New geometrical and dynamical techniques for problems in celestial mechanics

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New Geometrical and Dynamical Techniques for Problems in Celestial Mechanics

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Tiger got to hunt, bird got to fly;
Man got to sit and wonder 'why, why, why?'
Tiger got to sleep, bird got to land;
Man got to tell himself he understand.
Kurt Vonnegut, Cat's Cradle

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Summary

In this thesis, we study the application of symplectic geometry, regular and singular, to symplectic dynamical systems. We start with a motivating case: the relation between symplectic foliations and global transverse Poincaré sections, showing that meaningful dynamical information can be gleaned by simple observations on the geometry of the phase space - in this case, the existence of a symplectic foliation on a hypersurface of the phase space.

We then go on study dynamical systems of particular importance in geometry - those given by a group action on a manifold. In particular, we consider a singular symplectic manifold (specifically, a manifold equipped with a symplectic form which blows up in a controlled manner on a hypersurface of that manifold, namely, a b-symplectic form) with a group action preserving the geometry and give a b-symplectic slice theorem which provides an equivariant normal form of the b-symplectic form in the neighbourhood of an orbit. Particular examples of b-symplectic group symmetries are then explored: those given by the cotangent lift of group translation on so-called b-Lie groups.

The second part of this thesis focuses on symplectic and b-symplectic dynamical systems coming from celestial mechanics. In particular, the separatrix map of the stable and unstable manifolds of the fixed point at infinity of the planar circular restricted three-body problem is examined and an estimate of the width of the stochastic layer is given - that is the existence of a K.A.M. torus which acts as a boundary to bounded motions is proved. Due to the delicate nature of the problem - namely issues coming from the parabolic nature of the fixed point and exponentially small nature of the splitting, careful control of the errors of the separatrix map is paramount. This is achieved by employing geometric methods, namely, by taking full advantage of generating functions which exist by virtue of the symplectic nature of the system.

Finally, motivated by the important role of symplectic geometry in the systems of celestial mechanics in mind, we give examples of degenerate and singular symplectic structures occurring in systems of celestial mechanics which cannot be equipped with a symplectic form.

Resumen

En esta tesis, estudiamos una aplicación de la geometría simpléctica, regular y singular, a sistemas dinámicos simplécticos. Comenzamos con un caso motivador: la relación entre foliaciones simplécticas y secciones transversales de Poincaré globales, que muestra que se puede obtener información significativa de la dinámica mediante simples observaciones sobre la geometría del espacio de fase, en este caso, la existencia de una foliación simpléctica en una hipersuperficie del espacio de fase.

Luego continuamos estudiando sistemas dinámicos de particular importancia en geometría, dados por una acción de grupo sobre una variedad. En particular, consideramos una variedad simpléctica singular (específicamente, una variedad equipada con una forma simpléctica que explota de manera controlada en una hipersuperficie de esa variedad, es decir, una forma b-simpléctica) con una acción de grupo que preserva la geometría y damos un teorema de rebanada b-simpléctico que proporciona una forma normal equivariante de la forma b-simpléctica en una vecindad de una órbita. Luego se exploran ejemplos particulares de simetrías de grupo b-simplécticas: las dadas por el levantamiento cotangente de la translación (grupal) en los llamados b-grupos de Lie.

La segunda parte de esta tesis se enfoca en sistemas dinámicos simplécticos y b -simplécticos provenientes de la mecánica celeste. En particular, se examina el "separatrix map" de variedades estables e inestables del punto fijo al infinito del problema de los tres cuerpos restringido circular plano, y se da una estimación del ancho de la capa estocástica, es decir, la existencia de un toro K.A.M. que actúa como frontera para movimientos acotados. Debido a la naturaleza delicada del problema, es decir, los problemas que provienen de la naturaleza parabólica del punto fijo y la naturaleza exponencialmente pequeña de la separación, el control cuidadoso de los errores del "separatrix map" es primordial. Esto se logra empleando métodos geométricos, es decir, aprovechando al máximo las funciones generadoras que existen en virtud de la naturaleza simpléctica del sistema.

Finalmente, motivados por el importante papel de la geometría simpléctica en los sistemas de mecánica celeste en mente, damos ejemplos de estructuras simplécticas degeneradas y singulares que ocurren en sistemas de mecánica celeste que no pueden equiparse con una forma simpléctica.

Chapter 1

Introduction

It is difficult to avoid the impression that a miracle confronts us here, quite comparable in its striking nature to the miracle that the human mind can string a thousand arguments together without getting itself into contradictions, or to the two miracles of laws of nature and of the human mind's capacity to divine them.
Eugene Wigner, The Unreasonable Effectiveness of Mathematics in the Natural Sciences

Whether partial to a positivist or Platonic viewpoint, the effectiveness of mathematics in describing the "natural sciences" remains beyond doubt. Classical mechanics is governed by (or, at least, described with uncanny accuracy by) the rules of differential equations. In turn, the qualitative *dynamical* characteristics of the solutions of these equations can be decided by still more abstract *geometric* aspects of the system. Nowhere is this more apparent than in the study of Hamiltonian systems and their associated geometry, now an important subject of study in its own right. Symplectic geometry is responsible for the many striking characteristics of Hamiltonian mechanics - from the trivial dynamics of integrable systems to the stochastic layers surrounding split separatrices. Further abstractions of phase space as a geometric object (for example, reducing a phase space by a symmetry) leads naturally to Poisson geometry, in which the phase space is foliated by symplectic leaves on which the dynamics occur. Classical examples include the Lie-Poisson structures defined on the dual of the Lie algebras, which describe systems ranging from the movement of a rigid body to the flows of ideal fluids.

Recently, there has been a surge of interest in possible applications of other, somewhat less classical, Poisson structures to classical mechanics. Two of these – b-Poisson and cosymplectic manifolds – will feature here. This thesis, then, will explore Poisson and symplectic Hamiltonian dynamical systems ranging from the specific and quite-abstract to the general and more-abstract, loosely tied by the following theme, already considered an incontrovertible fact: an understanding of the geometry of any dynamical system can provide invaluable dynamical insight.

As the subject matter is somewhat diverse, the chapters of this thesis are designed to be self-contained: each come with their own necessary preliminaries, along with some cross-references. This is with the exception of Chapter 5 and Chapter 6, which should be read in order.

1.1 Outline

This thesis contains the following novel results:

1.1.1 Chapter 2: Preliminaries

Chapter 1 provides basic preliminaries of symplectic and Poisson geometry common to all following chapters. We start with a motivating example, which shows that the examination of more general Poisson structures can lead easily to dynamical results. In particular, the following result is given:

Theorem A. *Let (M, H, ω) be a Hamiltonian system and $Z = H^{-1}(c)$ a compact, regular level energy set. Then Z possesses a global transverse Poincaré section for the flow of H if and only if Z is cosymplectic.*

1.1.2 Chapter 3: Group Actions on b-Symplectic Manifolds

b-Symplectic forms and their symmetries have attracted considerable interest since their introduction in [1]. Chapter 2 continues the study of b-symplectic symmetries by giving a semi-local normal form for the structure in a neighbourhood of a group orbit, in the spirit of the symplectic slice theorem. In particular, we prove the following:

Theorem B. *Let (M, ω, G) be a b-symplectic manifold together with an effective b-symplectic action by a compact connected Lie group G. Let Z be the critical set of the b-symplectic form and assume that the orbits of G are transverse to the symplectic foliation of Z and Hamiltonian when restricted to the leaves of Z. Then*

1. *G is necessarily of the form $G = (S^1 \times H)/\mathbb{Z}_k$ where $\mathbb{Z}_k$ is a cyclic group and H is a compact Lie group.*

2. *The action of G lifts to an action of a product group $\tilde{G} = S^1 \times H$ on a finite cover $\mathcal{U}$ of a collar neighbourhood of Z, $\mathcal{U} := (-\epsilon, \epsilon) \times \tilde{Z}$, $\tilde{Z} \cong S^1 \times \mathcal{L}$, where S^1 acts on $\tilde{Z}$ by translations on the S^1-factor and H by symplectomorphisms on the symplectic leaf $\mathcal{L}$.*

3. *Let $\tilde{z} \in \tilde{Z}$. Denote by $\mathcal{O}^{\tilde{G}}_{\tilde{z}}$ the orbit of $\tilde{z}$ in $\tilde{Z}$ under the action of $\tilde{G}$ and by $Y^H_{\tilde{z}}$ the bundle of Theorem 3.1.2 associated to the action of H on $\mathcal{L}$. Then there is an equivariant b-symplectomorphism from a neighbourhood of the orbit $\mathcal{O}^{\tilde{G}}_{\tilde{z}} \cong S^1 \times \mathcal{O}^H_{\tilde{z}}$ to the zero section of the bundle $\tilde{E} = T^*S^1 \times Y^H_{\tilde{z}}$ where $\tilde{G}/H_{\tilde{z}}$ is embedded as the zero section and the b-symplectic form on $\tilde{E}$ is given by*

$$\omega = \omega_{tw,c} + \omega_{MGS}.$$

   Here $\omega_{tw,c}$ is the pullback of the twisted b-symplectic form of modular period c on the manifold T^*S^1 (see Definition 3.1.7) under the projection $\tilde{E} \to T^*S^1$ and ω_{MGS} is the pullback of the MGS normal form as given by Theorem 3.1.2 under the projection $\tilde{E} \to Y^H_{\tilde{z}}$

4. *There is an equivariant b-symplectomorphism from a neighbourhood of the orbit $\mathcal{O}^G_z$ to a neighbourhood of the zero section of the bundle $E = (T^*S^1 \times Y^H_{\tilde{z}})/(\mathbb{Z}_k)$ where $\mathbb{Z}_k$ is a finite cyclic group acting by the cotangent lifted action on T^*s^1.*

To achieve the b-symplectic slice theorem, we additionally give some minor results on cosymplectic and foliation preserving group actions which may be of independent interest. The normal form is then computed for several examples of b-symplectic actions.

This work is joint with Anna Kiesenhofer and Eva Miranda

1.1.3 Chapter 4: b-Lie Groups and b-Symplectic Reduction

Chapter 3 investigates a particular category of symmetric b-symplectic forms, those defined on the cotangent bundle of certain Lie groups known as *b-Lie groups*. Theses can be defined for and Lie group with a codimension-one subgroup. The existence of a codimension one Lie subgroup allows the definition of the canonical b-symplectic form on the tangent bundle. As in the symplectic case, the form is invariant under the cotangent lifted action of the group acting on itself by translations and the reduces to a Poisson structure on the quotient manifold.

Theorem 1.1.1. *Let ${}^bT^*G$ be endowed with the canonical b-Poisson structure. Then the Poisson reduction under the cotangent lifted action of H by left translations is*

$$(({}^bT^*G)/H, \Pi_{red}) \cong (\mathfrak{h}^* \times {}^bT^*(G/H), \Pi^-_{L\text{-}P} + \Pi_{b\text{-}can})$$

where $\Pi^-_{L\text{-}P}$ is the minus Lie-Poisson structure on $\mathfrak{h}^*$ and $\Pi_{b\text{-}can}$ is the canonical b-symplectic structure on ${}^bT^*(G/H)$, where G/H is viewed as a b-manifold with critical hypersurface the point $[e]_\sim$.

Some examples of b-Lie groups, their reduction and the associated reduced b-symplectic forms are computed.

This work is joint with Anna Kiesenhofer and Eva Miranda

1.1.4 Chapter 5: The Melnikov Potential and Poincaré Return Maps for the PCR3BP

Chapter 5 and Chapter 6 contains the necessary material to find rigorously the ”boundary of bounded motion” of the Planar Circular Restricted three-body Problem (PCR3BP). In order to achieve this, a particular Poincaré map of the problem is analyzed. Chapter 5 considers a modified Melnikov potential which can be employed to control errors of the associated separatrix map. This integral is calculated for the case of the parabolic fixed point “at infinity” of the PCR3BP. Upon a rescaling of the complex plane, this integral is shown to be close to the usual Melnikov potential taken along the separatrices (in this case, the parabolic orbits) of the Kepler problem. In particular, we consider the following “Melnikov potential”:

Consider the Kepler problem in rotating coordinates $(r, \phi) = (r, \alpha - t)$. Consider a parameterization $(v, \xi) \mapsto (r_e(v), \phi_e(v, \xi))$ of the orbits of the Kepler problem of eccentricity e in a level set of the Jacobi constant $\mathcal{J}^{-1}(\mathcal{J}_0)$. Consider the PCR3BP and define the Melnikov potential as the integral

$$\mathcal{L}(\xi; e, \mu, G) = G^{-2} \int_{-\frac{T}{2}}^{\frac{T}{2}} V(r_e(v), \phi_e(\xi); \mu)\, dv \tag{1.1}$$

where $T = T(\mathcal{J}_0, e)$ denotes the period of the Keplerian orbit of eccentricity e in $\mathcal{J}^{-1}(\mathcal{J}_0)$, μ is the ratio of the massive bodies in the PCR3BP and $G^{-2}V(r_e(v), \phi_e(v, \xi); \mu)$ is the difference in the potential between the Keplerian and restricted three body problem. The following is then proved for $G < \mathcal{J}_0$ and G large enough,

Theorem 1.1.2. *Let $\mathcal{L}(\xi; \mathcal{J}_0, G, \mu)$ be the Melnikov potential function of equation (5.63) for an orbit of angular momentum G and Jacobi constant $\mathcal{J}_0$. Then*

$$\mathcal{L}(\xi; \mathcal{J}_0, G, \mu) = \mathcal{L}^{[0]}(\xi; \mathcal{J}_0, G, \mu) + 2\sum_{\ell=1}^{+\infty} L^{[\ell]}(\xi; \mathcal{J}_0, G, \mu)\cos(\ell\xi) \tag{1.2}$$

where

$$L^{[1]}(G, \mu) = -\mu(1-\mu)\sqrt{\pi}\frac{1-2\mu}{4\sqrt{2}}G^{-3/2}e^{-\frac{G^3}{3}}\left(1 + \mathcal{O}\left(G^{-3/2}, \sqrt{1-e^2}\right)\right) \tag{1.3}$$

$$L^{[2]}(G, \mu) = -2\mu(1-\mu)\sqrt{\pi}G^{1/2}e^{-\frac{2G^3}{3}}\left(1 + \mathcal{O}\left(G^{-1/2}, \sqrt{1-e^2}\right)\right) \tag{1.4}$$

$$L^{[\ell]}(G, \mu) = \mathcal{O}\left(G^{\ell-3/2}e^{-\frac{\ell G^3}{3}}, \sqrt{1-e^2}\right), \text{ for } \ell \geq 2. \tag{1.5}$$

The integral is recast as the difference between approximate solutions to the Hamilton-Jacobi equation in certain domains of the complex plane, by which the difference between the Melnikov potential and the difference of these solutions can be shown to be exponentially small.

This work is joint with Amadeu Delshams

1.1.5 Chapter 6: Parabolic Return Maps and the PCR3BP

Separatrix maps suitable for modelling parabolic fixed points are examined and estimates for the stochastic layer width are given by applying a classical invariant curve theorem. A quantitative K.A.M. theorem is applied, which gives estimates for the appearance of invariant curves in the non-asymptotic regime. The study of the parabolic separatrix maps is then applied to the circular planar restricted three-body problem. The geometric properties of the system (in particular, the area-preserving nature of the Poincaré map) are used to prove the accuracy of a parabolic-type separatrix map in this case. In particular, we bound the "boundary of bounded motions" of the PCR3BP as follows:

Theorem 1.1.3. *Let $\mathbb{T}_{h,\mathcal{J}}$ be a torus of the Kepler equation in rotating coordinates with Keplerian energy h contained a level set $\mathcal{J}$ of the Jacobi constant, where the Jacobi constant is assumed to be negative. Let μ be fixed, $0 < \mu < 1$ and $\mu \neq 1/2$. Then for every $\epsilon_0 > \epsilon > 0$ and $h(\mu, \mathcal{J})$ satisfying*

$$h(\mu, \mathcal{J}) = -(c_1(\mu) f_1(\mathcal{J}))^{\frac{2}{5}-\epsilon}$$

where

- *$c_1(\mu)$ is a constant depending only on the mass ratio of the massive bodies given by*

$$c_1(\mu) = \sqrt{\frac{\pi}{32}} \left(\mu(1-\mu)^3 - (1-\mu)\mu^3\right)$$

- *$f_1(\mathcal{J})$ depends only on the Jacobi constant and is given by*

$$f_1(\mathcal{J}) = |\mathcal{J}|^{3/2} e^{-\frac{|\mathcal{J}|^3}{3}}$$

there exists some sufficiently large $\mathcal{J}^*$ such that for all $|\mathcal{J}| > |\mathcal{J}^*|$ there exists a Keplerian torus $\mathbb{T}_{h_0,\mathcal{J}}$ at some $|h_0| \leq |h|$ which continues to an invariant torus of the PCR3BP.

The above theorem bounds the stochastic layer width as $\mathcal{J} \to \infty$. In order to examine the problem in the non-asymptotic regime, we examine an application of a parameterized K.A.M. theorem to non-optimal estimates open to optimization by computer-assisted proofs are given using the parameterized K.A.M. theorem. Specifically, we prove the following:

Theorem 1.1.4. *Let $\mathbb{T}_{h_0,\mathcal{J}_0}$ be a torus of the Kepler equation in rotating coordinates with Keplerian energy h_0 contained a level set $\mathcal{J}_0$ of the Jacobi constant. Suppose that*

$$h_0 = \tilde{C}_* c_1(\mu)^{2/17} f(\mathcal{J}_0)^{2/17}$$

where

- $\tilde{C}_*$ *is of the form* $\tilde{C}_* = C_* + \mathcal{O}(\mathcal{J}_0^{-1})$*, where* C_* *is a constant which will be given in Theorem 6.4.13.*

- $c_1(\mu), f(\mathcal{J}_0)$ *are as in Theorem 1.1.3.*

Then for large $\mathcal{J}_0$ *the torus persists in the PC3BP.*

This work is joint with Amadeu Delshams

1.1.6 Chapter 7: Singular Symplectic Structures in Celestial Mechanics

This chapter gives some examples of b-symplectic forms and the more degenerate b^m-symplectic forms which occur in Hamiltonian systems, often as the result of singular coordinate transformations. The definition of *folded symplectic forms* is recalled, and several examples of these occurring in systems coming from celestial mechanics.

This work is joint with Amadeu Delshams, Eva Miranda, Cédric Oms and Arnau Planas

1.2 Publications Resulting from this Thesis

The results of this thesis can be found in the following articles:

- *An Invitation to Singular Symplectic Geometry*, joint with Cédric Oms, Amadeu Delshams, Eva Miranda, and Arnau Planas. International Journal of Geometric Methods in Modern Physics 16, no. supp01 (2019): 1940008.

- *b-Structures on Lie groups and Poisson reduction*, joint with Anna Kiesenhofer and Eva Miranda, (submitted for publication).

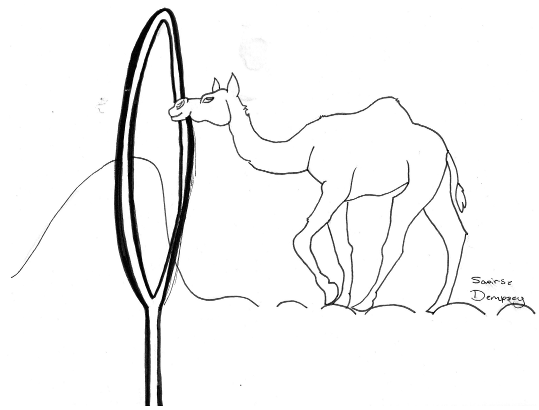
- *A b-Symplectic Slice Theorem*, joint with Anna Kiesenhofer and Eva Miranda, (submitted for publication).

- *Non-Existence of Global Transverse Poincaré Sections*, (submitted for publication).

- *The Boundary of Bounded Motions in the Restricted Planar Circular Three Body Problem*, joint with Amadeu Delshams (in preparation).

Chapter 2

Preliminaries



Let's start at the very beginning. A very good place to start.

Maria, The Sound of Music

An encompassing view of Hamiltonian systems might be "the study of transformations preserving symplectic width" - a seemingly reductive definition with many fundamental dynamical consequences.

The phase space of many classical systems is symplectic, a symplectic structure being equivalent to a "non-degenerate" Poisson structure, however, there are examples of dynamical systems occurring on more general Poisson manifolds. Two types of Poisson manifolds will play a leading role in this thesis: cosymplectic and b-Poisson.

Here we very briefly recall the basics symplectic geometry and Poisson geometry and recall the definitions

2.1 Symplectic manifolds

Symplectic geometry originated from the study of Hamiltonian dynamical systems but has become an important branch of mathematics in its own right.

Definition 2.1.1. *A Hamiltonian system is a triple (M, ω, H) where M is a manifold, ω is closed, non-degenerate 2-form, called a symplectic form and H is a smooth function, $H \in$*

$\mathcal{C}^\infty(M)$. *The vector field defined by* $\iota_{X_f}\omega = -dH$, *where* ι *denotes the interior product, is called a* Hamiltonian vector field.

Two important examples of symplectic forms are the *Darboux symplectic form*, and the *canonical symplectic form* on a cotangent bundle.

Definition 2.1.2. *Let* $M = \mathbb{R}^{2n}$ *have coordinates* $(x_1, y_1, \ldots, x_n, y_n)$. *The Darboux symplectic form* ω *is defined*

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

Definition 2.1.3. *Let* M *be a manifold of dimension* n *and consider the cotangent bundle* T^*M. *The Liouville one form* λ *is defined*

$$\langle \lambda_m, v \rangle = \langle m, (\pi_m)_*(v) \rangle$$

where we have $v \in T(T^*M), m \in T^*M$. *and* π *is the projection* $\pi : T^*M \to M$. *The canonical symplectic form on* T^*M *is defined*

$$\omega = -d\lambda$$

These two examples are especially important as they provide basic normal forms for symplectic structures. Proof of the local equivalence of symplectic forms is often achieved by the application of the (relative) Moser theorem:

Theorem 2.1.4. *Suppose* ω_0 *and* ω_1 *are two symplectic forms on a symplectic manifold* M, *that coincide on closed submanifold* V. *Then there is a neighbourhood* U *of* V *in* M *and a map* $\phi : U \to M$ *with* $\phi|_U = Id_U$ *and* $\phi^*(\omega_0) = \omega_1$.

The proof of this theorem is achieved by setting up an appropriate differential equation on the space of differential forms and using the Poincaré lemma to show the existence of a solution. An important corollary is the Darboux theorem, showing that all symplectic forms locally look alike:

Corollary 2.1.5. *Let* $p \in M$ *be a point in symplectic manifold* (M, ω) *There exists a system of local coordinates* $(x_1, y_1, \ldots, x_n, y_n)$ *centred at* p *such that* ω *is in Darboux form (2.1.2).*

2.2 Poisson manifolds

As in the symplectic case, Poisson structures associate dynamical systems to the underlying manifold by assigning vector fields, termed ”Hamiltonian vector fields”, to smooth functions on the manifold. In this case, a Hamiltonian vector field is assigned via a bracket on the algebra of smooth functions on the manifold:

Definition 2.2.1. *A Poisson manifold M is a smooth manifold equipped with a skew-symmetric $\mathbb{R}$-bilinear map*

$$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

that satisfies

1. *Skew symmetry:* $\{f, g\} = -\{g, f\}$
2. *The Jacobi identity:* $\forall f, g, h \in C^\infty(M), \{\{f, g\}, h\} + \{\{h, f\}, g\} + \{\{g, h\}, f\} = 0$ *and*
3. *Leibniz's rule:* $\forall f, g \in C^\infty(M), \{fg, h\} = f\{g, h\} + \{f, h\}g$.

Definition 2.2.2. *Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold and $f \in C^\infty(M)$ a smooth function on M. The Hamiltonian vector field, X_f of f is the derivation given by $X_f(g) = \{f, g\}$. f is then called the Hamiltonian function. $(M, \{\cdot,\cdot\}, H)$ is referred to as the Hamiltonian system.*

The Hamiltonian vector fields on a Poisson manifold define an involutive distribution which integrates to a (not necessarily regular) foliation of the manifold. The restriction of the Poisson structure to the leaves of the foliation defined by the Hamiltonian vector fields is non-degenerate and the associated isomorphism between the tangent and cotangent bundles of the manifold defines a symplectic form on each leaf of the foliation.

Poisson geometry, then, can be viewed as a combination of symplectic geometry, which exists on the symplectic leaves of the Poisson manifold with some extra theory not required in symplectic geometry, such as singularity theory. In general, it can be difficult to say very much about general Poisson manifolds. For this reason, one approach to the study of Poisson manifolds is to choose some simple classes of Poisson manifolds and try to describe their properties. One of the classes, known as, *b*-Poisson manifolds will be discussed in the next section.

2.2.1 b-Poisson Manifolds

In [1] the study of a certain class of mildly singular Poisson structures, known as b-Poisson structures was initiated. The 2-dimensional case had been previously studied in [2].

Definition 2.2.3. *Let (M^{2n}, Π) be an oriented Poisson manifold such that the map*

$$p \in M \mapsto (\Pi(p))^n \in \Lambda^{2n}(TM) \tag{2.1}$$

*is transverse to the zero section, then $Z = \{p \in M | (\Pi(p))^n = 0\}$ is a hypersurface and we say that Π is a b-**Poisson structure** on (M^{2n}, Z) and (M^{2n}, Z, Π) is a b-**Poisson manifold**. The hypersurface Z is called **singular hypersurface**.*

These Poisson manifolds were shown have a particularly simple symplectic foliation: the connected components of $M \backslash Z$ are open symplectic leaves of dimension $2n$. On Z there is a corank-1 Poisson structure, which in [3] was shown to be equivalent to a cosymplectic structure, as defined by Libermann [4].

2.2.2 Cosymplectic Manifolds

In [3] corank one Poisson structures were studied via foliated forms. The following were defined for codimension-one symplectic foliations:

Definition 2.2.4. *A defining 1-form of the foliation is a nowhere vanishing one-form $\alpha \in \Omega^1(M)$ such that $\iota_{\mathcal{L}}^* \alpha = 0$ for all leaves $\mathcal{L}$ of the foliation, where $\iota_{\mathcal{L}}$ is the canonical inclusion. A defining 2-form of the foliation is a 2-form $\omega \in \Omega^2(M)$ such that $\iota_{\mathcal{L}}^* \omega$ is the symplectic form on each leaf of the foliation.*

It was shown that such a Poisson structure is the singular hypersurface of a b-Poisson manifold if and only if such forms can be chosen to be closed. This is equivalent to the usual definition of cosymplectic structures:

Definition 2.2.5. *A cosymplectic manifold is a manifold Z equipped with a closed one-form $\eta \in \Omega^1(Z)$ and closed two-form ω such that $\eta \wedge (\iota^* \omega)^n$ is a volume form for Z, where ι is the embedding of Z in M.*

A cosymplectic manifold, then, has a codimension one foliation by symplectic leaves such that the symplectic form on each leaf is the restriction of a form which is closed on the ambient manifold.

When embedded as the critical set of a b-Poisson manifold, the cosymplectic manifold comes with a foliation preserving Poisson vector field transverse to the symplectic foliation given by the modular vector field of the Poisson structure:

Definition 2.2.6 (Modular vector field)**.** *Let M be a Poisson manifold and Ω a volume form on M. The associated* **modular vector field** *is defined as the derivation:*

$$C^\infty(M) \to \mathbb{R} :\ f \mapsto \frac{\mathcal{L}_{X_f}\Omega}{\Omega}.$$

It can be shown that the modular vector field is a Poisson vector field and that the modular vector fields associated to different volume forms only differ by a Hamiltonian vector field. In [3] the existence of this Poisson vector field was used to prove some important topological results for corank-1 Poisson structures, in particular, the following:

Definition 2.2.7. *Let (M, Z) be a b-symplectic manifold and suppose that Z is compact and connected and that its symplectic foliation has a compact leaf $\mathcal{L}$. Then Z is a mapping torus and taking any modular vector field v_{mod}, there exists a number $c > 0$ such that*

$$Z \cong \frac{[0, c] \times \mathcal{L}}{(0, x) \sim (c, \phi(x))}$$

where the time t-flow of v_{mod} corresponds to translation by t in the first coordinate. In particular, ϕ is the time c-flow of u. c is called the modular period of Z.

The modular vector field has an interpretation in terms of Poisson cohomology. For the case of b-Poisson manifolds, this class is given by the modular periods c of the connected components of the critical set. Having chosen a modular vector field v_{mod}, we can choose defining one and two-forms of the cosymplectic structure uniquely by imposing

$$\alpha(v_{mod}) = 1 \text{ and } \iota(v_{mod})\omega = 0. \tag{2.2}$$

Definition 2.2.8. *Defining one- and two-forms fulfilling this are referred to as* the *defining one- and two-forms of the foliation.*

2.3 b-Geometry

Much progress on the study of b-Poisson forms has been made by reinventing the mildly singular Poisson structure as a type of singular symplectic structure. This is achieved by utilising the concepts of b-calculus as given in [5].

We will now go on to discuss b-geometry, which is used to import some of the previous theorems on symplectic manifolds to the b-Poisson world.

Definition 2.3.1. *A b-manifold is a pair (M, Z) of an oriented manifold M and an oriented hypersurface $Z \subset M$.*

Definition 2.3.2. *A b-vector field on a b-manifold (M, Z) is a vector field which is tangent to Z at every point $p \in Z$.*

If f is a local defining function for Z on some open set $U \subset M$ and $(f, z_2, \ldots, z_n)$ is a chart on U, then the set of b-vector fields on U is a free $C^\infty(U)$-module with basis

$$(f\frac{\partial}{\partial f}, \frac{\partial}{\partial z_2}, \ldots, \frac{\partial}{\partial z_n}). \tag{2.3}$$

According to the Serre-Swan theorem ([6]), there exists a vector bundle whose local sections are b-vector fields.

Definition 2.3.3. *Let (M, Z) be a b-manifold. The b -tangent bundle ${}^bTM$ on (M, Z) is the vector bundle whose sections are the b -vector fields on (M, Z).*

Definition 2.3.4. *The b -cotangent bundle ${}^bT^*M$ is the dual bundle of ${}^bTM$ We call the vector bundle associated to this locally free C^∞_M-module the b-**tangent bundle** and denote it ${}^bTM$.*

The classical exterior derivative d on the complex of (smooth) k-forms extends to the complex of b-forms in a natural way. Indeed, any b^m-form ω can locally be written in the form

$$\omega = \alpha \wedge \frac{df}{f} + \beta$$

where $\alpha \in \Omega^{k-1}(M), \beta \in \Omega^k(M)$ where f is a local defining function of Z and $\frac{df}{f}$ is the b-one-form dual to $f\frac{\partial}{\partial f}$ in a frame of the form (2.3). We then define the exterior derivative $d\omega := d\alpha \wedge \frac{df}{f} + d\beta$ (see [1] for details).

2.3.1 b-Symplectic Forms

Definition 2.3.5. *Let (M, Z) be a b-manifold, where Z is the critical hypersurface as in Definition 2.3.1. Let $\omega \in {}^b\Omega^2(M)$ be a closed b-two-form. We say that ω is b-**symplectic** if ω_p is of maximal rank as an element of $\Lambda^2({}^bT_p^*M)$ for all $p \in M$.*

Remarkably we can show that, similar to the correspondence between non-degenerate Poisson structures and symplectic structures on a manifold, we can associate b-Poisson structures to b-symplectic structures. This turns out to be remarkably fortuitous - using this association one can prove parallels of important and useful results using techniques similar to those employed in symplectic geometry. We will now state analogues of the classic Moser theorem and the Darboux theorems found in [1].

Theorem 2.3.6. *Let ω_0 and ω_1 be two b-symplectic forms on (M, Z). If they induce on Z the same restriction of the Poisson structure and their modular vector fields differ on Z by a Hamiltonian vector field, then there exist neighbourhoods U_0, U_1 of Z in M and a diffeomorphism $\gamma : U_0 \to U_1$ such that the $\gamma|_Z = id_Z$ and $\gamma^*\omega_1 = \omega_0$.*

Theorem 2.3.7. *Let ω be a b-symplectic form on (M^{2n}, Z). Let $p \in Z$. Then we can find a local coordinate chart $(x_1, y_1, \ldots, x_n, y_n)$ centred at p such that hypersurface Z is locally defined by $y_1 = 0$ and the symplectic form is given by*

$$\omega = dx_1 \wedge \frac{dy_1}{y_1} + \sum_{i=2}^{n} dx_i \wedge dy_i.$$

In parallel with the case of local symplectic extensions of cosymplectic manifolds, any cosymplectic manifold can be locally extended to a b-symplectic manifold. The Moser path method ensures that any two such extensions are locally b-symplectomorphic as long as their modular periods are equal.

Theorem 2.3.8. *Let Π be a regular corank one Poisson structure on a compact manifold Z, and $\mathcal{F}$ the induced foliation by symplectic leaves. Then $c_{\mathcal{F}} = \sigma_{\mathcal{F}} = 0$ if and only if Z is the exceptional hypersurface of a b-symplectic manifold (M, Z) whose b-symplectic form induces on Z the Poisson structure Π. Furthermore, two such extensions (M_0, Z) and (M_1, Z) are b -symplectomorphic on a tubular neighbourhood of Z if and only if the image of their modular classes under the map $[p_*]$ is the same.*

Using the above theorem, on a collar neighbourhood of the critical set we have the following semi-local normal form for the b-symplectic form.

Theorem 2.3.9. *Let α and β be the defining one and two forms of the b-symplectic stucture as given by 2.2.8. Then in a collar neighbourhood of Z the b-symplectic form is equivalent to the following normal form:*

$$\omega = \pi_Z^*\alpha \wedge \frac{da}{a} + \pi_Z^*\beta. \tag{2.4}$$

where $\pi_Z : Z \times \mathbb{R} \to Z$ is the canonical projection and a the coordinate on $\mathbb{R}$.

2.4 Motivation: Global Transverse Surfaces of Section

As a motivation for studying more general Poisson structures in Hamiltonian dynamical systems, we will now give an application of cosymplectic geometry to the study of global transverse Poincaré sections, also known as global transverse *surfaces of section*. We recall the following definition:

Definition 2.4.1. *A global surface of section for a flow on a closed manifold M is a compact, embedded hypersurface Σ such that*

- *the boundary $\partial\Sigma$ consists of periodic trajectories (or invariant sets, for higher dimensions).*
- *$\Sigma \setminus \partial\Sigma$ is transverse to the flow.*
- *Every orbit of the flow intersects Σ in forward and backward time.*

In the case that M is a 3-manifold, then, the dynamics is essentially reduced to the study of an area-preserving map of the interior of Σ which greatly simplifies the study of the dynamics. Note, however, that the boundary $\partial\Sigma$ can be quite wild. In the ideal case, we can find a Σ such that the flow is everywhere transverse to Σ in which case studying the dynamics reduces to the simple case of an area-preserving diffeomorphism of a Riemann surface. We wish to address the latter situation.

Proposition 2.4.2. *Let Z be a compact energy level surface of a Hamiltonian system (H, ω, M) which possesses a global complete Poincaré section Σ. Then Z is a cosymplectic manifold.*

Proof. Let p be a point in Σ. Denote by $T(p)$ the first return time of p to the section sigma. By assumption, $T(p)$ is well-defined and finite. It is smooth as a function on Σ. We will denote by ϕ the Poincaré map of the global transverse section Σ, explicitly

$$\phi : \Sigma \to \Sigma \tag{2.5}$$

$$\phi(p) = \Phi_{X_H}^{T(p)}(p) \tag{2.6}$$

where Φ is the flow of the vector field X_H and $T(p)$ is the first return time. As noted in [7], the existence of a global Poincaré section equips Z with the structure of a mapping torus. Moreover, ϕ is symplectic (see e.g. [8]) and so Z has the structure of a symplectic mapping torus. Accordingly, as shown in [9] and [3], Z is cosymplectic. □

This puts immediate obvious restrictions on the topology of an energy surface possessing a global transverse Poincaré section using a well-known fact on cosymplectic manifolds.

Corollary 2.4.3. *Let (M, ω, H) be a Hamiltonian system and $Z = H^{-1}(c)$ a closed level energy set possessing a global transverse Poincaré section. The $H^i(Z)$ is non-trivial for all $0 \leq i \leq 2n-1$.*

Proof. Let α and β be the defining one and two forms of the symplectic folation. Then β^i and $\alpha \wedge \beta^i$ are nowhere vanishing for $0 \leq i \leq n-1$. □

Another easy consequence is the following:

Theorem 2.4.4. *Let (H, ω, T^*M) be a Hamiltonian system on T^*M equipped with the canonical symplectic form. Then no level energy surface of H possesses a global transverse Poincaré section.*

Motivated by the previous sections, we ask which hypersurfaces of a symplectic manifold possess a cosymplectic structure, and so a global transverse Poincaré section. It was noted in [10] that a cosymplectic structure exists on a level energy surface if and only if the hypersurface possesses a certain transverse vector field:

Proposition 2.4.5. *([10]) Let (M, ω) be a symplectic manifold and Z a codimension 1 hypersurface. Then Z has an induced cosymplectic structure if and only if Z possesses a transverse symplectic vector field.*

Interestingly, then, the existence of a global transverse Poincaré section depends only on the level energy set of the Hamiltonian and not on the Hamiltonian itself.

Theorem 2.4.6. *Let (M, H, ω) be a Hamiltonian system and $Z = H^{-1}(c)$ a regular, compact submanifold which is cosymplectic. Assume that $dH \neq 0$ on Z. Then Z possesses a global transverse Poincaré section for the flow of H.*

Proof. By Theorem 2.4.5, as Z is cosymplectic there exists a symplectic vector field X transverse to Z such that the defining one form of the symplectic foliation on Z is given by $\alpha = \iota_X(\omega)$. As $dH(Z) \neq 0$ and so $\alpha(X_H) = \iota_X\omega(X_H) = -\omega(X_H, X) = dH(X) \neq 0$. This implies that X_H is everywhere transverse to the symplectic foliation. As Z is compact $\inf_{z\in Z} \alpha(X_H)(z) = \varepsilon_0$, where $\varepsilon_0 > 0$. Note that a leaf of the foliation for a general α may not be compact. However, as in the proof of the Tischler theorem [11] we can approximate α up to arbitrary precision by the rational sum of pullbacks of generators of the circle. In particular, there exists some α' so that

$$\|\alpha - \alpha'\| < \varepsilon_0$$

and α' is of the form

$$\alpha' = \frac{1}{d}\sum_{i=1}^{N} n_i f_i^* d\theta$$

For $n_i \in \mathbb{N}, f_i : M \to \mathbb{S}^1$ where $d\theta$ is the canonical generator of $H^1(\mathbb{S}^1)$. Furthermore, as proved in [11], the foliation defined by such a one-form is a fibration, each fibre of which is necessarily compact as Z is. The sum

$$\alpha'_k = \frac{1}{d}\sum_{i=1}^{k} n_i f_i^* d\theta$$

defines a mapping torus for each $k \in [1, N]$ and this family of mapping tori tend the torus defined by α' as $k \to N$. As Z is compact and X_H is everywhere transverse the symplectic foliation, an orbit of X_H will intersect a chosen leaf $\mathcal{L}$ of the foliation in forward and backward time. Whence, $\mathcal{L}$ is a global transverse Poincaré section for X_H. □

Theorem 2.4.6 can be used to quickly infer the existence of a global transverse section. Consider a Hamiltonian H on the cotangent bundle T^*M of a manifold M

equipped with the canonical symplectic form ω. Them, no level energy set of H possesses a global Poincare section. This is not the case for twisted symplectic forms as the following example shows:

Example 2.4.7. *Recall that a* twisted symplectic form ω_σ *on a cotangent bundle of a manifold* M *is given by* $\omega_\sigma = \omega_0 + \pi^*\sigma$ *where* ω_0 *is the canonical form on* T^*M, π *is the projection* $\pi : T^*M \to M$ *and* $\sigma \in \Omega^2(M)$ *is some closed two form on* M, *called the* magnetic form.
Let M *be a* 2*-torus* $\mathbb{T}^2$ *with coordinates* (θ, ψ). *Let* $T^*\mathbb{T}^2$ *be the cotangent bundle equipped with a twisted symplectic form, where the magnetic term is given by* $\sigma = d\theta \wedge d\phi$. *Consider the usual kinetic Hamiltonian* $H = \frac{1}{2}(p_\theta^2 + p_\phi^2)$. *Recalling that the cotangent bundle of* $\mathbb{T}^2$ *is trivial,* $T^*\mathbb{T}^2 \cong \mathbb{T}^2 \times \mathbb{R}^2$ *it is easy to see that that the level energy sets* $H^{-1}(c)$ *for* $c \neq 0$ *are foliated by leaves* $\mathcal{L} \cong \mathbb{T}^2$ *which are symplectic, with symplectic form* σ. *As the manifold is cosymplectic,* $H^{-1}(c)$ *possesses a global transverse Poincaré section.*
Indeed, one example of a global section is simply the configuration space $\mathbb{T}^2$ *embedded as the zero section. To check this explicitly, one can calculate* $X_H = p_\theta\partial_\theta - p_\psi\partial_{p_\theta} + p_\psi\partial_\psi - p_\theta\partial_{p_\theta}$ *and remark that* $X_H \wedge \partial_\theta \wedge \partial_\psi \neq 0$.

