School of Electrical Engineering, Computing and Mathematical Sciences

Mathematical Models and Numerical Methods for Pricing Options on Investment Projects under Uncertainties

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This thesis is presented for the Degree of
Doctor of Philosophy
of
Curtin University

August 2020

Declaration

To the best of my knowledge and belief, this thesis contains no material previously published by any other person except where due acknowledgment has been made.

This thesis contains no material which has been accepted for the award of any other degree or diploma in any university.

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Abstract

Considering the significance of making appropriate decisions in business and investment, accurately valuing investment practice is crucial. In this work, we develop models and numerical methods for solving option pricing problems under real options theory. This research has been focused on establishing partial differential equation (PDE) models for pricing flexibility options on investment projects under uncertainties and numerical methods for solving these models. We assume uncertainty of a variable such as the underlying asset price of a project has a characteristic of a random fluctuation with constant or stochastic volatility.

Moreover, we develop a finite difference method and an advanced fitted finite volume scheme and combine with an interior penalty method, as well as their convergence analyses, to solve the PDE and LCP models developed. The MATLAB program is for implementing testing the models of numerical algorithms developed.

Our numerical experimental results show that the mathematical models and numerical methods we developed are able to predict financially sensible prices of options on project expansion, contraction, abandonment and deferral.

List of publications during PhD candidature

- N. Li and S. Wang. "Pricing Options on Investment Project Expansions Under Commodity Price Uncertainty.", *J. Ind. Manag. Optim.*, 15: 261-273, 2019.
- N. Li, S. Wang, and S.H. Zhang. "Pricing options on investment project contraction and ownership transfer using a finite volume scheme and an interior penalty method.", *J. Ind. Manag. Optim.*, 16: 1349-1368, 2020.
- N. Li, S. Wang, and K. Zhang. "Pricing Options on Investment Project Expansion under Asset Price Uncertainty with Stochastic Volatility.", submitted for publication.

Acknowledgements

The research reported in this thesis has been carried out from September 2015 to June 2020. During the period I have been enrolled in a PhD degree program in the Department of Mathematics & Statistics, Curtin University. I wish to express my sincere gratitude for the generous APA/CUPS scholarship from Curtin University. This financial assistance allows me to concentrate on my research.

I would like to express my very great appreciation to my supervisor, Prof. Song Wang, for his patience, understanding, and concern with the progress in my research. During my PhD study, Song spends a lot of time to guide me on many core issues. He provides me with precious and constructive suggestions for the planning and development of this research work. In particular, Prof. Wang offers a wide range of knowledge on numerical analysis, optimal control, convergence analysis, and software computation to guide me through the research. Besides, he comprehensively revises my papers to catch and correct errors. This thesis would not have been possibly completed without his help. I am deeply indebted to him.

I would like to thank my Chairperson of PhD, Prof. Yonghong Wu, for his encouragement and supervision throughout the past four years with remarkable patience and enthusiasm. My thanks are also extended to my co-supervisor, Prof. Benchawan Wiwatanpataphee, for her continuing encouragement during my PhD study. I would also like to thank Dr. Wen Li for her advice and assistance in my programming progress.

Special thanks would be given to the staff in the Department of Mathematics and Statistics of Curtin University for contributing to a friendly working environment.

Finally, I wish to thank everyone in my beloved family for their understanding, support, and encouragement throughout my study. I would also like to give thanks to all my friends for their support and friendship, particularly Ms Ling Jiang, Mr Chong Lai.

Contents

CHAPTER 1

Introduction

1.1 Background

The existence of unpredictable uncertainties increases the risk of investment in many industries, such as the natural resource sector. Flexible investment policies are effectively able to reduce these risks and therefore increase the values of business projects, such as adjusting investment size or duration, risks can be effectively reduced. These flexibilities are usually given as options that have time-values. Modern investors can make a proper decision by valuing investment practice accurately. This thesis investigates a range of real option pricing problems on investment projects, including to price European call option problems on options to expand, to price American option problems on option to defer and contract or abandon while their underlying asset prices are stochastic, and to price European-style options on investment project expansions under stochastic asset price and volatility.

1.2 Objectives of the Thesis

A range of uncertain factors derive investment risks such as the capital of investment, costs of financing and operations, the commodity price and its volatility, interest rate, etc. Flexible investment policies are effectively able to reduce these risks and therefore increase the values of business projects. To make appropriate decisions, we need to accurately assess these flexible options embedded in our investments.

The purpose of this work is to develop a range of mathematical pricing models on investment projects and numerical methods for solving these models. We will also validate the usefulness and reliability of the models and methods using the investment problem in the natural resource sector. The specific objectives of this work are as follows:

(1) Develop PDE-based mathematical models for pricing options on investment project expansion while the underlying commodity price follows a Geometric Brownian Motion (GBM). Also, present a finite difference scheme to numerically solve the models of normal and compound expansion options.

(2) Derive mathematical models for pricing options to contract, abandon (transfer part/all of the ownership of a project), and defer an investment project when the underlying asset price satisfies a GBM. Besides, a finite volume method is introduced for the discretization of the PDE models. A penalty approach is applied to the discretized Linear Complementarity Problem (LCP) when an option is of the American type.

(3) Develop PDEs-based mathematical models for pricing options to expand on investment projects under commodity price uncertainty with stochastic volatility. Moreover, adopt the nine-point stencil finite difference method for solving these 2-dimensional models.

1.3 Outline of the Thesis

In this thesis, we derive various types of pricing option models in one or two dimensions for valuing various real options when their underlying asset prices and the associated volatility are stochastic. We also propose numerical methods for solving these mathematical models. The thesis is structured as follows:

Chapter 1: This chapter outlines the objectives of research and clarifies the structure of the thesis.

Chapter 2: A literature review of the existing work is introduced. This includes the review of fundamental equations of pricing models, and numerical techniques presented.

Chapter 3: Apply real options theory to pricing investment projects embedded options to expand. The pricing models developed are of a similar form as the Black-Scholes model for pricing conventional European call options. In particular, the payoff conditions of the model are determined by a PDE system. We introduce an upwind finite difference scheme for numerically solving the models of normal and compound real options.

Chapter 4: We develop PDEs and the partial differential linear complementarity problems(LCPs) for pricing options of contraction, transfer and abandonment on investment projects when their underlying asset prices are stochastic. A finite volume method is used for the discretization of the PDE models, and an interior penalty approach is proposed for solving the discretized differential LCPs governing the values of the options of American type. Convergence analyses for both the finite volume and penalty methods have been provided.

Chapter 5: We derive models in the form of 2-dimensional and time-dependent PDEs to estimate the value of an investment project expansion option under uncertain underlying asset price stochastic volatility. We propose a 9-point finite differences method to numerically solve two dimensional PDE systems governing option pricing problems.

Chapter 6: Concludes the main results of this thesis. Lastly, a discussion of possible developments for further research is pointed out.

CHAPTER 2

Literature Review

2.1 General

In this chapter, a literature review of real options theory is introduced. We also introduce the fundamental mathematical models and numerical methods for pricing options on investment projects.

2.2 Real Options Theory

Real options are able to increase the value of a project by avoiding risks through flexible policies based on changing economic, technological, or market conditions. For example, managers have a right to optionally defer investment until a later date. Due to the time value of money, it is better for managers to pay the investment costs later rather than sooner. In addition, the value of the project can change before the option expires, for example if the price goes up, investments are better off [1]. But managers can limit losses because they can flexibly abandon partial or all of the unprofitable project at the predefined cost if things get worse.

2.2.1 The Definition of Real options

A real option is a contract under which managers of an enterprise have the flexibility to alter their investment with a fixed cost to either reduce the risks or catch expanding opportunities. It is the right to invest or transfer of some real world asset, whereas it is not an obligation to exercise that right in the future. A business project with flexible future choices will increase the value of business activity because the flexibility of the embedded option helps mangers to acquire the benefits of investments, but limits their losses.

Trigeorgis(1993) [2] presents an extensive classification of real options, which include Option to Defer, Staged Investment Option, Option to Alter Operating

Item	Call Option on Stock	Real option on project
Underlying asset	Stock	Investment project or Physical assets
Market value	Current value of stock	Gross present value of expected cash flows on investment projects
Strike price	Exercise price	Investment cost
Duration of the right	Time of expiration	Time until opportunity expires
Volatility	Stock value volatility	The volatility of project value
Interest rate	Risk-free rate of interest	Risk-free rate of interest

Table 2.1: The analogy between financial options and real options

Scale, Option to Abandon, Option to Switch, Growth Option, and Interacting Option. At each single decision point, the manager has an opportunity to abandon the option based on market circumstances or investors interests because a real option is the right, not the obligation.

2.2.2 Financial Options vs Real Options

Real options theory came from the pricing theory of financial options. In 1977, Myers [3] introduced real options as a new area of financial research. Similar to a financial option, a real option is the right but not the obligation, to undertake specific business decisions, for instance abandoning, staging, deferring, expanding or contracting investment. In the concept of real options, "real" refers typically to projects involving tangible or intangible assets against a financial instrument in financial options. Tangible assets are physical assets such as land, machine equipment, and plant, etc. Brand recognition and intellectual property are examples of intangible assets. It is one of the essential distinctions between the real option and the financial option that the option holder can directly influence the value of the option's underlying projects. For example, management can decide on choosing the time or scale of investment.

Trigeorgis(1996) [4] further investigated the analogy between financial options and real options. Savolainen (2016) [5] presented a simple correspondence between the parameters of call options and real options set as Table 2.1:

2.2.3 Application of Real Options on Natural Resources Investment

Natural resources are an important component of world wealth. The underlying commodity prices in the field of natural resources investments have often the characteristics of a high degree of random fluctuations. Besides these, there are other relevant factors such as exchange rate, interest rate which expose enterprises to huge risks. To avoid and hedge the risks caused by the uncertainties involved in the investment process, real options theory is widely applied by offering more flexible and alterable investment choices.

Brennan and Sehwaaz (1978) [6] studied the problem of how to estimate the value of a copper mining project with a high-risk cash-flow. Trigeorgis [2] analyzed the assessments of a multinational natural resource project. Costa Lima and Suslick (2006) [7], Dimitrakopoulos and Abdel Sabour (2007) [8], and Shockly (2007) [9] noted that real options theory is an effective method for evaluation of mining projects.

2.3 Valuation of Real Options

To make appropriate decisions in investment projects, investors need to consider options which may impact on the value of the project. The traditional Discounted Cash Flow (DCF) model has been widely adopted in the management of enterprise practices. Myers (1984) [3] put forward the limitations of DCF, and Hodder and Riggs (1985) [10] pointed out that the DCF method has been misused in practical applications. Traditional valuation tools do not capture the value of the options embedded in the project. Moreover, the real options approach extends the financial options theory. Therefore the options pricing models used for real options valuation(ROV) are derived from financial options pricing models.

Black and Scholes (1973) [11] introduced their famous Black-Scholes formula which computes the fair value of a European call option. In the process of pricing, the underlying asset can take only two possible values by creating a portfolio Π of stocks and bonds that achieves the same payoff in those two states of nature. The option price W is then the price of the replicating portfolio in a risk-neutral world. Geometric Brownian motion is stochastic processes which are often used to model the behaviour of stock prices. The process can be described by a stochastic differential equation (2.1) in 1996 [12].

$$dS = \mu S dt + \sigma S dz, \tag{2.1}$$

where dS and dt are the small changes in the price S of the underlying asset and the time t respectively, μ denotes a constant drift rate and σ is the constant volatility, and dz is a Wiener process, which has a drift of zero and a variance rate of 1. Using Ito's lemma (1951) [13], the value of an option on S satisfies

$$dW = \frac{\partial W}{\partial S} dS + \frac{\partial W}{\partial t} + \frac{1}{2} S^2 \sigma^2 \frac{\partial^2 W}{\partial S^2} dt + \frac{\partial W}{\partial S} dz \tag{2.2}$$

Let us consider a portfolio Π consisting of shorting one option W and longing $\frac{\partial W}{\partial S}$ of the underlying asset S. The portfolio risk has then been eliminated so that it earns a risk-free rate of interest r. Then, using the above Delta-hedging strategy, we can show that W satisfies the following equation:

$$\frac{\partial W}{\partial t} + \frac{1}{2} S^2 \sigma^2 \frac{\partial^2 W}{\partial S^2} + rS \frac{\partial W}{\partial S} - rW = 0. \tag{2.3}$$

Eq. (2.3) is the PDE developed by Black and Scholes [11] to value options. In the case of European Option, the Black-Scholes equation has the initial condition

$$W(S,T) = \begin{cases} \max(S - X, 0), & (\textit{call option}), \\ \max(0, X - S), & (\textit{put option}). \end{cases} \tag{2.4}$$

and boundary conditions

$$W(0,t) = \begin{cases} 0, & (\textit{call option}), \\ Xe^{-r(T-t)}, & (\textit{put option}), \end{cases} \tag{2.5}$$

$$W(S,t) = \begin{cases} S - Xe^{-r(T-t)}, & \textit{when} \quad S \to \infty, \quad (\textit{call option}), \\ 0, & \textit{when} \quad S \to \infty, \quad (\textit{put option}). \end{cases} \tag{2.6}$$

The exact solution for a European call is

$$W = S_0 N(d_1) - Xe^{-rT} N(d_2) \tag{2.7}$$

where

$$d_1 = \frac{\ln(S_0/X) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \tag{2.8}$$

$$d_2 = \frac{\ln(S_0/X) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T} \tag{2.9}$$

X = strike price
c = call option price
S_0 = the asset price at time zero
T = is the time to maturity of the option
σ = the standard deviation of the asset price in decimals
$N(\Delta)$ = cumulative standard normal probability distribution
r = the continuously compounded risk-free interest rate in decimals

As we mentioned earlier, the real options analysis is an extension of financial option theory which is also widely used in the field of real options valuation. For instance, in equations (2.7) –(2.9), S and X can be regarded as the underlying commodity price and the investment cost on real option pricing models, although they are often expressed as a stock price and an exercise price in a financial option. Another well-known example is the aforementioned δ-hedging principle in which the portfolio Π consists of two financial assets, i.e.a stock and a bond in (2.3). By contrast, the portfolio Π includes shorting one unit of real option W and longing $\frac{\partial W}{\partial S}$ unit of the underlying commodity price S in real options valuation.

2.3.1 The Existing Studies on Pricing Options under Asset Price Uncertainty

There are many existing mathematical models to value financial options and real options under various parameter uncertainties, such as Brennan and Schwartz [6], Trigeorgis [2], Moyen et al.(1996) [14], Kelly (1998) [15], Moel and Tufano (2002) [16], Monkhouse and Yeates (2005) [17], Abdel Sabour and Poulin (2006) [18], Samis et al. (2006) [19] and Shafiee et al. (2009) [20]. Brennan and Schwartz's developed a general model for valuing the cash flows from natural resource investments. In their research, the homogeneous commodity price S, is assumed to follow the continuous stochastic process:

$$\frac{dS}{S} = \mu dt + \sigma dz \tag{2.10}$$

Eq. (2.10) is similar to (2.1), in which
dz is the increment to a standard Gauss-Wiener process;
σ is the instantaneous standard deviation of the spot price;
μ is the local trend in the price.

The value of a mine H whose underlying asset is the aforementioned commodity will depend on S, the physical inventory in the mine Q, calendar time t, and the mine operating policy ϕ. Then we have

$$H \equiv H(S, Q, t; j, \phi), \quad j = 0, 1, \tag{2.11}$$

where $j = 0$ when the mine is closed, and $j = 1$ when it is operational. Applying Ito's lemma to Eq.(2.11), the instantaneous change of H can be written as

$$dH = \frac{\partial H}{\partial S} dS + \frac{\partial H}{\partial Q} dQ + \frac{\partial H}{\partial t} dt + \frac{1}{2} \frac{\partial^2 H}{\partial S^2} (dS)^2 \tag{2.12}$$

where

$$dQ = -q dt \tag{2.13}$$

Brennan and Schwartz constructed a financing portfolio including short-term assets in (H_s/F_s) futures contracts, and long-term assets of mineral resources which must be equal to the hedged return on the value of the investment. Then, they obtained the following PDE of copper values.

$$\frac{1}{2} S^2 \sigma^2 \frac{\partial^2 H}{\partial S^2} + (\rho S - C) \frac{\partial H}{\partial S} - q \frac{\partial H}{\partial Q} + \frac{\partial H}{\partial t} + q(S - A) - M(1 - j) - X - (\rho + \lambda_j) H = 0 \tag{2.14}$$

for $j = 0, 1$, where

A is the average cash cost rate of producing at the rate q at time t when the mine inventory is Q;

M is the after-tax fixed-cost rate of maintaining the mine at time t when mine is closed;

ρ is the riskless interest rate, therefore ρH is the riskless return;

C is the convenience yield of the marginal unit of inventory.

λ_j is proportional rate of tax on the value of the mine when it is opened and closed;

X is the total income tax and royalties of the mine when it is operating.

2.3.2 The Studies under Asset Price Uncertainty with Stochastic Volatility

Earlier option pricing studies often assumed that the volatility rate was constant, which is obviously not consistent with the actual situation. Stochastic volatility is another important source of uncertainty for an investment project. There are a number of studies about option pricing under uncertain volatility, such as

Hull and White (1978) [21] , Ncube (1996) [22], Zhu and Avellaneda (1998) [23], Dokuchaev and Savkin, Broadie, Detemple and Ghysels (1998) [24], Anderson (2002) [25], Alexander (2004) [26], Alexander and Korovilas (2013) [27], Jones (2003) [28], Cho, kim, kwon (2005) [29].

According to [21], to consider a derivative asset f with a price that depends upon some security price, S, and its instantaneous variance, $V = \sigma^2$, which are assumed to obey the following stochastic processes:

$$dS = \phi S dt + \sigma S dw, \tag{2.15}$$

$$dV = \mu V dt + \xi V dz, \tag{2.16}$$

the option value f satisfies the following partial differential equation:

$$\frac{\partial f}{\partial t} + \frac{1}{2}(S^2\sigma^2\frac{\partial^2 f}{\partial S^2} + 2\rho\sigma^3\xi S\frac{\partial^2 f}{\partial S\partial f} + \xi^2 V^2\frac{\partial^2 f}{\partial V^2}) - rf$$
$$= -rS\frac{\partial f}{\partial S} - \mu\sigma^2\frac{\partial f}{\partial V}, \tag{2.17}$$

where ϕ is a parameter that may depend on S, σ, and t. ρ is the correlation between dz and dw.

Another important basic pricing model for options on an asset with stochastic volatility is given by Steven Heston (1993) [30]. Heston assumed that the spot asset at time t follows the following processes

$$dS(t) = \mu S dt + \sqrt{y(t)} S dz_1(t) \tag{2.18}$$

$$dy(t) = \kappa[\theta - y(t)]dt + \sigma\sqrt{y(t)}dz_2(t), \tag{2.19}$$

the value W of an option whose underlying price and volatility satisfy (2.18)-(2.19) is governed by the following equation:

$$\frac{1}{2}yS^2\frac{\partial^2 W}{\partial S^2} + \rho\sigma v S\frac{\partial^2 W}{\partial S\partial y} + \frac{1}{2}\sigma^2 y\frac{\partial^2 W}{\partial y^2} + rS\frac{\partial W}{\partial S}$$
$$+ \{\kappa[\theta - y(t)] - \vartheta(S, y, t)\}\frac{\partial W}{\partial y} - rW + \frac{\partial W}{\partial t} = 0 \tag{2.20}$$

where $z_1(t)$ is a Wiener process, and $z_2(t)$ correlated to $z_1(t)$ with a correlation ρ. The term $\vartheta(S, y, t)$ denotes the price of volatility risk which is independent of the asset.

2.4 Numerical Schemes for Option-pricing Model

In general, it is challenging or impossible to solve a pricing problem analytically, and thus numerical methods are generally used in options valuation.

As one of the most straightforward and accessible method, the binomial tree has been used for option pricing widely and was first presented particularly by Cox, Ross, Rubinstein (CRR)(1979) based on risk-neutral valuation [31]. Comparing with the Monte Carlo simulation, the most significant advantage of CRR method is that it can easily handle pricing of American options [32]. Lattice methods (binomial or trinomial trees) have better computational speed than that of Monte Carlo simulation for solving simple models [33]. In contrast, the computation cost of the tree methods will increase exponentially for high dimensional variables, and the convergence rate of Monte Carlo methods is generally irrelevant with the number of state variables.

The lattice methods and Monte-Carlo simulations are accurate and straightforward to a certain degree and are suited to deal with some types of option pricing models. However, there are obvious limitations for both methods in handling some situations such as the more complicated option pricing formulas (options with high dimensional factors, portfolio of options, or real option with an optimal early exercise strategy etc.). In addition, the demand for speed in the trading market requires fast and accelerating convergence ways to process these calculations. To avoid the weaknesses contained within both the methods of binomial trees and Monte Carlo simulation, a possible solution could be the use of partial differential equations (PDEs) as an alternative numerical method for the option pricing model. The PDEs method was considered for pricing derivative by Brennan and Schwarz [6]. The stability and accuracy of the method were considered [34], and its convergence is faster than the other two methods along with the development of modern mathematics.

2.4.1 Finite Differences for PDEs

Finite differences method (FDM) is an applied numerical technique for solving option pricing models. It is one of the most practical methods to handle boundary value problems. In order to solve PDEs, each derivative in PDE has to be replaced by a suitable divided difference of function values at the chosen grid points. If $u = f(x)$ is a sufficiently smooth function, there are serveral typical forms of divided differences approximating $f'(x)$, $f''(x)$:

$$f'(x) \approx \frac{f(x+\triangle x)-f(x)}{\triangle x} = (\frac{\triangle f}{\triangle x})_f \qquad \text{(forward difference)}$$
$$f'(x) \approx \frac{f(x)-f(x-\triangle x)}{\triangle x} = (\frac{\triangle f}{\triangle x})_b \qquad \text{(backward difference)}$$
$$f'(x) \approx \frac{f(x+\triangle x)-f(x-\triangle x)}{2\triangle x} = (\frac{\triangle f}{\triangle x})_c \qquad \text{(centered difference)}$$
$$f''(x) \approx \frac{f(x+\triangle x)-2f(x)+f(x-\triangle x)}{2\triangle x^2} = (\frac{\triangle^2 f}{\triangle x^2})_c \qquad \text{(centered difference)}$$

By Taylor expansion, the following error estimate holds

$$|f'(x) - (\frac{\triangle f}{\triangle x})_f| = \mathcal{O}(\Delta x), \tag{2.21}$$
$$|f'(x) - (\frac{\triangle f}{\triangle x})_b| = \mathcal{O}(\Delta x), \tag{2.22}$$
$$|f'(x) - (\frac{\triangle f}{\triangle x})_c| = \mathcal{O}(\Delta x^2), \tag{2.23}$$
$$|f''(x) - (\frac{\triangle^2 f}{\triangle x^2})_c| = \mathcal{O}(\Delta x^2). \tag{2.24}$$

To implement the FDM on the Black-Scholes PDE, we introduce a price-time mesh for $(0, S_{max}) \times (0, T)$ and divide it into $M \times N$ rectangles with increments ΔS (the stock price increment) and Δt (the time increment), where M and N are any positive integers. Set $\Delta S = S_{i+1} - S_i$ and $\Delta t = t_{j+1} - t_j$

$$0 = S_0 < S_1 < ... < S_M = S_{max} \tag{2.25}$$
$$0 = t_0 < t_1 < ... < t_N = T \tag{2.26}$$

and $W_{i,j} = W(i\Delta S, j\Delta t)$, for $i = 0, 1, ..., M-1, \quad j = 0, 1, ..., N-1$.

We apply the above finite difference scheme, yielding the following approximations of the derivatives

$$\frac{\partial W}{\partial t} \approx \frac{W(S,t) - W(S, t-\Delta t)}{\Delta t} \tag{2.27}$$
$$\frac{\partial W}{\partial S} \approx \frac{W(S+\Delta S,t) - W(S-\Delta S, t)}{2\Delta S} \tag{2.28}$$
$$\frac{\partial^2 W}{\partial S^2} \approx \frac{W(S+\Delta S,t) - 2W(S,t) + W(S-\Delta S, t)}{\Delta S^2} \tag{2.29}$$

Using (2.27),(2.28),(2.29) in the Black-Scholes PDE (2.3), we obtain:

$$\frac{W_{i,j} - W_{i,j-1}}{\Delta t} + \frac{1}{2}\sigma^2 (i\Delta S)^2 \frac{W_{i+1,j} - 2W_{i,j} + W_{i-1,j}}{\Delta S^2} + ri\Delta S \frac{W_{i+1,j} - W_{i-1,j}}{2\Delta S} = rW_{i,j} \tag{2.30}$$

Then, the above equation can be re-written and simplifed to:

$$W_{i,j-1} = [\frac{1}{2}\Delta t(\sigma^2 i^2 - ri)]W_{i-1,j} + [1 - \Delta t(\sigma^2 i^2 + r)]W_{i,j} + [\frac{1}{2}\Delta t(\sigma^2 i^2 + ri)]W_{i+1,j}$$

2.4.2 Fitted Finite Volume for PDEs

Fitted finite volume method is a novel numerical method based on a fitted finite volume spatial discretization and an implicit time stepping technique. Wang (2020) [35, 36] approximated the flux of a given function locally by a constant, yielding a locally nonlinear approximation to the function. Let W denote the value of a European call/put option, and in [37] W satisfies the following Black-Scholes PDE:

$$\frac{\partial W}{\partial t} + \frac{\partial}{\partial S}[a(t)S^2\frac{\partial W}{\partial S} + b(S,t)SW] + c(S,t)W = f(S,t) \tag{2.31}$$

$$a = \frac{1}{2}\sigma^2 \tag{2.32}$$

$$b = r - d - \sigma^2 \tag{2.33}$$

$$c = 2r - \sigma^2 - \frac{\partial D}{\partial W} \tag{2.34}$$

where D is the dividend rate, $D(S,t) = d(S,t)S$, $d(S,t)$ is assumed being continuously differentiable. According to [38], integrating both sides of (2.31) where $J = (0, S_{max})$, $J_j := (S_j, S_{j+1})$, $j = 0, 1, ..., N-1$, and $S_{j\pm1/2} = (S_{j\pm1} + S_j)/2$, we have

$$-\int_{S_{j-1/2}}^{S_{j+1/2}} \frac{\partial W}{\partial t}dS - \left[S\left(aS\frac{\partial W}{\partial S} + bW\right)\right]_{S_{j-1/2}}^{S_{j+1/2}} + \int_{S_{j-1/2}}^{S_{j+1/2}} cW dS = \int_{S_{j-1/2}}^{S_{j+1/2}} f dS.$$

Applying the mid-point quadrature rule, and for $j = 1, 2..., N-1$, letting $l_j = S_{j+1/2} - S_{j-1/2}$, $c_j = c(S_j, t)$, the above equation can be approximated as

$$-\frac{\partial W_i}{\partial t}l_j - \left[S_{j+\frac{1}{2}}\rho(W)|_{S_{j+\frac{1}{2}}} - S_{j-\frac{1}{2}}\rho(W)|_{S_{j-\frac{1}{2}}}\right] + c_j W_j l_j = f_j l_j, \tag{2.35}$$

$$\rho(W) := aS\frac{\partial W}{\partial S} + bW.$$

Let $t_k (k = 0, 1, ..., K)$ be a set of partition points in [0,T] satisfying $T = t_0 > t_1 > ... > t_K = 0$. Under fitted finite volume method, we discretize equation

(2.35) as

$$(\theta \boldsymbol{E}^{k+1} + G^k)\boldsymbol{W}^{k+1} = \boldsymbol{f}^k + [G^k - (1-\theta)\boldsymbol{E}^k]\boldsymbol{W}^k$$

for $j = 1, 2, ..., N-1$ and $k = 0, 1, ..., K-1$,where $f_j = f(S_j, t_k)$, $G^k = \text{diag}(l_1/(-\Delta t_k), ..., l_{N-1}/(-\Delta t_k))$,

$$\hat{f}^k = \theta(f_1^{k+1} l_1, ..., f_{N-1}^{k+1} l_{N-1})^\top + (1-\theta)(f_1^k l_1, ..., f_{N-1}^k l_{N-1})^\top$$

and let $E_j, j = 1, 2, ..., N-1$, be $1 \times (N-1)$ matrices defined by

$$\begin{aligned} E_1 &= (e_{11}, e_{12}, 0, ..., 0), \\ E_j &= (0, ..., 0, e_{j,j-1}, e_{j,j}, e_{j,j+1}, 0, ..., 0), \quad j = 2, 3, ..., N-2, \\ E_{N-1} &= (0, ..., 0, e_{N-1,N-2}, e_{N-1,N-1}), \end{aligned}$$

where $\alpha_j = b_{j+1/2}/a$, and

$$e_{1,0} = -\frac{b_{1/2}S_1}{2}\frac{1-\text{sign}(b_{1/2})}{2}, \tag{2.36}$$

$$e_{1,1} = \frac{b_{1+1/2}S_1}{2}\frac{1+\text{sign}(b_{1+1/2})}{2} + \frac{b_{1+1/2}S_{1+1/2}S_1^{\alpha_1}}{S_2^{\alpha_1} - S_1^{\alpha_1}} + cl_1, \tag{2.37}$$

$$e_{1,2} = -\frac{b_{1+1/2}S_{1+1/2}S_2^{\alpha_1}}{S_2^{\alpha_1} - S_1^{\alpha_1}}, \tag{2.38}$$

and

$$e_{j,j-1} = -\frac{b_{j-1/2}S_{j-1/2}S_{j-1}^{\alpha_{j-1}}}{S_j^{\alpha_{j-1}} - S_{j-1}^{\alpha_{j-1}}}, \tag{2.39}$$

$$e_{j,j} = \frac{b_{j-1/2}S_{j-1/2}S_j^{\alpha_{j-1}}}{S_j^{\alpha_{j-1}} - S_{j-1}^{\alpha_{j-1}}} + \frac{b_{j+1/2}S_{j+1/2}S_j^{\alpha_j}}{S_{j+1}^{\alpha_j} - S_j^{\alpha_j}} + cl_j, \tag{2.40}$$

$$e_{j,j+1} = -\frac{b_{j+1/2}S_{j+1/2}S_{j+1}^{\alpha_j}}{S_{j+1}^{\alpha_j} - S_j^{\alpha_j}}, \tag{2.41}$$

The above time stepping scheme is unconditionally stable when $\theta \in [0.5, 1]$.

