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PERFORMANCE OF THE TRIANGULATION-BASED METHODS OF POSITIVITY-PRESERVING SCATTERED DATA INTERPOLATION

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Abstract

We present the result and accuracy comparison of generalized positivity-preserving schemes for triangular Bézier patches of C^1 and C^2 scattered data interpolants that have been constructed. We compare three methods of C^1 schemes using cubic triangular Bézier patches and one C^2 scheme using quintic triangular Bézier patches. Our test case consists of four sets of node/test function pairs, with node-count ranging from 26 to 100 data points. The absolute maximum and mean errors are computed using 33×33 evaluation points on a uniform rectangular grid.

1. Introduction

The problem of interpolating positive scattered data in the plane is described as follows: given a set of N arbitrarily distributed points $\{(x_i, y_i)\}_{i=1}^{N}$ in R^2, referred

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to as *nodes*, along with data values $z_i \geq 0$ at the nodes, we wish to construct a smooth function $F : R^2 \to R$ such that $F(x_i, y_i) = z_i$ for $i = 1, \ldots, N$ which preserves the positivity of data points. Since continuity of first partial derivatives is sufficient smoothness for most applications, and smoother surfaces are more costly to construct, most triangulation-based methods of positivity-preserving scattered data interpolation produce $F \in C^1(R^2)$. However, in this paper, we also consider one method that produces C^2 functions.

The criteria for assessing the effectiveness of a scattered-data interpolation method include accuracy in reproducing test functions, computational costs in both preprocessing and evaluation, storage requirements, flexibility in handling constraints, and appearance of the interpolatory surface. Furthermore, the most accurate method for one data set (or collection of data sets) may perform poorly on another [9]. The results reported in this paper are only focusing on testing of accuracy of the latest positivity-preserving triangulation-based method by using four different node sets/test functions pair.

We will consider three previous local methods of positivity-preserving based on C^1 continuous schemes using cubic triangular Bézier patches proposed by [1, 8] and [11] and a C^2 continuous scheme using quintic triangular Bézier patches as discussed in [10]. For a convenience, we denote the methods proposed in [1, 8, 11] and [10] as $CO\text{-}C^1$, $PGU\text{-}C^1$, $SPM\text{-}C^1$ and $SPM\text{-}C^2$, respectively.

In order to show the effectiveness of the shape preserving method, we have focused on the ability of the method to preserve the positivity of data points rather than to produce smoothness quality of the surfaces. This is due to the fact that in order to produce the positivity-preserving surfaces, we need to adjust some of the control points that might affect the quality of the overall constructed surface.

2. Triangulation-based Method of Positivity-preserving Scattered Data Interpolation

Consider a triangle T (as in Figure 1) with vertices $V_1(x_1, y_1)$, $V_2(x_2, y_2)$, $V_3(x_3, y_3)$ and barycentric coordinates u, v, w such that any point $V(x, y)$ on the triangle can be expressed as $V = uV_1 + vV_2 + wV_3$, where $u + v + w = 1$ and $u, v, w \geq 0$. A Bézier triangular patch P on T is defined as

$$P(u, v, w) = \sum_{\substack{i+j+k=n \\ i,\ j,\ k \geq 0}} b_{ijk} B_{ijk}^{n}(u, v, w),$$

where

$$B_{ijk}^{n}(u, v, w) = \frac{n!}{i!\,j!\,k!} u^{i} v^{j} w^{k},$$

and b_{ijk} are the Bézier ordinates or control points of P.

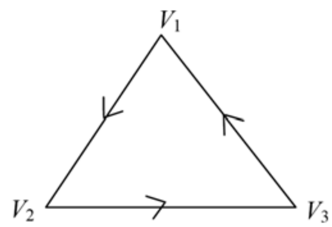


Figure 1. Triangle T.

Assume that the Bézier ordinates at vertices are assumed to be strictly positive, i.e., $b_{n00}, b_{0n0}, b_{00n} > 0$. Sufficient conditions on the remaining Bézier ordinates shall be derived to ensure the entire Bézier patch to be positive. For the purpose of this paper, we use the four algorithms [1, 8, 10, 11] and their underlying methods as in the following proposition.

