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ORIGINAL PAPER

# An extension of the entropic chaos degree and its positive effect

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## Abstract
The Lyapunov exponent is used to quantify the chaos of a dynamical system, by characterizing the exponential sensitivity of an initial point on the dynamical system. However, we cannot directly compute the Lyapunov exponent for a dynamical system without its dynamical equation, although some estimation methods do exist. Information dynamics introduces the entropic chaos degree to measure the strength of chaos of the dynamical system. The entropic chaos degree can be used to compute the strength of chaos with a practical time series. It may seem like a kind of finite space Kolmogorov-Sinai entropy, which then indicates the relation between the entropic chaos degree and the Lyapunov exponent. In this paper, we attempt to extend the definition of the entropic chaos degree on a *d*-dimensional Euclidean space to improve the ability to measure the stength of chaos of the dynamical system and show several relations between the extended entropic chaos degree and the Lyapunov exponent.

**Keywords** Entropic chaos degree · Chaos · Lyapunov exponent

**Mathematics Subject Classification** 65P20 · 37M25

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## 1 Introduction

A quantification of chaotic behavior is critical to understanding it. Some criteria to measure the strength of chaos of a dynamical system have been proposed such as the Lyapunov exponent, KS-entropy, fractal dimension, and so on. In particular, many researchers use the Lyapunov exponent to define the chaos, characterizing an exponential sensitivity on an initial point for the dynamical system. However, we cannot directly compute it for a dynamical system without its dynamical equation even if we know its time series, such as observation data of an experiment. For that reason, there exist some estimation methods of the Lyapunov exponent for the time series. [1, 9–13]

Information Dynamics (ID) was proposed for synthesizing the dynamics of state change and the complexity of a system, and introduced the entropic chaos degree (CD) [8]. Some trials have been carried out to characterize chaotic dynamics using the CD [2–5]. CD allows the strength of chaos of a practical time series to be computed. It may seem like a kind of finite space Kolomogorov-Sinai entropy (KS entropy).

In such situations, some authors have recently discussed the relation between CD and the Lyapunov exponent directly, without KS entropy [6]. They showed that in many cases, the CD for asymmetric tent maps takes a larger value than its Lyapunov exponent. Based on investigations of the difference between the CD and the Lyapunov exponent, they introduced an improved CD, which is the CD with an optional term corresponding to the above difference added [7]. They also showed that the improved CD coincides with the Lyapunov exponent for any one-dimensional chaotic maps under typical conditions.

In this paper, we show that an extended CD coincides with the sum of all Lyapunov exponents for any $d$-dimensional chaotic maps under a typical condition similar to that used for the one-dimensional chaotic maps. We also consider improving the computation algorithm of the extended CD to reduce its computational complexity while maintaining its computation accuracy.

## 2 Entropic chaos degree

In this section, we briefly review the definition of the entropic chaos degree for a difference equation.

Let $\mathbf{R}$ be the set of all real numbers and let $\mathbf{N}$ be the set of all natural numbers. Denote by $\mathbf{R}^d$ the $d$-dimensional Euclidean space. Let $f$ be a map $I$ to $I$ where

$$I \equiv [a,b]^d \subset \mathbf{R}^d,\ \ a,b \in \mathbf{R},\ \ d \in \mathbf{N}.$$

Now we consider a difference equation such that

$$x_{n+1} = f\left(x_n\right),\ \ n = 0, 1, \ldots, \quad x_0 \in I.$$

For an initial point $x_0$ and finite partitions $\{A_i\}$ of $I$ such that

$$I = \bigcup_{k=1}^{N} A_k, \quad A_i \cap A_j = \emptyset \ (i \neq j),$$

the probability distribution $\left(p_{i,A}^{(n)}(M)\right)$ of time $n$ and the joint distribution $\left(p_{i,j,A}^{(n,n+1)}(M)\right)$ of time $n$ and time $n+1$ are given as

$$p_{i,A}^{(n)}(M) = \frac{1}{M} \sum_{k=n}^{n+M-1} 1_{A_i}(x_k),$$

$$p_{i,j,A}^{(n,n+1)}(M) = \frac{1}{M} \sum_{k=n}^{n+M-1} 1_{A_i}(x_k) 1_{A_j}(x_{k+1})$$

where $1_A$ is the characteristic function of set $A$.