It is of value, then, to consider more general Poisson structures and their possible application to the theory of dynamical systems. In the next chapter, we go on to study the b-symplectic forms of Defintion 2.3.5.

Chapter 3

A b-Symplectic Slice Theorem

Think left and think right and think low and think high. Oh, the thinks you can think up if only you try!
Dr. Seuss, Oh the Places You'll Go!

A distinguishing feature of Hamiltonian mechanics, as contrasted with the Lagrangian or Newtonian formulations, is particular emphasis on the role of symmetries and the associated reductions of the phase space. It is of little surprise, then, that symplectic group actions are the object of intense study (and a fruitful source of results) in symplectic theory. One important facet of this theory is the *symplectic slice theorem* and its generalizations, which give a normal form for a symplectic structure in a neighbourhood of an orbit of a symplectic group action. The symplectic normal form is constructed by taking the symplectic quotient of a certain symplectic vector bundle. In the following chapter, we mimic this construction in the b-symplectic case.

b-Symplectic manifolds are particular instances of Poisson manifolds. Poisson reduction is somewhat easier to define than symplectic reduction: reducing a Poisson structure by a group symmetry always results in a Poisson structure on the quotient manifold. In contrast, in the case of symplectic reduction, one has to check the reduced form is, in fact, symplectic. However, the natural cost of a more flexible reduction theory is that the Poisson structure on the reduced space can look quite different from the Poisson structure on the cover. One example is the reduction of the canonical symplectic form (corresponding to a non-degenerate Poisson structure) on the cotangent bundle of a Lie group to the KKS form on the dual of the Lie Algebra, which is degenerate and, in the case of abelian Lie group, even trivial.

Poisson manifolds, then, have significantly more local structure than symplectic manifolds and questions on the local equivalence of Poisson structures can require sophisticated machinery to analyze. For general Poisson manifolds, normal form theorems are hard-won.

In contrast, the local flexibility of symplectic forms ensures that symplectic geometry is replete with local normal forms. Many of these are proved by applying some version of the Moser trick, which, given a manifold with symplectic forms satisfying some (not particularly stringent) conditions, determines a vector field which one can integrate to give a symplectomorphism. This has powerful consequences, such as the constant rank embedding theorem, which ensures that in a neighbourhood of a submanifold of a symplectic manifold, the symplectic form is determined by the restriction of the form to the submanifold N and its symplectic normal bundle in M.

By choosing vector fields given by the Moser trick to be invariant with respect to the group action, one can ensure that symplectomorphisms are equivariant with respect to the group symmetries leading to the symplectic slice theorem. This compares a symplectic form invariant under a group action in the neighbourhood of an orbit to a standard normal form on a symplectic twist product. The normal form for the symplectic form in is this case is constructed by standard symplectic reduction, from the canonical symplectic form on the cotangent bundle of a Lie group, the image of the momentum map (or, in the case of non-Hamiltonian symplectic actions the Chu map) which assigns to orbits of the group action Lie 2-cocycles, and a locally constant symplectic form on a so-called ”symplectic slice”. Using the constant rank embedding theorem, then, one gets the result.

As b-Poisson or b-symplectic manifolds have an analogous b-Moser theorem, a study of their symmetries puts them closer to the symplectic realm than that of more general Poisson structures. A study of their geometry in the presence of symmetries was initiated in [1] (see also [12]) which yielded global results on the structure of b-symplectic manifolds equipped with a class of toric actions preserving the b-symplectic form known as b-Hamiltonian actions. The result closely resembled the symplectic case.

Here, we focus on local results and reproduce the symplectic slice theorem for b-symplectic manifolds. As in the symplectic slice theorem, the b-symplectic slice theorem is given by defining a simple b-symplectic form on a trivial vector bundle over G and then reducing to get a b-symplectic form on a neighbourhood of the orbit. Using the Moser trick available for b-symplectic forms, one can conclude a b-symplectic slice theorem. The normal form and necessary reduction are easily achieved by noting the constricted nature of b-symplectic and cosymplectic symmetries. After proving some propositions on foliation preserving symmetries of codimension one foliations, the required normal form is given by the discrete reduction of a product b-symplectic structure.

3.1 Symmetries of Poisson manifolds

There is a rich and interesting theory of local and global symmetries of symplectic manifolds. The most important local result is the symplectic slice theorem. Consider the following group action on a manifold M.

$$\rho : G \times M \to M$$
$$\rho : (g, p) \to \rho_g(p)$$

Denote the orbit of $p \in M$ by $\mathcal{O}_p$ and the isotropy group of p by G_p. We have a natural representation G_p on the vector space $T_pM/T_p\mathcal{O}_p$ given by the action of $d\rho_h(p)$ on $T_pM/T_p\mathcal{O}_p$ for $h \in G_p$. Consider the twisting action of G_p on the product $G \times T_p(M)/T_p\mathcal{O}_p$

$$\sigma(h)(g, v) = (gh^{-1}, d\rho_h(p)(v)) \tag{3.1}$$

This action is free and therefore the quotient is a manifold (recall that the isotropy group is automatically compact). It is a principal bundle over G/G_p, denoted $G \times_{G_p} T_p(M)/T_p\mathcal{O}_p$. The slice theorem tells us that this bundle, the action on the fibre being the trivial action, is equivariantly diffeomorphic to a neighbourhood of $\mathcal{O}_p$.

Theorem 3.1.1. *Let M be a manifold equipped with a G action. The map $G/G_p \to M$, $[g] \mapsto g \cdot p$ extends to an invariant neighbourhood of G/G_p (viewed as a zero section) in $G \times_{G_p} T_pM/T_p\mathcal{O}_p$ to a neighbourhood of the orbit O_p.*

3.1.1 Symplectic Group actions

If the action of the group is symplectic then we can go further by equipping the twisted product of the classical slice theorem 3.1.1 with a symplectic form symplectomorphic to the invariant form in a neighbourhood of the orbit. This was originally formulated for Hamiltonian group actions independently by Marle [13] and Guillemin and Sternberg [14].

Construction of the slice coordinates

If W is a subspace of a symplectic vector space, let W^ω denote the symplectic orthogonal of W. Let G be a compact group acting on a manifold M, p a point in M and as before denote by $\mathcal{O}_p$ the orbit of p and by G_p the isotropy group of p. The vector space $V_p = T_p\mathcal{O}_p^\omega / T_p\mathcal{O}_p$ is symplectic, with the symplectic form induced by the form on T_pM i.e. $\omega([v],[w]) = \omega_p(v,w)$ for any $v = \pi(v)$, π the projection to the quotient. We have a natural representation G_p on the vector space V_p given by the action of $d\rho_h(p)$. Note that this action is automatically symplectic.

Consider the action of G_p on T^*G given by the cotangent lifted action of the right action of G_p on G and as before form the quotient bundle $T^*G \times_{G_p} V_p$. The quotient manifold $T^*G \times_{G_p} V_p$ is symplectic, with the obvious form. Moreover, the action of G on the bundle induced by the cotangent lifted action of G on T^*G is Hamiltonian with respect to the canonical symplectic form. In fact, according to the symplectic slice theorem, this construction gives a model of the action of a group G on M close to the orbit $\mathcal{O}_p$.

[15] for the more general symplectic case):

Theorem 3.1.2. *Let (M,ω) be a symplectic manifold and let H be a Lie group acting properly and by symplectomorphisms on M. Let $m \in M$. Denote the isotropy group of m by H_m and the orbit of m by $H \cdot m$. Let V_m be the symplectic normal space*

$$V_m := T_m(H \cdot m)^\omega / \left(T_m(H \cdot m)^\omega \cap T_m(H \cdot m)\right).$$

Let $\mathfrak{h}$ be the Lie algebra of H and define the following Lie subalgebra of $\mathfrak{h}$:

$$\mathfrak{k} := \{\eta \in \mathfrak{h} \mid \eta_M(m) \in T_m(H \cdot m)^\omega\}$$

where η_M is the generating vector field of η. Let $\mathfrak{i}$ be the Lie algebra of H_m. Then $\mathfrak{i} \subset \mathfrak{k}$. Denote by $\mathfrak{m}$ an Ad_{H_m}-invariant complement of $\mathfrak{i}$ in $\mathfrak{k}$. Then the twisted product

$$Y_m^H := H \times_{H_m} (\mathfrak{m}^* \times V_m) \tag{3.2}$$

is a symplectic H-space and can be chosen such that there is an H-invariant neighbourhood U of m in M, an H-invariant neighbourhood U' of $[e, 0, 0]$ in Y_m^H and an equivariant symplectomorphism $\phi : U \to U'$ satisfying $\phi(m) = [e, 0, 0]$. Equipping the bundle Y_m^H with coordinates $[k, \eta, v]$ for $k \in H, \eta \in \mathfrak{m}^*$ and $v \in V_m$, H acts on Y_m^H as $h \cdot [k, \eta, v] = [h \cdot k, \eta, v]$.

In the case that the action is Hamiltonian, the symplectic form on the quotient bundle is called the MGS-normal form[1] and denoted by ω_{MGS}. [13] We remark that the aim here is to show the rigidity of b-symplectic group actions, for which the group action and symplectic form are completely determined in a neighbourhood of an orbit by the isotropy group and its representation on the symplectic normal space. Therefore Theorem B does not reference the traditional moment map sometimes given as part of the symplectic slice theorem, although there does exist a generalization of the moment map to the b-symplectic case which could, in theory, be used to extend theorem further.

Definition 3.1.3. *Let G be a Lie group and let M be any smooth manifold. Given a group action $\rho : G \times M \longrightarrow M$, we define its cotangent lift as the action on T^*M given by $\hat{\rho}_g := \rho^*_{g^{-1}}$, $g \in G$. We then have a commuting diagram*

$$\begin{array}{ccc} T^*M & \xrightarrow{\hat{\rho}_g} & T^*M \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow{\rho_g} & M \end{array}$$

where π is the canonical projection from T^*M to M.

3.1.2 Poisson Group Actions

A Poisson group action (that is. a group action preserving the Poisson form) is defined in the obvious way:

[1] MGS for Marle [13] and Guillemin-Sternberg [16]

Definition 3.1.4. *Let $(M, \{\cdot, \cdot\})$ be a Poisson manifold and let G be a Lie group acting on M via the map $\Phi : G \times M \to M$. An action is called Poisson (or canonical) if for any $h \in G$ and*

$$f, g \in C^\infty(M) \text{ one has}$$
$$\{f \circ \Phi_h, g \circ \Phi_h\} = \{f, g\} \circ \Phi_h$$

If the quotient space M/G is a manifold (e.g. If the G-action is free and proper), then automatically has a quotient Poisson structure

Definition 3.1.5. *The quotient Poisson structure on the quotient manifold is the Poisson bracket $\{\cdot, \cdot\}^{M/G}$, uniquely characterised by the relation*

$$\{f, g\}^{M/G}(\pi(m)) = \{f \circ \pi, g \circ \pi\}(m)$$

b-Symplectic Group Actions

b-Symplectic actions (that is, actions which preserve the b-symplectic form) are Poisson in the sense of 3.1.4. b-symplectic actions, similar to symplectic actions, exhibit an interesting combination of flexibility and rigidity. This can be understood by way of their normal forms. In [17] b-symplectic form is discussed, called the **twisted b-symplectic form**. For the special case of a torus:

Definition 3.1.6. *Consider the cotangent bundle of the torus $T^*\mathbb{T}^n$ endowed with the standard coordinates (θ, a), $\theta \in \mathbb{T}^n$ and define the following one-form on the complement of the hypersurface $Z = \{a_1 = 0\}$ of $T^*\mathbb{T}^n$*

$$c \log |a_1| d\theta_1 + \sum_{i=2}^{n} a_i d\theta_i.$$

*This is called the **twisted Liouville one-form** (with modular period $c \in \mathbb{R}^+$).*

The negative differential of the twisted Liouville form is a b-symplectic form on $T^*\mathbb{T}^n$.

Definition 3.1.7. *The negative differential of this form extends to a b-symplectic form on $T^*\mathbb{T}^n$, which we call the* **twisted b-symplectic form** *on $T^*\mathbb{T}^n$ given in coordinates as*

$$\omega_{tw,c} := \frac{c}{a_1} d\theta_1 \wedge da_1 + \sum_{i=2}^{n} d\theta_i \wedge da_i.$$

As noted in [17], the construction can easily be generalised to the case $T^*(G)$, where G is a product group $G \cong S^1 \times H$. This will form the basis of the normal form for the b-symplectic slice theorem.

Cosymplectic Group Actions

In [18] it was observed that co-Kähler manifolds have a transverse S^1 action which renders the manifold a transversally equivariant fibration. This implies that the mapping torus of the co-Kähler structure possesses a finite cover with a product structure. Here we take the transverse S^1-action as a given. This allows us to apply a theorem of Sadowski [19] to give a trivialising cover where we can construct the slice theorem.

A bundle map $\pi : Z \to S^1$ is a *transversally equivariant fibration* if there is a smooth S^1 -action on Z such that the orbits of the action are transversal to the fibres of π and $\pi(t \cdot x) - \pi(x)$ depends on $t \in S^1$ only. The following is a specialization of a theorem by Sadowski that was applied to the case of co-Kähler manifolds in [18].

Theorem 3.1.8. *Let $Z \xrightarrow{\pi} S^1$ be a smooth bundle projection from a closed manifold Z to the circle. The following are equivalent:*

1. *$Z \xrightarrow{\pi} S^1$ is a mapping torus associated to a diffeomorphism of finite order*
2. *The bundle map π is transversally equivariant with respect to an S^1-action on Z, $\rho : S^1 \times Z \to Z$.*

Let $\mathcal{L}$ be the fibre of π. If the above conditions are satisfied then Z has a $\mathbb{Z}_k$-cover ($k \in \mathbb{N}$)

$$p : \tilde{Z} = S^1 \times \mathcal{L} \to Z$$

given by the action $(t, l) \mapsto \rho_t(l)$, where $\mathbb{Z}_k$ acts diagonally on $S^1 \times \mathcal{L}$ and by translations on S^1.

3.1.3 Construction of the Equivariant Cover

The $\mathbb{Z}_k$ action of 3.1.8 is described as follows: Consider the leaf-fixing subgroup of S^1,

$$\Gamma = \{s \in S^1 \,:\, \rho_s(\mathcal{L}) = \mathcal{L}\}. \tag{3.3}$$

Identifying $S^1 \cong \mathbb{R} \bmod 1$, the group Γ is of the form $\{0, \frac{1}{k}, \ldots, \frac{k-1}{k}\}$ for some $k \in \mathbb{N}$ and hence we can identify it with $\mathbb{Z}_k$ in the natural way. Then for $m \in \mathbb{Z}_k = \{0, 1, \ldots, k-1\}$, the action $\rho_{\frac{m}{k}}$ restricts to a leaf automorphism

$$\sigma_m : \mathcal{L} \to \mathcal{L}, \quad \sigma_m(l) = \rho_{\frac{m}{k}}(l). \tag{3.4}$$

The mapping torus Z is then the quotient of the cover $\tilde{Z}$ by the following action of $\mathbb{Z}_k$ on $\tilde{Z}$

$$\mu_m(t, l) = (t - \frac{m}{k}, \sigma_m(l)), \quad m \in \mathbb{Z}_k, (t, l) \in S^1 \times \mathcal{L}. \tag{3.5}$$

From the condition of transverse equivariance, it is clear that ρ maps leaves to leaves. It induces an action on the base S^1 given by translations $t \mapsto t + ks$ and the equivariance condition reads

$$\pi(\rho_s(l)) = ks, \quad l \in \mathcal{L} := \pi^{-1}(\{0\}).$$

There is an associated S^1-action $\tilde{\rho}$ on the cover $\tilde{Z}$ given by

$$\tilde{\rho}_s(t, l) = (t + s, l), \quad s \in S^1, (t, l) \in S^1 \times \mathcal{L}. \tag{3.6}$$

The projection $\tilde{Z} \to Z$ is equivariant with respect to this action.

The existence of a finite trivializing cover of the critical hypersurface Z will play a crucial role in the b-symplectic slice theorem.

3.2 A trivializing cover for the critical hypersurface

Now we consider (M, Z) a b-symplectic manifold. As we focus on a semi-local result, we will assume $M \cong Z \times (-\epsilon, \epsilon)$ where the critical hypersurface Z is compact and connected with b-symplectic form given by Equation (2.4). We recall that on a semi-local level the last assumption is not an additional restriction as b-symplectic manifold satisfying the previous conditions is of this form in some local coordinates on a tubular neighbourhood of its critical hypersurface. Finally, we will assume that the symplectic foliation on Z has a compact leaf $\mathcal{L}$.

Definition 3.2.1. *A group action on a b-symplectic manifold is called* transverse *if it is transverse to the symplectic foliation of the critical hypersurface. If the action, restricted to the critical hypersurface, preserves the cosymplectic structure we will call the action* cosymplectic. *Finally, if the action preserves the b-symplectic form we will call the action b-*symplectic.

When considering the manifolds with the associated Poisson structures cosymplectic and b-symplectic actions are special cases of Poisson actions.

As cosymplectic actions are automatically transversely equivariant the next proposition follows directly from Theorem 3.1.8:

Proposition 3.2.2. *Let Z be a cosymplectic manifold and suppose Z has a transverse S^1-action preserving the cosymplectic structure. Then Z has a finite cover $\tilde{Z} := S^1 \times \mathcal{L}$, $\mathcal{L}$ a leaf of the foliation, equipped with an S^1 action given by translation in the first coordinate for which the projection $p : S^1 \times \mathcal{L} \to Z$ is equivariant.*

To get a cosymplectic structure on the cover, one simply lifts the associated defining one and two-forms.

Proposition 3.2.3. *In the setting of the previous proposition, the cosymplectic structure on Z is given by the quotient of a cosymplectic structure on $\tilde{Z} = S^1 \times \mathcal{L}$ by the action of a finite cyclic group $\mathbb{Z}_k$.*

Proof. Let $p : \tilde{Z} \to Z$ be the finite cover given by Proposition 3.2.2. Denote the one and two forms of the cosymplectic structure by α and β respectively. Then $\tilde{\beta} = p^*\beta$ and $\tilde{\alpha} = p^*\alpha$ can easily be shown to define a cosymplectic structure on $S^1 \times \mathcal{L}$ and by construction, the cosymplectic structure on the quotient agrees with the cosymplectic structure on Z. □

To extend this cover to a b-symplectic neighbourhood of Z we simply use the extension theorem (Corollary 2.3.8.):

Corollary 3.2.4. *Let $M = Z \times (-\epsilon, \epsilon)$ come equipped with a transverse S^1-action preserving the b-symplectic form ω. Then the b-symplectic structure on M is b-symplectomorphic in a neighbourhood of Z to the quotient of a b-symplectic structure on $S^1 \times \mathcal{L} \times (-\epsilon, \epsilon)$ by the action of a finite cyclic group.*

Proof. As before let $p : \tilde{Z} \to Z$ be the finite cover. Let v_{mod} be some choice of modular vector field and denote the defining one and two-forms of Z fulfilling the condition in Equation (2.2.8) by α and β respectively. Denote by $\tilde{\alpha}, \tilde{\beta}$ the corresponding one and two forms defined in Proposition 3.2.3. By the extension theorem we can assume that the b-symplectic form on M is

$$\omega = \pi_Z^*\alpha \wedge \frac{da}{a} + \pi_Z^*\beta.$$

Let $\tilde{M} := \tilde{Z} \times (-\epsilon, \epsilon)$. Then we have a finite cover $p_M : \tilde{M} \to M$ for M given by the product map of the cover $p : \tilde{Z} \to Z$ and the identity on $(-\epsilon, \epsilon)$. Let $\pi_{\tilde{Z}} : \tilde{M} \to \tilde{Z}$ be the projection onto the first factor. Define for $a \in (-\epsilon, \epsilon)$ the b-symplectic form on $\tilde{M}$

$$\tilde{\omega} = \pi_{\tilde{Z}}^* \tilde{\alpha} \wedge \frac{da}{a} + \pi_{\tilde{Z}}^* \tilde{\beta}.$$

Then by construction $(p_M)^* \omega = \tilde{\omega}$. □

Remark 3.2.5. *Note that the modular period of the associated b-symplectic form on the $\mathbb{Z}_k$ cover is k times the modular period of the b-symplectic form on the base.*

Remark 3.2.6. *Similarly, any b-symplectic structure with defining one and two-forms $\tilde{\alpha}$ and $\tilde{\beta}$ equipped with a discrete b-symplectic group action gives a b-symplectic structure on the quotient. For such a group action there are well defined one and two-forms, α and β, on the base manifold defined by $p^*(\alpha) = \tilde{\alpha}$ and $p^*(\beta) = \tilde{\beta}$, where p is the projection to the quotient. Then α and β automatically fulfil the conditions to define a cosymplectic structure on the image of the critical hypersurface. As the group action is discrete, the quotient of the symplectic structure on leaves is likewise symplectic.*

3.3 The b-symplectic slice theorem for an S^1-action

First, we wish to simplify the expression of the b-symplectic form in the neighbourhood of an orbit. In the case that the leaf $\mathcal{L}$ is simply connected, the b-symplectic form has a particularly simple expression.

Proposition 3.3.1. *Let $M \cong Z \times (-\epsilon, \epsilon)$ be a b-symplectic manifold and suppose that Z is a product, $Z \cong S^1 \times \mathcal{L}$, $\mathcal{L}$ a leaf of the symplectic foliation. Suppose furthermore that $\mathcal{L}$ is simply connected. Then for a suitable defining function f of Z the b-symplectic form is given by*

$$\omega = c dt \wedge \frac{df}{f} + \pi_{\mathcal{L}}^*(\beta) \tag{3.7}$$

where t is the standard coordinate on S^1, β is the symplectic form on $\mathcal{L}$ and $\pi_{\mathcal{L}}$ is the projection $S^1 \times \mathcal{L} \to \mathcal{L}$.

Proof. A b-symplectic form on on $S^1 \times \mathcal{L} \times (-\epsilon, \epsilon)$ equipped with coordinates (t, l, a) can be written

$$\omega = c dt \wedge \frac{da}{a} + dt \wedge \eta + \pi_{\mathcal{L}}^*(\beta)$$

where β is the symplectic form on $\mathcal{L}$. Since $\mathcal{L}$ is simply connected, $\eta = dh$ for some $h \in C^\infty(M)$. The function $f = ae^h$ is then a defining function for Z and moreover

$$\frac{df}{f} = \frac{da}{a} + dh$$

Whence we have

$$\omega = cdt \wedge \frac{df}{f} + \pi_{\mathcal{L}}^*(\beta).$$

□

As in the symplectic slice theorem, the normal form of a b-symplectic form in the neighbourhood of an orbit is given by virtue of an equivariant Moser theorem. Equivariant b-Moser theorems for isotopic forms invariant under S^1-actions have been given in [20] and for more general groups in [21]. As we wish to compare b-symplectic forms in the neighbourhood of an orbit rather than on the whole of Z we require an equivariant b-Moser theorem of a slightly different nature:

Proposition 3.3.2. *Suppose that ω_1 and ω_0 are b-symplectic forms on M, invariant under an action of a group G on M which is transverse Poisson for ω_1 and ω_0. Denote by $\mathcal{O}_z$ the orbit of some $z \in Z$ and suppose that ω_1 and ω_0 coincide at z. Then ω_1 and ω_0 are equivariantly b-symplectomorphic in some neighbourhood $\mathcal{U}$ of $\mathcal{O}_z$.*

Proof. As the defining one and two-forms associated to ω_1 and ω_0 are invariant under the S^1 action, it follows that on $\mathcal{O}_z$ we have $\alpha_0 = \alpha_1$ and $\beta_0 = \beta_1$. By the relative Poincaré lemma, in a contractible neighbourhood of $\mathcal{O}_z$ we have that $\alpha_0 - \alpha_1 = dg$, an exact one-form on $\mathcal{U}$ and similarly $\beta_0 - \beta_1 = d\eta$, an exact two-form on $\mathcal{U}$. Whence $\omega_0 - \omega_1 = d(-g\frac{df}{f} + \eta)$. Then $\omega_t = \omega_0 + (1-t)\omega_1$ is non degenerate on $\mathcal{O}_z$ and so on a neighbourhood of $\mathcal{O}_z$. We use this to define a b-vector field v_t by $\iota_{v_t}\omega_t = g\frac{df}{f} - \eta$. As v_t is zero on $\mathcal{O}_z$, the time-one flow exists in a neighbourhood of $\mathcal{O}_z$ and gives the required b-symplectomorphism. As both b-symplectic forms are invariant under the group action, we can choose the b-symplectomorphism to be equivariant. □

Theorem 3.3.3. *Let $M \cong Z \times (-\epsilon, \epsilon)$ be a b-symplectic manifold equipped with a b-symplectic form ω of modular period c and a transverse b-symplectic S^1-action. Let $z \in \mathcal{L} \subset Z$, let $\mathcal{O}_z$ be its orbit under the S^1 action, let $V := T_z\mathcal{L}$ and let $\mathbb{Z}_l$ be the isotropy group of z. Then there exists an S^1-equivariant neighbourhood $\mathcal{V}$ of $\mathcal{O}_z$ in M and an S^1-equivariant mapping*

$$\phi : \mathcal{V} \to (T^*S^1 \times V)/\mathbb{Z}_l \tag{3.8}$$

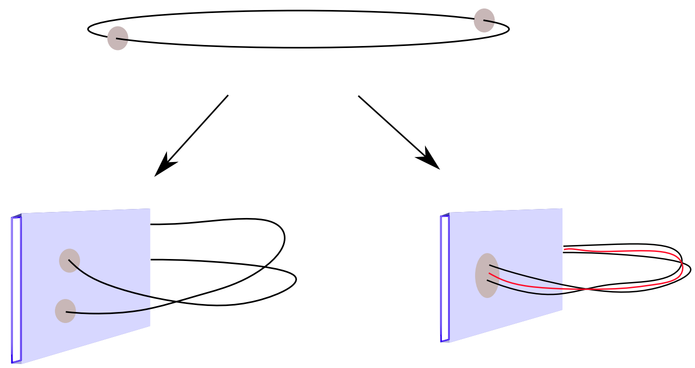


Figure 3.1: A scheme of the trivialising finite cover with a regular orbit ($\Gamma_{\tilde{z}} = 0$) in black and exceptional orbit ($\Gamma_{\tilde{z}} = \mathbb{Z}_2$) in red

where $\mathcal{O}_z$ is embedded as the zero section of the bundle $S^1 \times_{\mathbb{Z}_l} \mathbb{R} \times V \cong (T^*S^1 \times V)/\mathbb{Z}_l$ and where the action of $\mathbb{Z}_l$ is given by the cotangent lifted action on T^*S^1 and by the isotropy representation on V.

Moreover, if we equip the bundle $T^*S^1 \times V$ with the b-symplectic form:

$$\tilde{\omega}_0 = \omega_{c'} + \omega_V$$

where $\omega_{c'}$ the b-symplectic normal form on T^*S^1 as given in Definition 3.1.7 with modular period $c' = kc$ and ω_V the linear symplectic form on V, and the quotient $(T^*S^1 \times V)/\mathbb{Z}_l$ with the quotient b-symplectic form ω_0 (see Remark 3.2.6) then the mapping becomes an equivariant b-symplectomorphism onto its image.

Proof. Let $z \in Z$ be a point in the critical set and $\mathcal{O}_z$ the orbit of z under the S^1 action ρ. Denote by Γ_z the isotropy group of z. Note that Γ_z is automatically a subgroup of $\mathbb{Z}_k$ and so $\Gamma_z \cong \mathbb{Z}_l$ for some l. By the slice theorem there exists a neighbourhood $\mathcal{U}$ of $\mathcal{O}_z$ in Z equivariantly diffeomorphic to a neighbourhood of the zero section of the vector bundle $S^1 \times_{\Gamma_z} T_zZ/T_z\mathcal{O}_z$, where S^1 acts on the homogeneous space $S^1 \times_{\Gamma_z} T_zZ/T_z\mathcal{O}_z$ according to $s \cdot [t, v] = [t+s, v]$. By choosing the invariant Riemannian metric in such a way that $T_z\mathcal{L}$ is orthogonal to $T_z\mathcal{O}_z$, the slice theorem yields an equivariant diffeomorphism

$$S^1 \times_{\Gamma_z} T_z\mathcal{L} \to \mathcal{U} : [t, v] \mapsto \rho_t(\exp_z v).$$

Denote by ψ the corresponding diffeomorphism on the neighbourhood $(-\epsilon, \epsilon) \times \mathcal{U}$ of

$\mathcal{O}_z$ in M:

$$\psi : (-\epsilon, \epsilon) \times \mathcal{U} \to (-\epsilon, \epsilon) \times S^1 \times_{\Gamma_z} T_z\mathcal{L}.$$

Restricting the defining one and two forms of ω to $\mathcal{U}$, we have that $\mathcal{U}$ is a cosymplectic manifold with a cosymplectic S^1-action. The symplectic leaves of $\mathcal{U}$ are given by $\mathcal{L}_{\mathcal{U}} := \mathcal{U} \cap \mathcal{L}$ and the leaf fixing subgroup as defined by Equation (3.3) is Γ_z. By Proposition 3.2.2 there is a trivial Γ_z-cover $\tilde{\mathcal{U}} \cong S^1 \times \mathcal{L}_{\mathcal{U}}$ of $\mathcal{U}$. Then $\omega|_{(-\epsilon,\epsilon)\times\mathcal{U}}$ is the quotient of a unique b-symplectic form $\tilde{\omega}$ on $(-\epsilon, \epsilon) \times \tilde{\mathcal{U}}$ as given by Corollary 3.2.4. By Proposition 3.3.1 we may assume $\tilde{\omega}$ is of the form

$$\tilde{\omega} = ckdt \wedge \frac{da}{a} + \pi^*_{\mathcal{L}_{\mathcal{U}}}\beta$$

where $a \in (-\epsilon, \epsilon)$ and β is a symplectic two-form given on a leaf $\mathcal{U}_{\mathcal{L}}$. Consider the two form β_z on $T_z\mathcal{L}$. On $(-\epsilon, \epsilon) \times S^1 \times T_z\mathcal{L}$ define the b-symplectic form

$$\tilde{\omega}_0 = ckdt \wedge \frac{da}{a} + \beta_z.$$

By a linear change of basis we may assume $\beta_z = \sum_{i=1}^n dx_i \wedge dy_i$. Denote the quotient b-symplectic form on $((-\epsilon, \epsilon)\times S^1\times T_z\mathcal{L}))/\Gamma_z$ given in Remark 3.2.4 by ω_0. Finally consider the b-symplectic form $\psi^*(\omega_0)$ on $(-\epsilon, \epsilon) \times \mathcal{U}$. This is a b-symplectic structure, invariant under the S^1 action agreeing with ω at z. By Theorem 3.3.2, there is an equivariant b-symplectomorphism φ with $\varphi^*(\psi^*\omega_0) = \omega$. Whence we have $\phi = \psi \circ \varphi$ the required b-symplectomorphism given in the statement of theorem. □

Remark 3.3.4. *Note that the modular period of the form ω_0 is $\frac{k}{l}c$ where c is the modular period of the b-symplectic form. This is not necessarily the modular period of the original form ω.*

Example 3.3.5. *Consider the following symplectic mapping torus: take as a symplectic leaf a torus $\mathbb{T}^2$ with coordinates (φ, ψ), $\varphi, \psi \in \mathbb{R} \bmod 1$ equipped with the standard symplectic form and the holonomy map given by the diffeomorphism of $\mathbb{T}^2$ which descends from the diffeomorphism of $\mathbb{R}^2$ given by $\phi \in \mathrm{GL}(2, \mathbb{Z})$:*

$$\phi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The mapping on $\mathbb{R}^2$ corresponds to rotation by $\frac{\pi}{2}$ and so we have $\phi^4 = Id$. Denote the mapping torus $Z = ([0, 1] \times \mathbb{T}^2)/(0, x) \sim (1, \phi(x))$.

Consider the following b-symplectic form on $(t, \varphi, \psi, s) \in Z \times S^1$:

$$\omega = dt \wedge \frac{ds}{\sin(s)} + \beta$$

where β is the standard symplectic form on $\mathbb{T}^2$. Consider the action of S^1 on $Z \times S^1$ given by translation in the t-coordinate. Then there is a neighbourhood of a regular orbit contained in Z which is equivariantly diffeomorphic to a neighbourhood of the zero section $(t, \mathbf{0})$ of $S^1 \times \mathbb{R}^3$ where S^1 acts by translations on the S^1 factor of $S^1 \times \mathbb{R}^3$. Moreover, there exist coordinates (t, x, y, a) on $S^1 \times \mathbb{R}^3$ so that the equivariant diffeomorphism becomes a symplectomorphism where $S^1 \times \mathbb{R}^3$ is equipped with the b-symplectic form

$$\omega = 4dt \wedge \frac{da}{a} + dx \wedge dy \tag{3.9}$$

On the critical set there is also the exceptional orbit at $\phi = \psi = 0$. In a neighbourhood of the singular orbit the b-sympletic form is the quotient of the b-symplectic structure (3.9) *given above where the group action $\sigma_n \in GL(2, \mathbb{Z})$ on the vector space (x, y) is given by*

$$\sigma_n = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^n.$$

Example 3.3.6. *We can find examples from integrable systems having a naturally associated S^1-action model with non-trivial isotropy group.*

Take $M = T^*S^1 \times \mathbb{R}^2$ endowed with coordinates (p, t, x, y) and b-symplectic form $\omega = \frac{1}{p} dp \wedge dt + dx \wedge dy$. Consider the b-integrable system on M given by $F = (\log(p), xy)$. This b-integrable system has hyperbolic singularities. Now let $\mathbb{Z}/2\mathbb{Z}$ act on M in the following way: $(-1) \cdot (p, t, x, y) = (p, t, -x', -y')$ observe that this action leaves the hyperbola $xy = cnst$ invariant and switches its branches. The action clearly preserves the b-integrable system and induces a new integrable system on the quotient space $M/\sim$. Observe that the first component of the integrable system naturally induces an S^1-action given by the b-symplectic vector field associated to the singular Hamiltonian function $\log(p)$ (named as b-function, see [20] for a discussion). This circle action also descends to the quotient and the model for the circle action has non-trivial isotropy group of order two.

This twisted hyperbolic case in b-symplectic manifolds is reminiscent of the twisted hyperbolic construction in the symplectic case in [22] and [23][2] *and it is an invitation to study the*

[2] This example shows up in physical examples and corresponds to the 1:2 resonance (see for instance the example in page 32 of the monograph [24])

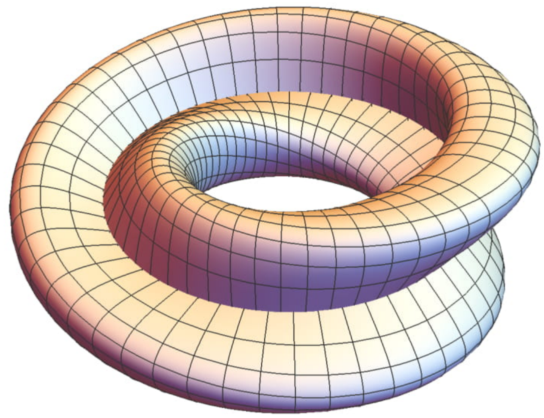


Figure 3.2: The Curled Torus. Source: Konstantinos Efstathiou.

invariants of a non-degenerate singularity of a b-symplectic manifold. This example can be extended to higher dimensions and the action of a $\mathbb{Z}/2\mathbb{Z}$ can be considered for every hyperbolic block added as long as the corank of the singularity is equal or bigger than one. The situation can be visualized using the curled torus*, the picture below showing the structure of the set $p = 0, xy = 0$.*

3.4 Actions of compact Lie Groups on cosymplectic manifolds

We treat the case of more general group actions on a b-symplectic manifold close to the critical set. First, we prove that only groups of a particular form can act on a b-symplectic manifold. For now, we will treat group actions on a mapping torus Z and then extend the results to a neighbourhood of the critical set.

In the following, we assume that the group G is compact and connected and acts on a mapping torus Z via a transverse, effective and foliation preserving action ρ. For a more general treatment of the lifting of group actions see [25].

Proposition 3.4.1. *Suppose an element $h \in G$ fixes a leaf of the mapping torus, $\rho_g(\mathcal{L}_0) = \mathcal{L}_0$. Then g fixes every leaf of the mapping torus.*

This is an easy consequence of the fact that compact connected subgroups of $\text{Diffeo}^+(S^1)$ are conjugate to $\text{SO}(2)$, which is itself a consequence of $\text{Diffeo}^+(S^1)$ having a unique maximal compact subgroup, see [26] for the case of orientation preserving homeomorphisms which can be adapted mutatis mutandis for the smooth case.

Proof. Let $\pi : Z \to S^1$ be the mapping torus projection. The action of the group G on a symplectic mapping torus Z induces an action of G on the base S^1 in the obvious fashion

$$\tau : G \times S^1 \to S^1 \tag{3.10}$$

$$(g, \pi(x)) \mapsto \pi(\rho_g(x)) =: \tau_g(x) \tag{3.11}$$

As G is compact and connected its image $\tau(G, \cdot)$ is a compact subgroup of $\text{Diffeo}^+(S^1)$, the group of orientation preserving diffeomorphisms of the circle. Whence $\tau(G, \cdot)$ is conjugate by some $w \in \text{Diffeo}^+(S^1)$ to $\text{SO}(2)$. Suppose $h \in G$ fixes a leaf $\mathcal{L}_0$. This corresponds to a fixed point of the induced action τ_h on S^1, and so a fixed point for $w\tau_h w^{-1} \in \text{SO}(2)$. Whence $w\tau_h w^{-1} = \text{Id}_{S^1}$ and so $\tau_h = \text{Id}_{S^1}$. This corresponds to h fixing all leaves of Z. □

It can be checked easily that this defines a subgroup of G. We call

$$H = \{h \in G \,|\, \rho_h(\mathcal{L}_0) = \mathcal{L}_0\}$$

the *leaf preserving* subgroup of G.

Proposition 3.4.2. *Let G be a group acting in a transverse and foliation preserving manner on a symplectic mapping torus. Let H be the leaf preserving subgroup of G. Then*

1. *H is a normal subgroup of G.*
2. *H is a closed Lie subgroup of G.*
3. *The codimension of H in G is one.*

Proof. 1. This follows immediately from the fact that for $h \in H$, $g \in G$ we have $\tau_{ghg^{-1}} = \tau_g \tau_h \tau_g^{-1} = \tau_g \tau_g^{-1} = \text{Id}_{S^1}$, hence $ghg^{-1} \in H$.

2. Consider the projection

$$\Phi : G \to \mathrm{SO}(2)$$
$$\Phi(g) = w\tau_g w^{-1}$$

corresponding to the map from G to SO(2) given in Proposition 3.4.1. It is clear that the level set $\Phi^{-1}(\mathrm{Id})$ consists precisely of the elements of G which are leaf preserving. Hence $\Phi^{-1}(\mathrm{Id}) = H$ is a closed subgroup of G.

3. The codimension of H is at most one since it is given as the level set $\Phi^{-1}(\mathrm{Id}) = H$. As G induces an action transverse to the foliation of Z it follows that the codimension of H is exactly one.