Proposition 1 [1]**.** *Let the cubic Bézier triangular patch be*

$$P(u, v, w) = \alpha\ell u^{3} + \beta\ell v^{3} + \ell w^{3} + 3b_{210}u^{2}v + 3b_{201}u^{2}w + 3b_{210}uv^{2}$$

$$+ 3b_{102}uw^{2} + 3b_{021}v^{2}w + 3b_{012}vw^{2} + 6b_{111}uvw$$

with $u, v, w \geq 0,\ u + v + w = 1,$ *where* $b_{300} = \alpha\ell,\ b_{030} = \beta\ell,\ b_{003} = \ell,\ \ell > 0$ *and* $\alpha \geq \beta \geq 1.$ *If* $b_{210}, b_{201}, b_{120}, b_{021}, b_{012}, b_{102}, b_{111} \geq -\dfrac{\ell}{3a},$ *where* a *is the unique solution in* $(1, 8/3]$ *of the equation* $16 - 8\alpha + (72\alpha - 27\alpha^{2})a + 54\alpha^{2}a^{2} - 27\alpha^{2}a^{3} = 0,$ *then* $P(u, v, w) \geq 0,\ \forall u, v, w \geq 0, u + v + w = 1.$

Proposition 1 gives sufficient condition for $P(u, v, w) \geq 0$ by prescribing the value of the Bézier ordinates in each triangle not smaller than the lower bounds

$-\frac{\ell}{3a}$ and this common lower bound of Bézier ordinates (except at vertices) is bounded below by the value of $-1/3$. Piah et al. [8] followed the similar approach as in [1] but offers more relaxed sufficient conditions by prescribing the value of the Bézier ordinates in each triangle not smaller than the lower bounds $-t$ $(t > 0)$ which are unbounded below compared [1]. This is stated in the following proposition.

Proposition 2 [8]**.** *Consider the cubic Bézier triangular patch* $P(u, v. w)$ *with* $b_{300} = A,$ $b_{030} = B,$ $b_{003} = C,$ $A, B, C > 0.$ *If* $b_{210}, b_{201}, b_{120}, b_{021}, b_{012}, b_{102},$ $b_{111} \geq -t_0 = -\frac{1}{s_0}$, *where* s_0 *is the unique solution of the* $G(s) = 1$ *with*

$$G(s) = \frac{1}{\sqrt{As+1}} + \frac{1}{\sqrt{Bs+1}} + \frac{1}{\sqrt{Cs+1}},$$

then $P(u, v, w) \geq 0, \forall u, v, w \geq 0, u + v + w = 1.$

The value of s_0 for the given values of A, B, C is obtained by false position method [3] with an initial estimate for the root is the value of s for which the line joining $8/N$ and $8/M$ has the value 1, where $M = \max(A, B, C)$ and $N = \min(A, B, C)$.

Both the schemes in [1] and [8] are constructed by considering the same value of the inner and edge Bézier ordinates lower bound. By extending the work of [8, 11], propose improved sufficient positivity conditions using cubic Bézier triangle where the lower bounds of the edge and inner Bézier ordinates are adjusted independently while still ensuring positivity of the triangular patches. This is stated in the following proposition.

Proposition 3 [11]**.** *Consider the cubic Bézier triangular patch* $P(u, v, w)$ *with* $b_{300} = A,$ $b_{030} = B,$ $b_{003} = C,$ $A, B, C > 0.$ *If* $b_{021}, b_{012} \geq -y_1,$ $b_{201},$ $b_{102} \geq -y_2,$ $b_{210}, b_{120} \geq -y_3$ *and* $b_{111} \geq -x_0$ (*where* $y_i, x_0 > 0$) *such that* y_1 *is the unique solution of* $3y_1^4 + 4(B+C)y_1^3 + 6BCy_1^2 - B^2C^2 = 0,$ y_2 *is the unique solution of* $3y_2^4 + 4(A+C)y_2^3 + 6ACy_2^2 - A^2C^2 = 0,$ y_3 *is the unique solution of* $3y_3^4 + 4(A+B)y_3^3 + 6ABy_3^2 - A^2B^2 = 0$ *and* x_0 *is given by*

$$\frac{(A+y_0)^{\frac{1}{3}}(B+y_0)^{\frac{1}{3}}(C+y_0)^{\frac{1}{3}}}{2}+\frac{5}{6}y_0$$

with $y_0 = \min(y_1, y_2, y_3)$, *then* $P(u, v, w) \geq 0$ *for all* $u, v, w \geq 0, \; u+v+w=1.$

All the above three methods concentrate only on generating the resulting C^1 smooth surfaces. Thus, motivated by previous works in [8, 10], we derive the sufficient conditions on Bézier points in order to ensure that surfaces comprising quintic Bézier triangular patches are always positive and satisfy C^2 continuity conditions as given in the following proposition.

