Proposed $\overline{X}$ and S Control Charts for Skewed Distributions

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Abstract – **This paper proposes a weighted variance method to compute the limits of the $\bar{X}$ and S charts for skewed distributions. The proposed charts extend the weighted variance $\bar{X}$ and R charts in [1] by enabling a process from a skewed distribution with moderate and large sample sizes to be monitored efficiently, hence producing more favourable Type-I and Type-II error rates than the charts in [1]. Note that the charts in [1] are only intended to be used for small sample sizes. The Type-I and Type-II error rates computed show that the proposed charts outperform the existing heuristic charts, as well as those in [1] for moderate and large sample sizes, involving cases with known and unknown parameters, when the distribution of a process is skewed.**

Keywords - **$\overline{X}$ chart, S chart, weighted variance, weighted standard deviation, skewness correction**

I. INTRODUCTION

In process monitoring, the assumption of a normally distributed process is often violated. A Shewhart chart is not applicable for skewed distributions as its Type-I error rate is inflated.

Nonparametric methods that do not rely on the assumption of normality can be used to deal with nonnormality. However, a major setback of nonparametric methods is that these methods are robust or insensitive to outliers [2]. Another popular strategy for dealing with nonnormality is to transform the data to achieve approximate normality. Among the transformation methods commonly used are the power transformation and Box-Cox procedure [2]. A potential drawback of transformation methods is that the samples are no longer plotted in the original scale of measurement.

To overcome the shortcomings faced by the nonparametric and transformation methods, heuristic $\bar{X}$ and R charts are suggested to address the problems of skewed underlying process distributions. These include the $\bar{X}$ and R charts using the weighted variance (WV) [1] and skewness correction (SC) [3] methods, and the $\bar{X}$ chart using the weighted standard deviation (WSD) method [4].

The standard deviation, σ of the heuristic charts in [1], [3] and [4] are estimated using the sample range, instead of the sample standard deviation. Since the range method in estimating σ loses statistical efficiency as the sample size, n increases [5], the use of the heuristic charts in [1], [3] and [4] is not desirable when n is moderate or large. As n increases, the sample standard deviation is more efficient than the sample range as an estimator of standard deviation. Thus, heuristic charts whose standard deviation are estimated from the sample standard deviation must be constructed for moderate or large n.

This paper proposes the $\bar{X}$ and S charts based on the WV approach, for use when $n \geq 10$. Unlike the standard $\bar{X}$ and S charts, the proposed charts provide asymmetric limits in accordance with the direction and degree of skewness by using different variances in computing the upper and lower limits. Thus, the proposed charts have lower false alarm rates than the standard $\bar{X}$ and S charts for a process from a skewed distribution. Contrary to the weighted variance $\bar{X}$ and R charts suggested in [1], the proposed charts are based on the standard deviation estimated from the sample standard deviation and not the sample range, like in the case of the charts in [1]. It will be shown in Section IV that the proposed charts outperform the existing heuristic charts for the various skewed distributions when process parameters are known and unknown by having lower Type-I and Type-II error rates when n is moderate or large.

In Section II, we review the existing heuristic charts for skewed distributions. The proposed weighted variance $\bar{X}$ and S charts are discussed in Section III. The performances of the proposed charts are compared with other heuristic charts for skewed distributions in Section IV. Finally, conclusions are drawn in Section V.

II. AN OVERVIEW OF HEURISTIC CHARTS FOR SKEWED DISTRIBUTIONS

A. WV $\bar{X}$ and R charts

The control limits of the $\text{WV}-\bar{X}$ chart are [1]

$$\text{UCL}_{\text{WV}-\bar{X}} = \mu_X + \frac{3\sigma_X}{\sqrt{n}}\sqrt{2P_X} \tag{1a}$$

$$\text{LCL}_{\text{WV}-\bar{X}} = \mu_X - \frac{3\sigma_X}{\sqrt{n}}\sqrt{2(1-P_X)}\,, \tag{1b}$$

where $P_X = \Pr(X \leq \mu_X)$. When parameters are unknown, the estimators used are $\hat{\mu}_X = \bar{\bar{X}}$, $\hat{\sigma}_X = \frac{\bar{R}}{d_2'}$ and $\hat{P}_X = \frac{\sum_{i=1}^{k}\sum_{j=1}^{n}\delta\left(\bar{\bar{X}} - X_{ij}\right)}{n \times k}$. Here, $\bar{\bar{X}}$ represents the grand average, $\bar{R}$ the average sample range, d_2' the constant for a given skewed population and $\delta(x) = 1$ for $x \geq 0$ or $\delta(x) = 0$ for $x < 0$.

