On a problem of Sárközy and Sós for multivariate linear forms

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Abstract

We prove that for pairwise co-prime numbers $k_1, \dots, k_d \geq 2$ there does not exist any infinite set of positive integers $\mathcal{A}$ such that the representation function $r_{\mathcal{A}}(n) = \#\{(a_1, \dots, a_d) \in \mathcal{A}^d : k_1 a_1 + \cdots + k_d a_d = n\}$ becomes constant for n large enough. This result is a particular case of our main theorem, which poses a further step towards answering a question of Sárközy and Sós and widely extends a previous result of Cilleruelo and Rué for bivariate linear forms (Bull. of the London Math. Society 2009).

Javier Cilleruelo, in Memoriam

1 Introduction

Let $\mathcal{A} \subseteq \mathbb{N}_0$ be an infinite set of positive integers and $k_1, \dots, k_d \in \mathbb{N}$. We are interested in studying the behaviour of the representation function

$$r_{\mathcal{A}}(n) = r_{\mathcal{A}}(n; k_1, \dots, k_d) = \#\{(a_1, \dots, a_d) \in \mathcal{A}^d : k_1 a_1 + \cdots + k_d a_d = n\}.$$

More specifically, Sárközy and Sós [5, Problem 7.1.] asked for which values of $k_1, \dots, k_d$ one can find an infinite set $\mathcal{A}$ such that the function $r_{\mathcal{A}}(n; k_1, \dots, k_d)$ becomes constant for n large enough. For the base case, it is clear that $r_{\mathcal{A}}(n; 1, 1)$ is odd whenever $n = 2a$ for some $a \in \mathcal{A}$ and even otherwise, so that the representation function cannot become constant. For $k \geq 2$, Moser [3] constructed a set $\mathcal{A}$ such that $r_{\mathcal{A}}(n; 1, k) = 1$ for all $n \in \mathbb{N}_0$. The study of bivariate linear forms was completely settled by Cilleruelo and the first author [1] by showing that the only cases in which $r_{\mathcal{A}}(n; k_1, k_2)$ may become constant are those considered by Moser.

The multivariate case is less well studied. If $\gcd(k_1, \dots, k_d) > 1$, then one trivially observes that $r(n; k_1, \dots, k_d)$ cannot become constant. The only non-trivial case studied so far was the following: for $m > 1$ dividing d, Rué [4] showed that if in the d-tuple of coefficients

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$(k_1, \ldots, k_d)$ each element is repeated m times, then there cannot exists an infinite set $\mathcal{A}$ such that $r_{\mathcal{A}}(n; k_1, \ldots, k_d)$ becomes constant for n large enough. This for example covers the case $(k_1, k_2, k_3, k_4, k_5, k_6) = (2, 4, 6, 2, 4, 6)$. Observe that each coefficient in this example is repeated twice, that is $m = 2$.

In this paper we provide a step beyond this result and show that whenever the set of coefficients is pairwise co-prime, then there does not exists any infinite set $\mathcal{A}$ for which $r(n; k_1, \ldots, k_d)$ is constant for n large enough. This is a particular case of our main theorem, which covers a wide extension of this situation:

Theorem 1.1. *Let $k_1, \ldots, k_d \geq 2$ be given for which there exist pairwise co-prime integers $q_1, \ldots, q_m \geq 2$ and $b(i,j) \in \{0, 1\}$, such that for each i there exists at least one j such that $b_{i,j} = 1$. Let $k_i = q_1^{b(i,1)} \cdots q_m^{b(i,m)}$ for all $1 \leq i \leq d$. Then, for every infinite set $\mathcal{A} \subseteq \mathbb{N}_0$ $r_{\mathcal{A}}(n; k_1, \ldots, k_d)$ is not a constant function for n large enough.*

In particular, if $m = d$ and for each $i \neq j$ $(q_i, q_j) = 1$ as well as $b(i,j) = 1$ if $i = j$ and $b(i,j) = 0$ otherwise, then this represents the case where $k_1, \ldots, k_d \geq 2$ are pairwise co-prime numbers. Other new cases covered by this result are for instance $(k_1, k_2, k_3) = (2, 3, 2 \times 3)$ as well as $(k_1, k_2, k_3, k_4) = (2^2 \times 3, 2^2 \times 5, 3 \times 5, 2^2 \times 3 \times 5)$.

Our method starts with some ideas introduced in [1] dealing with generating functions and cyclotomic polyomials (see Section 2). The main new idea in this paper is to use an inductive argument in order to be able to show that a certain multivariate recurrence relation is not possible to be satisfied unless some initial condition is trivial.