Proposition 4 [10]**.** *Consider the quintic Bézier triangular patch* $P(u, v, w)$ *with* $b_{500} = A$, $b_{050} = B$, $b_{005} = C$, *where* $A, B, C > 0$. *If* $b_{ijk} \geq -r_0 = -1/s_0$, $(i, j, k) \neq (5, 0, 0), (0, 5, 0)$ *and* $(0, 0, 5)$, *where* s_0 *is the unique solution of* $\frac{1}{\sqrt[4]{As+1}}+\frac{1}{\sqrt[4]{Bs+1}}+\frac{1}{\sqrt[4]{Cs+1}}=1$, *then* $P(u, v, w) \geq 0$, $\forall u, v, w \geq 0$, $u+v+w=1$.

Similar as in [8], the value of s_0 for the given values of A, B, C is obtained by false position method [3] but with an initial estimate for the root will be the value of s for which the line joining $80/N$ and $80/M$ has the value 1, where $M = \max(A, B, C)$ and $N = \min(A, B, C)$.

3. Test Result

We will illustrate our accuracy test using four test functions/node sets pair, i.e., test function 1 is evaluated on 36 node points [6], test function 2 is evaluated on 63 node points [5], test function 3 is evaluated on 26 node points [7], and test function 4 is evaluated on 100 node points [9]. Figure 2 shows the triangulated domains for the node sets and the test functions are displayed in Figure 3. The test functions are as follows:

$$F_1(x, y) = \begin{cases} 1.0, & \text{if } (y-x) \geq 0.5, \\ 2(y-x), & \text{if } 0.5 \geq (y-x) \geq 0.0, \\ \dfrac{\cos(4\pi\sqrt{(x-1.5)^2+(y-0.5)^2})+1}{2}, & \text{if } (x-1.5)^2+(y-0.5)^2 \leq \dfrac{1}{16}, \\ 0, & \text{elsewhere on } [0, 2]\times[0, 1], \end{cases}$$

$$F_2(x, y) = (x+1)^2(x-1)^2(y+1)^2(y-1)^2, \quad (x, y) \in (-1.4, 1.4)x(-1.4, 1.4),$$

$$F_3(x, y) = x^4 + y^4$$

and

$$F_4(x, y) = e^{-(5-10x)^2/2} + 0.75e^{-(5-10y)^2/2} + 0.75e^{-(5-10x)^2/2}e^{-(5-10y)^2/2}.$$

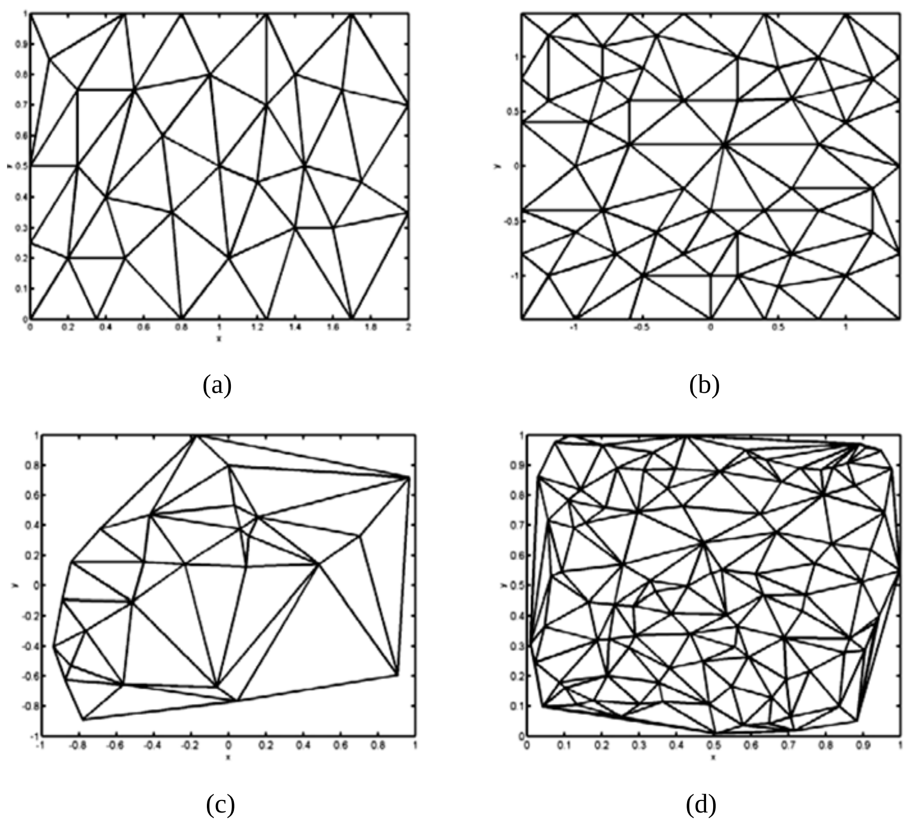
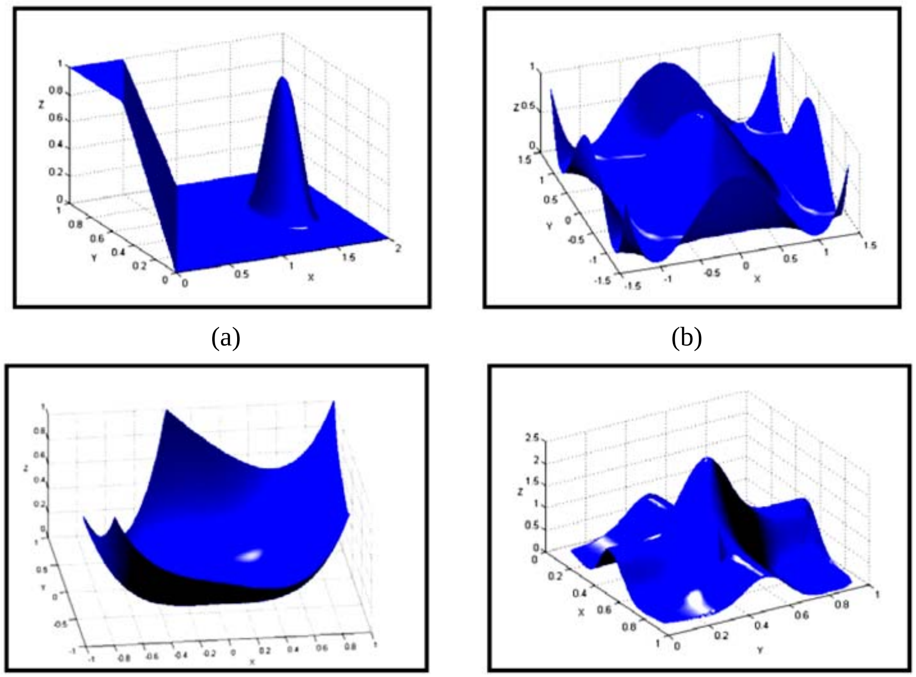


