An extension and an alternative characterization of May's theorem

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Abstract

The context of this work is a voting scenario in which each voter expresses his/her level of affinity about a proposal, by choosing a value in the set $\mathcal{J} = \{-j, \ldots, -1, 0, 1, \ldots, j\}$, and these individual votes produce a collective result, in the same set $\mathcal{J}$, through a decision function. The simple majority, defined for $j = 1$, is a widely used example of such a decision function.

In this paper, a set of independent axioms is proved to uniquely characterize the j-majority decision function. The j-majority decision is defined for any positive integer j, and it coincides with the simple majority decision when $j = 1$. In this way, this axiomatic characterization meets two goals: it gives a new characterization of the simple majority decision when $j = 1$ and it extends May's theorem to this broader context.

Key words: Simple majority decision; May's theorem; Multilevel decision functions; Extension of simple majority decision; Axiomatic characterization.

1 Introduction

Simple majority is a very common rule to make collective decisions in a binary voting context. Following this rule, a proposal is approved (or a candidate wins) if the number of votes in favor of it is strictly greater than the number of votes against it, and it is rejected in the opposite case. If there is the same number of votes in favor and against it, then the result is a tie. Under this rule, it does not matter if each individual has either two ('yes' or 'no') or three ('yes', 'no' or 'indifferent') voting options, because an indifferent vote is assimilated to an abstention and it is not taken into account for the collective decision.

In 1952, May [15] characterized simple majority by a set of four independent axioms: universal definition, anonymity, neutrality and positive responsiveness. Since then, many extensions and analysis on May's Theorem and the simple majority rule have been done, from different perspectives. Other axiomatic characterizations of the simple majority rule for two alternatives have been given in [2, 19, 13]. Some other studies weaken or ignore one or more of May's axioms and characterize the decision rules that verify a new set of conditions see, for instance, [20, 17, 11]. For three

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or more alternatives, Goodin and List [8] proved that the plurality rule can be uniquely characterized by a convenient extension of May's axioms, and an extension of May's theorem has also been established for three alternatives with some restrictions on the voter's preference relations [16]. Furthermore, simple majority rule has also been characterized under different preference domains, [18, 4, 14, 5, 6].

In the present paper we deal with two alternatives, say A and B, and ask the voters to express their degree of preference on them. For instance, a voter can be totally in favor of alternative A, more favorable to A than to B, indifferent between the two alternatives, less favorable to A than to B and totally against A (of course, each degree of vote for A is equivalent to the opposite degree of vote for B). In this example, the degree of preference of an individual is expressed by a number in the set $\mathcal{J}(2) = \{-2, -1, 0, 1, 2\}$, where 2 means the greatest degree of preference for A and -2 the least one. Similarly, the collective decision a degree of acceptance of the selected alternative, or even can result in a tie.

We extend May's axioms to this broader context, where the set of individual choices and the set of possible collective decisions is $\mathcal{J}(j) = \{-j, \ldots, -1, 0, 1, \ldots, j\}$, for $j \geq 1$. We prove that this new set of independent axioms uniquely characterize a decision rule, which we call j-majority rule, that coincides with the simple majority for $j = 1$. Thus, for $j = 1$ the result gives an alternative characterization of the simple majority decision by means of a set of independent axioms, different than those used in [15].

The decision functions in this paper are considered as ordinal aggregations of ordinal individual preferences, but they can also be seen as aggregation of cardinal evaluations in certain contexts. In particular, the j–majority functions defined in Section 4 are aggregation functions in the sense of Balinski and Laraki [3]. Some examples can shed light on this question.

Example 1.1 *Portia is tested in math, in history and language. She receives from her teachers a separate grade for each subject, which ranges from A (the best grade, which can be identified with 2) to E (the worst, which can be identified with -2). The C (is the minimum grade to pass the subject and is identified with 0). Portia is evaluated for the entire course taking into account the three subjects. She got A, B and D. The* 2*-majority gives her a B for the course taken.*

Example 1.2 *The credit rating is a financial indicator to potential investors of debt securities such as bonds. These are assigned by credit rating agencies such as Moody's, Standard & Poor's, and Fitch, which publish code designations (such as AAA, B, CC) to express their assessment of the risk quality of a bond. They use 21 ordered codes. Assume an evaluation is done by the three agencies using a common code scale. A form of aggregating the three evaluations should be done by a decision function with the same 21 code outputs. In particular, the* 10*-majority is one of these decision functions that potentially could serve for the aggregate evaluation of the three agencies.*

Note that nothing changes if the evaluations done from A to E are cardinal instead of qualitatively ordinal in Example 1.1 and the same occurs in Example 1.2. Thus, we can regard these examples as decision functions, but also as aggregation of cardinal evaluations, which evaluates the grade that Portia deserves for the entire course or the risk quality of a bond. Nevertheless, in most cardinal aggregation models [1, 3, 9, 10] the output set is the set of the real numbers (or a non-finite subset) instead of the discrete and finite set $\mathcal{J}(j)$.

The next example does not admit a cardinal interpretation as the previous ones do.

Example 1.3 *Voters express preferences by five integers, say* 2, 1, 0, -1, *and* -2, *where* 2 (-2) *means 'full support to candidate A' ('full support to candidate B'),* 1 (-1) *means 'partial support to candidate A' ('partial support to candidate B'), and* 0 *means indifference. If the result of the election gives an average of* 0.627 *of the votes cast and* 2*-majority is applied, then A gets* 1*: she is elected but with partial support, possibly with certain action restrictions that were not taken if she had been elected with a* 2.

The assumed professionalism of teachers and agencies in Examples 1.1 and 1.2 respectively, discards the consideration of strategic vote in them. However, it can occur in Example 1.3. In this article we assume that voters seek the truth and discard any possibility of voting strategically. In other words, it is assumed that voters are rational in the sense of Downs [7]: 'they always choose from among the possible alternatives that which ranks highest in his preference ordering'. Thus, we do not assume that voters use shortcuts, see [12] for more.