The entropic chaos degree $D$ of the orbit $\{x_n\}$ is then defined as in [8]

$$\begin{aligned} D^{(M,n)}(A,f) &= \sum_{i=1}^{N} \sum_{j=1}^{N} p_{i,j,A}^{(n)}(M) \log \frac{p_{i,A}^{(n)}(M)}{p_{i,j,A}^{(n,n+1)}(M)} \\ &= \sum_{i=1}^{N} p_{i,A}^{(n)}(M) \left( - \sum_{j=1}^{N} p_A^{(n)}(j|i)(M) \log p_A^{(n)}(j|i)(M) \right) \end{aligned} \tag{1}$$

where $p_A^{(n)}(j|i)$ is the conditional probability from component $A_i$ to $A_j$.

We simplify the denotation of $D^{(M,n)}(A,f)$ as $D^{(M)}(A,f)$ if the probability distribution $\left(p_{i,A}^{(n)}\right)$ is a stationary distribution. Moreover, we simplify the denotation of $D^{(M)}(A,f)$ as $D^{(M)}(A)$ if an orbit $\{x_n\}$ is regarded as a practical stationary time series without $f$.

## 3 An extension of the entropic chaos degree

In the following, we set

$$f : I \rightarrow I, \quad I = \prod_{k=1}^{d} \left[a_k, b_k\right] \subset \mathbf{R}^d,$$

$$f(\mathbf{x}) = \left(f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_d(\mathbf{x})\right)^t, \quad \mathbf{x} = (x_1, \ldots, x_d)^t.$$

Let the $L^d (= N)$-equipartitions $\{A_i\}$ of $I$ be

$$I = \bigcup_{i=0}^{L^d-1} A_i, \quad A_i = A_{(i_1 \cdots i_d)_L} = \prod_{k=1}^{d} A_{i_k}^{(k)}, \quad i_k = 0, 1, \ldots, L-1$$

where

$$A^{(k)}_{i_k} = \begin{cases} \left[a_k + \dfrac{b_k - a_k}{L} i_k,\ a_k + \dfrac{b_k - a_k}{L}(i_k + 1)\right) & (i_k = 0, 1, \ldots, L-2), \\ \left[a_k + \dfrac{b_k - a_k}{L}(L-1),\ b_k\right] & (i_k = L-1) \end{cases}$$

for any $k = 1, \ldots, d$.

Further, for any $A_i, A_j$, $i, j = 0, 1, \ldots, L^d - 1$, we divide $A_j$ into $\left(S_{i,j}\right)^d$ -equipartitions $\left\{B^{(i,j)}_l\right\}_{0 \le l \le (S_{i,j})^d - 1}$ such that

$$A_j = A_{(j_1 \cdots j_d)_L} = \bigcup_{l=0}^{(S_{i,j})^d - 1} B^{(i,j)}_l, \quad B^{(i,j)}_l = B^{(i,j)}_{(l_1 \cdots l_d)_{S_{i,j}}} = \prod_{k=1}^{d} B^{(i,j,k)}_{l_k}$$

where

$$B^{(i,j,k)}_{l_k} = \begin{cases} \left[\hat{a}_k + \dfrac{\hat{b}_k - \hat{a}_k}{S_{i,j}} l_k + \dfrac{\hat{b}_k - \hat{a}_k}{S_{i,j}}(l_k + 1)\right) & (l_k = 0, 1, \ldots, S_{i,j} - 2,\ S_{i,j} \ge 2) \\ \left[\hat{a}_k + \dfrac{\hat{b}_k - \hat{a}_k}{S_{i,j}}(S_{i,j} - 1), \hat{b}_k\right] & (l_k = S_{i,j} - 1) \end{cases}$$

and

$$\begin{aligned} \hat{a}_k &= a_k + \frac{b_k - a_k}{L} i_k, \\ \hat{b}_k &= \begin{cases} a_k + \dfrac{b_k - a_k}{L}(i_k + 1) & (i_k = 0, 1, \ldots, L-2) \\ b_k & (i_k = L-1) \end{cases} \end{aligned}$$

for $k = 1, \ldots, d$.