Figure 2. Triangulation domain: (a) 36 node points, (b) 63 node points, (c) 26 node points, and (d) 100 node points.



(a) (b) (c) (d)

Figure 3. Surfaces of test function: (a) F_1, (b) F_2, (c) F_3 and (d) F_4.

For each data set (node set/test function pair), the absolute maximum and mean errors are computed using 33×33 evaluation points of uniform rectangular grid. We have chosen the method described in [4] to estimate the first order derivatives at the data points for all data sets. The estimation of second order derivatives by C^2 quintic method is obtained by applying the method of quadratic fitting [10] with the six nearest neighbours for first data set, twenty nearest neighbours for the second data set and fifteen nearest neighbours for the third and fourth data sets.

The first test function F_1 is evaluated on 36 node points defined on rectangular domain $D = (0, 2) \times (0, 1)$ (Figure 2(a)), where the minimum value of $F_1(x, y)$ on D is 0. The estimated error for all of the methods is given in Table 1. It can be seen that among the C^1 interpolating surfaces, *SPM*-C^1 method is slightly better when compared to *CO*-C^1 [1] and *PGU*-C^1 [8] methods. However, the *SPM*-C^2 positivity-preserving method is the best performer among others.

Table 1. Maximum and mean errors (test function F_1)

Method	Maximum error	Mean error
$CO\text{-}C^1$	0.30962521823046	0.02028720197238
$PGU\text{-}C^1$	0.30962521823637	0.02025760108418
$SPM\text{-}C^1$	0.30337582787194	0.02023212419521
$SPM\text{-}C^2$	0.29790965400000	0.02017058643524

The second test function ([5]), F_2 is evaluated on 63 node points defined on rectangular domain $D = (-1.4, 1.4) \times (-1.4, 1.4)$ (Figure 2(b)), where the minimum value of $F_2(x, y)$ on D is 0. The estimated error for all the methods is given in Table 2. For this data set, $SPM\text{-}C^1$ cubic and $SPM\text{-}C^2$ quintic positivity-preserving methods perform almost similar and slightly better when compared to $CO\text{-}C^1$ [1] and $PGU\text{-}C^1$ [8] methods.