The paper is organized as follows. Section 2 recalls May's axioms and his characterization of the simple majority decision. Section 3 introduces a set of independent axioms in a broader context, i.e., for any $\mathcal{J}(j)$. It is proved that for $j = 1$ they uniquely characterize the simple majority rule, providing an alternative axiomatic characterization of this rule. In Section 4 it is proved that for any $j > 1$ this new set of axioms uniquely characterizes a decision rule, extending in this way May's theorem. In this Section 4 we also discuss other possible axiomatizations for the j–majority rule and prove an alternative characterization for it, extending a theorem by Llamazares. The paper concludes in Section 5 by summarizing the main achievements and proposing some open questions.

2 May's theorem

Assume that N is a set of n individuals, $N = \{1, \ldots, n\}$, that have to make a decision on a proposal. Each individual/voter has three balloting options: 1 (in favor of it), -1 (against it) or 0 (indifferent to it). The final decision is one of the same three options: 1, -1 or 0. The aggregated rule for the collective decision is formalized by a function $f : \mathcal{J}^n \to \mathcal{J}$, with $\mathcal{J} = \mathcal{J}(1) = \{-1, 0, 1\}$. [15] denotes to any such function as a *group decision function* (from now on, *decision function*). In his model, voters express preferences by three integers in $\mathcal{J}$, where 1 means 'A is preferred to B', -1 means 'B is preferred to A' and 0 means indifference.

Definition 2.1 ***Simple majority***
The simple majority decision is defined for all $\mathbf{x} = (x_1, \ldots, x_n) \in \mathcal{J}^n$ *by:*

$$f(\mathbf{x}) = \begin{cases} 1 & \textit{if } \sum_{r=1}^{n} x_r > 0 \\ 0 & \textit{if } \sum_{r=1}^{n} x_r = 0 \\ -1 & \textit{if } \sum_{r=1}^{n} x_r < 0 \end{cases}$$

May [15] proved that simple majority is the unique decision function

$$f : \mathcal{J}^n \to \mathcal{J}$$

which verifies the following axioms:

- **Universal Definition**: each set of individual votes must lead to a valid result. In mathematical terms, the domain of f is $\mathcal{J}^n$.
- **Neutrality**: if all individuals change their former vote to the opposite one, then the collective decision must be the opposite of the former one. Formally, $f(-\mathbf{x}) = -f(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{J}^n$.
- **Anonymity**: the vote of each individual must have exactly the same importance. Mathematically, $f(\pi(\mathbf{x})) = f(\mathbf{x})$ if $\mathbf{x} \in \mathcal{J}^n$ and $\pi(\mathbf{x}) = (x_{\pi(1)}, \ldots, x_{\pi(n)})$ for any permutation π of N.
- **Positive Responsiveness**: if, for some set of individual votes, the collective decision is 0 or 1 and one individual increases his/her vote then the collective decision must change to 1. Formally, $f(\mathbf{x} + \mathbf{e}^r) = 1$ if $f(\mathbf{x}) = 0$ or $f(\mathbf{x}) = 1$, for all $\mathbf{x} \in \mathcal{J}^n$ and all $r \in N$ such that $\mathbf{x} + \mathbf{e}^r \in \mathcal{J}^n$, where $\mathbf{e}^r = (0, \ldots, 0, \underbrace{1}_{r}, 0, \ldots, 0)$.

Theorem 2.2 ***(May, 1952)***
A decision function $f : \mathcal{J}^n \to \mathcal{J}$ verifies: Universal Definition, Neutrality, Anonymity and Positive Responsiveness if and only if f is the simple majority function.

If we assume that a function $f : \mathcal{J}^n \to \mathcal{J}$ only represents a decision function if it is universally defined then the axiom of universal definition does not need to be explicitly stated. Under this assumption we can say that May's axioms characterize the simple majority decision by three independent axioms: Neutrality, Anonymity and Positive Responsiveness. This last axiom can be thought as composed by two parts, see [11]: Monotonicity and Tie–breaking.

- **Monotonicity**: if an individual increases his/her vote then the collective decision cannot decrease. Formally, $f(\mathbf{x}) \leq f(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{J}^n$ such that $\mathbf{x} \leq \mathbf{y}$ (i.e., $x_r \leq y_r$ for all $r \in N$).
- **Tie–breaking** (Tb): if the collective decision is 0 and a single individual increases his/her vote then the collective decision must change to 1. Formally, $f(\mathbf{x} + \mathbf{e}^r) = 1$ if $f(\mathbf{x}) = 0$ and $x_r < 1$.

3 New axiomatic characterization

Our goal is to extend May's result when the inputs and outputs are elements of $\mathcal{J}(j) = \{-j, \ldots, -1, 0, 1, \ldots, j\}$, for some positive integer $j > 1$. $N = \{1, \ldots, n\}$ is the set of individuals that have to make a decision on a proposal. Each voter has an odd number $2j + 1$ of balloting options: j (absolutely in favor of it), $j - 1$ (in favor but a bit less convinced), and so on, lowering step by step the degree of agreement with the proposal, till 0 (indifferent to it), and following down from -1 (a little bit against the proposal) till $-j$ (absolutely opposed to it). The final decision, defined for any set of individual votes, is one of the same $2j + 1$ options. The aggregated rule for the collective decision is formalized by a decision function $f : \mathcal{J}^n(j) \to \mathcal{J}(j)$, defined for any $\mathbf{x} = (x_1, \ldots, x_n) \in \mathcal{J}^n(j)$ representing the set of individual choices.