For each $B^{(i,j)}_l$, we define a function $g_{i,j}$ by

$$g_{i,j}\left(B^{(i,j)}_l\right) \equiv \begin{cases} 1 & (B^{(i,j)}_l \cap f(A_i) \ne \emptyset) \\ 0 & (B^{(i,j)}_l \cap f(A_i) = \emptyset) \end{cases}.$$

For any $A_i, A_j,\ i \ne j$, we give a function $R(S_{i,j})$ using $g_{i,j}$ as

$$R(S_{i,j}) \equiv \frac{\sum_{l=0}^{(S_{i,j})^d - 1} g_{i,j}\left(B^{(i,j)}_l\right)}{\left(S_{i,j}\right)^d}. \tag{2}$$

Here, the numerator of $R(S_{i,j})$ is the number of components of $\{B^{(i,j)}_l\}$ included in $A_j \cap f(A_i)$, and the denominator of $R(S_{i,j})$ is the number of elements of $\{B^{(i,j)}_l\}$

included in $A_j$. Thus $R(S_{i,j})$ becomes the volume rate of $A_j \cap f(A_i)$ to $A_j$ under the $1/S_{i,j}$ scale.

We then define an extended entropic chaos degree as follows.

Definition 1

$$D_S^{(M,n)}(A,f) \equiv \sum_{i=0}^{L^d-1} p_{i,A}^{(n)}(M) \left( \sum_{j=0}^{L^d-1} p_A^{(n)}(j|i)(M) \log \frac{R(S_{i,j})}{p_A^{(n)}(j|i)(M)} \right) \tag{3}$$

where $S = (S_{i,j})_{0 \le i,j \le L^d-1}$.

***Remark 1*** If we set $S_{i,j} = 1$ $(i,j = 0,1,\ldots,L^d-1)$, then the extended chaos degree $D_S$ becomes the chaos degree $D$, or

$$D_1^{(M,n)}(A,f) = D^{(M,n)}(A,f)$$

where $1 = (1)_{0 \le i,j \le L^d-1}$.

***Remark 2*** To give an interpretation to the extension for the entropic chaos degree, we consider the information quantity of $f(A_i)$ included in $A_j$.

The entropic chaos degree $D$ includes an information quantity as

$$\log \frac{1}{p_A^{(n)}(j|i)(M)}. \tag{4}$$

Then the volume of $A_j \cap f(A_i)$ is treated as $m(A_j)$ where $m$ is the Lebesgue measure on $\mathbf{R}^d$.

On the other hand, we treat the volume of $A_j \cap (A_i)$ as $m(A_j)R(S_{i,j})$ under the scale $1/S_{i,j}$. Thus, under the scale $1/S_{i,j}$, we use

$$\log \frac{R(S_{i,j})}{p_A^{(n)}(j|i)(M)} \tag{5}$$

instead of Eq. (4). Because Eq. (5) can take a negative value, Eq. (5) is no longer any information quantities.

We regard the extended entropic chaos degree $D_S$ as the entropic chaos degree $D$ under the scale $1/S_{i,j}$.

Then we have the following theorem.

**Theorem 1** *Let $L$, $M$ be sufficiently large natural numbers. If a map $f$ is stable periodic with period $T$, then we have*

$$D_S^{(M,n)}(A,f) = -\frac{d}{T}\sum_{k=1}^{T} \log S_{i_k j_k}. \tag{6}$$

***Proof*** Let the number $M$ of orbit points be a sufficiently large natural number. If $f$ is stable periodic with period $T$, then there exist $i_k$ $(k = 1, \ldots, T)$ such that

$$P_{i,A}^{(n)}(M) = \begin{cases} \frac{1}{T} & (i = i_k) \\ 0 & (i \neq i_k) \end{cases} \tag{7}$$

For the same reason, there exist $j_k$ for $i_k = 1, \ldots, T$ such that

$$f(A_{i_k}) = A_{j_k} \quad (i_k \neq j_k). \tag{8}$$

From Eq. (8), we obtain

$$P_A^{(n)}(j|i)(M) = \begin{cases} 1 & (i \in i_k,\ j \in j_k) \\ 0 & (i \notin i_k \ or\ j \notin j_k) \end{cases}. \tag{9}$$

We also have

$$R(S_{i,j}) = \frac{1}{(S_{i,j})^d} \tag{10}$$

for any $A_j$ under the scale $1/S_{i,j}$.