2 Preliminaries

Generating functions. The language in which we will approach this problem goes back to [2]. Let $f_{\mathcal{A}}(z) = \sum_{a \in \mathcal{A}} z^a$ denote the *generating function* associated with $\mathcal{A}$ and observe that $f_{\mathcal{A}}$ defines an analytic function in the complex disc $|z| < 1$. By a simple argument over the generating functions, it is easy to verify that the existence of a set $\mathcal{A}$ for which $r_{\mathcal{A}}(n; k_1, \ldots, k_d)$ becomes constant would imply that

$$f_{\mathcal{A}}(z^{k_1}) \cdots f_{\mathcal{A}}(z^{k_d}) = \frac{P(z)}{1 - z}$$

for some polynomial P with positive integer coefficients satisfying $P(1) \neq 0$. To simplify notation, we will generally consider the d-th power of this equations, that is for $F(z) = f_{\mathcal{A}}^d(z)$ we have

$$F(z^{k_1}) \cdots F(z^{k_d}) = \frac{P^d(z)}{(1 - z)^d}. \tag{1}$$

Observe that $F(z)$ also defines an analytic function in the complex disk $|z| < 1$. This is the starting point of the proof of Theorem 1.1, mainly building upon the tools developed in [1] using properties relating to cyclotomic polynomials that we will in the following review briefly (see [1] for details).

Cyclotomic polynomials. Let us define the *cyclotomic polynomial of order* n as

$$\Phi_n(z) = \prod_{\xi \in \phi_n} (z - \xi) \in \mathbb{Z}[z]$$

where $\phi_n = \{\xi \in \mathbb{C} : \xi^k = 1, \, k \equiv 0 \mod n\}$ denotes the set of primitive roots of order $n \in \mathbb{N}$. Note that $\Phi_n(z) \in \mathbb{Z}[z]$, that is it has integer coefficients. Cyclotomic polynomials have the property of being irreducible over $\mathbb{Z}[z]$ and therefore it follows that for any polynomial $P(z) \in \mathbb{Z}[z]$ and $n \in \mathbb{N}$ there exists a unique integer $s_n \in \mathbb{N}_0$ such that

$$P_n(z) := P(z)\, \Phi_n^{-s_n}(z) \tag{2}$$

is a polynomial in $\mathbb{Z}[z]$ satisfying $P_n(\xi) \neq 0$ for all $\xi \in \phi_n$.

This factoring out of the roots is not guaranteed to hold for arbitrary functions F, that is it is possible that for a given $n \in \mathbb{N}$ there does not exist any $r_n \in \mathbb{R}$ satisfying

$$\lim_{z \to \xi} F(z)\, \Phi_n^{-r_n}(z) \notin \{0, \pm\infty\}$$

for all $\xi \in \phi_n$. One can easily verify however, that if such a number does exist, it is uniquely defined.

Let $q_1, \ldots, q_m$ be fixed co-prime integers. Given some $\mathbf{j} = (j_1, \ldots, j_m) \in \mathbb{N}_0^m$ we will use the following short-hand notation

$$\Phi_{\mathbf{j}}(z) := \Phi_{q_1^{j_1} \cdots q_m^{j_m}}(z), \;\; \phi_{\mathbf{j}}(z) := \phi_{q_1^{j_1} \cdots q_m^{j_m}}(z), \;\; s_{\mathbf{j}} := s_{q_1^{j_1} \cdots q_m^{j_m}} \;\text{ and }\; r_{\mathbf{j}} := r_{q_1^{j_1} \cdots q_m^{j_m}}.$$

The main strategy of the proof is to show that for a hypothetical function $F(z) = f_{\mathcal{A}}^d(z)$ satisfying Equation (1) the exponents $r_{\mathbf{j}}$ would have to exist for all $\mathbf{j} \in \mathbb{N}_0^m$ – at least with respect to some appropriate limit – and fulfil certain relations between them. The goal will be to find a contradiction in these relations, negating the possibility of such a function and therefore such a set $\mathcal{A}$ existing in the first place. Before formally stating this result, let us introduce the following observation regarding cyclotomic polynomials.

Lemma 2.1. *Given $k, n \in \mathbb{N}$ such that $k \mid n$ we have $\phi_{n/k} = \{\xi^k : \xi \in \phi_n\}$. Furthermore*

$$R(z) := \Phi_{n/k}(z^k)\, \Phi_n(z)^{-1} \in \mathbb{Z}[z]$$

satisfies $R(\xi) \neq 0$ for all $\xi \in \phi_n$.