Notice that, while the Neutrality and the Anonymity axioms still make sense when $\mathcal{J} = \{-1, 0, 1\}$ is substituted by $\mathcal{J}(j)$ for $j > 1$, this is not the case for the Positive Responsiveness axiom. But, if we split this axiom in two parts: Monotonicity and Tie–breaking, the first part makes sense for $j > 1$ but the last axiom is only defined for $j = 1$ and needs to be extended for $j > 1$. Thus, in order to extend May's theorem for $j > 1$, we first establish a new set of axioms, defined in the more general context $\mathcal{J}(j)$. In Section 4 another possible way of extending the Tie-breaking axiom for $j > 1$ is considered but we think that the approach we present here sheds light and clarifies the role of each one of the involved axioms.

Definition 3.1 ***Axioms***
Let $\mathcal{J}(j) = \{-j, \ldots, -1, 0, 1, \ldots, j\}$ and $f : \mathcal{J}^n(j) \to \mathcal{J}(j)$ be a decision function. The following properties are defined for f:

- ***Neutrality*** (N)*: if all individuals change their former vote to the opposite one, then the collective decision must be the opposite of the former one. Formally, $f(-\mathbf{x}) = -f(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{J}^n(j)$.*
- ***Anonymity*** (A)*: the vote of each individual must have exactly the same importance. Formally, $f(\pi(\mathbf{x})) = f(\mathbf{x})$ if $\mathbf{x} \in \mathcal{J}^n(j)$ and $\pi(\mathbf{x}) = (x_{\pi(1)}, \ldots, x_{\pi(n)})$ where π is a permutation on N.*
- ***Monotonicity*** (M)*: if an individual increases his/her vote then the collective decision cannot decrease. Formally, $\mathbf{x} \le \mathbf{y} \Longrightarrow f(\mathbf{x}) \le f(\mathbf{y})$, where $\mathbf{x}, \mathbf{y} \in \mathcal{J}^n(j)$ and $\mathbf{x} \le \mathbf{y}$ means that $x_r \le y_r$ for all $r \in N$.*
- ***Extended Tie–breaking*** (ETb)*: if all individuals vote the same non negative value except one of them who votes one level higher, then the collective result must be strictly higher than if all individuals voted the same non negative value. Formally, $f(\boldsymbol{a}+\mathbf{e}^r) > f(\boldsymbol{a})$ for all $r \in N$ with $x_r < j$ and all $\boldsymbol{a} = (a, \ldots, a)$ such that $a \in \{0, 1, \ldots, j-1\}$.*
- ***Balancedness*** (B)*: if an individual increases his/her vote one level and another one decreases his/her vote one level then the collective decision must remain the same. Formally, $f(\mathbf{x} + \mathbf{e}^r - \mathbf{e}^s) = f(\mathbf{x})$ whenever $\mathbf{x} \in \mathcal{J}^n(j)$ and $\mathbf{x} + \mathbf{e}^r - \mathbf{e}^s \in \mathcal{J}^n(j)$.*

The following proposition establishes another formulation for the Balancedness axiom, and proves that Balancedness implies, but is not equivalent to, Anonymity.

Proposition 3.2
Let $\mathcal{J}(j) = \{-j, \ldots, -1, 0, 1, \ldots, j\}$ and $f : \mathcal{J}^n(j) \to \mathcal{J}(j)$. Then,

i) f verifies Balancedness (B) if and only if

$$\sum_{r=1}^{n} x_r = \sum_{r=1}^{n} y_r \Rightarrow f(\mathbf{x}) = f(\mathbf{y})$$

for all $\mathbf{x}, \mathbf{y} \in \mathcal{J}^n(j)$.

ii) If f verifies Balancedness (B) then f verifies Anonymity (A). The converse is not true.

Proof:

i) Assume that f verifies (B), and let $\mathbf{x}, \mathbf{y} \in \mathcal{J}^n(j)$ be such that $\sum_{r=1}^n x_r = \sum_{r=1}^n y_r$. We will see that $f(\mathbf{x}) = f(\mathbf{y})$. Notice that, for any integer k and for any $r, s \in N$ such that $\mathbf{x} + k\mathbf{e}^r - k\mathbf{e}^s \in \mathcal{J}^n(j)$, we can conclude that $f(\mathbf{x} + k\mathbf{e}^r - k\mathbf{e}^s) = f(\mathbf{x})$, by applying (B) k times. Now, as $\sum_{r=1}^n x_r = \sum_{r=1}^n y_r$ we can write

$$\mathbf{x} = \mathbf{y} + \sum_{r=1}^{n}(x_r - y_r)\mathbf{e}^r = \mathbf{y} + \sum_{r=1}^{n-1}\left[\sum_{s=1}^{r}(x_s - y_s)\mathbf{e}^r - \sum_{s=1}^{r}(x_s - y_s)\mathbf{e}^{r+1}\right]$$

so that, by applying (B) $n-1$ times, we have $f(\mathbf{x}) = f(\mathbf{y})$.
Conversely, if $\mathbf{y} = \mathbf{x} + \mathbf{e}^r - \mathbf{e}^s$ for some $r, s \in N$ then $\sum_{r=1}^n x_r = \sum_{r=1}^n y_r$ and, therefore, $f(\mathbf{x}) = f(\mathbf{y})$.

ii) From part *i)* it is obvious that (B) implies (A). The converse is not true, for any j, as the following example shows. Let $n = 4$ and define f by

$$f(\mathbf{x}) = \begin{cases} 1 & \text{if } N_{\mathbf{x}}(1) > 2 \\ 0 & \text{if } N_{\mathbf{x}}(1) \leq 2 \end{cases}$$

where $N_{\mathbf{x}}(1)$ indicates the number of components of $\mathbf{x}$ equal to 1. Clearly, f verifies (A), but not (B) because, for instance, for $\mathbf{x} = (-1, 1, 1, 1)$ it is $f(\mathbf{x}) = 1$ but $f(\mathbf{x} + \mathbf{e}^1 - \mathbf{e}^2) = f(0, 0, 1, 1) = 0$.

□

The cancellation property, introduced in [13] can be easily extended to the context $\mathcal{J}(j)$ as follows.