From Eqs. (9) and (10), we get

$$\log \frac{R(S_{i,j})}{P_A^{(n)}(j|i)(M)} = \begin{cases} -d \log S_{i,j} & (i = i_k,\ j = j_k) \\ 0 & (i \neq i_k \ or\ j \neq j_k) \end{cases}. \tag{11}$$

Substituting Eqs. (7) and (11) into Eq. (3), we finally have

$$\begin{aligned} D_S^{(M,n)}(A,f) &= \sum_{k=1}^{T} P_{i_k,A}^{(n)}(M)\left( P_A^{(n)}(j_k|i_k)(M) \log \frac{R(S_{i_k j_k})}{P_A^{(n)}(j_k|i_k)(M)} \right) \\ &= \sum_{k=1}^{T} \frac{1}{T}\left(-d \log S_{i_k j_k}\right) \\ &= -\frac{d}{T}\sum_{k=1}^{T} \log S_{i_k j_k}. \end{aligned} \tag{12}$$

□

***Remark 3*** Theorem 1 means that the extended entropic chaos degree takes a quite small negative value if $f$ is stable periodic, i.e., from Eq. (6), we have

$$D_S^{(M,n)}(A,f) \longrightarrow -\infty \quad (S_{i_k j_k} \to \infty)$$

for any stable periodic orbits.

Thus, the extended entropic chaos degree $D_S^{(M,n)}(A,f)$ takes a smaller value as the scale $1/S_{i_k j_k}$ decreases.

Now we consider the relationship between the extended entropic chaos degree and the Lyapunov exponent. For any $\mathbf{x} = (x_1, x_2, \ldots, x_d)^t$, $\mathbf{y} = (y_1, y_2, \ldots, y_d)^t \in A_i$, we define an approximate Jacobian matrix $\widehat{J}$ by

$$\widehat{J}(\mathbf{x}, \mathbf{y}) \equiv \left( \frac{f_i(\mathbf{x}) - f_i(\mathbf{y})}{x_j - y_j} \right)_{1 \le i,j \le d}.$$

Further, we set that $r_k(\mathbf{x}, \mathbf{y})$, $k = 1, 2, \ldots, d$, is an eigenvalue of matrix $\sqrt{\widehat{J}^t(\mathbf{x}, \mathbf{y})\widehat{J}(\mathbf{x}, \mathbf{y})}$. Then we introduce the following assumption.

**Assumption 1** For sufficiently large natural numbers $L$, $M$, we assume that the following conditions are satisfied.

(1) Points in $A_i$ are uniformly distributed over the entirety of $A_i$.
(2) We have $r_k(\mathbf{x}, \mathbf{y}) = r_k^{(i)}$, $k = 1, 2, \ldots, d$ for any $\mathbf{x}, \mathbf{y} \in A_i$ where there exists at least one $r_j^{(i)}$ for any $A_i$ such that $r_j^{(i)} \ge 1$.

***Remark 4*** If we assume Assumption 1, then we have

$$R(S_{i,j}) \longrightarrow \frac{m\left(A_j \cap f(A_i)\right)}{m\left(A_j\right)} \quad \left(S_{i,j} \to \infty\right) \tag{13}$$

or

$$m(A_j \cap f(A_i)) = m(A_j)R(\infty),$$

where $m$ is the Lebesgue measure on $\mathbf{R}^d$.

Then we have the following theorem.

**Theorem 2** *For any $A_i$, $i = 0, 1, \ldots, L^d - 1$, we assume Assumption 1. Then we have*

$$\lim_{S\to\infty} \lim_{L\to\infty} \lim_{M\to\infty} D_S^{(M,m)}(A,f) = \sum_{k=1}^{d} \lambda_k$$

*where*

$$S \to \infty \Leftrightarrow S_{i,j} \to \infty \ (i,j = 0, 1, \ldots, L^d - 1)$$

*and $\{\lambda_1, \ldots, \lambda_d\}$ is the Lyapunov spectrum of a map $f$.*

***Proof*** $p^{(n)}(j|i)(M)$ is the rate of the number of points of $A_j \cap f(A_i)$ to the number of points of $f(A_i)$, i.e.,

$$p^{(n)}(j|i)(M) = \frac{\left|A_j \cap f(A_i)\right|}{\left|f(A_i)\right|}. \tag{14}$$

From Assumption 1, for sufficiently large natural numbers $L$, $M$, we have

$$\frac{\left|A_j \cap f(A_i)\right|}{\left|f(A_i)\right|} \simeq \frac{\mu\left(A_j \cap f(A_i)\right)}{\mu\left(f(A_i)\right)}, \tag{15}$$

where $\mu$ is the invariant measure of $f$.

