Total domination in plane triangulations

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Abstract

A total dominating set of a graph $G = (V, E)$ is a subset D of V such that every vertex in V is adjacent to at least one vertex in D. The total domination number of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G. A near-triangulation is a biconnected planar graph that admits a plane embedding such that all of its faces are triangles except possibly the outer face. We show in this paper that $\gamma_t(G) \leq \lfloor \frac{2n}{5} \rfloor$ for any near-triangulation G of order $n \geq 5$, with two exceptions.

Keywords: Total dominating sets, total domination number, near-triangulations, maximal planar graphs.

1. Introduction

Let $G = (V, E)$ be a simple graph. A *dominating set* of G is a subset $D \subseteq V$ such that every vertex not in D is adjacent to at least one vertex in D. The *domination number* of G, denoted by $\gamma(G)$, is defined as the minimum cardinality of a dominating set of G. Total dominating sets are defined in a similar way. A subset $D \subseteq V$ such that every vertex in V (including the vertices in D) is adjacent to a vertex in D is called a *total dominating set* (*TDS* for short) of G. The *total domination number*, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G. Since a total dominating set of a graph G is also a dominating set of G, the following inequality trivially holds $\gamma(G) \leq \gamma_t(G) \leq 2\gamma(G)$.

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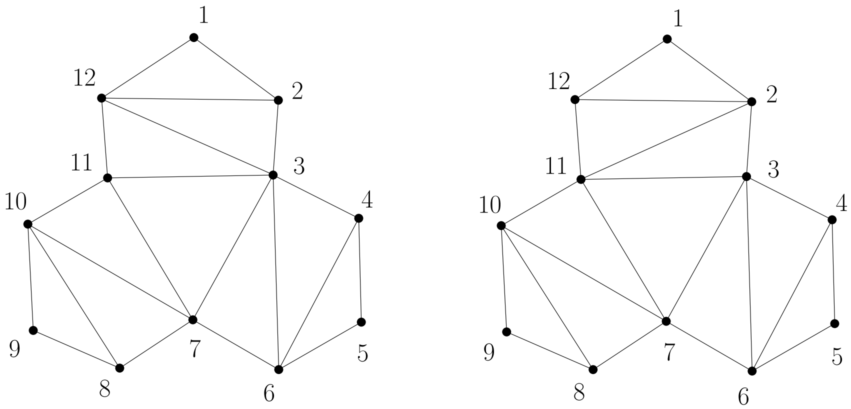


Figure 1: The two 12-vertex graphs H_1 and H_2

Domination and total domination in graphs have been widely studied in the literature. We refer the reader to [10, 9, 14] for excellent books on these topics and to [11] for a survey on total domination.

Given a graph G, it is well-known that computing $\gamma(G)$ or $\gamma_t(G)$ is an NP-hard problem, even when restricted to planar graphs. Hence, studying lower or upper bounds on the (total) domination number in some classes of graphs has been of interest during the last few years. In particular, for planar graphs, Matheson and Tarjan proved in [18] that $\gamma(G) \leq \lfloor \frac{n}{3} \rfloor$ for any n-vertex triangulated disc G. In the literature, triangulated discs are also called near-triangulations. A *near-triangulation* is a biconnected planar graph that has a plane embedding such that all of its faces are triangles except possibly the outer face. When the outer face is also a triangle, a near-triangulation is a *triangulation* or maximal planar graph. Note that removing an outer vertex from a triangulation gives a near-triangulation.

In the same paper [18], it is also conjectured that $\gamma(G) \leq \lfloor \frac{n}{4} \rfloor$ for any n-vertex triangulation G. King and Pelsmajer proved this conjecture in [15] for triangulations of maximum degree 6, and Plummer et al. proved in [20] that if G is an n-vertex Hamiltonian triangulation with minimum degree at least 4, then $\gamma(G) \leq \max\{\lceil 2n/7 \rceil, \lfloor 5n/16 \rfloor\}$. The upper bound $\frac{n}{3}$ for triangulations has been recently improved by Špacapan [22], showing that $\gamma(G) \leq \frac{17}{53}n$ for any n-vertex triangulation G, where $n > 6$.

Maximal outerplanar graphs are a special class of near-triangulations. A *maximal outerplanar graph*, MOP for short, is a near-triangulation such that all of its vertices belong to the boundary of the outer face. MOPs have additional properties that allow one to improve (or to prove) bounds for different types of problems on graphs. In [18], in addition to proving that $\gamma(G) \leq \frac{n}{3}$ for any n-vertex planar graph G, it is proved that this upper bound is tight for MOPs. In fact, the upper bound $\frac{n}{3}$ on the domination number in MOPs was already implicitly proved by Fisk [8]. In [1, 23], it is shown that $\gamma(G) \leq (n+k)/4$, where k is the number of vertices of degree 2 in a MOP G. Dorfling et al. proved in [6] that $\gamma_t(G) \leq \frac{n+k}{3}$ for a MOP G of order n with k vertices of degree 2. The same authors proved in [5] that apart from the graphs H_1 and H_2 shown in Figure 1,

$\gamma_t(G) \leq \lfloor \frac{2n}{5} \rfloor$ for a MOP G of order $n \geq 5$. In [17], Lemanska et al. presented an alternative proof of this last result. The reader is referred to [2, 3, 4, 12, 16] for other results in MOPs related to some variants of the domination concept.

In this paper, we extend the result proved in [5, 17] to the family of near-triangulations and we show that $\gamma_t(G) \leq \lfloor \frac{2n}{5} \rfloor$ for any near-triangulation G of order $n \geq 5$, apart from the graphs H_1 and H_2. Thus, we improve the best known upper bound $\frac{6}{11}n$ on the total domination number of n-vertex near-triangulations. This last bound follows from the fact that a near-triangulation is 2-connected and from the following result proved in [13]: If G is a 2-connected graph of order $n > 18$, then $\gamma_t(G) \leq \frac{6}{11}n$.

The upper bound $\lfloor \frac{2n}{5} \rfloor$ on the total domination number in near-triangulations is proved in Section 4. The proof is based on induction and combines common techniques used when proving results for MOPs, as the ones described in [17], with techniques related to what we call *reducible* and *irreducible* near-triangulations, and *terminal polygons* in irreducible near-triangulations. These concepts are defined in Section 4. In the induction process, the two exception graphs H_1 and H_2 can appear after removing some vertices or some edges from a near-triangulation. For these two graphs, induction cannot be applied since their total domination numbers are greater than $\lfloor \frac{2n}{5} \rfloor$. For this reason, we explain in Section 3 how to obtain suitable total dominating sets for some graphs involving H_1 and H_2 that will be used in the inductive proof. Section 2 is devoted to review some known properties for near-triangulations, and to show some special cases in which the removal of some vertices or the contraction of some edges from a near-triangulation, results in another near-triangulation. These cases will be needed in the inductive proof. We conclude the paper with some remarks in Section 5.

2. Near-triangulations and some of their properties

For the sake of simplicity, throughout the paper the term near-triangulation will refer to a near-triangulation $T = (V, E)$ that has been drawn in the plane without crossings, using straight-line segments, such that all of its faces are triangles except possibly the outer face (see Figure 2a). Such a drawing always exists by Fáry's Theorem [7]. We assume that the boundary of the outer face is given by the cycle $C = (u_1, u_2, \ldots, u_h, u_1)$, with its $h \geq 3$ vertices in clockwise order. In this way, we can refer to boundary edges and vertices (the edges and vertices of C), interior vertices (the vertices not in C), and diagonals (edges connecting two non-consecutive vertices of C). Recall that if $h = 3$, then T is a triangulation and if $h = |V|$, then T is a MOP.

In [17], the authors use induction to prove that $\gamma_t(T) \leq \lfloor \frac{2n}{5} \rfloor$ for a MOP T of order $n \geq 21$. The two main properties they use are that after contracting a boundary edge of T, the resulting graph is again a MOP, and that there is always a diagonal dividing T into two MOPs, leaving $5, 6, 7$ or 8 consecutive boundary edges of C in the smallest one. However, these two properties are not true for arbitrary near-triangulations. Sometimes, there are no diagonals dividing a near-triangulation T into smaller near-triangulations, and even in the case that such diagonals exist, a diagonal leaving $5, 6, 7$ or 8 consecutive boundary edges in the smallest near-triangulation cannot be chosen. Besides, in general, after contracting

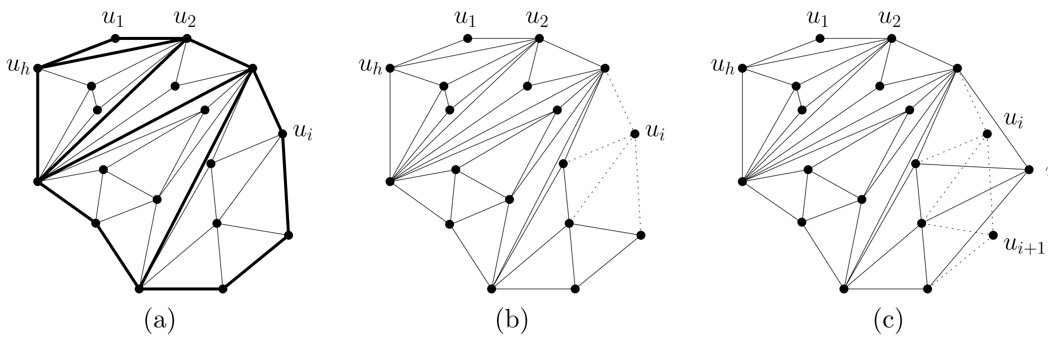


Figure 2: (a) A near-triangulation. The thick segments correspond to $T[C]$. (b) Removing a vertex of degree 2 in $T[C]$. (c) Contracting the edge (u_i, u_{i+1}) to the vertex u_i'.

a boundary edge, the resulting graph is not a near-triangulation. Therefore, we cannot follow in our inductive proof the same steps as described in [17], although we will use some of the ideas given in that paper.