Definition 3.3 ***Cancellation*** (C)
Let $\mathcal{J}(j) = \{-j, \ldots, -1, 0, 1, \ldots, j\}$ and $f : \mathcal{J}^n(j) \to \mathcal{J}(j)$ be a decision function.
f verifies the Cancellation property if

$$f(\mathbf{x} + t\mathbf{e}^r - t\mathbf{e}^s) = f(\mathbf{x})$$

for any $r \neq s$, for any $\mathbf{x} \in \mathcal{J}^n(j)$ with $x_r = x_s = 0$ and for any $t \in \{1, \ldots, j\}$.

Observe that Balancedness (B) implies Cancellation (C) for every value of j, but the converse is not true.

It is worth to observe that the Extended Tie-breaking axiom (ETB), given in Definition 3.1 is not equivalent to the Tie-breaking axiom (Tb), even for anonymous and monotonic functions for $j = 1$, as the following example shows:

Example 3.4 *Let $j = 1$ and $n = 2$. Tables 1 and 2 show the values of $f(\mathbf{x}) = f(x_1, x_2)$ for two different decision functions f. Both of them are anonymous and monotonic. The first one verifies (ETb) but not (Tb), and the second one verifies (Tb) but not (ETb).*

	-1	0	1
-1	-1	0	1
0	0	0	1
1	1	1	1

Table 1: f verifies (A), (M) and (ETb) but no (Tb).

	-1	0	1
-1	-1	0	1
0	0	1	1
1	1	1	1

Table 2: f verifies (A), (M) and (Tb) but no (ETb).

The following theorem provides an equivalent formulation of May's theorem by using new axioms.

Theorem 3.5
Let $\mathcal{J} = \{-1, 0, 1\}$ and $f : \mathcal{J}^n \to \mathcal{J}$ be a decision function which verifies Neutrality (N) and Monotonicity (M). Then,

f verifies Anonymity (A) and Tie–breaking (Tb)

if and only if

f verifies Balancedness (B) and Extended Tie–breaking (ETb).

Proof:
Let f verify (M) and (N). As a consequence of (N) we have $f(\mathbf{0}) = 0$, because $f(\mathbf{0}) = -f(\mathbf{0})$.

$\Rightarrow$) Assume that f verifies (A) and (Tb). From (A), the value of $f(\mathbf{x})$ depends only of the number of $1's$, $0's$ and $-1's$ in $\mathbf{x}$, denoted respectively by $N_{\mathbf{x}}(1)$, $N_{\mathbf{x}}(0)$ and $N_{\mathbf{x}}(-1)$. From (N) and (A), $N_{\mathbf{x}}(1) = N_{\mathbf{x}}(-1)$ implies $f(\mathbf{x}) = 0$.

To prove (ETb), note that, as $\mathcal{J} = \{-1, 0, 1\}$ and the only non–negative value a which can be increased is $a = 0$, we only need to prove that $f(\mathbf{e}^r) > f(\mathbf{0})$ for any $r \in N$. Thus, (ETb) is a direct consequence of (Tb), taking into account that $f(\mathbf{0}) = 0$.

To prove (B) we use the fact that, from (A), (N), (M) and (Tb), $N_{\mathbf{x}}(1) > N_{\mathbf{x}}(-1)$ implies $f(\mathbf{x}) = 1$ and $N_{\mathbf{x}}(1) < N_{\mathbf{x}}(-1)$ implies $f(\mathbf{x}) = -1$ (proved in May [15]). Let $\mathbf{x} \in \mathcal{J}^n$, $r \neq s \in N$ and $\mathbf{y} = \mathbf{x} + \mathbf{e}^r - \mathbf{e}^s$. There are four possibilities to consider:

a. If $x_r = -1$ and $x_s = 1$ then $N_{\mathbf{y}}(-1) = N_{\mathbf{x}}(-1) - 1$ and $N_{\mathbf{y}}(1) = N_{\mathbf{x}}(1) - 1$. Thus, if $N_{\mathbf{x}}(1) = N_{\mathbf{x}}(-1)$ it is $N_{\mathbf{y}}(1) = N_{\mathbf{y}}(-1)$ and $f(\mathbf{y}) = f(\mathbf{x}) = 0$, if $N_{\mathbf{x}}(1) > N_{\mathbf{x}}(-1)$ it is $N_{\mathbf{y}}(1) > N_{\mathbf{y}}(-1)$ and $f(\mathbf{y}) = f(\mathbf{x}) = 1$, and if $N_{\mathbf{x}}(1) < N_{\mathbf{x}}(-1)$ it is $N_{\mathbf{y}}(1) < N_{\mathbf{y}}(-1)$ and $f(\mathbf{y}) = f(\mathbf{x}) = -1$.

b. If $x_r = -1$ and $x_s = 0$ then $N_{\mathbf{y}}(-1) = N_{\mathbf{x}}(-1)$ and $N_{\mathbf{y}}(1) = N_{\mathbf{x}}(1)$. Thus, by (A), $f(\mathbf{y}) = f(\mathbf{x})$.

c. If $x_r = 0$ and $x_s = 1$ then $N_{\mathbf{y}}(-1) = N_{\mathbf{x}}(-1)$ and $N_{\mathbf{y}}(1) = N_{\mathbf{x}}(1)$. Thus, by (A), $f(\mathbf{y}) = f(\mathbf{x})$.

d. If $x_r = 0$ and $x_s = 0$ then $N_{\mathbf{y}}(-1) = N_{\mathbf{x}}(-1) + 1$ and $N_{\mathbf{y}}(1) = N_{\mathbf{x}}(1) + 1$. Thus, if $N_{\mathbf{x}}(1) = N_{\mathbf{x}}(-1)$ it is $N_{\mathbf{y}}(1) = N_{\mathbf{y}}(-1)$ and $f(\mathbf{y}) = f(\mathbf{x}) = 0$, if $N_{\mathbf{x}}(1) > N_{\mathbf{x}}(-1)$ it is $N_{\mathbf{y}}(1) > N_{\mathbf{y}}(-1)$ and $f(\mathbf{y}) = f(\mathbf{x}) = 1$, and if $N_{\mathbf{x}}(1) < N_{\mathbf{x}}(-1)$ it is $N_{\mathbf{y}}(1) < N_{\mathbf{y}}(-1)$ and $f(\mathbf{y}) = f(\mathbf{x}) = -1$.