□

Proposition 3.4.3. *The action of G on the mapping torus Z lifts to an action of a product group $\tilde{G} = S^1 \times H_0$ on a finite trivializing cover of Z. Moreover, G is necessarily of the form $G = (S^1 \times H_0)/\Gamma$ for a finite cyclic subgroup Γ (which might be trivial).*

Proof. Let $\mathfrak{h} \subset \mathfrak{g}$ be the Lie algebra of H, the leaf preserving subgroup of G and consider a complementary ideal $\mathfrak{k}$ of $\mathfrak{h}$ in $\mathfrak{g}$ such that the subgroup $K = \exp(\mathfrak{k})$ is closed. K is then a one dimensional closed subgroup of the compact group G and so $K \cong S^1$. The action of K is transverse to the foliation and so by Proposition 3.2.2 there exists a finite trivializing cover $\tilde{Z} \cong S^1 \times \mathcal{L}$ of Z, such that Z is the quotient of $\tilde{Z}$ by the action of the leaf fixing subgroup $\Gamma \cong \mathbb{Z}_k$ of K on $\tilde{Z}$ where Γ acts as

$$\mu_m(t, l) = (t - \frac{m}{k}, \sigma_m(l)), \quad m \in \Gamma, (t, l) \in S^1 \times \mathcal{L}$$

and σ is the leaf automorphism induced by the leaf-fixing elements of K on $\mathcal{L}$. Denote $\exp(\mathfrak{h}) \subset G$ by H_0. Denote by $\tilde{G}$ the group $K \times H_0$. Then we have an action $\tilde{\rho}$ of $\tilde{G}$ on $\tilde{Z}$ given by

$$\tilde{\rho} : \tilde{G} \times \tilde{Z} \to \tilde{Z}, \quad \tilde{\rho}_{(s,h)}(t, l) = (t + s, \rho_h(l)).$$

Suppose $\sigma_m = \rho_h$ for some $h \in H_0$, $m \in \mathbb{Z}_k \backslash \{0\}$. As H_0 is connected, $\sigma_1 = \sigma_{h'}$ for some $h' \in H_0$. Whence, the action μ of Γ on $\tilde{Z}$ is equivalent to the action $\tilde{\rho}$ of $\Gamma' \subset \tilde{G}$ on $\tilde{Z}$ where Γ' is the group

$$\Gamma' = \left\{ \left(-\frac{m}{k}, (h')^m \right) \middle| m = 0, \ldots, k-1 \right\}.$$

Letting $p_{\tilde{Z}}$ and $p_{\tilde{G}}$ denote the projections to $\tilde{Z}/\Gamma'$ and $\tilde{G}/\Gamma'$ respectively, we have a commutative diagram

$$\begin{array}{ccc} \tilde{G} \times \tilde{Z} & \xrightarrow{\tilde{\rho}} & \tilde{Z} \\ \Big\downarrow{\scriptstyle p_{\tilde{G}} \times p_{\tilde{Z}}} & & \Big\downarrow{\scriptstyle p_{\tilde{Z}}} \\ \tilde{G}/\Gamma' \times Z & \xrightarrow{\rho} & Z \end{array}$$

By construction, the action of $\tilde{G}/\Gamma'$ on Z and the action of G on Z possess the same fundamental vector fields. Moreover, the action of both groups is effective. Necessarily, then, $\tilde{G}/\Gamma' = G$.

Conversely, assume that $\sigma_1 \neq \rho_h$ any h in H_0. Then $\exp(\mathfrak{k}) \cap \exp(\mathfrak{h}) = 0$ and so $G \cong K \times H_0$. We can lift the action of this product group to an action of G on the finite trivializing cover $S^1 \times \mathcal{L}$ in the obvious way. □

Proposition 3.4.4. *Let $G = S^1 \times H$ be a product group acting on a mapping torus Z such that the S^1 factor acts transverse to the foliation. Let $z \in Z$ and denote by G_z the isotropy group of z. Then $G_z \cong \mathbb{Z}_l \times H_z$ where H_z the isotropy group of z under the H_0-action and $\mathbb{Z}_l$ is a cyclic subgroup.*

Proof. Let $\mathcal{L}_0$ be a leaf of Z and as before denote by H_0 the subgroup $(0, \exp(\mathfrak{h})) \subset G$, Denote by $\mathcal{O}_z^{H_0} \subset \mathcal{L}_0$ the orbit of z under the action of $(0, H_0) \subset G$. Denote the subgroup $(S^1, e_{H_0}) \subset G$ by K. Let ρ^K be the action of the K on Z. Let $\mathbb{Z}_k$ be the leaf preserving subgroup of K and $\frac{m}{k}$ an element $\frac{m}{k} \in \mathbb{Z}_k$. Note that $\rho^K(\frac{m}{k}, \mathcal{O}_z^{H_0}) \cap \mathcal{O}_z^{H_0} = \emptyset$ or $\rho^K(\frac{m}{k}, \mathcal{O}_z^{H_0}) \cap \mathcal{O}_z^{H_0} = \mathcal{O}_z^{H_0}$. Moreover, elements $\frac{m}{k} \in \mathbb{Z}_k$ satisfying $\rho^K(\frac{m}{k}, \mathcal{O}_z^{H}) \cap \mathcal{O}_z^{H_0} = \mathcal{O}_z^{H_0}$ form a subgroup $\mathbb{Z}_l$ of $\mathbb{Z}_k$.

If $\rho^K(\frac{m}{k}, z) \notin \mathcal{O}_z^{H_0}$ for all $\frac{m}{k} \in \mathbb{Z}_k$ then $\mathbb{Z}_l = \{0\}$ and $G_z = \{0\} \times H_z$ where H_z is the isotropy group of z under the action of $(0, H_0)$. Alternatively suppose $\mathbb{Z}_l \neq \{0\}$, so that $\rho^K(\frac{1}{l}, z) = h \cdot z$ for some $h \in H_0$. If $h \neq e_{H_0}$, we can find a new $K' \subset G$ which acts as the identity on z as follows: let η in $\mathfrak{k}$ be such that $K = \exp(t\eta)$ where $t \in [0, 1)$. Let $\nu \in \mathfrak{h}$ be such that $\exp(\frac{1}{l}\nu) = h^{-1}$. Consider the subgroup

$$K' = \{\exp(t(\eta + \nu)) | \, t \in [0, 1)\}$$

then the isotropy group of z is of the form $\mathbb{Z}_l \times H_z$ where $\mathbb{Z}_l = \{\exp\left(\frac{n}{l}(\eta + \nu)\right) | \, n = 0, \ldots, k-1\}$. □

3.5 A b-symplectic slice theorem

Let $(M \cong (-\epsilon, \epsilon) \times Z, \omega)$ be a b-symplectic manifold together with an effective b-symplectic action by a compact connected Lie group G acting transversely to the symplectic leaves inside the critical hypersurface Z. Moreover, assume that the action of the leaf-preserving subgroup H is Hamiltonian on each leaf.

First we will construct the b-symplectic models which will give us a normal form for the b-symplectic form about an orbit of G. By Proposition 3.4.3 there are two distinct cases

1. G is a group isomorphic to the product of Lie groups $G = S^1 \times H_0$.
2. $G = (S^1 \times H_0)/\Gamma$ where Γ is a non-trivial cyclic subgroup of $\tilde{G} = S^1 \times H_0$.

As there is a transverse S^1-action on the critical hypersurface Z, we have the existence of a trivial finite cover $\tilde{Z} = S^1 \times \mathcal{L}$ of Z equipped with a $\tilde{G} \cong S^1 \times H_0$-action which projects to the action of G on Z.

Let $z \in \mathcal{L}_0$ be a point in a symplectic leaf of Z and consider the orbit $\mathcal{O}_z^{H_0}$ of z given by the group action of $H_0 = \exp \mathfrak{h}$ on $\mathcal{L}_0$ and the symplectic form induced on $\mathcal{L}_0$. Denote the isotropy group of z by H_z. By the symplectic slice theorem (Theorem 3.1.2), there is an H_0-equivariant neighbourhood $\tilde{U}_{H_0}$ of $\mathcal{O}_z^{H_0}$ which is equivariantly symplectomorphic to a neighbourhood of the zero section of the vector bundle $(H_0 \times \mathfrak{m}^* \times V_z)/H_z$ with symplectic form ω_{MGS} as given by Theorem 3.1.2. Recall that $\mathfrak{m}$ is a certain Lie subalgebra of $\mathfrak{h}$, the vector space $V_z \subset T_z\mathcal{L}$ is the symplectic orthogonal $V_z = (T_m\mathcal{O}_z^{H_0})^{\omega}/T_m\mathcal{O}_z^{H_0}$ and H_z acts on V_z by the isotropy representation.

Definition 3.5.1 (b-Symplectic models)**.** *Consider the b-symplectic form on* $\tilde{E} = T^*S^1 \times (H_0 \times_{H_z} \mathfrak{m}^* \times V_z)$ *given by*

$$\tilde{\omega_0} = \omega_{ck} + \omega_{MGS} \tag{3.12}$$

where ω_{ck} *is the b-symplectic form on* T^*S^1 *given by Definition 3.1.7 of modular period* ck*, and* ω_{MGS} *is the symplectic form on* $H_0 \times_{H_z} \mathfrak{m}^* \times V_z$ *given by the symplectic slice theorem (Theorem 3.1.2). Consider the quotient b-symplectic structure on* $E = \tilde{E}/\mathbb{Z}_l$ *where* $\frac{m}{l} \in \mathbb{Z}_l$ *acts on* T^*s^1 *as the cotangent lift of* $\mathbb{Z}_l$ *acting by translations on* S^1 *and acts on the factor* $H_0 \times_{H_z} \mathfrak{m}^* \times V_z$ *equipped with the coordinates* $[k, \eta, v]$ *of Theorem 3.1.2 either by*

1. $\frac{m}{l} \cdot [k, \eta, v] = [k, \eta, \sigma^m(v)]$ *for a linear symplectomorphism* σ.

2. $\frac{m}{l} \cdot [k, \eta, v] = [h^m \cdot k, \eta, v]$ *where* h *is some element of* H_0.

Then E *has a unique* b*-symplectic structure such that the projection is a local* b*-symplectomorphism (see Remark 3.2.6). We call these normal forms* b***-symplectic models with symplectic slice*** V_z ***and modular period*** $c\frac{k}{l}$.

Let $\mathcal{O}_z$ be an orbit by the action of G contained in the critical set of (M, ω). We will now prove that there is a b-symplectomorphism from a neighbourhood of $\mathcal{O}_z$ to a neighbourhood of the zero section of the model above with symplectic slice V_z to, completing the proof of Theorem B, which we recall here in a succinct way:

Theorem 3.5.2. *Let* G *be a compact Lie group acting on a* b*-symplectic manifold* (M, ω) *transverse to the symplectic foliation. Suppose that the action of* G *is* b*-symplectic, effective and Hamiltonian when restricted to the symplectic leaves of* Z*. Let* $\mathcal{O}_z$ *be an orbit of the group action contained in the critical set of* M*. Then there is a neighbourhood* $\mathcal{V}$ *of* $\mathcal{O}_z$ *in* M *which is* b*-symplectomorphic to a neighbourhood of the zero section of a bundle given by the* b*-symplectic model* E *in Definition 3.5.1.*

Proof. By Proposition 3.4.3 there exists a finite cover $\tilde{Z} \cong S^1 \times \mathcal{L}$ which comes equipped with the action of a product group $\tilde{G} \cong S^1 \times H_0$ which covers the action of G on Z. Let $z \in Z$ and $\tilde{z} \in \tilde{Z}$ be a point projecting to z. Denote by $\mathcal{O}_{\tilde{z}}^{H_0}$ the orbit of $\tilde{z}$ under the action of the subgroup H_0 and by $\mathcal{O}_{\tilde{z}}^{\tilde{G}}$ the orbit of $\tilde{z}$ under the action of $\tilde{G} \cong S^1 \times H_0$ on the cover $\tilde{Z}$. Then an invariant open neighbourhood of $\mathcal{O}_{\tilde{z}}^{\tilde{G}}$ is of the form $\tilde{\mathcal{V}} = S^1 \times (-\epsilon, \epsilon) \times \mathcal{U}$ where $\mathcal{U}$ is an invariant open neighbourhood of $\mathcal{O}_{\tilde{z}}^{H_0}$. Recall that Z is the quotient of $\tilde{Z}$ by a cyclic subgroup Γ of $\tilde{G}$. By Proposition 3.4.3, we may assume that Γ is of the form

$$\Gamma = \left\{ \left(-\frac{m}{k}, h^m \right) | m = 0, \ldots, k-1 \right\} \tag{3.13}$$

We will distinguish the case $h = e_H$ and G is of the form $S^1 \times H_0$. We will treat the case $G \cong S^1 \times H_0$ first.

Let $\tilde{\omega}$ be the lift of ω to $\tilde{Z}$ as given by Proposition 3.2.2. By theorem 3.3.1 we may assume that $\tilde{\omega}(\tilde{z})$ is of the form

$$\tilde{\omega}(\tilde{z}) = ckdt \wedge \frac{da}{a} + \beta_{\tilde{z}}$$

where β is a symplectic form on the leaf $\mathcal{L}$. Denote by $H_{\tilde{z}}$ the isotropy group of $\tilde{z}$ under the action of H_0. By the symplectic slice theorem, Theorem 3.1.2, a neighbourhood $\mathcal{U}$ of $\mathcal{O}_{\tilde{z}}^{H_0}$ is equivariantly symplectomorphic to a neighbourhood of the zero section of the bundle $Y_{H_0} = H_0 \times_{H_z} \mathfrak{m}^* \times V_z$ on which the symplectic form on the leaf β is given by the MGS normal form. Consider the vector bundle $\tilde{E} = T^*S^1 \times (H_0 \times_{H_z} \mathrm{m}^* \times V_z)$ with symplectic form given by $\tilde{\omega}_0$ in (3.12), where c is the modular period of ω and k is the order of Γ. Let $\tilde{\psi}$ be the equivariant diffeomorphism given by the slice theorem. Then $\tilde{\psi}^*(\tilde{\omega}_0)$ is a b-symplectic form on a neighbourhood of $\mathcal{O}_{\tilde{z}}^{\tilde{G}}$ and therefore by the equivariant relative Moser Theorem 3.3.2, as $\tilde{\psi}^*\tilde{\omega}_0(\tilde{z}) = \tilde{\omega}(\tilde{z})$, we can conclude there is an equivariant b-symplectomorphism from a neighbourhood of $\mathcal{O}_{\tilde{z}}^{\tilde{G}}$ to a neighbourhood of the zero section of the bundle $\tilde{E} = S^1 \times \mathbb{R} \times Y_H$ equipped with the b-symplectic form $\tilde{\omega}_0$.

Denote the action of $m \in \Gamma$ by $\tilde{\rho}_m$. If $\tilde{\rho}_m(\mathcal{O}_z^{H_0}) \cap \mathcal{O}_z^{H_0} = 0$ then the action is free and, shrinking the neighbourhood if necessary, the projection $p : \tilde{Z} \to Z$ restricts to an equivariant symplectomorphism from a neighbourhood $\tilde{\mathcal{V}}$ of $\mathcal{O}_{\tilde{z}}^{\tilde{G}}$ to a neighbourhood $\mathcal{V}$ of $\mathcal{O}_z^G$.

Otherwise, $\rho_m(\mathcal{O}_z^H) \cap \mathcal{O}_z^H = \mathcal{O}_z^H$ for $m \in \mathbb{Z}_l$ some cyclic subgroup Γ_z of Γ. $\Gamma_z \subset \Gamma$ is then of the form

$$\Gamma_z = \left\{ \left(l, (h')^l\right) | m = 0, l-1 \right\}$$

for some $h' \in H_0$. Denote by $p_{\tilde{\mathcal{V}}}$, $p_{\tilde{E}}$ the projection to the quotients of $\tilde{\mathcal{V}}$ and $\tilde{E}$ by the action of Γ respectively. Define ψ by the condition that the following diagram commutes:

$$\begin{array}{ccc} \tilde{\mathcal{V}} & \xrightarrow{\tilde{\psi}} & \tilde{E} \\ {\scriptstyle p_{\tilde{\mathcal{V}}}}\downarrow & & \downarrow{\scriptstyle p_{\tilde{E}}} \\ \tilde{\mathcal{V}}/\Gamma & \xrightarrow{\psi} & E \end{array}$$

First, consider the case where $G \cong S^1 \times H_0$. We may assume that by Proposition 3.4.4 that the action of Γ_z on the orbit $\mathcal{O}_z^{H_0}$ and so on the base of the bundle Y_{H_0} is trivial. Moreover, it preserves the slice V_z and acts by linear symplectomorphisms and so ψ is b-symplectomorphism to Model (1) of Definition 3.5.1.

For $h \neq e_H$ in the group Γ from Equation 3.13 (that is, the case $G \cong (S^1 \times H_0)/\Gamma$ for Γ non trivial), we can assume by the symplectic slice theorem, Theorem 3.1.2, that the action of Γ_z on Y_{H_0} that given by Model (2) of Definition 3.5.1. $\square$

Example 3.5.3. *Let G be a compact Lie group with non-trivial centre. Let $\xi_1 \in Z(\mathfrak{g})$ be a central element of the Lie algebra and $\xi_2, \ldots, \xi_n \in \mathfrak{g}$ be such that $\xi_1, \ldots, \xi_n$ form a basis of the Lie algebra. Denote by η_i the basis of the Lie algebra dual such that $(\eta_i, \xi_j) = \delta_{ij}$. Denote the associated invariant vector fields $L_{g*}\xi_i$ by v_i and $L_g^*\eta_i$ by m_i respectively. At each point $g \in G$ these give a basis for the tangent and cotangent spaces at g. Consider the singular 2 -form on T^*G*

$$\omega = \pi^* m_1 \wedge \frac{d\left(\lambda\left(v_1\right)\right)}{\lambda\left(v_1\right)} + \sum_{i=2}^{n} \pi^* m_i \wedge d\left(\lambda\left(v_i\right)\right) \tag{3.14}$$

where π the canonical projection $T^*G \to G$. It can be checked directly that ω is a b-symplectic form on T^*G invariant under the cotangent lifted action of G on T^*G. By Proposition 3.4.3 G has a finite cover of the form $S^1 \times H$. The b-symplectic model for the action of G on T^*G is given by $\tilde{E} = T^*S^1 \times T^*H/\mathbb{Z}_k$ where $\mathbb{Z}_k$ acts diagonally on $T^*S^1 \times T^*H$ by the cotangent lift of translations on S^1.

$$\omega = \omega_{tw,c} + \omega_H \tag{3.15}$$

where

- *$\omega_{tw,c}$ is the twisted b-symplectic form of modular period c on the manifold T^*S^1, as given in Definition 3.1.7.*
- *ω_H is the canonical form on T^*H.*

Example 3.5.4. *Consider the symplectic mapping torus*

$$Z = \frac{\mathcal{L} \times [0,1]}{(l,0) \sim (\phi(l),1)} \tag{3.16}$$

where

- *$\mathcal{L} \cong S^2 \times S^2$, where S^2 is the two sphere equipped with the standard symplectic form and $\mathcal{L}$ is equipped with the product symplectic form.*
- *$\phi : \mathcal{L} \to \mathcal{L}$ is the diffeomorphism of $\mathcal{L}$ given by exchanging the S^2 factors of $\mathcal{L}$, i.e., $\phi(x,y) = (y,x)$.*

Consider the group $S^1 \times SO(3)$ where $SO(3)$ acts diagonally on the product $\mathcal{L} \cong S^2 \times S^2$, by rotation on each factor. Suppose $x \neq \pm y$. Then the action of $S_1 \times SO(3)$ on the orbit $\mathcal{O}_z$ is free. Denoting the orbit of z under the action of $SO(3)$ the leaf $\mathcal{L}$ by $\mathcal{O}_z^{\mathcal{L}}$ There is a neighbourhood $\mathcal{U}$ of $\mathcal{O}_z^{\mathcal{L}}$ and a neighbourhood V of $\mathcal{O}_z$ equivariantly b-symplectomorphic to the bundle $E = T^*\mathrm{S}^1 \times Y_z^{SO(3)}$, where E is equipped with the b-symplectic form

$$\omega = \omega_{tw,2} + \pi^* \left(\omega_{MGS}\right)$$

and ω_{MGS} is a symplectic form on $Y_z^{SO(3)}$ given by Theorem 3.1.2. Now let $x = y$. Then z has isotropy group $\mathbb{Z}_2 \times SO(2)$. The associated b-symplectic model is given by $E = T^*\mathrm{S}^1 \times F$, where $F = (SO(3) \times_{SO(2)} V)$ is a bundle over the homogenous space $SO(3)/SO(2) \cong S^2$, V a 2-dimensional vector space with Darboux symplectic form ω_V. The b symplectic form on E is given by

$$\omega = \omega_{tw,1} + 2\omega_{S^2} + \omega_V$$

where ω_{S^2} is the usual symplectic form on the sphere and ω_V the linear Darboux form on V. Finally, suppose $y = -x$, Let $\nu \in \mathfrak{k}$, $\mathfrak{k}$ the Lie algebra of S^1. Let $\exp(t\xi) \cong SO(2)$ be the 1-parameter subgroup of $SO(3)$ such that $g = exp(\xi)$ acts on S^2 by $g(x) = -x$ and $d\rho_g = -Id$. Then the subgroup $(exp(t\nu), exp(t\xi)) \hookrightarrow S^1 \times SO(3)$ acts as the identity on the orbit $\mathcal{O}_z \cong S^2$ and b-symplectic model is given by the quotient bundle $E = T^*\mathrm{S}^1 \times (SO(3) \times_{SO(2)} V)/\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts on $u, v \in V$ by $(u, v) \to (-u, -v)$ and E is equipped with the symplectic form

$$\omega = \omega_{tw,1} + 2\omega_{S^2} + \omega_V$$

Chapter 4

b-Lie Groups and b-Symplectic Reduction

Symmetry is economy.

Alan Lightman

b-Poisson structures represent an opportunity to try and import symplectic results and constructions to the Poisson case. In Chapter 3, the traditional symplectic slice theorem was shown to have a close analogue in b-symplectic setting. In this chapter, we will treat *b-Lie groups*. The cotangent bundle of a Lie group, as the cotangent bundle of any manifold, comes equipped with a canonical b-symplectic form. This form is invariant under the cotangent lifted action of the group acting on itself by (left) translations. The reduced space $T^*G/G \cong \mathfrak{g}^*$ has an associated linear Poisson structure. In the following the cotangent bundle of a Lie group is equipped rather with a b-symplectic form and the Poisson structure on the reduced space is examined.

4.1 The b-cotangent bundle of a Lie group

In [1] the canonical symplectic form (2.1.3) was noted to have a direct analogy in the b-symplectic world.

Definition 4.1.1. *Let (M, Z) be a b-manifold. Then we define a b-one-form λ on ${}^bT^*G$, considered as a b-manifold with critical hypersurface ${}^bT^*G|_Z$, in the following way:*

$$\langle \lambda_p, v \rangle := \langle p, (\pi_p)_*(v) \rangle, \qquad p \in {}^bT^*G, v \in {}^bT_p({}^bT^*G)$$

*We call λ the b-**Liouville form**. The negative differential*

$$\omega = -d\lambda$$

is the **canonical** b**-symplectic form** *on ${}^bT^*G$.*

Using this form the canonical b-cotangent lift was defined as follows [17]:

Definition 4.1.2. *Consider the b-cotangent bundle ${}^bT^*M$ endowed with the canonical b-symplectic structure. Moreover, assume that the action of G on M preserves the hypersurface Z, i.e. ρ_g is a b-map for all $g \in G$. Then the lift of ρ to an action on ${}^bT^*M$ is well-defined:*

$$\hat{\rho} : G \times {}^bT^*M \to {}^bT^*M : (g, p) \mapsto \rho^*_{g^{-1}}(p).$$

Moreover, it is b-Hamiltonian with respect to the canonical b-symplectic structure on ${}^bT^*M$. We call this action together with the underlying canonical b-symplectic structure the **canonical** b**-cotangent lift**.

Considering ${}^bT^*G$ instead of T^*G, Definition 4.1.1 gives is a natural way to construct a b-symplectic structure analogous to the definition of the canonical *symplectic* form on T^*G. This construction works for any b-manifold (M, Z), but in the case where $(M, Z) = (G, H)$ is a *b-Lie group* we can consider **reduction** of the canonical b-symplectic structure. This will be explored in Section 4.1.3. In the symplectic case, reducing T^*G by the action of G yields the Lie-Poisson structure on $\mathfrak{g}^*$. However, in order to lift an action to ${}^bT^*G$ (Definition 4.1.2), we have to demand that the action leaves the critical hypersurface invariant. This motivates us to consider the setting where the critical hypersurface is a Lie subgroup H and we consider the action of H on G by translations.

First we give the relevant definitions and preliminary results:

Definition 4.1.3. *A b-manifold (G, H), where G is a Lie group and $H \subset G$ is a closed subgroup is called a* b**-Lie group**.

Example 4.1.4. *The special Euclidean group of orientation-preserving isometries in the plane is the semidirect product*

$$\mathrm{SE}(2) \cong \mathrm{SO}(2) \ltimes T(2)$$

where $T(2)$ are translations in the plane. Recall that we can identify $\mathrm{SE}(2)$ *with matrices of the form*

$$\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}, \qquad A \in \mathrm{SO}(2), b \in \mathbb{R}^2$$

Then $T(2)$ (identified with $\{I\} \times T(2) \subset \mathrm{SE}(2)$) is a closed codimension 1 subgroup and the pair $(\mathrm{SE}(2), T(2))$ *is a b-Lie group.*

Example 4.1.5. *The Galilean group G is the group of transformations in space-time $\mathbb{R}^{3+1}$ (the first three dimensions are interpreted as spatial dimensions and the last one is time) whose elements are given by composition of a spatial rotation $A \in SO(3)$, uniform motion with velocity $v \in \mathbb{R}^3$ and translations in space and time by a vector $(a, s) \in \mathbb{R}^{3+1}$. As a matrix group, the elements are given by*

$$\begin{pmatrix} A & v & a \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}, \qquad A \in \mathrm{SO}(3), v, a \in \mathbb{R}^3, s \in \mathbb{R}$$

The subgroup H given by $s = 0$ (which corresponds to fixing time) is a closed codimension one subgroup and hence the pair (G, H) is a b-Lie group.

Example 4.1.6. *We consider the $(2n+1)$-dimensional Heisenberg group $H_{2n+1}(\mathbb{R})$ given by matrices of the form*

$$\begin{pmatrix} 1 & a & c \\ 0 & I_n & b \\ 0 & 0 & 1 \end{pmatrix}, \qquad a \in \mathbb{R}^{1\times n}, b \in \mathbb{R}^{n\times 1}, c \in \mathbb{R}$$

The subgroup Γ of matrices of the form

$$\begin{pmatrix} 1 & 0 & k \\ 0 & I_n & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad k \in \mathbb{Z}$$

is central, hence normal, and so we can consider $G := H_{2n+1}(\mathbb{R})/\Gamma$. This is a well-known example of a non-matrix Lie group. Now fixing one component $a_i = 0$ or $b_i = 0$ yields a closed codimension one subgroup of G.

Let us consider the action of H on G by left translations. This action is obviously free and since H is closed, it is also proper. Therefore, the left coset space G/H can be given the structure of a smooth manifold such that the projection $\pi : G \to G/H$ is a smooth submersion. Moreover, it is well-known that π turns G into a principal H-bundle.

For future reference we summarize these facts in the following lemma:

Lemma 4.1.7. *Let (G, H) be a b-Lie group. The projection $\pi : G \to G/H$ is a principal H-bundle; in particular G is semilocally around H a product*

$$\pi^{-1}(V) \cong V \times H, \quad [e]_\sim \in V \subset G/H,$$

where π corresponds to the projection onto the first component.

Note that by taking a coordinate φ on V centered at $[e]_\sim$, we obtain a global defining function $\varphi \circ \pi$ for the critical hypersurface H.

4.1.1 The H-action on ${}^bTG$ and ${}^bT^*G$

As in the previous section, let (G, H) be a b-Lie group and consider the action of H by left translations.

We can lift this action to TG in the obvious way:

$$H \times TG \to TG : (h, v_g) \mapsto (L_h)_* v_g.$$

This action is again proper and free; therefore the quotient space is a manifold, which we want to describe below.

Let us introduce the subbundle $\mathcal{H}$ of TG whose fibre $\mathcal{H}_g$ at $g \in G$ is given by the corresponding left-shift of the Lie algebra $\mathfrak{h}$ of H, $\mathcal{H}_g = (L_g)_*\mathfrak{h}$. Let $\pi_\mathcal{H} : TG \to \mathcal{H}$ be the projection onto $\mathcal{H}$. Recall that $\pi : G \to G/H$ induces a surjective bundle morphism $\pi_* : TG \to T(G/H)$ and at each fibre T_gG the kernel is $\mathcal{H}_g$.

Proposition 4.1.8. *There is a diffeomorphism*

$$
\begin{aligned}
(TG)/H &\xrightarrow{\sim} \mathfrak{h} \times T(G/H)\\
[v_g]_\sim &\mapsto ((L_{g^{-1}})_*(\pi_{\mathcal{H}}(v_g)), \pi_*(v_g))\,.
\end{aligned}
$$

Proof. The map is well-defined as it does not depend on the representative of $[v_g]_\sim = \{(L_h)_*(v_g) : h \in H\}$. It is obviously smooth and surjective. If $[v_g]_\sim$ and $[v'_{g'}]_\sim$ have the same image, then $\pi_*(v_g) = \pi_*(v'_{g'})$ implies $\pi(g) = \pi(g')$ so by choosing a different representative in $[v'_{g'}]_\sim$ we can assume $g = g'$. Then $v_g - v'_g \in \ker(\pi_*)_g = \mathcal{H}_g$ and combining this with $\pi_{\mathcal{H}}(v_g) = \pi_{\mathcal{H}}(v'_g)$ we see $v_g = v'_g$. □

The analogous result holds for the action of H on the b-tangent bundle,

$$H \times {}^bTG \to {}^bTG : (h, v_g) \mapsto (L_h)_* v_g.$$

Note that this action is well-defined since the left translation by $h \in \mathcal{H}$ preserves H i.e. it is a b-map. Moreover, we define the projection $\pi_{\mathcal{H}} : {}^bTG \to \mathcal{H}$ in the obvious way.

Proposition 4.1.9. *There is a diffeomorphism*

$$
\begin{aligned}
({}^bTG)/H &\xrightarrow{\sim} \mathfrak{h} \times {}^bT(G/H)\\
[v_g]_\sim &\mapsto ((L_{g^{-1}})_*(\pi_{\mathcal{H}}(v_g)), \pi_*(v_g))
\end{aligned}
$$

where ${}^bT(G/H)$ is the b-tangent bundle of the one-dimensional b-manifold G/H with critical hypersurface $[e]_\sim$. Note that $\pi : (G, H) \to (G/H, [e]_\sim)$ is a b-map and therefore $\pi_* : {}^bTG \to {}^bT(G/H)$ is well-defined.

The right hand sides of the diffeomorphisms in Proposition 4.1.8 and 4.1.9 are vector bundles over G/H. This makes TG/H resp. ${}^bTG/H$ vector bundles over G/H as well with bundle map $[v_g]_\sim \mapsto \pi(g) \in G/H$:

Corollary 4.1.10. *$(TG)/H$ (resp. $({}^bTG)/H$) is a vector bundle of rank n over G/H isomorphic to the direct sum of the trivial vector bundle $\mathfrak{h} \times G/H$ with $T(G/H)$ (resp. ${}^bT(G/H)$):*

$$(TG)/H \cong (\mathfrak{h} \times G/H) \oplus T(G/H), \quad ({}^bTG)/H \cong (\mathfrak{h} \times G/H) \oplus {}^bT(G/H).$$

4.1.2 The b-cotangent lift

In Definition 4.1.2 we introduced the b-cotangent lift; in the present setting this is given by the following action on the b-cotangent bundle ${}^bT^*G$:

$$H \times {}^bT^*G \to {}^bT^*G : (h, \alpha_g) \mapsto (L_{h^{-1}})^*\alpha_g.$$

The quotient space $({}^bT^*G)/H$ can be viewed as a vector bundle over G/H which is isomorphic to $\left(({}^bTG)/H\right)^*$ via the identification

$$({}^bT^*G)/H \xrightarrow{\sim} \left(({}^bTG)/H\right)^* : [\alpha_g]_\sim \mapsto \left([v_g]_\sim \mapsto \langle \alpha_g, v_g \rangle\right), \quad v_g \in {}^bT_gG.$$

Therefore we can dualize the result for ${}^bTG/H$ of the previous section to obtain an isomorphism of vector bundles

$$({}^bT^*G)/H \cong (\mathfrak{h}^* \times G/H) \oplus {}^bT^*(G/H).$$

As smooth manifolds,

$$({}^bT^*G)/H \cong \mathfrak{h}^* \times {}^bT^*(G/H),$$

where the isomorphism is given by identifying an element of the right hand side $(\alpha, \beta_{[g]_\sim}) \in \mathfrak{h}^* \times {}^bT^*_{[g]_\sim}(G/H)$ with the class of $L^*_{g^{-1}}(\alpha) + \pi^*\beta_{[g]_\sim} \in {}^bT^*_gG$ on the left hand side.

4.1.3 Reduction of the canonical b-symplectic structure

The cotangent bundle T^*G has a canonical symplectic structure, which under the action of G on itself by left translations reduces to the minus Lie-Poisson structure on $T^*G/G \cong \mathfrak{g}^*$.

In Definition 4.1.1 we have seen how to endow the b-cotangent bundle ${}^bT^*G$ with a canonical b-symplectic structure (with critical hypersurface ${}^bT^*G|_H$). What is the reduced Poisson structure on $({}^bT^*G)/H$?

Theorem 4.1.11. *Let ${}^bT^*G$ be endowed with the canonical b-Poisson structure. Then the Poisson reduction under the cotangent lifted action of H by left translations is*

$$(({}^bT^*G)/H,\, \Pi_{red}) \cong (\mathfrak{h}^* \times {}^bT^*(G/H),\, \Pi^-_{L\text{-}P} + \Pi_{b\text{-}can})$$

where $\Pi^{-}_{L\text{-}P}$ is the minus Lie-Poisson structure on $\mathfrak{h}^*$ and $\Pi_{b\text{-}can}$ is the canonical b-symplectic structure on ${}^bT^*(G/H)$, where G/H is viewed as a b-manifold with critical hypersurface the point $[e]_\sim$.

Proof. Let $V \subset G/H$ be open and such that G trivializes as a principal H-bundle over V (cf. Lemma 4.1.7), i.e.

$$G \supset U := \pi^{-1}(V) \xrightarrow{\sim} H \times V$$

where the projection onto the second component corresponds to the quotient projection π; in particular the critical hypersurface H gets mapped to $H \times [e]_\sim \subset H \times V$ and the b-cotangent bundle over U splits in the following way:

$${}^bT^*U \cong T^*H \times {}^bT^*V.$$

Then the canonical b-symplectic structure ω_0 on ${}^bT^*U$ is the product of the canonical symplectic structure ω_1 on T^*H and the canonical b-symplectic structure ω_2 on ${}^bT^*V$. Denoting the Poisson tensor corresponding to ω_i by Π_i,

$$\Pi_0 = \Pi_1 + \Pi_2.$$

The action of H on ${}^bT^*U \cong T^*H \times {}^bT^*V$ is given by the standard cotangent lift of left translations by H on T^*H times the identity on ${}^bT^*V$. For the corresponding quotient projections $\pi_0 : {}^bT^*U \to ({}^bT^*U)/H$ and $\pi_0' : T^*H \to (T^*H)/H$ we therefore have $\pi_0 = \pi_0' \times \mathrm{id}_{{}^bT^*V}$. Hence the reduced Poisson structure on $({}^bT^*U)/H$ is

$$\Pi_{\text{red}} = (\pi_0)_*\Pi_0 = (\pi_0)_*(\Pi_1 + \Pi_2) = (\pi_0')_*\Pi_1 + \Pi_2.$$

Now note that $(\pi_0')_*(\Pi_1)$ is the minus Lie Poisson structure on $\mathfrak{h}^*$ if we identify $(T^*H)/H \cong \mathfrak{h}^*$. □

Example 4.1.12. *We return to Example 4.1.5 of the special Euclidean group* SE(2)*. Since $T(2)$ is abelian, the Lie-Poisson structure on the dual of its Lie algebra is zero. Hence ${}^bT^*(\mathrm{SE}(2))$ reduces under the action of $T(2)$ to*

$$(({}^bT^*(\mathrm{SE}(2)))/T(2),\, \Pi_{red}) \cong (\mathbb{R}^2 \times {}^bT^*(\mathrm{SO}(2)),\, 0 + \Pi_{b\text{-}can}),$$

where $\Pi_{b\text{-}can}$ is the canonical b-Poisson structure on ${}^bT^*(\mathrm{SO}(2))$, i.e. identifying $\mathrm{SO}(2) \cong \mathbb{S}^1$ in the usual way and letting φ be the angle, (φ, p) a b-canonical chart in a neighbourhood of $\{\varphi = 0\}$, then in these coordinates

$$\Pi_{red} = \varphi \frac{\partial}{\partial \varphi} \wedge \frac{\partial}{\partial p}.$$

Chapter 5

Orbits Close to Parabolic in the PCR3BP

Internal time stretched and stilled, inattentive to the minutes and hours outside of itself.
Zadie Smith, NW

We now apply the techniques of symplectic geometry to dynamical systems of a more classical nature. The restricted three-body problem (R3BP) has played a central role in the development of the theory of Hamiltonian dynamical systems, both as an approximation to the full gravitational three-body problem and as a problem exhibiting interesting dynamical phenomena in its own right. In this chapter and the following, we will employ symplectic methods to bound the stochastic layer surrounding the split separatrices associated to the fixed point "at infinity" of the planar circular restricted three-body problem (PCR3BP).

5.1 Introduction

The PCR3BP models the motion of a massless body under the gravitational influence of two massive bodies that follow circular orbits about their mutual centre of mass. The two massive bodies are not affected by the gravitational force of the third and all three bodies are assumed to be coplanar.

The R3BP is often studied as a perturbation of the Kepler problem, where this integrable limit is given by setting the value of the mass ratio of the massive bodies equal to zero. The particular nature of Hamiltonian systems which are close to integrable is a consequence of the celebrated K.A.M. theorem, that states that in systems close to integrable systems whose action-angle coordinates satisfy a certain non-degeneracy condition, "most" invariant tori of the original integrable system survive [27, 28, 29]. In marked contrast, between these preserved tori, one expects chaotic motions.

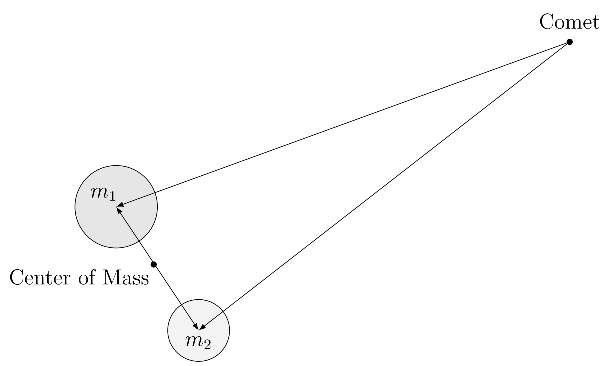


Figure 5.1: A schema of the R3BP with the massless body far from the primaries

One of the most classical mechanisms generating chaos is the transverse intersection of separatrices. The most studied case is that of a hyperbolic fixed point (or a multitude of these) together with their associated stable and unstable manifolds. In the integrable case, these invariant manifolds coincide along a "separatrix", so-called because it defines a boundary between domains with disparate dynamical behaviour. After perturbation, these stable and unstable manifolds may intersect in transversely. It was first shown by Poincaré [30] that such intersections can give rise to complicated dynamics. Later, is was shown by Smale [31] that the transverse intersection of stable and unstable manifolds implies the existence of a horseshoe and, in turn, chaos. In higher dimensions, the correspondence between the intersection of stable and unstable manifolds (now of dimension greater than one) and horseshoes remains.