We show in this section several cases in which the removal of some vertices or the contraction of some edges from a near-triangulation results in another near-triangulation. These cases will be enough for our purposes. Before stating them, we shall give some terminology and some properties.

Given a near-triangulation T with boundary cycle $C = (u_1, u_2, \ldots, u_h, u_1)$, we use $T[C]$ to denote the subgraph of T induced by the vertices in C (see Figure 2a). Observe that $T[C]$ is always Hamiltonian and outerplane (all the vertices belong to the boundary of the outer face). The following result for a Hamiltonian outerplanar graph is well-known.

Lemma 1. *Let G be a Hamiltonian outerplanar graph of order $n \geq 4$. Then, G contains at least two non-adjacent vertices of degree 2.*

Given a graph $G = (V, E)$, the graph obtained from G by deleting the vertices $\{v_1, \ldots, v_k\}$ and all their incident edges is denoted by $G - \{v_1, \ldots, v_k\}$. It is straightforward to prove the following lemma for near-triangulations (see Figure 2b).

Lemma 2. *Let T be a near-triangulation of order $n \geq 4$ with boundary cycle C. Then, $T - \{v\}$ is a near-triangulation if and only if v is an interior vertex of degree 3 or v is a vertex of degree 2 in $T[C]$.*

Let $G = (V, E)$ be a graph and let $e = (v_i, v_j)$ be an edge of G. We use $G - e$ to denote the graph obtained from G by removing e, and G/e to denote the graph obtained from G by contracting the edge e, that is, the simple graph obtained from G by deleting v_i, v_j and all their incident edges, adding a new vertex w and connecting w to each vertex v that is adjacent to either v_i or v_j in G (see Figure 2c). Observe that by Euler's formula, contracting an edge $e = (v_i, v_j)$ from a triangulation T results in another triangulation if and only if v_i and v_j have exactly two common neighbors. Besides, the two endpoints of an edge $e = (v_i, v_j)$ of T have exactly two common neighbors if and only if the edge e is not an edge of a *separating triangle* (a triangle containing vertices inside and outside).

We say that an edge e of a near-triangulation T is *contractible* if the graph T/e is also a near-triangulation. Since by adding a vertex w in the outer face of T and by connecting w to the vertices in C (the boundary cycle associated with T) we obtain a triangulation, then we have the following lemma.

Lemma 3. *Let T be a near-triangulation with boundary cycle C and let e be an edge of T. Then, the edge e is contractible if and only if e is neither a diagonal of T nor an edge of a separating triangle of T.*

The following lemma summarizes some of the cases in which we obtain new near-triangulations after removing vertices from a near-triangulation.

Lemma 4. *Let T be a near-triangulation with boundary cycle $C = (u_1, \dots, u_h, u_1)$. Suppose that T contains at least one interior vertex and has no diagonals. Let u_i be a vertex in C. Then:*

i) $T - \{u_i\}$ is also a near-triangulation.

ii) Assuming that T contains at least two interior vertices, there exists a vertex u_j with $i \le j < i - 1 + h \pmod h$ and an interior vertex v_j adjacent to u_j such that $T - \{u_i, u_{i+1}, \dots, u_j, v_j\}$ is a near-triangulation. In addition, the edge (u_j, v_j) is contractible in T.

iii) If the edge $e_i = (u_{i-1}, u_i)$ is not contractible in T, then there exists an interior vertex v_i adjacent to u_i such that $T - \{u_i, v_i\}$ is a near-triangulation.

Proof. Since the starting vertex of C is arbitrary, we may assume without loss of generality that u_i is u_2.

i) There are no diagonals in T, so the degree of u_2 in $T[C]$ is 2. Thus, the statement follows from Lemma 2.

ii) Let $u_1, w_1, \dots, w_k, u_3$ be the set of neighbors of u_2 in T, in counterclockwise order. Since there are no diagonals in T, u_2 is a vertex of degree 2 in $T[C]$ and $k \ge 1$. By Lemma 2, after removing u_2 we obtain a new near-triangulation $T_2 = T - \{u_2\}$ with boundary cycle $C_2 = (u_1, w_1, \dots, w_k, u_3, u_4, \dots, u_1)$ (see Figure 3b).

We repeat this operation and we remove from T_2 the first vertex w of degree 2 in $T_2[C_2]$, clockwise from u_1. By Lemma 2, we obtain again a near-triangulation $T_3 = T - \{u_2, w\}$ with boundary cycle C_3. Iterating this process, we obtain a sequence of near-triangulations $T_2, T_3, \dots, T_j, T_{j+1}$, where T_{i+1} is obtained from T_i, for $i = 2, \dots, j$, by removing from T_i the first vertex w of degree 2 in $T_i[C_i]$, clockwise from u_1, and where we have stopped the process the first time that w is an interior vertex in T. Hence, $T_{j+1} = T - \{u_2, u_3, \dots, u_j, v_j\}$, for some interior vertex v_j. See Figure 3 for an illustration of this process. Next we prove the following claim.

Claim 1. *For $i = 2, \dots, j$, the boundary cycle C_i of T_i consists of the following vertices and in this (clockwise) order: The vertex u_1, some vertices that are interior in T, and the boundary vertices $u_{i+1}, u_{i+2}, \dots, u_h$.*

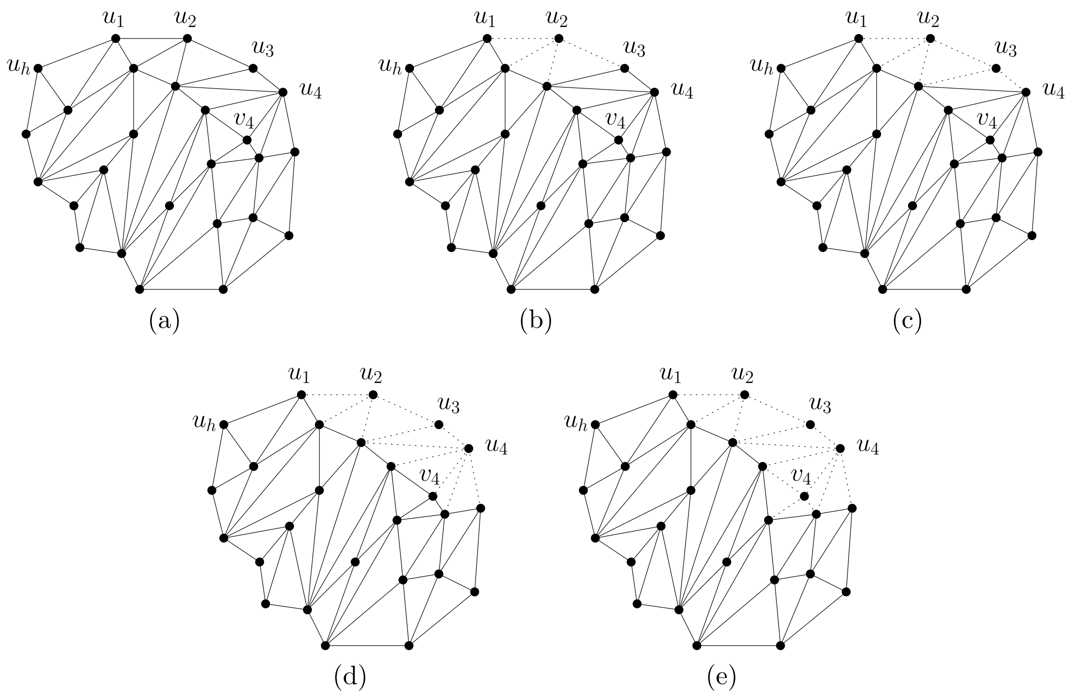


Figure 3: (a) A near-triangulation T without diagonals. (b), (c), (d) and (e) Obtaining the near-triangulations T_2, T_3, T_4 and T_5 by removing successively the vertices u_2, u_3, u_4 and v_4.

