AN *A*-STABLE EXPLICIT RATIONAL BLOCK METHOD FOR THE NUMERICAL SOLUTION OF INITIAL VALUE PROBLEM

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ABSTRACT: In this paper, a 2-point explicit rational block method for the numerical solution of first order initial value problem is proposed. The main reason to consider rational block method is to improve the numerical accuracy and absolute stability property of existing block multistep methods that are based on polynomial approximants. The proposed method is found to possess *A*-stability. Local truncation error is included as well. Numerical experimentations and results using some test problems are presented. Numerical results are satisfying in terms of numerical accuracy. Finally, a conclusion is included.

KEY WORDS: rational approximant; 2-point explicit rational block method; *A*-stable; initial value problem

1. INTRODUCTION

Conventional block multistep methods (BMMs) are very useful tools in terms of solvability. Firstly, BMMs can return several numerical approximations for a first order initial value problem within each integration step. Secondly, BMMs can easily be modified and extended to solve higher order initial value problems directly. Thirdly, BMMs can easily be implemented on a parallel machine. For excellence surveys and various perspectives of BMMs, see, for example, Sommeijer B.P. (1992), Watanabe D.S. (1978), Ibrahim Z.B. (2003), Ibrahim Z.B. (2005), Chollom J.P. (2007), Majid Z.A. (2009), Majid Z.A. (2012), Mehrkanoon S. (2009), Akinfenwa O.A. (2011), Ehigie J.O. (2011), Ibijola E.A. (2011), Majid Z.A. (2011), Badmus A.M. (2009), Olabode B.T. (2009) and Chartier P. (1994).

Despite the many great potential of conventional BMMs, they have several stability drawbacks. Implicit BMMs were introduced mainly to improve the order of consistencies and stability requirements suffered by most explicit BMMs. However, extra computations are required to solve the system of nonlinear equations arise from the implementations of implicit BMMs, which are very expensive in terms of computational costs when solving large scale problems. Alternatively, some of the researchers would prefer predictor-corrector BMMs because they allow the stage-by-stage implementations without the need to solve any system of nonlinear equations. The order of consistency is determined by the order of the correctors that are usually be the implicit BMMs, while the predictors are usually be the explicit BMMs. However, the stability requirements of predictor-corrector BMMs become more restricting when the order of the methods increases, which make the numerical solution of stiff problem impossible for larger step-sizes.

In order to overcome the stability drawbacks and at the same time, retain the advantages of conventional BMMs, Teh Y.Y. (2013a) suggested the idea of BMMs that are based on rational functions. This idea was first introduced as the concept of rational block multistep methods (RBMMs). Like conventional block multistep methods (BMMs), RBMMs can be considered as a set of simultaneously applied rational multistep methods to obtain several numerical approximations within each integration step. Why there is such an idea to search for BMMs based on rational functions? Our readings have shown that there exist some unconventional numerical methods that are based on rational functions, which possess strong stability conditions but yet explicit in nature. Hence, we expected that RBBMs are cheaper in computational costs compare to existing implicit BMMs and possess strong stability conditions such as L-stability. For excellence surveys and various perspectives, see, for example, Lambert J.D. (1965), Lambert J.D. (1974), Luke Y.L. (1975), Fatunla S.O. (1982), Fatunla S.O. (1986), van Niekerk F.D. (1987), van Niekerk F.D. (1988), Ikhile M.N.O. (2001), Ikhile M.N.O. (2002), Ikhile M.N.O. (2004), Ramos H. (2007), Okosun K.O. (2007a), Okosun K.O. (2007b), Teh Y.Y. (2009), Teh Y.Y. (2011), Yaacob N. (2010), Teh Y.Y. (2010), Teh Y.Y. (2013b) and Teh Y.Y. (2013c).

In Teh Y.Y. (2013a), a 2-point explicit rational block method was developed. However, the previously developed method is not A-stable but a method with finite region of absolute stability. In the next section, we develop a 2-point explicit rational block method which is A-stable. Section 3 presents the principal local truncation error terms and establish the absolute stability condition for the newly developed method. Some tests are carried out in order to verify the validity of the new RBMM in Section 4. Finally, a conclusion is included.