Proof. The inclusion $\{\xi^k : \xi \in \phi_n\} \subseteq \phi_{n/k}$ follows immediately by definition. For the other direction, note that

$$\phi_n = \{\xi \in \mathbb{C} : \xi^n = 1 \text{ but } \xi^k \neq 1 \text{ for all } k < n\} = \left\{ \exp\left(2\pi \frac{l}{n}\right) : 1 \leq \ell \leq n, \, \ell \nmid n \right\}.$$

It therefore follows that

$$\{\xi^k : \xi \in \phi_n\} = \left\{ \exp\left(2\pi \frac{lk}{n}\right) : 1 \leq \ell \leq n, \, \ell \nmid n \right\} \supseteq \left\{ \exp\left(2\pi \frac{l}{n/k}\right) : 1 \leq \ell \leq n, \, \ell \nmid (n/k) \right\} = \phi_{n/k}.$$

As $\Phi_{n/k}(z^k)$ is a polynomial in $\mathbb{Z}[z]$ and $\Phi_{n/k}(\xi^k) = 0$ for any $\xi \in \phi_n$ via the previous observation, it follows that $R(z)$ is a polynomial in $\mathbb{Z}[z]$ satisfying $R(\xi) \neq 0$ for all $\xi \in \phi_n$. Finally, the exponent s of $\Phi_{n/k}(z^k)$ in the equality $R(z) = \Phi_{n/k}(z^k)^s \, \Phi_n(z)^{-1}$ is equal to 1 because all roots of $\Phi_{n/k}(z^k)$ are simple. □

3 Recurrence relations

We can now give the statement and proof establishing the existence and relations of the values $r_{\mathbf{j}}$ for any hypothetical function $F(z)$ satisfying Equation (1). We will in fact state them for any $k_1, \ldots, k_d \in \mathbb{N}$ and later derive a contradiction from these relations in the specific case stated in Theorem 1.1.

For any $a, b \in \mathbb{N}_0$, $\mathbf{j} = (j_1, \ldots, j_m) \in \mathbb{N}_0^m$ and $\mathbf{b} = (b_1, \ldots, b_m) \in \mathbb{N}_0^m$, we will use the notation

$$a \ominus b = \max\{a - b, 0\} \quad \text{and} \quad \mathbf{j} \ominus \mathbf{b} = (j_1 \ominus b_1, \ldots, j_m \ominus b_m).$$

Furthermore, whenever we write some limit $\lim_{z \to \xi} F(z)$, where ξ is a unit root, we are referring to $\lim_{z \to 1} F(z\,\xi)$ where $0 \leq z < 1$ as F will always be analytic in the disc $|z| < 1$.

Proposition 3.1. *Let $k_1, \ldots, k_d \in \mathbb{N}$ and $q_1, \ldots, q_m \geq 2$ pairwise co-prime integers for which there exist $b(i,j) \in \mathbb{N}_0$ such that $k_i = q_1^{b(i,1)} \cdots q_m^{b(i,m)}$ for all $1 \leq i \leq d$. Furthermore, let $P \in \mathbb{Z}[z]$ be a polynomial satisfying $P(1) \neq 0$ and $F : \mathbb{C} \to \mathbb{C}$ a function analytic in the disc $|z| < 1$ such that*

$$F(z^{k_1}) \cdots F(z^{k_d}) = \frac{P^d(z)}{(1-z)^d}. \tag{3}$$

Then for all $\boldsymbol{j} \in \mathbb{N}_0^m$ there exist integers $r_{\boldsymbol{j}} \in \mathbb{N}_0$ such that

$$\lim_{z \to \xi} F(z)\, \Phi_{\boldsymbol{j}}^{-r_{\boldsymbol{j}}}(z) \notin \{0, \pm\infty\} \tag{4}$$

for any $\xi \in \phi_{\boldsymbol{j}}$. Writing $\boldsymbol{b}_i = (b(i,1), \ldots, b(i,m))$ for $1 \leq i \leq m$ as well as $s_{\boldsymbol{j}} \in \mathbb{N}_0$ for the integer satisfying $P(\xi)\, \Phi_{\boldsymbol{j}}^{-s_{\boldsymbol{j}}}(\xi) \neq 0$ for any $\xi \in \phi_{\boldsymbol{j}}$, these exponents satisfy the relations

$$r_{\mathbf{0}} = -1 \quad \textit{and} \quad r_{\boldsymbol{j} \ominus \boldsymbol{b}_1} + \cdots + r_{\boldsymbol{j} \ominus \boldsymbol{b}_d} = d s_{\boldsymbol{j}} \quad \text{for all } \boldsymbol{j} \in \mathbb{N}_0^m \setminus \{\mathbf{0}\} \tag{5}$$

and we have $r_{\boldsymbol{i}} \equiv -1 \mod d$ for all $\boldsymbol{i} \in \mathbb{N}_0^m$.

Proof. We start assuming that the set $\{r_{\mathbf{j}}\}_{\mathbf{j} \in \mathbb{N}_0^m}$ exists and show that the relations given by Equation (5) must be satisfied. After this, we will show that there is a unique way that defines the values $\{r_{\mathbf{j}}\}_{\mathbf{j} \in \mathbb{N}_0^m}$, proving their existences.