Proof of Claim 1. The proof is by induction. The claim is obviously true for T_2, the base case. Assume that the claim is true for $T_2, \ldots, T_i$, where $i < j$, and that C_i is $(u_1, x_1, \ldots, x_l, u_{i+1},\ u_{i+2}, \ldots, u_h, u_1)$, with $x_1, \ldots, x_l$ being interior vertices in T. Let us prove that T_{i+1} satisfies the claim. By the construction of T_{i+1} and since $i < j$, none of the vertices $x_1, \ldots, x_l$ has degree 2 in $T_i[C_i]$. Let us see that u_{i+1} is the first vertex of degree 2 in $T_i[C_i]$ from u_1. Assume to the contrary that its degree in $T_i[C_i]$ is greater than 2, so there is a diagonal (u_{i+1}, y) in $T_i[C_i]$. Since T has no diagonals, the vertex y necessarily is one of the vertices in $\{x_1, \ldots, x_{l-1}\}$, say x_k. But then, by Lemma 1, in the subgraph induced by the vertices $x_k, x_{k+1}, \ldots, x_l, u_{i+1}$ there is a vertex x_j of degree 2 different from x_k and u_{i+1}, that also is a vertex of degree 2 in $T_i[C_i]$, which is a contradiction. Hence, u_{i+1} is a vertex of degree 2 in $T_i[C_i]$. If we remove it from T_i, then the new cycle C_{i+1} corresponding to T_{i+1} is obtained from C_i by adding the neighbors of u_{i+1} in T_i between x_l and u_{i+2}. Therefore, the claim follows. □

From the claim, the set of boundary vertices removed to obtain T_j is $\{u_2, \ldots, u_j\}$, as required. Let us now see that during the previous process, there is always a first time in which an interior vertex in T can be removed. Assume that the process does not finish before removing u_{h-1}. By removing u_{h-1}, we obtain a near-triangulation T_{h-1} with boundary cycle $C_{h-1} = (u_1, x_1, \ldots, x_l, u_h, u_1)$. By hypothesis, T contains at least two interior vertices, so T_{h-1} is not a triangle. Thus, by Lemma 1, $T_{h-1}[C_{h-1}]$ contains a vertex x_j of degree 2, different from u_1 and u_h, that must be an interior vertex in T, and can be removed from T_{h-1} to obtain a near-triangulation by Lemma 2.

Let v_j be the interior vertex in T removed from T_j to obtain T_{j+1}. To finish this part of the proof, we need to show that u_j and v_j are adjacent and that (u_j, v_j) is contractible in T. Let $C_{j-1} = (u_1, y_1, \ldots, y_m, u_j, u_{j+1}, \ldots, u_h, u_1)$ be the boundary cycle of T_{j-1}. By hypothesis, none of the vertices $y_1, \ldots, y_m$, has degree 2 in $T_{j-1}[C_{j-1}]$. Since the vertex u_j has degree 2 in $T_{j-1}[C_{j-1}]$, it is not connected to any of $\{y_1, \ldots, y_{m-1}\}$, so when removing u_j from T_{j-1}, the only vertex among $\{y_1, \ldots, y_m\}$ that could decrease its degree in $T_j[C_j]$ in relation to its degree in $T_{j-1}[C_{j-1}]$ is precisely y_m. Therefore, v_j is either y_m or one of the new vertices that appear in C_j. Since all of these vertices are neighbors of u_j, then u_j and v_j are adjacent.

Let us prove that (u_j, v_j) is contractible in T. Assume to the contrary that the edge (u_j, v_j) is not contractible. T has no diagonals, hence there exists a separating triangle $\Delta = (u_j, v_j, v)$ in T by Lemma 3. The vertex u_j has degree 2 in $T_{j-1}[C_{j-1}]$ and T has no diagonals, so all the neighbors of u_j in T must belong to C_j except for u_{j-1}. The vertex v_j has degree 2 in $T_j[C_j]$, hence the only neighbors of u_j adjacent to v_j are the predecessor and the successor of v_j in C_j. Thus, v must be one of these two vertices. But in both cases, Δ would be empty, contradicting that Δ is separating. Therefore, (u_j, v_j) is contractible in T.

iii) Suppose that the edge (u_1, u_2) is not contractible. Since T contains no diagonals, by Lemma 3 this edge must belong to a separating triangle $\Delta = (u_1, u_2, w)$ containing some vertices inside, with w being an interior vertex in T. If Δ only contains a vertex w_i, then $T - \{u_2, w_i\}$ is clearly a near-triangulation by Lemma 2, because w_i is an interior vertex of degree 3 and u_2 is a vertex of degree 2 in $T[C]$. If Δ contains two or more vertices, then part ii) of this lemma can be applied to the triangulation T' induced by Δ and its interior vertices, so there is a vertex w_i inside Δ such that $T' - \{u_2, w_i\}$ is a near-triangulation. As a consequence, $T - \{u_2, w_i\}$ is also a near-triangulation. □

To finish this section, we show that for a boundary vertex, there is always a contractible edge incident to it.

Lemma 5. *Let T be a near-triangulation of order $n \geq 5$, with boundary cycle $C = (u_1, \ldots, u_h, u_1)$, and let u_i be a vertex in C. Then,*

i) If u_i has a neighbor not in C, then there exists an interior vertex v such that the edge (u_i, v) is contractible.

ii) If all neighbors of u_i are in C, then the edges (u_{i-1}, u_i) and (u_i, u_{i+1}) are contractible.

Proof. i) Suppose that the edge $e = (u_i, v)$ is not contractible, with $v \notin C$. Then e must be an edge of a separating triangle $\Delta = (u_i, v, w)$. All vertices inside Δ are interior vertices in T, and the subgraph induced by Δ and its interior vertices is a triangulation T'. If Δ contains at least two vertices, then, by Lemma 4(ii), there exists an interior vertex v' such that $T' - \{u_i, v'\}$ is a near-triangulation and (u_i, v') is contractible in T'. But this edge is also contractible in T. If Δ only contains an interior vertex z, then the edge (u_i, z) is clearly contractible in T.

ii) Suppose that the edge $e = (u_{i-1}, u_i)$ is not contractible. This edge is not a diagonal, hence there exists a separating triangle $\Delta = (u_{i-1}, u_i, u)$ containing at least one interior vertex. Thus, at least one of these interior vertices must be adjacent to u_i, which is a contradiction because we are assuming that all neighbors of u_i belong to C. Therefore, (u_{i-1}, u_i) is contractible. By the same argument, the edge (u_i, u_{i+1}) is also contractible. □

3. Dominating sets for some near-triangulations

In this section we show how to build (total) dominating sets in some special cases of near-triangulations. These dominating sets are needed in the proof of the main theorem. We first give the following results for triangulated pentagons and hexagons, and MOPs in general [5, 17].

Lemma 6 ([5, 17]). *Let T be a MOP of order 5 and let $C = (u_1, \ldots, u_5, u_1)$ be its boundary cycle. For every vertex u_i, there exists a TDS in T whose size is 2 and contains u_i.*

Lemma 7 ([5, 17]). *Let T be a MOP of order 6 and let $C = (u_1, \ldots, u_6, u_1)$ be its boundary cycle. For every pair u_i, u_{i+1} of consecutive vertices in C, there exists a TDS in T whose size is 2 and contains either u_i or u_{i+1}.*

Theorem 1 ([5, 17]). *If T is a MOP of order $n \geq 5$ and $T \notin \{H_1, H_2\}$, then $\gamma_t(T) \leq \lfloor \frac{2n}{5} \rfloor$.*

The following lemma provides total dominating sets in some cases that involve the graphs H_1 and H_2.

Lemma 8. *Let T be a near-triangulation with boundary cycle $C = (u_1, \ldots, u_h, u_1)$.*

I) For every vertex $u_i \in C$, T has a TDS of size 5 containing u_i if one of the following cases holds:

- *i) T is either H_1 or H_2.*
- *ii) $T - u_i$ is either H_1 or H_2.*
- *iii) $T - \{u_i, v_i\}$ is either H_1 or H_2 for some interior vertex v_i adjacent to u_i.*
- *iv) T/e is either H_1 or H_2 by contracting some edge e incident with u_i.*

II) For every edge $e_i = (u_i, u_{i+1})$ (where $i + 1$ is taken modulo h), T has a TDS of size 4 containing u_i or u_{i+1} if $T - e_i$ is H_1 or H_2.

Proof. We prove the lemma assuming that H_1 is T or the graph obtained from T. The analysis is totally analogous if H_2 is T or the graph obtained from T. Let Δ be the central triangle of H_1, consisting of the vertices w_1, w_2 and w_3. See Figure 4a. The three triangles that contain the three vertices of degree 2 are denoted by Δ_1, Δ_2 and Δ_3, respectively, where w_i is not adjacent to any vertex in Δ_i, for $i = 1, 2, 3$.

i) Suppose that $T = H_1$. If u_i belongs to Δ, say $u_i = w_1$, then w_1, its two neighbors in C and two arbitrary vertices in Δ_1 form a TDS D (see Figure 4a). If u_i belongs to one of

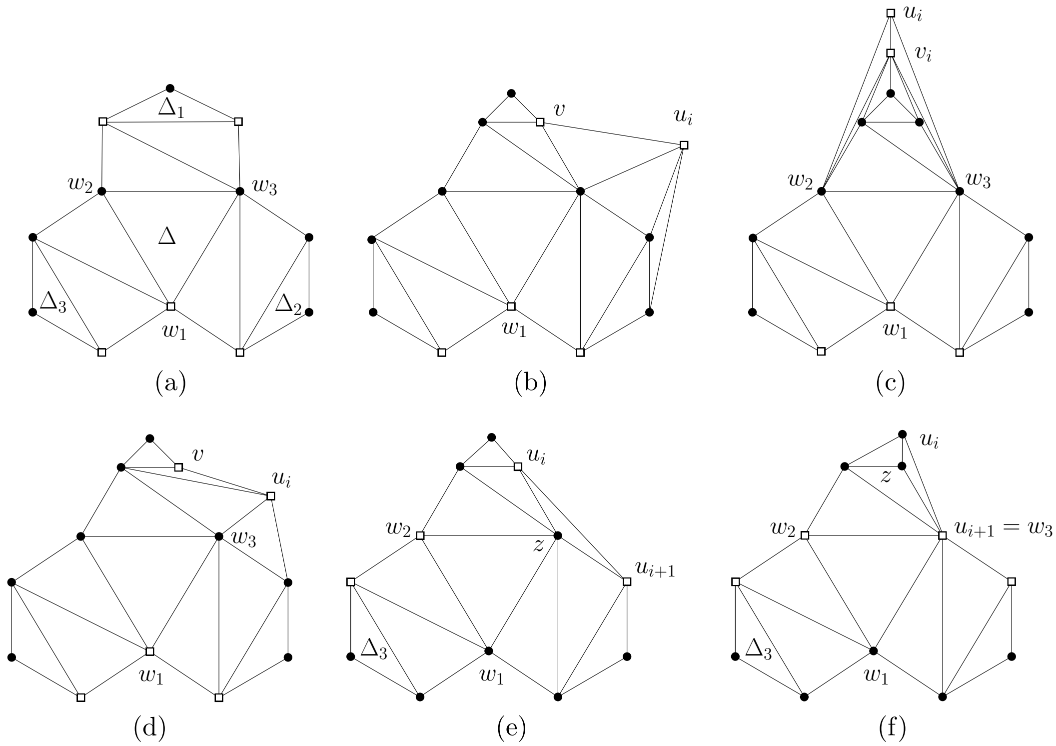


Figure 4: Illustrating Lemma 8. In each case, the squared vertices form a TDS of T. (a) The graph H_1. (b) Removing the vertex u_i. (c) Removing the vertices u_i and v_i. (d) The vertex w_3 is the vertex obtained by contracting the edge (u_i, v_i). (e) and (f) Removing the edge (u_i, u_{i+1}).