The WV–R chart is based on the following limits [1]:

$$\text{UCL}_{\text{WV}-R} = \mu_R + 3\,\sigma_R\sqrt{2P_X} \tag{2a}$$

$$\text{LCL}_{\text{WV}-R} = \left[\mu_R - 3\sigma_R\sqrt{2(1-P_X)}\right]^+, \tag{2b}$$

where μ_R and σ_R are the mean and standard deviation of the range R while $[a]^+$ denotes $\max\{0,\ a\}$. When parameters are unknown, μ_R, σ_R and P_X are estimated as $\bar{R}$, $\frac{\bar{R}d_3'}{d_2'}$ and $\hat{P}_X$, respectively, where d_3' is a constant for a given skewed population.

B. SC $\bar{X}$ and R charts

The SC $-\bar{X}$ chart employs the following limits [3]:

$$\text{UCL}_{\text{SC}-\bar{X}} = \mu_X + \left(3 + c_4^*\right)\frac{\sigma_X}{\sqrt{n}} \tag{3a}$$

$$\text{LCL}_{\text{SC}-\bar{X}} = \mu_X + \left(-3 + c_4^*\right)\frac{\sigma_X}{\sqrt{n}}. \tag{3b}$$

Here, $c_4^* = \frac{4\kappa_3(\bar{X})}{3\left[1 + 0.2\kappa_3^2(\bar{X})\right]}$, where $\kappa_3(\bar{X}) = \frac{\kappa_3}{\sqrt{n}}$ with κ_3 representing the skewness coefficient of X. μ_X, σ_X and κ_3 are estimated as $\bar{\bar{X}}$, $\frac{\bar{R}}{d_2^*}$ and $\frac{1}{nr-3}\sum_{i=1}^{r}\sum_{j=1}^{n}\left(\frac{X_{ij} - \bar{\bar{X}}}{S_{nr}}\right)^3$, respectively, when parameters are unknown, with r representing the number of in-control subgroups used in the estimation. Note that $S_{nr} = \sqrt{\frac{1}{nr-1}\sum_{i=1}^{r}\sum_{j=1}^{n}\left(X_{ij} - \bar{\bar{X}}\right)^2}$.

The limits for the SC – R chart are [3]

$$\text{UCL}_{\text{SC}-R} = \mu_R + \left(3 + d_4^*\right)\sigma_R \tag{4a}$$

$$\text{LCL}_{\text{SC}-R} = \left[\mu_R + \left(-3 + d_4^*\right)\sigma_R\right]^+, \tag{4b}$$

where $d_4^* = \frac{4\kappa_3(R)}{3\left[1 + 0.2\kappa_3^2(R)\right]}$. Here, $\kappa_3(R)$ denotes the skewness of R. When parameters are unknown, μ_R and σ_R are estimated as $\bar{R}$ and $\frac{\bar{R}d_3^*}{d_2^*}$, respectively. Values of d_2^* and d_3^* are given in [3].

C. WSD $\bar{X}$ chart

The WSD – $\bar{X}$ chart uses the following limits [4]:

$$\text{UCL}_{\text{WSD}-\bar{X}} = \mu_X + \frac{3\sigma_X}{\sqrt{n}}(2P_X) \tag{5a}$$

$$\text{LCL}_{\text{WSD}-\bar{X}} = \mu_X - \frac{3\sigma_X}{\sqrt{n}}[2(1-P_X)]. \tag{5b}$$

For the case of unknown parameters, μ_X, σ_X and P_X are estimated as $\bar{\bar{X}}$, $\frac{\bar{R}}{d_2^{\text{WSD}}}$ and $\hat{P}_X$, respectively, where $d_2^{\text{WSD}} = P_X d_2\left(2n(1-P_X)\right) + (1-P_X)\,d_2\left(2nP_X\right)$ and $d_2(n)$ is d_2 for the normal distribution corresponding to n.