Looking for chaos in a restricted three-body problem, then, usually involves searching for saddles of the Kepler problem and proving the transverse intersection of the associated stable and unstable manifolds upon perturbation. The Kepler problem comes with a family of saddles located at "parabolic infinity", defined by setting the distance from the origin $r = \infty$ and the associated momentum $y = P_r$ to 0. This cylinder is foliated by periodic orbits with constant angular momentum G, each with associated 2-dimensional stable and unstable manifolds. In the Kepler problem, the integrability of the system guarantees that these invariant manifolds coincide. Upon adding the perturbation giving the PCR3BP it has been shown that these invariant manifolds split and intersect transversally [32, 33].

Close to these split separatrices with transversal intersection the existence of chaotic motions is guaranteed. The area surrounding the split separatrices where one finds only chaotic motions is often referred to as the "stochastic layer". It is natural to ask about the width of the zone of chaotic motions, which generally extends to a width larger than that of the split separatrices. In order to study this, we can appeal to the K.A.M. theorem which guarantees the existence of some K.A.M. torus bounding this region, at least for small perturbations of the integrable system. We now wish to estimate the location of the last invariant torus before the onset of the stochastic layer. In the PCR3BP, such a torus has the following physical interpretation: the intersection of stable and unstable manifolds associated to fixed points "at infinity" leads to oscillatory motions, which leave every bounded region but return infinitely often to some bounded region. Conversely, the existence of a K.A.M. torus will prevent escape for all motions in the enclosed region of phase space. For this reason, such a torus is sometimes referred to as "the boundary of bounded motions" [34].

In order to specify this torus for the PCR3BP, we recall some concepts from the Kepler problem. Denote by (r, α, y, G) the coordinate chart on $T^*\mathbb{R}^2$ for which (r, α) are polar coordinates on $\mathbb{R}^2$ and $(y = P_r, G = P_\alpha)$ their associated conjugate momenta. Then all bounded solutions to the Kepler problem are ellipses specified uniquely by their eccentricity e and semi-major axis a, up to a constant angle α_0 giving the initial angle of the ellipse. In turn, we have that the eccentricity and semi-major axis of the orbit are functions of the (constant) angular momentum G of the orbit and Keplerian energy h. The PCR3BP then adds a time-dependent perturbation to the Keplerian Hamiltonian, so that the PCR3BP becomes, in principle, a two-and-a-half degree of freedom system. Therefore, 2-dimensional K.A.M. tori do not necessarily prevent escape. Moreover, as the Kepler problem is super-integrable and all orbits are periodic, the existence of a K.A.M. torus cannot be inferred in Hamiltonian perturbations of the system as no orbits are quasi-periodic. Both of these problems can be solved by viewing the system in a rotating coordinate system with a time-dependent angle $\phi = \alpha - t$. The transformed Hamiltonian, $\mathcal{J}$, also called the Jacobi constant can then be written as a function of the original Hamiltonian and the angular momentum of the orbit:

$$\mathcal{J} = H - G.$$

The K.A.M. theorem proves the persistence of tori in close to integrable systems by taking an invariant torus of the integrable system of irrational frequency and deforming it to a K.A.M. torus in the perturbed system, invariant under the flow of the perturbed Hamiltonian. Whence in every 3-dimensional level set of the Jacobi constant we look for the last torus which persists upon perturbing to the PCR3BP. This torus consists of rotated Keplerian ellipses of the lowest possible Keplerian energy (or, equivalently, longest possible semi-major axis).

It has been conjectured that for small mass ratio, μ (see [34]) a good cursory estimate for this boundary torus is given by the energy level

$$h \approx \left(\frac{3}{\kappa_G}\right)^{2/5} \frac{\pi^{3/5}}{2} \frac{\mathcal{J}_0^{14/5}}{32 - \mathcal{J}_0^3} \mu^{2/5} \exp\left(-\mathcal{J}_0^3/60\right)$$

where κ_G is Greene's constant. However, the purpose of this paper was to provide a general theoretical basis with which to compare numerical results and the estimate was not proven rigorously.

Our purpose here differs in two main ways. Firstly, the perturbation parameter is related to the value of the Jacobi constant and consequently, the estimate is valid for any mass ratio, μ. Secondly, we aim to prove the existence of such a torus rigorously. To achieve this, we overcome two main difficulties. Firstly, the saddle of the Keplerian problem "at infinity" is parabolic, rather than hyperbolic. To the author's knowledge, rigorous estimates for the width of the stochastic layer exist only in the hyperbolic case and methods need to be extended to the parabolic case. Secondly, the splitting of separatrices of the PCR3BP has been shown to be exponentially small with respect to the perturbation parameter $\mathcal{J}^{-1}$, which comes with some technical difficulties. To overcome these problems requires a study of the so-called "separatrix map", which expresses the Hamiltonian system close to split separatrices as a perturbation of an integrable twist map of the cylinder, allowing one to make inferences about the asymptotic width of the stochastic layer from the many available results on invariant circles of twist maps. With this general framework in mind, we examine an interpretation of the separatrix map via Hamilton-Jacobi theory, with an emphasis on rigorous results and careful control of exponentially small errors, achieved by taking full advantage of the symplectic nature of the system. Our main result comes in two flavours:

The first is an asymptotic statement giving the width of the layer as the Jacobi constant $\mathcal{J}$ goes to infinity. This is a consequence of a classic invariant curve theorem.

Theorem 5.1.1. *Let $\mathbb{T}_{h,\mathcal{J}}$ be a torus of the Kepler equation in rotating coordinates with Keplerian energy h contained a level set $\mathcal{J}$ of the Jacobi constant, where the Jacobi constant is assumed to be negative. Let μ be fixed and $\mu \neq 1/2$. Then, given any ϵ there exists some sufficiently large $\mathcal{J}^*$ such that for all $|\mathcal{J}| > |\mathcal{J}^*|$ and $h(\mu, \mathcal{J})$ satisfying*

$$h(\mu, \mathcal{J}) = -(c_1(\mu) f_1(\mathcal{J}))^{\frac{2}{5}-\epsilon}$$

there exists a Keplerian torus $\mathbb{T}_{h_0,\mathcal{J}}$ at some $|h_0| \leq |h|$ which continues to an invariant torus of the PCR3BP, where

- *$c_1(\mu)$ is a constant depending only on the mass ratio of the massive bodies given by*

$$c_1(\mu) = \sqrt{\frac{\pi}{32}}\mu(1-\mu)(1-2\mu)$$

- *$f_1(\mathcal{J})$ depends only on the Jacobi constant and is given by*

$$f_1(\mathcal{J}) = |\mathcal{J}|^{3/2} e^{-\frac{|\mathcal{J}|^3}{3}}$$

We remark that using the basic relations between the energy h, angular momentum, semi-major axis and eccentricity of Keplerian ellipses (see Appendix A), these tori correspond to rotated Keplerian ellipses of long semi-major axis

$$a \approx \frac{2^{2/5} e^{\frac{2|\mathcal{J}|^3}{15}}}{c_1(\mu)^{2/5} |\mathcal{J}|^{3/5}}$$

or high eccentricity

$$1 - e^2 \approx \frac{c_1(\mu)^{2/5}}{2^{2/5}} |\mathcal{J}|^{16/5} e^{\frac{2|\mathcal{J}|^3}{15}} \left(1 + \mathcal{O}\left(|\mathcal{J}|^{5/2} e^{\frac{2|\mathcal{J}|^3}{15}}\right)\right).$$

The second theorem provides an estimate on the layer width for all $|\mathcal{J}| > |\mathcal{J}^*|$ where $\mathcal{J}^*$ ensures that the separatrix mapping satisfies some hypothesis. While the second estimate is far from optimal, it gives a result where the perturbation parameter is not required to be arbitrarily small, a regime usually neglected when estimating stochastic layer width. Moreover, this method is suitable for optimization by computer-assisted proofs, both to lower $\mathcal{J}^*$ or to lower h_0. Specifically, we will prove:

Theorem 5.1.2. *Let $\mathbb{T}_{h_0,\mathcal{J}_0}$ be a torus of the Kepler equation in rotating coordinates with Keplerian energy h_0 contained a level set $\mathcal{J}_0$ of the Jacobi constant. Suppose that*

$$h_0 = \tilde{C}_* c_1(\mu)^{2/17} f(\mathcal{J}_0)^{2/17}$$

where

- *$\tilde{C}_*$ is of the form $\tilde{C}_* = C_* + \mathcal{O}(\mathcal{J}_0^{-1})$, where C_* is a constant given in 6.4.13.*
- *$c_1(\mu), f(\mathcal{J}_0)$ are as in Theorem 5.1.1.*

Then for large $\mathcal{J}_0$ the torus persists in the PC3BP.

5.1.1 The System

The Kepler Problem is a special case of the two-body problem, written in coordinates in which problem reduces to the motion of a body in a central field. In suitably scaled coordinates the Kepler problem possesses the Hamiltonian (see for instance [35])

$$H(r, y, \alpha, G) = \frac{y^2}{2} + \frac{G^2}{2r^2} - \frac{1}{r} \tag{5.1}$$

where (r, α) are the coordinates of the massless body in polar coordinates and (y, G) the corresponding canonical momenta. The associated equations of motion are given by Hamilton's equations, which for the Kepler problem are simply

$$\begin{aligned} \dot{r} &= y, & \dot{y} &= \frac{G^2}{r^3} - \frac{1}{r^2} \\ \dot{\alpha} &= \frac{G}{r^2} & \dot{G} &= 0. \end{aligned} \tag{5.2}$$

G is then a constant of motion. A general solution for the Kepler problem in configuration space is given by

$$\frac{1}{r} = \frac{1}{G^2}\left(1 + e\cos\left(\alpha - \alpha_0\right)\right) \tag{5.3}$$

which defines a conic section. Solutions with $e < 1$ corresponding to an ellipses, $e = 1$ to parabolas and $e > 1$ to hyperbolas. All bounded solutions are periodic. For details see appendix A.

The full PCR3P (see [35]) is given by adding a time dependent perturbation, given explicitly by

$$\begin{aligned} V(r,\alpha,t;\mu) =& \frac{1-\mu}{\left(r^2-2\mu r\cos(\alpha-t)+\mu^2\right)^{1/2}} \\ &+ \frac{\mu}{\left(r^2+2(1-\mu)r\cos(\alpha-t)+(1-\mu)^2\right)^{1/2}} - \frac{1}{r} \end{aligned} \tag{5.4}$$

where μ is a parameter given by the ratio of masses of the two massive bodies. The Hamiltonian (5.1) then becomes

$$H(r,y,\alpha,G) = \frac{y^2}{2} + \frac{G^2}{2r^2} - \frac{1}{r} - V(r,\alpha,\mu,t) \tag{5.5}$$

The associated equations of motion are

$$\begin{aligned} \dot{r} &= y \qquad & \dot{y} &= \frac{G^2}{r^3} - \frac{1}{r^2} + \partial_r V(r,\alpha,t;\mu) \\ \dot{\alpha} &= \frac{G}{r^2} \qquad & \dot{G} &= \partial_\alpha V(r,\alpha,t;\mu) \end{aligned} \tag{5.6}$$

In contrast to the Kepler problem, in the Hamiltonian system given by (5.6) neither energy nor angular momentum are conserved.

The Rotating Coordinate System

The PCR3BP is often studied as a perturbation of the Kepler problem, where the difference in potential energies between the Kepler problem and the full restricted three body problem, $V(r,\alpha,t;\mu)$, is assumed to be small. In (r,y,α,G,t) coordinates the R3BP is, in principle, a two and a half degree of freedom system, and so the existence of two-dimensional invariant torus is not guaranteed to provide a barrier to escape to infinity. This problem can be solved by observing that the perturbation potential $V(r,\alpha,t;\mu)$ depends only on the difference between the angular variables α and t. Therefore, upon putting the system in a rotating coordinate system rotating with the two massive bodies, the perturbation potential (5.4) becomes an explicitly time-independent function of a time-dependent angle $\phi = \alpha - t$. The Hamiltonian (5.5) becomes

$$\begin{aligned} \mathcal{J}(r,y,\phi,G) &= \frac{y^2}{2} + \frac{G^2}{2r^2} - \frac{1}{r} - V(r,y,\phi,G) - G \\ &= H(r,y,\phi,G) - G \end{aligned} \tag{5.7}$$

The system is then an autonomous 2 degree of freedom Hamiltonian system with conserved energy given by the Jacobi constant, $\mathcal{J}$. The existence of 2 dimensional K.A.M. tori in a 3 dimensional level energy set then provides a barrier to escape to infinity.

Moreover, upon viewing the Kepler problem in a rotating coordinate system, periodic orbits of the system whose period is not a rational multiple of 2π become quasiperiodic motions on a torus when viewed from the rotating system of coordinates, so that theory of Kolmogorov [27] can be applied directly in this case. Upon taking an appropriate Poincaré section in a high enough level set of the Jacobi constant, the system can be reduced to an area-preserving map of the annulus for which the integrable system is a monotone twist map, allowing the application of theorems on invariant circles of the perturbations of such maps to be applied [36].

The (ϕ, G) Poincaré section

Poincaré sections greatly reduce the complexity of a dynamical system by relating Hamiltonian flow on a level energy set to a symplectomorphism of a hypersurface in that level energy set. A well-chosen Poincaré section taking into account, for example, symmetry considerations, can simplify calculations. In [33], the angular coordinate ϕ was fixed $\phi = \phi_0$ and the system examined as a mapping of the (r, y) plane to itself. In our case we will take advantage of the fact that we are in the region inside the parabolic orbits (where all orbits of the Kepler problem are "rotated ellipses") and choose instead a Poincaré section $f = 0$ where f is the true anomaly. In the Keplerian case, this section is a cylinder with coordinates (ϕ, G) where $\phi = \xi + f - t$ for ξ an initial starting angle for the orbit. This confers one main advantage: in the region in which we are interested, that is, orbits which are rotated ellipses of high eccentricity, finding a time parameterization of orbits $(r(t), y(t), \alpha(t), G)$ involves solving the Kepler equation

$$M = E - e \sin E$$

for E, where M is the mean anomaly (a parameterization of time) and E is the eccentric anomaly (a parameterization of polar angle). Though one can get a series solution for E, in practise this converges extremely slowly for e close to 1 and so a good expression for $(r(t), y(t), \alpha(t), G)$, and so for the associated Poincaré mapping, is not available.

However, the period of a Keplerian orbit is simply (see appendix (A))

$$T(h) = \frac{\pi}{\sqrt{2}}(-h)^{-\frac{3}{2}} \tag{5.8}$$

where h is the Keplerian energy of the orbit, i.e. the value of (5.1). On a level set of the Jacobi constant $\mathcal{J}_0$, each Keplerian orbit is then defined uniquely by the initial angle ξ and by either the angular momentum of the orbit or its associated Keplerian energy. We have

$$T_{\mathcal{J}_0}(G) = \frac{\pi}{\sqrt{2}}(-\mathcal{J}_0 - G)^{-\frac{3}{2}} \tag{5.9}$$

and the Poincaré map for the chosen cylinder $f = 0$ is simply

$$\begin{aligned} \bar{\phi} &= \phi + \frac{\pi}{\sqrt{2}}\left(-\mathcal{J}_0 - G\right)^{-3/2} \\ \bar{G} &= G \end{aligned} \tag{5.10}$$

The PCR3BP can then be examined as a perturbation of this integrable twist map. This Poincaré section is similar to that used in the integrable limit in [34]. However, unlike the approach developed there we use in the perturbed problem a Poincaré section which is the same Keplerian case when restricted to configuration space, rather than the section $y = 0$. This is because our method of controlling errors relies on finding orbits in the perturbed case can as graphs over the Keplerian orbits and so we allow y to vary while keeping r fixed. Secondly, we use variables (ϕ, G) taking advantage of the fact that on a level set of the Jacobi constant we can define h, the Keplerian energy, uniquely as a function of $\mathcal{J}_0$ and G, which simplifies the mappings somewhat.

5.1.2 Perturbations of the Kepler Problem

The R3BP has been examined as a perturbation of the Kepler problem in various ways. The most traditional perhaps is by setting the mass ratio $\mu = 0$, for which the perturbation potential (5.4) is zero. The potential then reduces to the Kepler potential $U_K = \frac{1}{r}$. However, other cases have been treated. Another possibility, in the case that the massless body is far from collision, is to treat the distances between the two massive bodies as a perturbation parameter. In the case that the massless body is close to parabolic motion, the two massive bodies move quickly in comparison to the massless body and we are dealing with a fast-oscillating small perturbation. In [33] this is made clear by considering a perturbation parameter $\varepsilon = \mathcal{J}_0^{-1}$ where $\mathcal{J}_0$ is the Jacobi constant. Upon choosing a level energy set of the Jacobi constant $\mathcal{J}_0$ and performing a rescaling

$$r = \mathcal{J}_0^2 \widetilde{r}, \quad y = \mathcal{J}_0^{-1} \widetilde{y}, \quad \alpha = \alpha, \quad G = \mathcal{J}_0 \widetilde{G}, \quad t = \mathcal{J}_0^3 s$$

The Kepler Hamiltonian transforms to a Kepler-like Hamiltonian

$$\widetilde{H}\left(\widetilde{r}, \alpha, \widetilde{y}, \widetilde{G}, s; \mu, \mathcal{J}_0\right) = \frac{\widetilde{y}^2}{2} + \frac{\widetilde{G}^2}{2\widetilde{r}^2} - \frac{1}{\widetilde{r}}$$

with perturbation potential

$$U\left(\widetilde{r}, \alpha - \mathcal{J}_0^3 s; \mu, \mathcal{J}_0\right) = \mathcal{J}_0^2 V\left(\mathcal{J}_0^2 \widetilde{r}, \phi; \mu\right)$$

and it is now clear that we are dealing with a fast oscillating perturbation when $\mathcal{J}_0$ is very large. In [33], this rescaling was employed as it ensured that the parameterisation of the parabolic Kepler orbits was independent of $\mathcal{J}_0$, as the rescaled angular momentum $G\mathcal{J}_0^{-1} = \widetilde{G}$ is always equal to one. In our case, a similar rescaling will be applied where the rescaling variable is the angular moment of the now elliptic orbits which ensures that the parameterization of the orbits depends only on the ellipticity e. In this case, as we are dealing with orbits which have angular moment close to $\mathcal{J}_0$ again for $\mathcal{J}_0$ large, the perturbation will similarly be of fast oscillating nature. For fast oscillating perturbations, the most important terms of the perturbation are given by the low-order Fourier coefficients, as will be shown to be the case here.

5.1.3 The Surface at Parabolic Infinity

In the present study, the difference between the Keplerian potential energy and that of the full problem will be controlled by ensuring that the massless body, often called the comet, is always far from the two massive bodies. Additionally, we will be particularly interested in motions close to parabolic, representing the furthest that the comet can travel from the two bodies without escape. We recall the following object of interest which plays a fundamental role in the Keplerian problem and its perturbations in this regime: the manifold "at parabolic infinity" is defined by $(r, y) = (+\infty, 0)$. The set is then a cylinder foliated by periodic orbits corresponding to level sets of angular momentum

$$\Lambda_G = \{(r, \alpha, y, G) : r = \infty, y = 0, \alpha \in \mathbb{T}, G = G\}$$

To examine the nature of these periodic orbits, "McGehee variables" are often employed

McGeehee Variables

"McGeehee variables" [37] are defined by the non canonical transformation

$$r = \frac{2}{x^2}, \quad y = y, \quad \alpha = \alpha, \quad G = G$$

which sends $r = \infty$ to $x = 0$. The Keplerian Hamiltonian is then equivalent to that of a Duffing oscillator

$$H(x, \alpha, y, G, t; \varepsilon_{\mathrm{J}}) = \frac{y^2}{2} + \frac{x^4 G^2}{8} - \frac{x^2}{2}$$

One of the advantages of these variables is that the saddle nature of the fixed points now becomes clear. However, the symplectic form is now a non-canonical b^3-symplectic form (see [1]).

$$T = -\frac{1}{x^3} dx \wedge dy + d\alpha \wedge dG$$

which is associated to the singular Poisson bracket

$$\{f, g\} = -\frac{x^3}{4}\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right) + \frac{\partial f}{\partial \alpha}\frac{\partial g}{\partial G} - \frac{\partial f}{\partial G}\frac{\partial g}{\partial \alpha}$$

for which the Hamiltonian equations become

$$\frac{dx}{dt} = -\frac{1}{4}x^3 y \qquad \frac{dy}{dt} = \frac{1}{8}G^2 x^6 \tag{5.11}$$

$$\frac{d\alpha}{dt} = \frac{1}{4}x^4 G \qquad \frac{dG}{dt} = 0 \tag{5.12}$$

The fixed points, then, are degenerate topological saddles rather than true hyperbolic fixed points. The fact that these fixed points are parabolic rather than hyperbolic makes their study delicate, as e.g. standard theorems guaranteeing the form of their stable and unstable manifolds are not available. Nevertheless, the union of these stable and unstable manifolds of these parabolic fixed points has been shown to be analytic [37, 33].

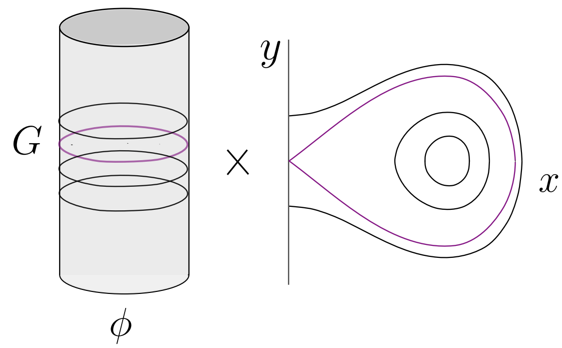


Figure 5.2: The phase space of the Kepler problem in McGeehee coordinates, union of the stable and unstable manifolds of the parabolic saddle in purple. On a level energy set of the Jacobi constant this orbit corresponds to a level energy set of G in the cylinder (ϕ, G)

5.1.4 Splitting of Separatrices

A separatrix is an invariant manifold composed of coincident stable and unstable manifolds of an invariant object of a dynamical system. When the system is integrable, these manifolds are coincident as long as they intersect. When the system undergoes a small perturbation, however, these separatrices generically split. If they intersect transversally, one can prove the existence of a Smale horseshoe, which implies the existence of chaos. The splitting of separatrices is usually proven using the Melnikov method.

The Case of Regular Splitting

The Melnikov method, sometimes known as the Poincaré-Melnikov method (see, e.g. [38] for a detailed exposition) has been applied by Poincaré and Melnikov to measure the distance between split separatrices [39, 30]. It reduces the problem of finding the distance between the stable and unstable manifolds of a hyperbolic fixed point, which are generally uncomputable globally, to an integral taken along the original separatrix known as the Melnikov integral. More specifically, consider a one and a half degree of freedom Hamiltonian system on $\mathbb{R}^2$ with coordinates $x = (x_1, x_2)$ with a Hamiltonian H of the form

$$H(x,t) = H_0(x) + \varepsilon H_1(x,t)$$

so that $H_0(x)$ is automatically integrable and possesses hyperbolic fixed points with associated homoclinic orbits. Parameterize these homoclinic orbits (which are the co-incident stable and unstable manifolds of the hyperbolic fixed points) by the variable v, referred to as "time along the unperturbed separatrix" so that

$$\varphi^t_{H_0}(x_h(v)) = x_h(v+t) \tag{5.13}$$

where $\varphi^t_{H_0}$ is the Hamiltonian flow associated to the integrable Hamiltonian H. Then the Melnikov function is given by

$$M(v,\tau) = \int_{-\infty}^{+\infty} \{H_0, H_1\}(x_h(v+s), \tau+s)\, ds$$

Due to the relation between τ and v given in equation (5.13), we can think of $M(v,\tau)$ in two equivalent ways. One can either fix a section in space (by choosing $v = v_0$) and observe that $M_{v_0}(\tau) = M(v_0,\tau)$ oscillates in time. Equivalently, one can fix time and move the section along the separatrix to see $M_\tau(v_0)$. We emphasize that both ways are completely equivalent as for Hamiltonian systems

$$M(v,\tau) = M(0,\tau - v) = \mathcal{M}(\tau - v).$$

Nevertheless it is useful for us to keep both formulations in mind.

It can be shown using perturbation theory that the difference between the unperturbed energies of the stable manifold $h^u(v,\tau)$ and the unstable manifold $h^s(v,\tau)$ is given by

$$h^u(v,\tau) - h^s(v,\tau) = \varepsilon M(v,\tau) + \mathcal{O}\left(\varepsilon^2\right)$$

Whence, if the Melnikov integral is independent of ε and oscillates about 0 in τ (equivalently with respect to v) one can infer the existence of transverse homoclinic points. Here, as we deal with Hamiltonian systems, we will work rather with the Melnikov potential.

$$L(v,\tau) = \int_{-\infty}^{+\infty} \left(H_1(x_h(v+s), \tau+s) - H_1(x^*, \tau+s)\right) ds$$

which satisfies

$$M(v,\tau) = \frac{\partial L}{\partial v}(v,\tau)$$

and so can be thought of as giving an approximation of a generating function describing the splitting of the separatrices.

Exponentially Small Splitting

The case of exponentially small splitting of separatrices is much more delicate. This typically occurs in the case of a fast oscillating small perturbation, e.g. one of the form $H_1(x,t) = f(x, \frac{t}{\varepsilon}, \varepsilon)$. In this case, the distance predicted by the Melnikov function is generally expected to be of the form

$$M(v,\tau) = M(v, \frac{\tau}{\varepsilon}, \varepsilon) = A(v, \frac{\tau}{\varepsilon})e^{-\frac{\rho}{\varepsilon}}$$

and as

$$h^u(v,\tau) - h^s(v,\tau) = \varepsilon M(v,\tau) + \mathcal{O}\left(\varepsilon^2\right)$$

we cannot conclude the transversal intersection of the manifolds by taking ε small enough. The question then arises if the Melnikov formula does indeed give an asymptotic expression for the distances between the stable and unstable manifolds. It has been shown, for example in [40], that indeed this is not always the case. In order to prove that the Melnikov function does, in fact, indicate transversal intersection of separatrices one needs to prove that that the Melnikov function gives the splitting up to an exponentially small error (smaller than that of the Melnikov function itself). Achieving these exponentially small bounds is an important problem, for which various parallel theories have been developed ([41],[42]).

5.1.5 Hamilton-Jacobi Parameterizations

One method of analysing the difference between the Melnikov function and the true splitting of separatrices is by examining the stable and unstable manifolds of the Hamiltonian system as solutions to the Hamilton-Jacobi equation. Recall that given an autonomous Hamiltonian system $H(p,q)$, $H \in \mathcal{C}^\infty(T^*M)$ equipped with the canonical symplectic form the Hamilton-Jacobi equation reads

$$H\left(x, \partial_x S\right) = c \tag{5.14}$$

solutions to the Hamilton-Jacobi equation are functions $S : M \to \mathbb{R}$ whose graphs $L_S = (q, \partial_q S(q))$ are Lagrangian submanifolds invariant under the flow of the Hamiltonian. In particular the stable and unstable manifolds, at least locally, are graphs of solutions of the Hamilton-Jacobi equation which satisfy the appropriate boundary conditions.

Suppose that the stable and unstable manifolds occur in a system close to integrable and the generating function of the separatrix is given by $S_0(q)$. To look for solutions $S(q)$ of the perturbed system one can reparameterise the Hamilton-Jacobi equation "through the unperturbed separatrix". That is, given the variable v representing time along the unperturbed separatrix defined by (5.13), one can write the Hamilton-Jacobi equation in terms of v and search for solutions $T(v) = S(q(v))$ close to that of the integrable case $T_0(v) = S_0(q(v))$. Writing the desired solution $T(v)$, as the sum of the generating function of coincident stable and unstable manifolds of the integrable system plus a perturbation term $T_1(v)$

$$T(v) = T_0(v) + T_1(v)$$

one can write a Hamilton-Jacobi type equation for $T_1(v)$. The generating functions for the stable manifold and unstable manifolds are then given by $T^s(v) = T_0(v) + T_1^s(v)$, where $T_1^{s,u}(v)$ satisfy different asymptotic boundary conditions.

Finally, to validate that the Melnikov potential gives the correct splitting of the stable and unstable manifolds, one bounds the difference between the splitting as given by the Melnikov integral and that given by the solutions to the Hamilton Jacobi equation above. Namely one tries to bound the expression

$$\partial_v(T_1^u(v,\tau) - T_1^s(v,\tau)) - \varepsilon \partial_v L(v,\tau) \tag{5.15}$$

Hamilton-Jacobi and Exponentially Small Splitting

To prove *exponentially small* splitting of separatrices, then, one needs to show that the Melnikov potential is exponentially close to the difference of solutions of the Hamilton-Jacobi equation. That is, one needs to show the expression in (5.15) is exponentially small. One method to achieve exponentially small bounds is by extending the Melnikov potential and aforementioned solutions of the Hamilton-Jacobi equation to the complex plane.

A consequence of the Cauchy theorem is that functions on the torus analytic in a complex strip possess exponentially decaying Fourier coefficients. Given an analytic function which oscillates very quickly, with a frequency $\tau = \frac{t}{\varepsilon}$ say, one can similarly show that the Fourier coefficients decay exponentially in ε.

More specifically, let $\Lambda(s)$ be a 2π periodic function and consider the Fourier series

$$\Lambda(s) = \sum_{k \in \mathbb{Z}} \Lambda_k e^{iks}.$$

Suppose that $\Lambda(s)$ extends to an analytic function in interior the complex strip

$$\mathcal{D}_\kappa = \{z \in \mathbb{C} \mid |\Im(z)| < \kappa\}.$$

Then one can bound

$$|\Lambda_k| \leq M\mathrm{e}^{-\kappa|k|}, \tag{5.16}$$

where $|\Lambda(z)| \leq M$ for all z in the complex strip of width $\kappa' < \kappa$.

Now let us consider a function which oscillates quickly in t. That is, let $g(v,\tau)$ be 2π periodic in τ with $\tau = \frac{t}{\varepsilon}$ for some small ε. Suppose $g(v,\tau)$ satisfies $g(v,\tau) = g(0, \tau - \frac{v}{\varepsilon})$. This implies that

$$g(v,\tau) = \Lambda\left(\tau - \frac{v}{\varepsilon}\right) \tag{5.17}$$

for some periodic function Λ. As g is 2π periodic in τ, Λ is 2π periodic in $s = \tau - \frac{v}{\varepsilon}$. Write $g(v,\tau)$ in terms of the Fourier coefficients of Λ:

$$g(v,\tau) = \sum_{k \in Z} \Lambda_k e^{-ik\frac{v}{\varepsilon}} e^{ik\tau}.$$

Finally we can conclude

$$|g(v,\tau) - \Lambda_0| \leq Me^{-\frac{\kappa'}{\varepsilon}} \tag{5.18}$$

where

$$|g(v,\tau)| \leq M$$

on the (closed) complex strip

$$\mathcal{D}_\kappa = \{z \in \mathbb{C} \mid |\Im(z)| \leq \kappa'\}.$$

and $\kappa' < \kappa$.

In order to take advantage of this observation when dealing with exponentially small splitting of separatrices, the unstable and stable manifolds are described as a graph over the unperturbed separatrices i.e. they are given by functions $T_1^u(v,\xi), T_1^s(v,\xi)$ which satisfy the Hamilton Jacobi equations and satisfy the initial conditions associated to the stable and unstable manifolds. These solutions are extended to some common of the complex plane which contains on the real axis two real values which correspond to consecutive homoclinic points and which extends to $\mathcal{O}(\varepsilon)$ of the singularities of the manifolds. Finally, the difference of these equations

$$\Delta(v,\tau) = T^u(u,\tau) - T^s(v,\tau)$$

is shown to obey a partial differential equation close to $(\varepsilon\partial_v + \partial_\tau)\, u(v,\tau) = 0$ i.e. one finds a change of variable defined on the domain, $\mathcal{C}(v,\tau)$ so that the function $\tilde{\Delta}$ defined by

$$\Delta(v,\tau) = \tilde{\Delta}(v + \mathcal{C}(v,\tau),\tau)$$

is in the kernel of the following linear operator $(\varepsilon\partial_v + \partial_\tau)$. This is equivalent to $\tilde{\Delta}(v,\tau) = \tilde{\Delta}(0,\tau - \frac{v}{\varepsilon})$ and so the Fourier coefficients of $\tilde{\Delta}$ can be bounded using (5.18). Moreover, as the Melnikov potential $\mathcal{L}(v,\tau)$ also satisfies (5.17),if the change of variable is sufficiently close enough to the identity, then by bounding the difference between $\Delta(v,\tau)$ and $\mathcal{L}(v,\tau)$ in the complex one can conclude that the Melnikov method gives the correct first order of the splitting.

5.1.6 Splitting of Separatrices in the PCR3BP

In the PCR3BP numerous challenges to proving the transversal intersection of the stable and unstable manifolds of the periodic orbits at infinity arise. The first difficulty is the parabolic nature of the fixed point. The second is that for very large Jacobi constant, the perturbation is a fast-oscillating small perturbation and so the Melnikov function is exponentially small in $\mathcal{J}_0^3$. Although, in this case, the correct first order of the splitting is given by Melnikov function, to prove this requires significant effort.

The first result on the transversal splitting of separatrices was obtained by Libre and Simo [32], but in order to conclude that the Melnikov function gave the correct splitting the mass ratio μ between the massive bodies was required to be exponentially small with respect to the Jacobi constant. A general result for arbitrary μ was given by Guardia, Martín and Seara in [33] and employed methods from Hamilton-Jacobi theory given in section 5.1.5.

The Reparameterized Hamilton-Jacobi Equation

In order to solve the Hamilton-Jacobi equation giving the generating functions of the perturbed stable and unstable manifolds, the first step is to write the generating functions of the perturbed invariant sets as perturbations of those generating the stable and unstable manifolds in the integrable case. This method was applied to the PCR3BP in [33]. Here we recall their derivation, which we will adapt to orbits of the Kepler problem for $e \neq 1$. The equations are as follows:

Let $\mathcal{J}_0$ be the Jacobi constant on the chosen level energy surface. Then the Hamilton-Jacobi equation for the integrable mapping reads

$$\mathcal{J}_0 = \frac{(\partial_r S)^2}{2} + \frac{(\partial_\phi S)^2}{2r^2} - \frac{1}{r} - \partial_\phi S$$

Now, as G is conserved, we know that a solution of the above partial differential equation is given by

$$S = G\phi + f(r)$$

where $f(r)$ is some solution of the equation

$$\frac{1}{2}\left(f'(r)\right)^2 + \frac{G^2}{2r^2} - \frac{1}{r} = h$$

Now one can look for solutions of the full Hamilton-Jacobi equation "close" to S_0, by writing $S = S_0 + S_1$, and substituting this into the energy equation. In the case of the PCR3BP find that S_1 satisfies the equation

$$\partial_r f \partial_r S_1 + \frac{1}{2}\left(\partial_r S_1\right)^2 - G\partial_\phi S_1 + \frac{G}{r^2}\partial_\phi S_1 + \frac{1}{2r^2}\left(\partial_\phi S_1\right)^2 - V\left(r, \phi; \mu, G\right) = 0 \tag{5.19}$$

where $V\left(r, \phi; \mu, G\right)$ is the potential given in (5.4).

In order to compare with the Melnikov potentiall, the Hamilton-Jacobi equation is reparameterized through the unperturbed separatrix as follows: let ξ parameterize the family of parabolic orbits in the chosen level set of the Jacobi constant and v be time along the unperturbed separatrix, i.e. $\xi \in \mathbb{S}^1, v \in \mathbb{R}$, $\phi = \xi + \alpha_{\mathrm{h}}(v)$ and $(r_{\mathrm{h}}(v), \alpha_{\mathrm{h}}(v), y_{\mathrm{h}}(v), G_{\mathrm{h}}(v)))$ satisfy the equations

$$
\begin{aligned}
&\frac{d}{dv} r_{\mathrm{h}}(v) = y_{\mathrm{h}}(v) \\
&\frac{d}{dv} y_{\mathrm{h}}(v) = \frac{\mathcal{J}_0^2}{r_{\mathrm{h}}(v)^3} - \frac{1}{r_{\mathrm{h}}(v)^2} \\
&\frac{d}{dv} \phi_{\mathrm{h}}(v,\xi) = \frac{\mathcal{J}_0}{r_{\mathrm{h}}(v)^2} - 1 \\
&\frac{d}{dv} G_{\mathrm{h}}(v) = 0
\end{aligned} \tag{5.20}
$$

with initial condition $\phi_{\mathrm{h}}(0,\xi) = \xi$, where we recall that $\mathcal{J}_0$ is the value of the angular momentum along the parabolic orbits in the chosen level set of the Jacobi constant $\mathcal{J}_0$

One then looks for a solution

$$
T_1\left(v,\xi;\mu,G\right) = S_1\left(\widetilde{r}_{\mathrm{h}}(v), \xi + \alpha_{\mathrm{h}}(v) - v; \mu, \mathcal{J}_0\right)
$$

of the reparameterized Hamilton-Jacobi equation

$$
\begin{aligned}
\partial_v T_1 - G^3 \partial_\xi T_1 + \frac{1}{2\widetilde{y}_{\mathrm{h}}^2}\left(\partial_v T_1 - \frac{1}{r_{\mathrm{h}}^2}\partial_\xi T_1\right)^2 + \frac{1}{2r_h^2}\left(\partial_\xi T_1\right)^2 \\
- V\left(r_{\mathrm{h}}\xi + \alpha_{\mathrm{h}} - v; \mu, G\right) = 0
\end{aligned} \tag{5.21}
$$

satisfying the initial conditions of the stable and unstable manifolds, given explicitly by

$$
\begin{aligned}
\lim_{v\to-\infty} \partial_v T_1^u(v,\xi;\mu,G) = 0, \lim_{v\to-\infty} \partial_v T_1^s(v,\xi;\mu,G) = 0 \\
\lim_{v\to\infty} \partial_\xi T_1^u\left(v,\xi;\mu,G\right) = 0, \lim_{v\to\infty} \partial_\xi T_1^s\left(v,\xi;\mu,G\right) = 0.
\end{aligned} \tag{5.22}
$$

The parameterized separatrices $(r_{\mathrm{h}}(v), \alpha_{\mathrm{h}}(v), y_{\mathrm{h}}(v), G_{\mathrm{h}}(v)))$ are then extended to the complex plane and the value of the Melnikov function is compared to solutions of the Hamilton-Jacobi equation in the complex domain, which allows one to achieve exponentially small bounds on the difference in the reals. In the case of the [33] several difficulties needed to be overcome. Firstly, solutions to the above equations cannot be extended analytically to the required domain, rather one has to extend Fourier coefficients and define the solution by way of a (formal but ultimately convergent) series. Moreover, the fact that solutions are not graphs over r came with inherent difficulties as the distance between separatrices as defined on a section $r = \text{constant}$ in the (r, y) plane becomes undefined for turning points $y = 0$. For this reason, the difference between the stable and unstable manifolds is examined on a *boomerang domain*. In [33] it is proven that

Theorem 5.1.3. *There exist* $0 < v_- < v_+, \mathcal{J}_0^* > 0$ *and* $K > 0$ *such that , for any* $\mathcal{J}_0 > \mathcal{J}_0^*$ *and* $\mu \in (0, 1/2]$ *the invariant manifolds of infinity have parameterizations of the form (25) for* $(v, \xi) \in (v_-, v_+) \times \mathbb{T}$ *Moreover, the corresponding generating functions satisfy*

$$|T_1^u(v,\xi) - T_1^s(v,\xi) - L(v,\xi) - E| \leq K\mu^2(1-2\mu)\mathcal{J}_0^{-2} e^{-\frac{\mathcal{J}_0^{-3}}{3}} + K\mathcal{J}_0^{-1/2}\mu^2 e^{\frac{2\mathcal{J}_0^{-3}}{3}}$$

for a constant $E \in \mathbb{R}$, *which might depend on* μ *and* $\mathcal{J}_0$, *and*

$$\begin{aligned} &\left|\partial_v^m \partial_\xi^n T_1^u(v,\xi) - \partial_v^m \partial_\xi^n T_1^s(v,\xi) - \partial_v^m \partial_\xi^n L(v,\xi)\right| \\ &\qquad \leq K\mu^2(1-2\mu)\mathcal{J}_0^{-2+3m} e^{\frac{\mathcal{J}_0^{-3}}{3}} + K\mathcal{J}_0^{-1/2+3m}\mu^2 e^{\frac{2\mathcal{J}_0^{-3}}{3}} \end{aligned} \tag{5.23}$$

where $L(v, \xi)$ *is the Melnikov potential of the PCR3BP*

$$L(v,\xi;\mu,\mathcal{J}_0) = \int_{-\infty}^{+\infty} V\left(\tilde{r}_{\mathrm{h}}(v+s), \xi - G_0^3 s + \alpha_{\mathrm{h}}(v+s); \mu, G_0\right) ds$$

In the current project, we generalize the method above to orbits of high eccentricity where the generating functions of the stable and unstable manifolds become the generating functions of circles of a fixed rotation number of the Poincaré section 5.1.1. The Hamilton-Jacobi equation along orbits of high eccentricity is close to the equation along the separatrix above in the complex plane and so we are able to deduce exponentially small bounds in the reals. The aim of the next section is to define a Melnikov-like integral as an approximate solution to the Hamilton-Jacobi equation.

