation are reduced.

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