Data Envelopment Analysis

Introduction & PIM software

Ali Emrouznejad

Aston University
Birmingham B4 7ET
UK

DEA Web:
www.DEAzone.com/

DEA Software:
www.DEAsoftware.co.uk

PI
FA
SFA
DEA

Data Envelopment Analysis

Data Envelopment Analysis

- Data Envelopment Analysis (DEA) was developed as a method for assessing the comparative efficiencies of organisational units such as the branches of a bank, schools, hospital departments or restaurants.

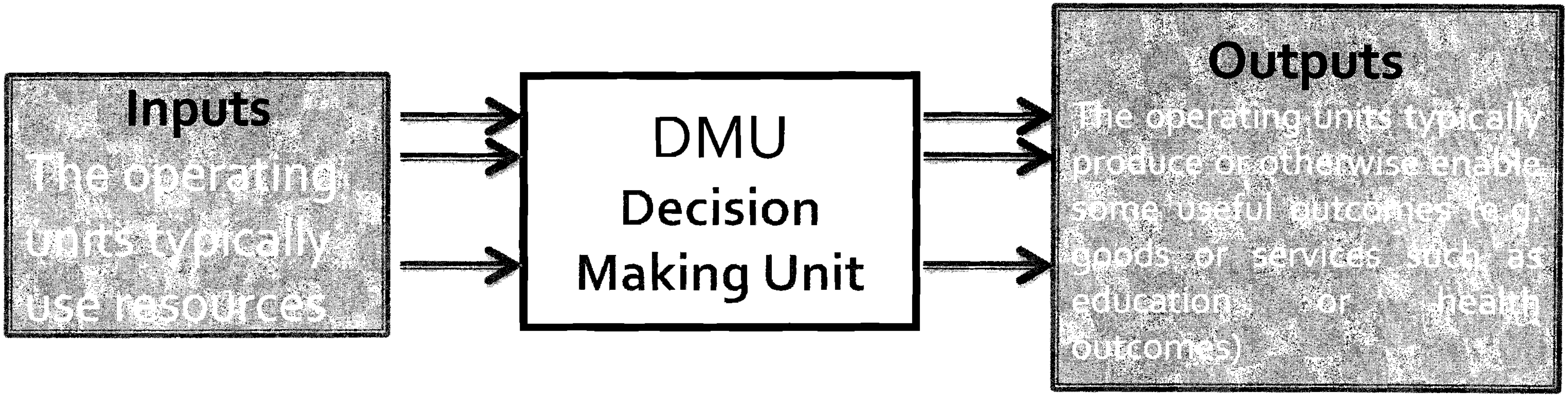
- The efficiencies assessed reflect the scope for resource conservation at the unit being assessed without detriment to its outputs, or alternatively, the scope for output augmentation without additional resources.

- In using DEA in practice we typically go far beyond the computation of a simple measure of the relative efficiency of a unit. We wish to know what operating practices, mix of resources, scale sizes, scope of activities and so on the operating units may adopt to improve their performance.

www.DEAzone.com

Production process

- DEA: a method to compare homogeneous operating units such as schools, hospitals, police forces, individuals etc on efficiency.



DEA makes it possible to compare the operating units on the levels of outputs they secure relative to their input levels.

Value Based DEA

DMU	Doctor	Nurse	Outpatient	Inpatient
H01	30	72	1200	360
H02	10	50	1000	1200
H03	35	20	1250	270
H04	33	44	1100	200
H05	52	91	1300	100
H06	24	40	800	200

Value Based DEA

- Relative efficiency of a unit from a value perspective,

$$\frac{\textit{Sum of weighted outputs of the unit concerned}}{\textit{Sum of weighted inputs of the unit concerned}}$$

- The weights are determined by a DEA model so as to maximise the efficiency of the operating unit concerned subject to no operating unit attaining an efficiency in excess of 1 when those same weights are applied to its inputs and outputs.

Input / output efficiency

- The measure of efficiency can be taken in the input or the output orientation:

  - Output orientation: The efficiency measure is the proportion of **observed output levels** to **maximum possible output levels** for given input levels (output efficiency).

  - Input orientation: The efficiency measure is the proportion to which the observed input levels can be reduced for given output levels (input efficiency).

A graphical approach

- The table below shows the number of doctors and outpatients in hospitals.

- What is the input efficiency of hospital Ho6?