III. PROPOSED WV $\bar{X}_S$ AND S CHARTS

The WV procedure splits a skewed distribution into two parts at its mean, where each part is used to create a new symmetric distribution. The two new symmetric distributions are used to set up the limits of the chart [1]. The construction of the proposed WV $-\bar{X}_S$ and WV–S charts are discussed in the following paragraphs.

Let $\bar{X}_i$, for i = 1, 2, …, be a sequence of sample means for a process. The WV $-\bar{X}_S$ chart is constructed by plotting $\bar{X}_i$ based on the limits in Equations (1a) and (1b), when parameters are known. When parameters are unknown, the limits of the WV $-\bar{X}_S$ chart differ from those in [1] and are given as follow:

$$\text{UCL}_{\text{WV}-\bar{X}_S} = \bar{\bar{X}} + \frac{3\bar{S}}{c_4'\sqrt{n}}\sqrt{2\hat{P}_X} = \bar{\bar{X}} + A_U\bar{S} \tag{6a}$$

$$\text{LCL}_{\text{WV}-\bar{X}_S} = \bar{\bar{X}} - \frac{3\bar{S}}{c_4'\sqrt{n}}\sqrt{2\left(1-\hat{P}_X\right)} = \bar{\bar{X}} - A_L\bar{S}. \tag{6b}$$

Here, $\bar{S} = \frac{\sum_{i=1}^{r} S_i}{r}$ is the average of the sample standard deviations, estimated from r preliminary subgroups while c_4' is a constant for a given skewed population.

The WV–S chart is set up by plotting the sample standard deviations, S_i, for i = 1, 2, …, based on limits

$$\text{UCL}_{\text{WV}-S} = \mu_S + 3\sigma_S\sqrt{2P_X} \tag{7a}$$

$$\text{LCL}_{\text{WV}-S} = \left[\mu_S - 3\sigma_S\sqrt{2(1-P_X)}\right]^+, \tag{7b}$$

where μ_S and σ_S are the mean and standard deviation of S, respectively. These limits are different from that of the WV–R chart in Equations (2a) and (2b). For the case of unknown parameters, the limits of the WV–S chart are

$$\text{UCL}_{\text{WV}-S} = \bar{S}\left[1 + \frac{3\sqrt{1-(c_4')^2}}{c_4'}\sqrt{2\hat{P}_X}\right] = B_U\bar{S} \tag{8a}$$

$$\text{LCL}_{\text{WV}-S} = \bar{S}\left[1 - \frac{3\sqrt{1-(c_4')^2}}{c_4'}\sqrt{2\left(1-\hat{P}_X\right)}\right]^+ = B_L\bar{S}. \tag{8b}$$

Here, $c_4' = \frac{E(S)}{\sigma_X}$. Values of c_4', A_U, A_L, B_U and B_L are estimated via simulation using the Statistical Analysis System (SAS) as follows: First, 200000 samples, each of size n, are generated from the Weibull, Burr and lognormal distributions based on a fixed level of κ_3 and its corresponding value, P_X. The average of the sample standard deviations from these 200000 samples are computed, followed by $\hat{\sigma}_X$ for 200000 × n observations. Next, c_4' is computed for each of the three distributions and the average is taken. Finally, A_U, A_L, B_U and B_L are computed. Table I gives values of c_4', A_U, A_L, B_U and B_L for selected combinations of n and P_X. Note that A_U for $P_X \le 0.5$ is similar to A_L for $1 - P_X$.

IV. PERFORMANCE COMPARISON

The $\mathrm{WV}-\bar{X}$ and WV–S charts are compared with the heuristic $\bar{X}$ and R charts discussed in Section II, in terms of their Type-I and Type-II error rates. The Type-I error rate is the probability of signaling a false out-of-control while the Type-II error rate is the probability of signalling a false in-control.

The Type-I error rate is computed based on the Weibull, gamma and normal distributions while the Type-II error rate based on an exponential distribution. The Weibull distribution reduces to an exponential distribution when its shape parameter, β is one.