5.2 Discussion of the Melnikov Integral

We note that in the parabolic case, the distance from the last invariant torus from the separatrix becomes large (with respect to the exponentially small splitting of separatrices). Therefore, upon return to the Poincaré section, it is no longer true that the difference in G is given to the required exactness by the splitting of the separatrices. In order to deal with this, we examine a family of Melnikov-like integrals, given by integrating the perturbation potential $V(r, \alpha, t; \mu)$ as given by (5.4) along Keplerian orbits contained in $\mathbb{T}_{G,\mathcal{J}_0}$. The result of these integrals will then be a two-parameter family of generating functions, indexed by the quantities G and $\mathcal{J}_0$ and expressed as functions of Keplerian time v and initial angle ξ which will give approximate solutions to the Hamilton-Jacobi equation. These approximate solutions can then be used to estimate the difference in the initial and final angular momentum of an orbit in a to-be-defined Poincaré section to a sufficient degree of accuracy.

5.2.1 Parameterizations of Keplerian Tori

Consider the Kepler problem in rotating coordinates and a level set of the Jacobi constant $\mathcal{J}^{-1}(\mathcal{J}_0)$, where we assume $\mathcal{J}_0$ is large and negative. In this level energy set, consider the set of orbits of positive angular moment $G \in \mathbb{R}$ satisfying $\mathcal{J}_0 + G \leq 0$. In the rotating coordinate system, the boundary of this set consists of a family of parabolic orbits satisfying $\mathcal{J}_0 + G = 0$ parameterised by their angle at (say) the aphelion ξ. The remaining phase space is foliated by a family of 2-tori, parameterized angular momentum G, which represent elliptic solutions to the Keplerian problem rotated in time. We write $\mathbb{T}_{G,\mathcal{J}_0}$ for the union of orbits of the Kepler problem in rotating coordinates with angular momentum G, contained in a level set of the Jacobi constant $\mathcal{J}^{-1}(\mathcal{J}_0)$. Such tori can be parameterized by (ξ, v), where ξ is the angle of the body at the aphelion and v the time along the Keplerian orbit. For convenience, we set $v = 0$ when the body is at the aphelion. A parameterization of $\mathbb{T}_{G,\mathcal{J}_0}$ is then given by

$$\mathbb{T}_{G,\mathcal{J}_0} = \{(r, y, \phi, G) : r = r_G(v), y = y_G(v), \phi = \xi + \alpha_G(v) - v\}$$

where we have

$$\xi \in \mathbb{R} \bmod 2\pi, \quad v \in \mathbb{R} \bmod T_{\mathcal{J}_0}(G)$$

where $T_{\mathcal{J}_0}(G)$ is the period of a Keplerian orbit of angular momentum G in the level set of the Jacobi constant $\mathcal{J}_0$ as defined in (5.9), and $r = r_G(v),\ y = y_G(v),\ \alpha_G(v)$ are are the unique solutions to the Keplerian equations of motion (5.2) with angular momentum G satisfying the initial condition $\alpha_G(0) = 0$ with Keplerian energy $h = \mathcal{J}_0 + G$.

We will work always in a fixed level set of the Jacobi constant. We recall that in a such a level set of the Jacobi constant each torus $\mathbb{T}_{G,\mathcal{J}_0}$ is specified uniquely by choosing one of:

- a specific value of angular momentum G
- the Keplerian energy of the orbit h
- the eccentricity of the orbit e
- the period of the Keplerian orbit T

With the previous discussion (section 5.1.6) in mind, we consider small perturbations of the Kepler problem and take advantage of the fact that perturbed orbits may be written (for some time) as a graph over the original Keplerian orbits. Whence, we fix the path in configuration space and use the Hamilton-Jacobi equation to describe the evolution of the conjugate variables. Accordingly, we will parameterize orbits in this section by their return time which is, by design, the same in both the perturbed and unperturbed case.

In later sections, we will often switch parameters depending on what is convenient, writing e.g. $(r_e(v), y_e(v))$ for the unique solutions to the integrable system 5.2 with eccentricity e in the level set of the Jacobi constant $\mathcal{J}^{-1}(\mathcal{J}_0)$. The conversions between these parameters can be found in appendix A.

5.2.2 The Integrable Poincaré Mapping

We now reduce the system to one of an area-preserving map of a cylinder by the use of a Poincaré section, which is chosen in the integrable case as follows: working in the level set $\mathcal{J}^{-1}(\mathcal{J}_0)$ consider the surface given by $y = 0$. This set comprises of two cylinders transverse to the Hamiltonian flow for which the invariant circle $\mathcal{J}_0 + G = 0$ corresponds to the parabolic orbit of the Kepler problem. Label the two cylinders $\mathcal{C}_a$, $\mathcal{C}_p$, where $\mathcal{C}_a$ is now a cylinder consisting of orbits at their aphelion and $\mathcal{C}_p$ a set of orbits at their perihelion. Recall that the values of r at the aphelion, r_a, and perhelion r_p are given by

$$r_a = \frac{G^2}{1-e^2} \quad \text{and} \quad r_p = \frac{G^2}{1+e^2} \tag{5.24}$$

respectively. Further recall that the eccentricity of the orbit e is given by $e = \sqrt{1-2h^2G}$ and that as we are in $\mathcal{J}^{-1}(\mathcal{J}_0)$ we have $G = h - \mathcal{J}_0$. Finally recalling that the period of the orbit is a function of energy only, $T = \frac{\pi}{\sqrt{2}} h^{-3/2}$, we can parameterise the value of r on $\mathcal{C}_a, \mathcal{C}_p$ as functions $r = r_a(T), r = r_p(T)$ respectively. Whence, in the Keplerian case, we have

$$\mathcal{C}_a = \{(r, y, \phi, G) | r = r_a(T), y = 0\} \tag{5.25}$$

$$\mathcal{C}_p = \{(r, y, \phi, G) | r = r_p(T), y = 0\} \tag{5.26}$$

In the integrable case, the Hamilton-Jacobi equation reads

$$H\left(r, \partial_r S, \phi, \partial_\phi s\right) - G = \mathcal{J}_0 \tag{5.27}$$

where $H(r, y, \phi, G)$ is the Keplerian Hamiltonian as defined by (5.1). As G is constant in the Keplerian case, finding a generating function for the integrable case is simply

$$S_0(r, \phi) = G\phi + f(r) \tag{5.28}$$

where where $f(r)$ is some solution of the equation

$$\frac{1}{2}\left(\partial_r f\right)^2 + \frac{G^2}{2r^2} - \frac{1}{r} = h$$

.

5.2.3 The Poincaré Section of the Perturbed Mapping

With the previous discussion in mind, we reformulate the Poincaré map following way: Let $\mathcal{C}_a$ be the cylinder contained in $\mathcal{J}^{-1}(\mathcal{J}_0)$ defined by

$$\mathcal{C}_a = \{(r, y, \phi, G) | r = r_a(T), y = g(\phi, r_a(T), G(T)), G = G(T)\}$$

where $g(\phi, r_a(T), G(T))$ is small and chosen so that the condition

$$\mathcal{J}(r_a(T), g(\phi, r_a(T), G(T)), \phi, G(T)) = \mathcal{J}_0. \tag{5.29}$$

is fulfilled. Define a family of circles $\mathcal{S}_T(\phi)$ in C_a by

$$\mathcal{S}_T(\phi) = \{(\phi, G) \in \mathcal{C}_a | G = G(T)\}.$$

We now define the "forward and backward circles" of each $\mathcal{S}_T$ as the images of $\mathcal{S}_T$ in forward and backward time respectively. Explicitly, denoting the Hamiltonian flow by Φ_H^t we have

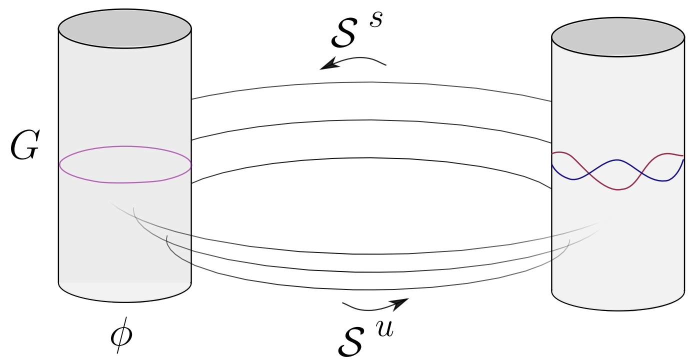


Figure 5.3: An impression of a level energy set of $\mathcal{J}_0$. The Hamiltonian flow maps $\mathcal{C}_a$ to $\mathcal{C}_p$. Each $\mathcal{S}_T$ is associated to an $\mathbb{S}^1$-parameterised set of Keplerian ellipses of eccentricity $e(T) = \frac{\sqrt{T^2-4\pi^2}}{T}$. We integrate along these orbits in forward and backward time to estimate the Poincaré mapping $\mathcal{C}_p \to \mathcal{C}_p$

$$\mathcal{S}_T^s(\phi, t) = \Phi_H^{-t}(\mathcal{S}_T(\phi)), \quad \mathcal{S}_T^u(\phi, t) = \Phi_H^{+t}(\mathcal{S}_T(\phi)) \tag{5.30}$$

We can now define the Poincaré section

$$\mathcal{C}_p = \{(r, y, \phi, G) | r = r_p(T), y = y(r, \phi, G)\} \tag{5.31}$$

Note that now we have G as a free variable, and $y = y_p(r, \phi, G)$ is then determined by the condition

$$H(r, y, \phi, G) - G = \mathcal{J}_0.$$

and denoting by $T_R^u(\phi)$ the time taken for $p \in \mathcal{S}_T(\phi)$ to reach the cylinder in forward time and $T_R^s(\phi)$ the time taken for $p \in \mathcal{S}_T(\phi)$ to reach the cylinder in backward time, the Poincaré map $F : \mathcal{C}_p \to \mathcal{C}_p$ as is given by

$$F(\phi_0, G_0) = (\phi_1, G_1) \tag{5.32}$$

where

$$G_0 = \mathcal{S}_T^s(\phi_0, T_R^s) \quad G_1 = \mathcal{S}_T^u(\phi_1, T_R^u) \tag{5.33}$$

As we expect solutions to be close to the integrable solutions, we expect $T_R^{s,u}$ to be close to $T/2$. Analogous to the setting in [33], we will search for the expression or these backward and forward circles by writing solutions to the Hamilton-Jacobi as graphs of generating functions close to that of the Kepler problem. That is, we will search for a generating function $S(\phi, r)$ which satisfies the Hamilton-Jacobi equation

$$\mathcal{J}\left(r, \phi, \partial_r s, \partial_\phi s; \mu\right) = \mathcal{J}_0.$$

where $S(\phi, r)$ is the unique solution to the Hamilton-Jacobi equation satisfying

$$\partial_\phi S(\phi, r_p(T), 0)|_{r_p(T), \phi_a} = \mathcal{S}_T(\phi_a) \tag{5.34}$$

$$\partial_r S(\phi, r_p(T), 0)|_{r_p(T), \phi_a} = f\left(\phi, r_a(T), G(T)\right) \tag{5.35}$$

and split the generating function into $S = S_0 + S_1$ where S_0 is a solution to the Hamilton Jacobi equation of the integrable system as given in (5.28), positing that S_1 is small. To search for the solution S_1, we will reparameterise the Hamilton-Jacobi equation through the unperturbed Keplerian orbits writing

$$\mathcal{T}_0(\xi, v) = S_0(\phi(\xi, v), r(\xi, v)), \quad \mathcal{T}_1(\xi, v) = S_1(\phi(\xi, v), r(\xi, v))$$

The change of variable $(r, \phi) \to (r(v), \phi(v, \xi))$ implies that the solutions of the equations of motion are given

$$r = r_e(v) \tag{5.36}$$

$$y = y_e(v) + y_e(v)^{-1}\left(\partial_v T_1^{u,s}\left(v, \xi; \mu, G_0\right) - r_e(v)^{-2}\partial_\xi T_1^{u,s}\left(v, \xi; \mu, G_0\right)\right) \tag{5.37}$$

$$\phi = \xi + \alpha_e(v) \tag{5.38}$$

$$G = G_0 + \partial_\xi T_1^{u,s}\left(v, \xi; \mu, G_0\right) \tag{5.39}$$

For each Keplerian torus consisting of Keplerian orbits of period T we will find a generating function $\mathcal{T}_T$ such that $\mathcal{T}_1 = \mathcal{T}_0 + \mathcal{T}_T$ is the unique solution to the Hamilton Jacobi equation with initial conditions

$$\begin{aligned}
&\lim_{v \to -T/2} y_e(v)^{-1}\partial_v \mathcal{T}_T^u\left(v, \xi; \mu, G_0\right) = f(\phi(\xi), r_a(T), T), \quad \partial_\xi \mathcal{T}_T^u\left(-T/2, \xi; \mu, G_0\right) = 0 \\
&\lim_{v \to T/2} y_e(v)^{-1}\partial_v \mathcal{T}_T^s\left(v, \xi; \mu, G_0\right) = f(\phi(\xi), r_a(T), T), \quad \partial_\xi \mathcal{T}_T^s\left(v, \xi; \mu, G_0\right) = 0
\end{aligned} \tag{5.40}$$

where $f(\phi(\xi), r_a(T), T)$ is the function given by (5.29) and we recall $T = T(G_0)$. We remark that $g(\phi, r_a(T), G(T))$ is very small, but cannot be set to zero as we need the Poincaré section to be contained in a level energy set.

5.2.4 The Reparameterized Hamilton-Jacobi Equation

Following the method of [33], as discussed in section 5.1.6 we write the Hamilton-Jacobi equation for a generating function $\mathcal{T}_T(\xi, v)$

$$\begin{aligned} \partial_v \mathcal{T}_T - G^3 \partial_\xi \mathcal{T}_T + \frac{1}{2y_T^2}\left(\partial_v \mathcal{T}_T - \frac{1}{r_T^2}\partial_\xi \mathcal{T}_T\right)^2 + \frac{1}{2r_T^2}\left(\partial_\xi \mathcal{T}_T\right)^2 \\ -V\left(r_T, \xi + \alpha_T - v; \mu, G\right) = 0 \end{aligned} \tag{5.41}$$

where (r_T, y_T, α_T) are the parameterizations of the r, y and α variables along the Keplerian orbit of period T in the level set $\mathcal{J}^{-1}(\mathcal{J}_0)$, which will given explicitly in section 5.3. Define the generating functions of the backward and forward circles on $\mathcal{C}_p$ as

$$\mathcal{T}_T^s(\xi) = -\mathcal{T}_T(\xi, -T/2), \quad \mathcal{T}_T^u(\xi) = \mathcal{T}_T(\xi, +T/2) \tag{5.42}$$

We can now write the Poincaré mapping (5.32) via the parameters T, ξ as

$$(\phi_0, G_0) \to (\phi_1, G_1) \tag{5.43}$$

where ϕ_0, ϕ_1, G_0, G_1 are given by

$$\phi_0 = \phi(\xi, -T/2), \quad \phi_1 = \phi(\xi, +T/2) \tag{5.44}$$

$$-G_0 = \frac{\partial \mathcal{T}_0}{\partial \xi}(\xi, -T/2) + \frac{\partial \mathcal{T}_T^s}{\partial \xi}(\xi), \quad G_1 = \frac{\partial \mathcal{T}_0}{\partial \xi}(\xi, T/2) + \frac{\partial \mathcal{T}_T^u}{\partial \xi}(\xi) \tag{5.45}$$

To express the Poincaré map as the difference of $\mathcal{T}_T^u(\xi, v)$, $\mathcal{T}_T^s(\xi, v)$ we simply write $G_1 = G_0 + (G_1 - G_0)$

$$\begin{aligned} \phi_1 &= \phi_0 + T(\phi_0, G_0) \\ G_1 &= G_0 + \frac{\partial}{\partial \xi}(\mathcal{T}_T^u(\xi, v) - \mathcal{T}_T^s(\xi, v)) \end{aligned} \tag{5.46}$$

Observe that the functions $\mathcal{T}_T^u(\xi, v)$ as $\mathcal{T}_T^s(\xi, v)$ above satisfy the following

$$\partial_\xi \mathcal{T}_T^u\left(-T/2, \xi; \mu, G\right) = 0 \quad \partial_\xi \mathcal{T}_T^s\left(T/2, \xi; \mu, G\right) = 0 \tag{5.47}$$

with conditions similar to (5.22).

5.2.5 The Parameterized Melnikov Function

With the similarity of our generating functions and those of [33] in mind, we now posit that the integrals $\mathcal{L}_T^s, \mathcal{L}_T^u$ are good approximations of the above generating functions $\mathcal{T}^s, \mathcal{T}^u$ where $\mathcal{L}_T^s, \mathcal{L}_T^u$ are defined by

$$\mathcal{L}_T^s(\xi) = \int_0^{-T/2} V\left(r_T\left(v_T\right), \xi + \alpha_T\left(v_T\right) - v_T\right) dv_T$$

$$\mathcal{L}_T^u(\xi) = \int_0^{T/2} V\left(r_T\left(v_T\right), \xi + \alpha_T\left(v_T\right) - v_T\right) dv_T$$

where $r_T(v), \alpha_T(v)$ are the solutions to the Keplerian equations of motion with initial conditions in the circle $\mathcal{S}_T(\phi) \in \mathcal{C}_p$. The difference

$$\mathcal{L}_T(\xi) = \mathcal{L}_T^u(\xi) - \mathcal{L}_T^s(\xi) = \int_{-T/2}^{T/2} V(r_T(v), \xi + \alpha_T(v) - v)dv$$

then gives a good approximation to the function $\mathcal{T}_T^u - \mathcal{T}_T^s$ necessary to find the Poincaré map (5.46). In the next section we concern ourselves with the calculation of the above integral.

5.3 Calculation of the Melnikov Potential

We will now, for convenience, switch the variable parameterizing the Keplerian tori to e. Consider the Kepler problem in rotating coordinates and a level energy set of the Jacobi constant $\mathcal{J}^{-1}(\mathcal{J}_0)$. As before, let v be a parameter parameterizing Keplerian orbits of angular momentum G contained in $\mathcal{J}^{-1}(\mathcal{J}_0)$, ξ the angle of the body at the apoapsis and $r(v), y(v), \phi(v, \xi)$ the corresponding solutions to the Kepler problem, i.e.

$$\frac{d}{dv}r(v) = y(v) \tag{5.48}$$

$$\frac{d}{dv}y(v) = \frac{G^2}{r(v)^3} - \frac{1}{r(v)^2} \tag{5.49}$$

$$\frac{d}{dv}\phi(v) = \frac{d\alpha(v)}{dv} - 1 \tag{5.50}$$

$$= \frac{G}{r(v)^2} - 1 \tag{5.51}$$

For convenience we set $v = 0$ at the perihelion (note that then $v(v) = v - T/2$). To simplify calculations, we reparameterize time by

$$\frac{dt}{d\tau} = 2Gr \tag{5.52}$$

so that the system (5.2) becomes

$$\frac{d}{d\tau}r = 2Gry \quad \frac{d}{d\tau}y = 2G\left(\frac{1}{r^2} - \frac{1}{r}\right) \tag{5.53}$$

$$\frac{d}{d\tau}\alpha = \frac{2G^2}{r} \quad \frac{d}{d\tau}G = 0 \tag{5.54}$$

The advantage of this is a good parameterization for the orbits of the Kepler problem

Lemma 5.3.1. *Let $e < 1$. Then for a Keplerian orbit of eccentricity e we have the following expression for $t(\tau)$, where we choose the convention $t = 0$ when the body is at the periapsis,*

$$t(\tau) = \frac{2G^3}{(1-e^2)^{3/2}}(\sqrt{1-e^2}\tau - e\sin(\sqrt{1-e^2}\tau))$$

and the solutions of the Kepler problem have the following parameterizations in terms of τ:

$$r(\tau) = \frac{G^2}{1-e^2}(1 - e\cos\sqrt{1-e^2}\tau) \tag{5.55}$$

$$\alpha(\tau) = 2\arctan\left(\sqrt{\frac{1+e}{1-e}}\tan\left(\sqrt{1-e^2}\frac{\tau}{2}\right)\right) + \alpha_0. \tag{5.56}$$

Proof. Remark that for $e < 1$ we have $E = \sqrt{1-e^2}\tau$ where E is the eccentric anomaly. The below equations giving parameterizations of elliptic Keplerian orbits in terms of the eccentric anomaly E are well known:

$$r = \frac{G^2}{1-e^2}(1 - e\cos E) \tag{5.57}$$

$$\tan\frac{\alpha}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}. \tag{5.58}$$

The equation

$$t(E) = \frac{2G^3}{(1-e^2)^{3/2}}(E - e\sin(E)) \tag{5.59}$$

can be found by direct integration, or by comparing $t(E)$ to the mean anomaly $M(E)$. For details see appendix A. We will switch between the parameters τ and E depending on convenience. □

Upon a similar rescaling to that in section 5.1.2 and translation of the angle

$$r = G^2 r_e, \quad y = G^{-1} y_e, \quad \alpha - \xi - \frac{T}{2} = \alpha_e, \quad t = G^3 t_e \tag{5.60}$$

The Keplerian equations of motion become

$$\frac{d}{d\tau} r_e(\tau) = y_e(\tau), \quad \frac{d}{d\tau} y_e(\tau) = \frac{1}{r_e(\tau)^3} - \frac{1}{r_e(\tau)^2} \tag{5.61}$$

$$\frac{d}{d\tau} \alpha_e(\tau, \xi) = \frac{1}{r_e(\tau)^2}, \quad \frac{d}{d\tau} \phi_e(\tau, \xi) = \frac{1}{r_e(\tau)^2} - G^3, \tag{5.62}$$

and the solutions $r_e(\tau), y_e(\tau), \alpha_e(\tau)$ become functions of e only. We note that α_e now represents the true anomaly of the orbit. In the rescaled variables we have that the potential is given by

$$V\left(r_e, \phi_e; \mu, G\right) = G^2 V\left(r, \phi; \mu\right).$$

We then define the Melnikov potential as the integral

$$\mathcal{L}\left(\xi; e, \mu, \mathcal{J}_0\right) = G^{-2} \int_{-\frac{T}{2}}^{\frac{T}{2}} V\left(r_e(v), \phi_e\left(\xi\right); \mu\right) dv, \tag{5.63}$$

where $T = T(\mathcal{J}_0, e)$ denotes the period of the Keplerian orbit of eccentricity e and $G^{-2} V\left(r_e(v), \phi_e(v, \xi); \mu\right)$ the difference in the potential between the Keplerian and restricted three-body problem as defined in equation (5.4), evaluated along the unperturbed orbit.

The value of these integrals is most readily expressed in terms of the Keplerian angular momentum G as a parameter rather than T or e. Therefore we will write $\mathcal{L}\left(\xi; e, \mu, \mathcal{J}_0\right)$ as a function $\mathcal{L}(\xi; \mathcal{J}_0, G, \mu)$.

In order to calculate the above integrals, we split $\mathcal{L}(\xi; \mathcal{J}_0, G, \mu)$ into the dominant terms of its Fourier series in ξ and a remainder term.

$$\begin{aligned} \mathcal{L}\left(\xi; \mathcal{J}_0, G, \mu\right) = \mathcal{L}^{[0]}\left(\mathcal{J}_0, G, \mu\right) &+ \mathcal{L}^{[1]}\left(\xi; \mathcal{J}_0, G, \mu\right) \\ &+ \mathcal{L}^{[2]}\left(\xi; \mathcal{J}_0, G, \mu\right) + \mathcal{L}^{[\geq 3]}\left(\xi; \mathcal{J}_0, G, \mu\right) \end{aligned}$$

where $\mathcal{L}^{[0]}, \mathcal{L}^{[1]}, \mathcal{L}^{[2]}$ represent the harmonics of order $0, 1$ and 2 respectively and $\mathcal{L}^{[\geq 3]}$ those of higher order. The rest of the this section will be devoted to the calculation of $\mathcal{L}\left(\xi; \mathcal{J}_0, G, \mu\right)$. We will show:

Theorem 5.3.2. *. Let $\mathcal{L}(\xi; \mathcal{J}_0, G, \mu)$ be the Melnikov potential function of equation (5.63) for an orbit of angular momentum G and Jacobi constant $\mathcal{J}_0$. Then*

$$\mathcal{L}(\xi; \mathcal{J}_0, G, \mu) = \mathcal{L}^{[0]}(\xi; \mathcal{J}_0, G, \mu) + 2\sum_{\ell=1}^{+\infty} L^{[\ell]}(\xi; \mathcal{J}_0, G, \mu)\cos(\ell\xi) \tag{5.64}$$

where

$$L^{[1]}(G, \mu) = -\mu(1-\mu)\sqrt{\pi}\frac{1-2\mu}{4\sqrt{2}}G^{-3/2}e^{-\frac{G^3}{3}}\left(1+\mathcal{O}\left(G^{-3/2}, \sqrt{1-e^2}\right)\right) \tag{5.65}$$

$$L^{[2]}(G, \mu) = -2\mu(1-\mu)\sqrt{\pi}G^{1/2}e^{-\frac{2G^3}{3}}\left(1+\mathcal{O}\left(G^{-1/2}, \sqrt{1-e^2}\right)\right) \tag{5.66}$$

$$L^{[\ell]}(G, \mu) = \mathcal{O}\left(G^{\ell-3/2}e^{-\frac{\ell G^3}{3}}(1+\mathcal{O}\sqrt{1-e^2})\right), \textit{ for } \ell \geq 2. \tag{5.67}$$

Here, $L^{[2]}$ will give the dominant term of the generating function for μ close to $1/2$. To do this, in Section 5.3.1 we perform a rescaling of the perturbation potential of the PCR3BP similar to that of [33]. Then in Section 5.3.3 we define the complex path along which the integral will be taken. In Section 5.3.4 we compare the integral to the usual Melnikov integral along the homoclinic orbit. Finally, in Sections 5.3.5 and 5.3.6 we calculate the integral by employing the results of [43] which calculate the integral along the homoclinic orbit and bounding the difference.

5.3.1 Rescaling and Expansion of the Potential Function

We note that as we have an integral of a quickly oscillating perturbation $V = V(\alpha_e(v) - G^3 v)$, the dominant terms will be given by the lower order terms of the Fourier series of the function. With this in mind, the potential function is expanded into a Fourier series rather than a power series in the perturbation parameter as found in [33].

Lemma 5.3.3. *The potential $G^{-2}V(r_e(v), \phi(v,\xi))$ admits the following expansion*

$$\frac{1}{G^2}V(r_e(v), \phi(v,\xi)) = \sum_{\ell\in\mathbb{Z}} e^{i\ell\phi(v,\xi)} \sum_{j\geq\max\{0,-\ell\}} c_j c_{j+\ell} \times \frac{(-1)^\ell(1-\mu)\mu^{2j+\ell} + \mu(1-\mu)^{2j+\ell}}{G^{4j+2\ell+2}r_e^{2j+\ell+1}(v)} - \frac{1}{G^2 r_e(v)},$$

where $c_j = \begin{pmatrix} -1/2 \\ j \end{pmatrix}$, $\delta_0(0) = 1$ *and* $\delta_0(\ell) = 0$ *for* $\ell \neq 0$.

Using $\phi_e(v,\xi) = \alpha_e(v) - G^3 t_e(v) + \xi$ the Melnikov potential 5.63 is given by

$$\mathcal{L}(\xi; G, e, \mu) = \sum_{\ell \in \mathbb{Z}} e^{i\ell\xi} \sum_{j \geq \max\{0,-\ell\}} \mathcal{N}_e(G,\mu)(\ell, j)$$

where

$$\mathcal{N}_e(G,\mu)(\ell,j) = \frac{c(\mu,\ell,j)}{G^{4(j+1)}} \int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{e^{i\ell\alpha_e(v)} e^{-i\ell G^3 t_e(v)}}{r_e^{j+2}(v)} dv \tag{5.68}$$

$$= \frac{2c(\mu,\ell,j)}{G^{4(j+1)-1}} \int_{-\frac{\pi}{\sqrt{1-e^2}}}^{\frac{\pi}{\sqrt{1-e^2}}} \frac{e^{i\ell\alpha_e(\tau)} e^{-i\ell G^3 t_e(\tau)}}{r_e^{j+1}(\tau)} d\tau \tag{5.69}$$

$$= \frac{2c(\mu,\ell,j)}{G^{4(j+1)-1}} \int_{-\frac{\pi}{\sqrt{1-e^2}}}^{\frac{\pi}{\sqrt{1-e^2}}} \frac{e^{i\ell\alpha_e(t)} e^{-i\ell G^3 t_e(\tau)}}{(1 + e\cos(\sqrt{1-e^2}\tau))^{j+1}} \tag{5.70}$$

$$= \frac{2c(\mu,\ell,j)}{G^{4(j+1)-1}} \mathcal{I}_e(\ell,j) \tag{5.71}$$

where we have defined for convenience

$$c(\mu,\ell,j) = c_j c_{j+\ell} \left((-1)^\ell (1-\mu)\mu^{2j+\ell} + \mu(1-\mu)^{2j+\ell} \right),$$

$$\mathcal{I}_e(\ell,j) = \int_{-\frac{\pi}{\sqrt{1-e^2}}}^{\frac{\pi}{\sqrt{1-e^2}}} \frac{e^{i\ell\alpha_e(t)} e^{-i\ell G^3 t_e(\tau)}}{(1 + e\cos(\sqrt{1-e^2}\tau))^{j+1}} \tag{5.72}$$

and we remark that the period of the Keplerian orbit can be found easily in the terms of τ, from the relation $E = \sqrt{1-e^2}\tau$ and $T(E,h) = 2\pi$.

This family of integrals becomes singular at $e = 1$. However, one can show that upon taking limits

$$\lim_{e \to 1} \mathcal{I}_e(\ell,j) = \int_{-\infty}^{+\infty} \frac{\mathrm{e}^{i\ell\left(\tau + \tau^3/3\right)G^3/2}}{(\tau - i)^{2j}(\tau + i)^{2j+2\ell}} d\tau := \mathcal{I}(\ell,j)$$

where $\mathcal{I}(\ell,j)$ is an integral corresponding to the usual Melnikov potential evaluated along the homoclinic orbits of the Kepler problem, the value of which was computed in [43].

5.3.2 Symmetries of the Potential Function

We note that the potential $V(r,\phi)$ satisfies the symmetry

$$V(r,-\phi) = V(r,\phi)$$

As $V(r,-\phi)$ is an even function, then, the Fourier coefficients satisfy

$$\widehat{V}^{[\ell]}(v) = \widehat{V}^{[-\ell]}(v)$$

and so

$$\mathcal{I}(\ell, j) = \mathcal{I}(-\ell, j)$$

Thus it is sufficient to compute the integrals for the case $\ell < 1$. We also note the symmetries

$$r_e(-v) = r_e(v), t_e(-v) = -t(v), \alpha_e(-v) = -\alpha(v) \tag{5.73}$$

which will be employed later.

5.3.3 The Complex Path

We recall that the presence of the term $\exp(i\ell G^3 t_e(\tau))$ in the integrals (5.68) means that we are dealing with a quickly oscillating integral for G large. To compute the integral, we will rely on the method of [43], which computed a similar integral using the method of steepest descent, which deforms the contour of integration so that it passes through a stationary point of the phase $t_e(\tau)$ in such a way that highly-oscillating nature of the integrand is eliminated. The dominant terms of the integral can then be derived from the form of the function at the stationary point $t'_e(\tau) = 0$.

To define such a path, we note the functions $t_e(\tau), \alpha_e(\tau), r_e(\tau)$ defined by the equations (5.60, 5.3.1) all have a natural analytic extension to the complex plane and we look for a path Γ in the complex plane satisfying

- $t_e(\tau)$ is purely imaginary
- Γ passes through the zeros of $t'_e(\tau)$

However, along this path the integrand has a singularity at $t'_e(\tau) = 0$ and it is necessary to estimate the integral by deforming Γ in such a way as to avoid the singularity. This is achieved by adding a circle Γ_3 of radius ε around the singularity, as in figure 5.4. Applying the Cauchy-Goursat theorem to the path in figure 5.4 we can conclude that

$$\mathcal{I}_e(\ell, j) = \int_{-\frac{T}{2}}^{\frac{T}{2}} \mathcal{F}_e(\ell, j) dt = \int_{\gamma_1 \cup \gamma_2} \mathcal{F}_e(\ell, j) d\tau + \int_{\Gamma} \mathcal{F}_e(\ell, j) d\tau$$

where we have defined for convenience

$$\mathcal{F}_e(\ell, j) = \frac{e^{i\ell\alpha_e(t)}e^{-i\ell t_e(\tau)}}{(1 + e\cos(\sqrt{1-e^2}\tau))^{j+1}} \tag{5.74}$$

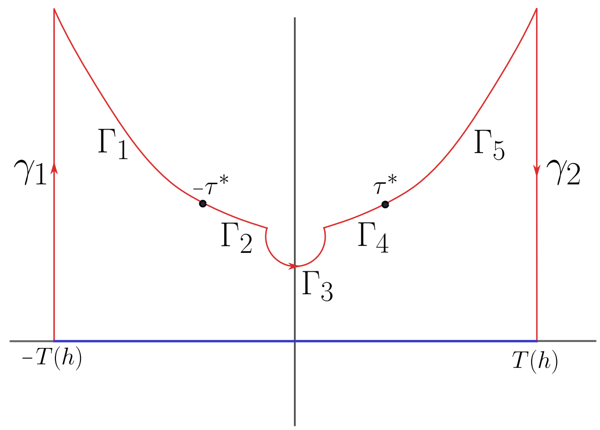


Figure 5.4: The chosen path

To define the path Γ we write

$$h_e(\tau) = it_e(\tau) = i\left(\frac{\sqrt{1-e^2}\tau - e\sin(\sqrt{1-e^2}\tau)}{(1-e^2)^{\frac{3}{2}}}\right) \tag{5.75}$$

Then, recalling the parameterization (5.52) we find

$$t_e'(\tau) = 0 \iff r_e(\tau) = 0$$

So that stationary points of the phase $t_e(\tau)$ then correspond to singularities of the potential function which occur for $r(\tau) = 0$. As noted in [33], these singularities then can be considered as collisions for complex values of time.

To find such stationary points we use the expression for $r(\tau)$ given in equation (5.55) to note that in the complex strip

$$B_T = \{\tau \in \mathbb{C} : |\Re(\tau)| < T\}$$

$r_e(\tau)$ has exactly two zeros

$$r_e(\tau) = 0 \iff \tau = \pm\tau_0(e)$$

where

$$\tau_0(e) = \frac{i\,\mathrm{arccosh}\left(\frac{1}{e}\right)}{\sqrt{1-e^2}} \tag{5.76}$$

We now define a function which will describe the phase along the contour Γ

$$u_{\mathrm{h}}(\tau) = h_e(\tau_0) - h_e(\tau)$$

where the path Γ is given by

$$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_5$$

with components

$$\Gamma_1 = \{\tau \in \mathbb{C} : \mathrm{Im}(u_e(\tau)) = 0\} \cap \left\{\tau \in \mathbb{C} : -\frac{\pi}{\sqrt{1-e^2}} \leq \mathrm{Re}(\tau) \leq \mathrm{Re}\left(-\tau^*\right)\right\}$$

$$\Gamma_5 = \{\tau \in \mathbb{C} : \mathrm{Im}(u_e(\tau)) = 0\} \cap \left\{\tau \in \mathbb{C} : \frac{\pi}{\sqrt{1-e^2}} \geq \mathrm{Re}(\tau) \geq \mathrm{Re}\left(\tau^*\right)\right\}$$

$$\Gamma_2 = \{\tau \in \mathbb{C} : \mathrm{Im}(u_e(\tau)) = 0\} \cap \{\tau \in \mathbb{C} : \mathrm{Re}\left(-\bar{\tau}^*\right) \leq \mathrm{Re}(\tau) \leq 0\} \cap \{\tau \in \mathbb{C} : |\tau - i| \geq c\varepsilon\}$$

$$\Gamma_4 = \{\tau \in \mathbb{C} : \mathrm{Im}(u_e(\tau)) = 0\} \cap \{\tau \in \mathbb{C} : 0 \leq \mathrm{Re}(\tau) \leq \mathrm{Re}\left(\tau^*\right)\} \cap \{\tau \in \mathbb{C} : |\tau - i| \geq c\varepsilon\}$$

$$\Gamma_3 = \{\tau \in \mathbb{C} : \mathrm{Im}(u_e(\tau)) \leq 0\} \cap \{\tau \in \mathbb{C} : |\tau - \tau_0| = c\varepsilon\}$$

and τ^* is a (positive) constant to be specified.

Finally we define γ_1 and γ_2 by

$$\gamma_1 = \{\tau \in \mathbb{C} : \mathrm{Re}(\tau) = -\frac{\pi}{\sqrt{1-e^2}}, \mathrm{Im}(\tau) \leq \tau^{**}\} \tag{5.77}$$

$$\gamma_2 = \{\tau \in \mathbb{C} : \mathrm{Re}(\tau) = \frac{\pi}{\sqrt{1-e^2}}, \mathrm{Im}(\tau) \leq \tau^{**}\} \tag{5.78}$$

where τ^{**} is simply the imaginary part of intersection of the line $\Re(\tau)$ and Γ_1 (equivalently Γ_5)

$$\tau^{**} = \Gamma_1 \cap \{\tau \in \mathbb{C} \mid \mathrm{Re}(\tau) = -\frac{\pi}{\sqrt{1-e^2}}\},$$

To compute an integral similar to 5.63, [43] noted the fact that the integral is ultimately ε independent and so the integral around Γ_3 is given in the limit as $\varepsilon \to 0$ by $2\pi i$ times the residue of the pole. For $\Gamma_1 \cup \Gamma_2$ and $\Gamma_3 \cup \Gamma_5$ one can then compare the ε-independent terms of the paths Γ_2 and Γ_3 to a standard integral (that of the Γ-function) and bounding the difference. These terms give the dominant terms of the integral and finally those of Γ_1 and Γ_5 are bounded. To compute the integrals $\mathcal{I}_e(\ell, j)$ found in (5.72) we will compare the integrands $\mathcal{F}_e(\ell, j)$ to those of [43] in the complex plane, and use the methods there to calculate $\mathcal{I}_e(\ell, j)$ up to terms of the form $e^{-\frac{G^3}{3}}\sqrt{1-e^2}$. For G large and $\sqrt{1-e^2}$ exponentially small in G, then, these terms can be safely discarded.