2.4.3 Interior Penalty Method for Linear Complementarity Problem (LCP)

An interior penalty method(IPM) for a finite-dimensional LCP is often arising from the discretization of stochastic optimal problems (2016) [39], in which constraints are penalized in a way that guarantees the strict interiority of the approaching solutions. The American option valuation problem is a typical example of stochastic optimal control problems so that IPMs could be used in this area. A standard form of such a problem is as follows:

Problem 2.1. *Find $x \in \mathbb{R}^n$ such that*

$$Lx \leq a, \quad Qx \leq b, \quad (Lx - a)^\top (Qx - b) = 0, \tag{2.42}$$

where L and Q are $n \times n$ matrices, and a and b are known vectors in $\mathbb{R}^n$ with $n \gg 1$ a positive constant.

We assume that L and Q are both irreducibly diagonally dominant and M-matrices, and $b = 0$ in (2.42). Then (2.42) is the KKT conditions of constrained optimization problem:

$$\min_{x \in \mathbb{R}^n} \left(\frac{1}{2} x^\top L x - a^\top x \right), \quad \text{subject to} \quad Qx \leq 0, \tag{2.43}$$

if L is also symmetric and positive-definite. Applying IPM, (2.43) is approximated by the following constrained problem:

$$\min_{x \in \mathbb{R}^n} \left(\frac{1}{2} x^\top L x - a^\top x - \mu \sum_{i=1}^{n} \ln(-(Qx)_i) \right), \tag{2.44}$$

where $\mu > 0$ and $x = (x_1, x_2, ..., x_n)^\top$. The first-order optimality condition for the solutions to (2.44) is

$$Lx_\mu - Q^\top (\mu./(Qx_\mu)) = a, \tag{2.45}$$

where $x_\mu = (x_{\mu,1}, x_{\mu,2}, ..., x_{\mu,n})^\top \in \mathbb{R}^n$ and ./ denotes the dot product. Thus, any negative solution to equation (2.45) is a minimum point of (2.44). Based on this idea, when L is un-symmetric, we define the approximation of Problem 2.1 by the following nonlinear equation:

$$Lx_\mu - \mu./(Qx_\mu) = a, \tag{2.46}$$

where $\mu > 0$ is a parameter. In [39], the following theorem 2.1 is proved.

Theorem 2.1. *For any $\mu > 0$, there exists a unique solution x_μ to (2.46) satisfying $Lx_\mu < 0$ and $Qx_\mu < 0$.*

CHAPTER 3

Pricing Options on Investment Project Expansions under Commodity Price Uncertainty

3.1 General

In this chapter we develop PDE-based mathematical models for valuing real options on investment project expansions when the underlying commodity price follows a geometric Brownian motion(GBM). The models developed are of a similar form as the Black-Scholes model for pricing conventional European call options, but, unlike the Black-Scholes' model, the payoff conditions are determined by a PDE system. An upwind finite difference scheme is developed for solving the models. Numerical experiments have been performed using two examples in pricing expansion options in the mining industry. The computation is done by MATLAB programs and demonstrates that our models are able to produce financially meaningful numerical results for the two non-trivial test problems.

Investments are always accompanied by risks as well as opportunities due to high uncertainties in our fast-changing economic environments and conditions. It is important for investors to cope with the uncertainties by flexible plans in the form of options. In [2] the author analyzed the value of a multinational natural resource project with real options. Costa Lima and Suslick [7] pointed out that real option theory is a useful method for evaluation of mining projects in which a real option has a value as a function of time and the underlying asset price. However, how to price such an option accurately has been a challenge in financial engineering. Valuation of real options has been studies by various authors such as Moyen [14], Moel and Tufano [16], Abdel Sabour and Poulin [18]. Brennan and Schwartz [6, 40] also studied the problem of how to estimate the value of a copper mining project with a high degree of uncertainty in output prices.

Although the traditional Discounted Cash Flow (DCF) model has been widely adopted in the management of enterprise practices, it has been pointed out in [3, 10, 41] that DCF has many limitations in practice. More specifically, DCF is unable to capture the value of flexibilities or options that is embedded in a project. On the other hand, such an option can be used as an instrument to represent value components resulted from the interaction of flexibility and uncertainty. However, unlike pricing conventional financial options, studies on how to price investment extension options using a PDE approach are very limited.

In what follows, we present a systematic study on a PDE approach to pricing 'options to expand'. More specifically, In Section 3.2, we will first derive a mathematical PDE model for valuing an expansion option of a project whose underlying commodity price follows a geometric Brownian motion. This model is of the form of a European call option based on the idea for valuing a resource project in [42]. We will then extend the model to pricing expansion-on-expansion, or call-on-call (CoC), compound options [43]. These models are of the form of Black-Scholes model for pricing conventional European options with a vanilla payoff function, but unlike the Black-Scholes model, the payoff condition of such a real option are defined by another PDE system. In Section 3.3, we also propose a finite difference scheme for numerically solving the developed models. A convergence analysis will be performed in Section 3.4 for the proposed discretization scheme which is shown to be stable, consistent and monotone. In Section 3.5, we demonstrate this approach using two non-trivial examples of iron-ore projects. The numerical results show that our models and numerical methods produce financially meaningful results. And the idea can easily be used for other types of projects.

3.2 The Pricing Models

In this section we will present mathematical or PDE models for valuing project expansion options when the volatility is also stochastic. For clarity, we will use an investment project in a natural resource (e.g., iron ore) industry to demonstrate our approach [44]. Meanwhile, the model can easily be applied for other types of project expansion options.

3.2.1 The Models for Pricing Expansion Options

Consider an investment project in a natural resource industry whose underlying commodity price P follows a geometric Brownian motion in time t. The value of the project is a function of P and t. In this work we assume that P satisfies the following stochastic differential equation proposed in [45]

$$dP = P(r - \delta)dt + P\sigma dz, \tag{3.1}$$

for $t \geq 0$, where dz is a Wiener process or geometric Brownian motion, σ is the instantaneous standard deviation, or volatility, of P, r is the risk free interest rate, and δ is the mean convenience yielded on holding one unit of the output. We assume that σ, r and δ are non-negative constants.

Now, we consider the project in the following two different cases.

C0. The project does not have any expansion options.

C1. There is an embedded production expansion option in the project that is exercisable at a fixed time $T > 0$ (expiry date) and requires the amount $K > 0$ (strike price) for the expansion.

We denote the value functions of the project in the above two cases as $V_0(P,t)$ and $V_1(P,t)$ respectively. We expect $V_1(P,t) \geq V_0(P,t)$ for all $t \geq 0$ and $P \geq 0$ if the investors can make the right decision on whether or not exercising the option. This is because Case C1 has the right, not an obligation, to expand the production rate at T and the option is exercised only when this decision adds value to the project. Whether the option is exercised or not at T depends on the commodity price P at maturity which is used to define the payoff of the option at $t = T$. When the payoff function is determined, the value of the option, denoted as $W(P,t)$, for $t < T$ is the difference between V_1 and V_0, i.e., $W(P,t) = V_1(P,t) - V_0(P,t)$ for $t < T$ and $P \geq 0$.

To determine $W(P,t)$ we let $q_k(t)$, $k = 0, 1$, be the production rates associated with Cases C0 and C1 respectively. Then the after-tax cash flow rates for C0 and C1 are respectively given by (cf., for example, [42])

$$D_k(P,t) := q_k(t)[P(t)(1 - R) - C(t)](1 - B), \quad k = 0, 1, \tag{3.2}$$

where $C(t)$ is the average cash cost rate of production per unit output, R is the rate of state royalties and B the company income tax rate.

Using a Taylor expansion and (3.1), and omitting terms of orders higher than

dt, we have the following instantaneous change dV_k in the value of the project

$$dV_k = \frac{\partial V_k}{\partial P}dP + \frac{\partial V_k}{\partial t}dt + \frac{1}{2}P^2\sigma^2\frac{\partial^2 V_k}{\partial P^2}dt, \quad k = 0, 1. \tag{3.3}$$

Following the Δ-hedging strategy in [46, 47], we construct a risk-free portfolio Π_k, consisting of a long position in the value of the project and a short position in $\frac{\partial V_k}{\partial P}$ units of the commodity at unit price P, i.e., $\Pi_k = V_k - \frac{\partial V_k}{\partial P}P$ for $k = 0, 1$. Then, when dt is sufficiently small, using the result in [42] we have that the return of the portfolio Π_k in time period dt is a combination of the changes in the values of the project and short position, given respectively by dV_k and $\frac{\partial V_k}{\partial P}dP$, the convenience yield of the short position given by $\delta\frac{\partial V_k}{\partial P}Pdt$, and the cash flow in the period given by $D_k(P,t)dt$, where D_k is defined in (3.2), i.e.,

$$d\Pi_k = dV_k - \frac{\partial V_k}{\partial P}dP - \delta\frac{\partial V_k}{\partial P}Pdt + D_k(P,t)dt, \tag{3.4}$$

for $k = 0, 1$. Replacing dV_k in (3.4) by the one defined in (3.3) and using the fact that Π_k is risk-less, i.e., $d\Pi_k = r\Pi_k dt$, we have the following PDE:

$$\mathcal{L}V_k := \frac{\partial V_k}{\partial t} + (r-\delta)P\frac{\partial V_k}{\partial P} + \frac{1}{2}P^2\sigma^2\frac{\partial^2 V_k}{\partial P^2} - rV_k = -D_k(P,t), \tag{3.5}$$

for $k = 0, 1$. Eq.(3.5) is satisfied by the value functions of the project in both Cases C0 and C1. Therefore, taking both sides of the equation for V_0 away from the corresponding sides of that for V_1, we have the following PDE for the value W of the expansion option:

$$\mathcal{L}W = D_0(P,t) - D_1(P,t), \tag{3.6}$$

for $P > 0$ and $t \in (0, T]$. In fact, for an expansion option, we usually have $D_0(P,t) - D_1(P,t) = 0$ for $t \in [0, T)$ since the change of operating rate starts from $t = T$. However, we still leave the above equation in its general form.

In computation, (3.6) needs to be solved on a finite domain $(P,t) \in (0, P_{\max}) \times [0, T)$, where $P_{\max}$ is a positive constant satisfying $P_{\max} \gg C(t)$ for any t in the life-time of the project. We now determine the boundary and payoff conditions for (3.6) at $P = 0$, $P = P_{\max}$, and $t = T$ in order for (3.6) to determine the correct value of the expansion option. When $P = 0$, it is clear that exercising the option will add negative value to the investment and thus the option is of no value, i.e.,

$$W(0,t) = 0, \quad t \in [0, T). \tag{3.7}$$

To determine the value (or payoff) of the option at $t = T$, we note that when $V_1(P,T) > V_0(P,T) + K$, the value $W(P,T)$ of the option is non-zero. Otherwise, $W(P,T) = 0$, as in this case, exercising the option does not add any value to the investment project. Thus, the terminal condition for W is defined as the following Vanilla payoff function

$$W(P,T) = [V_1(P,T) - V_0(P,T) - K]^+, \; P \in (0, P_{\max}). \tag{3.8}$$

For the boundary condition of W at $P = P_{\max}$ we define

$$W(P_{\max}, t) = V_1(P_{\max}, t) - V_0(P_{\max}, t) - e^{-r(T-t)}K, \tag{3.9}$$

for $t \in [0, T)$. From (3.7)–(3.9) we see that the payoff and boundary conditions satisfy the compatibility condition that $W(P,t)$ is continuous at the points $(0,T)$ and $(P_{\max}, T)$. Note that from (3.8) and (3.9) we see that $V_1(P,T) - V_0(P,T)$ and $V_1(P_{\max}, t) - V_0(P_{\max}, t)$ are needed to determine the payoff and boundary conditions. These functions can be determined using a different approach. We now discuss it.

To estimate the value of the expanding option for the investment project, we need to calculate the lifetime of the project. Let Q denote the reserve of the resource and T_k^* be life time of the project when the production rate is q_k. Then these quantities should satisfy

$$Q = \int_0^{T_k^*} q_k(t)dt, \quad k = 0, 1. \tag{3.10}$$

Since we expect the life-span of a production project is much longer than that of a real option , it is reasonable to assume that the expiry date of the option satisfies $T \ll \min\{T_0^*, T_1^*\}$. For simplicity, we let $T^* = \max\{T_0^*, T_1^*\}$ and use T^* as the life time of the project for both Cases C0 and C1. This requires that if the reserve is exhausted before T^*, we simply set the corresponding production rate in (3.2)to zero. For this expansion option, we have $T^* = T_0^*$,

$$q_0(t) = q_1(t), \; t \in [0,T), q_0(t) < q_1(t), \; t \in [T, T_1^*), \; q_1(t) = 0, \; t \in [T_1^*, T^*]. \tag{3.11}$$

When $P_{\max}$ is sufficiently large and fixed, we may use the 'Net Present Value' to estimate the value $V_k(P_{\max}, t)$ for $t \geq T$. Since the cash flow of the project is $D_k(P_{\max}, t)$, the rate of change at $\tau > t$ in the present value of the resource project at time t is

$$dV_k = D_k(P_{\max}, t)e^{-r(\tau-t)}d\tau, \quad \tau \geq t.$$

This expression is also true if $P_{\max}$ is replaced with 0. Therefore, the present values of the project at t when $P = 0$ and $P = P_{\max}$ are

$$V_k(P,t) = \int_t^{T^*} D_k(P,\tau)e^{-r(\tau-t)}d\tau, \tag{3.12}$$

for $P = 0, P_{\max}$, where D_k is defined in (3.2) for $k = 0, 1$. Using (3.12) we are able to define the boundary condition (3.9).

It now remains to determine $V_1(P,T) - V_0(P,T)$ in (3.8). Note that when $t = T^*$, the project has zero value for both of the two cases. Using ((3.12), we pose the following problem for the $\hat{W}$:

$$\begin{cases} \mathcal{L}\hat{W} = -(D_1(P,t) - D_0(P,t)), \ (P,t) \in (0, P_{\max}) \times [T, T^*), \\ \hat{W}(0,t) = \int_t^{T^*} [D_1(0,\tau) - D_0(0,\tau)]e^{-r(\tau-t)}d\tau, \ t \in [T, T^*), \\ \hat{W}(P_{\max},t) = \int_t^{T^*} [D_1(P_{\max},\tau) - D_0(P_{\max},\tau)]e^{-r(\tau-t)}d\tau, \ t \in [T, T^*), \\ \hat{W}(P,T^*) = 0, \ P \in (0, P_{\max}). \end{cases} \tag{3.13}$$

Solving (3.13), we can determine the payoff condition for (3.6) by replacing $V_1(P,T) - V_0(P,T)$ with $\hat{W}(P,T)$ in (3.8).

3.2.2 The Pricing Model for CoC Compound Options

We now extend the above pricing model to Call-on-Call compound expansion options [43]. In this discussion, we consider the following case.

C2 The investment project has a production expansion option with a given expansion rate and strike price $K_1 > 0$ exercisable at time $T_1 > 0$. This option has an embedded option of a further production expansion, exercisable at time $T_2 > T_1$, with another given operating rate and strike price $K_2 > 0$.

In this case, we have three different cash flow rates $D_k(P,t)$, $k = 0, 1, 2$, corresponding to three production rates $q_0(t), q_1(t)$ and $q_2(t)$ respectively. The life-spans T_k^* correspoding to q_k for $k = 0, 1, 2$ can be determined by (3.10) and we define $T^* = \max\{T_0^*, T_1^*, T_2^*\}$. Since we are only concerned with production expansions at the fixed time points T_1 and T_2, we have $T^* = T_0^* \geq T_1^* \geq T_2^*$.

More specifically, we assume that the rates satisfy

$$q_0(t) = q_1(t) = q_2(t),\ t \in [0, T_1),\ q_0(t) < q_1(t) = q_2(t),\ t \in [T_1, T_2),$$
$$q_0(t) < q_1(t) < q_2(t),\ t \in [T_2, T_2^*),\ q_1(t) = 0,\ t \in [T_1^*, T^*],\ q_2(t) = 0,\ t \in [T_2^*, T^*].$$

Let W_p denote the value of the option exercisable at T_2. Then, following the deduction of the model for pricing normal options in the previous subsection, we define the following problem to determine W_p.

$$\begin{cases} \mathcal{L}W_p = D_0(P,t) - D_2(P,t),\ (P,t) \in (0, P_{\max}) \times [T_1, T_2), \\ W_p(0,t) = 0, \\ W_p(P_{\max}, t) = V_2(P_{\max}, t) - V_0(P_{\max}, t) - e^{-r(T_2-t)}K_2, \quad t \in [T_1, T_2), \\ W_p(P, T_2) = [V_2(P, T_2) - V_0(P, T_2) - K_2]^+,\ P \in (0, P_{\max}). \end{cases} \tag{3.14}$$

The solution to the above problem defines the value of the compound option in $[T_1, T_2)$. However, the boundary and terminal conditions in (3.14) require the differences $V_2(P_{\max}, t) - V_0(P_{\max}, t)$ for $t \in [T_1, T_2)$ and $V_2(P, T_2) - V_0(P, T_2)$ for $P \in (0, P_{\max})$. The former can be calculated using (3.12) and the latter by solving (3.13) with D_1 and T replaced with D_2 and T_2 respectively.

To determine the value W_c of the compound option exercisable at T_1, we solve the following problem

$$\begin{cases} \mathcal{L}W_c = D_0(P,t) - D_1(P,t), \quad (P,t) \in (0, P_{\max}) \times [0, T_1), \\ W_c(0,t) = 0, \\ W_c(P_{\max}, t) = V_2(P_{\max}, t) - V_0(P_{\max}, t) - e^{-r(T_1-t)}K_1,\ t \in [0, T_1), \\ W_c(P, T_1) = [W_p(P, T_1) - K_1]^+, \quad P \in (0, P_{\max}), \end{cases} \tag{3.15}$$

where $W_p(P,t)$ used in the payoff and boundary conditions of equation (3.15) is the solution to equation (3.14).

3.3 Discretization

In this section we present a discretization scheme for (3.6) along with the boundary and terminal conditions (3.7)–(3.12). Various discretization scheme have been developed for solving Black-Scholes equations such as those in [35, 37, 48]. In this work we will develop a finite difference scheme with a unwinding technique for (3.6) based on that used in [38, 49–51]. Clearly, this discretization scheme is applicable to (3.13) as well as the problems governing the CoC option. We divide this discussion into two parts – discretisation of (3.6) and that of (3.7)–(3.12).

3.3.1 Discretization of the PDE model

For any positive integers M and N, we partition $(0, P_{\max}) \times (0, T)$ into $M \times N$ rectangles with nodes (P_i, t_j) for $i = 0, 1, ..., M$ and $j = 0, 1, ..., N$ satisfying

$$0 = P_0 < P_1 < ... < P_M = P_{\max},$$
$$T = t_0 > t_1 > ... > t_N = 0.$$

Set $h_i = P_{i+1} - P_i > 0$ for $i = 0, 1, ..., M-1$ and $\Delta t_j = t_{j+1} - t_j < 0$ for $j = 0, 1, ..., N-1$. On this mesh, we define the following finite difference equation approximating (3.6).

$$\begin{aligned}&\frac{W_i^{j+1} - W_i^j}{\Delta t_j} + \frac{P_i^2 \sigma^2}{h_i + h_{i+1}} \left(\frac{W_{i+1}^{j+1} - W_i^{j+1}}{h_i} - \frac{W_i^{j+1} - W_{i-1}^{j+1}}{h_{i-1}} \right) \\ &+ P_i(r - \delta) \left(\frac{\operatorname{sgn}(r-\delta) + 1}{2} \frac{W_{i+1}^{j+1} - W_i^{j+1}}{h_i} - \frac{\operatorname{sgn}(r-\delta) - 1}{2} \frac{W_i^{j+1} - W_{i-1}^{j+1}}{h_{i-1}} \right) \\ &- rW_i^{j+1} = f_i^{j+1}, \qquad (3.16)\end{aligned}$$

for $i = 1, 2, ..., M-1$ and $j = 0, 1, ..., N-1$, where $\operatorname{sgn}(\cdot)$ denotes the sign function, W_l^k denotes an approximation of $W(P_l, t_k)$ and $f_i^{j+1} = D_0(P_i, t_{j+1}) - D_1(P_i, t_{j+1})$. In (3.16), we used the backward difference to approximate the term $\frac{\partial W}{\partial t}$, the central difference to approximate $\frac{\partial^2 W}{\partial P^2}$, and the upwind difference for $\frac{\partial W}{\partial P}$. Upwind techniques have often been used for solving convection-diffusion problems (see, for example, [52]).

Re-arranging (3.16), we have

$$\alpha_i^j W_{i-1}^{j+1} + \beta_i^j W_i^{j+1} + \gamma_i^j W_{i+1}^{j+1} = -\frac{W_i^j}{\Delta t_j} - f_i^{j+1}, \qquad (3.17)$$

for $i = 1, 2, ..., M-1$ and $j = 0, 1, ..., N-1$, where

$$\alpha_i^j = -\frac{P_i^2\sigma^2}{h_{i-1}(h_i + h_{i-1})} - \frac{\mathrm{sgn}(r-\delta)-1}{2h_{i-1}} P_i(r-\delta), \tag{3.18}$$

$$\beta_i^j = -\frac{1}{\Delta t_j} + \frac{\sigma^2 P_i^2}{h_{i-1}h_i} + \left(\frac{\mathrm{sgn}(r-\delta)-1}{2h_{i-1}} + \frac{\mathrm{sgn}(r-\delta)+1}{2h_i}\right) P_i(r-\delta) + r, \tag{3.19}$$

$$\gamma_i^j = -\frac{P_i^2\sigma^2}{h_i(h_i + h_{i-1})} - \frac{\mathrm{sgn}(r-\delta)+1}{2h_i}(r-\delta)P_i. \tag{3.20}$$

Using (3.7)–(3.9) we define boundary and terminal conditions for (3.17) as follows:

$$W_0^j = 0,\ W_M^j = V_1(P_{\max}, t_j) - V_0(P_{\max}, t_j) - Ke^{-r(T-t_j)},\ W_i^0 = [\hat{W}(P_i, T) - K]^+, \tag{3.21}$$

for $i = 1, 2, ..., M-1$ and $j = 1, 2, ..., N$, where $\hat{W}$ denotes the solution to (3.13). In practice, the values of $V_1(P,t) - V_0(P,t)$ and $\hat{W}(P,T)$ at the boundary and terminal mesh points need to be approximated. We will discuss this in the subsection.

We write equation (3.17) along with (3.21) as the following matrix form:

$$A^j W^{j+1} = -\frac{1}{\Delta t_j} W^j + b^{j+1}, \tag{3.22}$$

where

$$A^j = \begin{bmatrix} \beta_1^j & \gamma_1^j & 0 & 0 & \cdots & 0 & 0 & 0 \\ \alpha_2^j & \beta_2^{j+1} & \gamma_2^j & 0 & \cdots & 0 & 0 & 0 \\ 0 & \alpha_3^j & \beta_3^j & \gamma_3^j & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \beta_{M-3}^j & \gamma_{M-3}^j & 0 \\ 0 & 0 & 0 & 0 & \cdots & \alpha_{M-2}^j & \beta_{M-2}^j & \gamma_{M-2}^j \\ 0 & 0 & 0 & 0 & \cdots & 0 & \alpha_{M-1}^j & \beta_{M-1}^j \end{bmatrix}, \tag{3.23}$$

(3.24)

$$b^{j+1} = \begin{bmatrix} -f_1^{j+1} - \alpha_1^j W_0^{j+1} \\ -f_2^{j+1} \\ \vdots \\ -f_{M-2}^{j+1} \\ -f_{M-1}^{j+1} - \gamma_{M-1}^j W_M^{j+1} \end{bmatrix}, \quad W^m = \begin{bmatrix} W_1^m \\ W_2^m \\ \vdots \\ W_{M-2}^m \\ W_{M-1}^m \end{bmatrix}, \tag{3.25}$$

for $m = j$ and $j + 1$.

The solution to the above linear system, along with the boundary and payoff conditions in (3.21) defines an approximation to $W(P, t)$ at the mesh nodes. In the following theorem, we prove that A^j in equation (3.22) is an M-matrix.

Theorem 3.1. *The matrix A^j is an M-matrix for $j = 0, 1, ..., N - 1$.*

Proof. To prove A^j is an M-matrix, it suffices to show that it is irreducible and strictly diagonally dominant with positive diagonal and non-positive off-diagonal entries.

It is obvious that A^j is irreducible, as otherwise (3.22) can be solved as two or more independent sub-problems. Note that A^j is a tri-diagonal matrix as defined in (3.23), where the non-zero entries are given in (3.18)–(3.20). Since $r \geq 0$ and $\Delta t_j < 0$, we have from equations (3.18)–(3.20)that $\alpha_i^j < 0$, $\beta_i^j > 0$ and $\gamma_i^j < 0$ for all $i = 1, 2, ..., M - 1$. Also,

$$\beta_i^j = |\alpha_i^j| + |\gamma_i^j| + r - \frac{1}{\Delta t_j} > |\alpha_i^j| + |\gamma_i^j|, \tag{3.26}$$

since $r \geq 0$ and $\Delta t_j < 0$. Therefore A^j is strictly diagonally dominant. By [53] we have that A^j is an M-matrix. □

Remark 3.1. We comment that in fact, both α_i^j and γ_i^j can be simply denoted as α_i and γ_i because from (3.18)–(3.20) we see both of them are independent of

j. However, we still keep the subscript j as the above scheme can also be used for problems in which any of σ, r and δ is time-dependent.

Remark 3.2. The upwind finite difference method has been used for solving various types of differential equations (see, for example, [38, 50–52, 54, 55]). In particular it is used for a nonlinear Black-Scholes equation arising in pricing a conventional option in [51] in which the authors also prove the convergence of the method by showing that it is consistent, monotone, stable. For brevity, we will omit the discussion on the convergence of the above method, but refer readers to the aforementioned papers.

3.3.2 Discretization of the Boundary and Payoff Conditions

The boundary and payoff conditions in (3.21) depend on the values of V_0 and V_1 at the mesh points on the boundaries of the solution domain. These values need to be calculated numerically using (3.10)–(3.13). Before considering approximations of (3.10)–(3.13), we first propose the following algorithm to determine T_k^*.

1. Let the reserve of the resource $Q \gg 0$ and $q_k(t)$ be given. Choose $\Delta t > 0$ sufficiently small. Set $j = 0$ and $Q_0 = 0$.
2. Evaluate $Q_{j+1} = Q_j + q_k((j + 1/2)\Delta t)\Delta t$.
3. If $Q_{j+1} \geq Q$, then set $T_k^* = (j + 1)\Delta t$ and stop. Otherwise, set $j = j + 1$ and go to Step 2.

In the above algorithm, we evaluate the integral in (3.10) using the mid-point quadrature rule until the total reserve Q is reached approximately. From the previous discussion we see that $T^* = \max\{T_0^*, T_1^*\} = T_0^*$ and q_1 is defined in (3.11).

To determine the payoff condition W_i^0, $i = 1, 2, ..., M - 1$ on (3.21), we need to solve equation (3.13) numerically to find approximations to $\hat{W}(P_i, T)$ for $i = 1, ..., M - 1$. To achieve this, we further divide (T, T^*) into N^* subintervals with the nodes $T^* = \tau_0 > \tau_1 > \cdots > \tau_{N^*} = T$ and solve (3.13) on the mesh with nodes (P_i, τ_j) for $i = 0, 1, ..., M$ and $j = 0, 1, ..., N^*$ using the discretization scheme (3.22) with the terminal and boundary conditions

$$\hat{W}_i^{N^*} = 0,\ \hat{W}_0^j = \hat{V}_{1,0}^j - \hat{V}_{0,0}^j,\ \hat{W}_M^j = \hat{V}_{1,M}^j - \hat{V}_{0,M}^j.$$

for $i = 0, 1, ..., M$ and $j = 1, 2, ..., N^*$, where $\hat{V}^j_{k,i}$ is the following approximation to $V_k(P_i, \tau_j)$ defined in equation (3.12):

$$\hat{V}^j_{k,i} = \sum_{l=1}^{j} (\tau_{l-1} - \tau_l) D_k \left(P_i, \bar{\tau}_{l-1}\right) e^{-r(\bar{\tau}_{l-1} - \tau_j)},$$

for $i = 0$ and M with $\bar{\tau}_{l-1} = (\tau_{l-1} + \tau_l)/2$. The above scheme is based on the composite mid-point quadrature rule for (3.12). It is easy to verify that the above expression can also be rewritten as the following recursive relation:

$$\hat{V}^0_{k,i} = 0, \quad \hat{V}^j_{k,i} = \hat{V}^{j-1}_{k,i} e^{-r(\tau_{j-1} - \tau_j)} + (\tau_{j-1} - \tau_j) D_k \left(P_i, \bar{\tau}_{j-1}\right) e^{-r(\bar{\tau}_{j-1} - \tau_j)}, \quad (3.27)$$

for $j = 1, 2, ..., N^*$, $i = 0, M$ and $k = 0, 1$.