Now let $c_k$ be

$$c_k = \frac{b_k - a_k}{L}, \quad k = 1, 2, \ldots, d.$$

Then the volume of $A_i$ is

$$m(A_i) = \prod_{k=1}^{d} c_k, \quad i = 0, 1, \ldots, L^d - 1, \tag{16}$$

where $m$ is the Lebesgue measure on $R^d$.

From Assumption 1, the volume of $f(A_i)$ becomes

$$m\left(f(A_i)\right) = \prod_{k=1}^{d} r_k^{(i)} c_k. \tag{17}$$

Under Assumption 1, we have

$$\frac{\mu\left(A_j \cap f(A_i)\right)}{\mu\left(f(A_i)\right)} \simeq \frac{m\left(A_j \cap f(A_i)\right)}{m\left(f(A_i)\right)}. \tag{18}$$

Using Eqs. (14), (15), (17), and (18), we obtain

$$p^{(n)}(j|i)(M) \simeq \frac{m\left(A_j \cap f(A_i)\right)}{\prod_{k=1}^{d} r_k^{(i)} c_k}. \tag{19}$$

On the other hand, for sufficiently large natural numbers $L$, $M$, from Eqs. (13) and (16), we have

$$R(S_{i,j}) \longrightarrow \frac{m\left(A_j \cap f(A_i)\right)}{\prod_{k=1}^{d} c_k} \quad (S_{i,j} \to \infty). \tag{20}$$

Thus from Eqs. (19) and (20), we obtain

$$\begin{aligned} \log \frac{R(S_{i,j})}{p^{(n)}(j|i)(M)} &\longrightarrow \log\left(\prod_{k=1}^{d} r_k^{(i)}\right) \quad (S_{i,j}\to\infty) \\ &= \sum_{k=1}^{d} \log\left(r_k^{(i)}\right). \end{aligned} \tag{21}$$

Finally, for sufficiently large natural numbers $L$, $M$, from Eqs. (3) and (21), we have

$$\begin{aligned} &D_S^{(M,n)}(A,f) \\ &\longrightarrow \sum_{i=0}^{L^d-1} p_{i,A}^{(n)}(M)\left\{\sum_{j=0}^{L^d-1} p_A^{(n)}(j|i)(M)\left(\sum_{k=1}^{d}\log\left(r_k^{(i)}\right)\right)\right\}(S_{i,j}\to\infty) \\ &= \sum_{i=0}^{L^d-1} p_{i,A}^{(n)}(M)\left(\sum_{k=1}^{d}\log\left(r_k^{(i)}\right)\right) \\ &= \sum_{i=0}^{L^d-1}\left(\sum_{k=1}^{d}\log\left(r_k^{(i)}\right)\right)p_{i,A}^{(n)}(M) \\ &\longrightarrow \int_{\mathbf{x},\mathbf{y}}\left(\sum_{k}\log\left(r_k(\mathbf{x},\mathbf{y})\right)\right)\rho(\mathbf{x},\mathbf{y})\prod_{k=1}^{d} dx_k dy_k \quad (L,M\to\infty) \\ &= \sum_{k=1}^{d}\lambda_k. \end{aligned} \tag{22}$$

Here, $\rho(\mathbf{x},\mathbf{y})$ is the density function of $(\mathbf{x},\mathbf{y})$, and $\{\lambda_1,\lambda_2,\ldots,\lambda_d\}$ is the Lyapunov spectrum of $f$. □

## 4 An improvement of the numerical calculation method for the extended entropic chaos degree

Theorem 2 implies that under Assumption 1, the extended entropic chaos degree $D_S$ for a map $f$ goes to the sum of all Lyapunov exponents for the map $f$ as $L$, $M$, and $S_{i,j}$ go to $\infty$. However, we must treat $M$ and $L$ as finite natural numbers in the numeric calculation of the entropic chaos degree.