Δ_1, Δ_2 and Δ_3, say Δ_1, then D is also a TDS by choosing u_i as one of the vertices of Δ_1 in D.

ii) Suppose that $T - u_i$ is H_1. In this case, u_i has at least two neighbors in T that necessarily are consecutive vertices on the boundary of H_1 (see Figure 4b). Hence, u_i has a neighbor v in one of the triangles Δ_1, Δ_2 and Δ_3, say triangle Δ_1. Then, u_i, v and the three vertices of a TDS of the MOP $H_1 - \Delta_1$ of order 9 define a TDS of T.

iii) Suppose that $T - \{u_i, v_i\}$ is H_1 for some interior vertex v_i adjacent to u_i. Assume first that u_i has a neighbor v in one of the triangles Δ_1, Δ_2 and Δ_3, say triangle Δ_1. As in the previous case, u_i, v and the three vertices of a TDS of the MOP $H_1 - \Delta_1$ of order 9 define a TDS of T.

Assume now that none of the vertices in Δ_1, Δ_2 and Δ_3 is adjacent to u_i. In this case, since T is a near-triangulation and u_i a boundary vertex, then v_i must be adjacent to all the vertices of at least one of the triangles Δ_1, Δ_2 and Δ_3, say Δ_1 (see Figure 4c for an example). Therefore, u_i, v_i, and the three vertices of a TDS of the MOP $H_1 - \Delta_1$ define a TDS of T.

iv) Suppose that H_1 is obtained from T by contracting an edge $e = (u_i, v_i)$ incident with u_i, and let w be the new vertex obtained after contracting this edge. If w is one of the vertices of Δ_1, Δ_2 or Δ_3, say Δ_1, then the set formed by u_i, v_i and the three vertices of a TDS of $H_1 - \Delta_1$ is a TDS of T.

On the contrary, suppose that w is one of the vertices of Δ, say w_3 (see Figure 4d for an example). In this case, u_i has a neighbor v in T belonging to either Δ_1 or Δ_2. Assume that v belongs to Δ_1. The set formed by u_i, v and the three vertices of a TDS of $H_1 - \Delta_1$ is a TDS of T.

II) Suppose that $T - e$ is H_1 for some edge (u_i, u_{i+1}). Let z be the third vertex of the triangle in T containing e. Then z belongs to one of the triangles Δ, Δ_1, Δ_2 or Δ_3. Suppose first that z belongs to Δ. We may assume that $z = w_3$ (see Figure 4e). Then, u_i belongs to Δ_1 and u_{i+1} to Δ_2 or viceversa. According to Lemma 6, there is a TDS D of size 2 containing w_2 in the triangulated pentagon defined by w_1, w_2 and Δ_3. Therefore, u_i, u_{i+1} and D define a TDS of size 4 in T.

Now suppose that z belongs to one of the triangles Δ_1, Δ_2 or Δ_3, say Δ_1 (see Figure 4f). Thus, one of the vertices of $\{u_i, u_{i+1}\}$ is the vertex of degree 2 of Δ_1 and the other one is w_2 or w_3, say w_3. If D is a TDS of size 2 containing w_2 in the triangulated pentagon defined by w_1, w_2 and Δ_3, then D together with w_3 and a vertex in Δ_2 adjacent to w_3 form a TDS of size 4 in T. $\square$

To finish this section, we give some bounds on the size of a (total) dominating set of a near-triangulation under the contraction operation. Given a simple graph $G = (V, E)$, we say that a vertex $v \in V$ dominates a vertex $u \in V$ if v and u are adjacent in G. Thus, a vertex $v \in V$ dominates all its neighbors in G but not itself.

Lemma 9. *Let T be a near-triangulation of order $n \geq 5$ with boundary cycle $C = (u_1, \ldots, u_h, u_1)$. Suppose that for some vertex u_i there is a contractible edge $e = (u_i, v_i)$ of T such that T/e has a TDS of size s. Then:*

I) T has a set of vertices D satisfying one of the following conditions:

i) D is a TDS of size $s+1$ in T such that u_i and v_i belong to D,

ii) D is a set of vertices of size s such that neither u_i nor v_i belong to D and D dominates all vertices of T except possibly one of u_i or v_i.

II) There is a dominating set D of size $s+1$ in T such that D contains u_i and either D is a TDS of T or D dominates all vertices of T except possibly u_i.

Proof. I) The result follows from the same well-known result for abstract graphs: If G/e is the graph obtained by contracting an edge $e = (u_i, v_i)$ of G to a new vertex w, according to whether w belongs to a TDS D' of size s in G/e or not, either i) the set $D = \{D' - w\} \cup \{u_i, v_i\}$ is a TDS of G or ii) $D = D'$ dominates all vertices of G except possibly u_i or v_i.

II) As before, if the new vertex w belongs to a TDS D' of size s in G/e, then $D = \{D' - w\} \cup \{u_i, v_i\}$ is a TDS of G. Otherwise, the set $D = D' \cup \{u_i\}$ dominates all vertices of T except possibly u_i. $\square$

4. Upper bound for near-triangulations

In this section we prove the main result of this paper: the upper bound $\lfloor \frac{2n}{5} \rfloor$ on the total domination number in near-triangulations of order n. Before proving it, we define the two main concepts required in its proof: reducible near-triangulations and terminal polygons.

Let T be a near-triangulation with some interior vertices and boundary cycle $C = (u_1, u_2, \ldots, u_h, u_1)$. We say that T is *reducible* if it contains a triangle (u_i, u_{i+1}, v) with v a vertex not in C. In this case, by removing the boundary edge $u_i u_{i+1}$, we obtain a new near-triangulation T' with boundary cycle $C' = (u_1, \ldots, u_i, v, u_{i+1}, \ldots, u_h, u_1)$. Obviously, $\gamma_t(T) \leq \gamma_t(T')$, and T' contains fewer interior vertices than T. If T' is also reducible, then we can obtain a new near-triangulation T'' with fewer interior points than T'. Iterating this process, we reach either a near-triangulation without interior vertices (a MOP), or a near-triangulation with interior vertices that is *irreducible*, that is, a near-triangulation with interior vertices such that for every boundary edge (u_i, u_{i+1}), the vertex v in the triangular face (u_i, u_{i+1}, v) adjacent to (u_i, u_{i+1}) is also in C. See Figure 5 for some examples of irreducible near-triangulations.

With these definitions, note that if T is a near-triangulation, then T is either reducible, or irreducible, or a MOP. Also note that if a near-triangulation T is irreducible, then an interior vertex together with its neighbors induce a wheel of order at least 4 such that every boundary edge of the wheel is an internal edge in T (otherwise T would be reducible). Therefore, the order of an irreducible near-triangulation is at least 7. Figure 5a shows the simplest irreducible near-triangulation. As a consequence, if $n \leq 6$ then every near-triangulation is either reducible or a MOP.

Let T be an irreducible near-triangulation with boundary cycle $C = (u_1, u_2, \ldots, u_h, u_1)$. The diagonals of the subgraph $T[C]$ divide the interior of C into several regions whose interiors are disjoint. These regions are simple polygons that can be non-empty or empty, depending on whether they contain interior vertices of T or not (see Figure 5b). Let $P_1, \ldots P_k$ denote the polygons obtained in this way such that they contain some interior vertex of T. The irreducible near-triangulation shown in Figure 5b contains five non-empty polygons P_1, P_2, P_3, P_4 and P_5. Observe that, by definition, every side d of a polygon P_i has to be a diagonal of $T[C]$, and that P_i has no diagonals. Therefore, a side d of a polygon P_i divides T into two non-empty near-triangulations $T_{in}(P_i, d)$ and $T_{out}(P_i, d)$ sharing d, where $T_{in}(P_i, d)$ denotes the near-triangulation containing the polygon P_i. In Figure 5b, $T_{in}(P_5, (u, u'))$ is the near-triangulation of order 6 containing P_5.