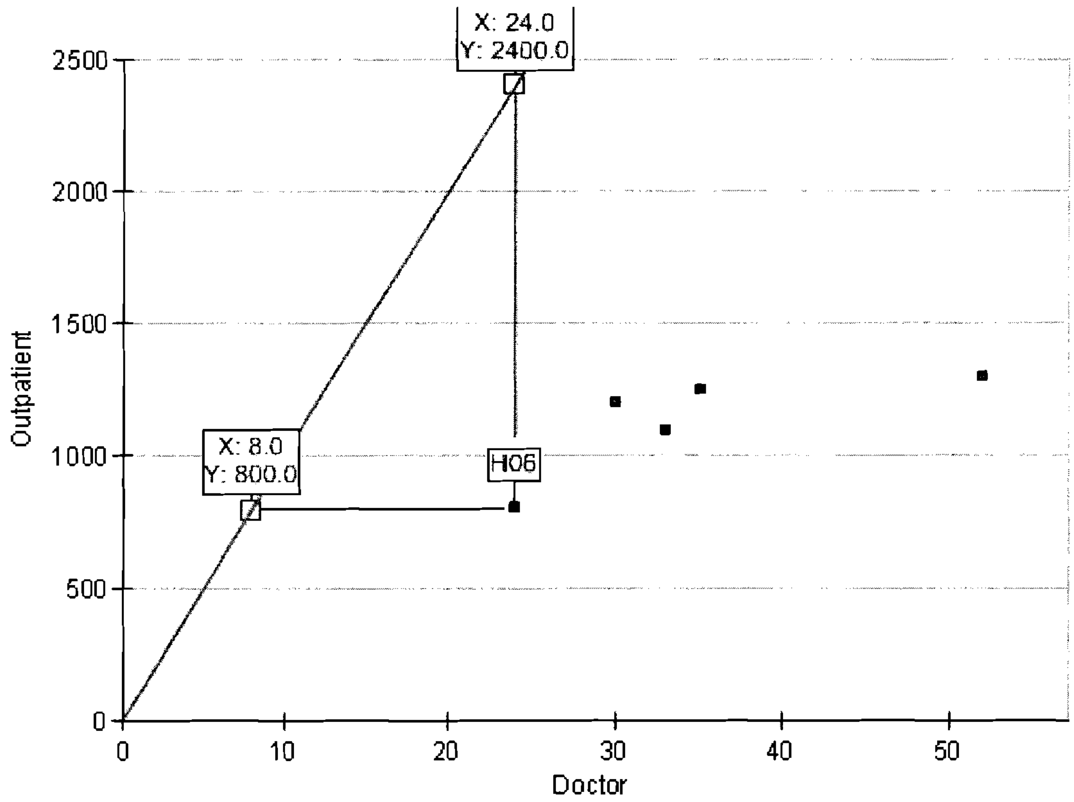
DMU	Doctor	Outpatient
H01	30	1200
H02	10	1000
H03	35	1250
H04	33	1100
H05	52	1300
H06	24	800

Performance Improvement Management

Single input- single output - CRS

- Single input: Doctor
- Single output: Outpatient

DMU	Doctor	Outpatient
H01	30	1200
H02	10	1000
H03	35	1250
H04	33	1100
H05	52	1300
H06	24	800



$$\text{The relative efficiency of H06} = \frac{\text{Target of doctor}}{\text{Actual doctor}} = \frac{8}{24}$$

$$\frac{\text{Actual outpatients}}{\text{Target outpatients}} = \frac{800}{2400}$$