Let us start with $r_{\mathbf{0}} = -1$. For $F_{\mathbf{0}}(z) := F(z)(1-z)$ we wish to show that $\lim_{z \to 1} F_{\mathbf{0}}(z) \notin \{0, \pm\infty\}$. Inserting the equality $F(z) = (1-z)^{-1} F_{\mathbf{0}}(z)$ into Equation (3) and observing that $(1-z)/(1-z^{k_\ell}) = (1 + z + \cdots + z^{k_\ell - 1})^{-1}$, we get that $F_{\mathbf{0}}(z)$ satisfies

$$\prod_{\ell=1}^{d} (1 + z + \cdots + z^{k_\ell - 1})^{-1} F_{\mathbf{0}}(z^{k_\ell}) = P^d(z).$$

As $P^d(1) \neq 0$ as well as $(1+1+\cdots+1^{k_\ell-1})^{-1} = 1/k_\ell \neq 0$ for $1 \leq \ell \leq d$ it follows that $\lim_{z\to 1} F_{\mathbf{0}}(z) \notin \{0, \pm\infty\}$ as desired. Next, let us show that if for a given $\mathbf{j} \in \mathbb{N}_0^m \setminus \{\mathbf{0}\}$ the values $r_{\mathbf{j}\ominus\mathbf{b}_1}, \ldots, r_{\mathbf{j}\ominus\mathbf{b}_d}$ exist, then they must satisfy the relation given by Equation (5). For $1 \leq i \leq d$ let

$$F_{\mathbf{j}\ominus\mathbf{b}_i} := F(z)\,\Phi_{\mathbf{j}\ominus\mathbf{b}_i}^{-r_{\mathbf{j}\ominus\mathbf{b}_i}}$$

and rewrite Equation (3) as

$$\Phi_{\mathbf{j}\ominus\mathbf{b}_1}^{r_{\mathbf{j}\ominus\mathbf{b}_1}}(z^{k_1})\,F_{\mathbf{j}\ominus\mathbf{b}_1}(z^{k_1})\;\cdots\;\Phi_{\mathbf{j}\ominus\mathbf{b}_d}^{r_{\mathbf{j}\ominus\mathbf{b}_d}}(z^{k_d})\,F_{\mathbf{j}\ominus\mathbf{b}_d}(z^{k_d}) = \frac{\Phi_{\mathbf{j}}^{ds_{\mathbf{j}}}(z)\,P_{\mathbf{j}}^d(z)}{(1-z)^d}.$$

Writing $R_{\mathbf{j},i}(z) := \Phi_{\mathbf{j}\ominus\mathbf{b}_i}(z^{k_i})\,\Phi_{\mathbf{j}}^{-1}(z)$ we can restate Equation (3) as

$$\Phi_{\mathbf{j}}^{r_{\mathbf{j}\ominus\mathbf{b}_1}+\cdots+r_{\mathbf{j}\ominus\mathbf{b}_d}-ds_{\mathbf{j}}}(z)\,\Big(R_{\mathbf{j},1}^{r_{\mathbf{j}\ominus\mathbf{b}_1}}(z)\,F_{\mathbf{j}\ominus\mathbf{b}_1}(z^{k_1})\;\cdots\;R_{\mathbf{j},d}^{r_{\mathbf{j}\ominus\mathbf{b}_d}}(z)\,F_{\mathbf{j}\ominus\mathbf{b}_d}(z^{k_d})\Big) = \frac{P_{\mathbf{j}}^d(z)}{(1-z)^d}.$$

We observe that by Lemma 2.1 for $z \to \xi \in \phi_{\mathbf{j}}$ where $\mathbf{j} \neq \mathbf{0}$ all involved factors but the first one converge neither to 0 nor to $\pm\infty$, so the desired relation must hold.

It remains to be shown that these basic relations recursively 'build up' $\mathbb{N}_0^m$ so that all values $r_{\mathbf{j}}$ are indeed well defined and exist. From now on, let us – for simplicities sake – redefine the value $s_{\mathbf{0}}$ (which previously was 0 as $P(0) \neq 0$) to be $s_{\mathbf{0}} = -1$, so that the initial relation $r_{\mathbf{0}} = -1$ is now included in the general relation for the case $\mathbf{j} = \mathbf{0}$. We observe that if for some $1 \leq \ell \leq d$ all values $r_{\mathbf{j}\ominus\mathbf{b}_1}, \ldots, r_{\mathbf{j}\ominus\mathbf{b}_d}$ except for $r_{\mathbf{j}\ominus\mathbf{b}_\ell}$ are determined, then through the already established Equation (5) it is clear that setting

$$r_{\mathbf{j}\ominus\mathbf{b}_\ell} = ds_{\mathbf{j}} - \sum_{i\neq\ell} r_{\mathbf{j}\ominus\mathbf{b}_i}$$

would give the desired $\lim_{z\to\xi} F(z)\,\Phi_{\mathbf{j}\ominus\mathbf{b}_\ell}^{-r_{\mathbf{j}\ominus\mathbf{b}_\ell}}(z) \notin \{0, \pm\infty\}$ for all $\xi \in \phi_{\mathbf{j}\ominus\mathbf{b}_\ell}$. We therefore wish to show inductively that for all $\mathbf{i} \in \mathbb{N}_0^m$ there exists a $\mathbf{j} \in \mathbb{N}_0^m$ and $1 \leq \ell \leq d$ such that $\mathbf{i} = \mathbf{j}\ominus\mathbf{b}_\ell$ and all other involved values $\mathbf{j}\ominus\mathbf{b}_1, \ldots, \mathbf{j}\ominus\mathbf{b}_{\ell-1}, \mathbf{j}\ominus\mathbf{b}_{\ell+1}, \ldots, \mathbf{j}\ominus\mathbf{b}_d$ have already been determined by the inductive hypothesis.