Now, for any $A_i, A_j,\ i,j = 0,1,\ldots,L^d-1$, we define

$$S_{i,j}^{\max} \equiv \left\lfloor \sqrt[d]{\left|A_j \cap f(A_i)\right|} \right\rfloor. \tag{23}$$

Using $S_{i,j}^{\max}$, we compute the extended entropic chaos degree $D_{S^{\max}}$ as follows.

$$D_{S^{\max}}^{(M,n)}(A,f) = \sum_{i=0}^{L^d-1} p_{i,A}^{(n)}(M)\left(\sum_{j=0}^{L^d-1} p_A^{(n)}(j|i)(M)\log\frac{R(S_{i,j}^{\max})}{p_A^{(n)}(j|i)(M)}\right), \tag{24}$$

where

$$S^{\max} = \left(S_{i,j}^{\max}\right)_{0\le i,j\le L^d-1}.$$

In the definition of the extended entropic chaos degree $D_S$ (Eq. 3), from Eq. (21), we have

$$h(i,j) \equiv \log\frac{R(S_{i,j})}{p^{(n)}(j|i)(M)} \simeq \sum_{k=1}^{d}\log\left(r_k^{(i)}\right) \tag{25}$$

if all of $S_{i,j}$ $(i,j=0,1,\dots,L^d-1)$ are sufficiently large natural numbers. Noticing that $h(i,j)$ does not depend on $j$, we introduce a simplified form $\overline{D}_{S^{\max}}$ of the extended entropic chaos degree $D_{S^{\max}}$ by

$$\overline{D}_{S^{\max}}^{(M,n)}(A,f) \equiv \sum_{i=0}^{L^d-1} p_{i,A}^{(n)}(M)\left(\log\frac{R(S_{i,j_{\max}}^{\max})}{p_A^{(n)}(j_{\max}|i)(M)}\right), \tag{26}$$

where $j_{\max}$ is the number $j\in\{0,1,\dots,L^d-1\}$ such that

$$p_A^{(n)}(j_{\max}|i)(M) = \max_{0\le j\le L^d-1} p_A^{(n)}(j|i)(M). \tag{27}$$

That means that the simplified form $\overline{D}_{S^{\max}}$ uses $h(i,j_{\max})$ instead of the average of $h(i,j)$ in the definition of the extended entropic chaos degree $D_{S^{\max}}$.

Then we have the following relation.

$$\overline{D}_{S^{\max}}^{(M,n)}(A,f) \longrightarrow \sum_{k=1}^{d}\lambda_k \quad (L,M\to\infty) \tag{28}$$

## 5 Numerical computation results

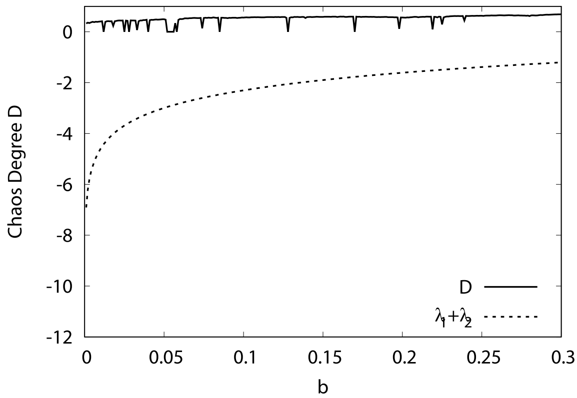
In this section, we attempt to numerically compute the extended entropic chaos degree for a typical two-dimensional chaotic map. We set $M = 1{,}000{,}000$, $L = \left\lfloor\sqrt{M}\right\rfloor$ to satisfy that $L^2 < M$.

### 5.1 Two-dimensional chaotic map

We consider the Hénon map as a typical two-dimensional chaotic map.

The Hénon map $f_{a,b}$ is given as

**Fig. 1** $D$ versus $b$ for Hénon map $f_b$




$$f_{a,b}(\mathbf{x}) = \left(a - x_1^2 + bx_2, x_1\right)^t \tag{29}$$

where $\mathbf{x} = (x_1, x_2)^t \in [a_1, b_1] \times [a_2, b_2]$.

For $a = 1.4, 0 < b \leq 0.3$, we have

$$a_k = -1.8, \quad b_k = 1.8, \quad k = 1, 2.$$

Then the Jacobi matrix $Df_{a,b}(\mathbf{x})$ of the map $f_{a,b}$ becomes

$$Df_{a,b}(\mathbf{x}) = \begin{pmatrix} 2x_1 & b \\ 1 & 0 \end{pmatrix}. \tag{30}$$

Thus $Df_{a,b}(\mathbf{x})$ depends on $\mathbf{x}$ and the parameter $b$. We also cannot keep the orthonormal system with the map $f$.