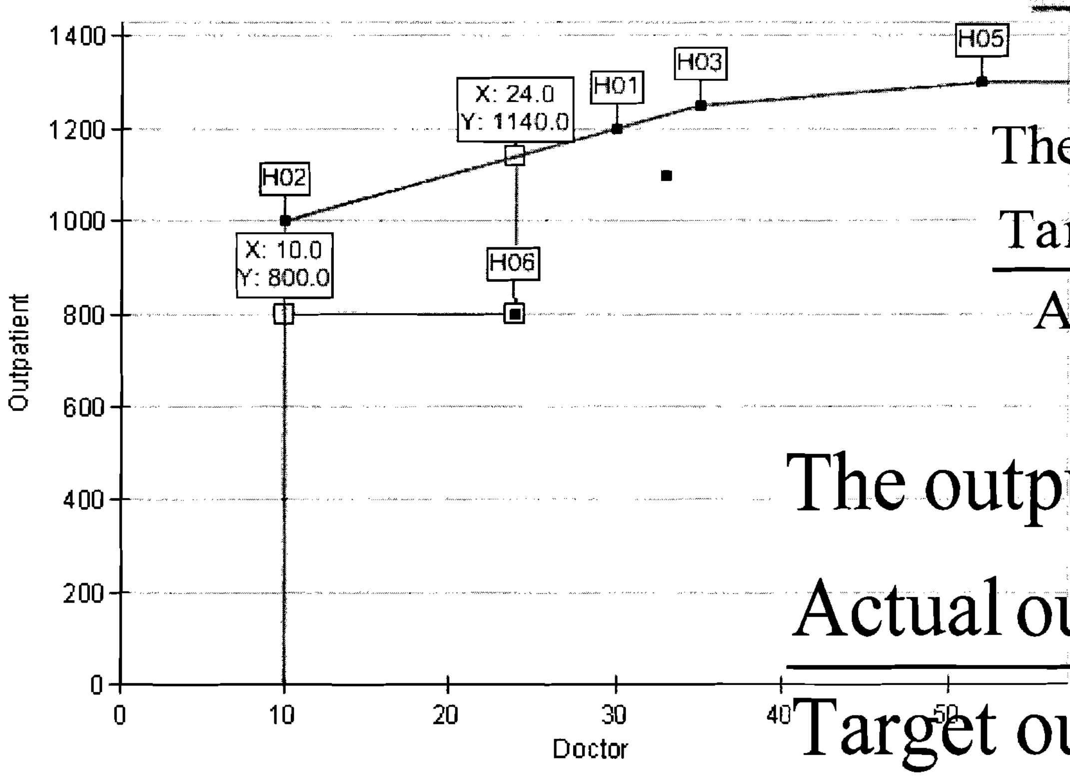
$$= 0.33 = 33\%$$

Single input- single output - VRS

- Single input: Doctor
- Single output: Outpatient

DMU	Doctor	Outpatient
H01	30	1200
H02	10	1000
H03	35	1250
H04	33	1100
	52	1300
	24	800

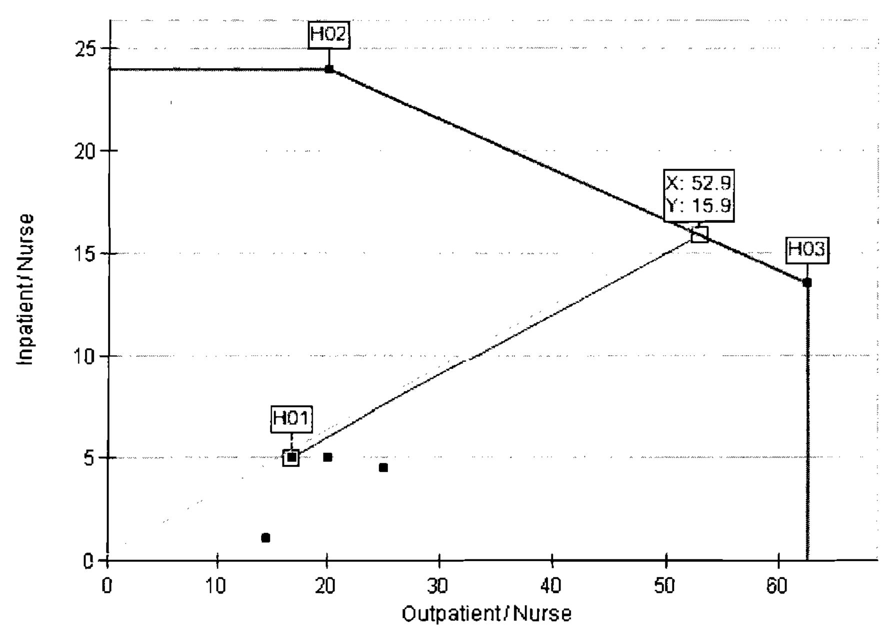
VRS



$$\text{The input efficiency of H06} = \frac{\text{Target of doctor}}{\text{Actual doctor}} = \frac{10}{24} = 0.42 = 42\%$$

$$\text{The output efficiency of H06} = \frac{\text{Actual outpatients}}{\text{Target outpatients}} = \frac{800}{1140} = 0.70$$