We say that a non-empty polygon P_i is *terminal* if at most one of the near-triangulations $T_{out}(P_i, d)$ corresponding to the sides d of P_i (diagonals in $T[C]$) contains interior vertices. Hence, if P_i is a terminal polygon with k sides, then at least $k-1$ of the near-triangulations $T_{out}(P_i, d)$ are MOPs with at least three vertices. The irreducible near-triangulation shown in Figure 5b contains three terminal polygons P_2, P_3, P_5. The following lemma shows that an irreducible near-triangulation has at least one terminal polygon.

Lemma 10. *Let T be an irreducible near-triangulation of order $n \geq 7$ with boundary cycle C. Then, T contains at least one terminal polygon.*

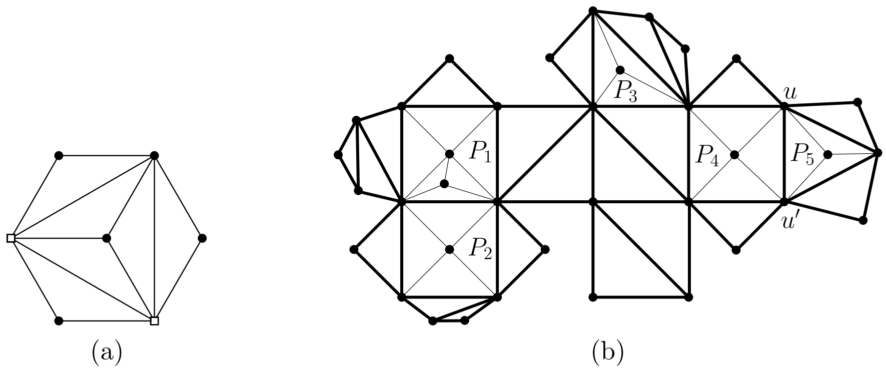


Figure 5: (a) The simplest irreducible near-triangulation H. The squared vertices form a total dominating set. (b) An irreducible near-triangulation T. Thick lines correspond to the subgraph $T[C]$. The diagonals of $T[C]$ define a set of adjacent polygons, five of which are non-empty and three are terminal, P_2, P_3 and P_5.

Proof. Since T is irreducible, it must contain non-empty polygons. Consider the dual graph $G = (V, E)$ associated with $T[C]$, where the vertices of G are the faces defined by $T[C]$ and two vertices are adjacent in G if their corresponding faces are adjacent. Since $T[C]$ is a Hamiltonian outerplane graph, G must be a tree. Note that each non-empty polygon of $T[C]$ is a vertex of G.

If there is only one non-empty polygon, then it is terminal. Otherwise, observe that terminal polygons correspond to the leaves of the minimal subtree of G containing all the vertices corresponding to non-empty polygons. Since every non-trivial tree has at least two leaves, then the lemma follows. □

We are now ready to prove the main result of the paper, Theorem 2. To this end, we also need the following two lemmas. The first one was proved in [19, 21] and the proof of the second one is straightforward.

Lemma 11 ([19, 21])**.** *Given a MOP G of order $n \geq 10$ and a boundary edge (u_i, u_{i+1}) of G, there exists a diagonal d of G that partitions G into two MOPs, one of which contains exactly $6, 7, 8$ or 9 vertices of G and does not contain (u_i, u_{i+1}).*

Lemma 12. *Let n, k, d be positive integers. If $n-k \geq 5$ and $d/k \leq 2/5$, then $\lfloor \frac{2(n-k)}{5} \rfloor + d \leq \lfloor \frac{2n}{5} \rfloor$.*

Theorem 2. *If $T = (V, E)$ is a near-triangulation of order $n \geq 5$, different from H_1 and H_2, then $\gamma_t(G) \leq \lfloor \frac{2n}{5} \rfloor$.*

Proof. Let $f(n) = \lfloor \frac{2n}{5} \rfloor$. Thus, $f(n-k) + d \leq f(n)$ if $n - k \geq 5$ and $d/k \leq 2/5$.

We proceed by induction on the number m of interior vertices of T and the number n of vertices of T. For $m = 0$, the base of the induction, T is a MOP and the result is true by Theorem 1.

Let T be a near-triangulation of order n, with $m > 0$ interior vertices and boundary cycle $C = (u_1, \ldots, u_h, u_1)$. Suppose that $\gamma_t(T') \leq f(n')$ for any near-triangulation T' of order $n' \geq 5$ such that one of the following conditions holds:

- T' is different from H_1 and H_2, and T' has $m' < m$ interior vertices.
- T' has $m' = m$ interior vertices and $n' < n$ vertices.

Under this assumption, we need to prove that $\gamma_t(T) \leq f(n)$. To make further reasoning easier, we prove the following claim.

Claim 2. *Let T be a near-triangulation of order $n \geq 6$, with m interior vertices and with boundary cycle $C = (u_1, \ldots, u_h, u_1)$. Assume that the previous induction hypotheses hold, that is, $\gamma_t(T') \leq f(n')$ for any near-triangulation T' of order $n' \geq 5$ such that one of the following conditions holds:*

- *T' is different from H_1 and H_2, and T' has $m' < m$ interior vertices.*
- *T' has $m' = m$ interior vertices and $n' < n$ vertices.*

Then, for any vertex $u_i \in C$, there exists a dominating set D of size at most $f(n-1)+1$ such that D contains u_i and all the vertices of T are dominated except possibly u_i.

Proof of the claim. Assume that T is neither H_1 nor H_2. By Lemma 5, there is always a contractible edge $e = (u_i, v_i)$ with u_i as one of its endpoints. Note that T/e has either fewer interior vertices than T or the same number of interior vertices but $n-1$ vertices. Thus, if T/e is not H_1 or H_2, then $\gamma_t(T/e) \leq f(n-1)$ by the induction hypotheses. In this case, the result follows from Lemma 9 (II). If T/e is H_1 or H_2, then the order of T is 13 and the result follows from Lemma 8 (I) (iv), since $f(12)+1 = 5$. Finally, if T is H_1 or H_2, then Lemma 8 (I) (i) ensures the result because $f(11)+1 = 5$. □

Let us go into the details of the proof of the theorem. Assume first that T is reducible. Hence, by removing a suitable boundary edge (u_i, u_{i+1}) we obtain a near-triangulation T' of order n with $m-1$ interior vertices. If T' is H_1 or H_2, then T has 12 vertices and Lemma 8 (II) guarantees that $\gamma_t(T) = 4 = f(12)$. Otherwise, the induction hypothesis can be applied to T', so $\gamma_t(T) \leq \gamma_t(T') \leq f(n)$.

Assume then that T is irreducible, hence $n \geq 7$ and T contains at least one terminal polygon P by Lemma 10, with $k \geq 3$ sides $d_1 = (u'_1, u'_2), d_2 = (u'_2, u'_3), \ldots, d_k = (u'_k, u'_1)$. Note that the vertices $u'_1, \ldots, u'_k$ of P correspond to vertices in C and assume that they are in clockwise order. Without loss of generality, we may assume that $u'_1 = u_1$. For $j = 1, \ldots, k$, every near-triangulation $T_{out}(P, d_j) = M_j$ is a MOP, except possibly one of them, say $T_{out}(P, d_k) = M_k$. Let $\overline{M_j}$ denote the near-triangulation $T_{in}(P, d_j)$, so $|M_j| + |\overline{M_j}| = n+2$. Observe that, since P is non-empty and has no diagonals, $\overline{M_j}$ is a reducible near-triangulation, for $j = 1, \ldots, k$, because d_j can be removed from $\overline{M_j}$ (see Figure 6).

We prove that $\gamma_t(T) \leq f(n)$ by applying induction to a suitable near-triangulation obtained after some graph operations. We distinguish cases according to the sizes of the MOPs M_j.

Removing vertices from one MOP

We begin analyzing the cases when there is a MOP M_j such that either $|M_j| \in \{4, 6, 7, 8\}$, or $|M_j| = 9$ and d_j is contractible in $\overline{M_j}$, or $|M_j| > 9$. These cases are the same as those described in [17], except for the case $|M_j| = 4$, and the analysis is totally analogous. For the sake of completeness, we include them. Note that $\overline{M_j}$ contains interior vertices, so it is neither H_1 nor H_2, and has at least 6 vertices (because T is irreducible). Therefore, the induction hypothesis can be applied to $\overline{M_j}$, if necessary.

Case 1: $|M_j| = 4$.

Suppose that there is a MOP M_j of order 4 (M_5 in Figure 6)[1]. One of u'_j or u'_{j+1} is a dominating set of M_j (the vertex u'_6 in Figure 6). Suppose that u'_{j+1} is such a vertex (the same reasoning can be applied in the other case). Note that $\overline{M_j}$ has $n-2$ vertices and is reducible because the edge (u'_j, u'_{j+1}) can be removed from $\overline{M_j}$. Let (u'_{j+1}, u_i) be the other boundary edge of $\overline{M_j}$ incident with u'_{j+1}.

From $\overline{M_j}$, we build another reducible near-triangulation $\overline{M_j}'$ of order n, by adding two vertices w_1 and w_2 and the edges $(u'_{j+1}, w_1), (u'_{j+1}, w_2), (w_2, w_1)$ and (u_i, w_2) in the outer face, that is, a MOP of order 4 is joined to the edge (u'_{j+1}, u_i). Since $\overline{M_j}'$ is reducible, the induction hypothesis can be applied to $\overline{M_j}'$, so it has a TDS D of size at most $f(n)$. Recall that Lemma 8 (II) guarantees the same bound for D, even in the case that either H_1 or H_2 is obtained after the reduction.