Note that P_X for the Weibull [1] and gamma [6] distributions are

$$P_X = 1 - \exp\left[-\left(\Gamma\left(1 + 1/\beta\right)\right)^{\beta}\right] \tag{9}$$

and

$$P_X = F\left(\eta\right), \tag{10}$$

respectively, where β and η are shape parameters, while Γ(·) and F(·) are the gamma function and gamma distribution function, respectively.

The in-control means of the Weibull and gamma distributions are

$$\mu_X = \Gamma\left(1 + \frac{1}{\beta}\right) \tag{11}$$

and

$$\mu_X = \eta, \tag{12}$$

respectively, and the in-control standard deviations are

$$\sigma_X = \sqrt{\Gamma\left(1 + \frac{2}{\beta}\right) - \left[\Gamma\left(1 + \frac{1}{\beta}\right)\right]^2} \tag{13}$$

and

$$\sigma_X = \sqrt{\eta}, \tag{14}$$

respectively.

The mean of an out-of-control process is $\mu_1 = \mu_X + a\sigma_X$, where $a \in \{0.25,\ 0.5(0.25),\ 1.5,\ 2\}$. Here, μ_X and σ_X are the in-control mean and standard deviation, respectively. The out-of-control standard deviation is $\sigma_1 = b\sigma_X$, where $b \in \{1.5,\ 2(0.5),\ 4.5\}$. For the in-control process, the skewness coefficients used for the Weibull and gamma distributions are $\kappa_3 = \{0, 1, 2, 3\}$.

When parameters are known, the Type-I and Type-II error rates are obtained based on 10000 simulation trials and sample sizes, $n \in \{10,\ 15,\ 20\}$. For the case with unknown parameters, first, 30 in-control samples, each of size, n are generated and the limits of the chart estimated. Then to compute the Type-I error rate, another 1000 in-control samples, each of size, n are generated in a phase-II process and the proportion of points falling beyond the limits computed. On the contrary, for computing the Type-II error rate, 1000 out-of-control samples, each containing n observations, are generated in a phase-II process and the proportion of points plotting within the limits computed. The procedure of generating 30 in-control samples followed by 1000 in-control/out-of-control samples is repeated for 10000 simulation trials and the average Type-I/Type-II error rate is recorded.

Tables II and III give the Type-I error rates for the heuristic charts for mean and variance, respectively, while Tables IV and V, show the Type-II error rates for those charts. The $\mathrm{WV}-\bar{X}_S$ and $\mathrm{WV}-\bar{X}$ charts have similar limits when parameters are known, hence they have the same error rates in Tables II and IV. Table II shows that when parameters are known, the Type-I error rates of the charts are comparable, though that of the $\mathrm{WSD}-\bar{X}$ chart tends to be higher when n and κ_3 increase. When parameters are unknown, the $\mathrm{WV}-\bar{X}_S$ chart has the lowest Type-I error rate (see Table II). Overall, Table III shows that the WV–S chart has a lower Type-I error rate than the other charts for variance when parameters are known and unknown. When parameters are known, the Type-II error rates of both the $\mathrm{WV}-\bar{X}_S$ and $\mathrm{WV}-\bar{X}$ charts are lower than that of the $\mathrm{WSD}-\bar{X}$ and $\mathrm{SC}-\bar{X}$ charts (see Table IV). For the case of unknown parameters in Table IV, the $\mathrm{WV}-\bar{X}_S$ chart has a lower Type-II error rate than the $\mathrm{WSD}-\bar{X}$ and $\mathrm{SC}-\bar{X}$ charts but slightly higher than that of the $\mathrm{WV}-\bar{X}$ chart. Table V shows that the WV–S chart has the lowest Type-II error rate when parameters are known and unknown.

V. CONCLUSION

Numerous charts for skewed distributions have been suggested, besides those described in Section II (see [6] – [11], to name a few). The $\mathrm{WV}-\bar{X}_S$ and WV–S charts proposed in this paper for skewed populations are found to be superior to their existing counterparts when the sample size is moderate or large.