We will examine the integral only for those Fourier coefficients of $V(r, \phi; \mu, G)$ giving the dominant terms in G and then bound the rest. As the integral is quickly oscillating, the dominant terms will be associated to the lowest order terms in the Fourier series, i.e. $\mathcal{N}_e(G, \mu)(\pm 1, j)$. Furthermore, these integrals will be shown to decrease with increasing j. For $\mu \neq \frac{1}{2}$ the dominant term is then associated to the integral.

$$\mathcal{I}_e(\pm 1, 1) = \int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{e^{-i\phi_e(\tau)}}{r_e(\tau)^3} d\tau \tag{5.79}$$

For μ very close to $1/2$ one needs to investigate higher order terms. The dominant Fourier term will be given by

$$\mathcal{I}_e(\pm 2, 1) = \int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{e^{-i2\phi_e(\tau)}}{r_e(\tau)^5} d\tau \tag{5.80}$$

5.3.4 Comparisons to the Homoclinic Orbit

In order to calculate the integrals (5.72) we will compare the parameterizations $(r_e(v), \alpha_e(v))$ to those of the homoclinic solution $(r_h(v), \alpha_h(v))$. We can then compare the integrals (5.72) to those of the components of the true Melnikov potential

$$L(0, \xi) = \mathcal{L}(\xi) = \int_{-\infty}^{\infty} V(r_{\rm h}(v), \xi + \alpha_{\rm h}(v) - v; \mu) dv$$

as computed in ([43]). The expressions $r_h(v), \alpha_{\rm h}(v)$, as solutions to the system 5.20 are given by the following Lemma:

Lemma 5.3.4. *Let $\tau(v)$ be the unique analytic function defined by*

$$v = \frac{1}{2}\left(\frac{1}{3}\tau^3 + \tau\right)$$

with the convention that τ is real for real values of v that is,

$$\tau(v) = (3v + \sqrt{9v^2+1})^{1/3} - (3v + \sqrt{9v^2+1})^{-1/3}.$$

Then, the homoclinic orbit satisfying (5.20) has the following properties:

$$r_{\rm h}(v) = r_{\rm h}(\tau(v)) \tag{5.81}$$

and $\alpha_{\rm h}(v) = \alpha_{\rm h}(\tau(v))$, *where*

$$r_{\rm h}(\tau) = \frac{1}{2}\left(\tau^2 + 1\right) \tag{5.82}$$

$$\alpha_{\rm h}(\tau) = 2\arctan(\tau) = -i\log\left(\frac{i-\tau}{i+\tau}\right). \tag{5.83}$$

time along the homoclinic solution is then given by

$$t(\tau) = v = \frac{1}{2}\left(\frac{1}{3}\tau^3 + \tau\right)$$

We then have the following expressions for the quantities expressed in (5.3.3)

$$\begin{aligned} h_{\rm h}(\tau) &= i\left(\frac{\tau^3}{3} + \tau\right) \\ u_{\rm h}(\tau) &= h_{\rm h}(i) - h_{\rm h}(\tau) = -\frac{2}{3} - i\left(\frac{\tau^3}{3} + \tau\right) \\ &= (\tau - i)^2 - \frac{i}{3}(\tau - i)^3 \end{aligned} \tag{5.84}$$

and we label the integrand along the homoclinic orbit as

$$\mathcal{F}_{\rm h}(\ell, j)(\tau) = \frac{e^{i\ell\alpha_{\rm h}(\tau)}e^{-i\ell\mathcal{J}_0^3 t_{\rm h}(\tau)}}{r_{\rm h}^{j+1}(\tau)} = \frac{{\rm e}^{i\ell\left(\tau+\tau^3/3\right)G^3/2}}{(\tau-i)^{2j}(\tau+i)^{2j+2\ell}} \tag{5.85}$$

Expressing the difference between $\mathcal{F}_{\rm h}(\ell, j)$ and $\mathcal{F}_e(\ell, j)$ comes with minor difficulties not least because the expressions (5.55, 5.56, 5.3.1) all become singular at $e = 1$. Moreover, the singularity of the Keplerian orbits are located close to, but not at the singularities of the separatrix. For this reason, to compare the integrals in the section of the complex path it is necessary to rescale the complex plane in a way that makes the singularities coincident and compare the expressions of the elliptic and parabolic integrands about their (now mutual singularities).

5.3.5 The Integral along Γ

We now expand the parameterizations of the Keplerian orbits (5.55,5.56,5.3.1) around the singularity τ_0.

Lemma 5.3.5. *$u(\tau)$ has the following expansion about $\tau = \tau_0$*

$$\sqrt{u(\tau)} = \frac{\tau - \tau_0}{\sqrt{e}}\tilde{u}(\tau) + (\sqrt{1-e^2})\left(\mathcal{O}\frac{(\tau-\tau_0)^2}{\tilde{u}(\tau)}\right)$$

where

$$\tilde{u}(\tau) = \sqrt{1 - \frac{i}{3}(\tau - \tau_0)}. \tag{5.86}$$

Proof. By construction $u(\tau_0) = 0$. Also, $u'(\tau_0) = 0$ as

$$u'(\tau) = \sqrt{1-e^2}\frac{du}{dE} = -2i\left(\frac{1 - e\cos(E)}{1-e^2}\right) = 2ir(\tau)$$

and so $u'(\tau)$ has a zero whenever $r(\tau)$ has. Writing u as a Taylor series and taking the square root we have

$$\sqrt{u(\tau)} = \sqrt{\frac{iu''(\tau_0)}{\sqrt{1-e^2}}(\tau-\tau_0)^2 + \frac{iu'''(\tau_0)}{3}(\tau-\tau_0)^3 + (\sqrt{1-e^2})\mathcal{O}(\tau-\tau_0)^4} \tag{5.87}$$

$$= (\tau-\tau_0)\sqrt{-\frac{i\sin(\tau_0)}{2\sqrt{1-e^2}} - \frac{i\cos(\tau_0)(\tau-\tau_0)}{3} + (\sqrt{1-e^2})\mathcal{O}((\tau-\tau_0)^2)} \tag{5.88}$$

$$= (\tau-\tau_0)\tilde{u}(\tau) + (\sqrt{1-e^2})\left(\mathcal{O}\frac{(\tau-\tau_0)^2}{\tilde{u}(\tau)}\right) \tag{5.89}$$

where evaluating

$$\sin\left(i\cosh^{-1}\left(\frac{1}{e}\right)\right) = \frac{i\sqrt{1-e^2}}{e}$$

and

$$\cos\left(i\cosh^{-1}\left(\frac{1}{e}\right)\right) = \frac{1}{e}$$

gives expression (5.86) for $\tilde{u}(\tau)$. □

We now perform a rescaling of the complex plane, This ensures that the singularities of our integrand are at the same location as those of the homoclinic $\tau_0 = \pm i$ and that our integral is comparable to that of [43].

Lemma 5.3.6. *Define $\tilde{\tau} = -i\tau\tau_0$ Then we have the following expressions*

$$u(\tilde{\tau}) = (\tilde{\tau} - i)^2 \left(1 - \frac{i}{3}(\tilde{\tau} - i)\right) + \mathcal{O}(\sqrt{1 - e^2}) \tag{5.90}$$

$$r_e(\tilde{\tau}) = \frac{1}{2}(\tilde{\tau} - i)(\tilde{\tau} + i) + \mathcal{O}(\sqrt{1 - e^2}) \tag{5.91}$$

$$\alpha_e(\tilde{\tau}) = \frac{\tilde{\tau} - i}{\tilde{\tau} + i} + \mathcal{O}(\sqrt{1 - e^2}). \tag{5.92}$$

Proof. The expression (5.90) is immediate upon considering the expression for u given in Lemma 5.3.5 and the expansion for

$$-i\tau = \frac{\operatorname{arccosh}\left(\frac{1}{e}\right)}{\sqrt{1 - e^2}}$$

given in Lemma B.0.1.

For equation (5.91) it is sufficient to note that

$$u(\tilde{\tau}) = -i\tau_0^3 \left(\frac{\tilde{\tau}^3}{3} + \tilde{\tau} - \frac{2}{3}\right) + \mathcal{O}(1 - e^2)$$

and recalling that $r(\tau) = \frac{1}{2i}u'(\tau)$ the result follows. Finally, for equation (5.92) we simply note using equation (5.62)

$$\frac{d\alpha_e}{d\tilde{\tau}} = \frac{i}{\tau_0}\frac{2}{r(\tilde{\tau})} \tag{5.93}$$

$$\Longrightarrow \alpha_e = \frac{2i}{\tau_0}\int \frac{1}{r(\tilde{\tau})} = \frac{i}{\tau_0}2\arctan(\tilde{\tau}) + \mathcal{O}(\sqrt{1 - e^2}) \tag{5.94}$$

$$= \frac{1}{\tau_0}\log\left(\frac{i - \tau}{i + \tilde{\tau}}\right) + \mathcal{O}(\sqrt{1 - e^2}) \tag{5.95}$$

And again using Lemma B.0.1 to get $\frac{i}{\tau_0} = 1 + \mathcal{O}(\sqrt{1 - e^2})$ we have get (5.92) as required. □

Lemma 5.3.7. *Let $h_e(\tau)$ be the function defined in 5.75 and τ_0 the zero of r defined by expression 5.76. Then we have the following $h_e(\tau_0) = -\frac{2}{3} + \mathcal{O}(1 - e)$.*

Proof. We have

$$h(\tau_0) = 2i\left(\frac{\sqrt{1-e^2}\tau_0 - e\sin\left(\sqrt{1-e^2}\tau_0\right)}{(1-e)^{3/2}}\right) \tag{5.96}$$

$$= \frac{-2}{(1-e^2)}\left(\frac{\operatorname{arccosh}\left(\frac{1}{e}\right)}{\sqrt{1-e^2}} - 1\right) \tag{5.97}$$

$$= \frac{-2}{(1-e^2)}\left(\frac{1-e^2}{3} + \mathcal{O}\left(\left(1-e^2\right)^2\right)\right). \tag{5.98}$$

where we have have used the expansion for $\frac{\operatorname{arccosh}\left(\frac{1}{e}\right)}{\sqrt{1-e^2}}$ found in B.0.1. □

We can now express the integrand $\mathcal{F}_e(\ell, j)(\tau)$ of equation (5.74) as an integrand close the $\mathcal{F}_{\mathrm{h}}(\ell, j)(\tau)$ of equation (5.85). First define the functions

$$\tilde{u}_e(\tau) = u_e(\tilde{\tau}), \tilde{r}_e(\tau) = r_e(\tilde{\tau}), \tilde{\alpha}_e(\tau) = \alpha_e(\tilde{\tau})$$

and the rescaled integrand as

$$\tilde{\mathcal{F}}_e(\ell, j)(\tau) = \mathcal{F}_e(\ell, j)(\tilde{\tau}) \tag{5.99}$$

where $\tilde{\tau}$ is the rescaling given in Lemma 5.3.1.

Lemma 5.3.8. *Let $\tilde{\mathcal{F}}_e(\ell, j)(\tau)$ be as defined by equation (5.99). Let $\mathcal{F}_{\mathrm{h}}(\ell, j)$ be as defined by equation (5.85). Then we have*

$$\tilde{\mathcal{F}}_e(\ell, j)(\tau) = \mathcal{F}_{\mathrm{h}}(\ell, j)(\tau) + e^{-2G^3/3}(\mathcal{O}(\sqrt{1-e^2})). \tag{5.100}$$

Proof. We write

$$\tilde{\mathcal{F}}_e(\ell, j)(\tau) = \frac{e^{i\ell\alpha_e(\tilde{\tau})}e^{-i\ell t_e(\tilde{\tau})}}{(r_e(\tilde{\tau}))^{j+1}} \tag{5.101}$$

$$= e^{-2\ell G^3/3}\left(\frac{e^{i\ell\alpha_e(\tilde{\tau})}e^{\ell u_e(\tilde{\tau})}}{(r_e(\tilde{\tau}))^{j+1}} + \mathcal{O}(1-e^2)\right) \tag{5.102}$$

$$= e^{-2\ell G^3/3}\left(\frac{e^{i\ell\tilde{\alpha}_e(\tau)}e^{\ell\tilde{u}_e(\tau)}}{(\tilde{r}_e(\tau))^{j+1}} + \mathcal{O}(1-e^2)\right) \tag{5.103}$$

and the result then follows from Lemma 5.60. □

We now bound the difference between the rescaled integrals

$$\tilde{\mathcal{I}}_e(\ell,j)=\int_{-\frac{T}{2}}^{\frac{T}{2}}\tilde{\mathcal{F}}_e(\ell,j)d\tau$$

and the usual Melnikov function as computed in [43].

Lemma 5.3.9. *Let $\tilde{\mathcal{F}}_e(\ell,j)(\tau)$ be as defined in equation (5.74). Then*

$$\int_{\tilde{\Gamma}_1\cup\tilde{\Gamma}_2}\tilde{\mathcal{F}}_e(\ell,j)(\tau)d\tau=\int_{\tilde{\tau}^{**}}^{+\infty}\frac{e^{i\ell\left(\tau+\tau^3/3\right)G^3/2}}{(\tau-i)^{2j}(\tau+i)^{2j+2\ell}}d\tau+\mathcal{E}_{\ell,j} \tag{5.104}$$

where $\mathcal{E}_{\ell,j}$ satisfies

$$|\mathcal{E}_{\ell,j}|<\frac{2\mathrm{e}^{-\ell\frac{G^3}{2}}}{\ell G^3}\left(\mathcal{O}(\sqrt{1-e^2})\right). \tag{5.105}$$

Proof. We note that

$$\int_{\Gamma_5}\tilde{\mathcal{F}}_e(\ell,j)(\tau)d\tau=\int_{u(\tilde{\tau}^*)}^{\infty}\tilde{\mathcal{F}}_e(\ell,j)(u)du-\int_{u(\tau^{**})}^{\infty}\tilde{\mathcal{F}}_e(\ell,j)(u)du \tag{5.106}$$

where τ^{**} is the intersection of the path γ_1 with Γ_5. We can bound the second term easily by

$$\begin{aligned}\left|e^{\ell h(i)}\mathcal{E}_{\ell,j}\right|=\left|\int_{u(T)}^{\infty}\tilde{\mathcal{F}}_{j-1,\ell}(u)du\right|&\leq\int_{\tilde{\tau}(T)}^{\infty}\frac{\mathrm{e}^{-\ell G^3u/2}}{|(\tilde{\tau}(u)-i)^{j-\ell+1}(\tilde{\tau}(u)+i)^{j+\ell+1}|}du\\&\leq\frac{2\mathrm{e}^{-\ell\frac{G^3}{2}(\tilde{\tau}(h))}}{\ell G^3}\frac{1}{|\tilde{\tau}(T)-i|^{j-\ell+1}}\frac{1}{|\tilde{\tau}(T)+i|^{j+\ell+1}}\\&\leq\frac{2\mathrm{e}^{-\ell\frac{G^3}{2}(\tilde{\tau}(h))}}{\ell G^3}\end{aligned}$$

and using $T=T(e)=\frac{\pi}{\sqrt{1-e^2}}$ we get result (5.105). □

The result for the integral along Γ_1 is equivalent. Applying the Cauchy-Goursat theorem, then, and the expression for $\mathcal{F}_e(\ell,j)$ given in Lemma (5.3.9) we have that

Lemma 5.3.10. *Let $\tilde{\mathcal{I}}_e(\ell,j)$ be defined by*

$$\tilde{\mathcal{I}}_e(\ell,j)=(-1)^{\ell}2^{2j+\ell}\int_{\tilde{\Gamma}}\tilde{\mathcal{F}}(\ell,j)d\tau$$

where $\tilde{\Gamma}=\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4\cup\Gamma_5$ then

$$\tilde{\mathcal{I}}_e(\ell,j)=(-1)^{\ell}2^{2j+\ell}\int_{-\infty}^{+\infty}\tilde{\mathcal{F}}(\ell,j)d\tau+\mathcal{E}_{\ell,j} \tag{5.107}$$

where $\mathcal{E}_{\ell,j}$ satisfies

$$|\mathcal{E}_{\ell,j}|<\frac{2\mathrm{e}^{-\ell\frac{G^3}{2}}}{\ell G^3}\left(\mathcal{O}(\sqrt{1-e^2})\right). \tag{5.108}$$

We can now use the previous expansions to show that the integral along orbits of high-eccentricity are similar to those of the homoclinic orbit:

Lemma 5.3.11. *Let* $\mathcal{I}_e(\ell, j)$ *be as defined in 5.72 and* $\mathcal{I}_e(\tilde{\ell}, j)$ *be as defined in Lemma 5.3.10. Then we have*

$$\mathcal{I}_e(\ell, j) = \tilde{\mathcal{I}}_e(\ell, j) + \mathcal{O}(\sqrt{1-e^2}) \tag{5.109}$$

Proof. Using the expansion of Lemma B.0.1) we have

$$\frac{d\tau}{d\tilde{\tau}} = 1 + \mathcal{O}(\sqrt{1-e^2}). \tag{5.110}$$

Using the definition of $\tilde{\mathcal{F}}(\ell, j)$ in 5.99 we have

$$\int_{-T/2}^{+T/2} \mathcal{F}(\ell, j)(\tau) d\tau = \int_{-T/2}^{+T/2} \mathcal{F}(\ell, j)(\tilde{\tau}) \frac{d\tau}{d\tilde{\tau}} d\tilde{\tau} \tag{5.111}$$

$$= \int_{-T/2}^{+T/2} \tilde{\mathcal{F}}(\ell, j)(\tau) d\tau + \mathcal{O}(\sqrt{1-e^2}) \tag{5.112}$$

□

5.3.6 The Integral along $\gamma_1 \cup \gamma_2$

We now turn our attention to the integral along γ_1, γ_2 as defined by equations (5.77,5.78). Let $q(v) = (r_e(v), y_e(v), \phi_e(v), G(v))$ represent a simple periodic orbit of the Kepler problem in rotating coordinates, i.e. in the chosen level set of the Jacobi constant assume that G is such that

$$T(h(\mathcal{J}_0, G)) = 2n\pi \tag{5.113}$$

Then the integral defined by equation (5.63) corresponds to the subharmonic Melnikov integral (see, e.g. [38]). Note that as $\alpha_e(T) = \pi = -\alpha_e(-T)$ we have

$$\phi(\gamma_1(t)) - \phi(\gamma_2(t)) = t_e(\gamma_1(t)) - t_e(\gamma_2(t)) \tag{5.114}$$

In this case that condition (5.113) is satisfied the integrals along γ_1 and γ_2 cancel,

Lemma 5.3.12. *Let* $\mathcal{I}_e(\ell, j)$ *be the integrals defined by equation (5.72) evaluated along a periodic orbits of the Kepler problem in rotating coordinates. Then*

$$\mathcal{I}_e(\ell, j) = \int_\Gamma \mathcal{F}_e(\ell, j) d\tau.$$

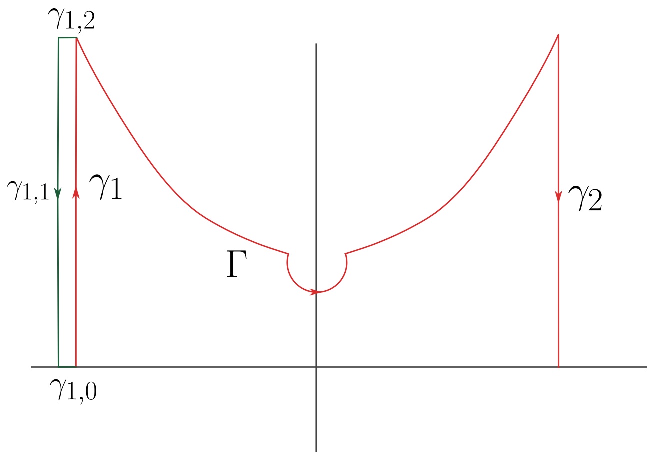


Figure 5.5: extending the path

Proof. For $\Im(\tau) = n\pi$ we have $V(\tau) = V(-\bar{\tau})$. Therefore, when T satisfies condition (5.113), $V(\tau)$ satisfies

$$V(\gamma_1(t)) = V(-\gamma_2(-t))$$

and so

$$\int_{\gamma_1 \cup \gamma_2} V(\tau)d\tau = \int_0^{\gamma^*} V(\tau(t))dt - \int_{\gamma^*}^0 V(-\bar{\tau}(t))dt = 0$$

□

Looking at equation (5.114) it is clear that in the case that if condition (5.113) is not fulfilled we have that $V(\gamma_1(t)) \neq V(-\gamma_2(-t))$. In this case, in order to bound the integral over $\gamma_1 \cup \gamma_2$ we deform the path of integration slightly. Declare $\delta = \phi(T/2) - \phi(-T/2)$ and define the following piece-wise smooth path

$$\gamma_{1,0}(t) = \{-T/2 - t | t \in [0, \delta]\} \tag{5.115}$$

$$\gamma_{1,1}(t) = \{-T/2 - \delta + it | t \in [0, \gamma^*]\} \tag{5.116}$$

$$\gamma_{1,2}(t) = \{\gamma^* - \delta + t | t \in [0, \delta]\} \tag{5.117}$$

Clearly

$$\int_{\gamma_1} \mathcal{F}_e(\ell, j)d\tau = \int_{\gamma_{1,0} \cup \gamma_{1,1} \cup \gamma_{1,2}} \mathcal{F}_e(\ell, j)d\tau$$

First we bound the integral over $\gamma_{1,0} \cup \gamma_{1,2}$ and then compare the integral of $\mathcal{F}_e(\ell, j)$ over $\gamma_{1,1}$ is close to that over γ_2.

Lemma 5.3.13. *Let $\mathcal{F}_e(\ell, j)$ be as defined in equation (5.74) and $\gamma_{1,0}, \gamma_{1,2}$ be defined by equations (5.115,5.117) respectively. Then*

$$\left|\int_{\gamma_{1,2}} \mathcal{F}_e(\ell, j) d\tau\right| < \left|\int_{\gamma_{1,0}} \mathcal{F}_e(\ell, j) d\tau\right| < \delta\left(\left(\sqrt{1-e^2}\right)^{2j}\right). \tag{5.118}$$

$$\tag{5.119}$$

Proof. First, we note that for $\ell > 1$ we have

$$\left|\mathrm{e}^{-\ell i G^3 t_e(-T/2-\delta+it)/2}\right| < \left|\mathrm{e}^{-\ell i G^3 t_e(T/2+t)/2}\right|$$

For the first inequality we simply note that for all $t \in [0, \delta]$

$$|\mathcal{F}_e(\ell, j)(\gamma_{1,0}(t))| \tag{5.120}$$

$$= \left|\frac{\mathrm{e}^{-\ell i G^3 t_e(-T/2-\delta+it)/2}}{(-T-\delta+i(t-1))^{j-\ell}(-T-\delta+i(t+1))^{j+\ell}}\right| \tag{5.121}$$

$$< \frac{\left|\mathrm{e}^{-\ell i G^3 t_e(T+t)/2}\right|}{|(T+t-i)^{j-\ell}(T+t+i)^{j+\ell}|} \tag{5.122}$$

$$= |\mathcal{F}_e(\ell, j)(\gamma_{1,2}(t))| \tag{5.123}$$

while a simple base by the height bound suffices for the integral over $\gamma_{0,1}$

$$\left|\int_{\gamma_{1,0}} \mathcal{F}_e(\ell, j) d\tau\right| = \int_{\frac{T}{2}}^{\frac{T}{2}+\delta} \frac{\mathrm{e}^{-\ell i G^3 t_e(\tau)/2}}{|(\tilde{\tau}-i)^{j-\ell}(\tilde{\tau}+i)^{j+\ell}|} d\tau \tag{5.124}$$

$$\leq \int_{\frac{T}{2}}^{\frac{T}{2}+\delta} \frac{\mathrm{e}^{-\ell i G^3 t_e(T/2+s)/2}}{|(T+t-i)^{j-\ell}(T+t+i)^{j+\ell}|} ds \tag{5.125}$$

$$\leq \delta\left(\frac{1}{|T|^{2j}}\right) \tag{5.126}$$

and recalling

$$T(e) = \frac{\pi}{\sqrt{1-e^2}} \tag{5.127}$$

gives the result. □

Lemma 5.3.14. *Let $F_e(\ell, j)$ be as defined in (5.74),γ_1 be as defined in equation (5.77) and $\gamma_{1,1}$ as defined in (5.116). Then*

$$F_e(\ell, j)(\gamma_{1,1})(t) = F_e(\ell, j)(\gamma_1(t)) + \mathcal{O}\left(\left(\sqrt{1-e^2}\right)^{j+1}\right).$$

Proof. We have

$$
\begin{aligned}
&F_e(\ell, j)(\gamma_{1,1}(t)) && (5.128)\\
&= \frac{\mathrm{e}^{-\ell i G^3 t_e(\gamma_{1,1}(t))}}{(-T-\delta+i(t-1))^{j+l}(-T-\delta+i(t+1))^{j-l}} && (5.129)\\
&= \frac{1}{T^{2j}} \frac{\mathrm{e}^{-\ell i G^3 t_e(\gamma_1(t))}\mathrm{e}^{-\ell i G^3 \delta}}{(-1-(\delta/T)+i(t-1)/T)^{j+l}(-1-\delta/T+i(t+1)/T)^{j-l}} && (5.130)\\
&= F_e(\ell, j)(\gamma_1(t)) + \frac{1}{T^{2j+1}} && (5.131)
\end{aligned}
$$

□

Lemma 5.3.15. *Let $\mathcal{F}_e(\ell, j)$ be defined by equation (5.74), γ_2 be as defined in equation (5.78) and $\gamma_{1,1}$ as defined in (5.116).*

$$\int_{\gamma_1 \cup \gamma_2} \mathcal{F}_e(\ell, j) d\tau = \mathcal{O}\left(\left(\sqrt{1-e^2}\right)^{2j-1}\right). \tag{5.132}$$

Proof. Using Lemma 5.3.13 we have

$$\int_{\gamma_1} \mathcal{F}_e(\ell, j) d\tau = \int_{\gamma_{1,0}} \mathcal{F}_e(\ell, j) d\tau + \mathcal{O}(T^{-2j})$$

By construction $t_e(\gamma_{1,1}(t)) = t_e(\gamma_2(t)) \bmod 2\pi$. Using Lemma 5.3.14 then, we have

$$\mathcal{F}_e(\ell, j)\left(\gamma_2(t)\right) = \mathcal{F}_e(\ell, j)\left(\gamma_{1,1}(t)\right) + \mathcal{O}\left(T^{3j+2|\ell|}\right)$$

Finally, noting that $|\gamma_2| = \mathcal{O}(T)$ we have bound (5.132) and we simply need to recall relation (5.127) to conclude. □

We can now employ directly the calculations in [43]. Define

$$N(\ell, m, n) = \frac{2^{m+n}}{G^{2m+2n-1}} \begin{pmatrix} -1/2 \\ m \end{pmatrix} \begin{pmatrix} -1/2 \\ n \end{pmatrix} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{i\ell G^3\left(\tau+\tau^3/3\right)/2}}{(\tau-i)^{2m}(\tau+i)^{2n}} d\tau \tag{5.133}$$

Now we take the following results from [43] (Lemma 30, Proposition 19)

Lemma 5.3.16. *Let $N(\ell, m, n)$ be defined by equation (5.133) and let $G > 32$. Then*

$$N(1,2,1) = \frac{1}{4}\sqrt{\frac{\pi}{2}}G^{-1/2}\mathrm{e}^{-G^3/3} + E^1 \tag{5.134}$$
$$N(1,2,0) = \sqrt{\frac{\pi}{2}}G^{3/2}\mathrm{e}^{-G^3/3} + E^2 \tag{5.135}$$
$$\tag{5.136}$$

where $|E^1| \leq 2^6 9 G^{-2}\mathrm{e}^{-G^3/3}, \quad |E^2| \leq 2^5 9 \mathrm{e}^{-G^3/3}$.

Lemma 5.3.17. *Let* $N(\ell, m, n)$ *as defined in 5.133 for* $\ell \geq 1, m, n \geq 0, m+n > 0, G > 1$. *Then*

$$|N(\ell, m, n)| \leq 2^{n+m+3}\mathrm{e}^q G^{m-2n-1/2}\mathrm{e}^{-\ell G^3/3}.$$

Using the form of the rescaled integral given in Lemma 5.3.9, bounding the integral along the different path components using Lemmas 5.3.10, 5.3.13, 5.3.15 and finally bounding the difference between the rescaled integral as per Lemma 5.3.11 we finally have:

Lemma 5.3.18. *Let* $\mathcal{N}_e\left(G, \mu\right)\left(\ell, j\right)$ *be as defined in 5.68 and* $N(\ell, j)$ *be as defined in 5.133. Then*

$$\mathcal{N}_e(\ell, j) = N(l, j, j+l) + \mathrm{e}^{-\ell\frac{G^3}{2}}\left(\mathcal{O}\left(\sqrt{1-e^2}\right)\right). \tag{5.137}$$

Now using Lemma 5.3.18 to bound the difference between the integral along the homoclinic orbit and orbits of high eccentricity and Lemmas 5.3.16 and 5.3.17 giving the value of these integrals, we can conclude the value of the Melnikov integral is given by Theorem 5.3.2.

5.4 Controlling Errors with Hamilton Jacobi

In order to conclude Theorem 5.4.1, that is to get a Theorem similar to that of 5.1.3 of [33], we need to show that the generating functions $\mathcal{S}^s, \mathcal{S}^u$ approximate the generating functions of the images of the circle $\mathcal{C}_G$ with sufficient accuracy and to bound errors in the return time. As noted in section (5.1.5), an effective strategy to prove the Melnikov-fuction correctly predicts the splitting of separatrices is to compare the result to solutions of the Hamilton-Jacobi equation on the appropriate energy level. similarly, we posit that our Melnikov-like function predicts differences in the return energy of orbits with sufficient accuracy. More specifically, we require a theorem of the following nature, close to Theorem 3.2 of [33]

Theorem 5.4.1. *There exist $0 < v_- < v_+, G^* > 0$ and $K > 0$ such that , for any $G > G^*$ and $\mu \in (0, 1/2]$ the invariant manifolds of infinity have parameterizations of the form (25) for $(v, \xi) \in (v_-, v_+) \times \mathbb{T}$ Moreover, the corresponding generating functions satisfy*

$$|T_1^u(v,\xi) - T_1^s(v,\xi) - L(v,\xi) - E| \leq K\mu^2(1-2\mu)G^{-2} \\ e^{\frac{G^{-3}}{3}}(1+\mathcal{O}(1-e^2)) + KG^{-1/2}\mu^2 e^{\frac{2G^{-3}}{3}}(1+\mathcal{O}(1-e^2)) \quad (5.138)$$

for a constant $E \in \mathbb{R}$, which might depend on μ and G, and

$$|\partial_v^m \partial_\xi^n T_1^u(v,\xi) - \partial_v^m \partial_\xi^n T_1^s(v,\xi) - \partial_v^m \partial_\xi^n L(v,\xi)| \\ \leq K\mu^2(1-2\mu)G^{-2+3m} e^{\frac{G^{-3}}{3}}(1+\mathcal{O}(1-e^2)) \\ + KG^{-1/2+3m}\mu^2 e^{\frac{2G^{-3}}{3}}(1+\mathcal{O}(1-e^2)) \quad (5.139)$$

for $0 < m + n \leq 2, 0 \leq m, n,$.

Here we note that the Hamilton Jacobi equation along the separatix is close to the Hamilton Jacobi equation along unperturbed orbits of the Kepler problem of sufficiently high eccentricity, allowing us to conclude that the Melnikov integral (1.1.2) correctly gives the true splitting. It is important to note that this method works as the the expression for r_h (equation (5.81)) and r_e (equation (5.91)) are close also in the complex plane and therefore using the methods of 5.1.5, one can use this to obtain super exponentially small bounds in the reals.

5.4.1 The Fixed Point Equation

In many cases of exponentially small Fourier splitting, in order to extend the solutions of the Hamilton-Jacobi equation to the complex plane one recasts the Hamilton-Jacobi equation so that the generating functions of the stable and unstable manifolds correspond to the fixed point of a certain operator. By showing that this operator is contractive on some appropriate Banach space of functions, then, one can conclude the existence of an analytic extension of the solution of the Hamilton-Jacobi equation to the desired region of the complex plane. Instead of writing the complete procedure, we will just trace the necessary changes to be made to the method of [33] which would allow the proof of exponentially small splitting of separatrices to be generalized to prove the exponential closeness of the Melnikov function 5.3.2 and the generating functions of the circles (5.30).

The Equations of the Homoclinic Orbit

We recall the constructions of [33] in order to guess the necessary changes for the subharmonic Melnikov case. In [33] the following operator was defined:

$$\mathcal{L}(h) = \partial_v h - G^3 \partial_\xi h \tag{5.140}$$

together with the two inverse operators

$$\mathcal{G}^u(h)(v,\xi) = \int_{-\infty}^{0} h\left(v+s, \xi - G^3 s\right) ds \tag{5.141}$$

$$\mathcal{G}^s(h)(v,\xi) = \int_{+\infty}^{0} h\left(v+s, \xi - G^3 s\right) ds \tag{5.142}$$

The potential function $\hat{U}$ was split as $\widehat{U} = \widehat{U}_0 + \widehat{U}_1$ where

$$\widehat{U}_0(v,\phi) = -\frac{\mu(1-\mu)}{2}\left(1 - 3\cos^2\phi\right)\frac{1}{G_0^4 \widetilde{r}_{\mathrm{h}}(v)^3}$$

the potential $\widehat{U}_0$ along the separatrix inverted is then inverted

$$Q_0^* = \mathcal{G}^*\left(\widehat{U}_0\left(v, \xi + \alpha_{\mathrm{h}}(v)\right)\right), \quad * = u, s$$

and finally the following operator is defined:

$$\mathcal{F}_h^*(h) = -\frac{1}{2}\widetilde{y}_{\mathrm{h}}^2\left(\partial_v Q_0^* + \partial_v h - \frac{1}{\widetilde{r}_{\mathrm{h}}^2}\left(\partial_\xi Q_0^* + \partial_\xi h\right)\right)^2 - \frac{1}{2\widetilde{r}_{\mathrm{h}}^2}\left(\partial_\xi Q_0^* + \partial_\xi h\right)^2 + \widehat{U}_1\left(v, \xi + \widetilde{\alpha}_{\mathrm{h}}(v)\right) \tag{5.143}$$

Solutions of the Hamilton-Jacobi equation (5.21) are then solutions of the equation

$$\mathcal{L}(h) = \mathcal{F}_h(h) \tag{5.144}$$

A solution of the above equation, then, corresponds to a fixed point of the equation

$$h = \tilde{\mathcal{F}}(h) \tag{5.145}$$

where

$$\tilde{\mathcal{F}}_h = \mathcal{G}^u \circ \mathcal{F}.$$

In order to prove the existence of solutions to equation (5.144) on complex domains extending to infinity, that is, on domains

$$\begin{aligned} D^u_{\infty,\rho} &= \{v \in \mathbb{C}; \operatorname{Re} v < -\rho\} \\ D^s_{\infty,\rho} &= \{v \in \mathbb{C}; \operatorname{Re} v > \rho\} \end{aligned}$$

then one examines the operator $\tilde{\mathcal{F}}$ and notes that it is contraction in some ball $\mathcal{B}_r$ of radius r in a suitable Banach space. One can conclude existence of a solution to the equation (5.145) in the ball $\mathcal{B}_r$. Finally, the "solution" of the Hamilton Jacobi equation corresponding to the unstable manifold is extended to a "boomerang domain" as a formal Fourier series and the exponential bounds are achieved.

Comparisons to Orbits of High Eccentricity

We posit that that theorem 5.3 follows, mutatis mutandis from the proof of [33].

In order to set up an argument similar to that of [33], one defines two inverses of $\mathcal{L}$ by

$$\begin{aligned} \mathcal{G}^u(h)(v,\xi) &= \int_{-T/2}^{0} h\left(v+s, \xi - G^3 s\right) ds \\ \mathcal{G}^s(h)(v,\xi) &= \int_{+T/2}^{0} h\left(v+s, \xi - G^3 s\right) ds \end{aligned} \tag{5.146}$$

The Hamilton-Jacobi equation (5.21) then becomes

$$\mathcal{L}\left(T_1\right) = -\frac{1}{2y_e^2}\left(\partial_v T_1 - \frac{1}{r_e^2}\partial_\xi T_1\right)^2 - \frac{1}{2r_e^2}\left(\partial_\xi T_1\right)^2 + \widehat{U}\left(v, \xi + \alpha_e(v)\right) \tag{5.147}$$

Define the operator

$$\mathcal{F}(h) = -\frac{1}{2\widetilde{y}_e^2}\left(\partial_v h - \frac{1}{\widetilde{r}_e^2}\left(\partial_\xi h\right)\right)^2 - \frac{1}{2\widetilde{r}_e^2}\left(\partial_\xi h\right)^2 + \widehat{U}_1\left(v, \xi + \widetilde{\alpha}_e(v)\right) \tag{5.148}$$

Solutions of the Hamilton-Jacobi equation, then, are solutions of the equation

$$\mathcal{L}(h) = \mathcal{F}(h) \tag{5.149}$$

In order to prove the existence of solutions to the Hamilton-Jacobi equation, one can define the domains

$$\mathcal{D}^u_{T/2,\rho} = \{v \in \mathbb{C}; \mathrm{Re}\, -(T/2+\epsilon) < v < -\rho\}$$
$$\mathcal{D}^s_{T/2,\rho} = \{v \in \mathbb{C}; \mathrm{Re}\, T/2+\epsilon > v > \rho\}$$

and where consider functions $h : \mathcal{D}^*_{T/2,\rho} \times \mathbb{T}_\sigma \to \mathbb{C}$ where h satisfies conditions (5.40). Defining analogous norms and function spaces to those found in [33], one proves that the operator $\mathcal{G}$ has similar properties of $\mathcal{G}^{u,s}_h$.

Finally, using Lemmas (5.91), (5.92) and noting

$$y_e\tilde{\tau} = y_h(\tilde{\tau}) + \mathcal{O}(\sqrt{1-e^2}) \tag{5.150}$$

we can write the operator $\mathcal{F}$ in (5.148) as $\mathcal{F} = \mathcal{F}_h + \mathcal{F}_e$ where $\mathcal{F}_h$ corresponds to the operator of [33] and $\mathcal{F}_e = (1-e^2)\mathcal{F}'$ and $\mathcal{F}'$ can shown to be contractive for G_0 large enough.

Inverting the operator $\mathcal{L}$, then, solutions of (5.149) with appropriate initial conditions correspond to a fixed point of the operator

$$\tilde{\mathcal{F}} = \mathcal{G} \circ (\mathcal{F} + \mathcal{F}_e) \tag{5.151}$$

Using similar bounds to those of [33], then, taking G_0 large enough and orbits of eccentricity close to one, one can conclude that the operator 5.151 is contractive. As the solution will be close to that of the parabolic case everywhere in the complex plane, we should be able to achieve exponentially small bounds in the reals.

Chapter 6

Parabolic Separatrix Maps

Ok, one last time. These are small, *but the ones out there are* far away.

– Father Ted

Estimating the width of the stochastic layer surrounding stable and unstable manifolds with transversal intersections is achieved by studying the separatrix map, which approximates the dynamics close to separatrix by assuming that motions are dominated by those of the stable and unstable manifolds. Usually, the separatrix map is assumed to be associated to a hyperbolic fixed point. In the following, we examine *parabolic models* which are suitable for the examination of dynamics close more degenerate fixed points. A rescaling of such a map is approximated as a "generalised separatrix map " which gives asymptotic estimates for the width of the stochastic layer. Finally, we apply general K.A.M. theorem which leaves the estimates in the non-asymptotic case open to optimisation.

6.1 Introduction

The separatrix map was introduced by Zaslavsky and Filonenko [44] for near-integrable Hamiltonian systems and Shilnikov [45] for more general systems. Since then, the separatrix map has been studied in a wide variety of contexts with numerous applications, see, for instance, [46, 47]. In particular, the question of estimating the stochastic layer for the case of exponentially small splitting has first been explored by Latzukin and collaborators for the important case of the standard map. A survey can be found in [41].

One embodiment of the separatrix map is as follows:

$$\mathrm{Sep}\, M : \begin{pmatrix} t \\ h \end{pmatrix} \rightarrow \begin{pmatrix} \bar{t} = t - T(\bar{h}, \varepsilon) + \mathcal{E}_t(h, \varepsilon) \ (\mathrm{mod} 2\pi) \\ \bar{h} = h + a(\varepsilon)\phi(t) + \mathcal{E}_h(h, \varepsilon) \end{pmatrix}$$

where $\mathcal{E}_t(h, \varepsilon)$ is a general expression for the error incurred in the return time (which is generally approximated by the return time of the unperturbed system, possibly with extra terms to reflect errors in the varying energy). Similarly $\mathcal{E}_h(h, \varepsilon)$ is a general error in h, reflecting, for instance, the fact that the difference in energies is not given exactly by the splitting of separatrices. Using the variational equations these errors can be shown to be $\mathcal{O}(h, \varepsilon)$ (see [48] for a general discussion or [49] for a discussion from the point of view of the gluing map formalism). Though these errors are usually small, for our purposes, a detailed estimate of their effects is necessary and so we will have to adjust the separatrix map somewhat.