2 outputs & 1 input

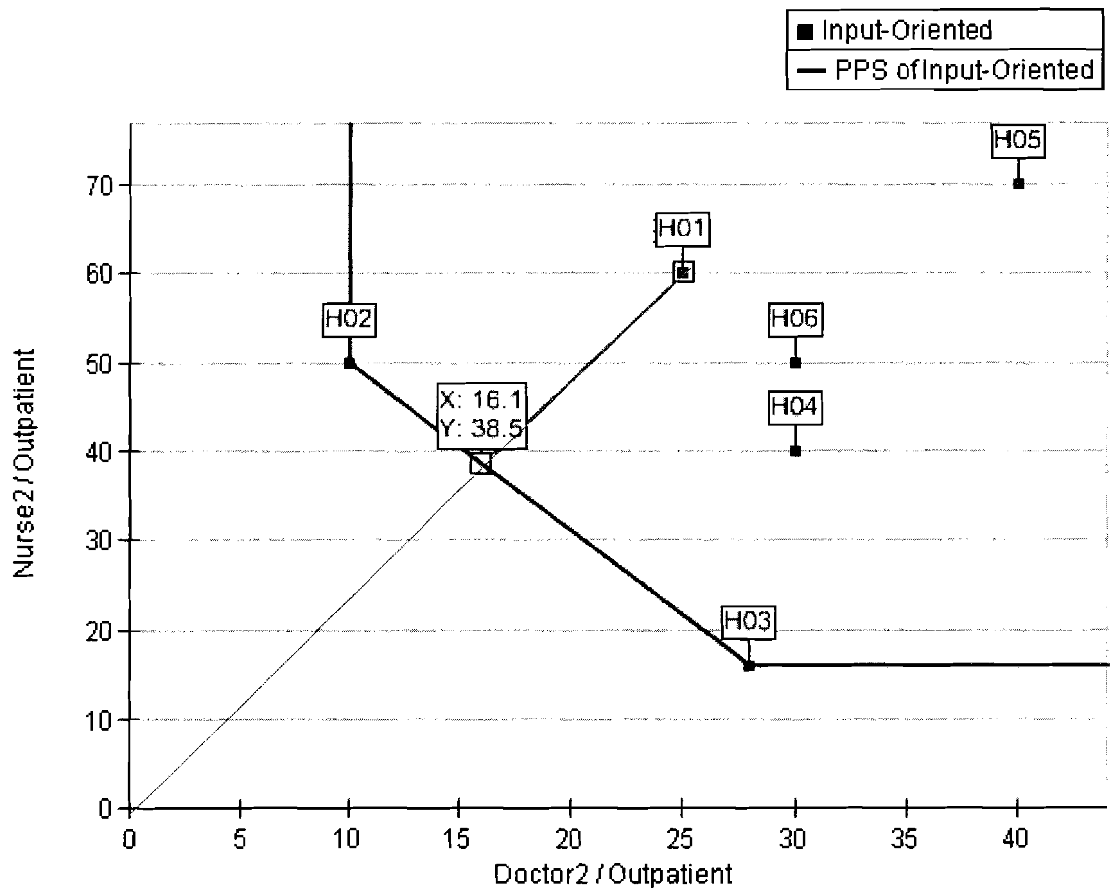


DMU	Nurse	Outpatient	Inpatient
H01	72	1200	360
H02	50	1000	1200
H03	20	1250	270
H04	44	1100	200
H05	91	1300	100
H06	40	800	200

Normalise outputs per 1 nurse

DMU	Nurse	Outpatient	Inpatient
H01	1	16.7	5
H02	1	20	24
H03	1	62.5	13.5
H04	1	25	4.5
H05	1	14.3	1.1
H06	1	20	5

2 inputs & 1 output



DMU	Doctor	Nurse	Outpatient
H01	30	72	1200
H02	10	50	1000
H03	35	20	1250
H04	33	44	1100
H05	52	91	1300
H06	24	40	800

Normalise inputs per 1000 outpatients

DMU	Doctor	Nurse	Outpatient
H01	25	60	1000
H02	10	50	1000
H03	28	16	1000
H04	30	40	1000
H05	40	70	1000
H06	30	50	1000

Feasible units (hospitals)

- The number of doctors & nurses per 1000 outpatients.
- Input efficiency for Hospital H01?

DMU	Doctor	Nurse	Outpatient
H01	25	60	1000
H02	10	50	1000
H03	28	16	1000
H04	30	40	1000
H05	40	70	1000
H06	30	50	1000

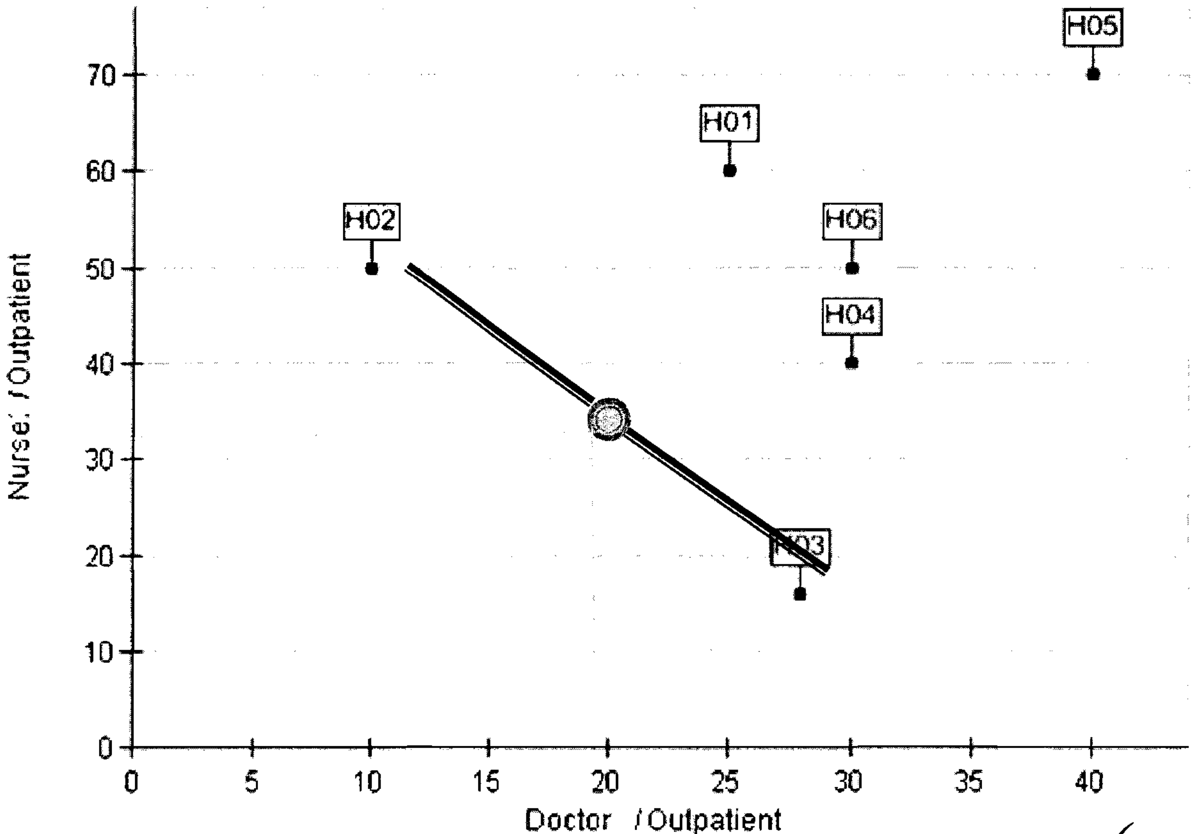
- Compute the proportion to which its input levels can be lowered without detriment to its output level.
- Identify the minimum possible input levels for H01