For this we will give the indices $\mathbf{j} \in \mathbb{N}_0^m$ inducing these relations an appropriate ordering. More preciesly, for each $\mathbf{j} = (j_1, \ldots, j_m) \in \mathbb{N}_0^m$ let $\mathbf{j}^{\leq} = (j_1^{\leq}, \ldots, j_m^{\leq})$ denote the *ordered version*, that is $j_1^{\leq} \leq j_2^{\leq} \leq \cdots \leq j_m^{\leq}$ and there exists some permutation σ on m letters such that $\mathbf{j} = \big(j_{\sigma(1)}^{\leq}, \ldots, j_{\sigma(m)}^{\leq}\big)$. Consider the ordering on $\mathbb{N}_0^m$ given by $\mathbf{j} \prec \mathbf{j}'$ if $\mathbf{j}^{\leq}$ lexicographically comes before $\mathbf{j}'^{\leq}$. In this situation, ties are broken arbitrarily. We want to show that going through the indices $\mathbf{j}$ in that order and considering the relation $r_{\mathbf{j}\ominus\mathbf{b}_1} + \cdots + r_{\mathbf{j}\ominus\mathbf{b}_d} = ds_{\mathbf{j}}$, then at most one of the $r_{\mathbf{j}\ominus\mathbf{b}_\ell}$ will not have occurred in any of the previous relations given by some $\mathbf{j}' \prec \mathbf{j}$.

Assume to the contrary that there exist $\mathbf{i} \neq \mathbf{i}' \in \mathbb{N}_0^m$ such that, for both of them, $\mathbf{j} \in \mathbb{N}_0^m$ is the first index for which there exist $1 \leq \ell, \ell' \leq d$ satisfying $\mathbf{i} = \mathbf{j}\ominus\mathbf{b}_\ell$ and $\mathbf{i}' = \mathbf{j}\ominus\mathbf{b}_{\ell'}$. Note that $\mathbf{b}_\ell \neq \mathbf{b}_{\ell'}$ and therefore at least one of the two statements $\mathbf{j}\ominus(\mathbf{b}_\ell - \mathbf{b}_{\ell'}) \prec \mathbf{j}$ and $\mathbf{j}\ominus(\mathbf{b}_{\ell'} - \mathbf{b}_\ell) \prec \mathbf{j}$

must hold. To see this, assume without loss of generality that $\mathbf{j} = (j_1, \dots, j_m)$ is already in ordered form. Note that $\mathbf{b}_\ell - \mathbf{b}_{\ell'} \neq \mathbf{0}$ as $\mathbf{i} \neq \mathbf{i}'$. Writing $\mathbf{b}_\ell = (b_1, \dots, b_m)$ and $\mathbf{b}_{\ell'} = (b'_1, \dots, b'_m)$, and letting $1 \leq i \leq m$ be the first index such that $b_i \neq b'_i$ and $j_i > 0$, then we clearly have that either

$$j_i \ominus (b_i - b_{i'}) = \max\{j_i - (b_i - b_{i'}), 0\} < j_i \quad \text{or} \quad j_i \ominus (b_{i'} - b_i) = \max\{j_i + (b_i - b_{i'}), 0\} < j_i,$$

meaning that at least one of the two values $\mathbf{j} \ominus (\mathbf{b}_\ell - \mathbf{b}_{\ell'})$ and $\mathbf{j} \ominus (\mathbf{b}_{\ell'} - \mathbf{b}_\ell)$ must lexicographically come before $\mathbf{j}$. Note that such index i must exist since if $j_i = 0$ whenever $b_i - b'_i \neq 0$ then we would have had $\mathbf{i} = \mathbf{j} \ominus \mathbf{b}_\ell = \mathbf{b}_{\ell'} = \mathbf{i}'$ in contradiction to our assumption that $\mathbf{i} \neq \mathbf{i}'$.