From D, we build as follows another TDS D' of $\overline{M_j}'$ such that $|D'| \leq f(n)$, D' contains u'_{j+1} and does not contain w_1 or w_2. The degree of w_1 in $\overline{M_j}'$ is 2, hence at least one of u'_{j+1} and w_2 must belong to D so that w_1 is dominated. Suppose that u'_{j+1} belongs to D. If neither w_1 nor w_2 belongs to D, we are done. Otherwise, since the neighbors of w_1 and w_2 are also neighbors of u'_{j+1}, by removing w_1 and w_2 from D (at least one belongs to D) and by adding a neighbor of u'_{j+1} to D (if no neighbor of u'_{j+1} different from w_1 and w_2 belongs to D), we obtain such a set D'. On the contrary, suppose that u'_{j+1} does not belong to D but w_2 does. Thus, by removing w_2 from D and by adding u'_{j+1} to D (and removing w_1 and adding a neighbor of u'_{j+1} different from w_1 and w_2 if w_1 belongs to D), such a set D' is obtained. Since u'_{j+1} dominates the vertices of M_j, then D' is a TDS of T and $\gamma_t(T) \leq f(n)$.

Case 2: $|M_j| = 6$.

Suppose that there is a MOP M_j of order 6 (M_3 in Figure 6). Since M_j is a triangulated hexagon, by Lemma 7, either u'_j and one of its neighbors, or u'_{j+1} and one of its neighbors form a TDS of the triangulated hexagon M_j. Assume that $\{u'_j, u\}$ is such a set (the other case is analyzed in the same way). By Claim 2, $\overline{M_j}$ has a set D of size at most $f(n-5)+1$ containing the vertex u'_j and dominating all the vertices of $\overline{M_j}$ except possibly u'_j. But

[1]We remark that particular graphs M_i are drawn in Figures 6 and 7, but our claims apply for all possible graphs and drawings.

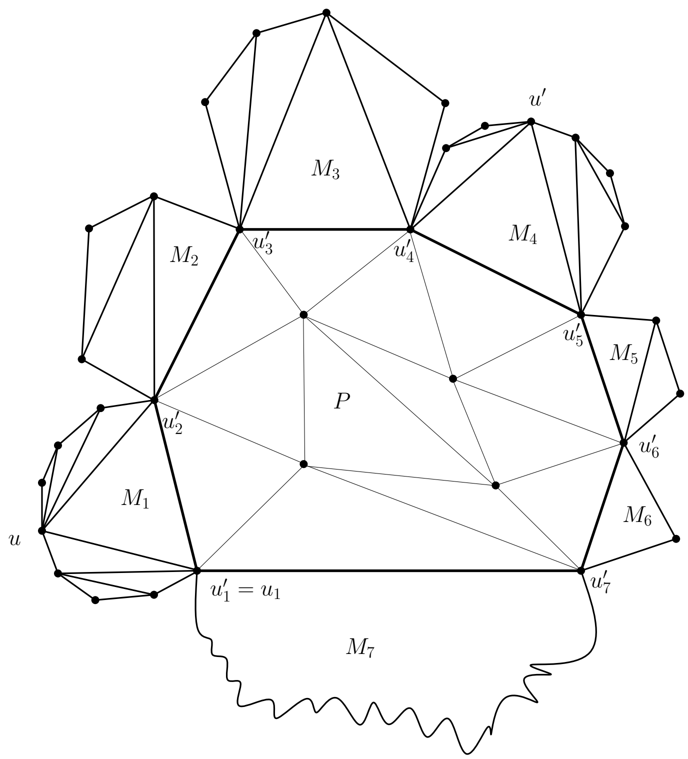


Figure 6: A terminal 7-gon P with 6 MOPs M_1, M_2, M_3, M_4, M_5 and M_6 of orders $9, 5, 6, 8, 4$ and 3, respectively, around it.

then, the set $D \cup \{u\}$ is a TDS of T with size at most $f(n-5)+2 = f(n)$.

Case 3: $|M_j| = 7$.

Suppose that there is a MOP M_j of order 7. In this case, a TDS of $\overline{M_j}$ has size at most $f(n-5)$ by the induction hypothesis. This set can be transformed into a TDS of T by adding a TDS of M_j that consists of two vertices by Theorem 1. Therefore, $f(n-5)+2 = f(n)$, so $\gamma_t(T) \leq f(n)$.

Case 4: $|M_j| = 8$.

Suppose that there is a MOP M_j of order 8 (M_4 in Figure 6). Let $\{u'_j = u_k, \ldots, u_{k+7} = u'_{j+1}\}$ denote the vertices of M_j. Let $\Delta = (u'_j, u'_{j+1}, u')$ be the triangle adjacent to the edge (u'_j, u'_{j+1}) in M_j. If u' is u_{k+1}, u_{k+2}, u_{k+5} or u_{k+6}, then either (u'_j, u') or (u'_{j+1}, u') defines a MOP of order 6 or 7, and we can argue as in Cases 2 or 3, respectively.

Assume that $u' = u_{k+3}$ (the case $u' = u_{k+4}$ is symmetric). By removing the vertices $u_{k+1}, u_{k+2}, u_{k+4}, u_{k+5}, u_{k+6}$ from T, we obtain a new near-triangulation T' of order $n-5 \geq 7$ and m interior vertices. By the induction hypothesis, T' has a TDS D' of size at most $f(n-5)$ that necessarily contains either u'_j or u'_{j+1} since the degree of u' in T' is 2.

If D' contains u'_j, then by adding u' and a suitable vertex v adjacent to u' in the triangulated pentagon $\{u', u_{k+4}, u_{k+5}, u_{k+6}, u'_{j+1}\}$, we obtain a TDS of T with size at most $f(n-5)+2 = f(n)$. If D' contains u'_{j+1}, applying Lemma 6 to the triangulated pentagons

$\{u'_{j+1}, u', u_{k+4}, u_{k+5}, u_{k+6}\}$ and $\{u'_{j+1}, u'_j, u_{k+1}, u_{k+2}, u'\}$, we can then obtain a TDS in T of size at most $f(n-5)+2$, by adding one additional vertex in each one of these two triangulated pentagons.

Case 5: $|M_j| = 9$ and d_j is contractible in $\overline{M_j}$.

Suppose that there is a MOP M_j of order 9 (M_1 in Figure 6). Let $\Delta = (u'_j, u'_{j+1}, u)$ be the triangle adjacent to the edge (u'_j, u'_{j+1}) in M_j and let $\{u'_j = u_k, \ldots, u_{k+8} = u'_{j+1}\}$ denote the vertices of M_j. If u is u_{k+1}, u_{k+2}, u_{k+3}, u_{k+5}, u_{k+6} or u_{k+7}, then either (u'_j, u) or (u'_{j+1}, u) defines a MOP of order 6, 7 or 8, and we can argue as in Cases 2, 3 or 4, respectively.

Assume that $u = u_{k+4}$. In this case, the sets of vertices $\{u'_j, u_{k+1}, u_{k+2}, u_{k+3}, u\}$ and $\{u, u_{k+5}, u_{k+6}, u_{k+7}, u'_{j+1}\}$ induce two triangulated pentagons. Since d_j is contractible in $\overline{M_j}$, then $\overline{M_j}/d_j$ is a near-triangulation of order $n-8 \geq 5$ with m interior vertices. Thus, $\overline{M_j}/d_j$ is different from H_1, H_2 and has a TDS of size at most $f(n-8)$ by the induction hypothesis.

As a consequence, by Lemma 9(I), $\overline{M_j}$ has either a TDS D of size at most $f(n-8)+1$ containing u'_j and u'_{j+1}, or a set D of size at most $f(n-8)$, not containing either u'_j or u'_{j+1}, and dominating every vertex of $\overline{M_j}$ except possibly u'_j or u'_{j+1}. In the first case, by Lemma 6 we can add to D a suitable vertex in each one of the two previous triangulated pentagons, so that the resulting set is a TDS of T of size at most $f(n-8)+3 \leq f(n)$. In the second case, by Theorem 1, there is a TDS D' of size 3 in M_j. Therefore, $D \cup D'$ is a TDS in T of size $f(n-8)+3 \leq f(n)$.

Case 6: $|M_j| > 9$.

Suppose that there is a MOP M_j of order greater than 9. By Lemma 11, there is a diagonal d in M_j such that it partitions M_j into two MOPs, one of which, M', has $6, 7, 8$ or 9 vertices and does not contain the edge (u'_j, u'_{j+1}). Therefore, we can also argue as in Cases 2, 3, 4 and 5 by removing M' from T, since d is contractible in the near-triangulation obtained after removing M'.

Removing vertices from two or more MOPs

We now study irreducible near-triangulations where all MOPs M_j are of order $3, 5$ or 9. Besides, the case of a MOP M_j of order 9 must be analyzed only when d_j is not contractible in $\overline{M_j}$. In this situation, we have to remove vertices from more than one MOP. Most of the cases can be solved by removing vertices from two consecutive MOPs M_j and M_{j+1} around the terminal polygon P. We recall that M_k can be a MOP or not. Since we are studying irreducible near-triangulations, if M_k is not a MOP, then it has interior vertices and is either irreducible or reducible such that d_k is the only boundary edge that can be removed from M_k. This implies that necessarily $|M_k| \geq 6$. If M_k is a MOP, we can assume that it is the largest one, among all MOPs M_j adjacent to P (by renumbering them if necessary).