To determine the boundary condition W^j_M for $j = 1, 2, ..., N$ in (3.21), we may use a formula similar to (3.27) which approximates (3.12). However, note that $D_0(P, t) = D_1(P, t)$ for $t \in [0, T)$. We may simply use the following formula to determine the boundary condition of W^j_M:

$$W^j_M = \left(\hat{W}^{N^*}_M - K\right) e^{-r(T - t_j)}, \quad j = 0, 1, ..., N,$$

where $\hat{W}^{N^*}_M = \hat{V}^{N^*}_{1,M} - \hat{V}^{N^*}_{0,M}$ with $\hat{V}^{N^*}_{0,M}$ and $\hat{V}^{N^*}_{1,M}$ determined by (3.27).

3.4 Convergence Analysis

In this section we will prove that the above discretization scheme is consistent, stable and monotone. The convergence of the numerical solution to the exact one is then guaranteed by the theoretical result in [56].

For $i = 1, 2, ...M - 1$ and $j = 0, 1, ...N - 1$, we introduce a function H^{j+1}_i defined by

$$H^{j+1}_i(W^{j+1}_i, W^{j+1}_{i+1}, W^{j+1}_{i-1}, W^j_i) := \alpha^j_i W^{j+1}_{i-1} + \beta^j_i W^{j+1}_i + \gamma^j_i W^{j+1}_{i+1} + \frac{1}{\Delta t^j} W^j_i. \quad (3.28)$$

Then, (3.17) can be rewritten as

$$H^{j+1}_i(W^{j+1}_i, W^{j+1}_{i+1}, W^{j+1}_{i-1}, W^j_i) = 0.$$

For the function H^{j+1}_i defined above, we have the following lemma.

Lemma 3.1. *The discretization is monotone, i.e., for any* $\varepsilon > 0$ *and* $i =$

$1, 2, ..., M-1$,

$$H_i^{j+1}(W_i^{j+1}, W_{i+1}^{j+1}+\varepsilon, W_{i-1}^{j+1}+\varepsilon, W_i^j+\varepsilon) \leqslant H_i^{j+1}(W_i^{j+1}, W_{i+1}^{j+1}, W_{i-1}^{j+1}, W_i^j), \tag{3.29}$$

$$H_i^{j+1}(W_i^{j+1}+\varepsilon, W_{i+1}^{j+1}, W_{i-1}^{j+1}, W_i^j) \geq H_i^{j+1}(W_i^{j+1}, W_{i+1}^{j+1}, W_{i-1}^{j+1}, W_i^j). \tag{3.30}$$

Proof. Recall $\alpha \leqslant 0$, $\gamma \leqslant 0$, and $\beta > 0$, and $\frac{1}{\Delta t_j} < 0$. From (3.28) we have

$$\begin{aligned}
&H_i^{j+1}(W_i^{j+1}, W_{i+1}^{j+1}+\varepsilon, W_{i-1}^{j+1}+\varepsilon, W_i^j+\varepsilon)\\
&\quad = \alpha_i^j(W_{i-1}^{j+1}+\varepsilon) + \beta_i^j W_i^{j+1} + \gamma_i^j(W_{i+1}^{j+1}+\varepsilon) + \frac{1}{\Delta t_j}(W_i^j+\varepsilon))\\
&\quad = \alpha_i^j W_{i-1}^{j+1} + \beta_i^j W_i^{j+1} + \gamma_i^j W_{i+1}^{j+1} + \frac{1}{\Delta t_j} W_i^j + \left(\alpha_i^j + \gamma_i^j + \frac{1}{\Delta t_j}\right)\varepsilon\\
&\quad = H_i^{j+1}(W_i^{j+1}, W_{i+1}^{j+1}, W_{i-1}^{j+1}, W_i^j) + \left(\alpha_i^j + \gamma_i^j + \frac{1}{\Delta t_j}\right)\varepsilon.
\end{aligned}$$

From (3.18) and (3.20) we see that $\alpha_i^j + \gamma_i^j \leq 0$. Also, $\frac{1}{\Delta t_j} < 0$ and $\varepsilon > 0$. Thus the from the above equation we see that (3.29) holds true.

Similarly, since $\beta_i^j > 0$ by (3.19) and $\varepsilon > 0$, we have

$$\begin{aligned}
&H_i^{j+1}(W_i^{j+1}+\varepsilon, W_{i+1}^{j+1}, W_{i-1}^{j+1}, W_i^j)\\
&\qquad = \alpha_i^j W_{i-1}^{j+1} + \beta_i^j(W_i^{j+1}+\varepsilon) + \gamma_i^j W_{i+1}^{j+1} + \frac{1}{\Delta t^j} W_i^j\\
&\qquad = \alpha_i^j W_{i-1}^{j+1} + \beta_i^j W_i^{j+1} + \gamma_i^j W_{i+1}^{j+1} + \frac{1}{\Delta t^j} W_i^j + \beta_i^j \varepsilon\\
&\qquad = H_i^{j+1}(W_i^{j+1}, W_{i+1}^{j+1}, W_{i-1}^{j+1}, W_i^j) + \beta_i^j \varepsilon\\
&\qquad \geqslant H_i^{j+1}(W_i^{j+1}, W_{i+1}^{j+1}, W_{i-1}^{j+1}, W_i^j).
\end{aligned}$$

This is (3.30). Thus, we have proved this lemma. □

In the following lemma we show that the discretization scheme is unconditionally stable.

Lemma 3.2. *The discretization equation* (3.22) *is unconditional stable.*

Proof. From (3.22) we have

$$W^{j+1} = \frac{1}{|\Delta t_j|}(A^j)^{-1} W^j + (A^j)^{-1} b^{j+1}.$$

(Recall $\Delta t_j < 0$). Taking ℓ^∞-norm on both side of the above equation we have the following estimate.

$$\|W^{j+1}\|_\infty \le \frac{1}{|\Delta t_j|}\|(A^j)^{-1}\|_\infty\|W^j\|_\infty + \|(A^j)^{-1}\|_\infty\|b^{j+1}\|_\infty. \tag{3.31}$$

It has been shown in [53] that, if $G = (g_{ij})$ is a $K \times K$ strictly diagonally dominant matrix for a positive integer K, then G^{-1} has the following estimate.

$$\|G^{-1}\|_\infty \le \frac{1}{\min\limits_{0\leqslant i\leqslant K}(|g_{ii}| - \sum_{j\neq i}|g_{ij}|)}.$$

Applying this result to A^j defined in (3.23) and using (3.26) we get

$$\|(A^j)^{-1}\|_\infty \le \frac{1}{\min\limits_{0\le i\le M-1}(\beta_i^j - |\alpha_i^j| - |\gamma_i^j|)} \le \frac{1}{r - 1/\Delta t_j} \le |\Delta t_j|,$$

since $r \ge 0$. (Recall $\Delta t_j < 0$.) Therefore, combining the above estimate and (3.31) gives

$$\begin{aligned}
\|W^{j+1}\|_\infty &\le \|W^j\|_\infty + |\Delta t_j|\|b^{j+1}\|_\infty \\
&\le \|W^{j-1}\|_\infty + |\Delta t_{j-1}|\|b^j\|_\infty + |\Delta t_j|\|b^{j+1}\|_\infty \\
&\;\;\vdots \\
&\le \|W^0\|_\infty + \sum_{k=0}^{j}|\Delta t_k|\|b^{k+1}\|_\infty \\
&\le \|W^0\|_\infty + \max_{1\le k\le N}\|b^k\|_\infty,
\end{aligned}$$

for $j = 0, 1, ..., N-1$. From (3.25) we see that b^j contains only the nodal values of the boundary conditions and the RHS function, and thus $\max_{1\le k\le N}\|b^k\|_\infty$ should be bounded above since these all these given functions are bounded. Therefore, we have proved that the numerical scheme is unconditionally stable. □

Lemma 3.3. *The discretization is consistent.*

Since (3.22) is based on some standard finite difference techniques which are known to be consistent. Therefore, the combined scheme is also consistent. Therefore, we omit the proof of this lemma.

Remark 3.3. For conventional problems, stability and consistency established in Lemmas 3.2 and 3.3 imply that the solution to (3.22) converges to that of (3.6)–(3.9).

$Q = 10^4$ million tons	$B = 30\%$ per annum
$C_0 = \text{US\$}35$	$C(t) = C_0 \times e^{0.005t}$
$R = 5\%$ per annum	$r = 0.06$ per annum
$K = \text{US\$}10^4$ million	$T = 2$ years
$\sigma = 30\%$	$\delta = 0.02$
$q_0 = 0.01Q \times e^{0.007t}$	$q_1 = \begin{cases} q_0 & t < T \\ \kappa \times q_0 & t \geq T \end{cases}$

Table 3.1: Project and market data used in Test 1.

However, the PDE systems we have established in this work may only have viscosity solutions since the payoff condition (3.8) is non-smooth. In this case, we need to use the theoretical result in [56] that if a numerical scheme for a general nonlinear 2nd-order PDE is, in addition to stability and consistency, also monotone, then the numerical solution generated by the scheme converges to the viscosity solution of the PDE. Therefore, if we let $h = \max h_i$ and $\Delta t = \max |\Delta t_j|$, then using the result in [56] we see that the numerical solution from (3.22) converges to the viscosity solution of (3.6)–(3.9) when $(h, \Delta t) \to (0, 0)$.

3.5 Numerical Experiments

We now solve two non-trivial iron-ore project expansion test problems using our models and numerical method developed in this paper to demonstrate that our method is able to produce numerical solutions which are financially meaningful.

Test 1: Option to Expand

This test problem is to price the value of an option to expand an investment project in the iron ore industry. All data for this project and option is given in Table 3.1.

To price the option, we choose $P_{\max} = 100$ (USD) and using the algorithm in Subsection 3.2.1 we find $T^* \approx 75$ (years). Now, the problems equation (3.13) and equations (3.6)–(3.9) are solved on uniform meshes with mesh sizes $h = 1$ and $\Delta t = 0.02$. The numerical solution for $\kappa = 2$ is depicted in Figure 3.1 in which the option price is a million USD. From the figure we see that this option as a function of (P, t) behaves like a conventional option. To further investigate the behaviours of the option, we solve the problem on the meshes for various values of expansion rate κ and volatility σ and plot the numerical results at $t = 0$ in Figure 3.2 and 3.3. From Figure 3.2 we see that the value of the option is an increasing function of κ, while Figure 3.3 demonstrates that, for a fixed κ, the value of the option increases when σ increases from 0.35 to 0.7. However, as σ decreases from 0.35 to 0.1 , the value of the option increases (respectively decreases) when P is larger (respectively smaller) than some critical values. This may represent the fact that the commodity price has more chance to decrease (respectively increase) for $\sigma = 0.35, 0.55$ and 0.7 than for $\sigma = 0.2$ and 0.1 when P is larger (respectively smaller) than some critical values. Thus, the option corresponding to $\sigma = 0.1$ and 0.2 are more (respectively less) expensive than the others when P is greater (respectively smaller) than some critical values.

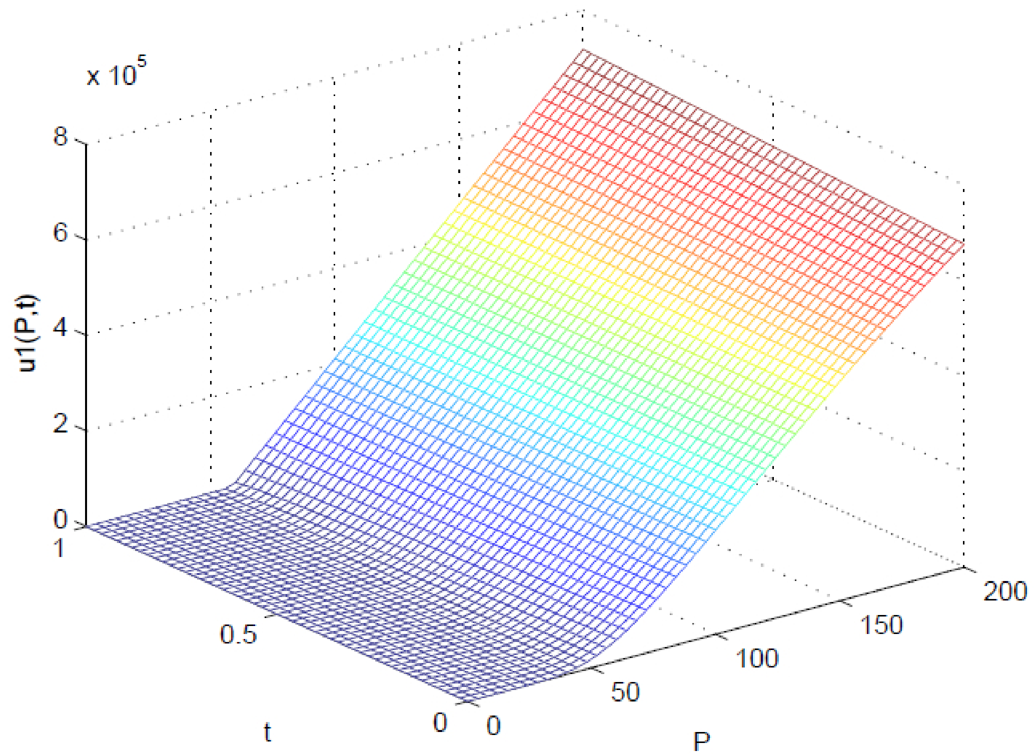


Figure 3.1: The computed option value for Test 1.

Test 2: Compound Expansion Option

As described in Case C2 in Subsection 3.2.2, this compound expansion option consists of two sequential options – one exercisable at T_1 with strike price K_1 and one at $T_2 > T_1$ with strike price K_2. The latter option is available only if the former is exercised. In addition to the market and project constants and functions defined in Table 3.1, other required parameters and functions are listed in Table 3.2.

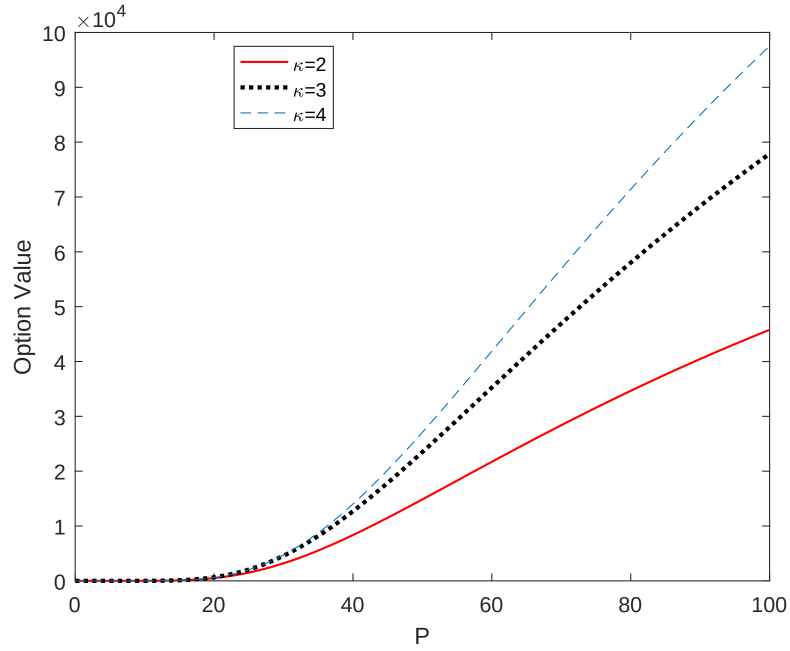


Figure 3.2: Computed option values at $t = 0$ for the different values of κ

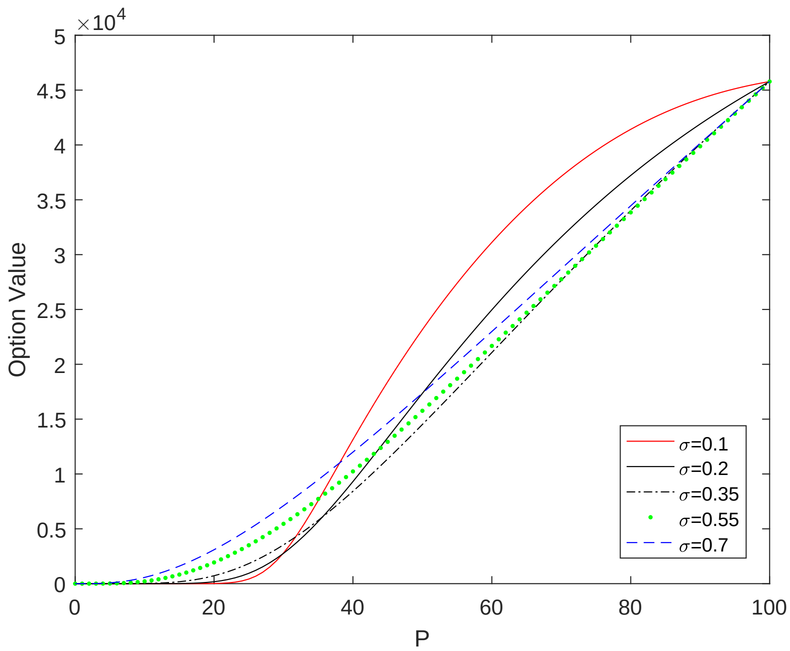


Figure 3.3: Computed option values at $t = 0$ for the different values of σ

$T_1 = 2$ years $K_1 =$ US\$$10^4$ million $q_1 = \begin{cases} q_0 & t < T_1 \\ 2q_0 & t \geq T_1 \end{cases}$	$T_2 = 4$ years $K_2 =$ US\$$2 \times 10^4$ million $q_2 = \begin{cases} q_1 & t < T_2 \\ 2q_1 & t \geq T_2 \end{cases}$

Table 3.2: Project and market data used in Test 2.

To solve this problem numerically, we choose $P_{\max} = 100$ (USD). Note that the numerical solution of the problem is divided into three sequential sub-problems which are (3.13) for determining the payoff condition at $t = T_2$, equation (3.14) for W_p and (3.15) for W_c. The sub-regions corresponding to the three sub-problems are partitioned into uniform meshes with mesh sizes $(h, \Delta t) = (1, 0.02)$.

The computed value W_c of the compound option for $t \in (0, T_1)$ is displayed in Figure 3.4. For comparison, we also compute the value W of the normal option with maturity $T = 2$, $\kappa = 4$ and strike price $K = 3 \times 10^4$ (million USD) and plot the result in Figure 3.5. Both of the expansion options require the same amount of capital. However, in the compound option case, the investment is in two stages. The difference $W_c - W$ in the region $(0, P_{\max}) \times (0, 2)$ is depicted in Figure 3.6 from which we see that the compound option is more valuable than the normal one when P is around the average cost function $C(t)$ and cheaper than the normal option when P is higher. To further demonstrate this, we plot the values of the two options and their difference at $t = 0$ in Figure 3.7. From the figure we see that $W_c \approx W$ when $P < 20$, $W_c > W$ when P is roughly between 20 to 60 and $W_c < W$ when $P > 60$. This observation is meaningful as the compound option has the flexibility for investors to avoid risks by not to exercise the 2nd component of option at T_2 when P is in a certain range near C_0 (in this case $P \in (20, 60)$) in which the return of the investment is very uncertain due to commodity price changes. Therefore, W_c is more valuable than W in a range. However, when P is large, the normal option is more valuable than the compound one which is financially true as in this case investors should increase the production rate as soon as possible.

Finally, we comment that since the exact solutions are unknown to the above test problems, we are unable to calculate the computational errors from our method. However, from the above numerical results produced by our method we see that our approach is able to calculate project extension option prices which are financially very meaningful.

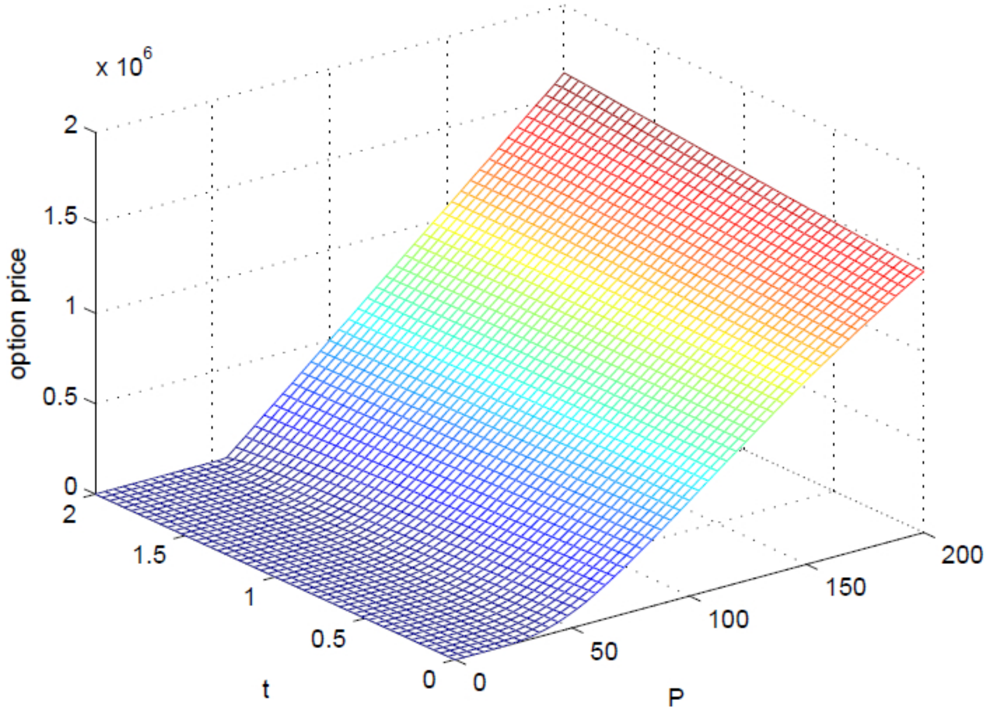


Figure 3.4: Computed values of normal options (a)

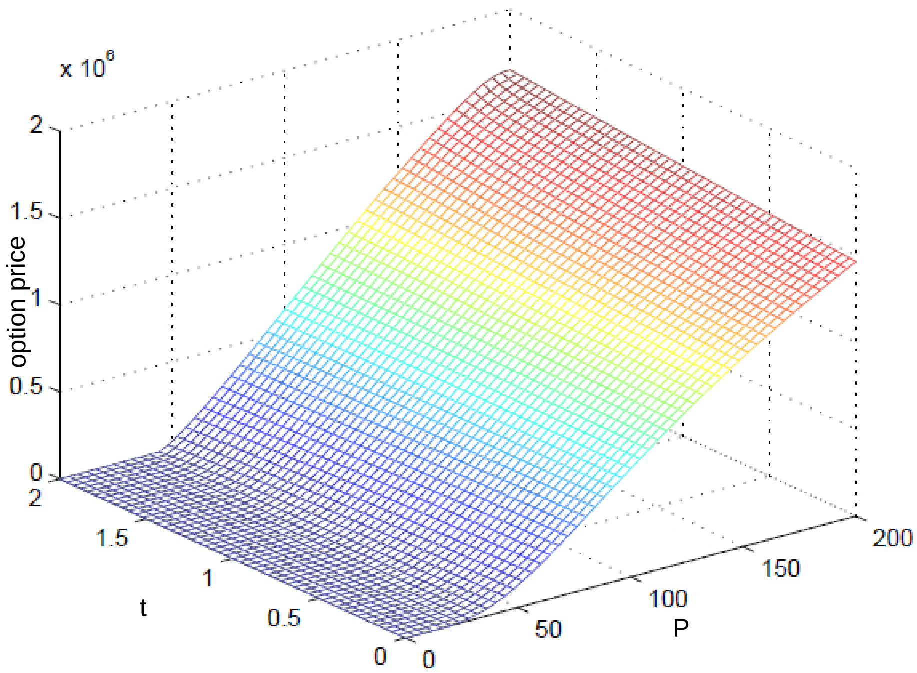


Figure 3.5: Computed values of compound options (b)

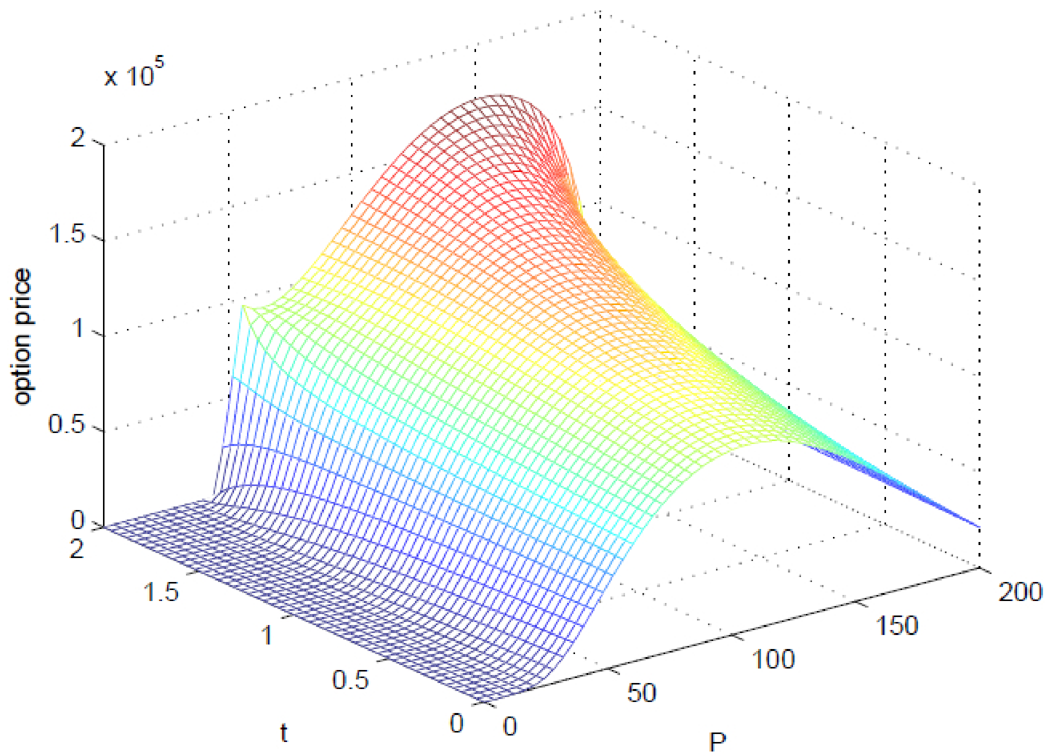


Figure 3.6: $W_c - W$, The differences of option values for Test 2.

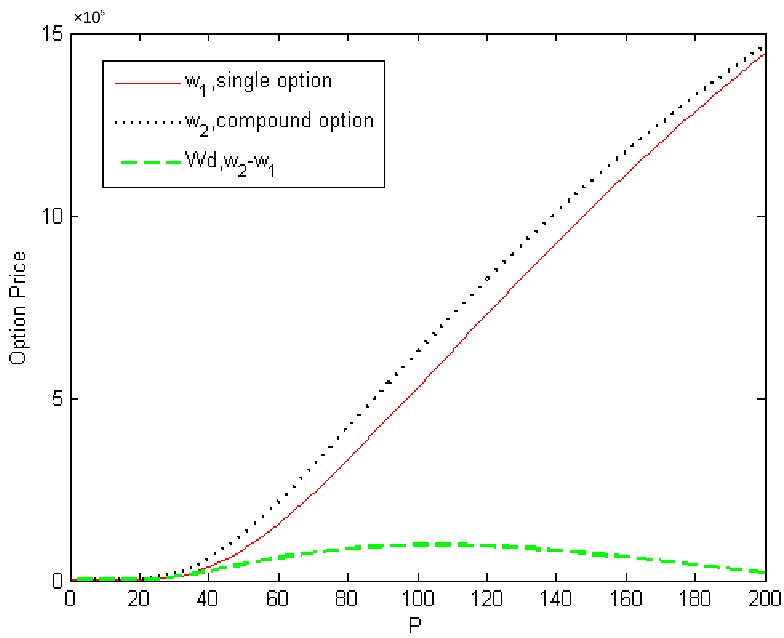


Figure 3.7: $W_c - W$ at $t = 0$, The differences of option values for Test 2.

3.6 Concluding Remarks

In this chapter we have proposed new PDE-based models to price real options and compound real options to expand investment projects when the underlying asset price follows geometric Brownian motion. An upwind finite difference method has been developed to numerically solve the pricing models. We have also proved the convergence of the finite difference method. Numerical experiments have been performed to demonstrate that the models and numerical methods produce financially meaningful results. Through our case studies we have found that the value of an expansion option of a project is affected notably by the volatility and the expanding rate. Also, a compound option on a project expansion is more valuable than the corresponding normal one when the commodity price is not far from the average unit operating cost, because in that commodity price range, the former provide more flexibility than the latter to avoid investment risks.