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TABLE I. VALUES OF c_4', A_U, A_L, B_U AND B_L

	P_X										
	0.50	0.52	0.54	0.56	0.58	0.60	0.62	0.64	0.66	0.68	0.70
						$n = 10$					
c_4'	0.97	0.97	0.97	0.96	0.96	0.94	0.92	0.89	0.87	0.83	0.76
A_U	0.96	0.98	1.01	1.02	1.04	1.10	1.15	1.20	1.22	1.31	1.43
A_L	0.96	0.95	0.93	0.92	0.91	0.90	0.90	0.90	0.89	0.91	0.95
B_U	1.71	1.73	1.81	1.91	2.00	2.16	2.47	2.77	2.95	3.36	4.02
B_L	0.29	0.30	0.25	0.20	0.15	0.05	0.00	0.00	0.00	0.00	0.00
						$n = 15$					
c_4'	0.98	0.98	0.98	0.97	0.97	0.96	0.94	0.91	0.90	0.86	0.80
A_U	0.78	0.79	0.80	0.82	0.85	0.86	0.91	0.95	0.97	1.02	1.12
A_L	0.78	0.76	0.74	0.73	0.72	0.70	0.70	0.71	0.70	0.72	0.74
B_U	1.56	1.58	1.65	1.73	1.82	1.96	2.24	2.52	2.67	3.04	3.67
B_L	0.44	0.44	0.40	0.35	0.30	0.21	0.03	0.00	0.00	0.00	0.00
						$n = 20$					
c_4'	0.99	0.99	0.98	0.98	0.98	0.97	0.95	0.93	0.92	0.88	0.83
A_U	0.68	0.69	0.70	0.71	0.72	0.74	0.77	0.80	0.83	0.85	0.93
A_L	0.68	0.66	0.64	0.64	0.63	0.62	0.61	0.60	0.60	0.61	0.63
B_U	1.48	1.50	1.56	1.64	1.71	1.84	2.10	2.36	2.49	2.84	3.42
B_L	0.52	0.52	0.48	0.44	0.39	0.31	0.14	0.00	0.00	0.00	0.00

TABLE II. TYPE-I ERROR RATES FOR MEAN CHARTS

	Distribution	κ_3	$n = 10$				$n = 15$				$n = 20$			
			$WV-\overline{X}_S$	$WV-\overline{X}$	$WSD-\overline{X}$	$SC-\overline{X}$	$WV-\overline{X}_S$	$WV-\overline{X}$	$WSD-\overline{X}$	$SC-\overline{X}$	$WV-\overline{X}_S$	$WV-\overline{X}$	$WSD-\overline{X}$	$SC-\overline{X}$
Parameters Known	Normal	0.0	0.0027	0.0027	0.0027	0.0027	0.0027	0.0027	0.0027	0.0027	0.0027	0.0027	0.0027	0.0027
	Weibull	0.0	0.0025	0.0025	0.0025	0.0025	0.0026	0.0026	0.0026	0.0026	0.0026	0.0026	0.0026	0.0026
		1.0	0.0025	0.0025	0.0025	0.0025	0.0027	0.0027	0.0028	0.0025	0.0026	0.0026	0.0031	0.0025
		2.0	0.0030	0.0030	0.0027	0.0026	0.0030	0.0030	0.0038	0.0026	0.0027	0.0027	0.0048	0.0027
		3.0	0.0048	0.0048	0.0028	0.0029	0.0038	0.0038	0.0049	0.0030	0.0032	0.0032	0.0071	0.0032
	Gamma	0.0	0.0027	0.0027	0.0027	0.0027	0.0027	0.0027	0.0027	0.0027	0.0027	0.0027	0.0027	0.0027
		1.0	0.0028	0.0028	0.0027	0.0028	0.0027	0.0027	0.0029	0.0027	0.0027	0.0027	0.0031	0.0027
		2.0	0.0034	0.0034	0.0027	0.0026	0.0029	0.0029	0.0038	0.0026	0.0027	0.0027	0.0048	0.0027
		3.0	0.0042	0.0042	0.0027	0.0021	0.0032	0.0032	0.0068	0.0024	0.0027	0.0027	0.0102	0.0026
Parameters Unknown	Normal	0.0	0.0034	0.0034	0.0037	0.0034	0.0034	0.0036	0.0035	0.0039	0.0034	0.0033	0.0035	0.0034
	Weibull	0.0	0.0031	0.0033	0.0035	0.0033	0.0031	0.0038	0.0037	0.0042	0.0031	0.0037	0.0038	0.0037
		1.0	0.0032	0.0035	0.0042	0.0037	0.0031	0.0039	0.0048	0.0040	0.0030	0.0040	0.0052	0.0040
		2.0	0.0040	0.0044	0.0060	0.0046	0.0034	0.0043	0.0068	0.0050	0.0031	0.0038	0.0074	0.0044
		3.0	0.0051	0.0061	0.0083	0.0059	0.0041	0.0051	0.0095	0.0062	0.0033	0.0040	0.0099	0.0056
	Gamma	0.0	0.0034	0.0034	0.0036	0.0034	0.0034	0.0036	0.0035	0.0039	0.0033	0.0033	0.0035	0.0034
		1.0	0.0036	0.0037	0.0042	0.0037	0.0035	0.0037	0.0043	0.0037	0.0033	0.0035	0.0042	0.0034
		2.0	0.0041	0.0045	0.0060	0.0047	0.0032	0.0042	0.0067	0.0049	0.0029	0.0037	0.0074	0.0043
		3.0	0.0039	0.0057	0.0093	0.0056	0.0028	0.0045	0.0124	0.0060	0.0023	0.0034	0.0134	0.0057