In the following, we simplify the denotation of $f_{1.4,b}$ as $f_b$ .

### 5.2 Extended entropic chaos degree for a two-dimensional chaotic map

First, we show the computation results of the entropic chaos degree $D$ for the Hénon map $f_b$ in Fig. 1. The entropic chaos degree $D$ for any map always takes a non-negative value. Therefore, if the sum $\lambda_1 + \lambda_2$ is negative, then the entropic chaos degree $D$ for $f_b$ takes a different amount from the sum $\lambda_1 + \lambda_2$ of all Lyapunov exponents for $f_b$.

Secondly, we show the computation results of the extended entropic chaos degree $D_{S\max}$ for $f_b$ in Fig. 2. The extended entropic chaos degree $D_{S\max}$ for $f_b$ has almost the same change on $b$ in (0, 0.3] as the sum $\lambda_1 + \lambda_2$ of all Lyapunov exponents for $f_b$. That is, the extended entropic chaos degree $D_{S\max}$ for $f_b$ takes nearly the same value at most points on $b$ in (0, 0.3] as the sum $\lambda_1 + \lambda_2$ of all Lyapunov exponents for $f_b$. However, there exist some points on $b$ such that the extended entropic chaos degree $D_{S\max}$ for $f_b$ takes a quite small amount relative to $\lambda_1 + \lambda_2$ for $f_b$.

Now we show the bifurcation diagram of the Hénon map and the computation results of Lyapunov exponents $\lambda_1$ and $\lambda_2$ for $f_b$ in Figs. 3 and 4, respectively.