2. FORMULATION OF 2-POINT EXPLICIT RATIONAL BLOCK METHOD

The 2-point explicit rational block method is formulated to solve the following first order initial value problem given by

$$y' = f(x,y), y(a) = \eta, \tag{1}$$

where $f(x,y): \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ and $f(x,y)$ is assumed to satisfy all the required conditions such that problem (1) possesses a unique solution. Suppose that the interval of numerical integration is $x \in [a,b] \subset \mathbb{R}$ and is divided into a series of blocks with each block containing two points as shown in Figure 1.

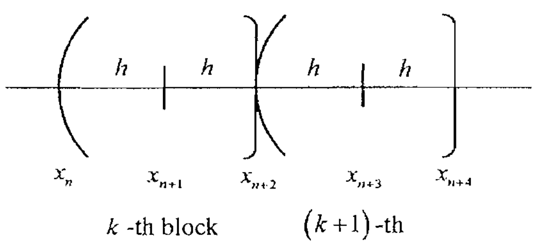

Figure 1 2-point explicit rational block method.

From Figure 1, we have observed that k-th block contains three points i.e. x_n, x_{n+1} and x_{n+2}, and each of these points is separated equidistantly by a constant step-size h. The next $(k+1)$-th block also contains three points. In the k-th block, we want to use the values y_n at x_n to compute the approximation values of y_{n+1} and y_{n+2} simultaneously. In the $(k+1)$-th block, previously computed values of y_{n+2} is used to generate the approximations values of y_{n+3} and y_{n+4}. The same computational procedure is repeated to compute the solutions for the next few blocks until the end-point i.e. $x = b$ is reached. The evaluation information from the previous step in a block could be used for other steps of the same block (Teh Y.Y. (2013a)).

Along the x-axis, we consider the points x_n, x_{n+1} and x_{n+2} to be given by

$$x_n = x_0 + nh, \tag{2}$$

$$x_{n+1} = x_0 + (n+1)h, \tag{3}$$

and

$$x_{n+2} = x_0 + (n+2)h. \tag{4}$$

where h is the step-size. Let us assume that the approximate solution of (1) is locally represented in the range $[x_n, x_{n+1}]$ by the rational approximant

$$R(x) = \frac{a_0 + a_1 x}{b_0 + x}, \tag{5}$$

where a_0, a_1 and b_0 are undetermined coefficients. This rational approximant in equation (5) is required to pass through the points (x_n, y_n) and (x_{n+1}, y_{n+1}), and moreover, must assume at these points the derivatives given by $y' = f(x,y)$ and $y'' = f'(x,y)$. Altogether, there are four equations to be satisfied i.e.

$$R(x_n) = y_n, \tag{6}$$

$$R(x_{n+1}) = y_{n+1}, \tag{7}$$

$$R'(x_n) = f_n, \tag{8}$$

and

$$R''(x_n) = f_n', \tag{9}$$

where $f_n = f(x_n, y_n)$ and $f_n' = f'(x_n, y_n)$. On using *MATHEMATICA 8.0*, the elimination of the three undetermined coefficients i.e. a_0, a_1 and b_0 from equations (6) – (9) is the one-step second order rational method found in Lambert J.D. (1974),

$$y_{n+1} = y_n + \frac{2h(f_n)^2}{2f_n - hf_n'}. \tag{10}$$

Equation (10) is the formula to approximate y_{n+1} by using the information at the previous point (x_n, y_n). To approximate y_{n+2}, we have to assume that the approximate solution of (1) is locally represented in the range $[x_n, x_{n+2}]$ by the same rational approximant given in equation (5). Now, we required the rational approximant (5) to pass through the points (x_n, y_n), (x_{n+1}, y_{n+1}) and (x_{n+2}, y_{n+2}), and moreover, must assume at these points the derivative given by $y' = f(x,y)$. There are five equations to be satisfied i.e.