Assume now without loss of generality that $\mathbf{j} \ominus (\mathbf{b}_\ell - \mathbf{b}_{\ell'}) \prec \mathbf{j}$. Since for $a, b, c \geq 0$ we trivially have that $\max\{\max\{a - b + c, 0\} - c, 0\} = \max\{\max\{a - b, -c\}, 0\} = \max\{a - b, 0\}$, it follows that

$$\big(\mathbf{j} \ominus (\mathbf{b}_\ell - \mathbf{b}_{\ell'})\big) \ominus \mathbf{b}_{\ell'} = \mathbf{j} \ominus \mathbf{b}_\ell = \mathbf{i}.$$

This is however in contradiction to the requirement that $\mathbf{j}$ was the smallest index with respect to the ordering $\prec$ for which the relation given by Equation (5) involves $r_{\mathbf{i}}$, giving us the desired result.

Finally, note that from the previous argument it also inductively follows that $r_{\mathbf{i}} \equiv -1 \mod d$ for all $\mathbf{i} \in \mathbb{N}_0^m$ as in the base case we have that $r_{\mathbf{0}} = -1$. □

4 Proof of Theorem 1.1

We will now use the proposition established in the previous section to prove Theorem 1.1 by contradiction. We start by introducing some necessary notation and definitions. We write $\mathbf{c}_i = (c(i,1), \dots, c(i,m))$ and for any $1 \leq \ell \leq m$ we use the notation

$$S_\ell = \{1 \leq i \leq d : c(i,\ell) = 0\} \quad \text{and} \quad S'_\ell = \{1, \dots, d\} \setminus S_\ell.$$

We will also use the following notation: for any $\mathbf{i} = (i_1, \dots, i_{m-1}) \in \mathbb{N}_0^{m-1}$ and $1 \leq \ell \leq m$ let

$$\Delta_{\mathbf{i},\ell} = v_{(i_1,\dots,i_{\ell-1},1,i_\ell,\dots,i_{m-1})} - v_{(i_1,\dots,i_{\ell-1},0,i_\ell,\dots,i_{m-1})}.$$

Finally, for $1 \leq l \leq m$, we write $\mathbb{1}_\ell \in \mathbb{N}_0^m$ for the vector whose entries are all equal to 0 except for the l-th entry, which is equal to 1.

Definition 4.1. *For $m \geq 1$, we define an m-*structure *to be any set of values $\{v_{\boldsymbol{j}} \in \mathbb{Q}\}_{\boldsymbol{j} \in \mathbb{N}_0^m}$ for which there exist $\boldsymbol{c}_1, \dots, \boldsymbol{c}_d \in \mathbb{N}_0^m$ and $\{u_{\boldsymbol{j}} \in \mathbb{Z}\}_{\boldsymbol{j} \in \mathbb{N}_0^m \setminus \{\mathbf{0}\}}$ so that the values satisfy the relation*

$$v_{\boldsymbol{j} \ominus \boldsymbol{c}_1} + \cdots + v_{\boldsymbol{j} \ominus \boldsymbol{c}_d} = u_{\boldsymbol{j}} \quad \textit{for all } \boldsymbol{j} \in \mathbb{N}_0^m \setminus \{\mathbf{0}\}.$$

Additionally, we define the following:

1. *We say that an m-structure is* regular *if we have that the corresponding vectors $\mathbf{c}_1, \dots, \mathbf{c}_d \in \{0,1\}^m \setminus \{\mathbf{0}\}$ for all $1 \leq i \leq d$ as well as $S_\ell \neq \emptyset$ for all $1 \leq \ell \leq m$.*

2. *We say that an m-structure is* homogeneous outside $\mathbf{t} = (t_1, \dots, t_m) \in \mathbb{N}_0^m$ *if the corresponding vectors $\{u_{\boldsymbol{j}} \in \mathbb{Z}\}_{\boldsymbol{j} \in \mathbb{N}_0^m \setminus \{\mathbf{0}\}}$ satisfy $u_{\boldsymbol{j}} = 0$ for all $\boldsymbol{j} \in \mathbb{N}_0^m \setminus [0, t_1] \times \cdots \times [0, t_m]$.*

The first lemma shows a key ingredient in the inductive step developed later by reducing the value of m:

Lemma 4.2. *For any m-structure $\{v_{\boldsymbol{j}} \in \mathbb{Q}\}_{\boldsymbol{j} \in \mathbb{N}_0^m}$ that is homogeneous outside $\boldsymbol{t} = (t_1, \dots, t_m) \in \mathbb{N}_0^m$ and for which there exists $1 \leq \ell \leq m$ such that $|S_\ell| \neq 0$, the values $\{\Delta_{\boldsymbol{i},\ell}\}_{\boldsymbol{i} \in \mathbb{N}_0^{m-1}}$ define an $(m-1)$-structure that is homogeneous outside $\boldsymbol{t}_\ell = (t_1, \dots, t_{\ell-1}, t_{\ell+1}, \dots, t_m)$.*

Proof. To simplify notation, assume without loss of generality that $\ell = m$. Let $\mathbf{c}_1, \dots, \mathbf{c}_d \in \mathbb{N}_0^m$ and $\{u_{\mathbf{j}} \in \mathbb{Z}\}_{\mathbf{j} \in \mathbb{N}_0^m \setminus \{\mathbf{0}\}}$ the corresponding sets of vectors given by the definition of m-structure.