CHAPTER 4

Pricing Options on Investment Project Contraction, Ownership Transfer and Deferral Using a Fitted Finite Volume Scheme and an Interior Penalty Method

4.1 General

In this chapter, we develop partial differential equation (PDE) based computational models for pricing real options to contract the production or to transfer part/all of the ownership of a project, and defer the investment project expansion when the underlying asset price of the project satisfies a geometric Brownian motion. The developed models are similar to the Black-Scholes equation for valuing conventional European-style options or the partial differential linear complementarity problem (LCP) for pricing American-style options. A finite volume method is used for the discretization of the PDE models. And a penalty approach is applied to the discretized LCP while a smooth Newton method is proposed for solving the penalty equation. We show that the coefficient matrix of the discretized systems is a positive-definite M-matrix. It guarantees that the solution from the penalty equation converges to that of the discretized LCP. Numerical experiments, performed to demonstrate the usefulness of our methods, show that our models and numerical methods are able to produce financially meaningful numerical results for the two non-trivial test problems.

The world economies are always full of uncertainties. In particular, commodity pries such as that of iron ore, one of the most influential factors in our economies, are always uncertain. These uncertainties pose significant risks to businesses and investors who own investment projects linked to commodities. In order to hedge risks of a project from commodity price uncertainties, a manager or an investor can buy an option which allows its holder to sell part or all of the project to another party at a fixed cost. Some projects may also have 'built-in' flexibilities, called real options, allowing their owners to divest part or all of the projects.

Such flexibilities can usually add value to a project as they can be used as an instrument to hedge the risk due to the fluctuation of the project's underlying asset price. There are many types of real options with the nature of conventional put options [4] such as project abandonment or closure options. These real options are beneficial in businesses since they are able to improve the investors' wealth by reducing the operating loss or hedging the investment risk when the output market becomes weak.

Brennan and Schwartz [57] and Haque, Topal, and Lilford [58] introduce the models for the evaluation of a natural resource investment project when the underlying commodity price is stochastic. And McDonald and Siegel present a general model of investment with the option to wait [59]. However, the existing methods do not, in general, provide concrete PDE based pricing model with well-defined boundary and payoff conditions for real options arising in investment projects involving commodity price uncertainties. Thus, how to determine the value of American style options accurately has still been a challenge [1].

Since the introduction of real options in [3], the valuation of these has been investigated by various researchers (see, for example, [5, 12, 14, 16, 18, 60–62]). To best of our knowledge, unlike the Black-Scholes pricing models for conventional options, there are limited PDE-based computational models for valuing real options, particularly for options of an American type. Some notable PDE-based methods for pricing real options are given in [58, 60, 62]. However, as we can see later, pricing models of real options involved in a commodity price sensitive investment need judicious consideration of boundary and payoff conditions, and thus the methods in the aforementioned works do not apply to many real option pricing models such as those developed below.

In this work, we consider options to contract or abandon a project of both European and American types and develop PDE-based computational models for valuing these options. The developed models are of the form of either a parabolic PDE system for options of European put type, a partial differential LCP or a variational inequality for American put options with appropriate boundary and terminal/payoff conditions. Both are usually not solvable analytically except in some trivial cases. In addition, the above infinite-dimensional LCP is extended to govern American-style call option, i.e. the option to defer which is also a typical example of a stochastic optimal control problem. We will also investigate the numerical solution of these models. Numerical solution of Black-Scholes equations and LCPs involved discretization of differential operators and approximation of LCPs. There are many existing discretization schemes and optimization techniques for solving these equations and LCPs, such as finite difference, finite volume and finite element methods, and penalty approaches for linear and non-linaer complementarity problems. See for exmaple, [6, 35, 48, 63–70]. We will adopt the fitted finite volume scheme proposed and analyzed in [48,66] for the discretization of our models and the interior penalty approach to the complementarity problem governing the real options of the American type.

As follows, we will develop models for pricing real options of both European and American types and numerical methods for solving these models. More specifically, in Section 4.2 we will develop models for pricing partial closure or contracting production, abandoned and deferring options in which underlying asset price follows a GBM as used in [58]. These include options exercisable only on maturity (European type) and at any time prior to or on maturity (American type). In Section 4.3, we will use a fitted finite volume method for the discretization of the partial differential complementarity problem to value the American type real options. The numerical method developed for pricing an American-style real option can also be used for valuing its European counterpart. The application of the finite volume scheme to the models yields a finite dimensional LCP. In Section 4.4 we will apply the interior penalty method proposed in [39] to the resulting finite-dimensional LCP, and prove that the approximate solutions converges ultimately to the exact one as the penalty constant goes to zero. In Section 4.5, we conduct our numerical experiments on non-trivial test problems of the options to contract, abandon and defer, to demonstrate the performance, correctness and usefulness of our models and numerical techniques. The numerical results illustrate that our models and numerical methods always produce financially meaningful solutions.

4.2 Mathmatical Modeling

In this section, we will develop models for pricing real options of both European and American types. Let us first consider an investment project whose value V depends on the underlying commodity price P and time t. We look into the value of the project in the following two alternative cases.

C0. The project does not have any options on contraction or abandonment of its production or on transfer of partial/full ownership of the project.

C1. There is an embedded option which allows the option holder to either sell part or all of the the project ownership to the option issuer at the strike price $K > 0$ or to scale down the production rate from q_0 to $q_1 < q_0$ at the cost $C > 0$, or both. This option is exercisable either at the maturity $T > 0$ (European type), or at any time before or on T (American type).

In what follows, we use $V_0(P, t)$ and $V_1(P, t)$ to denote the values of the project for Cases C0 and C1 respectively. For Case C1 containing an option of either a European or American type, we use $U(P, t)$ and $W(P, t)$ to denote respectively the values of the European and American put options of either production contraction with the cost C or ownership transfer (abandonment) with strike price K. The rate of cash flow reduction for the option holder after exercising the option of either ownership transfer or production contraction is characterised by $(1 - \lambda)$, where $\lambda \in [0, 1)$. Then, we establish PDE models for determining U and W. We start this discussion with the European put option pricing model.

4.2.1 European Put Options

Intuitively, the value of a European put option is the difference between V_1 and V_0. However, in determining U, we also need to consider the influence of the strike price K, the cash flow retention rate λ and the cost C for production contraction in the form of boundary and payoff conditions.

A General Modeling

From their definitions we see that $V_1(P, t) \geq V_0(P, t)$ for all $t \in [0, T)$ and $P \geq 0$ as V_1 is equal to V_0 plus the value of the flexibility of either scaling down the production or transferring part of the project to a third party, or both at T. This flexibility is a right, not an obligation, and thus the flexibility should always be value-adding for the project. To derive the pricing model, we follow the standard Delta-hedging technique, particularly the one used in [58], for deriving equations of Black-Scholes' type.

We assume that the commodity price follows the following continuous stochastic process as proposed in [45]

$$dP = P(r - \delta)dt + P\sigma dz, \tag{4.1}$$

where dz is the increment to a geometric Brownian motion of P, σ denotes the volatility of the commodity (underlying asset) price P, r is the risk free interest rate, and δ is the mean convenience yield on holding one unit of the output. We also assume that the after-tax cash flow rates for the project under Assumptions C0 and C1 are respectively given by

$$D_k(P,t) := q_k(t)[P(t)(1-R)-c(t)](1-B), \quad k=0,1, \tag{4.2}$$

where $c(t)$ is the average unit production cost, R is the rate of state royalties, B is the income tax rate, and $q_0(t)$ and $q_1(t)$ are the production rates respectively for C0 and C1. It has been shown that, using the Delta(Δ)-hedging technique proposed in [11,61] and equations (4.1)–(4.2), the value functions V_0 and V_1 satisfy the following equation (see, for example, [44, 58]):

$$\frac{\partial V_k}{\partial t} + (r-\delta)P\frac{\partial V_k}{\partial P} + \frac{1}{2}P^2\sigma^2\frac{\partial^2 V_k}{\partial P^2} - rV_k = -D_k(P,t), \quad k=0,1, \tag{4.3}$$

for $(P,t) \in (0,\infty) \times [0,T)$. Theoretically, P is unbounded. However, in practice, it is normally bounded above by a large positive number in a specific period of time. Thus, in computation, the range $(0,\infty)$ for P is usually truncated into $(0,P_{\max})$, where $P_{\max}$ denotes a sufficiently large positive number.

The value U of the option is the premium of the project with the flexibility in Case C1 over that in Case C0, taking into account of K and C. Note that both K and C are receivable/payable only at the terminal time T. Therefore, when $(P,t) \in (0,P_{\max}) \times [0,T)$, U satisfies the difference between the equations in (4.3), i.e.,

$$\mathcal{L}U := -\frac{\partial U}{\partial t} - (r-\delta)P\frac{\partial U}{\partial P} - \frac{1}{2}P^2\sigma^2\frac{\partial^2 U}{\partial P^2} + rU = D_1 - D_0 = 0, \tag{4.4}$$

for $t \in [0,T)$ as $q_0(t) = q_1(t)$ when $t \in [0,T)$, where $\mathcal{L}$ represents the differential operator as self-explained in the above expression.

We comment that the above pricing model is based on that the commodity price follows a Geometric Brownian Motion as given in (4.1). Other models are also proposed for pricng real options, for example, the geometric mean-reverting pricing model proposed in [71]

B Boundary and Payoff Condition

We now determine the boundary and payoff conditions for (4.4) at $P=0$, $P=P_{\max}$, and $t=T$. To achieve this, we first introduce the life span of the project $T^* := \max\{T_0^*, T_1^*\}$, where T_0^* and T_1^* are the life-spans of the project under the production rates q_0 and q_1 respectively, i.e., T_0^* and T_1^* satisfy

$$Q = \int_0^{T_k^*} q_k(t)dt, \quad k = 0, 1. \tag{4.5}$$

Note that in this work we only consider the case that $q_1 < q_0$ when $t > T$. Thus, $T^* = T_1^*$ and we extend the production of Case 0 to $(T_0^*, T_1^*]$ by defining $q_0(t) = 0$ for $t \in (T_0^*, T_1^*]$.

For $P \geq 0$, let

$$\zeta(P,t) = \begin{cases} 0, & t < T, \\ \lambda D_1(P,t) - D_0(P,t), & t \in [T, T^*) \end{cases} \tag{4.6}$$

be the difference between the cash flow rates corresponding to C1 and C0, where $\lambda \in [0,1]$ is a constant representing the percentage of the project ownership held by the option holder after the option is exercised. When $P = 0$ and $P_{\max}$, we use $\zeta(P,t)$ to calculate the present value of the difference $V_1(P,t) - V_0(P,t)$ as follows

$$\begin{aligned} V_1(P,t) - V_0(P,t) &= \int_t^{T^*} \zeta(P,\tau)e^{-r(\tau-t)}d\tau \\ &= \int_{\min\{t,T\}}^{T} \zeta(P,\tau)e^{-r(\tau-t)}d\tau + \int_{\max\{t,T\}}^{T^*} \zeta(P,\tau)e^{-r(\tau-t)}d\tau \\ &= \int_{\max\{t,T\}}^{T^*} \zeta(P,\tau)e^{-r(\tau-t)}d\tau \\ &= e^{r(t-T)} \int_{\max\{t,T\}}^{T^*} \zeta(P,\tau)e^{-r(\tau-T)}d\tau, \end{aligned} \tag{4.7}$$

for $t \geq 0$, where $e^{-r(\tau-t)}$ is the discount factor with r the nominal interest rate. From (4.7) and (4.6) we see that when $t < T$,

$$\begin{aligned} V_1(P,t) - V_0(P,t) &= e^{r(t-T)} \int_T^{T^*} \zeta(P,\tau)e^{-r(\tau-T)}d\tau \\ &= e^{r(t-T)}(V_1(P,T) - V_0(P,T)), \end{aligned} \tag{4.8}$$

for $P = 0$ and $P_{\max}$.

Let $[z]^+ = \max\{0, z\}$ denote the positive part of z for any function z. Therefore, using (4.7) and taking into consideration of the strike price K and cost for production contraction C, we define the boundary condition of the option at

$P = 0$, as

$$\begin{aligned} U(0,t) =&[(K-C)e^{r(t-T)} + (V_1(0,t) - V_0(0,t))]^+ \\ =&\left[K + (V_1(0,T) - V_0(0,T)) - C\right]^+ e^{r(t-T)} \\ =&\left[K + (V_1(0,T) - V_0(0,T)) - C\right] e^{r(t-T)}, \end{aligned} \tag{4.9}$$

for $t \in [0, T)$, since we expect that $K + (V_1(0,T) - V_0(0,T)) - C > 0$, i.e., the option should be exercised when the underlying stock price $P = 0$.

When $P_{\max}$ is sufficiently large, the option has no value at $P = P_{\max}$ as its holder should not exercise it to either scale down the production or transfer part/all of the ownership to the option issuer. In this case we expect that $K + (V_1(0,T) - V_0(0,T)) - C < 0$, and thus the boundary condition for U at $P_{\max}$ is defined as

$$U(P_{\max}, t) = \left[K + (V_1(P_{\max}, T) - V_0(P_{\max}, T)) - C\right]^+ e^{r(t-T)} = 0, \tag{4.10}$$

for $t \in [0, T)$.

At $t = T$, we define the payoff condition for U as follows

$$U(P,T) = [K + (V_1(P,T) - V_0(P,T)) - C]^+, \quad P \in (0, P_{\max}). \tag{4.11}$$

Clearly, from (4.9)–(4.11) we see that the boundary and payoff conditions are consistent at the corner points $(0, T)$ and $(P_{\max}, T)$, i.e, $U(P, T)$ is continuous at $(0, T)$ and $(P_{\max}, T)$.

Note that $V_1(P,T) - V_0(P,T)$ used in (4.9)–(4.11) is unknown and needs to be determined by solving another PDE with appropriate boundary and terminal conditions. To define this PDE system, we first need to calculate the lifespan T^* of the project by (4.5). In fact, since we consider options of production contraction, we expect $q_1 \leq q_0$. Thus, $T_0^* \leq T_1^*$,and $T^* = T_1^*$. In this case, $q_0(t) = 0$ for $t \in [T_0^*, T^*)$.

In addition, when $t = T^*$, the project has zero value for both of the two cases. We denote the difference between the values of the project in the two cases by $\hat{U}$. Then, combining this observation, T^* determined above and (4.7), $\hat{U}$ is determined by the following problem:

$$\begin{cases} \mathcal{L}\hat{U} = \zeta(P,t), \ (P,t) \in (0, P_{\max}) \times [T, T^*), \\ \hat{U}(0,t) = \displaystyle\int_t^{T^*} \zeta(0,\tau) e^{-r(\tau-t)} d\tau, \ t \in [T, T^*), \\ \hat{U}(P_{\max},t) = \displaystyle\int_t^{T^*} \zeta(P_{\max},\tau) e^{-r(\tau-t)} d\tau, \ t \in [T, T^*), \\ \hat{U}(P, T^*) = 0, \ P \in (0, P_{\max}). \end{cases} \tag{4.12}$$

Using the solution to (4.12) we are able to determine the payoff condition for equation (4.4) by setting $V_1(P,T) - V_0(P,T) = \hat{U}(P,T)$ for $P \in (0, P_{\max})$, where $\hat{U}(P,T)$ is the solution to (4.12) evaluated at T.

We comment that the above real option pricing model is based on the risk-neutral measure Q used in original Black-Scholes pricing model.

4.2.2 American Options

An American Option is a type of options contract that its holders have the right to exercise at any time before including the day of expiration. In contrast, a European option cannot exercise the option until the option expire although it may be more favorable to be exercised earlier. Real options can be call options, such as options to expand which can be both European style options [44] or American style. For instance, option to defer, is an American-style option. In addition, options to contract [36], and options to abandon refer to put options,and they could be classified as European or American style depending on exercise time exactly at or at any time prior to the expiration date.

4.2.2.1 American Put Options

We now consider options to contract the production rate of a project requiring the capital C or to transfer part or all of the ownership of the project with the strike price K, or both. This option is exercisable any time prior to or at maturity $T > 0$. Using the same argument as for the conventional put options of American type, called Vanilla American put options [65], and noting that $\zeta(P,t) = 0$ for $t < T$, we see that the value of American option satisfies the following linear complementarity problem (LCP) ([65]).

$$\mathcal{L}W(P,t) \geq 0, \tag{4.13}$$

$$W(P,t) - W^*(P) \geq 0, \tag{4.14}$$

$$\mathcal{L}W(P,t) \cdot (W(P,t) - W^*(P)) = 0, \tag{4.15}$$

for $(P,t) \in (0, P_{\max}) \times [0,T)$ with boundary and payoff conditions, where $\mathcal{L}$ and ζ are defined respectively in (4.4), and (4.6) and $W^*(P)$ is a given function which usually equals the payoff condition evaluated at $t = T$.

We now define boundary and payoff conditions for the above LCP. At $t = T$, the payoff condition (4.11) should be satisfied by W as well, i.e.,

$$\begin{aligned} W^*(P) = W(P,T) &= [K + (V_1(P,T) - V_0(P,T)) - C]^+ \\ &= [K + \hat{U}(P,T) - C]^+, \end{aligned} \tag{4.16}$$

for $P \in [0, P_{\max}]$.

Noting that when $P_{\max}$ is sufficiently large, we expect that the option is of no value as neither contracting the production nor transferring part/all of the ownership of project to the option issuer will not add any value to the option for the option holder. Therefore, similar to (4.10), we have

$$W(P_{\max}, t) = 0, \quad t \in [0,T). \tag{4.17}$$

To define the boundary condition for W at $P = 0$, we note that when $P \to +0$, using the definition of $\mathcal{L}$ we see that equations (4.13)–(4.14) reduces to the following LCP

$$\begin{aligned} -\frac{dg(t)}{dt} + rg(t) &\geq 0, \\ g(t) - W^*(0) &\geq 0, \\ \left(-\frac{dg(t)}{dt} + rg(t)\right)(g(t) - W^*(0)) &= 0, \end{aligned}$$

for $t \in [0,T)$ with the terminal condition $g(T) = W^*(0)$, where W^* is defined in equation (4.16). It is easy to see that the above 1D continuous LCP has the solution $g(t) = W^*(0)$. Therefore, using equation (4.16) and equation (4.7), we define

$$\begin{aligned} W(0,t) = W^*(0) &= K + (V_1(0,T) - V_0(0,T)) - C \\ &= K + \int_T^{T^*} \zeta(0,\tau) e^{-r(\tau - T)} d\tau - C, \end{aligned} \tag{4.18}$$

for $t \in [0,T)$.

To summarise, the value of the American put option is governed by (4.13)–(4.15) subject to the boundary and payoff conditions (4.17), (4.18) and (4.16). Continuous differential LCPs such as (4.13)–(4.15) are hardly solvable analytically except for some trivial cases. Therefore, numerical methods are usually used for the approixmation of their solution. In the next section we develop a fitted finite volume method for the dicretization of (4.13)–(4.15).

4.2.2.1 American Call Options

We now study the pricing model on the option to defer. Consider an investment project in a natural resource industry where the underlying asset price P follows a geometric Brownian motion in time t. The value of the project is a function of P and t. Then, we consider the project in the following two althernate cases.

A0. The project does not have any expansion options.

A1. The project embedded an option to defer. In the production expansion project that is exercisable at any time before or on $T > 0$ (expiry date), and requires the amount $K > 0$ (strike price) for the expansion.

We denote the value functions of the project in the above two cases as $V_{a0}(P,t)$ and $V_{a1}(P,t)$ respectively. We expect $V_{a1}(P,t) \geq V_{a0}(P,t)$ for all $t \geq 0$ and $P \geq 0$ if the investors can make the right decision on whether or not to exercise the option. The reason is that the embedded option to defer can increase the value of the project in Case A1 which has the right, not an obligation to increase the production rate on optimal timing. We also denote the difference between the values of the project in the two cases by U_a.The option also satisfies the following LCP (4.13)–(4.15) for $(P,t) \in (0, P_{\max}) \times [0,T)$. The boundary and payoff conditions are defined in the following equations: at $t = T$, the payoff condition (3.8) should be satisfied by W_a as well, i.e.,

$$W_a^*(P) = W_a(P,T) = [V_{a1}(P,T) - V_{a0}(P,T) - K]^+, \ P \in (0, P_{\max}), \qquad (4.19)$$

for $P \in [0, P_{\max}]$. According to (3.7-3.9) in [44] and [36], the boundary condition for W_a at $P = 0$ and $P = P_{max}$, for $t \in [0,T)$,

$$W_a(0,t) = 0, \quad t \in [0,T), \qquad (4.20)$$

$$\begin{aligned} W_a(P_{\max},t) = W_a^*(P_{\max}) &= (V_{a1}(P_{\max},T) - V_{a0}(P_{\max},T)) - K \\ &= \int_T^{T_a^*} \zeta_a(P_{\max},\tau)e^{-r(\tau-T)}d\tau - K, \end{aligned} \qquad (4.21)$$

where for $P \geq 0$, the life-spans $T_a k^*$ are corresponding to q_{ak} for $k = 0, 1$. We defined $T_a^* = \max\{T_{a0}^*, T_{a1}^*\}$, and T_{ak}^* is determined by (4.5). Then, we have

$$q_{a0}(t) = q_{a1}(t),\ t \in [0,T), q_{a0}(t) < q_{a1}(t),\ t \in [T, T_{a1}^*),\ q_{a1}(t) = 0,\ t \in [T_{a1}^*, T_a^*], \tag{4.22}$$

$$\zeta_a(P,t) = \begin{cases} 0, & t < T, \\ D_1(P,t) - D_0(P,t), & t \in [T, T_a^*), \end{cases} \tag{4.23}$$

ζ_a be the difference between the cash flow rates corresponding to A1 and A0 where D_k is defined in (4.2). To determine the value W_a of the option exercisable at T , we solve above problem represented in (4.13)–(4.15) and (4.20)-(4.21).

4.3 The Fitted Finite Volume Method

We now consider the numerical solution of (4.4), (4.12) and (4.13)–(4.15). It may be possible to find analytical solutions to (4.4) and (4.12). However, solving the LCP (4.13)–(4.15) analytically is a real challenge, and thus we choose to seek numerical solutions to them in this work.

We now apply the fitted finite volume method proposed in [35,48,66] to (4.4), (4.12) and (4.13)-(4.15). Since these equations and inequalities involve the same differential operator $\mathcal{L}$, we only consider the discretization of (4.13)-(4.15) and, as we can see later, the developed scheme is applicable to both (4.4) and (4.12).

We first rewrite the LHS of (4.13) in the following form:

$$\mathcal{L}W = -\frac{\partial W}{\partial t} - \frac{\partial}{\partial P}\left(\alpha P^2 \frac{\partial W}{\partial P} + \beta P W\right) + \gamma W,$$

where

$$\begin{aligned} \alpha &= \frac{1}{2}\sigma^2, \\ \beta &= r - \delta - \sigma^2, \\ \gamma &= r + \beta = 2r - \sigma^2 - \delta. \end{aligned}$$

In what follows we develop a discretization scheme for the following more general case:

$$\mathcal{L}W(P,t) - f(P,t) \geq 0, \tag{4.24}$$

$$W(P,t) - W^*(P) \geq 0, \tag{4.25}$$

$$(\mathcal{L}W(P,t) - f(P,t)) \cdot (W(P,t) - W^*(P)) = 0, \tag{4.26}$$

where $f(P,t)$ is a known function. Clearly, (4.13)-(4.15) is a special case of the above when $f = 0$. The above LCP also contains both (4.4) and (4.12) as special cases when $W^*(P) = 0$ and f is chosen properly. Thus, the developed scheme is applicable to (4.4) and (4.12) as well.

To discretize (4.24)–(4.26), we first divide the interval $J := (0, P_{\max})$ into N sub-intervals

$$J_i := (P_i, P_{i+1}), \quad i = 0, 1, ..., N-1,$$

with $0 = P_0 < P_1 < \cdots < P_N = P_{\max}$. For each $i = 0, 1, ..., N-1$, let $h_i = P_{i+1} - P_i$. We also define a mesh dual to the one defined above by choosing the mesh nodes $\{P_{i-1/2} : i = 0, 1, ..., N+1\}$, where $P_{i-1/2} = (P_{i-1} + P_i)/2$ for $i = 1, 2, ..., N$, $P_{-1/2} = P_0 = 0$ and $P_{N+1/2} = P_N = P_{\max}$. Thus, integrating both sides of (4.24) over $(P_{i-1/2}, P_{i+1/2})$ we have

$$-\int_{P_{i-1/2}}^{P_{i+1/2}} \frac{\partial W}{\partial t} dP - \left[P\left(\alpha P \frac{\partial W}{\partial P} + \beta W\right)\right]_{P_{i-1/2}}^{P_{i+1/2}} + \int_{P_{i-1/2}}^{P_{i+1/2}} \gamma W dP - \int_{P_{i-1/2}}^{P_{i+1/2}} f dP \geq 0,$$

for $i = 1, 2, ..., N-1$. Applying the mid-point quadrature rule to the first, third and last terms we define the following approximation to the above inequality

$$-\frac{\partial W_i}{\partial t} l_i - \left[P_{i+\frac{1}{2}} \rho(W)|_{P_{i+\frac{1}{2}}} - P_{i-\frac{1}{2}} \rho(W)|_{P_{i-\frac{1}{2}}}\right] + \gamma W_i l_i - f_i l_i \geq 0, \qquad (4.27)$$

for $i = 1, 2, ..., N-1$, where, for any feasible i, $l_i = P_{i+1/2} - P_{i-1/2}$, $f_i = f(P_i, t)$, W_i denotes a nodal approximation to $W(P_i, t)$ to be determined and $\rho(W)$ is a flux associated with W defined by

$$\rho(W) := \alpha P \frac{\partial W}{\partial P} + \beta W.$$

It now remains to approximate the above flux ρ at $P_{i-1/2}$ and $P_{i+1/2}$. In [35, 48], the authors approximate $\rho(W)|_{P_{i+\frac{1}{2}}}$ by ρ_i which solves the following 2-point boundary value problem on J_i.

$$\frac{d\rho_i}{dP} = (\alpha P v' + \beta v)' = 0, \ P \in J_i,$$
$$v(P_i) = W_i, \ v(P_{i+1}) = W_{i+1}.$$

Solving this 2-point boundary value problem exactly yields the following approximation [35].

- When $i \geq 1$,
$$\rho(W)|_{P_{i+\frac{1}{2}}} \approx \rho_i = \beta \frac{P_{i+1}^a W_{i+1} - P_i^a W_i}{P_{i+1}^a - P_i^a}, \tag{4.28}$$
where $a = \beta/\alpha$.

- When $i = 0$,
$$\rho(W)|_{P_{\frac{1}{2}}} \approx \rho_0 = \beta \frac{1 - \mathrm{sign}(\beta)}{2} W_0 + \beta \frac{1 + \mathrm{sign}(\beta)}{2} W_1. \tag{4.29}$$

Replacing $\rho(W)|_{P_{i-\frac{1}{2}}}$ and $\rho(W)|_{P_{i+\frac{1}{2}}}$ with the approximations defined in (4.28) and (4.29), (4.27) can further be approximated by the following difference inequality

$$-\frac{\partial w_i}{\partial t} l_i + e_{i,i-1} w_{i-1} + e_{i,i} w_i + e_{i,i+1} w_{i+1} - f_i l_i \geq 0, \; i = 1, 2, ..., N-1, \tag{4.30}$$

where

$$e_{1,0} = -\frac{\beta P_1}{2} \frac{1 - \mathrm{sign}(\beta)}{2}, \tag{4.31}$$
$$e_{1,1} = \frac{\beta P_1}{2} \frac{1 + \mathrm{sign}(\beta)}{2} + \frac{\beta P_{1+1/2} P_1^a}{P_2^a - P_1^a} + \gamma l_1, \tag{4.32}$$
$$e_{1,2} = -\frac{\beta P_{1+1/2} P_2^a}{P_2^a - P_1^a}, \tag{4.33}$$

and

$$e_{i,i-1} = -\frac{\beta P_{i-1/2} P_{i-1}^a}{P_i^a - P_{i-1}^a}, \tag{4.34}$$
$$e_{i,i} = \frac{\beta P_{i-1/2} P_i^a}{P_i^a - P_{i-1}^a} + \frac{\beta P_{i+1/2} P_i^a}{P_{i+1}^a - P_i^a} + \gamma l_i, \tag{4.35}$$
$$e_{i,i+1} = -\frac{\beta P_{i+1/2} P_{i+1}^a}{P_{i+1}^a - P_i^a}, \tag{4.36}$$

for $i = 2, 3, ..., N-1$. It is easily seen that (4.30) is a semi-discrete system of $(N-1) \times (N-1)$ linear inequalities in $\boldsymbol{W}(t) := (W_1(t), ..., W_{N-1}(t))^\top$ with $W_0(t)$ and $W_N(t)$ being given boundary conditions defined in equations (4.17) and (4.18) respectively.

We now consider the time-discretization of (4.30). We first rewrite (4.30) in a matrix form. Let $E_i, i = 1, 2, ..., N-1$, be $1 \times (N-1)$ matrices defined by

$$\begin{aligned} E_1 &= (e_{11}, e_{12}, 0, ..., 0), \\ E_i &= (0, ..., 0, e_{i,i-1}, e_{i,i}, e_{i,i+1}, 0, ..., 0), \quad i = 2, 3, ..., N-2, \\ E_{N-1} &= (0, ..., 0, e_{N-1,N-2}, e_{N-1,N-1}), \end{aligned}$$

where $e_{i,i-1}, e_{i,i}$ and $e_{i,i+1}$ are defined in e(4.31)–(4.36) from which we see that the above coefficient matrices are independent of t. Using these matrices, we rewrite (4.30) in the following matrix form.

$$-\frac{\partial W_i(t)}{\partial t} l_i + E_i \boldsymbol{W}(t) \geq f_i(t) l_i + p_i(t), \tag{4.37}$$

for $i = 1, 2, ..., N-1$, where $p_1(t) = -e_{1,0} W_0(t)$, $p_{N-1}(t) = -e_{N-1,N} W_N(t)$ and $p_i(t) = 0$ for $i = 2, 3, .., N-2$.