TABLE III. TYPE-I ERROR RATES FOR VARIANCE CHARTS

	Distribution	κ_3	$n = 10$			$n = 15$			$n = 20$		
			WV–S	WV–R	SC–R	WV–S	WV–R	SC–R	WV–S	WV–R	SC–R
Parameters Known	Normal	0.0	0.0030	0.0043	0.0041	0.0029	0.0045	0.0045	0.0027	0.0046	0.0047
	Weibull	0.0	0.0023	0.0026	0.0026	0.0024	0.0028	0.0038	0.0025	0.0029	0.0034
		1.0	0.0043	0.0055	0.0093	0.0037	0.0057	0.0108	0.0034	0.0059	0.0116
		2.0	0.0070	0.0080	0.0092	0.0061	0.0077	0.0112	0.0055	0.0076	0.0115
		3.0	0.0089	0.0095	0.0040	0.0078	0.0091	0.0051	0.0074	0.0088	0.0061
	Gamma	0.0	0.0028	0.0042	0.0023	0.0028	0.0043	0.0027	0.0027	0.0044	0.0027
		1.0	0.0056	0.0069	0.0069	0.0049	0.0071	0.0088	0.0045	0.0073	0.0096
		2.0	0.0069	0.0081	0.0088	0.0061	0.0077	0.0109	0.0055	0.0078	0.0109
		3.0	0.0074	0.0082	0.0041	0.0065	0.0080	0.0071	0.0059	0.0078	0.0088
Parameters Unknown	Normal	0.0	0.0035	0.0047	0.0032	0.0032	0.0047	0.0035	0.0032	0.0049	0.0039
	Weibull	0.0	0.0015	0.0014	0.0013	0.0015	0.0012	0.0013	0.0014	0.0010	0.0012
		1.0	0.0041	0.0041	0.0048	0.0028	0.0041	0.0054	0.0022	0.0037	0.0054
		2.0	0.0060	0.0074	0.0118	0.0040	0.0057	0.0128	0.0031	0.0047	0.0121
		3.0	0.0064	0.0080	0.0167	0.0084	0.0067	0.0177	0.0035	0.0061	0.0160
	Gamma	0.0	0.0032	0.0046	0.0030	0.0032	0.0048	0.0034	0.0030	0.0048	0.0038
		1.0	0.0069	0.0069	0.0064	0.0054	0.0074	0.0073	0.0044	0.0069	0.0074
		2.0	0.0060	0.0075	0.0121	0.0039	0.0055	0.0119	0.0031	0.0046	0.0124
		3.0	0.0032	0.0050	0.0211	0.0019	0.0038	0.0208	0.0012	0.0031	0.0183