In any of the four cases we deduce that $f(\mathbf{y}) = f(\mathbf{x})$.

$\Leftarrow$) Assume that f verifies (ETb) and (B). Then, from Proposition 3.2, f also verifies (A), so that we only need to prove that f verifies (Tb).

The first step is to prove that if $N_{\mathbf{x}}(1) \geq 1$ and $N_{\mathbf{x}}(-1) = 0$ then $f(\mathbf{x}) = 1$. By (ETb) it is $f(\mathbf{e}^r) > f(\mathbf{0}) = 0$ for any $r \in N$, so that $f(\mathbf{e}^r) = 1$. Thus, by (A), if $N_{\mathbf{x}}(1) = 1$ and $N_{\mathbf{x}}(-1) = 0$ it is $f(\mathbf{x}) = 1$, and, by (M), if $N_{\mathbf{x}}(1) \geq 1$ and $N_{\mathbf{x}}(-1) = 0$ it is $f(\mathbf{x}) = 1$.

The second step is to prove that $N_{\mathbf{x}}(1) = s > N_{\mathbf{x}}(-1) = t$ implies $f(\mathbf{x}) = 1$, by the following argument: Let $\mathbf{y} = (\underbrace{-1, \ldots, -1}_{t}, \ldots, \underbrace{1, \ldots, 1}_{s})$. By (A), $f(\mathbf{y}) = f(\mathbf{x})$. Let $\mathbf{z} = \mathbf{y} + \sum_{s=1}^{t} (\mathbf{e}^s - \mathbf{e}^{n-s+1})$. By applying (B) t times we have $f(\mathbf{z}) = f(\mathbf{y})$, but $N_{\mathbf{z}}(1) \geq 1$ and $N_{\mathbf{z}}(-1) = 0$ so that $f(\mathbf{x}) = f(\mathbf{z}) = 1$.
As a consequence, if $N_{\mathbf{x}}(1) < N_{\mathbf{x}}(-1)$ then $f(\mathbf{x}) = -1$, because, by (N), $N_{-\mathbf{x}}(1) > N_{-\mathbf{x}}(-1)$, thus $f(-\mathbf{x}) = 1$ and $f(\mathbf{x}) = -1$. Therefore, $f(\mathbf{x}) = 0$ implies that $N_{\mathbf{x}}(1) = N_{\mathbf{x}}(-1)$.

Let us prove now that f verifies (Tb). If $f(\mathbf{x}) = 0$ then $N_{\mathbf{x}}(1) = N_{\mathbf{x}}(-1)$, and there are two possibilities for $\mathbf{y} = \mathbf{x} + \mathbf{e}^r$: a) if $x_r = 0$ then $N_{\mathbf{y}}(1) = N_{\mathbf{x}}(1) + 1 > N_{\mathbf{x}}(-1) = N_{\mathbf{y}}(-1)$; b) if $x_r = -1$ then $N_{\mathbf{y}}(-1) = N_{\mathbf{x}}(-1) - 1 < N_{\mathbf{x}}(1) = N_{\mathbf{y}}(1)$. In both cases it is $N_{\mathbf{y}}(1) > N_{\mathbf{y}}(-1)$ so that $f(\mathbf{y}) = 1$. □

As a consequence of Theorem 3.5 we can give a characterization of the simple majority decision by a new set of axioms, which is proved to be an independent set in Proposition 3.7.

Corollary 3.6
A decision function $f : \{-1, 0, 1\}^n \to \{-1, 0, 1\}$ verifies: Neutrality (N), Monotonicity (M), Extended Tie–breaking (ETb) and Balancedness (B) if and only if f is the simple majority decision.

Proposition 3.7
Neutrality (N), Monotonicity (M), Extended Tie–breaking (ETb) and Balancedness (B) are independent axioms for $\mathcal{J} = \{-1, 0, 1\}$.

Proof:
For each one of these axioms we exhibit a function which verifies the rest of axioms but not the selected one.

i) The following is a decision function that verifies axioms (M), (ETb) and (B) but not (N) for $n \geq 1$:

$$f(\mathbf{x}) = \begin{cases} 1 & \text{if } \sum_{r=1}^{n} x_r > 0 \\ 0 & \text{if } \sum_{r=1}^{n} x_r \leq 0 \end{cases}$$

ii) The following is a decision function that verifies axioms (N), (ETb) and (B) but not (M) for $n \geq 2$:

$$f(\mathbf{x}) = \begin{cases} 1 & \text{if } \sum_{r=1}^{n} x_r < -n/2 \quad \text{or} \quad 0 < \sum_{r=1}^{n} x_r \leq n/2 \\ 0 & \text{if } \sum_{r=1}^{n} x_r = 0 \\ -1 & \text{if } \sum_{r=1}^{n} x_r > n/2 \quad \text{or} \quad -n/2 \leq \sum_{r=1}^{n} x_r < 0 \end{cases}$$

iii) The function $f \equiv 0$ is a decision function that verifies axioms (N), (M) and (B) but not (ETb) for $n \geq 1$.

iv) The following is a decision function that verifies axioms (N), (M) and (ETb) but not (B) for $n \geq 3$:

$$f(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} > \mathbf{0} \\ 0 & \text{if } \mathbf{x} = \mathbf{0} \text{ or } \mathbf{x} \text{ is not componentwise comparable with } \mathbf{0} \\ -1 & \text{if } \mathbf{x} < \mathbf{0} \end{cases}$$

where $\mathbf{x} > \mathbf{0}$ (or $\mathbf{x} < \mathbf{0}$) means that $x_r \geq 0$ (or $x_r \leq 0$) for all $r \in N$ and $\mathbf{x} \neq \mathbf{0}$.