To discretize this system, we let, for a given positive integer K, t_k ($k = 0, 1, ..., K$) be a set of points in $[0, T]$ satisfying $T = t_0 > t_1 > \cdots > t_K = 0$. We put $\Delta t_k = t_{k+1} - t_k < 0$ for $k = 0, 1, ..., K-1$. Using the two-level implicit time-stepping method with a splitting parameter $\theta \in [1/2, 1]$ for (4.37) we define the following inequality system approximating (4.37):

$$\frac{W_i^{k+1} - W_i^k}{-\Delta t_k} l_i + E_i \left(\theta \boldsymbol{W}^{k+1} + (1-\theta) \boldsymbol{W}^k\right) \geq \theta (f_i^{k+1} l_i + p_i^{k+1}) + (1-\theta)(f_i^k l_i + p_i^k),$$

for $i = 1, 2, ..., N-1$ and $k = 0, 1, ..., K-1$, where $f_i^l = f(x_i, t_l)$, $p_i^l = p_i(t_l)$ and $\boldsymbol{W}^l$ denotes the approximation of $\boldsymbol{W}$ at $t = t_l$ for $l = k$ and $k+1$. Let E be the $(N-1) \times (N-1)$ matrix given by $E = (E_1^\top, E_2^\top, ..., E_{N-1}^\top)^\top$. Then, the above linear system of inequalities can be re-written as

$$(\theta E + G^k) \boldsymbol{W}^{k+1} \geq \hat{f}^k + [G^k - (1-\theta) E] \boldsymbol{W}^k, \tag{4.38}$$

for $k = 0, 1, ..., K-1$, where $G^k = \mathrm{diag}(l_1/(-\Delta t_k), ..., l_{N-1}/(-\Delta t_k))$ is an $(N-1) \times (N-1)$ diagonal matrix and

$$\begin{aligned} \hat{f}^k =& \theta (f_1^{k+1} l_1 + p_1^{k+1}, ..., f_{N-1}^{k+1} l_{N-1} + p_{N-1}^{k+1})^\top \\ &+ (1-\theta)(f_1^k l_1 + p_1^k, ..., f_{N-1}^k l_{N-1} + p_{N-1}^k)^\top. \end{aligned}$$

The above time stepping scheme is unconditionally stable when $\theta \in [0.5, 1]$ and two special cases often used in practice are Crank-Nicolson scheme when $\theta = 1/2$ and the backward Euler scheme when $\theta = 1$. They are of second- and first-order accuracy, respectively.

Applying the above discretization schemes to equations (4.25) and (4.26) yields the following discretized inequality and equation:

$$\boldsymbol{W}^{k+1} - \boldsymbol{W}^* \geq 0, \quad (4.39)$$

$$(\boldsymbol{W}^{k+1} - \boldsymbol{W}^*)^\top \left((\theta E + G^k)\boldsymbol{W}^{k+1} - (\hat{f}^k + [G^k - (1-\theta)E]\boldsymbol{W}^k) \right) = 0, \quad (4.40)$$

where $\boldsymbol{W}^* = (W^*(P_1), ..., W^*(P_{N-1}))^\top$. The systems of inequalities and equations equations (4.38)–(4.40) form an LCP in $\mathbb{R}^{N-1}$ for each of $k = 1, 2, ..., K$. The payoff condition for this discrete LCP is $\boldsymbol{W}^0 = (W(P_1, T), W(P_2, T), ..., W(P_{N-1}, T))^\top$, where $W(P, T)$ is defined in equation (4.16).

We comment that a convergence analysis for the above finite volume method has been performed in [66] in which the authors have established the first order convergence rate in the maximal mesh sizes $h = \max_i h_i$ and $|\Delta t| = \max_k |\Delta t_k|$. Also, we will show below that the system matrix of (4.38) is a positive-definite M-matrix.

We also remark that though we use the fitted finite volume method for the discretization equations (4.13)–(4.15), the super convergent fitted finite volume scheme in [35] can also be used for this LCP to improve the accuracy of the flux approximation. Since the formulatios in both of the methods are the same except for the choices of $P_{i\pm\frac{1}{2}}$, we will not discuss the method in [35] here, but refer the reader to [35].

4.4 Solution of the Finite-dimensional LCP

We now consider the numerical solution of the LCP (4.38)–(4.40) for any fixed $k \in \{1, 2, ..., N-1\}$. There are various methods for solving generic LCPs such as those discussed in [72–79]. Clearly, these generic methods do not take full advantage of the properties the system matrix of (4.38) satisfies. Power penalty methods have been developed for discretized LCPs or continuous LCPs arising in various American option pricing problems (see, for example, [50, 55, 68, 80, 81]). However, these methods are 'exterior' in the sense that approximate solutions generated by them do not satisfy (4.39) strictly. Interior point methods have been used extensively in constrained optimization (see, for example, [82]). In recent years, interior penalty methods in infinite and finite dimensions have been proposed in [39, 83–85] for solving large-scale linear and nonlinaer complementarity problems arising in financial engineering. These methods have the merit that the approximate solutions from these methods always satisfy the constraint (4.39).

In this work we will apply the interior penalty method in [39] to (4.38)–(4.40) and show that all the conditions which guarantee the convergence of the solutions to the penalty equations are satisfied by our discretized LCP.

Introducing the substitution $w^k = -(\boldsymbol{W}^{k+1} - \boldsymbol{W}^*)$, we transform (4.38)–(4.40) into the following standard LCP:

$$A^k w^k \leq b^k, \quad w^k \leq 0, \quad \text{and} \quad (A^k w^k - b^k)^\top w^k = 0, \tag{4.41}$$

where $A^k := (\theta E + G^k)$ and $b^k := A^k \boldsymbol{W}^* - [\hat{f}^k + [G^k - (1-\theta)E]\boldsymbol{W}^k]$. Following [39], we approximate the solution w^k to equation (4.41) by the negative solution to the following nonlinear system

$$A^k w^k_\mu - \mu./w^k_\mu = b^k, \tag{4.42}$$

for $k = 1, 2, ..., K$, where $\mu > 0$ is the penalty constant and ./, as used in Matlab, denotes the element-by-element (Hadamard) division of either a scalar and a matrix, or two matrices of the same size. It has been proved in [39] that under certain conditions (4.42) is uniquely solvable and its solution converges to the solution to (4.41) as $\mu \to +0$. In the following theorem, we show that the conditions which guarantee the existence, uniqueness and convergence of the solution to (4.42) are satisfied by our discrete LCP (4.41).

Theorem 4.1. *For any $k \in \{1, 2, ..., K\}$, if Δt_k is chosen such that*

$$\frac{1}{|\Delta t_{k-1}|} + \frac{\theta(r+\gamma)}{2} > 0, \tag{4.43}$$

then there exists a unique solution w^k_μ to (4.42) satisfying $w^k_\mu < 0$. Furthermore, let w^k be the solution to (4.41). Then w^k and w^k_μ, satisfy

$$\|w^k_\mu - w^k\|_2 \leq L\sqrt{(N-1)\mu}, \tag{4.44}$$

for $k = 1, 2, ..., N$, where $\|\cdot\|_2$ denotes the Euclidean norm on $\mathbb{R}^{N-1}$ and L is a positive constant, independent of both N and μ.

Proof. For notation clarity, we will omit the superscript k in this proof, as k does not play any role in the proof. It has been proved in [39] that if A is an M-matrix and is positive-definite, then (4.44) is satisfied. In what follows we show that both of these properties are satisfied by A.

We first prove $A = (a_{ij})$ is an M-matrix by showing that it is irreducibly diagonally dominant with positive diagonal and non-positive off-diagonal entries, i.e., $a_{ii} > 0$, $a_{ji} \leq 0$ when $i \neq j$ and $a_{ii} \geq \sum_{\substack{j=1,...,N-1 \\ i \neq j}} |a_{ji}|$ for $i = 1, 2, ..., N-1$, and this inequality is strict for at least one i (see, for example, [53]). Since $A = \theta E + G$ and G is a diagonal matrix with positive diagonal entries, to show A is irreducibly diagonally dominant, it suffices to prove that E is irreducibly diagonally dominant.

From (4.34)–(4.36) it is easy to verify (see [48]) that, for any feasible i,

$$\frac{\beta}{P_{i+1}^a - P_i^a} = \alpha \frac{a}{P_{i+1}^a - P_i^a} = \frac{\alpha}{P_i^a} \frac{a}{\left(\frac{P_{i+1}}{P_i}\right)^a - 1} > 0,$$

if $a = \beta/\alpha \neq 0$, because $\alpha > 0$, $\left(\frac{P_{i+1}}{P_i}\right)^a > 1$ when $a > 0$ and $\left(\frac{P_{i+1}}{P_i}\right)^a < 1$ when $a < 0$. When $a \to 0$, using L'Hospital's rule we have

$$\lim_{a \to 0} \frac{P_{i+1}^a - P_i^a}{\beta} = \frac{1}{\alpha} \lim_{a \to 0} \frac{P_{i+1}^a - P_i^a}{a} = \frac{1}{\alpha} (\ln P_{i+1} - \ln P_i) > 0.$$

Thus, from (4.31)–(4.36) we have that, for all feasible i and j, $e_{i,i} > 0$ and $e_{i,j} \leq 0$ when $j \neq i$.

Furthermore, from (4.31)–(4.36) we also see that, for $i = 2, ..., N-2$,

$$\begin{aligned} e_{i,i} - \sum_{\substack{j=1,...,N-1 \\ j \neq i}} |e_{i,j}| &= e_{i,i} + e_{i,i-1} + e_{i,i+1} \\ &= \frac{\beta P_{i-1/2} P_i^a}{P_i^a - P_{i-1}^a} - \frac{\beta P_{i-1/2} P_{i-1}^a}{P_i^a - P_{i-1}^a} + \frac{\beta P_{i+1/2} P_i^a}{P_{i+1}^a - P_i^a} - \frac{\beta P_{i+1/2} P_{i+1}^a}{P_{i+1}^a - P_i^a} + \gamma l_i \\ &= \beta P_{i-1/2} - \beta P_{i+1/2} + \gamma l_i \\ &= r l_i, \end{aligned} \tag{4.45}$$

since $P_{i+1/2} - P_{i-1/2} = l_i$ and $\gamma - \beta = r$ by their definitions in the previous section. When $i = N-1$, from the deduction of (4.45) we have

$$e_{N-1,N-1} - \sum_{j=1}^{N-2} |e_{N-1,j}| = e_{N-1,N-1} + e_{N-1,N-2} > r l_{N-1}. \tag{4.46}$$

Similarly, when $i = 1$, from (4.32)–(4.33) and the above analysis we obtain

$$
\begin{aligned}
e_{1,1} - \sum_{j=2}^{N-2} |e_{1,j}| &= e_{1,1} + e_{1,2} \\
&= \frac{\beta P_1}{2}\frac{1+\text{sign}(\beta)}{2} + \frac{\beta P_{1+1/2}P_1^a}{P_2^a - P_1^a} + \gamma l_1 - \frac{\beta P_{1+1/2}P_2^a}{P_2^a - P_1^a} \\
&= \frac{\beta P_1}{2}\frac{1+\text{sign}(\beta)}{2} - \beta P_{1+1/2} + \gamma l_1 \\
&= \frac{\beta P_1}{2}\frac{1+\text{sign}(\beta)}{2} - \frac{\beta P_1}{2} + (\gamma - \beta) l_1 \\
&= \frac{\beta P_1}{2}\frac{\text{sign}(\beta)-1}{2} + r l_1 \\
&= \frac{P_1}{2}\frac{|\beta| - \beta}{2} + r l_1 \\
&\geq r l_1.
\end{aligned} \tag{4.47}
$$

In the above we used the relation $P_{1+1/2} = P_1/2 + l_1$. Therefore, from (4.45)–(4.47) we see that E is strictly diagonally dominant. Clearly, E is irreducible, as otherwise, (4.41) can be decomposed into two decoupled sub-problems. Thus, E is an M-matrix, and so is A.

We now show that A is positive-definite. To show this, it suffices to prove that the symmetric part of A, $M := \frac{1}{2}(A + A^\top)$, is positive-definite. From the definition of A we have

$$
\begin{aligned}
M_{i,i-1} = M_{i-1,i} &= \frac{1}{2}(a_{i,i-1} + a_{i-1,i}) = \frac{1}{2}\theta(e_{i,i-1} + e_{i-1,i}) \leq 0, \\
M_{i,i+1} = M_{i+1,i} &= \frac{1}{2}(a_{i+1,i} + a_{i,i+1}) = \frac{1}{2}\theta(e_{i+1,i} + e_{i,i+1}) \leq 0, \\
|M_{i,i-1}| + |M_{i,i+1}| &= -\frac{1}{2}\theta\left(e_{i,i-1} + e_{i-1,i} + e_{i+1,i} + e_{i,i+1}\right).
\end{aligned}
$$

Using the above and equation (4.45) we have

$$
\begin{aligned}
M_{i,i} = a_{i,i} &= \frac{l_j}{|\Delta t_{k-1}|} + \theta e_{i,i} \\
&= \frac{l_i}{|\Delta t_{k-1}|} + |M_{i,i-1}| + |M_{i,i+1}| + \theta\left[e_{i,i} + \frac{1}{2}(e_{i,i-1} + e_{i-1,i} + e_{i+1,i} + e_{i,i+1})\right] \\
&= \frac{l_i}{|\Delta t_{k-1}|} + \sum_{\substack{j=1,\dots,N-1 \\ j\neq i}} |M_{i,j}| + \frac{1}{2}\theta(e_{i,i} + e_{i,i-1} + e_{i,i+1} + \gamma l_i) \\
&= \sum_{\substack{j=1,\dots,N-1 \\ j\neq i}} |M_{i,j}| + \left(\frac{\theta(r+\gamma)}{2} + \frac{1}{|\Delta t_{k-1}|}\right) l_i,
\end{aligned} \tag{4.48}
$$

for $i = 2, ..., N-2$. In the above deduction we have used the relation $e_{i,i} = -e_{i-1,i} - e_{i+1,i} + \gamma l_i$ by equations (4.34)–(4.36).

When $i = 1$ and $i = N-1$, using the same argument as for equations (4.48) and (4.46)–(4.47) we have

$$M_{1,1} \geq \sum_{j=2}^{N-1} |M_{1,j}| + \left(\frac{1}{|\Delta t_{k-1}|} + \frac{\theta(r+\gamma)}{2} \right) l_1, \tag{4.49}$$

$$M_{N-1,N-1} > \sum_{j=1}^{N-2} |M_{N-1,j}| + \left(\frac{1}{|\Delta t_{k-1}|} + \frac{\theta(r+\gamma)}{2} \right) l_{N-1}. \tag{4.50}$$

From (4.48)–(4.50) we see that M is strictly diagonally dominant when the condition in the theorem is satisfied. Note that since M is symmetric, any of its eigenvalues, denoted as $\lambda(M)$, is real. Therefore, by Gershgorin Circle Theorem (see, for example, [86]), there is an index $n \in \{1, 2, ..., N-1\}$ such that $\lambda(M)$ satisfies

$$\lambda(M) \geq M_{n,n} - \sum_{\substack{j=1,...,N-1 \\ j \neq n}} |M_{n,j}| \geq \left(\frac{\theta(r+\gamma)}{2} + \frac{1}{|\Delta t_{k-1}|} \right) l_n > 0,$$

because of the condition (4.43). Thus, M is positive-definite , and so A is also positive-definite.

To summarize, we have proved in the above that, when (4.43) is satisfied, the coefficient matrix $A^k = \theta E + G^k$ of both (4.41) and (4.38)–(4.40) is a positive-definite matrix. Using the convergence result in [39] and Theorem 4.1 we conclude that (4.44) holds for some positive constant L, independent of μ. Therefore, we have proved this theorem. □

Note that (4.43) is not a restrictive condition. This is because in computation, $|\Delta t_k|$ is usually chosen to be very small so that (4.43) is always satisfied. Also, from the above proof we see that Theorem 4.1 also holds true when $P_{i+\frac{1}{2}}$ of the dual mesh is chosen to be any point in (P_i, P_{i+1}), not just when $P_{i+\frac{1}{2}} = \frac{1}{2}(P_i + P_{i+1})$, for all feasible i. Therefore, since the superconvergent finite volume method proposed in [35] differs from the method proposed above by the choices of $P_{i+\frac{1}{2}}$ for all feasible i, Theorem 4.1 also holds true when the superconvergent finite volume method in [35] is used for the discretization of (4.13)–(4.15).

For each $k = 1, 2, ..., K$, (4.42) is a nonlinear system in w_μ^k and smooth in the interior of the region $\mathcal{K} = \{x \in \mathbb{R}^{N-1} : x \leq 0\}$. The following smooth Newton method, originally proposed in [39], can be used for solving (4.42) for each $k = 0, 1, ..., N-1$.

Algorithm Newton

(i) Choose positive constants μ, ε and $\delta > 0$ sufficiently small and an initial guess u^0 in the interior of $\mathcal{K}$. Set $j = 0$.

(ii) Solve the following linear system for p^j.

$$[A^k + \mu H(u^j)]p^j = -(A^k u^j - \mu./u^j - b^k),$$

where $H(u) = \text{diag}(u_1^{-2}, u_2^{-2}, ..., u_{N-1}^{-2})$.

(iii) Set $u^{j+1} = \min\{u^j + p^j, -\delta \mathbf{1}_{N-1}\}$, where $\mathbf{1}_{N-1} = (1, 1, ..., 1)^\top \in \mathbb{R}^{N-1}$.

(iv) If $\|u^{j+1} - u^j\|_2 < \varepsilon$, then put $w^k = u^{j+1}$ and stop. Otherwise, increment j and go to Step 2.

Step 3 of the above algorithm is to project the approximate solution into the interior of $\mathcal{K}$ when overshooting occurs due to linearization in Newton's algorithm. Also, from the results established in the proof of Theorem 4.1, it is easy to see that the coefficient matrix of the system in Step 2 is a positive-definite M-matrix as H is a diagonal matrix with all positive diagonal entries.

4.5 Numerical Experiments

In this section we present three numerical examples to demonstrate the usefulness of the pricing models and the numerical methods developed. The first examples are options to contract the production, and second are the options transfer ownership of an iron ore mining project either at any time before and including the day of or only on the day of expiration. The last example is an option to defer expansion before or on maturity.

Test 1: Option to contract the production of a project

This option gives a company the right to scale down its iron-ore mining operation rate from q_0 to q_1 either at the end of the first year (European type) or any time within the first year (American type). The cost for the contracting production is C. The market, production and option data are listed in Table 4.1 in which $\kappa \in (0, 1)$ represents the rate of production reduction. Note that this option does not involve ownership transfer and thus $\lambda = 1$ and $K = 0$. All computations have been performed in double precision under Matlab environment.

$Q = 10^4$ million tons	$B = 30\%$ per annum
$c_0 = \text{US}\$25$	$c(t) = c_0 \times e^{0.005t}$
$R = 5\%$ per annum	$r = 0.06$ per annum
$C = \text{US}\$5 \times 10^2$ million	$T = 1$ year
$\sigma = 30\%$	$\delta = 0.02$
$q_0 = \begin{cases} 0.01Q \times e^{0.007t}, & t \leq T_1^*, \\ 0, & t \in (T_1^*, T^*) \end{cases}$	$q_1 = \begin{cases} 0.01Q \times e^{0.007t}, & t < T, \\ \kappa \times 0.01Q \times e^{0.007t}, & t \in [T, T^*) \end{cases}$
$q_{a0} = 0.01Q \times e^{0.007t}$	$q_{a1} = \begin{cases} q_{a0} & t < T \\ \vartheta \times q_{a0} & t \geq T \end{cases}$

Table 4.1: Project, production and market data and functions used in Test 1.

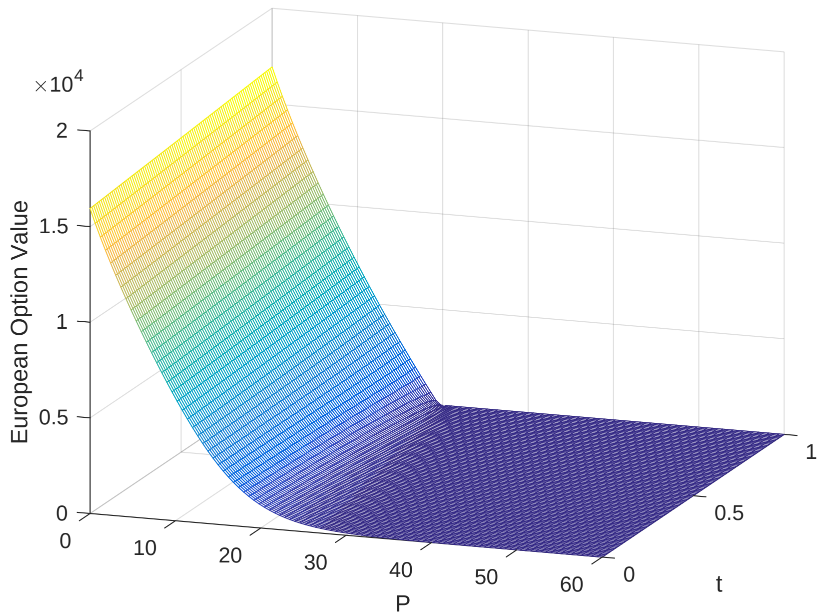


Figure 4.1: $\boldsymbol{U}$, The values of European options when $\kappa = 0.5$ for Test 1.

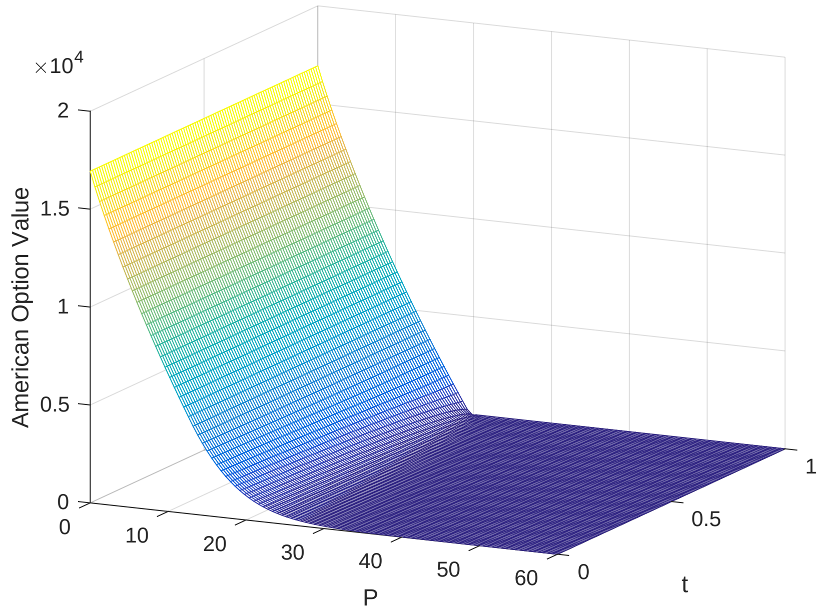


Figure 4.2: $\boldsymbol{W}$, The values of American options when $\kappa = 0.5$ for Test 1.

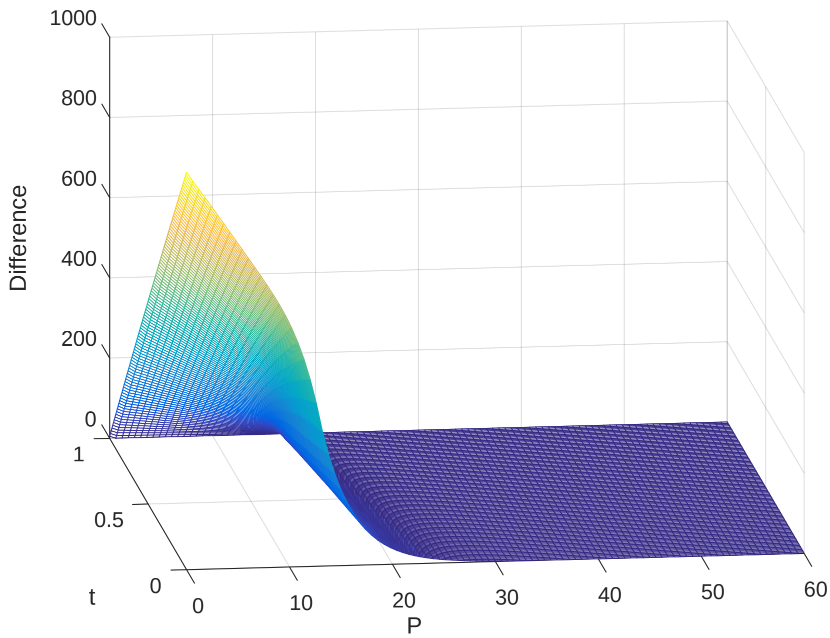


Figure 4.3: $W - U$, The difference between the values of the European and American contracting options for Test 1.

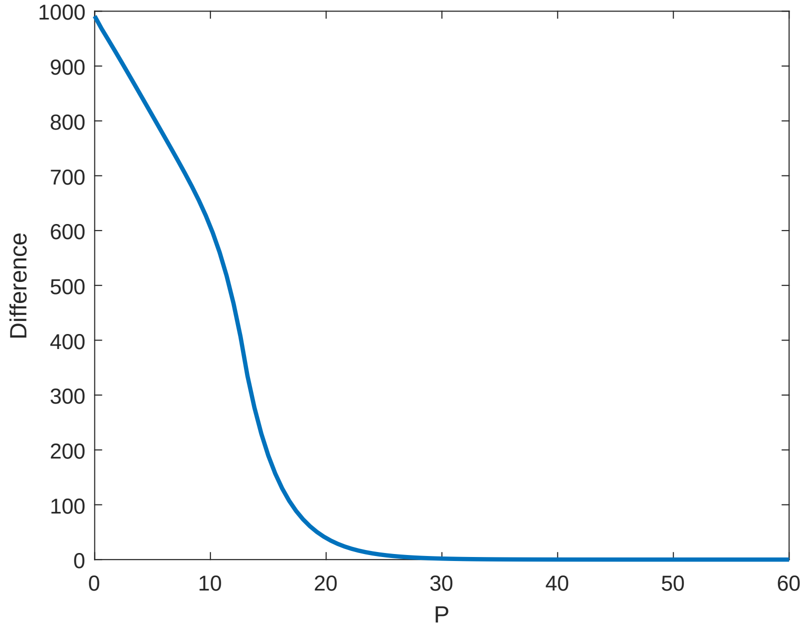


Figure 4.4: $W - U$ at $t = 0$ The difference between the values of the European and American contracting options for Test 1.

For this test, we choose $\kappa = 0.5$ in q_1, i.e., the production rate is halved after the option is exercised. Before we solve this problem numerically, we need to estimate T_1^* and T^* defined in equation (4.5) for the project defined in Section 4.2. A simple algorithm based on the 2nd-order accurate mid-point quadrature rule is proposed in [44] to calculate both numerically. Following this algorithm, we have found that $T_0^* \approx 75$ years and $T^* = T_1^* \approx 124$ years.

To solve this pricing problem we choose $P_{\max} = 60$USD and the solution domain $(0, 60) \times [0, 1)$ for (4.13)–(4.15) is partitioned into a uniform mesh with the constant mesh sizes $h = 0.6$ and $|\Delta t| = 0.01$ in P and t directions respectively. Note that since maturity of this option is 1 (year), we use a mesh size in the t-direction much smaller than in the P-direction to ensure the numerical solution is of reasonably high resolution. The boundary conditions are calculated by integrating (4.7) and (4.10) numerically using a quadrature rule on the uniform mesh in t with mesh size $|\Delta t| = 0.01$. The payoff condition (4.12) is solved by the discretization scheme proposed in Section 4.2 also on the uniform mesh with $h = 0.6$ and $|\Delta t| = 0.01$. This option pricing problem is solved as both a European put and an American put. For the numerical solution of the American put, Algorithm Penalty in the previous section is used in which we choose $\mu = 10^{-12}$,

$\varepsilon = 10^{-6}$ and $\delta = 10^{-12}$. The computed values of the European and American options are plotted in Figure 4.1 and 4.2 respectively. The difference between the values of European and American puts are depicted in Figure 4.3 and 4.4 from which we see that American option is more valuable than its European counterpart. This is financially correct as the former is more flexible then the latter and the premium of the American put over its European counterpart is depicted in Figure 4.3 and 4.4.

We comment that usually $T^* \ll T$. This is because when T is close to T^* it is not viable to issue such an option as the project is close to its end of life. Nevertheless, our pricing methods still work even if T is close to T^*.