If there exist at least two MOPs of different sizes, then we can assume that there are two consecutive MOPs M_j and M_{j+1} such that $\{|M_j|, |M_{j+1}|\}$ are either $\{5, 3\}$, or $\{9, 3\}$, or $\{9, 5\}$ Otherwise, all the MOPs are of order either 3 or 5 or 9. For the sake of clarity and since, as we will see, the reasoning used in the proof holds for every pair of consecutive MOPs of different order, we assume that these MOPs of different order, whenever they exist, are M_1 and M_2 and that $|M_1| > |M_2|$.

Let $\overline{M}$ denote the near-triangulation obtained by removing from T the vertices of M_1 and M_2 that are not in P. Hence, $|\overline{M}| = n - (|M_1| - 2) - (|M_2| - 2)$ and (u'_1, u'_2) and (u'_2, u'_3) are boundary edges of $\overline{M}$. Observe that $|\overline{M}| \geq 2 + |M_k|$, since P is at least a triangle containing at least one interior vertex and M_k is included in $\overline{M}$. Therefore, the induction hypothesis can be applied to $\overline{M}$ when necessary, since $|\overline{M}| \geq 5$ and it is not either H_1 or H_2 ($\overline{M}$ contains interior vertices). Next, we analyze all possible combinations of the sizes of M_j's.

Case 7: $|M_1| = 5$ and $|M_2| = 3$.

Since M_1 is a triangulated pentagon, M_1 has a TDS formed by the vertex u'_2 and one of its neighbors u' by Lemma 6 (see Figure 7a). Besides, P does not contain diagonals, so there is no diagonal incident to u'_2 in $\overline{M}$. By Lemma 2, $\overline{M} - \{u'_2\}$ is a near-triangulation of order $n - 5$. Recall that if M_k is a MOP, then $|M_k| \geq 5$ and if it is not a MOP, then $|M_k| \geq 6$. It follows that the induction hypothesis can be applied to $\overline{M} - \{u'_2\}$ because $n - 5 = |\overline{M}| - 1 \geq |M_k| + 1 \geq 6$ (the interior vertices of P belong to $\overline{M}$ and there is at least one).

Suppose that $\overline{M} - \{u'_2\}$ is neither H_1 nor H_2, so it has a TDS D of size at most $f(n-5)$ by the induction hypothesis. Thus, $D \cup \{u'_2, u'\}$ is a TDS of T of size at most $f(n-5) + 2 = f(n)$. On the contrary, if $\overline{M} - \{u'_2\}$ is either H_1 or H_2, then Lemma 8 (I) (ii) guarantees that $\overline{M}$ has a TDS D' of size 5 containing u'_2. Therefore, $D' \cup \{u'\}$ is a TDS in T of size 6, so $\gamma_t(T) \leq f(n)$ since the order of T is 17 and $f(17) = 6$.

Case 8: $|M_1| = 9$, $|M_2| = 3$ and $d_1 = (u'_1, u'_2) = (u_1, u_9)$ is not contractible.

Arguing as in Case 5, we may assume that $\Delta = (u_1, u_9, u_5)$ is the triangle adjacent to the edge (u_1, u_9) in M_1, because otherwise a MOP of order 6,7 or 8 could be removed. Thus, the vertices $\{u_1, u_2, u_3, u_4, u_5\}$ and $\{u_5, u_6, u_7, u_8, u_9\}$ induce two triangulated pentagons, P' and P'', respectively (Figure 7b). Applying Lemma 6 to P' and P'', there exist two vertices $u' \in P'$ and $u'' \in P''$ such that $\{u_5, u', u''\}$ is a TDS of M_1.

Since $d_1 = (u'_1, u'_2)$ is not contractible in $\overline{M_1}$, then it is also not contractible in the near-triangulation T' induced by the vertices of the terminal polygon P and the vertices inside P. T' has no diagonals, hence there exists a vertex v_2 inside P by Lemma 4(iii), such that v_2 is adjacent to $u'_2 = u_9$ and $T' - \{u'_2, v_2\}$ is a near-triangulation. As a consequence, $\overline{M} - \{u'_2, v_2\}$ is a near-triangulation of order $n - 10 \geq 7$ (recall that $|M_k| \geq 6$). If $\overline{M} - \{u'_2, v_2\}$ is neither H_1 nor H_2, then it has a TDS D of size at most $f(n-10)$ by the induction hypothesis. Thus, $D \cup \{u_5, u', u'', u'_2\}$ is clearly a TDS of T with size at most $f(n-10) + 4 = f(n)$, so $\gamma_t(T) \leq f(n)$. On the contrary, if $\overline{M} - \{u'_2, v_2\}$ is either H_1 or H_2, then $n = 22$ and Lemma 8 (I) (iii) ensures that there exists a TDS D containing u'_2 of size 5 in $\overline{M}$. The set $D \cup \{u_5, u', u''\}$ is a TDS of T with size $8 = f(22)$.

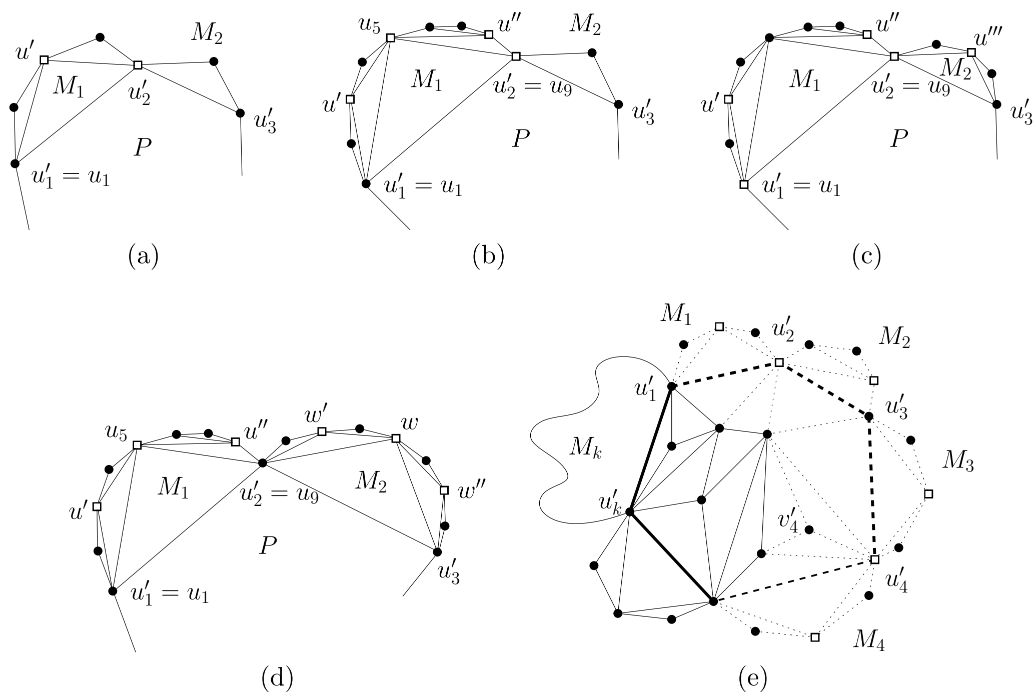


Figure 7: (a) Case 7: u'_2 and u' define a TDS in M_1. (b) Case 8: u_5, u' and u'' form a TDS in M_1. (c) Case 9: u'_1, u'_2, u', u'' and u''' are a TDS in $M_1 \cup M_2$. (d) Case 11: u_5, u', u'', w, w' and w'' form a TDS in $M_1 \cup M_2$. (e) Case 12: Removing the MOPs M_1, M_2, M_3 and M_4, and the vertices u'_2, u'_3, u'_4 and v'_4 to obtain the near-triangulation $\overline{M}$. The squared vertices form a TDS of $\{M_1 \cup M_2\} \cup \{M_3 \cup M_4\}$.

Case 9: $|M_1| = 9$, $|M_2| = 5$ and $d_1 = (u'_1, u'_2) = (u_1, u_9)$ is not contractible.

Arguing as in Case 8, we may assume that $\Delta = (u_1, u_9, u_5)$ is the triangle adjacent to the edge (u_1, u_9) in M_1 (so $\{u_1, u_2, u_3, u_4, u_5\}$ and $\{u_5, u_6, u_7, u_8, u_9\}$ induce two triangulated pentagons, P' and P''), and that $\overline{M} - \{u'_2, v_2\}$ is a near-triangulation of order $n - 12 \geq 7$.

By Claim 2, $\overline{M} - \{u'_2, v_2\}$ has a set D of size $\leq f(n-13) + 1$ containing the vertex u'_1 and dominating all the vertices of $\overline{M} - \{u'_2, v_2\}$ except possibly u'_1. We add u'_2 to D and, by Lemma 6, we can also add to D a vertex u' to dominate P', a vertex u'' to dominate P'' and a vertex u''' to dominate M_2 (see Figure 7c). Therefore, $D \cup \{u'_2, u', u'', u'''\}$ is a TDS of T with size at most $f(n-13) + 5 \leq f(n)$.

Case 10: All MOPs M_j are of order 3, so $|M_1| = |M_2| = 3$.

This case is similar to Case 1. $\overline{M}$ is reducible (any of (u'_1, u'_2) and (u'_2, u'_3) can be removed), hence the graph $\overline{M}'$ of order n, obtained from $\overline{M}$ by adding two vertices w_1, w_2 in the outer face and the edges $(u'_2, w_1), (u'_2, w_2), (w_2, w_1), (u'_3, w_2)$, is also reducible by removing (u'_1, u'_2). Arguing as in Case 1, $\overline{M}'$ has a TDS D' of size at most $f(n)$ containing the vertex u'_2 and not containing either w_1 or w_2, even in the case that $\overline{M}'$ is reducible to either H_1 or H_2[2]. Therefore, $\gamma_t(T) \leq f(n)$ since D' is also a TDS of T.