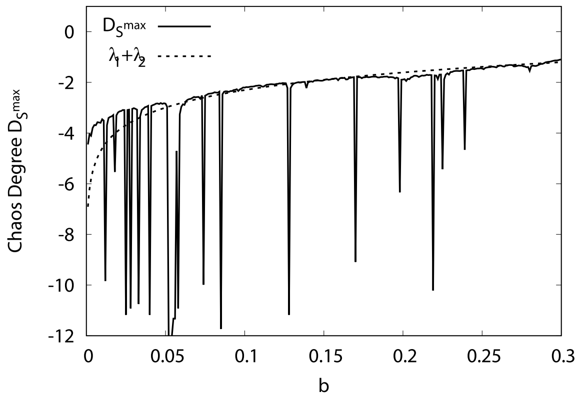



**Fig. 2** $D_{S^{\max}}$ versus $b$ for Hénon map $f_b$

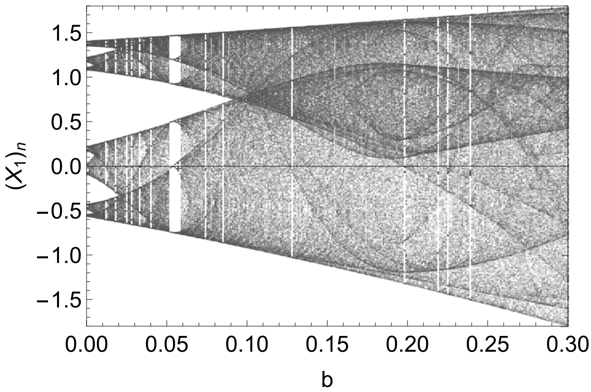



**Fig. 3** $(x_1)_n$ versus $b$ for Hénon map $f_b$

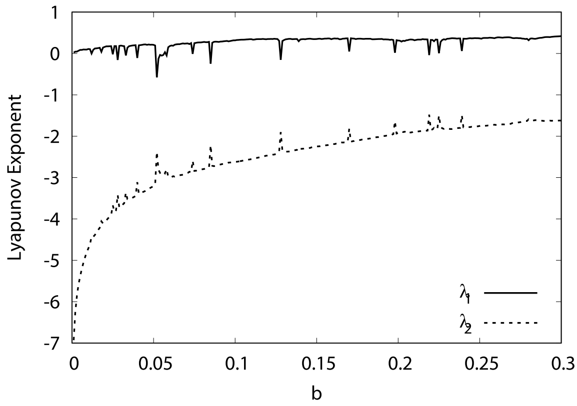
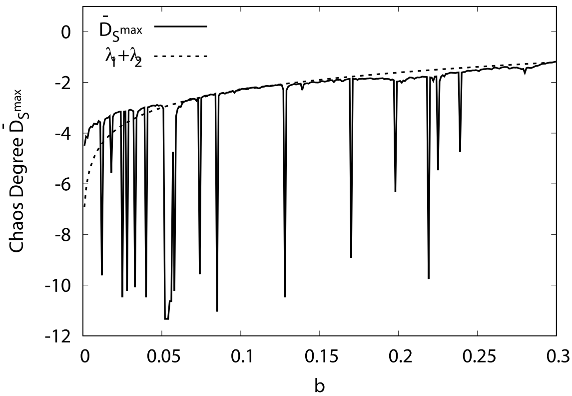



**Fig. 4** $\lambda_k$ $(k = 1, 2)$ versus $b$ for Hénon map $f_b$

At many bifurcation points of the Hénon map, the Lyapunov exponents diverge to negative infinity, or do not exist, while the sum of its Lyapunov exponents is always $\log b$. The extended CD and the Lyapunov exponents also do not work well at many bifurcation points of the Hénon map.

**Fig. 5** $\overline{D}_{S^{\max}}$ versus $b$ for Hénon map




The above implies the following. (1) For any chaotic map $f$, the extended entropic chaos degree $D_{S^{\max}}$ takes almost the same value as the sum of all of its Lyapunov exponents. (2) However, for any stable periodic map, the extended entropic chaos degree $D_S^{\max}$ does not work well because it takes a finite negative value as a numerical finiteness of an infinitely negative amount.

Finally, we show the computation results of the simplified form $\overline{D}_{S^{\max}}$ of the extended entropic chaos degree $D_{S^{\max}}$ in Fig. 5.

One finds that the simplified form $\overline{D}_{S^{\max}}$ takes almost the same value as the extended entropic chaos degree $D_{S^{\max}}$. Therefore, the simplified form $\overline{D}_{S^{\max}}$ can reduce the computation time, while maintaining nearly the same computation accuracy as the extended entropic chaos degree $D_{S^{\max}}$ .

Though only dissipative systems are directly considered in this paper, the extended CD can be computed for any discrete dynamics, including conservative systems, if the dynamics satisfies the conditions of Assumption 1. For dynamics that do not satisfy Assumption 1, an appropriate setting for computing the extended CD is necessary and will be discussed in future works.

## 6 Conclusion

In this paper, we introduced an extended chaos degree $D_S$ for a $d$-dimensional map $f$, where $f$ maps from $I$ to $I$. Firstly, we showed that the extended chaos degree $D_S$ for a stable periodic orbit takes a quite small value. Secondly, we showed that under a typical condition, the extended entropic chaos degree $D_\infty$ for a chaotic map $f$ becomes the sum of all Lyapunov exponents of map $f$ where $M$, $L$, and $S_{i,j}$ for $i,j, = 1, 2, \ldots, L^d - 1$ are infinite. Here, $M$ is the number of mapped points, $L$ is the number of partitions on each orthogonal axis, and $1/S_{i,j}$ is the scale of $A_j \cap f(A_i)$ for the components $A_i$, $A_j$ of the finite partitions $\{A_i\}$ of $I$.

However, we must treat $M$, $L$, $S_{i,j}$ as finite numbers in the computation of the extended entropic chaos degree $D_S$. Thus we introduced the extended entropic chaos degree $D_{S^{\max}}$ rather than $D_\infty$. We confirmed that the extended entropic chaos degree $D_{S^{\max}}$ for the Hénon map $f_b$, which is a typical two-dimensional chaotic map, takes almost the same value as the sum of all Lyapunov exponents for $f_b$. On the other hand, the extended entropic chaos degree $D_{S\max}$ for the Hénon map $f_b$ takes a quite small value at some points on $b$. For any stable periodic map, the extended entropic chaos degree $D_S^{\max}$ does not work well because it takes a finite negative value as a numerical finiteness of an infinitely negative amount.

Further, we introduced the simplified form $\overline{D}_{S\max}$ of $D_{S\max}$. The simplified form $\overline{D}_{S\max}$ can reduce the computation time while maintaining the computation accuracy of the approximate form $D_{S\max}$.



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