Test 2: Option to Transfer Ownership

This real option gives its holder the right, not obligation, to transfer $100 \times (1-\lambda)$ percent of the ownership of the project to the option issuer with the strike price K, where $\lambda \in [0, 1)$ is the percentage of the ownership the option holder keeps after the option is exercised. The strike price is chosen to be $K = 2(1-\lambda) \times 10^4$ million USD. Since this option does not involve production reduction, we have $C = 0$, $\kappa = 1$ and $q_1(t) = q_0(t) = 0.01Q \times e^{0.007t}$ for $t \in (0, T^*)$ with $T^* = T_0^* = T_1^* = 75$ years. Other market and option parameters used in this test are listed in Table 4.1. We assume that this option can be exercised any time before or on maturity T and thus the option can be regarded as an American put.

To solve Test 2, we choose $P_{\max} = 80$USD. The solution domain $(0, 80) \times [0, 1)$ is partitioned into the uniform mesh with the mesh sizes $h = 0.8$ and $|\Delta t| = 0.01$ and the finite volume method on this mesh is applied to (4.24)–(4.26). The boundary condition PDE system (4.12) is also solved on the uniform mesh with $h = 0.8$ and $|\Delta t| = 0.01$. The discretized LCP is then solved by Algorithm Penalty in the previous section with the same values of μ, ε and δ as in Test 1. The computed option values for $\lambda = 0.5$ and $\lambda = 0$, representing 50% and 100% ownership transfers to the option issuer respectively, are plotted in Figure 4.5 and 4.6, from which we see that when P is small, i.e., the iron-ore market has a downturn, the option for 100% ownership transfer is more valuable than that with only 50% transfer. We also plot the difference between the value of the option for $\lambda = 0.5$ and the lower bound W^* in Figure 4.7 and the partial derivatives $\Delta := \frac{\partial W}{\partial P}$ and $\Gamma := \frac{\partial^2 W}{\partial P^2}$ in Figure 4.8 and Figure 4.9 respectively. From Figure 4.7 we see that the numerical solution satisfies the lower bound constraint (4.25) in the solution region. From the figure we also see that the optimal exercise curve (or free boundary in the engineering context) is clearly shown in the plots for Δ and Γ.

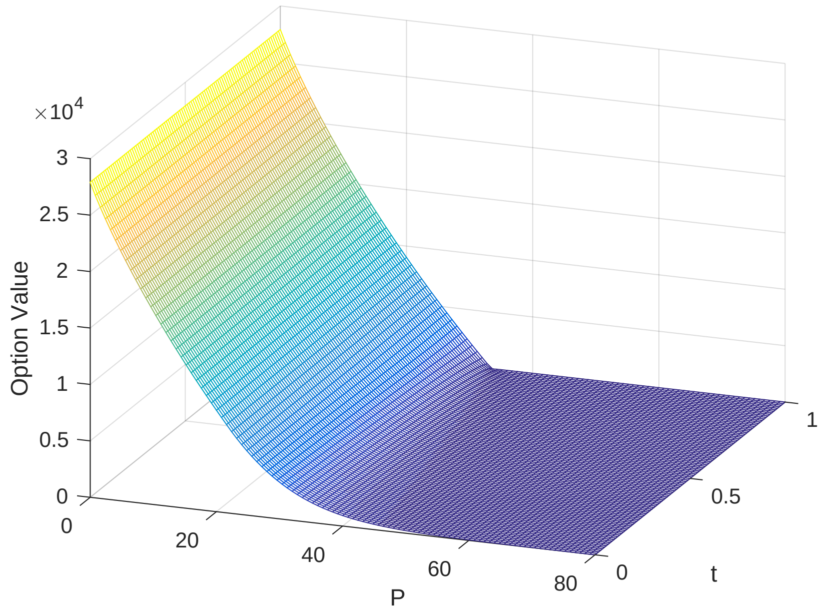


Figure 4.5: $\lambda = 0.5$, The value of the option to abandon for Test 2.

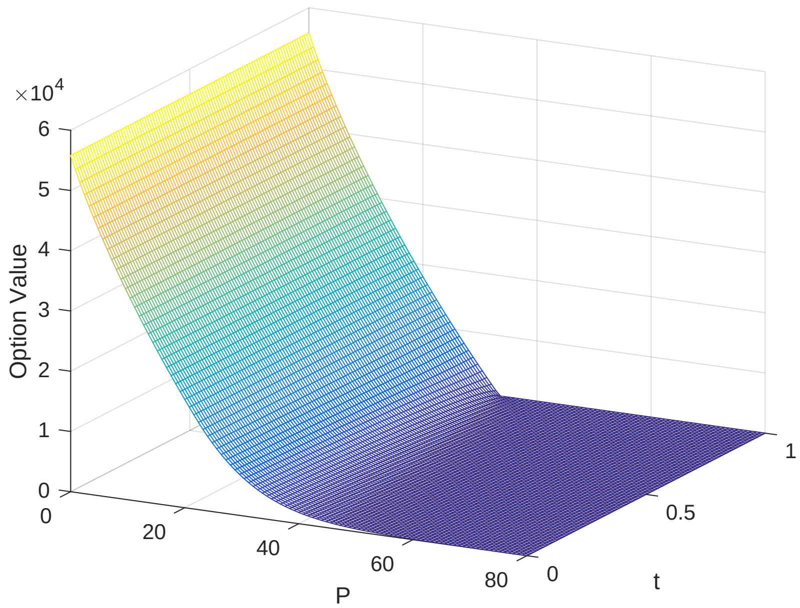


Figure 4.6: $\lambda = 0$, The value of the option to abandon for Test 2.

‘To further demonstrate the optimal exercise curve, we solve the problem using the uniform mesh with $h = 0.32$ and $|\Delta t| = 0.004$ and use the numerical solution to estimate the optimal exercise curve for the American option. From Figure 4.8 we see that Δ is non-smooth across the optimal exercise curve and Γ has a jump when P goes across the optimal exercise curve, as seen in Figure 4.9. Thus, we calculate the pointwise values of Γ and find the coordinates of the largest jump in Γ at each t_k for $P_i < K$ for all feasible i. The results are depicted in Figure 4.10 from which we can see the optimal exercise curve relative to the option value when $\lambda = 0.5$. This is similar to those in standard American puts (see, for example, [39]).

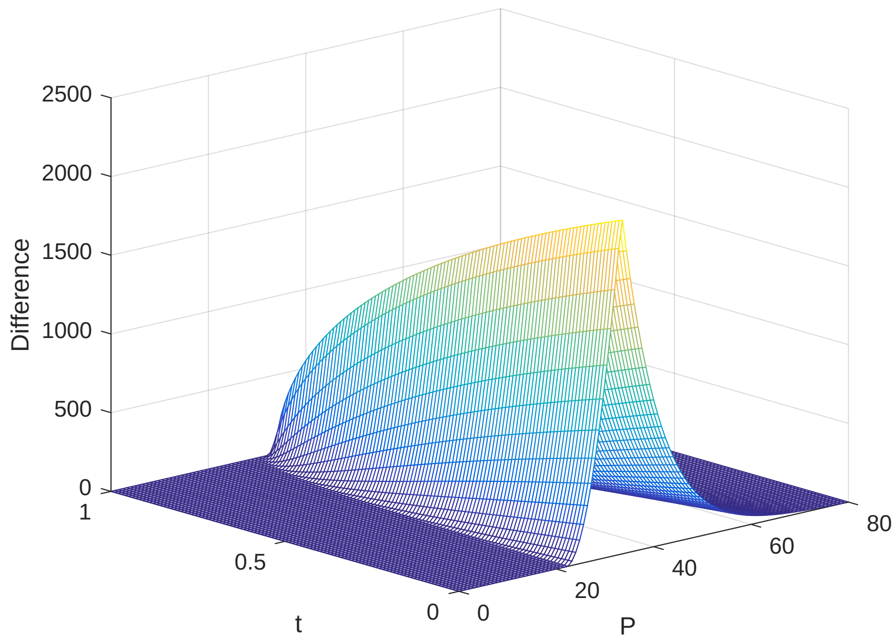


Figure 4.7: Computed $W - W^*$ when $\lambda = 0.5$ for Test 2.

Test 3: Option to Defer

This option gives a company the right to scale up its iron-ore mining operation rate from q_{a0} to q_{a1} deferring to any time within T (American type). The market, production and option data are also listed in Table 4.1 in which $\vartheta = 2$ represents the rate of production expansion, $T = 1$ (year) . We choose $P_{\max} = 60(USD)$ and using the algorithm in (4.5) we find $T^* \approx 75$(years). From the figure we see that the option value W_a as a function of (P, t),and this option can be exercised any

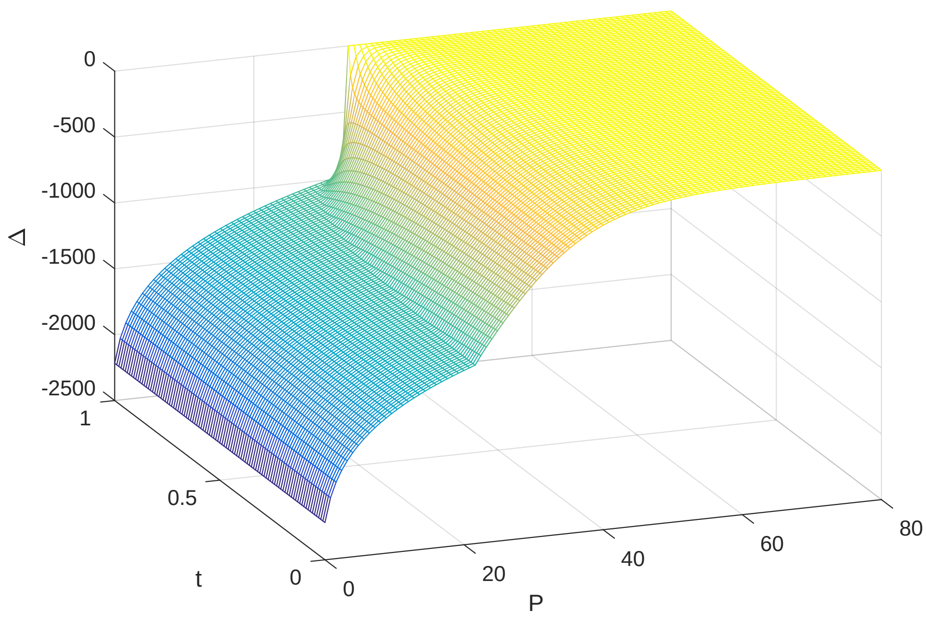


Figure 4.8: Δ of W when $\lambda = 0.5$ for Test 2.

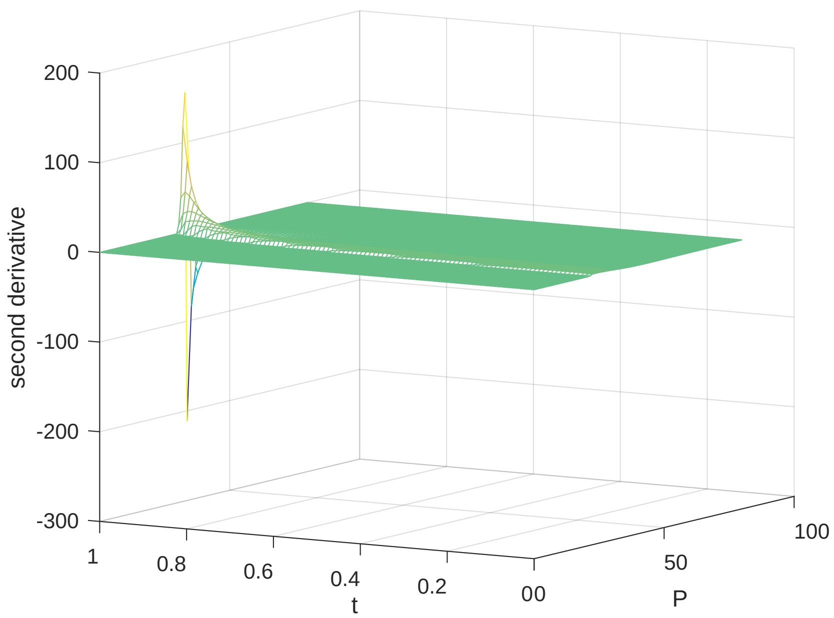


Figure 4.9: Γ of W when $\lambda = 0.5$ for Test 2.

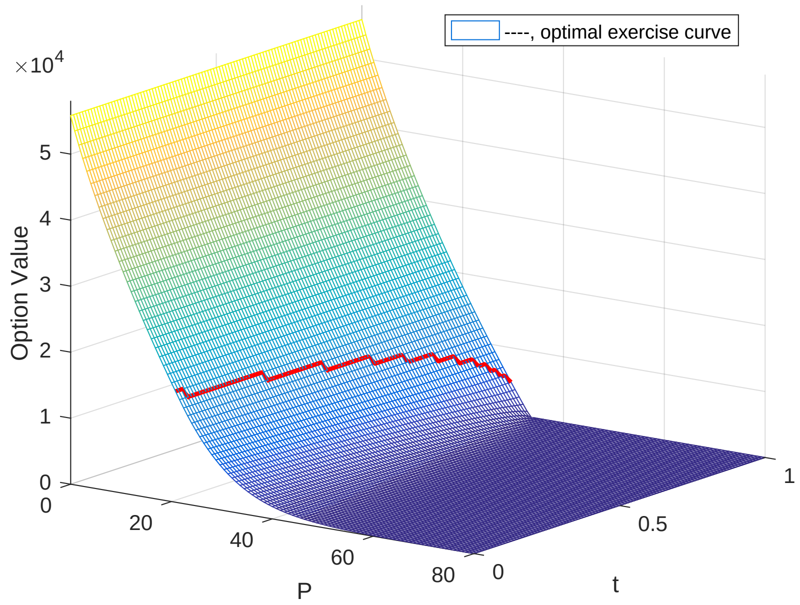


Figure 4.10: The American option value and its optimal exercise curve for $\lambda = 0.5$ for Test 2.

time before or on maturity T behaving like an American-style call option where mangers of the project can exercise at any time and receive the payoff $W_a^*(P)$ (4.19). In this case, the schemes (4.24)–(4.26) and the payoff and boundary condition PDE system (4.16), (4.20) and (4.21)are solved on the uniform mesh with mesh sizes $h = 0.6$ and $\Delta t = 0.01$. The numerical solution of the option to defer for $\vartheta = 2$ is depicted in Figure 4.11.

We calculate Δ and Γ of the value W_a in Test 3 showing in figures 4.12 and 4.13 respectively. From figure 4.14, we have identified the optimal exercise curve for the option to defer deriving from the numerical solution Δ and Γ. The optimal exercise curve determines the optimal early exercise curve of the defer option.

4.6 Concluding Remarks

Based on the Delta-hedging technique used in the Black-Scholes option pricing model, we have developed PDE and LCP for pricing real options of European and American types to contract the production or to fully/partially transfer the ownership of a project whose underlying commodity price is uncertain. Meanwhile in order to determine the pay-off condition for such an option at maturity, a

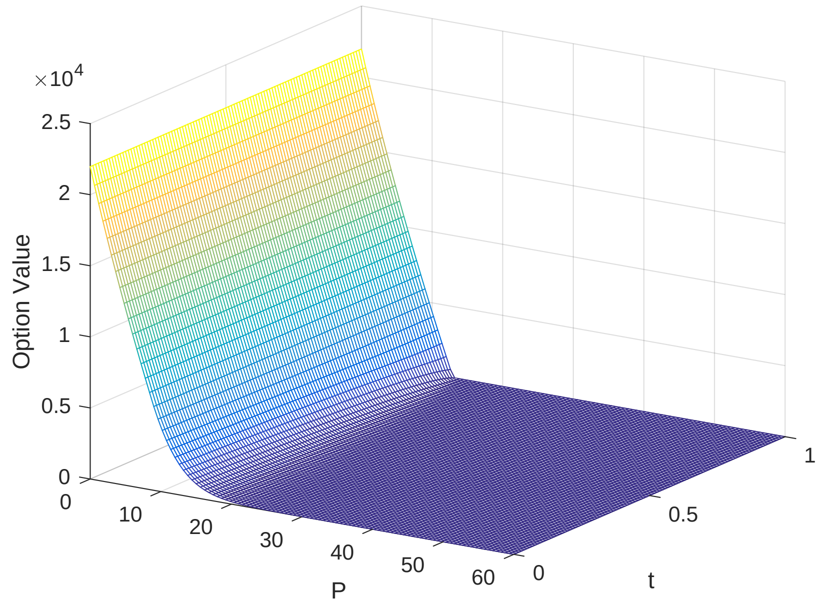


Figure 4.11: $\vartheta = 2$ The value of the option to defer for Test 3.

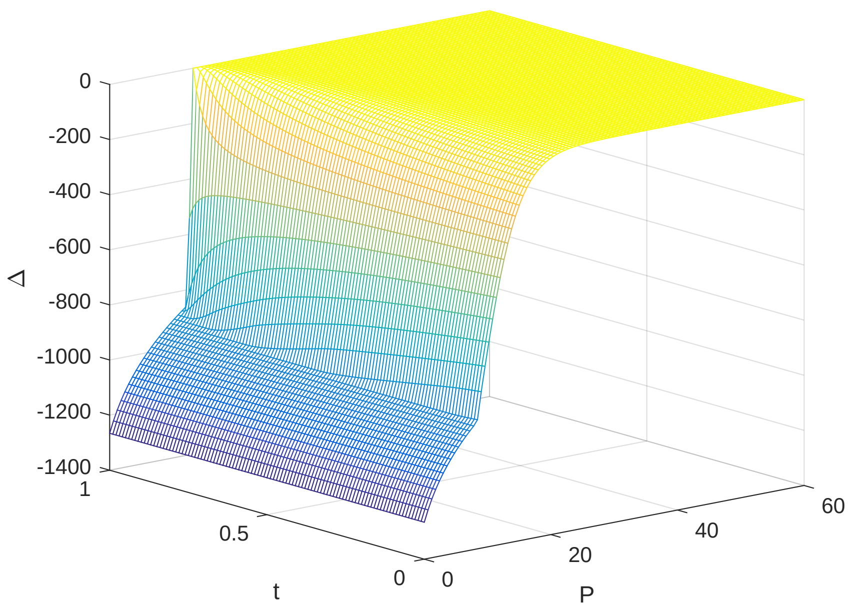


Figure 4.12: Δ of W_a with $\vartheta = 2$ for Test 3.

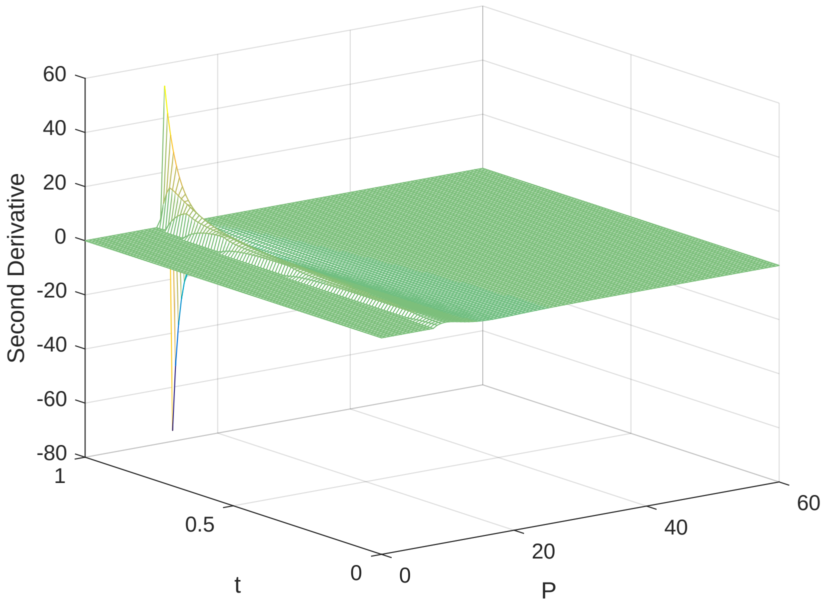


Figure 4.13: Γ of W_a with $\vartheta = 2$ for Test 3.

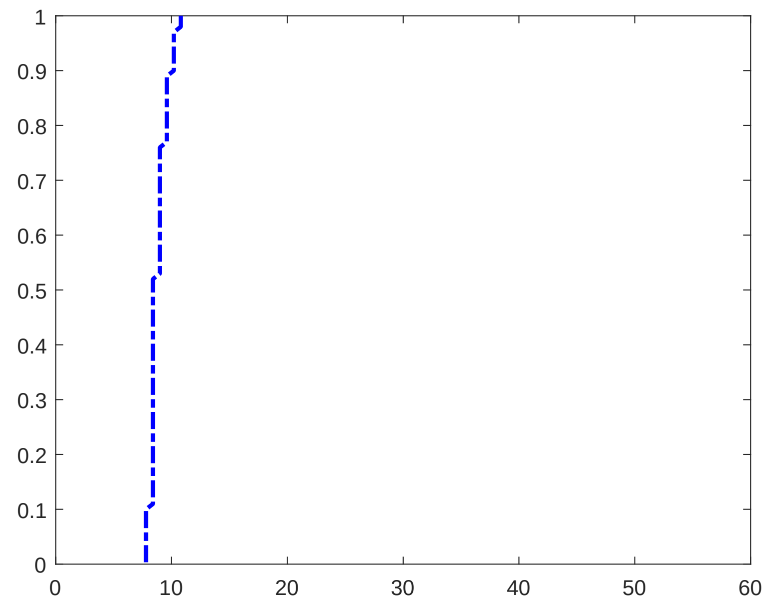


Figure 4.14: The optimal curve of W_a with $\vartheta = 2$ for Test 3.

partial differential system also needs to be solved in the time horizon beyond the maturity. Stable and convergent discretization and optimization methods have also been proposed for the numerical solution of these infinite-dimensional models so that they can be used to price real-world real options. Three non-trivial test examples have been used to demonstrate the effectiveness and usefulness of our models and numerical methods. The numerical experiments show that our methods are able to produce financially meaningful results. More specifically, the numerical results demonstrate that the computed prices of the options are intuitively correct as they are comparable to those from the Black-Scholes call/put option pricing models. Also, the outcome shows that an American type real option is more valuable than its European counterpart.

CHAPTER 5

Pricing Options on Investment Project Expansion under Asset Price Uncertainty with Stochastic Volatility

5.1 General

The research on pricing investment projects is a practical application of the real options theory. In this chapter, we develop a PDE-based mathematical model for valuing real options on the expansion of an investment project whose underlying commodity price and its variance follow their respective geometric Brownian motions(GBMs). This mathematical model is of the form of a 2-dimensional Black-Scholes equation whose payoff condition is determined also by a PDE system. A 9-point finite difference scheme is introduced for the discretization of the spatial derivatives and the fully implicit time-stepping scheme is used for the time discretization of the PDE systems. To demonstrate the usefulness and effectiveness of the mathematical model and numerical method, we present a case study on a real option pricing problem in the iron mining industry. The numerical results show that our model is able to produce numerical results which are financially meaningful.

Risks always accompany investments due to uncertainties in investment conditions, stock/commodity prices and market parameter estimations. Real options may hedge the risks in part or whole in such investment projects. In [4] the author introduces a classification of real options. One type of real option is the 'Growth option' which is also known as expansion option. The real options theory is a very useful method to evaluate capital investments, as studied and discussed in [3, 6, 7, 14, 40–42]. In [44], we have proposed a PDE-based approach to valuing an expansion option of an investment project where only the underlying commodity price follows a GBM, which behaves like a European option. We extend this approach in [36] to pricing American options governing the value of a real option on project contraction or abandonment.

Just like the work [11] by Black and Scholes, The volatility of the underlying stock price was assumed to be constant in the aforementioned papers. However, in practice, the volatility is normally uncertain. In [87], the authors proposed a model examining option pricing with stochastic volatility. Other notable models for pricing conventional options under stochastic volatilities can be found in [30], [21], and [88]. Computational methods have also been developed for solving some of these models; for example, the finite volume methods in [89–91]. However, to our best knowledge, there are very few works on pricing real options with stochastic volatility.

In this chapter, we develop a PDE-based mathematical model for pricing expansion options on an investment project in which underlying commodity price and its variance are both stochastic. The developed pricing model is of the form of a set of Black-Scholes PDEs in two spatial dimensions. We also develop a finite difference method based on that in [92] for solving the two-dimensional partial differential equations which involve mixed 2nd-order derivatives.

The rest of this chapter is organized as follows. In section 5.2, we will derive two-dimensional Black-Scholes equations for the valuation of an option on investment project expansion assuming both the underlying commodity price and its volatility follow their respective geometric Brownian motions. In Section 5.3, we will develop a nine-point finite difference method, along with a time-stepping scheme, for the 2nd-order PDEs model developed in Section 5.2. More specifically, we price expansion options for a mining investment project in which underlying commodity is iron-ore, and demonstrate the numerical results are practically sensible. Numerical experiments on two model problems are carried out in Section 5.4 to demonstrate the correctness and usefulness of the pricing model and the numerical method developed in Sections 5.2 and 5.3.

5.2 Mathmatical Modeling

In this section, we develop a mathematical model in the form of PDEs for valuing project expansion options when both the underlying asset price and its volatility are stochastic. For clarity, we will use an investment project in a natural resource (e.g., iron ore) industry to demonstrate our approach. Clearly, the idea and technique developed for options in an iron-ore industry can also be used for other types of project expansion options.

5.2.1 The Models for Pricing Expansion Options

Consider an investment project in a natural resource industry whose underlying commodity price P and its instantneous variance, $y = \sigma^2$, follow their respective GBMs in time t [34]. In this work we assume that P and y satisfy the following stochastic differential equations respectively [21]:

$$dP = P\mu_p dt + P\sqrt{y}dz_1, \tag{5.1}$$

$$dy = \mu_y y dt + \sigma_y y dz_2, \tag{5.2}$$

for $t \geq 0$, where z_1 and z_2 are two correlated GBM with instantaneous correlation ρ, μ_p and μ_y are constant drift rates for P and y respectively, and σ_y is a constant volatility of the volatility $\sqrt{y}$.

Following [44], we now consider the project in the following two althernative cases:

C0. The project does not have any expansion options.

C1. There is an embedded production expansion option in the project that is exercisable at a fixed time $T > 0$ (expiry date) and requires capital $K > 0$ (strike price) for the expansion.

We denote the value functions of the project in the above two cases as $V_0(P, y, t)$ and $V_1(P, y, t)$ respectively. Clearly, $V_1(P, y, t) \geq V_0(P, y, t)$ for all $t \geq 0$ and $P \geq 0$. This is because the expansion option in Case C1 is 'value adding', and $V_1 = V_0$ trivially holds if the option is not exercised. Therefore, the value of the option, denoted as $W(P, y, t)$, is the difference between V_1 and V_0, i.e., $W(P, y, t) = V_1(P, y, t) - V_0(P, y, t)$ for $P \geq 0$, $y \geq 0$ and $t \in [0, T)$.

To determine $V(P, y, t)$ let $q_k(t)$ be the production rates associated with Cases C0 and C1 respectively. Then the after-tax cash flow rates [44] for C0 and C1 are respectively given by

$$D_k(P, t) := q_k(t)[P(t)(1 - R) - C(t)](1 - B), \quad k = 0, 1 \ , \tag{5.3}$$

where $C(t)$ is the average cash cost rate of production per unit output, R is the rate of state royalties and B is the income tax rate.

Using Ito's lemma, we have the following instantaneous change dV_k in the value of the project k

$$\begin{aligned} dV_k =& \frac{\partial V_k}{\partial P} dP + \frac{\partial V_k}{\partial t} dt + \frac{\partial V_k}{\partial y} dy + \frac{1}{2}(dP)^2 \frac{\partial^2 V_k}{\partial P^2} + \frac{1}{2}(dy)^2 \frac{\partial^2 V_k}{\partial y^2} + dydt \frac{\partial^2 V_k}{\partial y \partial t} + dPdy \frac{\partial^2 V_k}{\partial P \partial y} \\ =& \left(\mu_p P \frac{\partial V_k}{\partial P} + \frac{\partial V_k}{\partial t} + \mu_y y \frac{\partial V_k}{\partial y} + \frac{1}{2}(P^2 y \frac{\partial^2 V_k}{\partial P^2} + \sigma^2 y^2 \frac{\partial^2 V_k}{\partial y^2} + 2\rho\sigma P y^{\frac{3}{2}} \frac{\partial^2 V_k}{\partial P \partial y}) \right) dt \\ &+ P\sqrt{y} \frac{\partial V_k}{\partial P} dz_1 + \sigma y \frac{\partial V_k}{\partial y} dz_2. \end{aligned} \tag{5.4}$$

where ρ is the instantaneous correlation between P and y.