Case 11: All MOPs M_j are of order 9 and all d_j are not contractible.

We have $|M_1| = |M_2| = 9$ and d_1 and d_2 are not contractible. As in case 8, we may assume that $\Delta = (u_1, u_9, u_5)$ is the triangle adjacent to the edge (u_1, u_9) in M_1, so $\{u_1, u_2, u_3, u_4, u_5\}$ and $\{u_5, u_6, u_7, u_8, u_9\}$ induce two triangulated pentagons, P' and P''. Therefore, there exist two vertices $u' \in P'$ and $u'' \in P''$ such that $D_1 = \{u_5, u', u''\}$ is a TDS of M_1. The same happens in M_2, so M_2 has a TDS $D_2 = \{w, w', w''\}$ of size 3 (see Figure 7d).

Since P contains no diagonals, $\overline{M}' = \overline{M} - \{u_9\}$ is a near-triangulation of order $n-15 \geq 7$ by Lemma 2. We claim that $\overline{M}'$ is neither H_1 nor H_2. We recall that $\overline{M}'$ must contain M_k. If M_k is not a MOP, then it contains interior vertices, so $\overline{M}'$ is neither H_1 nor H_2. Assume to the contrary that M_k is a MOP, so $|M_k| \geq 9$ by hypothesis, and that $\overline{M}'$ is H_1 (the same reasoning applies if $\overline{M}'$ is H_2). P is terminal, hence some vertices of H_1 must be interior vertices in $\overline{M}$, implying that d_k is a diagonal of H_1. Thus, by the symmetry of H_1 (see Figure 1), d_k can only be one of the edges $(3,7), (3,6)$ and $(4,6)$. If d_k is $(3,6)$ or $(4,6)$, then it defines a MOP of size at least 10 and we are in Case 6. If d_k is $(3,7)$, then it defines a MOP of size 9, where $(3,7)$ would be contractible in $\overline{M_k}$ and we would be in Case 5. Hence, $\overline{M}'$ is neither H_1 nor H_2.

As a consequence, $\overline{M}'$ has a total dominating set D of size at most $f(n-15)$ by the induction hypothesis. Therefore, $D \cup D_1 \cup D_2$ is a TDS in T of size at most $f(n-15)+6 = f(n)$.

Case 12: All MOPs M_j are of order 5.

The case $|M_j| = 5$ for every MOP M_j is the only case left. We recall that $(u'_1, u'_2), \ldots, (u'_k, u'_1)$ denote the diagonals $d_1, \ldots, d_k$ of T defining the terminal polygon P, and that M_k can also be a MOP when P is the only non-empty polygon of T. If it is the case, then M_k must also have 5 vertices. Next, we explain how to obtain a TDS of size at most $f(n)$, by removing vertices from several consecutive MOPs.

Let T' be the near-triangulation induced by P and its interior vertices. We distinguish whether T' has one interior vertex or more than one.

Assume first that T' has at least two interior vertices. By Lemma 4(ii), there is a vertex u'_j in T', $2 \leq j < k$, and an interior vertex v'_j adjacent to u'_j such that the graph $T' - \{u'_2, \ldots, u'_j, v'_j\}$ is a near-triangulation. As a consequence, by removing the vertices in the MOPs $M_1, M_2, \ldots, M_j$ that do not belong to P, and the vertices $u'_2, u'_3, \ldots, u'_j, v'_j$, we obtain a near-triangulation $\overline{M}'$ of size $|\overline{M}'| = n - 3j - j = n - 4j \geq 6$ (see Figure 7e). Since every MOP M_i is a triangulated pentagon, observe that the vertex u'_i, $2 \leq i \leq j$, a neighbor v_{i-1} of u'_i in M_{i-1} and another neighbor v_i of u'_i in M_i, form a TDS of size 3 of $M_{i-1} \cup M_i$.

Suppose that j is an even number. If $\overline{M}'$ is neither H_1 nor H_2, it contains a TDS D of

[2] In fact, a detailed analysis of cases shows that $\overline{M}'$ can be neither H_1 nor H_2.

size at most $f(n-4j)$ by the induction hypothesis. If $\overline{M}'$ is either H_1 or H_2, by Lemma 8 (I) (ii) there exists a TDS D in $\overline{M}' \cup \{v'_j\}$ of size 5 containing v'_j. Therefore, the set D together with the 3-vertex sets $\{v_{i-1}, u'_i, v_i\}$, for $i = 2, 4, \ldots, j$, form a TDS of T with size at most $f(n-4j) + 3j/2$ in the first case and with size $5 + 3j/2$ in the second case. By Lemma 12, $f(n-4j)+3j/2 \le f(n)$ because $\frac{3j/2}{4j} < \frac{2}{5}$, and trivially $5+3j/2 \le \lfloor \frac{2}{5}(12+4j) \rfloor$ for even $j \ge 2$. Hence, $\gamma_t(T) \le f(n)$.

Suppose now that j is an odd number. By Claim 2, even if $\overline{M}'$ is either H_1 or H_2, we can obtain a set D of vertices in $\overline{M}'$ such that the size of D is at most $f(n-4j-1)+1$, D contains the vertex u'_1 and D dominates all vertices of $\overline{M}'$ except possibly u'_1. Since M_1 is a triangulated polygon, u'_1 and one of its neighbors, say v_1, form a TDS of M_1. Thus, by adding to D the vertex v_1 and the 3-vertex sets $\{v_{i-1}, u'_i, v_i\}$, for $i = 3, 5, \ldots, j$, we obtain a TDS of size at most $f(n-4j-1)+1+1+\frac{3}{2}(j-1)$. By Lemma 12, $f(n-4j-1)+1+1+\frac{3}{2}(j-1) \le f(n)$ because $\frac{2+3(j-1)/2}{4j+1} \le 2/5$, hence $\gamma_t(T) \le f(n)$. Note that v_j is dominated by u'_j.

Finally, assume that T' has only one interior vertex v, so T' is a wheel. By removing the vertices in $M_1, \ldots, M_{k-1}$ not in P, the vertices $u'_2, \ldots, u'_{k-1}$ and the vertex v, we obtain a near-triangulation $\overline{M}'$ that coincides with M_k, and we argue as in the previous paragraphs depending on the parity of j. We remark that $\overline{M}'$ can be neither H_1 nor H_2, and that if j is an odd number and $\overline{M}' = M_k$ is a triangulated pentagon (so Claim 2 cannot be applied), then we chose a TDS of size 2 including u'_1 in $\overline{M}'$. □

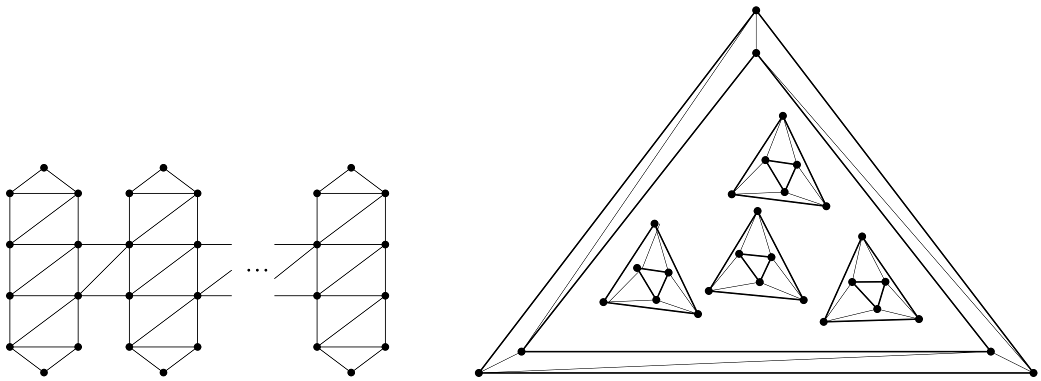
5. Final remarks

In this paper, we proved that the total domination number for any n-vertex near-triangulation is at most $\lfloor \frac{2n}{5} \rfloor$ with two exceptions. The proof is by induction and is based on a new decomposition of some near-triangulations (the irreducible ones) into several near-triangulations, using what we call terminal polygons.

To finish this paper, we give the following conjecture.

Conjecture 1. *For any triangulation T of order $n \ge 6$, $\gamma_t(T) \le \lfloor \frac{n}{3} \rfloor$.*

The conjecture is based on the following. The bound $\lfloor \frac{2n}{5} \rfloor$ on the total domination number in near-triangulations is tight, since there are near-triangulations achieving the bound. Figure 8a shows one of these near-triangulations. However, all the examples reaching the bound that we know are MOPs. For triangulations, we feel that the total domination number should be smaller and close to $n/3$. This bound would be tight because there are triangulations reaching it. Figure 8b shows one of them. It consists of an octahedron containing in its interior other $k-1$ octahedra. The inter-octahedra region can be triangulated in any way. It is not difficult to see that any TDS for this triangulation must contain at least two vertices of each octahedron.



(a) (b)

Figure 8: (a) A MOP T of order n such that $\gamma_t(T) = \lfloor \frac{2n}{5} \rfloor$. Any TDS must contain at least two vertices of each MOP of order 5. (b) Triangulating in any way the inter-octahedra region, a triangulation T of order n is obtained such that $\gamma_t(T) = \lfloor \frac{n}{3} \rfloor$.

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