$$R(x_n) = y_n, \tag{11}$$

$$R(x_{n+1}) = y_{n+1}, \tag{12}$$

$$R(x_{n+2}) = y_{n+2}, \tag{13}$$

$$R'(x_n) = f_n, \tag{14}$$

and

$$R'(x_{n+1}) = f_{n+1}, \tag{15}$$

where $f_n = f(x_n, y_n)$ and $f_{n+1} = f(x_{n+1}, y_{n+1})$. On using *MATHEMATICA 8.0*, the elimination of the four undetermined coefficients i.e. a_0, a_1, b_0 and f_n from equations (11) – (15) is the two-step third order rational method found in Lambert J.D. (1974),

$$y_{n+2} = y_{n+1} + \frac{hf_{n+1}(y_{n+1} - y_n)}{2(y_{n+1} - y_n) - hf_{n+1}}. \tag{16}$$

Equation (16) is the formula to approximate y_{n+2} by using the information at the previous points (x_n, y_n) and (x_{n+1}, y_{n+1}). Hence, the 2-point explicit rational block method based on the rational approximant (5) consists of two formulae i.e. formulae (10) and (16).

The implementation of the 2-point explicit rational block method is rather simple: with y_n is known, compute the approximate solution y_{n+1} using formula (10); and then compute the approximate solution y_{n+2} using formula (16) with the value of y_{n+1} obtained from formula (10).

3. LOCAL TRUNCATION ERRORS AND STABILITY ANALYSES

Since formulae (10) and (16) are used in the same block to solve for the approximate solutions at x_{n+1} and x_{n+2}, we wish to have both formulae possess the same order of accuracy. However, calculations of the principal truncation errors revealed that formula (10) possessed second order of accuracy, while formula (16) possessed third order of accuracy. We note that the local truncation errors (LTE) for formula (10) and formula (16) are

$$\text{LTE}_{(10)} = h^3\left(\frac{1}{2}(y_n'')^2 - \frac{1}{3}y_n'y_n'''\right) + O(h^4), \tag{17}$$

and

$$\text{LTE}_{(16)} = h^4\left(\frac{1}{2}(y_n'')^2 - \frac{1}{3}y_n'y_n'''\right) + O(h^5), \tag{18}$$

respectively. Therefore, the order of consistency of the entire 2-point explicit rational block method ,fluctuates' between second order and third order.

To investigate the linear stability condition for formulae (10) and (16) in the same block, we need to combine both formulae and apply the Dahquist's test equation

$$y' = \lambda y, y(a) = y_0, \text{Re}(\lambda) < 0, \tag{19}$$

to both formulae. With $f_{n+1} = \lambda y_{n+1}, f_n = \lambda y_n$ and $f_n' = \lambda^2 y_n$, we can obtain the following difference equation

$$y_{n+2} = \left(\frac{h\lambda + 2}{h\lambda - 2}\right)^2 y_n. \qquad (20)$$

On setting $h\lambda = z$, $y_{n+2} = \zeta^2$ and $y_{n+2} = \zeta^0 = 1$ in equation (20), then the stability polynomial for the 2-point explicit rational block method is

$$\zeta^2 - \left(\frac{z + 2}{z - 2}\right)^2 = 0. \qquad (21)$$

Here, ζ can be interpreted as the roots of stability polynomial (21). By taking $z = x + iy$ in the roots of equation (21), we have plotted the region of absolute stability of the 2-point explicit rational block method in Figure 2.

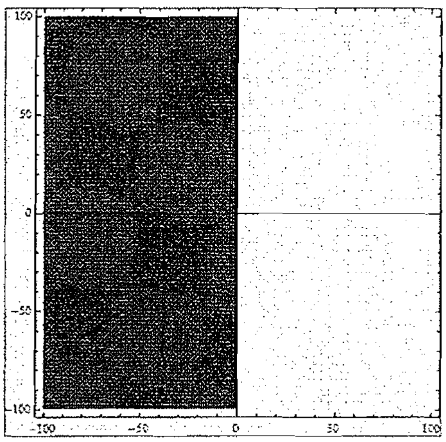

Figure 2 Absolute stability region of 2-point explicit rational block method.

The shaded region in Figure 2 is the region of absolute stability of the 2-point explicit rational block method. Hence, this shaded region can also be viewed as the ‚combined' region of absolute stability of formulae (10) and (16). The shaded region is the place where the absolute value of each root of equation (21) is less than or equal to 1. From Figure 2, we can see that the region of absolute stability contains the whole left-hand half plane which suggests that our proposed rational block method is A-stable.