We now consider constructing a suitable portfolio to hedge the risks in V_k for $k = 0, 1$. For clarity, we let $\hat{V}_1 = V_k$ and rewrite (5.4) as the following form:

$$d\hat{V}_1 = \hat{\mu}_1 \hat{V}_1 dt + \hat{\sigma}_{1,1} \hat{V}_1 dz_1 + \hat{\sigma}_{1,2} \hat{V}_1 dz_2, \tag{5.5}$$

where $\hat{\mu}_{1,1}$, $\hat{\sigma}_{1,1}$ and $\hat{\sigma}_{1,2}$ satisfy

$$\begin{cases} \hat{\mu}_1 \hat{V}_1 = \mu_p P \frac{\partial V_k}{\partial P} + \frac{\partial V_k}{\partial t} + \mu_y y \frac{\partial V_k}{\partial y} + \frac{1}{2}(P^2 y \frac{\partial^2 V_k}{\partial P^2} + \sigma_y^2 y^2 \frac{\partial^2 V_k}{\partial y^2} + 2\rho\sigma_y P y^{\frac{3}{2}} \frac{\partial^2 V_k}{\partial P \partial y}), \\ \hat{\sigma}_{1,1} \hat{V}_1 = P\sqrt{y} \tfrac{\partial V_k}{\partial P}, \\ \hat{\sigma}_{1,2} \hat{V}_1 = \sigma_y y \tfrac{\partial V_k}{\partial y}. \end{cases} \tag{5.6}$$

Unlike the case in [44], we need to construct a portfolio of 3 assets whose prices depend on P and y to hedge two risk sources dz_1 and dz_2. Now, we choose a portfolio

$$\Pi_k = \gamma_1 \hat{V}_1 + \gamma_2 \hat{V}_2 + \gamma_3 \hat{V}_3, \tag{5.7}$$

where recall that $\hat{V}_1$ is defined as the project values, $\hat{V}_2$ and $\hat{V}_3$ are two securities dependent on P and y, and γ_i, $(i = 1, 2, 3)$ are positive weights of the portfolio. Using Ito's lemma, both $\hat{V}_2$ and $\hat{V}_3$ should also satisfy an relation similar to (5.5), i.e.,

$$d\hat{V}_i = \hat{\mu}_i \hat{V}_i dt + \hat{\sigma}_{i,1} \hat{V}_i dz_1 + \hat{\sigma}_{i,2} \hat{V}_i dz_2 \tag{5.8}$$

for $i = 2, 3$, where $\hat{\mu}_i, \hat{\sigma}_{i,1}$ and $\hat{\sigma}_{i,2}$ have the same forms as those in (5.6) .

When dt is sufficiently small, the portfolio increment $d\Pi_k$ is a combination of the changes in $\hat{V}_i$, $(i = 1, 2, 3)$ and the after-tax cash flow rate D (recall the index k is omitted) for the project $\hat{V}_1$ in dt defined in (5.3). More specifically, using

(5.5) and (5.8), we have

$$d\Pi = \left(\sum_{i=1}^{3} \gamma_i \hat{\mu}_i \hat{V}_i\right) dt + \left(\sum_{i=1}^{3} \gamma_i \hat{\sigma}_{i,1} \hat{V}_i\right) dz_1 + \left(\sum_{i=1}^{3} \gamma_i \hat{\sigma}_{i,2} \hat{V}_i\right) dz_2 + \gamma_1 D dt. \quad (5.9)$$

Since Π is instantaneously riskless, we have

$$d\Pi = r\Pi dt = r \sum_{i=1}^{3} \gamma_i \hat{V}_i dt. \quad (5.10)$$

Thus, comparing this equality with (5.9) we obtain

$$\begin{aligned} \gamma_1 \Big((\hat{\mu}_1 - r)\hat{V}_1 + D\Big) + \gamma_2(\hat{\mu}_2 - r)\hat{V}_2 + \gamma_3(\hat{\mu}_3 - r)\hat{V}_3 &= 0, \\ \gamma_1 \hat{\sigma}_{11} \hat{V}_1 + \gamma_2 \hat{\sigma}_{21} \hat{V}_2 + \gamma_3 \hat{\sigma}_{31} \hat{V}_3 &= 0, \\ \gamma_1 \hat{\sigma}_{12} \hat{V}_1 + \gamma_2 \hat{\sigma}_{22} \hat{V}_2 + \gamma_3 \hat{\sigma}_{32} \hat{V}_3 &= 0. \end{aligned}$$

The above equations can be regarded as a linear homogeneous system in γ_1, γ_2 and γ_3. Since $\gamma_i > 0$ for $i = 1, 2, 3$, the above homogeneous system has non-trivial solutions, and thus the coefficient matrix of this system is singular. Therefore, there exist λ_1 and λ_2 such that

$$(\hat{\mu}_1 - r)\hat{V}_1 + D = \lambda_1 \hat{\sigma}_{11} \hat{V}_1 + \lambda_2 \hat{\sigma}_{12} \hat{V}_1, \quad (5.11)$$

$$(\hat{\mu}_2 - r)\hat{V}_2 = \lambda_1 \hat{\sigma}_{21} \hat{V}_2 + \lambda_2 \hat{\sigma}_{22} \hat{V}_2, \quad (5.12)$$

$$(\hat{\mu}_3 - r)\hat{V}_3 = \lambda_1 \hat{\sigma}_{31} \hat{V}_3 + \lambda_2 \hat{\sigma}_{32} \hat{V}_3. \quad (5.13)$$

In fact, λ_1 and λ_2 are called the market prices of risks of P and y respectively. It is also worth noting that the above analysis follows the same line as in [61]. Combining (5.6) and (5.11) and rearranging the resulting equation, we have

$$\begin{aligned} \frac{\partial V_k}{\partial t} + \frac{1}{2}\left(P^2 y \frac{\partial^2 V_k}{\partial P^2} + 2\rho\sigma_y P y^{\frac{3}{2}} \frac{\partial^2 V_k}{\partial P \partial y} + \sigma_y^2 y^2 \frac{\partial^2 V_k}{\partial y^2}\right) \\ + (\mu_p - \lambda_1\sqrt{y}) P \frac{\partial V_k}{\partial P} + (\mu_y - \lambda_2 \sigma_y) y \frac{\partial V_k}{\partial y} - rV_k + D_k = 0. \quad (5.14) \end{aligned}$$

The above pricing equation holds for any security whose price depends on P and y and follows (5.8). One particular security is the underlying commodity itself whose price P satisfies (5.1). Thus, from (5.1) and (5.12) we have

$$\mu_p - r = \lambda_1 \sqrt{y}. \quad (5.15)$$

(This also means in our portfolio Π, we use P as one of $\hat{V}_2$ and $\hat{V}_3$.) Therefore, replacing λ_1 in (5.14) with the one obtained from (5.15) and omitting the subscript in λ_2, we finally have

$$\frac{\partial V_k}{\partial t}+\frac{1}{2}\left(P^2 y\frac{\partial^2 V_k}{\partial P^2}+2\rho\sigma_y P y^{\frac{3}{2}}\frac{\partial^2 V_k}{\partial P\partial y}+\sigma_y^2 y^2\frac{\partial^2 V_k}{\partial y^2}\right)$$
$$+rP\frac{\partial V_k}{\partial P}+(\mu_y-\lambda\sigma_y)y\frac{\partial V_k}{\partial y}-rV_k+D_k=0. \tag{5.16}$$

The above equation is true for both $k = 0$ and 1, i.e., values of the project in both Cases C0 and C1 governed by (5.16). The value W of the option is the premium of the projet with the flexibility in Case C1 over that in Case C0, taking into account of K which are payable only at the terminal time T. Therefore, taking both sides of the equation for V_1 away from the corresponding sides of that for V_0 and rearranging the resulting equation, we have the following PDE for in W:

$$\mathcal{L}W = f(P,t), \tag{5.17}$$

for $P > 0$, $y > 0$ and $t \in (0,T)$, where

$$\mathcal{L} := -\frac{\partial}{\partial t}-\frac{1}{2}\left(P^2 y\frac{\partial^2}{\partial P^2}+2P\rho\sigma_y y^{\frac{3}{2}}\frac{\partial^2}{\partial P\partial y}+\sigma_y^2 y^2\frac{\partial^2}{\partial y^2}\right)-rP\frac{\partial}{\partial P}-(\mu_y-\lambda\sigma_y)y\frac{\partial}{\partial y}+r,$$

and $f(P,t) = D_0(P,t) - D_1(P,t)$.

For an expansion option, we usually have $f(P,t) = 0$ for $t \in [0,T)$ since the change of operating rate starts from $t = T$. However, we still leave the above equation in its general form.

Note that (5.17) is defined on an infinite domain for (P,y,t). In computation, we need to truncate it and solve (5.17) over a finite domain and define boundary and payoff conditions on the boundary of the solution domain. To achieve this, we let chose positive constant $P_{\max}$, η and Y satisfying $P_{\max} \gg 0$ and $\eta < y < Y$ and consider the following problem .

$$\begin{cases}\mathcal{L}W = f(P,t), \quad (P,y,t)\in(0,P_{\max})\times(\eta,Y)\times[0,T),\\ W(0,y,t)=G_1(y,t),\ W(P_{\max},y,t)=G_2(y,t), \quad (y,t)\in(\eta,Y)\times[0,T),\\ W(P,\eta,t)=G_3(P,t), \quad W(P,Y,t)=G_4(P,t), \quad (P,t)\in(0,P_{\max})\times[0,T),\\ W(P,y,T)=G_5(P,y), \quad (P,y)\in(0,P_{\max})\times(\eta,Y),\end{cases} \tag{5.18}$$

where G_j, $j = 1,2...,5$ are to be determined.

Below we will discuss the determination of G_i, $i = 1, 2, ..., 5$ one by one. Before this discussion, we first introduce the lifespans of the project. Let Q denote the reserve of the resource and T_k^* be life time of the project when the production rate is q_k for $k = 0, 1$. In [44], we propose to estimate T_k^* by following equation:

$$Q = \int_0^{T_k^*} q_k(t)dt, \quad k = 0, 1.$$

We then define $T^* = \max\{T_0^*, T_1^*\}$ and T^* is the life time of the project for both Cases C0 and C1. Note that for this production expansion option, we have

$$\begin{cases} q_0(t) = q_1(t), \ t \in [0, T), \\ q_0(t) < q_1(t), \ t \in [T, T_1^*), \\ q_1(t) = 0, \ t \in [T_1^*, T^*]. \end{cases} \tag{5.19}$$

5.2.2 Determination of the Boundary Conditions G_i's

Following the cash flow argument in [44], we define the boundary conditions G_1 and G_2 in (5.18) as follows.

$$G_1(y, t) = 0 \text{ and } G_2(y, t) = \bar{V}_1(P_{\max}, t) - \bar{V}_0(P_{\max}, t) - Ke^{-r(T-t)},$$

for $(y, t) \in (\eta, Y) \times [0, T)$, where $\bar{V}_k(z, t)$ is calculated using the following net present value formulate:

$$\bar{V}_k(z, t) = \int_t^{T^*} [D_1(z, \tau) - D_0(z, \tau)]e^{-r(T-\tau)}dt, \tag{5.20}$$

for $z = 0$ and $P_{\max}$. $t \in [0, T^*]$,and $k = 0$ and 1.

It is not possible to explicitly define $G_3(P, t)$ and $G_4(P, t)$ in (5.18). Following [44], we define the boundary conditions using the solutions to the following 1D Black-Scholes equation and boundary conditions.

$$\begin{cases} \dfrac{\partial G_i}{\partial t} + rP\dfrac{\partial G_i}{\partial P} + \dfrac{1}{2}P^2\omega_i\dfrac{\partial^2 G_i}{\partial P^2} - rG_i = f(P, t) \ (P, t) \in (0, P_{\max}) \times [0, T) \\ G_i(0, t) = 0, \ G_i(P_{\max}, t) = \bar{V}_1(P_{\max}, t) - \bar{V}_0(P_{\max}, t) - Ke^{-r(T-t)}, \ t \in [0, T), \\ G_i(P, T) = [\hat{G}_i(P, T) - K]^+, \ P \in (0, P_{\max}), \end{cases} \tag{5.21}$$

for $i = 3, 4$, where $\omega_3 = \eta$ and $\omega_4 = Y$, $\bar{V}_j(P_{\max}, t)$, $j = 0, 1$ are defined in (5.20),

$[z]^+ = \max\{0, z\}$ for any z and $\hat{G}_i(P, T)$, for $i = 3, 4$ and $P \in (0, P_{\max})$, will be determined later.

Finally, G_5 in (5.18) is defined as follows:

$$G_5(P, y, T) = [\hat{W}(P, y, T) - K]^+, \ (P, y) \in (0, P_{\max}) \times (\eta, Y), \tag{5.22}$$

where $\hat{W}(P, y, T)$ is to be determined in the next subsection.

5.2.3 Determination of the Payoff Conditions $\hat{G}$ and $\hat{W}$

We first determine the payoff conditions used in (5.21) and (5.22). Follwing [44], $\hat{G}$ is defined as follows.

$$\begin{cases} \dfrac{\partial \hat{G}_i}{\partial t} + rP\dfrac{\partial \hat{G}_i}{\partial P} + \dfrac{1}{2}P^2\omega_i\dfrac{\partial^2 \hat{G}_i}{\partial P^2} - r\hat{G}_i = f(P, t), \ (P, t) \in (0, P_{\max}) \times [T, T^*), \\ \hat{G}_i(0, t) = \bar{V}_1(0, t) - \bar{V}_0(0, t), \ t \in [T, T^*), \\ \hat{G}_i(P_{\max}, t) = \bar{V}_1(P_{\max}, t) - \bar{V}_0(P_{\max}, t), \ t \in [T, T^*), \\ \hat{G}_i(P, T^*) = 0, \ P \in (0, P_{\max}), \end{cases} \tag{5.23}$$

for $i = 3, 4$, where $\bar{V}_k$, $k = 1, 2$, are defined in (5.20).

We now define $\hat{W}$ based on the ideas:

1) when $t = T^*$, the values of the project under both conditions C0 and C1 are zero;

2) the values of the project at time t in both cases C0 and C1 equal the present value of the cash flows generated by the project when $P = 0$ and $P = P_{\max}$ with $P_{\max} \gg 0$. Thus, we define the following problem governing $\hat{W}$:

$$\begin{cases} \mathcal{L}\hat{W} = f(P, t), \ (P, t) \in (0, P_{\max}) \times (\eta, Y) \times [T, T^*), \\ \hat{W}(z, y, t) = \bar{V}_1(z, t) - \bar{V}_0(z, t) \text{ for } z = 0, P_{\max}, \ (y, t) \in (\eta, Y) \times [T, T^*), \\ \hat{W}(P, \eta, t) = \hat{G}_3(P, t), \ \hat{W}(P, Y, t) = \hat{G}_4(P, t), \ (P, t) \in (0, P_{\max}) \times [T, T^*), \\ \hat{W}(P, y, T^*) = 0, \ (P, y) \in (0, P_{\max}) \times (\eta, Y), \end{cases} \tag{5.24}$$

where $\bar{V}_k$ is defined in (5.20) for $k = 0, 1$, and $\hat{G}_i$ is the solution to (5.23) for $i = 3$ and 4.

5.3 Discretization

In this section, we propose a finite difference method for the numerical solution of (5.18). We start this discussion by defining a uniform mesh for $(0, P_{\max}) \times (\eta, Y) \times [0, T)$ as follows. For any positive integers N_P, N_y and N_t, let $\Delta P = 1/N_P$, $\Delta y = 1/N_y$ and $\Delta t = -1/N_t$. We then choose the following sets of points on P, y and t axes

$$P_m = m\Delta P,\ y_n = \eta + n\Delta y,\ t_k = T + k\Delta t,$$

for $m = 0, .., N_P$, $n = 0, 1, ..., N_y$ and $k = 0, 1, ..., N_t$. Clearly, (P_m, y_n, t_k), $m = 0, .., N_P$, $n = 0, 1, ..., N_y$ and $k = 0, 1, ..., N_t$, form a partition on $(0, P_{\max}) \times (\eta, Y) \times [0, T)$.

We now construct a finite difference scheme for (5.18) on the mesh constructed above. On a typical local spatial 9-point stencil shown in 5.1 and a timestep k, we define the following finite difference approximation to (5.18):

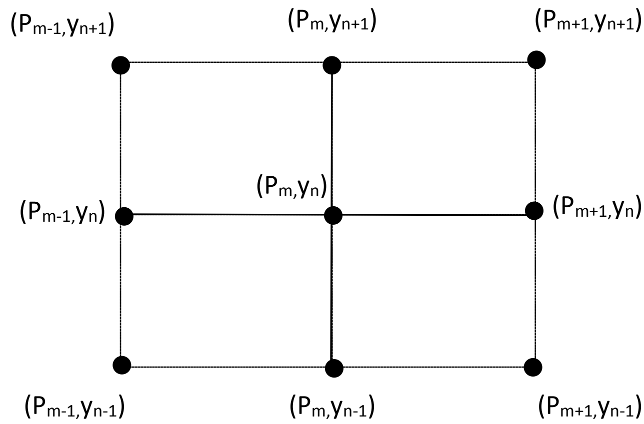


Figure 5.1: A local nine-point stencil.

$$\begin{aligned}
\mathcal{L}W \approx & -a_{1,m,n}\left(W^{k+1}_{m+1,n} - 2W^{k+1}_{m,n} + W^{k+1}_{m-1,n}\right) \\
& - a_{2,m,n}\left(W^{k+1}_{m,n+1} - 2W^{k+1}_{m,n} + W^{k+1}_{m,n-1}\right) \\
& + a_{3,m,n}\left(W^{k+1}_{m+1,n+1} + W^{k+1}_{m-1,n-1} - W^{k+1}_{m+1,n-1} - W^{k+1}_{m-1,n+1}\right) \\
& - a_{4,m,n}\left(W^{k+1}_{m+1,n} - W^{k+1}_{m,n}\right) - a_{5,m,n}\left(W^{k+1}_{m,n+1} - W^{k+1}_{m,n}\right) \\
& + a_{6,m,n}W^{k+1}_{m,n} + \frac{1}{|\Delta t|}\left(W^{k+1}_{m,n} - W^{k}_{m,n}\right), \\
& = f^{k+1}_{m,n}
\end{aligned} \tag{5.25}$$

for $m = 1, 2, ..., N_P - 1$, $n = 1, 2, ..., N_y - 1$ and $k = 0, 1, ..., N_k - 1$, where $W^{k+1}_{m,n}$ denotes an approximation to $W(P_m, y_n, t_k)$, $f^{k+1}_{m,n} = D_0(P_m, t^{k+1}) - D_1(P_m, t^{k+1})$

and

$$a_{1,m,n} = \frac{P_m^2 y_n}{2\Delta P^2}, \quad a_{2,m,n} = \frac{y_n^2 \sigma_y^2}{2\Delta y^2}, \quad a_{3,m,n} = \frac{P_m \sigma_y y_n^{1.5} \rho}{4\Delta P \Delta y}, \tag{5.26}$$

$$a_{4,m,n} = \frac{rP_m}{\Delta P}, \quad a_{5,m,n} = \frac{(\mu_y - \lambda_2 \sigma_y) y_n}{\Delta y}, \quad a_{6,m,n} = r. \tag{5.27}$$

(Recall $\Delta t < 0$.)

In (5.25), we used the backward finite differences to approximate $\frac{\partial W}{\partial t}$, $\frac{\partial W}{\partial P}$ and $\frac{\partial W}{\partial y}$, the central finite differences to approximate $\frac{\partial^2 W}{\partial P^2}$, $\frac{\partial^2 W}{\partial y^2}$, and a difference scheme based on the 9-points to approximate $\frac{\partial^2 W}{\partial P \partial y}$, as described in [92]. We now rewrite (5.25) as a matrix. To achieve this, we introduce an index $i := m + (n-1)(N_P - 1)$ for $m = 1, 2, ..., N_P - 1$ and $n = 1, 2, ..., N_y - 1$. This transformation maps the double index (m, n) in space into a single index $i = 1, 2, ..., (N_P - 1) \times (N_y - 1) =: M$. Under this transformation, we rewrite the sequences $W_{m,n}^k$ and $f_{m,n}^k$ for $m = 1, 2, ..., N_P - 1$ and $n = 1, 2, ..., N_y - 1$ as column vectors $(\bar{W}_1^k, \bar{W}_2^k, ..., \bar{W}_M^k)^\top$ and $(\bar{f}_1^k, \bar{f}_2^k, ..., \bar{f}_M^k)^\top$ respectively for any feasible k. Thus, rearranging (5.25) and using this notation, we have the following system:

$$\begin{aligned} & e_{i,i-N_P} \bar{W}_{i-N_p}^{k+1} + e_{i,i-N_P+1} \bar{W}_{i-N_P+1}^{k+1} + e_{i,i-N_P+2} \bar{W}_{i-N_P+2}^{k+1} + e_{i,i-1} \bar{W}_{i-1}^{k+1} \\ & \quad + \left(e_{i,i} + \frac{1}{|\Delta t|} \right) \bar{W}_i^{k+1} + e_{i,i+1} \bar{W}_{i+1}^{k+1} + e_{i,i+N_P-2} \bar{W}_{i+N_P-2}^{k+1} \\ & \quad + e_{i,i+N_P-1} \bar{W}_{i+N_P-1}^{k+1} + e_{i,i+N_P} \bar{W}_{i+N_P}^{k+1} = \frac{1}{|\Delta t|} \bar{W}_{i,i}^k + \bar{f}_i^{k+1}, \end{aligned} \tag{5.28}$$

for $i = 1, 2, ..., M$, where

$$e_{i,i-N_P} = a_{3,m,n}, \; e_{i,i-N_P+1} = -a_{2,m,n}, \; e_{i,i-N_P+2} = -a_{3,m,n}, \; e_{i,i-1} = -a_{1,m,n}, \tag{5.29}$$

$$e_{i,i} = 2a_{1,m,n} + 2a_{2,m,n} + a_{4,m,n} + a_{5,m,n} + a_{6,m,n}, \tag{5.30}$$

$$e_{i,i+1} = -a_{1,m,n}, \; e_{i,i+N_P-2} = -a_{3,m,n}, \; e_{i,i+N_P-1} = -a_{2,m,n}, \; e_{i,i+N_P} = a_{3,m,n}, \tag{5.31}$$

with $a_{l,m,n}$ for $l = 0, 1, ..., 6$ defined in (5.26)-(5.27).

Let $\bar{\boldsymbol{W}}^l := (\bar{W}_1^l, \bar{W}_2^l, ..., \bar{W}_M^l)^\top$ for $l = k$ and $k+1$, and $\bar{\boldsymbol{f}}^{k+1} := (\bar{f}_1^{k+1}, \bar{f}_2^{k+1}, ..., \bar{f}_M^{k+1})^\top$. Then (5.28) can be written as the following matrix form:

$$\left(E + \frac{1}{|\Delta t|} I \right) \bar{\boldsymbol{W}}^{k+1} = \frac{1}{|\Delta t|} \bar{\boldsymbol{W}}^k + \bar{\boldsymbol{f}}^{k+1} + \boldsymbol{b}^{k+1}, \tag{5.32}$$

for $k = 0, 1, ..., N_t - 1$, where I denotes the $M \times M$ identity matrix, $\boldsymbol{b}^{k+1}$ denotes

$Q = 10^4$ million tons	$B = 30\%$ per annum
$C_0 =$ US\$35	$C(t) = C_0 \times e^{0.005t}$
$R = 5\%$ per annum	$r = 6\%$ per annum
$K =$ US\$$10^4$ million	$T = 2$ years
$\eta = 0.0001$	$Y = 1$
$\mu_y = 0.1$	$\sigma_y = 10\%$
$\lambda = 0.1$	$\rho = 0.1$
$q_0 = 0.01Q \times e^{0.007t}$	$q_1 = \begin{cases} q_0 & t < T \\ \kappa \times q_0 & t \geq T \end{cases}$

Table 5.1: Project and market data used in Test 1.

the contribution of the boundary conditions G_1, G_2, G_3 and G_4 in (5.18) and E is the $M \times M$ sparse matrix with non-zero entries defined in (5.29)–(5.31). The initial condition for We comment that though the scheme (5.32) is developed for (5.18), it also applies to the PDE system (5.24), the both involve the same differential operator $\mathcal{L}$. The discretization of the 1-dimensional PDE systems (5.21) and (5.23) has been discussed before in our previous works [44] and thus we omit this discussion.

We also comment that from its definition it is easy to see that the E on the left-hand side of (5.32) is not an M-matrix, though the system matrix $E + \frac{1}{|\Delta t|}I$ of (5.32) is a positive-definite matrix when $|\Delta t|$ is sufficiently small. This is because $\frac{1}{|\Delta t|}I$ is a diagonal matrix with sufficiently large diagonal entries when $|\Delta t|$ is sufficiently small. As mentioned in [92], it is not possible to derive a finite difference scheme for (5.18), so that resulting coefficient matrix is an M-matrix due to the mixed term $\frac{\partial^2 W}{\partial P \partial y}$. However, the application of a proper finite volume scheme to (5.18) may result in a linear system with an M system matrix [90,91]. The discussion and implementation of such a finite volume scheme will be much more complex than the current one.

5.4 Numerical Experiments

We now use an iron-ore project as our test problem to demonstrate the effectiveness and benefits of the models and numerical method developed in the previous sections.

Our example is to estimate the value of an expanding option within an investment project in the iron-ore industry. All data for the test is given in Table 5.1 in which $\kappa > 1$ is a parameter representing the ratio of the production rates. In the following numerical experiments, we choose $\kappa = 2$.

To price the option, we choose $P_{\max} = 80$ (USD). Using the algorithm in [44], we find $T^* \approx 75$ (years). To approximate (5.18) by (5.32), we choose a uniform mesh for $(0, P_{\max}) \times (\eta, Y) \times [0, T)$ with $N_P = 40$, $N_y = 40$ and $N_t = 40$. Note that solving (5.18) requires the numerical solutions to (5.21), (5.23) and (5.24), which are obtained using the finite difference/volume methods in Section 5.3 and [44] on appropriate uniform meshes.

Test 1: The Computed Option Values

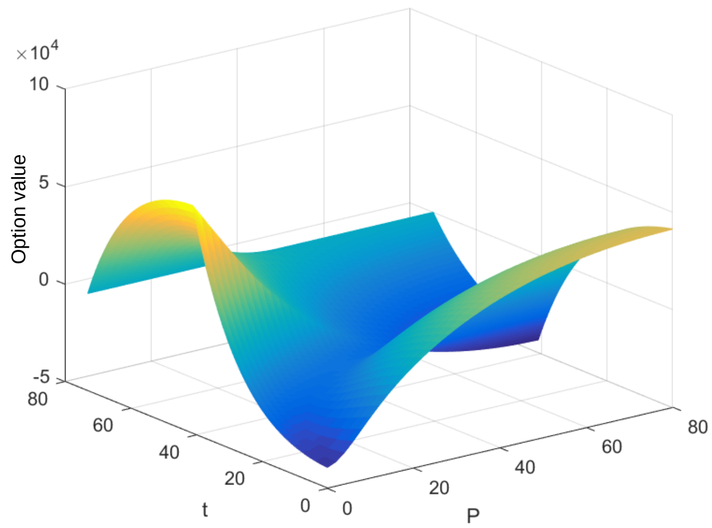
We now look into the numerical solution to this option valuation problem. In order to solve the payoff condition of such a real option W at $t = T$, we need to solve two models with $y = \eta, Y$. The models are of the form of the Black-Scholes model for pricing conventional European options with a vanilla payoff function defined by (5.21) [44] where $P \in (0, P_{max})$, $t \in (T, T^*)$ shown in figures 5.2.

Then, the pay-off condition G_5 at $t = T$ for $y \in (\eta, Y)$ and $P \in (0, P_{\max})$ is computed and depicted in Figure 5.4. The computed boundary conditions G_3 and G_4 are plotted in Figure 5.6. For comparison, we also compute the cross-section of the value W of the option at $t = 0, 0.5, 1, 1.5$ and plot the result in Figure 5.3, in which the option price is in a million USD with a product expanding rate $\kappa = 2$.

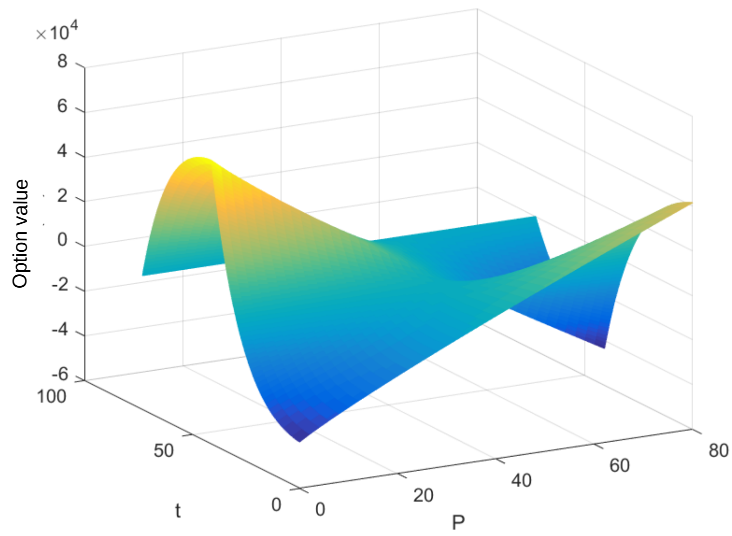
In addition, in Figure 5.5 we test the how time t influences the option value W under a certain $y = 0.25$ at $t = 0$. Figure 5.5 shows that the more time closes to the expiry time by comparing $t = 0, T$, the higher option value is realized when underlying commodity price P below approximate 68(USD). But when P becomes larger than 68(USD), the situation is the opposite. And the price P is approximate 66(USD), when we compare the option values at $t = 0, 0.5$.