For $i \in S_\ell$, let $\mathbf{c}'_i = (c(i,1), \dots, c(i, m-1))$. Furthermore for $\mathbf{j}' = (j_1, \dots, j_{m-1}) \in \mathbb{N}_0^{m-1}$ let $\mathbf{j} = (j_1, \dots, j_{m-1}, 0)$ and let $u_{\mathbf{j}'} = u_{\mathbf{j}+\mathbb{1}_\ell} - u_{\mathbf{j}}$. Using this notation, we have

$$\begin{aligned}\sum_{i \in S_\ell} \Delta_{\mathbf{j}' \ominus \mathbf{c}'_i, \ell} &= \sum_{i \in S_\ell} v_{(\mathbf{j}+\mathbb{1}_\ell) \ominus \mathbf{c}_i} - \sum_{i \in S_\ell} v_{\mathbf{j} \ominus \mathbf{c}_i}, \\ &= \Big(u_{\mathbf{j}+\mathbb{1}_\ell} - \sum_{i \in S'_\ell} v_{(\mathbf{j}+\mathbb{1}_\ell) \ominus \mathbf{c}_i} \Big) - \Big(u_{\mathbf{j}} - \sum_{i \in S'_\ell} v_{\mathbf{j} \ominus \mathbf{c}_i} \Big) = u_{\mathbf{j}+\mathbb{1}_\ell} - u_{\mathbf{j}} = u_{\mathbf{j}'}.\end{aligned}$$

Here we have used the fact that for $i \in S'_\ell$ we have $(\mathbf{j} + \mathbb{1}_\ell) \ominus \mathbf{c}_i = \mathbf{j} \ominus \mathbf{c}_i$ as $c(i,\ell) \neq 0$. It follows that the values $\{\Delta_{\mathbf{i},\ell}\}_{\mathbf{i} \in \mathbb{N}_0^{m-1}}$ form an $(m-1)$-structure with $\{\mathbf{c}'_i\}_{i \in S_\ell}$ and $\{u_{\mathbf{j}'}\}_{\mathbf{j}' \in \mathbb{N}_0^{m-1} \setminus \{\mathbf{0}\}}$. As $u_{\mathbf{j}'} = u_{\mathbf{j}+\mathbb{1}_\ell} - u_{\mathbf{j}} = 0$ for $\mathbf{j}' \in \mathbb{N}_0^{m-1} \setminus [0, t_1] \times \cdots \times [0, t_{m-1}]$, it follows that the structure is homogeneous outside $\mathbf{t}_\ell$. ☐

Lemma 4.3. *A regular m-structure that is homogeneous outside $\boldsymbol{t} = (t_1, \dots, t_m) \in \mathbb{N}_0^m$ satisfies $v_{\boldsymbol{i}} = 0$ for all $\boldsymbol{i} \in \mathbb{N}_0^m \setminus [0, t_1] \times \cdots \times [0, t_m]$.*

Proof. We will prove the statement by induction on m. Let us start by showing the statement for $m = 1$. In this case, $\mathbf{c}_1, \dots, \mathbf{c}_d$ are non-zero, positive integers satisfying $\mathbf{c}_1 = \cdots = \mathbf{c}_d = 1$ as the structure is regular. It follows that the relations defining the structure are of the type $d\, v_{\mathbf{j} \ominus 1} = u_{\mathbf{j}}$ for all $\mathbf{j} \in \mathbb{N}$. Since $u_{\mathbf{j}} = 0$ for $\mathbf{j} > \mathbf{t} = t_1$, we have $v_{\mathbf{i}} = 0$ for all $\mathbf{i} \in \mathbb{N}_0 \setminus [0, t_1 \ominus \mathbf{c}_1] \subseteq \mathbb{N}_0 \setminus [0, t_1]$ as desired.

Now assume that the statement is true for all $(m-1)$-structures and let us show that then it must also hold for any m-structure. As the structure is regular, we have $S_\ell \neq \emptyset$ for all $1 \leq \ell \leq m$ and Lemma 4.2 shows that $\{\Delta_{\mathbf{i},\ell}\}_{\mathbf{i} \in \mathbb{N}_0^{m-1}}$ is an $(m-1)$-structure that is homogeneous outside $\mathbf{t}_\ell$ for any $1 \leq \ell \leq m$. Let us without loss of generality assume that $\ell = m$ to simplify notation. By the inductive assumption it follows that $\Delta_{\mathbf{i},\ell} = 0$ for all $\mathbf{i} \in \mathbb{N}_0^{m-1} \setminus [0, t_1] \times \cdots \times [0, t_{m-1}]$.