- In DEA we do this by assuming interpolation of hospitals yields input output levels which are feasible in principle.

- Example: take 0.5 of the input-output levels of hospital 2 and 0.5 of of the input-output levels of hospital 3 and combine:

Feasible units (hospitals)

- (H02+H03)/2



DMU	Doctor	Nurse	Outpatient
H01	25	60	1000
H02	10	50	1000
H03	28	16	1000
H04	30	40	1000
H05	40	70	1000
H06	30	50	1000

$$0.5 \times \text{Hospital}\,2 + 0.5 \times \text{Hospital}\,3 = 0.5 \times \begin{pmatrix} 10 \\ 50 \\ 1000 \end{pmatrix} + 0.5 \times \begin{pmatrix} 28 \\ 16 \\ 1000 \end{pmatrix} = \begin{pmatrix} \boxed{19} \\ \boxed{33} \\ 1000 \end{pmatrix}$$

Feasible units (Hospitals)

- The interpolation assumption means that 19 doctors and 33 nurses, <u>though observed at no hospital</u>, are capable of handling 1000 outpatients.

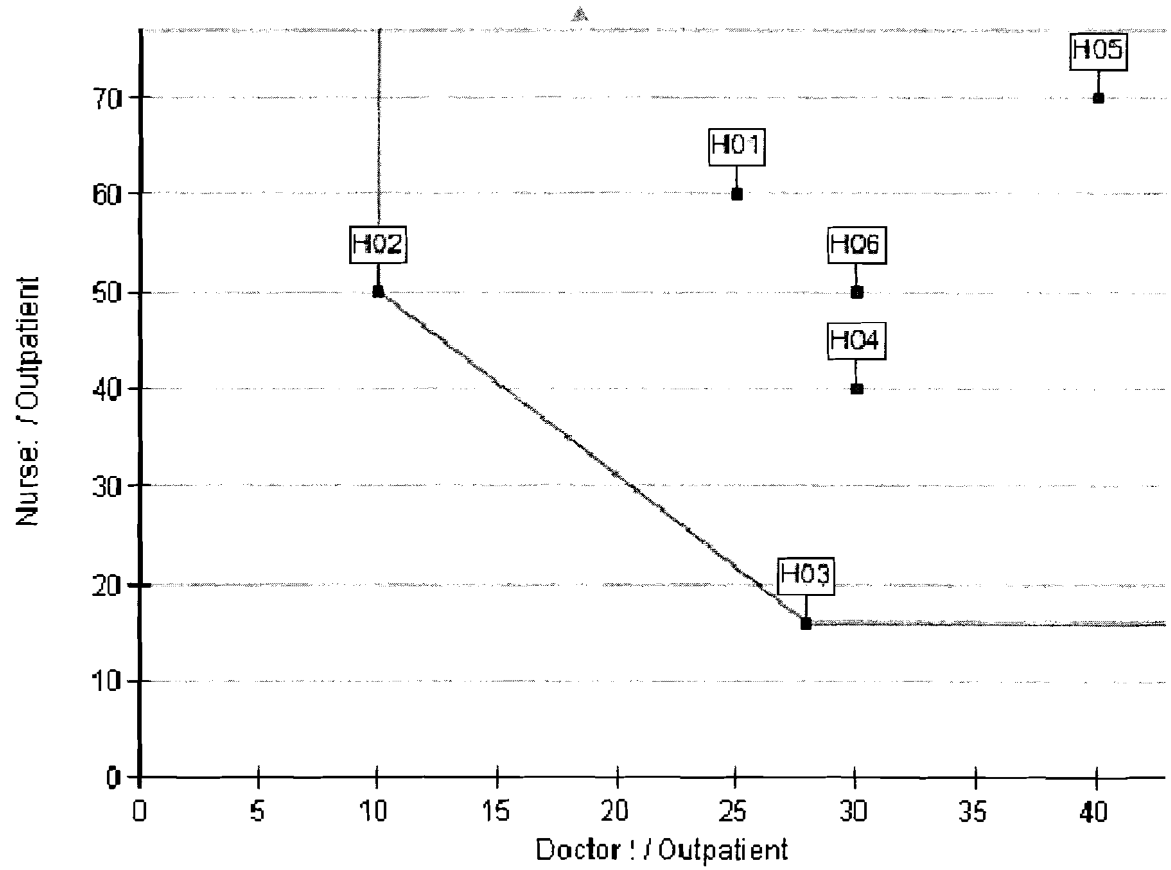
To the extent that any other hospital is also feasible if its input -output levels satisfy

number of doctors ≥ 19.