TABLE IV. TYPE-II ERROR RATES FOR MEAN CHARTS

	a	$n = 10$				$n = 15$				$n = 20$			
		WV – $\overline{X}_S$	WV – $\overline{X}$	WSD – $\overline{X}$	SC – $\overline{X}$	WV – $\overline{X}_S$	WV – $\overline{X}$	WSD – $\overline{X}$	SC – $\overline{X}$	WV – $\overline{X}_S$	WV – $\overline{X}$	WSD – $\overline{X}$	SC – $\overline{X}$
Parameters Known	0.25	0.9860	0.9860	0.9933	0.9931	0.9827	0.9827	0.9921	0.9899	0.9790	0.9790	0.9907	0.9854
	0.50	0.9492	0.9492	0.9738	0.9733	0.9151	0.9151	0.9562	0.9452	0.8701	0.8701	0.9299	0.9024
	0.75	0.8453	0.8453	0.9123	0.9108	0.7040	0.7040	0.8203	0.7864	0.5377	0.5377	0.6904	0.6146
	1.0	0.6210	0.6210	0.7573	0.7538	0.3308	0.3308	0.5013	0.4453	0.1301	0.1301	0.2610	0.1862
	1.25	0.3030	0.3030	0.4773	0.4724	0.0532	0.0532	0.1427	0.1060	0.0041	0.0041	0.0206	0.0090
	1.5	0.0632	0.0632	0.1674	0.1638	0.0007	0.0007	0.0070	0.0037	0.0000	0.0000	0.0000	0.0000
	2.0	0.0000	0.0000	0.0001	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Parameters Unknown	0.25	0.9838	0.9827	0.9932	0.9884	0.9799	0.9776	0.9866	0.9829	0.9743	0.9725	0.9852	0.9775
	0.5	0.9429	0.9413	0.9743	0.9588	0.9079	0.8991	0.9349	0.9194	0.8555	0.8488	0.9060	0.8708
	0.75	0.8364	0.8332	0.9123	0.8766	0.7000	0.6802	0.7692	0.7289	0.5305	0.5175	0.6460	0.5656
	1.0	0.6241	0.6200	0.7573	0.7000	0.3532	0.3314	0.4486	0.3938	0.1475	0.1397	0.2462	0.1777
	1.25	0.3356	0.3337	0.4773	0.4320	0.0807	0.0721	0.1366	0.1051	0.0095	0.0087	0.0290	0.0153
	1.5	0.1042	0.1056	0.1674	0.1736	0.0051	0.0044	0.0145	0.0095	0.0000	0.0000	0.0005	0.0002
	2.0	0.0007	0.0009	0.0001	0.0041	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

TABLE V. TYPE-II ERROR RATES FOR VARIANCE CHARTS

	b	$n = 10$			$n = 15$			$n = 20$		
		WV–S	WV–R	SC–R	WV–S	WV–R	SC–R	WV–S	WV–R	SC–R
Parameters Known	1.5	0.8778	0.9193	0.9586	0.8720	0.9086	0.9952	0.8588	0.9007	0.9993
	2.0	0.7273	0.7611	0.9889	0.6435	0.7152	0.9968	0.5593	0.6793	0.9962
	2.5	0.5252	0.5704	0.9883	0.3853	0.4874	0.9852	0.2721	0.4250	0.9796
	3.0	0.3535	0.4009	0.9705	0.2057	0.2991	0.9542	0.1129	0.2312	0.9375
	3.5	0.2298	0.2721	0.9357	0.1039	0.1728	0.8992	0.0437	0.1150	0.8642
	4.0	0.1496	0.1819	0.3003	0.0527	0.0973	0.1935	0.0171	0.0543	0.1263
	4.5	0.0960	0.1203	0.2138	0.0262	0.0536	0.1182	0.0064	0.0251	0.0667
Parameters Unknown	1.5	0.9289	0.9310	0.9487	0.9266	0.9334	0.9492	0.9194	0.9334	0.9488
	2.0	0.7820	0.7962	0.8350	0.7358	0.7810	0.8182	0.6820	0.7637	0.8038
	2.5	0.5998	0.6274	0.6785	0.5008	0.5820	0.6312	0.4055	0.5383	0.5943
	3.0	0.4329	0.4678	0.5202	0.3064	0.3974	0.4460	0.2070	0.3364	0.3911
	3.5	0.3025	0.3373	0.3843	0.1769	0.2558	0.2964	0.0969	0.1939	0.2372
	4.0	0.2080	0.2389	0.2776	0.0992	0.1593	0.1896	0.0437	0.1062	0.1357
	4.5	0.1419	0.1677	0.1985	0.0549	0.0970	0.1185	0.0197	0.0562	0.0753