It follows that $\{v'_{\mathbf{i}} = v_{\mathbf{i}+\mathbb{1}_\ell}\}_{\mathbf{i}\in\mathbb{N}_0^m}$ is an m-structure where the corresponding $\{u'_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{N}_0^m}$ satisfying

$$u'_{\mathbf{j}} = \begin{cases} u_{\mathbf{j}+\mathbb{1}_\ell} & \text{for } \mathbf{j} = (j_1, \ldots, j_m) \text{ s.t. } j_\ell \neq 0, \\ u_{\mathbf{j}+\mathbb{1}_\ell} + \sum_{i\in S'_\ell} \Delta_{\mathbf{j}\ominus\mathbf{c}_i,\ell} & \text{for } \mathbf{j} = (j_1, \ldots, j_m) \text{ s.t. } j_\ell = 0. \end{cases} \tag{6}$$

Note that this structure is homogeneous outside $(t_1, \ldots, \ldots, t_m - 1)$, that is we have reduced the size of the inhomogeneous part. Repeated application of this principle along all dimensions $1 \leq \ell \leq d$ gives us that

$$v_{\mathbf{i}} = v_{\mathbf{i}+\mathbb{1}_\ell} \text{ for all } \mathbf{i} \in \mathbb{N}_0^m \setminus \big([0, t_1] \times \cdots \times [0, t_{m-1}] \times \mathbb{N}_0\big) \text{ and } 1 \leq \ell \leq m. \tag{7}$$

Considering the relation given by $\mathbf{j} = (2t_1, \ldots, 2t_m)$, which states that

$$d\, v_{\mathbf{j}} = v_{\mathbf{j}\,\ominus\mathbf{c}_1} + \cdots + v_{\mathbf{j}\,\ominus\mathbf{c}_d} = u_{\mathbf{j}} = 0.$$

Note that the choice of the constant 2 was arbitrary, it just needs to be 'large enough'. It follows that $v_{\mathbf{j}} = 0$ and hence, again by relation (7), it follows that $v_{\mathbf{i}} = 0$ for all $\mathbf{i} \in \mathbb{N}_0^m \setminus [0, t_1] \times \cdots \times [0, t_m]$ as desired. □

Proof of Theorem 1.1. We write $F(z) = f_{\mathcal{A}}(z)^d$. Recall that the existence of a set $\mathcal{A}$ for which $r_{\mathcal{A}}(n; k_1, \ldots, k_d)$ is a constant function for n large enough would imply the existence of some polynomial $P(z) \in \mathbb{Z}[z]$ satisfying $P(1) \neq 0$ such that

$$F(z^{k_1}) \cdots F(z^{k_d}) = \frac{P^d(z)}{(1-z)^d}.$$

Using Proposition 3.1 we see that if a such a function $F(z)$ were to exist, then the values $\{r_{\mathbf{i}}\}_{\mathbf{i}\in\mathbb{N}_0^m}$ together with $\mathbf{b}_1, \ldots, \mathbf{b}_m$ and $\{s_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{N}_0^m\setminus\{\mathbf{0}\}}$ would define an m-structure. By the requirements of the theorem we have $\mathbf{b}_i \in \{0,1\}^m$ and since $k_1, \ldots, k_d \geq 2$ we have $\mathbf{b}_i \neq \mathbf{0}$. We may also assume that $S_\ell \neq \emptyset$ for all $1 \leq \ell \leq d$ as otherwise there exists some ℓ' such that $q_{\ell'} \mid k_i$ for all $1 \leq i \leq d$, in which case the representation function clearly cannot become constant, so that this m-structure would be regular. It would also be homogeneous outside some appropriate $\mathbf{t} \in \mathbb{N}_0^m$ as $P(z)$ is a polynomial and hence $s_{\mathbf{j}} \neq 0$ only for finitely many $\mathbf{j} \in \mathbb{N}_0^m$. Finally, since $r_{\mathbf{i}} \equiv -1 \mod d$ for all $\mathbf{i} \in \mathbb{N}_0^m$, this would contradict the statement of Lemma 4.3, proving Theorem 1.1. □

5 Concluding Remarks

We have shown that under very general conditions for the coefficients $k_1, \ldots, k_d$ the representation function $r_{\mathcal{A}}(n; k_1, \ldots, k_d)$ cannot be constant for n sufficiently large. However, there are cases that our method does not cover. This includes those cases where at least one of the k_i is equal to 1. The first case that we are not able to study is the representation function $r_{\mathcal{A}}(n; 1, 1, 2)$.

On the other side, let us point out that Moser's construction [3] can be trivially generalized to the case where $k_i = k^{i-1}$ for some integer value $k \geq 2$. In view of our results and this construction, we state the following conjecture:

Conjecture 5.1. *There exists some infinite set of positive integers $\mathcal{A}$ such that $r_{\mathcal{A}}(n; k_1, \ldots, k_d)$ is constant for n large enough if and only if, up to permutation of the indices, $(k_1, \ldots, k_d) = (1, k, k^2, \ldots, k^{d-1})$, for some $k \geq 2$.*

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