number of nurses ≥ 33

and
outpatients ≤ 1000

are also feasible.



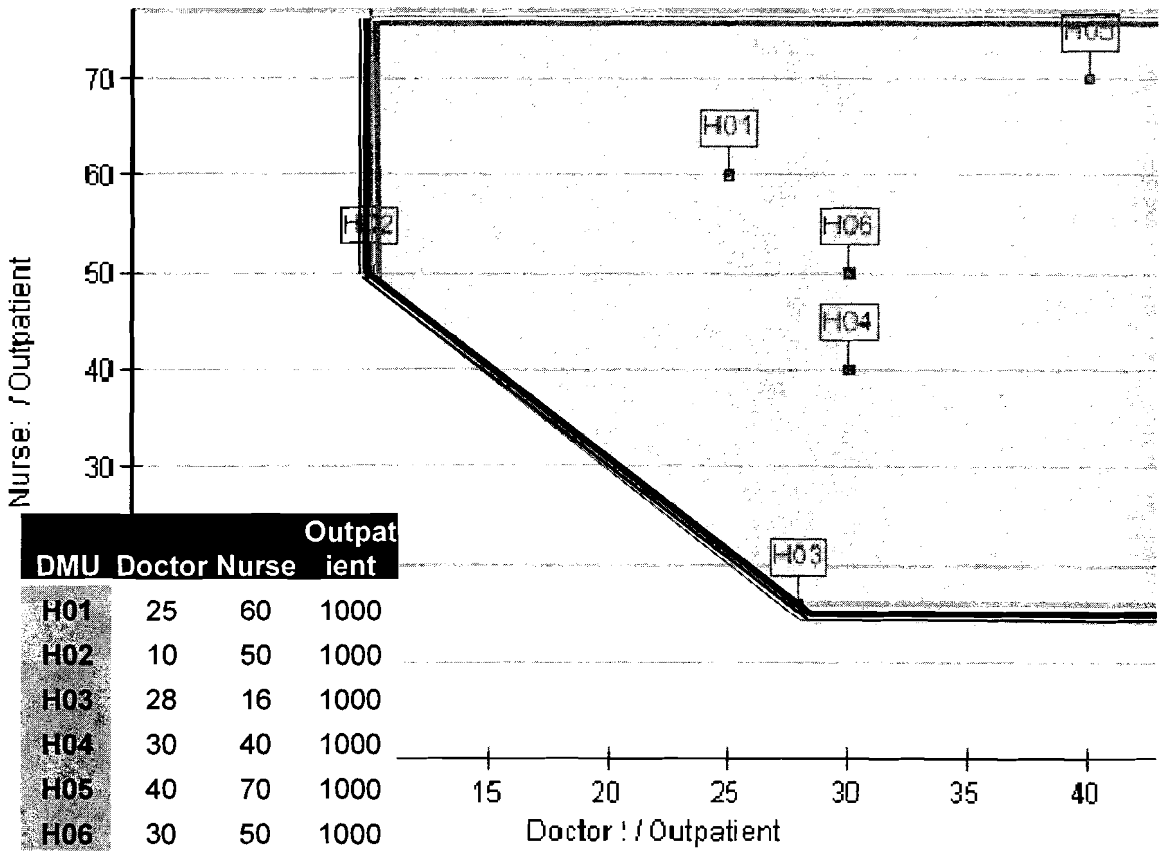
A linear programming approach

Min θ
Such that:

$$
\begin{aligned}
\text{Possible doctor} &= 25\lambda_1 + 10\lambda_2 + 28\lambda_3 + 30\lambda_4 + 40\lambda_5 + 30\lambda_6 \le 25\,\theta \\
\text{Possible nurse} &= 60\lambda_1 + 50\lambda_2 + 16\lambda_3 + 40\lambda_4 + 70\lambda_5 + 50\lambda_6 \le 60\,\theta \\
\text{Possible outpatients} &= 1000\lambda_1 + 1000\lambda_2 + 1000\lambda_3 + 1000\lambda_4 + 1000\lambda_5 + 1000\lambda_6 \ge 1000 \\
& \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 >= 0 .
\end{aligned}
$$