$\square$

In the characterization of the simple majority given in Corollary 3.6, the Balancedness axiom can be replaced by the Cancellation property given in Definition 3.3, as an immediate consequence of Theorem 10 in [13]. This result is stated in the following Corollary.

Corollary 3.8
A decision function $f : \{-1, 0, 1\}^n \to \{-1, 0, 1\}$ verifies: Neutrality (N), Monotonicity (M), Extended Tie–breaking (ETb) and Cancellation (C) if and only if *f is the simple majority decision.*

Observe that Cancellation implies Anonymity for $j = 1$, but the converse is not true, as the example in the part *ii*) of the proof of Proposition 3.2 shows.

4 The j-majority function

Now we are ready to consider the context of decision functions with the same set of inputs and outputs $\mathcal{J}(j) = \{-j, \ldots, -1, 0, 1, \ldots, j\}$ for $j > 1$. A natural question to ask is whether for $j \geq 1$ there are many decision functions verifying Neutrality, Monotonicity, Extended Tie–breaking and Balancedness or there is only one of them as for $j = 1$. The answer is given in the following theorem.

Theorem 4.1
Let $j \geq 1$. There exists a unique decision function $f : \mathcal{J}^n(j) \to \mathcal{J}(j)$ verifying Neutrality (N), Monotonicity (M), Extended Tie–breaking (ETb) and Balancedness (B).

Proof:
Assume that f verifies these axioms. Notice that, by using (N) and (ETb) we can deduce that, if $a \in \{-j+1, \ldots, 0\}$ then $f(\boldsymbol{a} - \mathbf{e}^r) < f(\boldsymbol{a})$ for any $r \in N$, with $\boldsymbol{a} = (a, \ldots, a) \in \mathcal{J}^n(j)$.

The first step is to prove that, for all $a \in \mathcal{J}(j)$, $f(\boldsymbol{a}) = a$. By applying (N), (ETb) and (M) we can write:

$$-j \leq f(-\mathbf{j}) < \cdots < f(\mathbf{0}) < \cdots < f(\mathbf{j}) \leq j$$

and the only possibility is that $f(\boldsymbol{a}) = a$ for all $a \in \mathcal{J}(j)$.

The second step is to prove that, for all $a \in \{1, \ldots, j\}$, if $\mathbf{x} \in \mathcal{J}^n(j)$ verifies $(a-1)n < \sum_{r=1}^n x_r \leq an$ then $f(\mathbf{x}) = a$. By Proposition 3.2, axiom (B) implies that, for all $\mathbf{x} \in \mathcal{J}^n(j)$, $f(\mathbf{x})$ depends only on $\sum_{r=1}^n x_r$. But the possible values of $\sum_{r=1}^n x_r$ range from $(a-1)n+1$ to an. The increasing finite sequence $(a, a-1, \ldots, a-1) < (a, a, a-1, \ldots, a-1) < \cdots < (a, \ldots, a, a-1) < (a, \ldots, a) = \boldsymbol{a}$ gives a set of vectors whose respective sums are all these possible values. But, from (ETb), it is $f(a, a-1, \ldots, a-1) > f(a-1, \ldots, a-1) = a-1$, so that $f(a, a-1, \ldots, a-1) \geq a$. Thus, by (M), we can write

$$a \leq f(a, a-1, \ldots, a-1) \leq f(a, a, a-1, \ldots, a-1) \leq \cdots \leq f(a, \ldots, a, a-1) \leq f(\boldsymbol{a}) = a$$

This proves that all the vectors in the sequence have image a and, by (B), it is $f(\mathbf{x}) = a$.

Finally, using axiom (N) we deduce that if $a \in \{-j, \ldots, -1\}$ and $\mathbf{x} \in \mathcal{J}^n(j)$ verifies $an \leq \sum_{r=1}^n x_r < (a+1)n$ then $f(\mathbf{x}) = a$. Thus, f is defined on any $\mathbf{x} \in \mathcal{J}(j)$ so that it is the unique decision function verifying the proposed axioms. $\square$

The unique decision function $f_j : \mathcal{J}^n(j) \to \mathcal{J}(j)$, verifying Neutrality, Monotonicity, Extended Tie–breaking and Balancedness, defined in the proof of Theorem 4.1, will be called j-majority function. It coincides with the classical simple majority decision for $j = 1$.

Definition 4.2 ***j-majority function***
Let $j \geq 1$ and $\mathbf{x} = (x_1, \ldots, x_n) \in \mathcal{J}^n(j)$. The j-majority function *f_j is defined as:*

$$f_j(\mathbf{x}) = \begin{cases} \lceil \overline{x} \rceil & \text{if } \overline{x} \geq 0 \\ \lfloor \overline{x} \rfloor & \text{if } \overline{x} < 0 \end{cases}$$

where $\overline{x} = \frac{1}{n} \sum_{r=1}^{n} x_r$.

Proposition 4.3
Neutrality (N), Monotonicity (M), Extended Tie–breaking (ETb) and Balancedness (B) are independent axioms for any $\mathcal{J}(j) = \{-j, \ldots, -1, 0, 1, \ldots, j\}$ with $j > 1$.