4. NUMERICAL EXPERIMENTS

In this section, some test problems are used to verify the validity of the new 2-point explicit rational block method shown in formulae (10) and (16). We present the maximum absolute errors over the integration interval given by $\max_{0 \le n \le N}\{|y(x_n) - y_n|\}$ where N is the number of integration steps. We note that $y(x_n)$ and y_n are the theoretical solution and numerical solution of a test problem at point x_n, respectively.

Problem 1

$$y'(x) = -10y(x), y(0) = 1, x \in [0,1]$$

The theoretical solution is given by $y(x) = e^{-10x}$.

N	**2-point Explicit Rational Block Method**
32	3.02055(-03)
64	7.48959(-04)
128	1.87214(-04)
256	4.67803(-05)

Table 1 Maximum absolute errors with respect to number of integration steps, N (*Problem 1*)

Problem 2 (Yaakub A.R. (2003))

$$y''(x) + 101y'(x) + 100y(x) = 0, y(0) = 1.01, y'(0) = -2, x \in [0,1]$$

The theoretical solution is given by $y(x) = 0.01e^{-100x} + e^{-x}$. *Problem 2* can also be written as a system i.e.

$$y_1'(x) = y_2(x), y_1(0) = 1.01, x \in [0,1]$$
$$y_2'(x) = -100y_1(x) - 101y_2(x), y_2(0) = -2, x \in [0,1]$$

The theoretical solutions of this system are given by $y_1(x) = y(x) = 0.01e^{-100x} + e^{-x}$, $y_2(x) = y'(x) = -e^{-100x} - e^{-x}$.

N	**2-point Explicit Rational Block Method**
32	1.78416(-02)
64	3.98233(-03)
128	9.39539(-04)
256	2.32928(-04)

Table 2 Maximum absolute errors with respect to number of integration steps, N (*Problem 2*)

Problem 3 (Ramos H. (2007))

$$y'(x) = 1 + y(x)^2, y(0) = 1, x \in [0,1]$$

Problem 3 is a problem whose solution possesses singularity. The theoretical solution is $y(x) = \tan(x + \pi/4)$. From the theoretical solution, we have noticed that the solution becomes unbounded in the neighbourhood of the singularity at $x = \pi/4 \approx 0.785398163367448$.

N	**2-point Explicit Rational Block Method**
32	1.39181(+01)
64	3.63857(+00)
128	1.20080(+00)
256	6.71306(+01)

Table 3 Maximum absolute errors with respect to number of integration steps, N (*Problem 3*)

Results from Table 1 and Table 2 showed a consistent pattern i.e. the maximum absolute error decreases whenever the number of integration step increases. This also means, whenever the step-size becomes smaller, the numerical solution also approaches the exact solution and therefore convergent. However results from Table 3 showed fluctuations in the maximum absolute errors even if the number of integration step increases. We believed the fluctuations were caused by the inconsistent in the accuracy exhibited by formula (10) and formula (16).

5. CONCLUSION

In this paper, a 2-point explicit rational block method was introduced. This rational block method was able to approximate two successive solutions at the points x_{n+1} and x_{n+2} defined in the same block (see Figure 1), within every single integration step. This rational block method also contained two rational formulae: formula (10) possesses second order of accuracy, while formula (16) possesses third order of accuracy. Stability analysis showed that the proposed method is A-stable. Hence, the proposed method is suitable to solve stiff problems.

Numerical experiments showed that the proposed rational block method generated converging numerical solution when solving general initial value problems such as *Problem 1* and *Problem 2*. However, this is not the case in solving *Problem* 3, which is a problem whose solution possesses singularity. Numerical solution did not converge as the step-size approaches zero, but the solution was still considered as stable solution. This was most probably caused by the inconsistent in the accuracy exhibited by formula (10) and formula (16). The inconsistency in the accuracy also affected *Problem 1* and *Problem 2*, but these are not as obvious as *Problem 3*. Future study should look into this matter, and redesign rational block method which had exactly one order of accuracy in a same block. Numerical comparison with other existing block methods should also be included in future study. Last but not least, since A-stable explicit rational block method is possible, L-stable explicit rational block method will be introduced in the near future.

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