H02 and H03 are efficient, all other hospitals are inefficient. Target for hospital 1 is (16.1, 38.5)

$$\text{The relative efficiency of H01} = \frac{\text{Target of doctor}}{\text{Actual doctor}} = \frac{\text{Target of nurse}}{\text{Actual nurse}} = \frac{16.5}{25} = \frac{38.5}{60} = 0.64 = 64\%$$



DMU	Doctor	Nurse	Outpatient
H01	25	60	1000
H02	10	50	1000
H03	28	16	1000
H04	30	40	1000
H05	40	70	1000
H06	30	50	1000

A linear programming approach

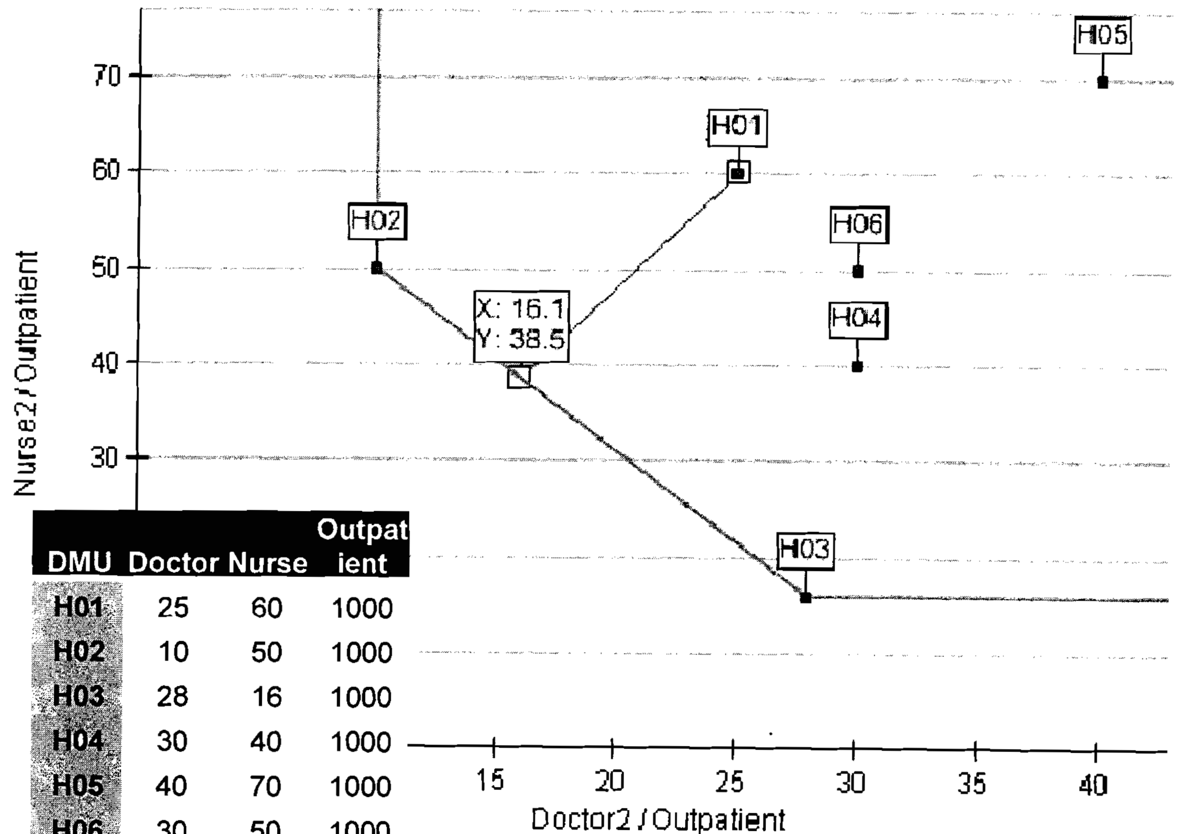
Min θ
Such that:

$$
\begin{aligned}
&\text{Possible doctor} &&= 25\lambda_1 + 10\lambda_2 + 28\lambda_3 + 30\lambda_4 + 40\lambda_5 + 30\lambda_6 \le 25\theta\\
&\text{Possible nurse} &&= 60\lambda_1 + 50\lambda_2 + 16\lambda_3 + 40\lambda_4 + 70\lambda_5 + 50\lambda_6 \le 60\theta\\
&\text{Possible outpatients} &&= 1000\lambda_1 + 1000\lambda_2 + 1000\lambda_3 + 1000\lambda_4 + 1000\lambda_5 + 1000\lambda_6 \ge 1000
\end{aligned}
$$

$\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 >= 0$.