Proof:
For each one of these axioms we exhibit a function which verifies the rest of axioms but not the selected one. In all the examples $\overline{\mathbf{x}} = \frac{1}{n} \sum_{r=1}^{n} x_r$.

i) The following function verifies axioms (M), (ETb) and (B) but not (N):
For all $a \in \{1, \ldots, j\}$:

$$f(\mathbf{x}) = \begin{cases} a & \text{if} \quad a-1 < \overline{x} \leq a \\ 0 & \text{if} \quad \overline{x} \leq 0 \end{cases}$$

ii) The following function verifies axioms (N), (ETb) and (B) but not (M):

$$f(\mathbf{x}) = \begin{cases} 0 & \text{if} \quad \mathbf{x} \in \{-\mathbf{j}, \mathbf{j}\} \\ f_j(\mathbf{x}) & \text{otherwise} \end{cases}$$

iii) The following function verifies axioms (N), (M) and (B) but not (ETb):
For all $a \in \{1, \ldots, j-1\}$:

$$f(\mathbf{x}) = \begin{cases} a & \text{if} \quad a < \overline{x} \leq a+1 \\ 0 & \text{if} \quad -1 \leq \overline{x} \leq 1 \\ -a & \text{if} \quad -(a+1) \leq \overline{x} < -a \end{cases}$$

iv) The following function verifies axioms (N), (M) and (ETb) but not (B):
For all $a \in \{1, \ldots, j\}$:

$$f(\mathbf{x}) = \begin{cases} a & \text{if} \quad \mathbf{x} > \boldsymbol{a} - \mathbf{1} \quad \text{and} \quad \mathbf{x} \not> \boldsymbol{a} \\ 0 & \text{if} \quad \mathbf{x} = \mathbf{0} \quad \text{or} \quad \mathbf{x} \text{ is not componentwise comparable with } \mathbf{0} \\ -a & \text{if} \quad \mathbf{x} < \textbf{-}\boldsymbol{a} + \mathbf{1} \quad \text{and} \quad \mathbf{x} \not< \textbf{-}\boldsymbol{a} \end{cases}$$

□

4.1 Another way of extending the Tie-breaking axiom

The Tie–breaking axiom can be extended to $\mathcal{J}(j)$ for $j > 1$ in other different ways. For instance, the following alternative seems reasonable.

- **Extended Tie-breaking Alternative** $(ETbA)$: $f(\mathbf{0}) = 0$ and $f(\mathbf{x}) = a + 1$ for every $\mathbf{x}$ such that $\mathbf{a} < \mathbf{x} \leq \mathbf{a} + \mathbf{1}$ and $a \in \{0, \ldots, j-1\}$.

Note that Extended Tie–breaking alternative $(ETbA)$ is stronger than the Extended Tie–breaking axiom (ETb) as shown below.

Proposition 4.4
f verifies Extended Tie–breaking Alternative $(ETbA)$, then f verifies Extended Tie–breaking (ETb).

Proof: Assume that f verifies $(ETbA)$ and let $a \in \{0, \ldots, j-1\}$. Then, since $\mathbf{a} < \mathbf{a} + \mathbf{e}^r \leq \mathbf{a} + \mathbf{1}$, by $(ETbA)$ we have $f(\mathbf{a} + \mathbf{e}^r) = a + 1 > a = f(\mathbf{a})$. □

The converse is not true, i.e., we can find decision functions verifying (ETb) but not $(ETbA)$, as shown in the following example.

Example 4.5
Let $j = 2$ and $n = 2$. Tables 3 and 4 show the values of $f(\mathbf{x}) = f(x_1, x_2)$ for two different decision functions f which verify (ETb) but not $(ETbA)$. Both of them are anonymous. The first one verifies also (M) but not (N), and the second one verifies (N) but not (M).

	-2	-1	0	1	2
-2	-2	-2	-2	-1	-1
-1	-2	-1	-1	0	0
0	-2	-1	-1	0	1
1	-1	0	0	0	1
2	-1	0	1	1	2

Table 3: f verifies (A), (M) and (ETb) but neither $(ETbA)$ nor (N).

	-2	-1	0	1	2
-2	0	-2	-1	0	0
-1	-2	0	-2	0	0
0	-1	-2	0	2	1
1	0	0	2	0	2
2	0	0	1	2	0

Table 4: f verifies (A), (N) and (ETb) but neither $(ETbA)$ nor (M).

These two possible extensions of the Tie-breaking axiom are equivalent if we assume both (N) and (M), as a consequence of the following proposition.

Proposition 4.6
If a decision function f verifies Neutrality (N), Monotonicity (M) and Extended Tie–breaking (ETb), then it verifies Extended Tie–breaking Alternative $(ETbA)$.

Proof:
Assume that f verifies (N), (M) and (ETb). Then, we have

$$f(\mathbf{0}) = 0 < f(\mathbf{1}) < \cdots < f(\mathbf{j}) \leq j$$

and the only possibility is that $f(\boldsymbol{a}) = a$ for all $a \in \{0, \ldots, j\}$. Furthermore, if $a \in \{0, \ldots, j-1\}$ and $\mathbf{a} < \mathbf{x} \leq \mathbf{a} + \mathbf{1}$ then $\mathbf{a} < \mathbf{a} + \mathbf{e}^r \leq \mathbf{x} \leq \mathbf{a} + \mathbf{1}$ for some $r \in N$. Thus, by (ETb) and (M),

$$f(\mathbf{a}) < f(\mathbf{a} + \mathbf{e}^r) \leq f(\mathbf{x}) \leq f(\mathbf{a} + \mathbf{1}) = a + 1$$

and it is $f(\mathbf{x}) = a + 1$. $\square$

Thus, we can give a characterization of the j–majority function different from the one established in Theorem 4.1.

Corollary 4.7
Let $j \geq 1$. There exists a unique decision function $f : \mathcal{J}^n(j) \to \mathcal{J}(j)$ verifying Neutrality (N), Monotonicity (M), Extended Tie–breaking Alternative $(ETbA)$ and Balancedness (B).

But neither the Extended Tie–breaking (ETb) axiom nor the Extended Tie–breaking Alternative $(ETbA)$ play, for $j > 1$, a role equivalent to the (Tb) axiom for $j = 1$, in the sense that for $j > 1$ there are many decision functions verifying Anonymity, Neutrality, Monotonicity and Extended Tie–breaking. In the following example we show three different functions verifying these axioms for $j = 2$.