6.1.1 Estimates for the Hyperbolic Case

Most of the literature concerning the separatrix map is concerned with the splitting of stable and unstable manifolds of hyperbolic points of one and a half degree of freedom systems or two degrees of freedom systems. Some general estimates for the width w of the stochastic layer of a two degree of freedom system satisfying certain symmetry conditions have been given in the case that the fixed point is hyperbolic and the splitting of the separatrices is $\mathcal{O}(\varepsilon)$. In particular, for many systems the following relation is satisfied (see, for instance, [50] for the case of 2 d.o.f. symplectic mappings)

$$w/d \sim 1/\lambda$$

where d is the width of the lobe domain and λ is the logarithm of the larger multiplier of the hyperbolic fixed point.

However, in the case of exponentially small splitting of separatrices, the width of the stochastic layer is usually greater than the width of the lobe domain. It has been calculated in several cases and it is conjectured ([49]) that the following estimate holds

$$\lim_{\varepsilon \to 0} \frac{w(\varepsilon)\lambda(\varepsilon)}{d(\varepsilon)} = \frac{4\pi}{K}$$

where K is the largest value such that the standard map

$$F_K : \begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} \bar{u} = u + v - \frac{K}{2\pi}\sin(2\pi u) \\ \bar{v} = v - \frac{K}{2\pi}\sin(2\pi u) \end{pmatrix} \tag{6.1}$$

has an invariant curve.

6.1.2 Generalizations to the Parabolic Case

When applying this theory to the current problem we encounter two main issues: firstly the splitting is exponentially small, which means care has to be taken to ensure the splitting is well approximated by the Melnikov integral. Secondly, proofs for the above estimates often rely on the hyperbolic nature of the fixed point. The case of defining separatrix mappings associated to more degenerate saddles has been treated in [48]. Here, an estimate of the width of the stochastic layer in the case that the stable and unstable manifolds are those of a parabolic point is given by considering the separatrix map as a return map

$$\mathrm{Sep}\, M : \begin{pmatrix} t \\ h \end{pmatrix} \to \begin{pmatrix} \bar{t} = t - T(\bar{h},\varepsilon)(\mathrm{mod} 2\pi) \\ \bar{h} = h + a(\varepsilon)\phi(t) \end{pmatrix} \tag{6.2}$$

where $T(\bar{h},\varepsilon)$ is the return time to some domain, $a(\varepsilon)$ is a measure of the amplitude of the splitting of separatrices and $\phi(t)$ is a function giving the shape (which is often independent of ε). In the case of a parabolic point satisfying some natural conditions (see the above-mentioned [48] for details), the separatrix map is well modelled by mapping (6.2) with return time $T(h)$ given by

$$T(h) = c_{l,k,r} h^{-m/(2lk)}(1 + \mathcal{O}(h)) \tag{6.3}$$

where $c_{l,k,r}$ is a constant depending only on l, k, r Moreover, in many cases, for example when the first term of Fourier series describing the splitting of the stable and unstable manifolds dominates, we can assume

$$a(\varepsilon)\phi(t) = \varepsilon \sin(\phi) \tag{6.4}$$

Errors in the separatrix mapping occur proportional to the size of the perturbation ε. To get a general estimate for the width of the stochastic layer in the case of parabolic points, the above mapping is compared to a "standard-like map" by taking a Taylor series approximation about given level energy set h_0.

This method has already been applied in the case of the R3BP in [34] however the results there differ from the current aims in two fundamental ways. Firstly, the estimates given depended on taking μ to be a small parameter. In our case, we wish to find the location of the last invariant torus as a function of G_0 only, i.e. the results apply for μ arbitrary. Secondly, [34] focused on giving estimates, not necessarily rigorous, to compare to numerical estimates. To make the estimates rigorous, careful analysis of the errors incurred by the above method is called for. Nevertheless, to demonstrate the method, we derive the estimate in the case of the PCR3BP :

The chosen Poincaré section (see Section 5.1.1) is a cylinder with coordinates $\phi = \alpha - t$. Here, ϕ will play the role of "time" and angular momentum G the role of "energy". Suppose that in some level set of the Jacobi constant $\mathcal{J}^{-1}(\mathcal{J}_0)$ and for some fixed value of the mass ratio of the massive bodies we have a good approximation of the splitting of the stable and unstable manifolds associated to the parabolic fixed point at infinity, $f(\mathcal{J}_0, \mu)\sin(\phi)$. Then, following the method of [34] we approximate the Poincaré map of the cylinder as follows:

$$\begin{aligned} \bar{\phi} &= \phi + 2\pi(-2(\mathcal{J}_0 + G))^{-3/2} \\ \bar{G} &= G + f(\mathcal{J}_0, \mu)\sin(\bar{\phi}) \end{aligned}$$

where to facilitate the comparison to the standard map we then map backwards in time, relabelling ϕ as $\bar{\phi}$ and δ as $\bar{\delta}$.

$$\begin{aligned} \bar{\phi} &= \phi - 2\pi(-2(\mathcal{J}_0 + \bar{G}))^{-3/2} \\ \bar{G} &= G - f(\mathcal{J}_0, \mu)\sin(\phi) \end{aligned}$$

Note that this mapping is **not** canonically symplectic. We now localize around a level set of angular momentum G_0 (equivalently around a energy level set $h = h_0$) by declaring $\delta = h - h_0$ and then Taylor expanding with respect to δ. Using $h = \mathcal{J}_0 + G_0$ we find that the mapping is given

$$\bar{\phi} = \phi - \frac{\pi}{\sqrt{2}} h_0^{-3/2} + \frac{3\pi}{2\sqrt{2}} \frac{\bar{\delta}}{h_0^{-5/2}} + \mathcal{O}(\delta^2) \tag{6.5}$$

$$\bar{\delta} = \delta - f(\mathcal{J}_0, \mu)\sin(\phi) \tag{6.6}$$

Discarding terms of order 2 and higher and performing the change of variables

$$(u(\phi,\delta), v(\phi,\delta)) = \left(\phi, \frac{\pi}{\sqrt{2}} h_0^{-3/2} + \frac{3\pi}{2\sqrt{2}} \frac{\bar{\delta}}{h_0^{-5/2}}\right)$$

we then get the standard map

$$\bar{u} = u + \bar{v} \tag{6.7}$$

$$\bar{v} = v - \frac{3\pi}{2\sqrt{2}} \delta_0^{-5/2} f(G_0,\mu)\sin(u) \tag{6.8}$$

One can then conclude the existence of invariant circles for the above mapping at a distance h_0 satisfying

$$\frac{3\pi}{2\sqrt{2}} \delta_0^{-5/2} f(G_0,\mu) < \kappa_G$$

where κ_G is the Greene's constant given by

$$\kappa_G \approx 0.971635406. \tag{6.9}$$

In [34] this was shown to give good agreement with numerical estimates for μ small. To achieve rigorous results, however, it is necessary to bound the errors in the separatrix map, including the error incurred by dropping terms of $\mathcal{O}(\delta^2)$. In the next section, we prove the existence of an invariant curve via a more general invariant curve theorem, rather than using the standard map, which eliminates minor problems encountered when trying to prove that more general parabolic maps are close to the standard map. We study the Poincaré section of the PCR3BP via following general *parabolic model*.

Definition 6.1.1. *Let $F_{\varepsilon,q}$, $q \in \mathbb{Q}$ be an exact symplectic map of the cylinder $\mathbb{C} = \mathbb{T} \times (0,\infty)$ such that there exist coordinates (ϕ,δ) in which the mapping $F_{\varepsilon,q}(\phi,\delta) = (\bar{\phi},\bar{\delta})$ is given by the expression*

$$\begin{aligned} \bar{\phi} &= \phi - \bar{\delta}^{-q} + g(\phi,\bar{\delta},\varepsilon) \\ \bar{\delta} &= \delta - \varepsilon\sin(\phi) + h(\phi,\bar{\delta},\varepsilon) \end{aligned} \tag{6.10}$$

where $h(\phi,\bar{\delta})$ is assumed small compared to ε. Then we call $F_{\varepsilon,q}$ a parabolic separatrix map.

Similar to the analysis carried out in [34] and [48] an integral part of the study of these parabolic models is a rescaling, which allows the effects of the higher-order singularity seen in the return time to be appreciated. However, the *linearized* parabolic model will be of slightly more general nature that that proposed in [48], which helps when trying to discard errors in the expansion about a level energy set. We then study these linearized parabolic models via two methods: the first, given in Section 6.2 is by applying an invariant curve theorem, which gives asymptotic results. The second method, given in the Section, is by applying a K.A.M. theorem, which will give results for invariant curves of the above model maps for general $\mathcal{J}_0$, provided $\mathcal{J}_0$ is large enough so that Theorem 5.3.2 and 6.17 are valid.

6.2 Invariant Curves of Linearised Parabolic Models

To conclude rigorous results for parabolic-type motions we will Taylor expand the model in Definition 6.1.1 around an invariant curve of the Kepler problem. We then rescale the mapping according to a rescaling factor η which depends on the distance from the separatrix. Rather than employing estimates for the standard map, we use rather the following invariant curve theorem formulated for analytic maps by Kolmogorov. Then, examining a cure in parameter space $\gamma(t) : (t_0, \infty) \to \mathbb{R}^3 = (\mathcal{J}, \delta, \eta)$ one can show that the associated family of rescaled parabolic separatrix maps validate the hypotheses of the invariant curve theorem, with a small parameter ϵ, where $\epsilon \to 0$ as $t \to \infty$. This shows rigorously the asymptotic width of the stochastic layer for the PCR3BP estimated in [34].

6.2.1 The Invariant Curve theorem

The statement below was taken from [35] and more details on the history are given there. The proof is somewhat technical, see [51] and [52] for details.

Theorem 6.2.1. *Consider a mapping $F : (I, \phi) \to (I', \phi')$ which has the following form:*

$$I' = I + \varepsilon^{s+r} c(I, \phi, \varepsilon) \tag{6.11}$$

$$\phi' = \phi + \omega + \varepsilon^s h(I) + \varepsilon^{s+r} d(I, \phi, \varepsilon) \tag{6.12}$$

where

- *c and d are smooth for $0 \leq a \leq I < b < \infty, 0 \leq \varepsilon \leq \varepsilon_0$, and all ϕ*

- *c and d are 2π-periodic in ϕ*
- *r and s are integers with $s \geq 0, r \geq 1$*
- *h is smooth for $0 \leq a \leq I < b < \infty$*
- *$dh(I)/dI \neq 0$ for $0 \leq a \leq I < b < \infty$*
- *if Γ is any continuous closed curve of the form $\Xi = \{(I, \phi) : I = \Theta(\phi), \Theta : \mathbb{R} \to [a, b]$ continuous and 2π -periodic $\}$, then $\Xi \cap F(\Xi) \neq \emptyset$.*

Then for sufficiently small ε, there is a continuous F -invariant curve Γ of the form $\Gamma = \{(I, \phi) : I = \Phi(\phi), \Phi : \mathbb{R} \to [a, b]$ continuous and $2\pi -$ periodic$\}$.

6.2.2 Linearized Parabolic Models

Following the method of [48], to study the parabolic models 6.1.1 choose some level set $\delta = \delta_0$ to linearize the map in a neighbourhood of $\delta = \delta_0$. Write $\delta = \delta_0 + \frac{\eta\nu}{q}$ with $\eta = \delta_0^{q+1}$ and rescale the mapping (6.1.1) as

$$\begin{aligned} \bar{\phi} &= \phi - \frac{1}{\delta_0^q} + \frac{\eta\bar{\nu}}{\delta_0^{q+1}} + \delta_0^{q+1}(l(\phi, \nu, \delta_0, \varepsilon)) + g(\phi, \nu, \delta_0, \varepsilon) \\ \bar{\nu} &= \nu - \frac{q\varepsilon}{\eta}\sin(\phi) + \frac{1}{\delta_0^{q+1}} h(\phi, \nu, \delta_0, \varepsilon) \end{aligned} \tag{6.13}$$

We can easily then conclude:

Lemma 6.2.2. *Let $F_{\varepsilon,q}$ be as defined by (6.1.1). Let $a = 1 - \epsilon$, for some $\epsilon \in (0, 1)$. Let $\eta = \delta_0^{q+1} = \varepsilon^a$. Let the functions $\tilde{g}(\nu, \phi, \varepsilon)$ and $\tilde{h}(\nu, \phi, \varepsilon)$ defined by*

$$\begin{aligned} \tilde{g}(\nu, \phi, \varepsilon) &= \varepsilon^{-1/2} g\left(\phi, \nu, \eta(\varepsilon), \delta_0(\varepsilon), \varepsilon\right) \\ \tilde{h}(\nu, \phi, \varepsilon) &= \varepsilon^{-1} h\left(\phi, \nu, \eta(\varepsilon), \delta_0(\varepsilon), \varepsilon\right) \end{aligned}$$

be smooth for all $0 \leq \varepsilon \leq \varepsilon_0$. Let $\delta_0 = \varepsilon^a$. Then $F_{\varepsilon,q}$ has an invariant curve for sufficiently small $\varepsilon > 0$.

Proof. The mapping (6.2.4) takes the form

$$\begin{aligned} \bar{\phi} &= \phi - \frac{1}{\delta_0^q} + \nu - \frac{\varepsilon}{\delta_0^{q+1}}\sin(\phi) + g\left(\eta, \nu, \delta_0, \phi, \varepsilon\right) + \frac{\eta^2}{\delta_0^{q+2}}\left(l\left(\eta, \nu, \delta_0, \phi, \varepsilon\right)\right) \\ \bar{\nu} &= \nu - \frac{\varepsilon}{\delta_0^{q+1}}\sin(\phi) + \frac{1}{\eta} h\left(\eta, \nu, \delta_0, \phi, \varepsilon\right) \end{aligned} \tag{6.14}$$

Upon setting $\delta_0^{q+1} = \varepsilon^{1-\epsilon}$ we see that mapping 6.2.2 satisfies the conditions of theorem (6.2.1) with $\tilde{\varepsilon} = \varepsilon^{\epsilon}$ and therefore the mapping 6.2.2 has an invariant curve for sufficiently small $\tilde{\varepsilon}$. □

Remark 6.2.3. *The choice $\tilde{g}(\eta, \nu, \delta_0, \phi, \varepsilon) = \varepsilon^{-1/2} g(\eta, \nu, \delta_0, \phi, \varepsilon)$ is clearly not the only one. Here it was chosen to deal with the case where $g(\eta, \nu, \delta_0, \phi, \varepsilon)$ is slightly larger than the splitting, as is the case for the PCR3BP, see Lemma 6.3.2. Due to the rescaling, these errors in the return time will not be significant.*

Definition 6.2.4. *We call the following map defined by $F(\phi, \nu) = (\bar{\phi}, \bar{\nu})$*

$$\begin{aligned} \bar{\phi} &= \phi - \frac{1}{\delta_0^q} + \frac{\eta\bar{\nu}}{\delta_0^{q+1}} \\ \bar{\nu} &= \nu - \frac{q\varepsilon}{\eta}\sin(\phi) \end{aligned} \tag{6.15}$$

a linearized parabolic separatrix map.

Note that here η plays the role of a parameter controlling both the size of the perturbation of the integrable map and the size of the errors. By choosing $\eta = \delta_0^{q+1}$ the map (6.2.4) is equivalent to a perturbed standard map with a "kick" of size δ_0^{-q}.

6.3 The Linearized Parabolic Separatrix Map of the PCR3BP

We now estimate the difference between the linearized parabolic model 6.2.4 and the true Poincaré map of the PCR3BP. In doing so we show that the hypotheses of Lemma 6.2.2 are true for PCR3BP and so we can conclude Theorem 5.1.1.

6.3.1 Generating Functions for Parabolic Models

Consider the parabolic model given in Definition 6.1.1, where it is assumed that we know the difference in momenta accurately, i.e., we assume $v(\phi, \bar{\delta})$ is small in comparison to ε. As the mapping is an exact symplectic mapping of the cylinder, it is given by a generating function $W(\phi, \bar{\delta})$ satisfying

$$F(\phi, \delta) = (\bar{\phi}, \bar{\delta}) \iff \frac{\partial W}{\partial \phi} = \delta, \quad \frac{\partial W}{\partial \bar{\delta}} = \bar{\phi}$$

In the case of the parabolic model 6.1.1 this reads:

$$\delta = \frac{\partial W}{\partial \phi} = \bar{\delta} - \varepsilon \sin(\phi)$$

And so

$$W(\phi, \bar{\delta}) = \bar{\delta}\phi - \varepsilon \cos(\phi) + \psi(\bar{\delta})$$

Observing that

$$\bar{\phi} = \frac{\partial W}{\partial \bar{\delta}} = \phi - \bar{\delta}^{-1} + g(\phi, \bar{\delta})$$

we can conclude

$$g(\phi, \bar{\delta}) = \int \frac{\partial h\left(\phi, \bar{\delta}\right)}{\partial \delta} d\phi + \frac{d\varphi(\bar{\delta})}{d\bar{\delta}}$$

where

$$\psi(\bar{\delta}) = -\frac{\delta^{1-q}}{1-q} + \varphi(\bar{\delta})$$

We now wish to estimate the norms for the particular case of the parabolic model given by the PCR3BP.

Remark 6.3.1. *For technical reasons, in what follows we assume that μ is not very close to $\frac{1}{2}$. This ensures that the dominant term of the splitting is given by the first harmonic of the Melnikov function. A more detailed analysis could extend the results to the case $\mu = \frac{1}{2}$.*

We wish to expand around a fixed energy level and show that the difference between the return map of the PCR3BP and the linearized parabolic models in Definition 6.2.4 is small. First, denote $\delta = -\left(2\pi^2\right)^{1/3} h$ where h is the Keplerian energy of the orbit as $h = \mathcal{J} + G$, the difference in h upon the return to the section is simply $\left(2\pi^2\right)^{1/3} (\bar{G} - G)$. We show the dependence of the splitting on the Jacobi constant $\mathcal{J}$ we express the splitting in terms of $\mathcal{J}$ plus an error term which depends on the distance from the separatrix and the eccentricity of the orbit and the parameter η in the rescaled mapping 6.14. This shows that even though we are relatively far away from the split separatrices the splitting of the circles 5.30 which corresponds to the quantity ε in the parabolic models of Definition 6.1.1, is still well approximated by the splitting of the separatrices.

Lemma 6.3.2. *Let δ_0 be close to 0. Consider the Melnikov potential as given in 5.3.2. Then*

$$L^{[1]}\left(G, \mu\right) = -\mu(1-\mu)\sqrt{\pi}\frac{1-2\mu}{4\sqrt{2}}\mathcal{J}_0^{-3/2} e^{-\frac{\mathcal{J}_0^3}{3}} \left(1 + \mathcal{O}(\mathcal{J}_0^{-2}, \delta_0, \sqrt{1-e^2})\right) \tag{6.16}$$

And similar for $L^{[n]}, n > 1$. Let T_1^u, T_1^s be the generating functions of the circles $\mathcal{S}^u, \mathcal{S}_1^s$ as defined by 5.30. Then

$$\left|\partial_\xi T_1^u(v,\xi) - \partial_\xi T_1^s(v,\xi) - \partial_\xi^n L(v,\xi)\right| \\ \leq K\mu^2(1-2\mu)\mathcal{J}_0^{-2} e^{-\frac{\mathcal{J}_0^{-3}}{3}}\left(1+\mathcal{O}(\mathcal{J}_0^{-2},\delta_0,\sqrt{1-e^2})\right). \quad (6.17)$$

Proof. The Lemma follows immediately from Theorem 5.3.2, Theorem 5.4.1 and the definition of the Jacobi constant $\mathcal{J} = h - G$. □

We can now describe Poincaré map of the PCR3BP as a linearized parabolic model and bound the error terms:

Lemma 6.3.3. *Consider the rescaled Poincaré map $F(\phi,\delta)$ of the PCR3BP as given in Definition 6.25. Let $\mathcal{J}_0$ be a value of the Jacobi constant large enough to guarantee the validity of Theorem 1.1.2 and Theorem 6.17. Then around a level energy set $\delta = \delta_0$ satisfying $\delta_0 \ll \mathcal{J}_0^{-1}$ the Poincaré map is given by a linearized parabolic model in the sense of definition 6.1.1 with the splitting given by*

$$\varepsilon(\mathcal{J}_0) = -\mu(1-\mu)\sqrt{\pi}\frac{1-2\mu}{4\sqrt{2}}\mathcal{J}_0^{3/2} e^{-\frac{\mathcal{J}_0^3}{3}} \quad (6.18)$$

where $g'(\delta,\phi) = g(\delta,\phi) - \varphi(\delta_0,\mathcal{J}_0)$ for some function φ independent of ϕ and $g'(\delta,\phi)$, $h(\delta,\phi)$ satisfy the following inequalities:

1. $\|g'(\delta,\phi)\| < 2K\mu^2(1-2\mu)\mathcal{J}_0 e^{-\frac{\mathcal{J}_0^{-3}}{3}}$
2. $\|h(\delta,\phi)\| < 2K\mu^2(1-2\mu)\mathcal{J}_0^{-2} e^{-\frac{\mathcal{J}_0^{-3}}{3}}$

where K is the constant given in Theorem 6.17.

Proof. Denoting $\delta = -\left(2\pi^2\right)^{1/3}(\mathcal{J}_0 + G)$ and using Theorem 5.3.2 we can express the Poincaré map for the PCR3BP (5.46) as $F(\phi,\nu,\mathcal{J}_0) = (\bar{\phi}(\phi,\nu,\mathcal{J}_0), \bar{\nu}(\phi,\nu,\mathcal{J}_0))$:

$$\begin{aligned} \bar{\phi} &= \phi - (\delta - \varepsilon(\mathcal{J}_0)\sin(\phi) + h(\delta,\phi))^{-3/2} + g(\delta,\phi,\mathcal{J}_0) \\ \bar{\delta} &= \delta - \varepsilon(\mathcal{J}_0)\sin(\phi) + h(\delta,\phi,\mathcal{J}_0) \end{aligned} \quad (6.19)$$

where $h(\delta,\phi,\mathcal{J}_0)$ contains the error incurred from discarding the higher order Fourier coefficients of the Melnikov function and the difference between the Melnikov function and the true difference of the circles $\mathcal{S}^u$, $\mathcal{S}^s$ as defined in 5.30 and $g(\delta, \phi, \mathcal{J}_0)$ is a correction to the return time which will be shown to small. To form the linearized parabolic model 6.2.4 around $\delta = \delta_0$, write $\delta = \delta_0 + \frac{2\eta\nu}{3}$ use the change of coordinates

$$\Phi : (\phi, \delta) \mapsto (\phi, \nu = (\delta - \delta_0)/\eta) \tag{6.20}$$

$h(\delta, \phi, \mathcal{J}_0)$ is then given by

$$h(\delta, \phi, \mathcal{J}_0) = h(\delta_0, \phi, \mathcal{J}_0) + \mathcal{O}(\eta) \tag{6.21}$$

where using Theorem 5.64, Theorem 5.4.1 and the expression for the eccentricity e given in A similar to the method of Lemma 6.3.2 we have

$$h(\delta_0, \phi, \mathcal{J}_0) = \mu^2(1 - 2\mu)\mathcal{J}_0^{-2} e^{-\frac{\mathcal{J}_0^3}{3}} (1 + \mathcal{O}(\mathcal{J}_0^{-1}, \delta_0, \sqrt{1 - e^2})) \tag{6.22}$$

Using the existence of generating functions we have

$$g(\delta_0, \phi, \mathcal{J}_0) = \int \frac{\partial h(\phi, \delta, \mathcal{J}_0)}{\partial \delta}\Big|_{\delta = \delta_0} d\phi + \varphi(\delta_0, \mathcal{J}_0) + \mathcal{O}(\eta) \tag{6.23}$$

and so

$$g(\delta_0, \phi) = \mu^2(1 - 2\mu)\mathcal{J}_0 e^{-\frac{2\mathcal{J}_0^3}{3}} (1 + \mathcal{O}(\mathcal{J}_0^{-1}, \delta_0, \sqrt{1 - e^2})) + \varphi(\delta_0, \mathcal{J}_0) + \mathcal{O}(\eta) \tag{6.24}$$

We now restrict to the case where δ_0 is close to the separatrix, $\delta_0^{q+1} \ll \mathcal{J}_0^{-1}$. For large $\mathcal{J}_0$, then, $\delta_0, \sqrt{1 - e^2} \ll \mathcal{J}_0^{-1}$ and we can conclude the Lemma. □

Definition 6.3.4. *We call the map $\tilde{F}(\phi, \nu) = (\bar{\phi}, \bar{\nu})$ given by the Taylor expanded, rescaled mapping*

$$\begin{aligned}
\bar{\phi} &= \phi - \frac{1}{\delta_0^{3/2}} + \nu - \frac{\varepsilon}{\delta_0^{5/2}} \sin(\phi)) + \varphi(\delta_0) + g\left(\eta, \nu, \delta_0, \phi, \varepsilon\right) + \delta_0^{3/2} \left(l\left(\eta, \nu, \delta_0, \phi, \varepsilon\right)\right) \\
\bar{\nu} &= \nu - \frac{\varepsilon}{\delta_0^{5/2}} \sin(\phi) + \frac{1}{\delta_0^{5/2}} h\left(\eta, \nu, \delta_0, \phi, \varepsilon\right)
\end{aligned} \tag{6.25}$$

the rescaled Poincaré map of the PCR3BP.

Remark 6.3.5. *The choice $\delta_0 < \mathcal{J}_0^{-3/2} e^{-\frac{\mathcal{J}_0^3}{3}}$ is somewhat arbitrary, it is necessary simply to ensure that we are close enough to the separatrix to ensure that the errors are not too big.*

To prove Theorem 5.1.1, then, it suffices to note:

Lemma 6.3.6. *Let $F(\phi, \delta)$ be the expression of the Poincaré map of the PCR3BP in the (ϕ, δ) variables as given by 6.19. Consider the path in parameter space beginning at $(\mathcal{J}_0, \delta = \varepsilon(\mathcal{J}_0), \eta = \varepsilon(\mathcal{J}_0)^{a(q+1)})$ where $\varepsilon(\mathcal{J}_0)$ is given by is given in* (6.18). *Then*

$$\gamma : (t_0, \infty) \to \mathbb{R}^2 \ni (\mathcal{J}, \delta, \eta) \tag{6.26}$$

$$t \mapsto (\mathcal{J}_0 + t, \varepsilon(\mathcal{J}_0 + t)^a, \varepsilon(\mathcal{J}_0 + t)^{a(q+1)}) \tag{6.27}$$

is the expression given in 6.3.3 and $a = 1 - \epsilon$ for $\epsilon \in (0, 1)$. Then the family of parabolic models given by $F_{\kappa(\mathcal{J}_0), \frac{3}{2}}$ satisfy the hypotheses of 6.2.2.

6.4 A Quantitative K.A.M. theorem and Results for G_0 Large

We now study the parabolic map associated to the PCR3BP by applying a quantitative K.A.M. theorem, based on the seminal theorem of Kolmogorov, Moser and Arnold. A wordy statement of one version of the celebrated K.A.M. theorem can be found in Arnold's [53].

Theorem 6.4.1 (Kolmogorov's theorem)**.** *If the unperturbed system is nondegenerate or isoenergetically nondegenerate, then for a sufficiently small Hamiltonian perturbation most nonresonant invariant tori do not vanish but are only slightly deformed, so that in the phase of the perturbed system there are invariant tori densely filled with conditionally-periodic phase curves winding around them, with a number of independent frequencies equal to the number of degrees of freedom. These invariant tori form a majority in the sense that the measure of the complement of their union is small when the perturbation is small. In the case of isoenergetic nondegeneracy, the invariant tori form a majority on each level manifold of the energy.*

The crux of K.A.M. theory is the continuation of K.A.M. tori under small perturbations of the system. The persistence of these K.A.M. tori is intimately linked to their frequency vector. More specifically, one expects tori with fundamental frequencies that are "far" from commensurable to survive. The original proof of the K.A.M. theorem hinges on action-angle variables. One writes the perturbed Hamiltonian in action-angle variables as

$$H(I, \theta) = H_0(I) + \varepsilon H_1(I, \theta)$$

and then searches for a generating function $S(\theta, J)$ which would transform the system into new action-angle variables (ϕ, J) with Hamiltonian $K(J)$, resulting in a Fourier series function for $S(\theta, J)$ which converges when the frequencies are ”sufficiently irrational”.

6.4.1 Summary of the Parameterization Method

The above method of searching for K.A.M. tori is not suitable in our case, not least because the perturbation of the Kepler problem giving the R3BP is not amenable to expansion in action-angle variables. For this reason, we use rather the parameterization method, which searches for invariant objects. This leads to K.A.M. without action-angle variables [54]. The method has developed quickly and has been applied to look for different types of invariant objects in numerous different contexts (see for instance [55, 56] for two examples among many) and is particularly successful as it is well suited for finding numerical estimates and also can be adapted for computer-assisted proofs. Even though numerical calculations are not a concern, we take advantage of the easily checkable conditions for the existence of an invariant torus here. A good introduction to the parameterization method for K.A.M. tori unifying the many different approaches can be found in [57] and we will take the statement of the quantitative K.A.M. theorem verbatim from the same. Here, the quantitative K.A.M. theorem is applied to the standard map to prove the persistence of the invariant curve of the integrable limit of frequency the golden mean. The application can be mildly adjusted to prove the persistence of this torus in the parabolic models given in 6.1.1. We will follow the derivation giving remarks on the various required adjustments and thus conclude Theorem 5.1.2. The rest of this section is devoted to defining the objects needed to state the K.A.M. theorem in 6.4.13 following the derivation for the standard map given in [57].

The Parameterization of the Torus

The parameterization method for K.A.M. tori considers a model invariant torus

$$\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$$

homotopic to the zero section of the torus in an ambient annulus

$$\mathscr{A} \subset \mathbb{T}^n \times \mathbb{R}^{2n}$$

with coordinates $\{(x, y)|x \in \mathbb{T}^n, y \in \mathbb{R}^n\}$ and the image of such a torus under an exact symplectic mapping homotopic to the identity

$$F : \mathscr{A} \to \mathscr{A} \tag{6.28}$$

$$(x, y) \mapsto (F^x(x, y), F^y(x, y)) \tag{6.29}$$

The parameterization of the torus is the embedding $K : \mathbb{T}^d \to \mathscr{A}$ satisfying the invariance equation

$$F \circ K = K \circ R_\omega$$

where R_ω is a rigid rotation of the torus. To search for such an invariant parameterization, one first uses an ansatz parameterization and computes the error function

$$E = F \circ K - K \circ R_\omega \tag{6.30}$$

Then, if the system satisfies a certain non-degeneracy condition and the error is small in a certain norm, one can conclude the existence of a true invariant torus close by, by virtue of an algorithm iteratively adding a correction to the ansatz torus. More specifically, the parametrization method defines a constant C which depends in a polynomial fashion on system-specific constants and the initial ansatz torus and for which, if the condition

$$C\|E\| < 1 \tag{6.31}$$

is satisfied, then there exists an actual invariant torus close to the ansatz torus. To get rigorous results, one only needs to check an inequality involving (many) constants depending on the particulars of the system, which then guarantees the convergence of the K.A.M. process. Analogous to traditional K.A.M algorithm, the correction to the approximately invariant torus is found by solving a "cohomological equation" and the convergence of this process hinges on the error of the corrected torus being quadratic in the error of the original torus, which allows the scheme to overcome the problem of small divisors occurring in the construction of the solution.

The following sections follow [57] to detail the construction of the error term to explain the origin of the constant C in (6.31) and to be able to state the K.A.M. theorem, Theorem 6.4.13.

Construction of the Correction Term

The K.A.M. theorem studies a linearization of the invariance equation via the construction of an approximately symplectic frame for the approximately Lagrangian ansatz torus $\mathscr{K} = K(\theta)$.

Definition 6.4.2. *Let F fulfill the conditions of (6.28). Let $\mathscr{K}$ be an approximately invariant torus of F, K an embedding of $\mathscr{K}$ with error equation (6.30). Then the linearized error equation is given by*

$$\mathrm{D}F(K(\theta))\Delta K(\theta) - \Delta K(f(\theta)) - \mathrm{D}K(f(\theta))\Delta f(\theta) = -E(\theta) \tag{6.32}$$

and the quantity $E(\theta)$ is referred to as the error of the torus.

Geometric Objects

To solve the above equation (6.32) the following linear operators are introduced

Definition 6.4.3. *Let $\mathscr{K}$ be a Lagrangian torus homotopic to the zero section of the torus in an ambient annulus $\mathscr{A}$ with an embedding*

$$K : \mathbb{T}^n \to \mathscr{A} \tag{6.33}$$

then the linearized embedding of $\mathscr{K}$ is an operator L defined by

$$L : \mathbb{T} \to \mathbb{R}^{2n\times n} \tag{6.34}$$

$$L(\theta) = DK(\theta) \tag{6.35}$$

The equations in the normal direction are linearized using a type of "adapted frame" which is specialized to the symplectic nature of the system

Definition 6.4.4. *An adapted frame for the torus $\mathscr{K}$ is a mapping $P(\theta) = L(\theta)N(\theta)$ for some*

$$N : \mathbb{T} \to \mathbb{R}^{2n\times n} \tag{6.36}$$

with the property that

$$P : \mathbb{T} \to \mathbb{R}^{2n\times 2n} \tag{6.37}$$

is invertible for all $\theta \in \mathbb{T}$. $N(\theta)$ is then called a complementary frame *for $L(\theta)$.*

An adapted symplectic *frame for a Lagrangian torus is an adapted frame satisfying*

$$P(\theta)^\top \Omega(K(\theta))P(\theta) = \Omega_0$$

where $\Omega(K(\theta))$ *is the symplectic matrix of the symplectic form* ω *at* $K(\theta)$ *and* Ω_0 *the standard symplectic matrix.*

$$\Omega_0 = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$$

This symplectic frame is constructed via a linear map parameterizing the normal directions. The error in a symplectic frame measures how far the transformed symplectic form is from the canonical one.

Definition 6.4.5. *The error in a symplectic frame is given by*

$$E_{\text{sym}}(\theta) = P(\theta)^\top \Omega(K(\theta))P(\theta) - \Omega_0$$

Errors in the Linearized Dynamics

Suppose $\mathscr{K}$ was a true invariant Lagrangian torus. the linearized dynamics around an invariant torus is upper-triangular and symplectic in some appropriate coordinates, a feature called *automatic reducibility*. Specifically, one could choose a symplectic adapted frame $P(\theta)$ satisfying

$$P(\theta+\omega)^{-1}\mathrm{D}F(K(\theta))P(\theta) = \Lambda(\theta), \quad \Lambda(\theta) = \begin{pmatrix} I_n & T(\theta) \\ O_n & I_n \end{pmatrix}$$

where the submatrix $T(\theta)$ which gives the twisting around the invariant torus is known as the *torsion matrix*, which has the following expression:

Definition 6.4.6. *Let* F *be an exact symplectic mapping of the annulus,* $K(\theta)$ *be an invariant torus as defined by* (6.33) *and* N *a complementary subspace as given by Definition 6.4.4. Then the torsion matrix* $T(\theta)$ *is defined*

$$T(\theta) = N(\theta+\omega)^\top \Omega(K(\theta+\omega))\, \mathrm{DF}(K(\theta))N(\theta)$$

Likewise, an approximately invariant torus exhibits linear dynamics approximately reducible to a linear skew-product. The *error in reducibilty* measures how far the dynamics around the approximate torus are from those of a true invariant torus.

Definition 6.4.7. *The error in reducibility is the difference*

$$E_{\text{red}}(\theta) = P(f(\theta))^{-1}\mathrm{D}F(K(\theta))P(\theta) - \Lambda(\theta) \tag{6.38}$$

Construction of the Symplectic Frame

In the case that there is a plane field transversal to the chosen embedded torus the construction of an adapted symplectic frame as given by Definition 6.4.4 proceeds as follows: By assumption, there is a map

$$N^0 : \mathbb{T}^n \to \mathbb{R}^{2n\times n}$$

such that $(L(\theta)N^0(\theta))$ has non vanishing determinant. To find a complementary subspace one writes

$$N(\theta) = L(\theta)A(\theta) + N^0(\theta)B(\theta) \tag{6.39}$$

where $N(\theta)$ should satisfy the conditions

$$L(\theta)^\top \Omega(K(\theta))N(\theta) = -I_n, \quad N(\theta)^\top \Omega(K(\theta))N(\theta) = O_n$$

leading to

$$B(\theta) = -\left(L(\theta)^\top \Omega(K(\theta))N^0(\theta)\right)^{-1} \tag{6.40}$$

and

$$A(\theta) = -\frac{1}{2}\left(B(\theta)^\top N^0(\theta)^\top \Omega(K(\theta))N^0(\theta)B(\theta)\right) \tag{6.41}$$

A symplectic frame is obtained by juxtaposing the map $L(\theta)$ with a map $N(\theta)$ giving the complementary directions so that

$$P(\theta) = (L(\theta)N(\theta))$$

Extensions to a Complex Strip

The parameterized K.A.M. Theorem, as the original, takes advantage of the fact that by extending a mapping analytically to the complex strip, one can control errors in the reals.

Definition 6.4.8. *A complex strip of a torus $\mathbb{T}^n$ of width ρ is the following subset of $\mathbb{C}^n/\mathbb{Z}^n$*

$$\mathbb{T}^n_\rho = \{\theta \in \mathbb{C}^n/\mathbb{Z}^n : |\mathrm{Im}\,\theta_i| < \rho, i = 1, \dots, n\}$$

Definition 6.4.9. *A complex strip $\mathscr{B}$ of an annulus $\mathscr{A}$ is a connected open neighbourhood of $\mathscr{A}$ in $(\mathbb{C}^n/\mathbb{Z}^n)\times\mathbb{C}^n$*

$$\mathscr{A} \subset \mathscr{B} \subset (\mathbb{C}^n/\mathbb{Z}^n)\times\mathbb{C}^n$$

We define the following norms

Definition 6.4.10. *Let $u : \mathbb{T}^n_\rho \to \mathbb{C}$ be an analytic function, continuous on the boundary of $\mathbb{T}^n_\rho$ and real for real values of z. Then the ρ-norm of u is defined*

$$\|u\|_\rho = \sup_{\theta \in \mathbb{T}^n_\rho} |u(\theta)|$$

Definition 6.4.11. *For functions $u : \mathscr{B} \to \mathbb{C}$ analytic on $\mathscr{B}$, continuous on the boundary and real on $\mathscr{A}$ define the $\mathscr{A}$-norm of u*

$$\|u\|_{\mathscr{B}} = \sup_{z \in \mathscr{B}} |u(z)|$$

Definition 6.4.12. *For matrices of such functions we define the following norm*

$$\|A\|_\rho = \max_{\{1,\ldots,n_1\}} \sum_{j=1}^{m_2} \|A_{i,j}\|_\rho$$

and similar for $\|A\|_{\mathscr{B}}$

6.4.2 Statement of the K.A.M. Theorem

We are now in a position to state the quantitative K.A.M. theorem.