H02 and H03 are efficient, all other hospitals are inefficient.
Target for hospital 1 is (16.1, 38.5)

$$\text{The relative efficiency of H01} = \frac{\text{Target of doctor}}{\text{Actual doctor}} = \frac{\text{Target of nurse}}{\text{Actual nurse}} = \frac{16.5}{25} = \frac{38.5}{60} = 0.64 = 64\%$$



DMU	Doctor	Nurse	Outpatient
H01	25	60	1000
H02	10	50	1000
H03	28	16	1000
H04	30	40	1000
H05	40	70	1000
H06	30	50	1000

2 inputs / 2 outputs – input orientation

DMU	Doctor	Nurse	Outpatient	Inpatient
H01	30	72	1200	360
H02	10	50	1000	1200
H03	35	20	1250	270
H04	33	44	1100	200
H05	52	91	1300	100
H06	24	40	800	200

Min θ
Such that:

$$30\,\lambda_1 + 10\,\lambda_2 + 35\,\lambda_3 + 33\,\lambda_4 + 52\,\lambda_5 + 24\,\lambda_6 \le 30\,\theta$$
$$72\lambda_1 + 50\,\lambda_2 + 20\,\lambda_3 + 44\,\lambda_4 + 91\,\lambda_5 + 40\,\lambda_6 \le 72\,\theta$$

$$1200\,\lambda_1 + 1000\,\lambda_2 + 1250\,\lambda_3 + 1100\,\lambda_4 + 1300\,\lambda_5 + 800\,\lambda_6 \ge 1200$$
$$360\,\lambda_1 + 1200\,\lambda_2 + 270\,\lambda_3 + 200\,\lambda_4 + 100\,\lambda_5 + 200\,\lambda_6 \ge 360$$

$$\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 >= 0\,.$$

2 inputs / 2 outputs – output orientation

DMU	Doctor	Nurse	Outpatient	Inpatient
H01	30	72	1200	360
H02	10	50	1000	1200
H03	35	20	1250	270
H04	33	44	1100	200
H05	52	91	1300	100
H06	24	40	800	200

Max θ
Such that:

$$30\lambda_1 + 10\lambda_2 + 35\lambda_3 + 33\lambda_4 + 52\lambda_5 + 24\lambda_6 \leq 30$$
$$72\lambda_1 + 50\lambda_2 + 20\lambda_3 + 44\lambda_4 + 91\lambda_5 + 40\lambda_6 \leq 72$$

$$1200\lambda_1 + 1000\lambda_2 + 1250\lambda_3 + 1100\lambda_4 + 1300\lambda_5 + 800\lambda_6 \geq 1200\theta$$
$$360\lambda_1 + 1200\lambda_2 + 270\lambda_3 + 200\lambda_4 + 100\lambda_5 + 200\lambda_6 \geq 360\theta$$

$$\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 >= 0 .$$

Slacks – input orientation

Name	Doctor	Nurse	Outpatient	Inpatient
H01	0	40.24	0	96
H02	0	0	0	0
H03	0	0	0	0
H04	0	0	54.49438202	425.28089888
H05	0	0	0	0
H06	0	0	325	535

Min θ

Such that:

$$30\,\lambda_1 + 10\,\lambda_2 + 35\,\lambda_3 + 33\,\lambda_4 + 52\,\lambda_5 + 24\,\lambda_6 + S_d = 30\,\theta$$
$$72\lambda_1 + 50\,\lambda_2 + 20\,\lambda_3 + 44\,\lambda_4 + 91\,\lambda_5 + 40\,\lambda_6 + S_n = 72\,\theta$$

$$1200\,\lambda_1 + 1000\,\lambda_2 + 1250\,\lambda_3 + 1100\,\lambda_4 + 1300\,\lambda_5 + 800\,\lambda_6 - S_o = 1200$$
$$360\,\lambda_1 + 1200\,\lambda_2 + 270\,\lambda_3 + 200\,\lambda_4 + 100\,\lambda_5 + 200\,\lambda_6 - S_i = 360$$

$$\lambda_1 + \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 >= 1 \qquad \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 >= 0\,.$$

Lambdas & Target – output orientation

Max θ

$$1200\,\lambda_1 + 1000\,\lambda_2 + 1250\,\lambda_3 + 1100\,\lambda_4 + 1300\,\lambda_5 + 800\,\lambda_6 \geq 1200\,\theta$$
$$360\,\lambda_1 + 1200\,\lambda_2 + 270\,\lambda_3 + 200\,\lambda_4 + 100\,\lambda_5 + 200\,\lambda_6 \geq 360\,\theta$$

Target **Outpatient for H01** $=1.22 \times 1000 + 0.56 \times 1250 = 1914$
Target **Inpatient for H01** $= 1.22 \times 1200 + 0.56 \times 270 = 1611$

Name	H02	H03
H01	1.21714286	0.55714286
H02	1	0
H03	0	1
H04	0.53428571	0.86428571
H05	1.33714286	1.20714286
H06	0.57142857	0.57142857

	Outpatient Value	Outpatient Target	Outpatient Gain(%)	Inpatient Value	Inpatient Target	Inpatient Gain(%)
H01	1200	1913.57	59.46	360	1611	347.5
H02	1000	1000	0	1200	1200	0
H03	1250	1250	0	270	270	0
H04	1100	1614.64	46.79	200	874.5	337.25
H05	1300	2846.07	118.93	100	1930.5	1830.5
H06	800	1285.71	60.71	200	840	320