Example 4.8
Let $j = 2$ and $n = 2$. If a function f verifies (A), (N), (M) and (ETb) then its values on most vectors $\mathbf{x} = (x_1, x_2)$ are completely defined, as shown in Table 5:

	-2	-1	0	1	2
-2	-2	-2	-2		0
-1	-2	-1	-1	0	
0	-2	-1	0	1	2
1		0	1	1	2
2	0		2	2	2

Table 5: f verifies (A), (N), (M) and (ETb).

But there are three possible different values for $f(2, -1)$: either 0, 1 or 2 (once one of these options is selected the images of the remaining vectors are determined by (N) and (A)). It is clear that all three functions verify (A), (N), (M) and (ETb).

Notice that the decision function in Example 4.8 satisfies Cancellation (C) for all possible values of $f(2, -1)$. This fact proves that Cancellation (C) cannot be used instead of Balancedness (B) in the statement of Theorem 4.1.

4.2 Extending the Llamazares characterization of simple majority

Theorem 10 in [13] uses Cancellation, $f(\mathbf{0}) = 0$ and strong Pareto to characterize the simple majority decision (for $j = 1$) where

- **Strong Pareto** (SP): $f(\mathbf{x}) = 1$ for every $\mathbf{x}$ such that $\mathbf{x} > \mathbf{0}$ and $f(\mathbf{x}) = -1$ for every $\mathbf{x}$ such that $\mathbf{x} < \mathbf{0}$.

Note that $f(\mathbf{0}) = 0$ and (SP) implies $(ETbA)$ for $j = 1$ but the converse is not true.

A natural extension of (SP) to $j > 1$ is the following.

Definition 4.9 ***Extended strong Pareto*** (ESP)

$$f(\mathbf{x}) = a \quad \textit{for every } \mathbf{x} \textit{ such that } \begin{cases} \mathbf{a} - \mathbf{1} < \mathbf{x} \leq \mathbf{a} & \textit{for} \quad a \in \{1, \dots, j\} \\ \quad \textit{or} & \\ \mathbf{a} \leq \mathbf{x} < \mathbf{a} + \mathbf{1} & \textit{for} \quad a \in \{-j, \dots, -1\} \end{cases}$$

If Cancellation is replaced by Balancedness in Theorem 10 in [13] we have a new characterization of the j-majority function for $j \geq 1$.

Proposition 4.10
A decision function $f : \mathcal{J}^n(j) \to \mathcal{J}(j)$ is the j-majority function if and only if *it verifies Balancedness (B), $f(\mathbf{0}) = 0$ and Extended strong Pareto (ESP).*

Proof:
It is obvious that the j-majority function satisfies the three axioms. Conversely, let f be a decision

function satisfying the three axioms. Balancedness implies that $f(\mathbf{x}) = f(\mathbf{y})$ whenever $\mathbf{x}$ and $\mathbf{y}$ have the same components' sum. The achievable components' sums for the elements in $\mathcal{J}^n(j)$ are all the integers in the interval $[-nj, nj]$. From (B) and $f(\mathbf{0}) = 0$ it follows that $f(\mathbf{x}) = 0$ for all $\mathbf{x}$ with a zero sum. Let $C_m = \{\mathbf{y} \in \mathcal{J}^n(j) : \sum_{i=1}^n y_i = m\}$ for each integer $m \in [-nj, nj]$. For every value of m there is at least a vector $\mathbf{x} \in C_m$ verifying $\lfloor \overline{x} \rfloor \leq x_r \leq \lceil \overline{x} \rceil$ for all $r \in N$. Then, (ESP) sets the value of $f(\mathbf{x})$, and by (B) the value of f on any other vector $\mathbf{y} \in C_m$ coincides with $f(\mathbf{x})$. $\square$

Note that Cancellation, as defined in Definition 3.3 for $j \geq 1$ can only be replaced by Balancedness in Proposition 4.10 if $j = 1$, which leads to Theorem 10 in [13].

5 Conclusion and some open questions

In this paper we give a set of independent necessary and sufficient conditions which uniquely characterize j-majority, a decision function defined for any integer $j \geq 1$ which coincides with the simple majority decision when $j = 1$. The restriction of this characterization for $j = 1$ provides an equivalent formulation of May's theorem by using some new axioms. It is worth to recall that in our broader context the positive responsiveness property considered by May must be formulated in a more general way. In summary, although the j–majority function is not strategy–proof for $j \geq 2$, we consider that this extension of the simple majority decision is worthwhile whenever voters prioritize their sincere vote in search of a truthful collective decision. An alternative characterization has been provided for the j-majority decision, whose restriction to $j = 1$ coincides with the Llamazares characterization of the simple majority [13].

The decision functions, and in particular the j-majority, can be interpreted as aggregation functions or as object evaluation functions. They can be used in both ordinal and cardinal sense and therefore they have a wide field of application.

The work done in this paper motivates the consideration of future research lines:

As a first objective, we may look for different sets of axioms, alternative to those considered in this paper, with the purpose to uniquely characterize a function for $j > 1$, different from the j-majority function, whose restriction to $j = 1$ leads to the simple majority decision function.

A second goal would be the enumeration of the decision functions, for $j > 1$, whose restriction to $j = 1$ gives the simple majority function, i.e., decision functions that *extend* the simple majority function.

A third, more complex, goal would be to study the case in which voters act strategically with the two-candidate situation described, with two or more different possible levels of support for each of them. In this situation, a voter who prefers A (with a low degree of support) to B may even vote for B in order A does not get all the support that the voter thinks A can get. Some empirical experiments done with students suggest that manipulability is less frequent in this context than in multiparty systems.

Finally, it would be interesting to enumerate the decision functions that verify a list of properties in the context $j > 1$. For instance, we would like to enumerate all decision functions that verify neutrality and monotonicity plus the properties in each of the following items:

- Anonymity and Extended Tie-breaking.

- Balancedness.
- Anonymity (for $j = 1$ this number is $\binom{n+1}{\lfloor \frac{n}{2} \rfloor + 1}$, computation done in [20]).

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