Table 2. Maximum and mean errors (test function F_2)

Method	Maximum error	Mean error
$CO\text{-}C^1$	0.38121498881100	0.04886427357678
$PGU\text{-}C^1$	0.38121498879810	0.04880146871918
$SPM\text{-}C^1$	0.37559422208724	0.04856107463984
$SPM\text{-}C^2$	0.37399672900514	0.04672331328514

In third example, we use the data points which are defined on a sparse non-rectangular domain consisting of 26 node points (Figure 2(c)), where $\min x = -0.9375$, $\max x = 0.9688$, $\min y = -0.8906$ and $\max y = 1$ taken from [7]. The data values are evaluated from function $F_3(x, y)$ whose minimum value of F_3 in this domain is 0.0003. For this sparse data set, the evaluation points are defined on rectangular grid $[-0.9375, 0.9688] \times [-0.8906, 1.0000]$. Of the 1089 evaluation points, 317 are not contained in the convex hull of the nodes and therefore required extrapolation. To evaluate the point beyond the convex hull of the rectangular grid, a similar extrapolation procedure which provides C^1 extrapolation is used. The extrapolation points are determined by considering the function values from triangles having edges along convex hull of data set. The area of triangles formed by boundary edges with extrapolation points is used as a weight of this extrapolation procedure. Table 3 gives the maximum and mean absolute interpolation errors (772 interpolation points) for all four methods. It can be seen that for this sparse node of ungridded scattered data, $SPM\text{-}C^1$ cubic is the best performer among others.

Table 3. Maximum and mean errors (test function F_3)

Method	Maximum error	Mean error
$CO\text{-}C^1$	0.31863914920889	0.04146193757220
$PGU\text{-}C^1$	0.31863914918925	0.04295779521786
$SPM\text{-}C^1$	0.30875649903190	0.04123935175095
$SPM\text{-}C^2$	0.31129191572976	0.04701430313475

The fourth data set also has been defined on a dense non-rectangular grid domain consisting of 100 node points (Figure 2(d)), where $\min x = 0.0096$, $\max x = 0.9983$, $\min y = 0.009$ and $\max y = 0.9982$ taken from [9]. The data values are evaluated from function F_4, where the minimum value of F_4 in this domain is 0.0001. The grids of evaluation points are defined on rectangular grid $[0.0096, 0.9983] \times [0.0090, 0.9982]$ with 178 evaluation points not in the convex hull of the nodes and also required extrapolation as described earlier. The maximum and mean absolute interpolation errors (911 interpolation points) for all the methods are given in Table 4. It can be seen that for this dense node ungridded scattered data, $SPM\text{-}C^2$ positivity-preserving method is the best performer among others. However, as stated in [9], this test function is quite challenging due to its multiple feature and abrupt transitions, and therefore in terms of maximum errors, it is clearly seen that maximum error for fourth data set is the biggest compared to the other data sets.

Table 4. Maximum and mean errors (test function F_4)

Method	Maximum error	Mean error
$CO\text{-}C^1$	0.53071047400598	0.03592742156490
$PGU\text{-}C^1$	0.53223521271406	0.03581209909939
$SPM\text{-}C^1$	0.48909435271271	0.03329492880695
$SPM\text{-}C^2$	0.44332571028901	0.03318843060028

We also summarized our accuracy test by giving the minimum values of the test functions for the non-positivity and positivity preserved of all the methods as in Table 5.

Table 5. Minimum value of test functions for non-positivity and positivity-preserved methods

<table>
<tr><th rowspan="3">Test function</th><th rowspan="3">No. of points</th><th colspan="4">Surface and methods</th></tr>
<tr><th colspan="2">Non-positivity-preserved</th><th colspan="2">Positivity-preserved</th></tr>
<tr><th>C^2 quintic</th><th>C^1 cubic</th><th>C^2 quintic</th><th>C^1 cubic</th></tr>
<tr><td>F_1</td><td>36</td><td>–0.09714</td><td>–0.07095</td><td>0</td><td>0</td></tr>
<tr><td>F_2</td><td>63</td><td>–0.08955</td><td>–0.08727</td><td>0</td><td>0</td></tr>
<tr><td>F_3</td><td>26</td><td>–0.09208</td><td>–0.00244</td><td>0</td><td>0</td></tr>
<tr><td>F_4</td><td>100</td><td>–0.02777</td><td>–0.01415</td><td>0</td><td>0</td></tr>
</table>

4. Conclusion

From Table 5, we conclude that $SPM\text{-}C^2$ quintic has an advantage to preserve the positivity of surfaces although the minimum value is more negative when compared to C^1 cubic. However, as previously stated, the inner and edge Bézier ordinates in this method may be assigned the same lower bound when compared to $SPM\text{-}C^1$ cubic positivity-preserved method where the lower bounds of the edge and inner Bézier ordinates can be adjusted independently while still ensuring positivity of the triangular patches.

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