Theorem 6.4.13. *Consider an exact symplectic mapping $F : \mathscr{A} \to \mathscr{A}$ on an annulus $\mathscr{A}$ equipped with an exact symplectic structure $\omega = d\alpha$ such that the following conditions are satisfied*

1. *The map F the 1-form α and the 2-form ω are real-analytic and can be analytically extended to some complex strip $\mathscr{B}$ and continuously up to the boundary. Moreover, there are constants $c_{F,1}, c_{F,2}, c_{\Omega,0}, c_{\Omega,1}, c_{a,1}$, and $c_{a,2}$ such that*

$$\|\mathrm{DF}\|_{\mathscr{B}} \leq c_{F,1}, \left\|\mathrm{D}^2 F\right\|_{\mathscr{B}} \leq c_{F,2}, \|\Omega\|_{\mathscr{B}} \leq c_{\Omega,0}, \|\mathrm{D}\Omega\|_{\mathscr{B}} \leq c_{\Omega,1}, \|\mathrm{D}a\|_{\mathscr{B}} \leq c_{a,1}$$

   and

$$\left\|\mathrm{D}^2 a\right\|_{\mathscr{C}} \leq c_{a,2}$$

2. *There exists $K : \mathbb{T}^n \to \mathscr{A}$, homotopic to the section, that can be analytically extended to $\mathbb{T}^n_\rho$ with $\rho > 0$, and continuously up to the boundary. Moreover, there exist constants σ_L and σ_L^* such that*

$$\|\mathrm{D}K\|_\rho < \sigma_L, \quad \left\|\mathrm{D}K^\top\right\|_\rho < \sigma_L^*, \quad \mathrm{dist}\left(K\left(\mathbb{T}^n_\rho\right), \partial\mathscr{B}\right) > 0$$

3. *There exists a map $N^0 : \mathbb{T}^n \to \mathbb{R}^{2n\times n}$ that is real-analytic and can be analytically extended to $\mathbb{T}^n_\rho$, and continuously up to the boundary. Moreover, there exist constants $c_{N^0}, c^*_{N^0}, \sigma_B$, and σ_B^* such that*

$$\left\|N^0\right\|_\rho \leq c_{N^0}, \quad \left\|\left(N^0\right)^\top\right\|_\rho \leq c^*_{N^0}, \quad \|B\|_\rho < \sigma_B, \quad \left\|B^\top\right\|_\rho < \sigma_B^*$$

where $B(\theta) = -\left(DK(\theta)^\top \Omega(K(\theta))N^0(\theta)\right)^{-1}$

4. *There exists a constant σ_T such that the torsion of F restricted to K satisfies the non-degeneracy condition $|\langle T\rangle^{-1}| < \sigma_T$.*

5. *The frequency vector ω satisfies Diophantine conditions of type (γ, τ)*

Then, for every $0 < \rho_\infty < \rho$ there exists a constant $\hat{C}_*$ such that if

$$\frac{\hat{C}_*\|E\|_\rho}{\gamma^4\rho^{4\tau}} < 1$$

then there exists a F-invariant torus $\mathscr{K}_\infty = K_\infty\left(\mathbb{T}^n\right)$, with the same frequency ω, analytic in $\mathbb{T}^n_{\rho_\infty}$, that satisfies

$$\|\mathrm{D}K_\infty\|_{\rho_\infty} < \sigma_L, \quad \left\|\mathrm{D}K_\infty^\top\right\|_{\rho_\infty} < \sigma_L^*, \quad \mathrm{dist}\left(K_\infty\left(\mathbb{T}^n_{\rho_\infty}\right), \partial\mathscr{B}\right) > 0$$

and $\mathscr{K}_\infty$ is close to the ansatz torus in a sense to be made precise later.

6.5 Application of the K.A.M. Theorem to Rescaled Parabolic Models

We will apply the parameterized K.A.M. theorem, Theorem 6.4.13 to the models 6.1.1. This section is devoted to defining the constructions of the previous case for the case of the parabolic models. In particular, our aim is to give quantitative estimates for when the following proposition is true:

Proposition 6.5.1. *Let $F : \mathcal{C}_a \to \mathcal{C}_a$ be the rescaled Poincaré map of the PCR3BP as given by* (5.46). *Let K_{ω_0} be a parameterization of the planar torus*

$$K(\theta) = \begin{pmatrix} \theta \\ \omega_0 \end{pmatrix}, \quad \mathrm{D}K(\theta) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{6.42}$$

with rotation number $\omega_0 = (\sqrt{5}-1)/2$ the golden mean. Then for $\mathcal{J}_0$ and $\delta_0(\mathcal{J}_0)$ sufficiently large, K_{ω_0} continues to an invariant curve of the PCR3BP.

First we define some auxiliary constants used to calculate the constant C_* in the K.A.M. theorem 6.4.13.

Definition 6.5.2. *Let $g(\phi,\delta) = g(\phi,\bar{\delta}(\phi,\delta))$, $h(\phi,\delta) = h(\phi,\bar{\delta}(\phi,\delta))$ be as in Definition 6.1.1 and assume they have an analytic extension to some complex strip $\mathscr{B}$. Denote by c_g and c_h constants satisfying*

$$\|g\|_{\mathscr{B}} \leq c_g \quad \textit{and} \quad \|h\|_{\mathscr{B}} \leq c_h$$

Denote by $c_{g,\phi}$ and $c_{h,\phi}$ constants satisfying

$$\|\partial_\phi g\|_{\mathscr{B}} \leq c_{g,\phi} \quad \textit{and} \quad \|\partial_\phi h\|_{\mathscr{B}} \leq c_{h,\phi}$$

respectively. Denote by $c_{g,\delta}$ and $c_{h,\nu}$ constants satisfying

$$\|\partial_\nu g\|_{\mathscr{B}} \leq c_{g,\nu} \quad \textit{and} \quad \|\partial_\delta h\|_{\mathscr{B}} \leq c_{h,\nu}.$$

For the particular case of the linearized parabolic model describing the PCR3BP then, we use the estimates from Lemma 6.3.2 and we have the following:

Lemma 6.5.3. *Let $F(\phi, v)$ be the linearized parabolic model in a level set of the Jacobi constant $\mathcal{J}^{-1}(\mathcal{J}_0)$ associated to the PCR3BP where we choose $\eta = \delta_0^{5/2}$. Then we can choose the constants in definition 6.5.2 as the following:*

- $c_g = 2K\mu^2(1-2\mu)\mathcal{J}_0^2 e^{\frac{-\mathcal{J}_0^3}{3}} \cosh(2\pi\tilde{\rho})$

- $c_h = \mathcal{J}_0^{-3} c_g$

- $c_{g,\nu} = \mathcal{J}_0^3 c_g$

- $c_{h,\nu} = c_g$

- $c_{g,\phi} = c_g$
- $c_{h,\phi} = \mathcal{J}_0^{-3} c_g$

We can now use the estimates of Lemma 6.5.3 to give constants which will be needed to show the validity of Theorem 6.4.13.

Lemma 6.5.4. *Consider the Poincaré map of the PCR3BP. Consider the associated rescaled parabolic model $F(\phi, \nu, \mathcal{J}_0)$ as defined in* (6.14). *Denote $\varepsilon/\delta_0^{q+1}$ by κ. Let $K(\theta)$ be the planar torus at $\delta = \delta_0$ as given by 6.42 where δ_0. Then we have*

$$\|\mathrm{D}F\|_{\mathscr{B}} \leq c_{F,1} = 2 + \kappa \cosh(2\pi\tilde{\rho}) + c_{g,\phi} + c_{g,\nu} \tag{6.43}$$

$$\left\|\mathrm{D}^2 F\right\|_{\mathscr{B}} \leq c_{F,2} = 2\pi\varepsilon \cosh(2\pi\bar{\rho}) + \tilde{c}_{F,2} \tag{6.44}$$

where $\tilde{c}_{F,2} = 2\mathcal{J}_0^6 c_g$. Let the rescaling factor η be given by $\eta = \delta_0^{q+1}$. Then the error for the torus $K(\theta)$ is given

$$\|E\|_\rho \leq \left(\kappa + \mathcal{O}(\mathcal{J}_0^{-2})\right) \cosh(2\pi\rho). \tag{6.45}$$

Now in order to show the existence of a K.A.M. torus it is sufficient to show that the parabolic return map together with the flat torus satisfies the hypotheses of Theorem 6.4.13.

Theorem 6.5.5. *Let the map $F(\phi, \nu, \mathcal{J}_0) = (\bar{\phi}(\phi, \nu, \mathcal{J}_0), \bar{\nu}(\phi, \nu, \mathcal{J}_0))$ be the Poincaré return map of the PCR3BP as defined by* (6.14) *and denote $\kappa(\mathcal{J}_0, \delta^{-3/2}) = \varepsilon\left(\mathcal{J}_0/\delta^{-3/2}\right)$. Let $\mathcal{J}_0$ be large enough to guarantee the validity of Theorem 5.3.2 and Theorem 5.4.1. Let δ_0 be such that*

$$\delta_0^{-3/2} + \varphi(\delta_0) \bmod 2\pi = \omega_0 \tag{6.46}$$

where $\varphi(\delta_0)$ is the ϕ independent difference in the return time between the Kepler problem and the PCR3BP as defined in the statement of Lemma 6.3.2 and ω_0 is the golden mean

$$\omega_0 = \frac{1+\sqrt{5}}{2}.$$

Let $\kappa_0 = \varepsilon(\mathcal{J}_0)/\delta_0^{3/2}$ satisfy

$$\kappa_0 < \tilde{C} \tag{6.47}$$

where $\tilde{C}$ is a constant defined up to $\mathcal{O}(\mathcal{J}_0^{-2})$ and given explicitly by $\tilde{C} = 10^{-7} + \mathcal{O}\left(\mathcal{J}_0^{-2}\right)$. Then the invariant curve $(\phi, \delta = \delta_0)$ persists[1] *in the PCR3BP.*

[1] Here, "persists" means that there is an invariant curve close to the stated one, see Remark 6.5.6

Proof. We choose as the complex strip $\mathscr{B}$ of the annulus $\mathscr{A}$ the following:

$$\mathscr{B} = \mathscr{B} = \mathbb{T}_{\tilde{\rho}} \times \mathcal{D}$$

where $\mathcal{D}$ is a domain in the complex plane given by

$$\mathcal{D} = \{z \in \mathbb{C} \,|\, \Re(z) < 1, \Im(z) < \tilde{\rho}\}.$$

where $\tilde{\rho}$ will be given later. Then, referring to the hypotheses of Theorem 6.4.13 we have

1. H_1 is clearly satisfied for map 6.1.1 where $\omega = d\alpha = d(\phi d\nu)$. For the constants listed in H_1 associated to the forms ω and α we have $c_{\Omega,0} = 1, c_{\Omega,1} = 0, c_{a,1} = 1$, and $c_{a,2} = 0$. The values of the constants $c_{F,1}$ and $c_{F,2}$ are given by Lemma 6.5.4.

2. For H_2 we note that choosing the planar torus (6.42) which has a complex extension

$$K(\mathbb{T}_\rho) = \mathbb{T}_\rho \times \{\omega_0\}.$$

   where $\rho < \tilde{\rho}$. We can then choose $\|\mathrm{D}K\|_\rho = 1$ and $\left\|\mathrm{D}K^\top\right\|_\rho = 1$ for any for any $\rho > 0$ and so we choose $\sigma_L = \|\mathrm{D}K\|_\rho \sigma = \sigma$ and $\sigma_L^* = \left\|\mathrm{D}K^\top\right\|_\rho \sigma = \sigma$ where σ is chosen to optimize the constant C_* and given in (6.48).

3. For hypothesis H_3 a symplectic frame is given by

$$N^0(\theta) = N^0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

   and so we can take $\|N^0\|_\rho = c_{N^0} = 1$ and $\left\|(N^0)^\top\right\|_\rho = c^*_{N^0} = 1$. Looking at the definition of B given in 6.40 have $B(\theta) = 1$ and so we can take

$$\sigma_B = \|B\|_\rho \sigma = \sigma, \quad \sigma_B^* = \left\|B^\top\right\|_\rho \sigma = \sigma.$$

4. For hypothesis H_4 we note that the torsion of F restricted to K is given by 6.4.6 and so we choose $|\langle T\rangle^{-1}| < \sigma$. See Appendix C for a discussion of the torsion.

5. Hypothesis H_5 is satisfied for the golden mean with the constants

$$\gamma = \frac{3 - \sqrt{5}}{2} \text{ and } \tau = 1$$

   (see Appendix B in [58]).

The constant C_* is computed explicitly C up to $\mathcal{O}(\mathcal{J}_0^{-2})$ and the condition

$$\frac{\hat{C}_*(\delta_0, \mathcal{J}_0)\|E\|_\rho}{\gamma^4\rho^{4\tau}} < 1$$

is validated. □

In order to give explicitly the constant C_* of the K.A.M. theorem 6.4.13 and show the continuation of the invariant curve in the case of the Poincaré map of the PCR3BP for the associated value of $\kappa(\mathcal{J}_0, \delta_0)$, the method of [57] for the proof of the invariant curve of the standard map 6.1 with frequency the golden mean is followed with the relevant adjustments. Some constants require some extra discussion due to the difference in torsion and the constants $c_{F,1}$ and $c_{F,2}$. However the dominant errors come from the error function $E(\theta)$ which is greater in the case of the linearized parabolic models due to the fact that we do not know the splitting exactly. In the specific case of the Planar Restricted three-body Problem, this error can be given as approximately $\mathcal{J}_0^{-2}\kappa$ where κ is a measure of the size of the splitting of separatrices as approximated by the first harmonic of the Melnikov integral. A detailed summary of the application of the K.A.M. theorem to the stardard map taken from [57] detailing the difference between the standard map and the linearized parabolic model of the PCR3BP is given in Appendix C.

In this case, the K.A.M. theorem 6.4.13 was shown to be satisfied for a values of

$$\rho = 0.42264, \quad \delta = 0.087909, \quad \sigma = 1.0706, \quad \tilde{\rho} = 0.42284 \tag{6.48}$$

To prove that the K.A.M. Theorem converges, it is necessary that the following inequality is satisfied:

$$\frac{\hat{C}_*(\kappa + \frac{c_h}{\delta_0^{5/2}})\cosh(2\pi\rho)}{2\pi\gamma^4\rho^{4\tau}} < 1$$

So it is necessary to find a constant

$$\tilde{C}_*$$

satisfying

$$\frac{\tilde{C}_*\kappa(\mathcal{J}_0, \delta_0)\cosh(2\pi\rho)}{2\pi\gamma^4\rho^{4\tau}} < 1 + \mathcal{O}(\mathcal{J}_0^{-1})$$

In [57] it was shown that for the standard map, the planar torus 6.33, the bounds on the geometric quantities given in Appendix C and the values given in (6.48) that

$$\frac{\tilde{C}_* \cosh(2\pi\rho)}{2\pi\gamma^4\rho^{4\tau}} \approx 2 \times 10^5. \tag{6.49}$$

Therefore we may assume that given a value of κ which satisfies

$$\kappa(\mathcal{J}_0, \delta_0) < 10^{-7} + \mathcal{O}(\mathcal{J}_0^{-2})$$

there is continuation of the invariant curve for $\mathcal{J}_0$ large enough. Now, as for the linearized parabolic model the constant $\kappa(\delta_0, \mathcal{J}_0)$ is given by

$$\kappa(\mathcal{J}_0, \delta_0) = \frac{\varepsilon(\mathcal{J}_0)}{\delta_0^{5/2}}$$

to ensure the continuation of the invariant curve one only needs to guarantee $\delta_0 > \mathcal{J}_0$.

Remark 6.5.6. *The meaning of the word "persists" in the statement of the previous theorem is as follows: Let $K(\mathbb{T}_\rho)$ be the complex extension of the planar torus specified in Theorem 6.4.13, where ρ is chosen in* (6.48)*. Then there exists an invariant curve K_∞ which is analytic in a complex strip of width ${\rho_\infty}$[2] such that there exists an invariant curve which satisfies*

$$\|K_\infty - K\|_{\rho_\infty} < \frac{\hat{C}_{**}}{\gamma^2\rho^{2\tau}}\|E\|_\rho$$

where the constant $\hat{C}_{**}$ is a computable constant which depends polynomially on the quantities given in appendix C. We do not calculate the constant $\hat{C}_{**}$.

Remark 6.5.7. *In [57] the value of κ_0 can be greatly improved in the case of the standard map by taking an ansatz torus defined via a Lindstedt series. By a similar method, one can dramatically increase the value of κ_0 given in 6.5.5. Via computer-assisted proofs, the value of κ in the standard map 6.1 for which the map has an invariant curve has been shown in to be approximately the value given in* (6.9)*. The discussion of this chapter both opens the quantities $\mathcal{J}_0$ and $\delta_0(\mathcal{J}_0)$ to optimization by computer-assisted proof and motivates the study of soft invariant curves of the "linearized parabolic maps" of the form given in 6.2.4.*

[2] In [57] this ρ_∞ was given as $\rho_\infty = 0.00042264$

Chapter 7

Singular Symplectic Structures in Celestial Mechanics

An astronomer must be the wisest of men; his mind must be duly disciplined in youth; especially if mathematical study is necessary; both an acquaintance with the doctrine of number, and also with that other branch of mathematics, which, closely connected as it is with the science of the heavens, we very absurdly call geometry, the measurement of the earth.

Plato, The Laws

The phase space of many systems is symplectic, the non-degeneracy of the symplectic structure ensuring that there is an isomorphism between the tangent and cotangent spaces which can be used to associate vector fields to Hamiltonians uniquely[1]. However, there are important natural examples of dynamical systems occurring on more general Poisson manifolds: systems defined on a "reduced" phase space (see Chapter 3) are one important source of examples. This chapter will focus on another interesting source of examples: degenerate symplectic forms in celestial mechanics.

It was noted in [59] that structures which are symplectic almost everywhere can arise as the result of non-canonical and singular changes of coordinates. The resulting "singular symplectic forms" can be b-symplectic, b^m-symplectic or m-folded symplec-

[1] up to an additive constant

tic. Analogous to the symplectic case, level energy sets of singular symplectic manifolds satisfying certain conditions have b-contact structures associated with them.

[60] gave an introduction to these singular symplectic geometries, reviewed previous examples and gave several new ones. Applications to the dynamics of some of the examples given here are given in [61].

7.1 Preliminaries

In Chapter 3, b-symplectic (Definition 2.3.5), or equivalently b-Poisson (Definition 2.2.3) structures were considered. These are examples of Poisson structures almost everywhere symplectic whose degeneration is the "mildest possible". A generalization of these structures, "b^k-symplectic structures", includes the case of higher order dengenerations of the Poisson structure. "b^k-symplectic structures" were introduced in [16].

7.1.1 b^k-symplectic structures

Analogous to the definition of b-tangent bundles and b-forms, it is possible consider bundles with higher order tangency to some critical manifold and correspondingly dual forms of higher order singularities. In this case, in order to make the notion well defined, one needs to consider the k-jets, J^k_Z, of the critical hypersuface Z.

Definition 7.1.1. *For $k \geq 1$, a b^k -manifold is a triple (M, Z, j_Z) where $\boldsymbol{M}$ is an oriented manifold. $Z \subseteq M$ is an oriented hypersurface and j_z an element of J^{k-1}_Z that can be represented by a positively oriented local defining function y for Z.*

Around each point $p \in Z$ we can find coordinates in which the space of b^k vector fields is generated

$$\left\{ \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{n-1}}, x_n^k \frac{\partial}{\partial x_n} \right\}$$

similar to the case of b-geometry we can conclude that such fields are the sections of a vector bundle, called the b^k-manifold. A nondegerate closed two-form on the dual bundle, then, is a b^k-symplectic form.

Example 7.1.2. *Consider $\mathbb{R}^{2n}$ with coordinates $(x_1, y_1, \cdots, x_n, y_n)$. A prototypical b^k-symplectic manifold is the b^k-manifold $(\mathbb{R}^{2n}, x_1 = 0, \mathbf{0})$ with b^k-Darboux form*

$$\left(x_1^k \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right)$$

7.1.2 Folded Symplectic Forms

Analogous to the definition of b-Poisson structures *folded symplectic forms* can be defined as symplectic forms which degenerate along a codimension-one hypersurface in a mild manner.

Definition 7.1.3. *Let (M^{2n}, ω) be a manifold with ω a closed 2-form such that the map*

$$p \in M \mapsto (\omega(p))^n \in \Lambda^{2n}(T^*M)$$

*is transverse to the zero section , then $Z = \{p \in M | (\omega(p))^n = 0\}$ is a hypersurface and we say that ω defines a **folded symplectic structure** on (M, Z) if additionally its restriction to Z has maximal rank. We call the hypersurface Z **folding hypersurface** and the pair (M, Z) is a **folded symplectic manifold**.*

There is a folded Darboux theorem [62] given by

Theorem 7.1.4. *Let ω be a folded symplectic form on (M^{2n}, Z) and $p \in Z$. Then we can find a local coordinate chart $(x_1, y_1, \ldots, x_n, y_n)$ centered at p such that the hypersurface Z is locally defined by $y_1 = 0$ and*

$$\omega = y_1 dx_1 \wedge dy_1 + \sum_{i=2}^{n} dx_i \wedge dy_i.$$

In [59] examples of dynamical systems with naturally occurring singular symplectic structures were given, with examples coming from celestial mechanics and projective dynamics. In [60] the set of examples were extended.

7.2 Point Transformations and Singular Symplectic Forms

Given configuration space $\mathbb{R}^2$ and phase space $T^*\mathbb{R}^2$ as is seen, for example, in the Kepler problem, the traditional (canonical) Levi-Civita transformation is the following: identify $\mathbb{R}^2 \cong \mathbb{C}$ so that $T^*\mathbb{R}^2 \cong T^*\mathbb{C} \cong \mathbb{C}^2$ and treat (q, p) as complex variables $(u :=$

$q_1 + iq_2, v := p_1 + ip_2)$. Take the following change of coordinates $(q,p) = (u^2/2, v/\bar{u})$, where $\bar{u}$ denotes the complex conjugation of u. The resulting coordinate change can easily be seen to be canonical. However this canonical change of coordinates can result in more difficult equations of motion, or a more difficult Hamiltonian, which can both obscure certain aspects of the dynamics of the system.

7.2.1 The Kepler Problem

In suitable coordinates in $T^*\left(\mathbb{R}^2 \setminus \{0\}\right)$, the Kepler problem has Hamiltonian

$$H(q,p) = \frac{\|p\|^2}{2} - \frac{1}{\|q\|}. \tag{7.1}$$

With the canonical Levi-Civita transformation $(q,p) = (u^2/2, v/\bar{u})$, this becomes

$$H(u,v) = \frac{\|v\|^2}{2\|\bar{u}\|^2} - \frac{1}{\|u\|^2}. \tag{7.2}$$

Sometimes, as in this case, canonical changes lead to a more difficult system, so it may be desirable to leave the momentum unchanged and examine instead the transformation $(q,p) = (u^2/2, p)$ which can result in a simpler Hamiltonian. Now the transformation is not a symplectomorphism and the symplectic form on $T^*\mathbb{R}^2$ pulls back under the transformation to a two-form symplectic almost everywhere, but degenerate on a hypersurface of $T^*\mathbb{R}^2$.

Explicitly, the Liouville one-form $p_1 dq_1 + p_2 dq_2 = \mathrm{Re}(p d\bar{q})$ pulls back to

$$\begin{aligned} \theta = \mathrm{Re}\left(pd\left(\frac{\bar{u}^2}{2}\right)\right) &= \mathrm{Re}\left(p\bar{u}d\bar{u}\right) \\ &= p_1(u_1du_1 - u_2du_2) + p_2(u_2du_1 + u_1du_2) \end{aligned}$$

and computing $-d\theta$ we get the almost everywhere symplectic form

$$\omega = u_1 du_1 \wedge dp_1 - u_2 du_1 \wedge dp_2 + u_2 du_2 \wedge dp_1 + u_1 du_2 \wedge dp_2.$$

Wedging this form with itself we find

$$\omega \wedge \omega = (u_1^2 - u_2^2) du_1 \wedge dp_1 \wedge du_2 \wedge dp_2$$

which is degenerate along the hypersurface given by $u_1 = \pm u_2$.

7.2.2 The Problem of Two Fixed Centers

Related to the folded symplectic form found in the Levi-Civita transformation is the folded form associated with elliptic coordinates, employed while regularizing the problem of two fixed centers. This describes the motion of a satellite moving in a gravitational potential generated by two fixed massive bodies. We assume also that the motion of the satellite is restricted to the plane in $\mathbb{R}^3$ containing the two massive bodies. The Hamiltonian in suitable coordinates is given by

$$H = \frac{p^2}{2m} - \frac{\mu}{r_1} - \frac{1-\mu}{r_2} \tag{7.3}$$

where μ is the mass ratio of the two bodies (i.e. $\mu = \frac{m_1}{m_1+m_2}$).

Euler first showed the integrability of this problem using *elliptic* coordinates, where the coordinate lines are confocal ellipses and hyperbola. Explicitly, consider a coordinate system in which the two centers are placed at $(\pm 1, 0)$, in which the (Cartesian) coordinates are given by (q_1, q_2). Then the elliptic coordinates of the system are given by

$$q_1 = \sinh\lambda\cos\nu \tag{7.4}$$

$$q_2 = \cosh\lambda\sin\nu \tag{7.5}$$

for $(\lambda, \nu) \in \mathbb{R} \times S^1$. Thus lines of $\lambda = c$ and $\nu = c$ are given by confocal hyperbola and ellipses in the plane, respectively. Similar to the Levi-Civita transformation this results in a double branched covering with branch points at the centers of attraction.

Pulling back the canonical symplectic structure $\omega = dq \wedge dp$ we find

$$\omega = \cosh\lambda\cos\nu(d\lambda \wedge dp_1 + d\nu \wedge dp_2) - \sinh\lambda\sin\nu(d\nu \wedge dp_1 + d\lambda \wedge dp_2) \tag{7.6}$$

which is degenerate along the hypersurface (λ, ν) satisfying $\cosh\lambda\cos\nu = \sinh\lambda\sin\lambda$.

7.3 Escape Singularities and b-Symplectic forms

The restricted elliptic 3-body problem describes the behaviour of a massless object in the gravitational field of two massive bodies, orbiting in elliptic Keplerian motion. The planar version assumes that all motion occurs in a plane. The associated Hamiltonian of the particle is given by

$$H(q,p) = \frac{\|p\|^2}{2} + \frac{1-\mu}{\|q-q_1\|} + \frac{\mu}{\|q-q_2\|} = T + U \tag{7.7}$$

where μ is the reduced mass of the system.

After making a change to polar coordinates $(q_1, q_2) = (r\cos\alpha, r\sin\alpha)$ and the corresponding canonical change of momenta we find the Hamiltonian

$$H(r,\alpha,P_r,P_\alpha) = \frac{P_r^2}{2} + \frac{P_\alpha^2}{2r^2} + U(r\cos\alpha, r\sin\alpha) \tag{7.8}$$

where P_r, P_α are the associated canonical momenta and $U(r\cos\alpha, r\sin\alpha)$ is the potential energy of the system in the new coordinates.

The McGehee change of coordinates is traditionally employed to study the behaviour of orbits near infinity, see also [43]. This non-canonical change of coordinates is given by

$$r = \frac{2}{x^2}. \tag{7.9}$$

The corresponding change for the canonical momenta is easily seen to be

$$P_r = -\frac{x^3}{4} P_x. \tag{7.10}$$

The Hamiltonian is then transformed to

$$H(r,\alpha,P_r,P_\alpha) = \frac{x^6 P_x^2}{32} + \frac{x^4 P_\alpha^2}{8} + U(x,\alpha). \tag{7.11}$$

By dropping the condition that the change is canonical and simply transforming the position coordinate (7.9), we are left with a simpler Hamiltonian, however the pull-back of the symplectic form under the non-canonical transformation is no longer symplectic, but rather b^3-symplectic:

$$\omega = \frac{4}{x^3} dx \wedge dP_r + d\alpha \wedge dP_\alpha. \tag{7.12}$$

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Appendix A

Solutions to the Kepler Problem

A.1 The Orbital Equations

As finding a solution of the equations of motion of the Keplerian equations of motion parameterised by time is difficult, one often expresses a solution for the equations of motion as a function of the polar angle α. From the Hamiltonian equations of motion for the Kepler problem (5.2) we have

$$\frac{dr}{d\alpha} = \frac{yx^2}{G} \quad \text{and} \quad \frac{dy}{d\alpha} = \frac{G}{r} - \frac{1}{G}.$$

Using the substitution $u = 1/r$ then, we find

$$\frac{du}{d\alpha} = -\frac{y}{G} \quad \text{and} \quad \frac{dy}{d\alpha} = G\alpha - \frac{1}{G}.$$

Whence

$$\frac{d^2u}{d\alpha^2} + u = \frac{1}{G},$$

which can be solved to give the general solution

$$u(\alpha) = c_2 \sin(\alpha) + c_1 \cos(\alpha) + \frac{1}{G^2}.$$

Without loss of generality, we can choose initial conditions

$$u_0 = u(0) = c_1 + \frac{1}{G^2}, \quad \left.\frac{du}{d\alpha}\right|_{\alpha=0} = 0.$$

The full solution then being

$$u = \frac{1}{r} = \frac{1}{G^2}\left(1 + e\cos\left(\alpha\right)\right)),$$

where e is related to a constant of integration and gives the eccentricity of the so-defined conic section. To find $e(h, G)$, one evaluates the Hamiltonian (5.1) at $y = 0, r = r_p = \frac{G^2}{1+e}$ (note that this is clearly a turning point for r) to find

$$e = \sqrt{1 - 2h^2 G}.$$

Likewise, the relation

$$h = -\frac{1}{2a}$$

for a the semi-major axis of the orbit can be derived from substituting the equations for r_a, r_p into the Hamiltonian and evaluating them.

The relation between the period of an orbit and its energy is then a consequence of Kepler's third law. This can be derived in a number of different ways, here we will use the solutions of the Keplerian equation in the reparameterized variable τ.

A.2 Reparameterized Solutions

Looking at the Keplerian equations in the reparameterized variable τ we find that

$$\frac{d\alpha}{d\tau} = \frac{2G^2}{r}. \tag{A.1}$$

Substituting

$$\frac{1}{r} = \frac{1}{G^2}(1 + e\cos(\alpha)), \tag{A.2}$$

we find

$$\tau = \int \frac{d\alpha}{(1 + e\cos(\alpha))}. \tag{A.3}$$

Whence

$$\frac{\tau}{2} = \frac{\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\alpha}{2}\right)\right)}{\sqrt{1-e^2}}, \tag{A.4}$$

which can be inverted to give

$$\alpha = 2\arctan\left(\sqrt{\frac{1+e}{1-e}}\tan\left(\sqrt{1-e^2}\frac{\tau}{2}\right)\right). \tag{A.5}$$

Comparing with the equation for the eccentric anomaly

$$\tan\frac{\alpha}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}, \tag{A.6}$$

we find

$$\sqrt{1-e^2}\tau = E. \tag{A.7}$$

Substituting the equation for $\alpha(E)$, one can derive the classic expression for $r(E)$ as given by

$$r = a(1 - e\cos E). \tag{A.8}$$

Appendix B

Expansion of the Singularity of the Parameterized Elliptic Orbits

Lemma B.0.1. *Let* $e < 1$*. Then*

$$\frac{\operatorname{arccosh}\left(\frac{1}{e}\right)}{1-e^2} = 1 + \frac{1-e^2}{3} + \mathcal{O}((1-e^2)^2).$$

Proof. As

$$\tanh\left(\operatorname{arcosh}(x)\right) = \frac{\sqrt{x^2-1}}{x}$$

for $|x| > 1$ for $x = \frac{1}{e}$, $e < 1$ we have

$$\operatorname{arcosh}\left(\frac{1}{e}\right) = \operatorname{arctanh}\left(\sqrt{1-e^2}\right).$$

Using the series expansion for arctanh at $\sqrt{1-e^2} \approx 0$ we have

$$\operatorname{arctanh}(\sqrt{1-e^2}) = \sqrt{1-e^2} + \frac{(1-e^2)^{\frac{3}{2}}}{3} + \mathcal{O}((1-e^2)^{\frac{5}{2}}),$$

which gives the result. □

Appendix C

The Constants of the Quantitative K.A.M. theorem

The constant C^* of the K.A.M. theorem (6.4.13) depends on intermediate constants depending polynomially on the following constants intrinsic to the system: $c_{F,1}$, $c_{F,2}$, $c_{\Omega,0}$, $c_{\Omega,1}$, $c_{a,1}$ and $c_{a,2}$. It depends also on the following quantities, which are affected, for example, by the choice of the geometric objects given in section 6.4.1: $(\sigma_L - \|\mathrm{D}K\|_\rho)^{-1}$, $\left(\sigma_L^* - \|\mathrm{D}K^\top\|_\rho\right)^{-1}$, $(\sigma_B - \|B\|_\rho)^{-1}$, $(\sigma_B^* - \|B\|_\rho)^{-1}$, $(\sigma_T - |\langle T\rangle^{-1}|)^{-1}$ and the distance from the ansatz invariant torus to the boundary of the complex strip $\mathrm{dist}\left(K\left(\mathbb{T}^n_\rho\right), \partial\mathscr{B}\right)^{-1}$. It further depends on the bounds σ_L, σ_L^*, σ_B, σ_B^*, and σ_T given in the statement of the K.A.M. theorem 6.4.13 which can be chosen to optimise the value of C^* after the ansatz torus has been chosen. Following the proof of the persistence of torus of the frequency the golden ration in [57], we note that all estimates given there are satisfied also for the case of the linearized parabolic map with the exception of the constants $C_9, C_{14}, \hat{C}^4, \hat{C}^5, \hat{C}^6$ and constants which depend polynomially the previous terms. We will trace the calculation of C_* given in [57] (Section 4.4.1) and we will discuss the terms $C_9, C_{14}, \hat{C}^4, \hat{C}^5, \hat{C}^6$ as they arise. It turns out the necessary adjustments to the aforementioned constants are negligible in comparison to the error incurred by the difference between the error function associated to the planar torus of the standard map and the error function of the planar torus of the rescaled Poincaré map of the PCR3BP. First we discuss the differences in the choice of complex strip:

In the application of the K.A.M. theorem 6.4.13 to the standard map in [57], the complex strip of the annulus $\mathscr{B}$ was taken to be

$$\mathscr{B} = \mathscr{B} = \mathbb{T}_{\tilde{\rho}} \times \mathbb{C}$$

with $\tilde{\rho} > \rho$. This was possible as for the standard map the constants $c_{F,1}$ and $c_{F,2}$ did not depend on the momentum and so one could pick and unbounded domain in momentum. Note that for the parabolic separatrix maps 6.1.1 we are studying, this is not technically possible as we need the Taylor series of δ^{-q} about δ_0 to converge in order for the linearized approximation given in 6.15 to be valid. Technically, this affects the constants involved in calculating C_*, as some constants depend on the quantity $\mathrm{dist}\left(K\left(\mathbb{T}^n_\rho\right), \partial\mathscr{B}\right)^{-1}$. However, due to the rescaling we can take the variable ν to be in a complex strip

$$\mathscr{B} = \mathscr{B} = \mathbb{T}_{\tilde{\rho}} \times \mathcal{D}$$

where $\mathcal{D}$ is a domain in the complex plane given by, for example

$$\mathcal{D} = \{z \in \mathbb{C} \quad | \quad \Re(z) < 1, \Im(z) < \tilde{\rho}\}$$

and so in practise, considering the constants chosen in (6.48) this does not affect the value of the constant $\hat{C}^*$.

As the bounds for the quantities given in the hypothese of the K.A.M. theorem 6.4.13 we choose, exactly as in the exposition of [57].

$$c_N = c_{N^0}\sigma_B = \sigma, \tag{C.1}$$

$$c^*_N = \sigma^*_B c^*_{N^0} = \sigma, \tag{C.2}$$

$$c_P = \sigma_L + c_N = 2\sigma, \tag{C.3}$$

$$\sigma_T = \sigma, \tag{C.4}$$

$$c_R =| \frac{1}{4\pi}\sqrt{\frac{\pi^2}{3} - 2}, \tag{C.5}$$

where c_R is a constant depending only on the constant τ associated frequency of the planar torus $K(\theta)$ (see Lemma 4.50 in [57]) and c_P controls the size of the matrix $P(\theta)$ given in 6.4.4.

Remark C.0.1. *In fact, in the case of the linearized parabolic separatrix map for the planar torus the torsion is given by:*

$$T(\theta) = 1 - c_{h,\nu} + \mathcal{O}(\eta) \tag{C.6}$$

Therefore, for values of $\mathcal{J}_0$ not large enough, it is possible that the quantity σ does not bound the torsion in the problem and it is necessary to deal with σ_T separately. However, for the eventual value of the constants given in 6.48, for large $\mathcal{J}_0$ this does not present a problem. Likewise for the constant c_T controlling the norm of the torsion matrix (Definition C.6) can be chosen as similar to the bound given in [57] for the standard map

$$c_T \leq \sigma^2,$$

as long as $\mathcal{J}_0$ is large enough.

Errors in the Lagrangian Nature of the Frame

Constants C_1 and C_2 control the error in the Lagrangian nature of the torus. As for the parabolic separatrix map the torus is a one-dimensional submanifold of a two dimensional annulus, in this case we have $c_A = 0$ and

$$C_1 = C_2 = 0 \tag{C.7}$$

Errors in the Symplectic Nature of Frame

As the ansatz torus is automatically Lagrangian, the constant C_3 can controlling the error in the symplectic nature of the frame can be set to zero

$$C_3 = 0 \tag{C.8}$$

The error in reducibility (see Definition 6.4.7) is controlled via the constants C_4, C_5, C_6 and $C_7 = \max\{C_4, C_5 + C_6\}$ which can be taken as:

$$C_4 \leq n c_N^* c_{\Omega,0} \gamma \delta^\tau + c_A C_2 \leq \gamma \delta^\tau \sigma \tag{C.9}$$

$$C_5 \leq C_2 + n \sigma_L^* c_{\Omega,0} \gamma \delta^\tau \leq \gamma \delta^\tau \sigma \tag{C.10}$$

$$C_6 \leq 2\gamma \delta^\tau \sigma \tag{C.11}$$

$$C_7 \leq 3\gamma \delta^\tau \sigma. \tag{C.12}$$

Russman Estimates

The constants associated with the cohomological equations are given by:

$$C_8 \leq c_R\sigma \tag{C.13}$$

$$C_9 \leq c_R\sigma + \sigma^2\left(\gamma\delta^\tau + c_R\sigma^2\right) \tag{C.14}$$

$$C_{10} \leq c_R\sigma\left(1+\sigma^3\right)\left(\gamma\delta^\tau + c_R\sigma^2\right) \tag{C.15}$$

Controlling the Size of the Correction

The constant controlling the correction of the parameterization can then be calculated:

$$\hat{C}_2 \leq \left(c_R\sigma^2\left(1+\sigma^3\right) + \gamma\delta^\tau\sigma^3\right)\left(\gamma\delta^\tau + c_R\sigma^2\right) + c_R\gamma\delta^\tau\sigma^2 \tag{C.16}$$

We can then control the correction to the object $B(\theta)$ as defined in 2.3 via

$$C_{11} \leq 2\hat{C}_2, \quad C_{11}^* \leq \hat{C}_2, \quad \hat{C}_3 \leq 4\sigma^2\hat{C}_2, \quad \hat{C}_3^* \leq 2\sigma^2\hat{C}_2$$

In the proof of the persistance of the planar torus for the standard map the constants C_{14} and $\hat{C}_4$ which control the error in the difference of the torsion between the ansatz torus and the new torus which were taken to be the quatities

$$C_{14} \leq 2\sigma_B\hat{C}_3 \text{ and } \hat{C}_4 \leq 2\sigma_T^2 C_{14}$$

These are different in the case of the parabolic separatrix map as they involve the quatity $c_{F,2}$ given in our case by Lemma 6.5.4. However, these errors are again proportional to $\mathcal{O}(\mathcal{J}_0^{-2})$ and can safely be ignored, as the constant $\hat{C}_*$ will be defined only up to orders of $\mathcal{J}_0^{-2}$.

As $A(\theta) = 0$ we have $C_{12} = 0$. Similarly C_{13} and C_{13}^* are not used. Finally the constant $\hat{C}_*$ is given by

$$\hat{C}_* = \max\left\{\left(a_1a_3\right)^{4\tau}\hat{C}_5, \frac{16\sigma^5\left(a_3\right)^{2\tau+1}\gamma^2\rho^{2\tau-1}\hat{C}_2}{(\sigma-1)\left(1-a_1^{1-2\tau}\right)}, \frac{\left(a_3\right)^{2\tau}\gamma^2\rho^{2\tau}\hat{C}_2}{(\tilde{\rho}-\rho)\left(1-a_1^{-2\tau}\right)}\right\}$$

Where the constants a_1 and a_2 control the reduction in the domain of analyticity of the of the iterated torus and are defined by:

$$\rho_0 = \rho, \quad \delta_0 = \frac{\rho_0}{a_3}, \quad \rho_s = \rho_{s-1} - 3\delta_{s-1}, \quad \delta_s = \frac{\delta_0}{a_1^s}, \quad \rho_\infty = \lim_{s\to\infty}\rho_s = \frac{\rho_0}{a_2}$$

where

$$a_3 = 3\frac{a_1}{a_1 - 1}\frac{a_2}{a_2 - 1} \tag{C.17}$$

Here, as in [57], we take $a_2 = 1000$. a_3 is then set to $a_3 = \rho/\delta$ and the constant a_1 is then calculated using the relation (C.17). Evaluating the quantity $\hat{C}_*$ at these values and using the constants given in (6.48) gives the estimate in Theorem 6.5.5.