Test 2: The Comparison of the Numerical Results Between W_{2D} and W_{1D}

We now compare the results from our above stochastic volatility 2D model and the non-stochastic 1D model in [44]. For ease of discussion, we denote the numerical solutions to these 2D and 1D models by $W_{2D}(P, y, t)$ and $W_{1D}(P, t; \sigma)$ respectively, where σ denotes value of the volatility used in the 1D model. By definition, $y = \sigma^2$. The numerical solutions to (5.18) at the cross-section $y = 0.25$, $W_{2D}(P, 0.25, t)$ and $W_{1D}(P, t; 0.5)$ are displayed in Figure 5.7(a) and (b). As can be seen, the two solutions are qualitatively similar to each other, showing the

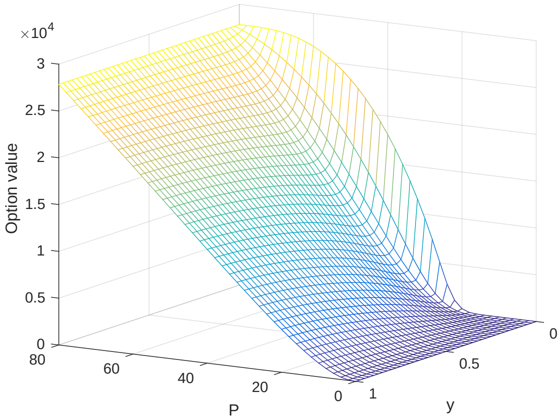


(a)$y = \eta$

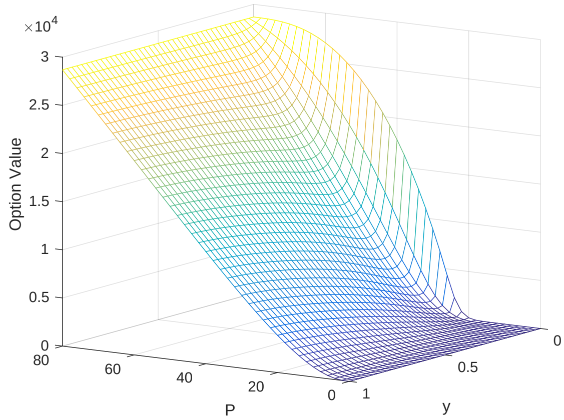


(b) $y = Y$

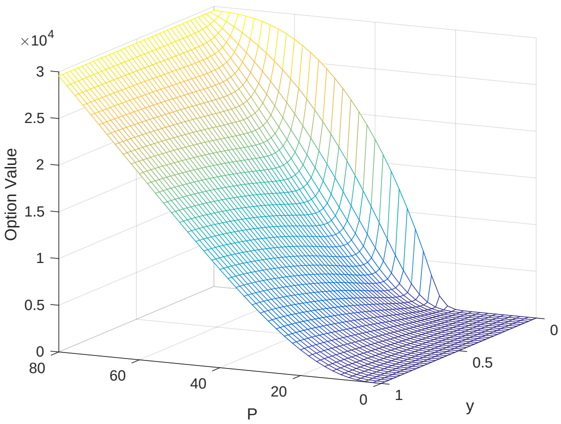
Figure 5.2: The condition of option value at $y = \eta$ where $P \in (0, P_{\max})$, $t \in (T, T^{*})$



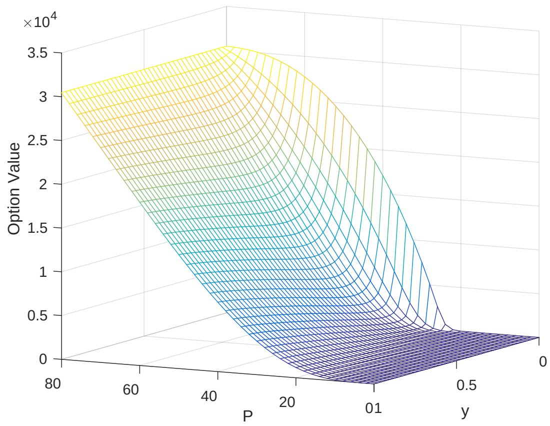
(a)$t = 0$



(b) $t = 0.5$



(c)$t = 1$



(d)$t = 1.5$

Figure 5.3: The computed option value at $t = 0, 0.5, 1, 1.5$ with $\kappa = 2$ for Test 1.

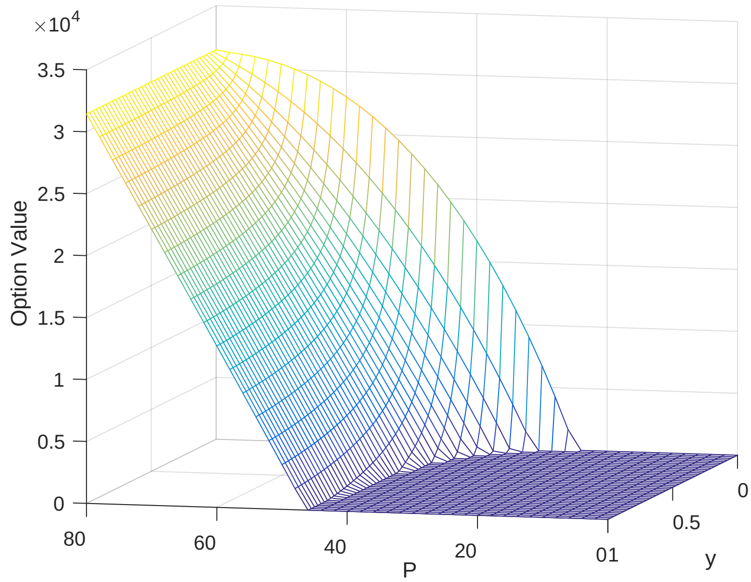


Figure 5.4: The computed option value at t=2 with $\kappa = 2$ for Test 1.

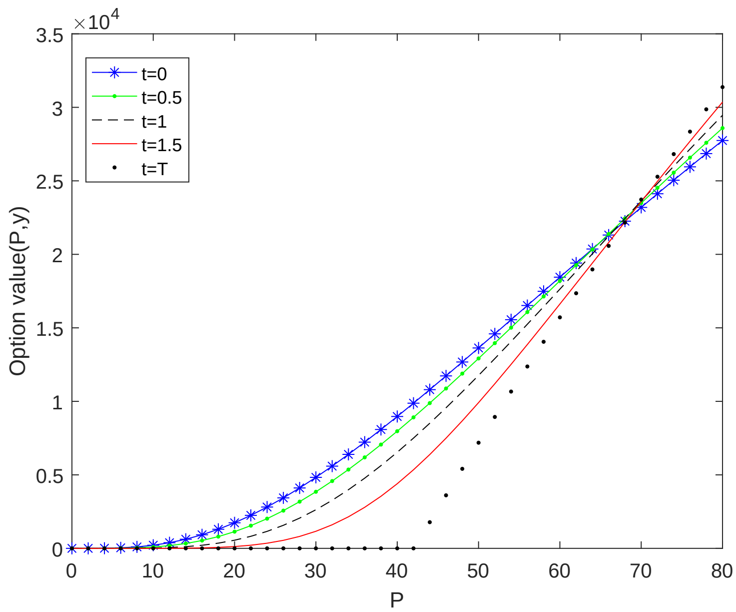


Figure 5.5: The computed option value W at $t = 0, 0.5, 1, 1.5, 2$ with $y = 0.25$ and $\kappa = 2$ for Test 1.

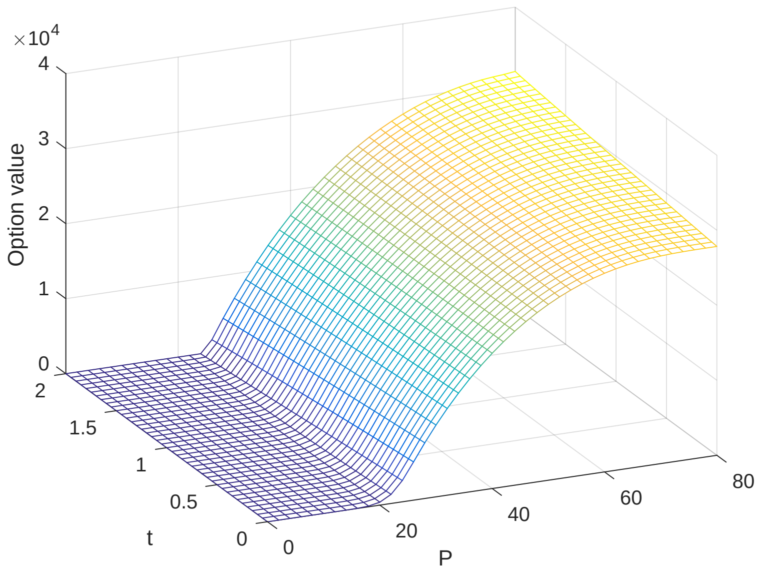
consistency of our 1-dimensional model in [44] and the current 2-dimensional model. The difference between the two numerical solutions, $W_{2D}(P, 0.25, t) - W_{1D}(P, t; 0.5)$, is depicted in 5.7(c)

From the PDE (5.18) we see that the value W_{2D} of this option is a function of (P, y, t), we also depicted the numerical solution W_{2D} when $y = \eta, Y$ and $t \in [0, T)$ respectively in figure 5.6 which behave like European call options. Then we test to calculate the option values by pricing model presented in equation (5.18).

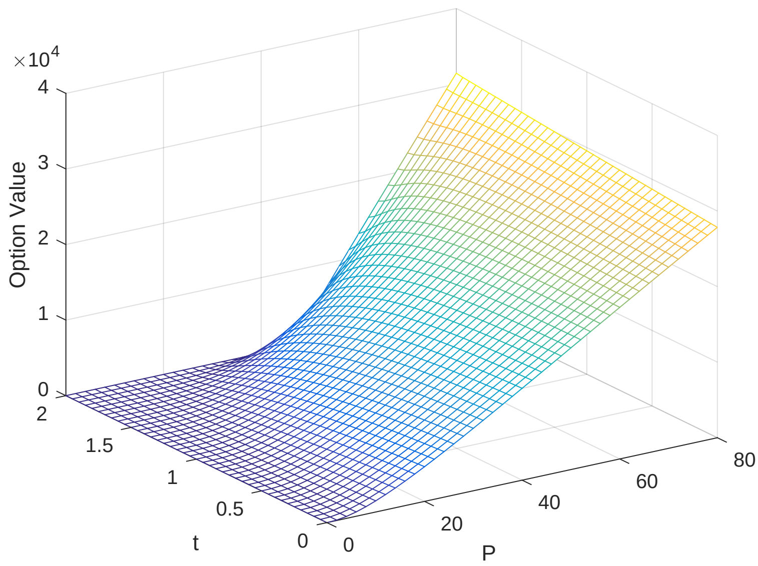
From Figure 5.7 (a) and (b), we compare the result of values which are similar in shape. We continue to calculate the difference of the values then found the maximum absolute difference appearing when $P \approx \$40$ at $t = T$ shown in figure 5.7 (c). Then, from Figures 5.8 and 5.9, we compare the result of values which are also similar in shape between various y and its corresponding σ. We continue to calculate the difference of the values displayed in figures (c) of 5.7 and 5.8 and 5.9. More clearly figure 5.10(a) notes that the solution of W_{2D} at $t = 0$ when $y = 0.25, 0.5, 0.75$ are almost greater than that W_{1D} presented in [44] when $\sigma = 0.5, 0.707, 0.866$ correspondingly when underlying commodity price P from 0 to $P_{\max}$. Then we found the result in contrast to the solution of W at $t = 0$ which the max absolute difference appears when $P \approx \$42$ which is shown in figure5.10(b), and when $P \approx \$44$ respectively, at $t = T$.

Moreover, the numerical results of W_{2D} when $y = 0.25$ are still larger than corresponding that W_{1D} when $\sigma = 0.5$ except when P is below \$64 .These may represent the fact that our model may get larger results in most case of options' present value, comparing with that in our previous study [44]. The reason for the difference can be because the earlier pricing model often assumed that the volatility rate of P was constant.

In contrast, the underlying commodity price follows a GBM with stochastic volatility, which may result in higher uncertainty that impact the options' value. Further analysis on the matter demonstrates that the values of options are larger when y increases from 0.25 to 1 at various P under a fixed κ from Figure 5.11. It proves the increased uncertainty will raise the value of the options, although the influence on the option price decreases as y increases gradually. Only $y = \eta$ where $\eta = 0$ as a special case differs from all others in which the value of the option increases (respectively decreases) when P is larger (respectively smaller) than some critical values with other various y.

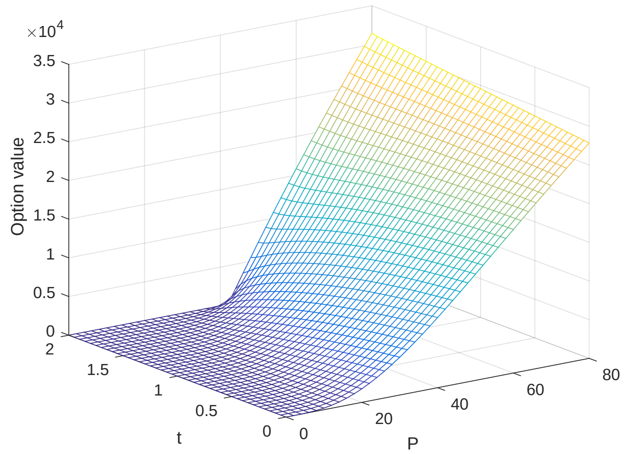


(a)$W_{2D}, y = \eta$

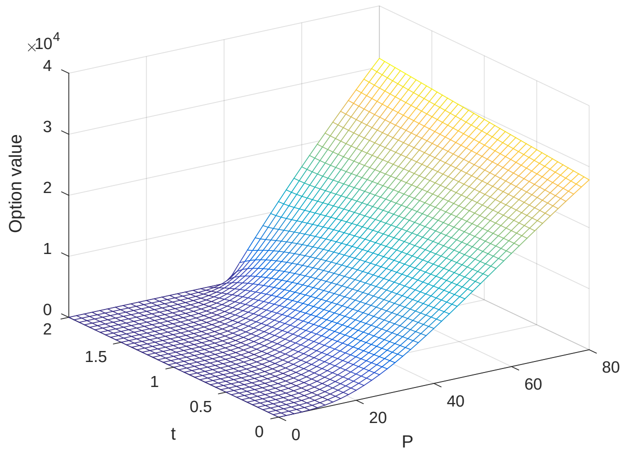


(b) $W_{2D}, y = Y$

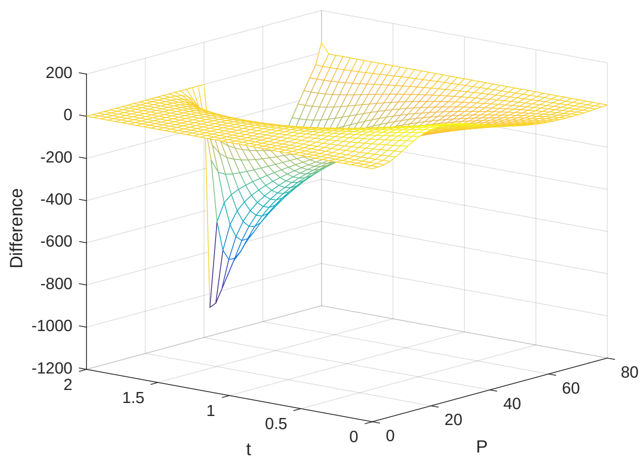
Figure 5.6: The option value of two boundary conditions at $y = \eta$ and $y = Y$ where $P \in (0, P_{\max})$, $t \in (0, T)$



(a)$W_{2D}, y = 0.25$

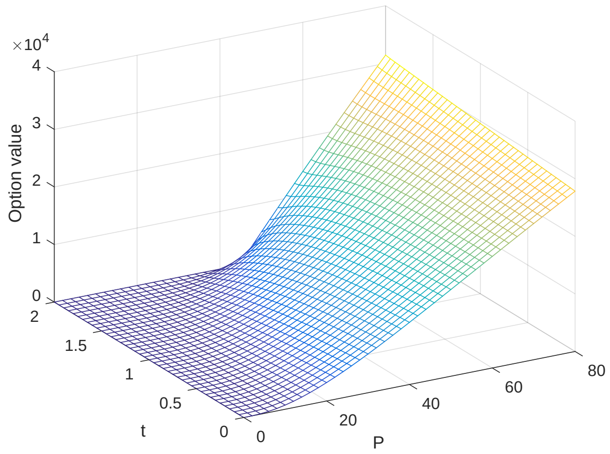


(b) $W_{1D}, \sigma = 0.5$

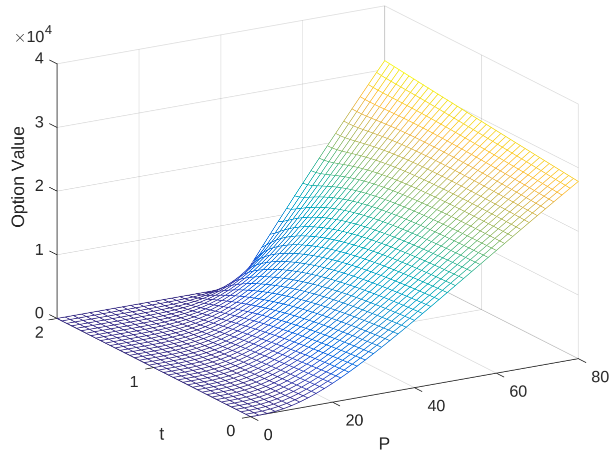


(c) $WD = W_{2D} - W_{1D}$

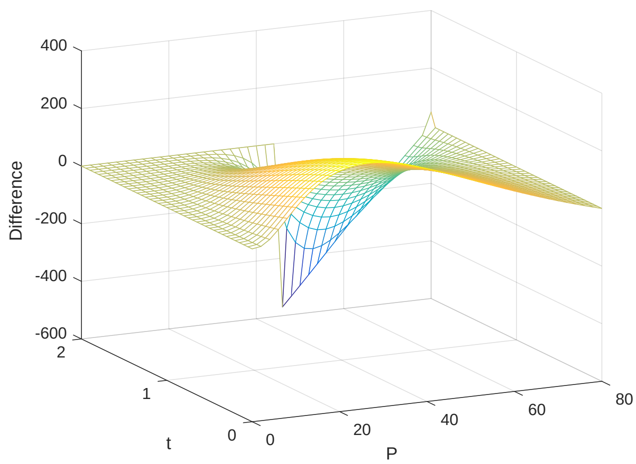
Figure 5.7: The option value W with $y = 0.25$, $\sigma = 0.5$, and Ws' Difference between W_{2D} and W_{1D}



(a)$W_{2D}, y = 0.5$

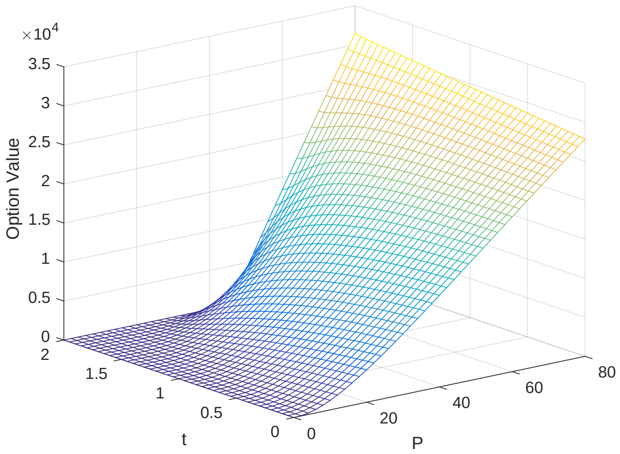


(b) $W_{1D}, \sigma = 0.707$

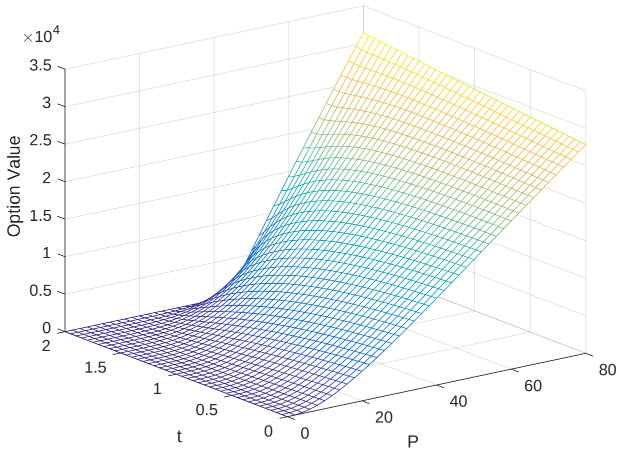


(c) $WD = W_{2D} - W_{1D}$

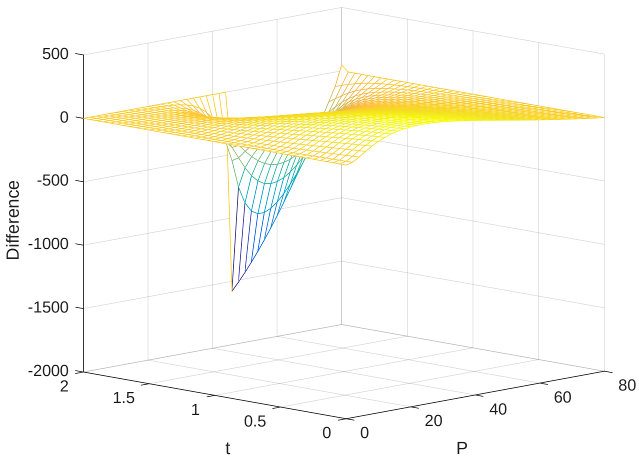
Figure 5.8: The option value W with $y = 0.5$, $\sigma = 0.707$, and Ws' Difference between W_{2D} and W_{1D}



(a)$W_{2D}, y = 0.75$

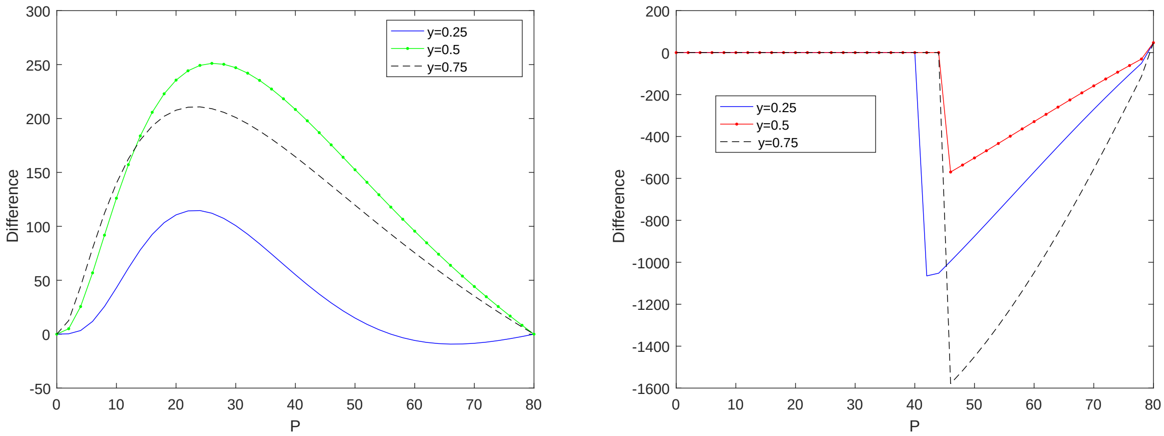


(b) $W_{1D}, \sigma = 0.866$



(c) $WD = W_{2D} - W_{1D}$

Figure 5.9: The option value W with $y = 0.75$, $\sigma = 0.866$, and Ws' Difference between W_{2D} and W_{1D}



(a)WD at $t = 0$ (b) WD at $t = T$

Figure 5.10: Ws' Difference between various $y = 0.25, 0.5, 0.75 = \sigma^2$ at $t = 0$ and $t = T$

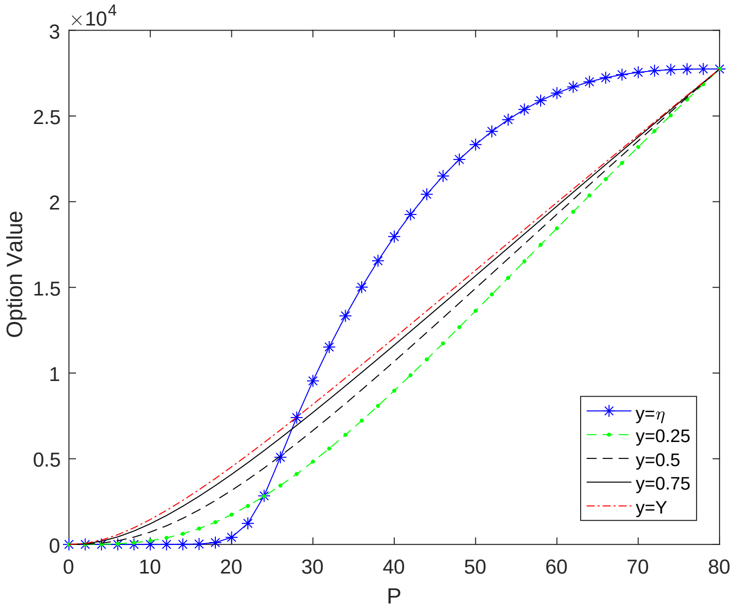


Figure 5.11: The option value W_{2D} with different y where $y = \eta, 0.25, 0.5, 0.75, 1$ at $t = 0$

5.5 Concluding Remarks

In this chapter, we developed a model for pricing project expansion options when both the underlying commodity price and its variance follow GBMs. The model is in the form of a 2D Black-Scholes PDE system with boundary and payoff conditions also defined by a few PDE systems. A finite difference scheme is proposed to solve the PDE system. Numerical experiments on a model option problem in an iron-ore industry have performed to demonstrate that the numerical results from this approach are financially correct. The results show that the value of the option decreases as the time approaches the expiration date when underlying commodity price P below a relatively high price. Furthermore, the increased uncertain will aggrandize the value of the options although the influence on option price decreases as the volatility increases gradually .

CHAPTER 6

Summary and Future Research Directions

6.1 Summary

In this thesis, we have studied various types of options embedded in investment projects, including the option to expand, the option to defer, options on investment contraction and ownership transfer. We have developed partial differential equation(PDE) and inequality based models for pricing European and American types of real options on investment projects. To solve these models, we propose several numerical methods, i.e. Finite difference method, Fitted finite volume method, and Penalty approach. Convergence analyses for some of these numerical methods have been performed. Several test problems with iron-ore as their underlying asset have been solved using the models and methods developed to demonstrate the correctness and effectiveness of our advances. More detailed summaries of our achievements are given below.

In the first systematic study, an upwind finite difference scheme is presented for a one-dimensional PDE with the underlying commodity price P following a Geometric Brownian Motion(GBM) to price European-style options on investment project expansions. The convergence analyses and numerical solutions of two non-trivial expansion options examples have demonstrated that our methods can efficiently and effectively solve practical problems. Through our case studies, we have found that the value of an expansion option is affected notably by the volatility σ and the expanding rate κ. Also, a compound option on an investment project expansion is more valuable than the corresponding normal one when the commodity price is not far from the average unit operating cost, because in that commodity price range, the former provides more flexibility than the latter to avoid investment risks.

In the second part of our study, we derive new PDEs based computational models for pricing options to contract/defer to invest, or to transfer part/all of the ownership of a project at or before a certain time. The developed models are similar to the Black-Scholes equation for valuing conventional European put options or the partial differential linear complementarity problem(LCP) for pricing American put/call options. We use a finite volume method for the approximation to the PDE, and an interior penalty approach to the discretized LCP. A smooth Newton's algorithm is present to solve the nonlinear system arising from the penalty method. We also prove that the approximate solution converges to that of the discretized LCP if the coefficient matrix of discretized systems is a positive-definite M-matrix. Numerical experiments are performed to demonstrate the usefulness and effectiveness of our methods which are able to produce financially meaningful results. The results confirm that an American type real option is more valuable than its European counterpart.

In the last part of our research, we extend the first part of the study from pricing options on one asset to the valuation of expansion option problems whose underlying asset price P and its volatility y follow their respective GBMs. We show that the value of such an option is governed by a PDE-based mathematical model in the form of a 2D Black-Scholes PDE system with boundary and payoff conditions also defined by some PDE systems. A nine-point finite difference scheme is used to solve the PDE system in two spatial dimensions. Numerical experiments on the iron-ore industry have been carried out to show that the results from this approach are practical and financial correct.

6.2 Future Research Directions

In real economics, flexible options can effectively remedy unpredictable uncertainties which are significant drivers to the risks in investment activities. In view of options which affect the valuation of potential investments, further work will focus on developing PDE-based models and numerical methods for pricing more realistic real options. Firstly, we may extend our study to more complex cases such as new option pricing models with more uncertain parameters, e.g. free interest rate, exchange rate, or other factors which may affect the value of the real option.

Another possible extension is to consider different types of real options. We have taken into account of pricing models for the option to expand, contract, abandon, and option to defer. In the future, we can also study some other options, such as, option to switch, both European and American types. Finally, development of more accurate and efficient numerical methods for solving the existing and future real option pricing models can also be future research topics.

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Appendix

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I, Nan Li, contributed 80% work to the publication entitled 'PRICING OPTIONS ON INVESTMENT PROJECT EXPANSIONS UNDER COMMODITY PRICE UNCERTAINTY', JOURNAL OF INDUSTRIAL AND MANAGEMENT OPTIMIZATION, Vol:15 Year: 2019 Page: 261–273.

Attribution Statement

Percentage contribution by field of activity: Indicate the percentage time each author contributed

Co-Author	Conception and Design	Acquisition of Data and Method	Data Conditioning and Manipulation	Analysis and Statistical Method	Interpret--ation and Discussion	Final Approval	Total % contribution
Song Wang	√	√		√	√	√	20%

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Attribution Statement

Percentage contribution by field of activity: Indicate the percentage time each author contributed

Co-Author	Conception and Design	Acquisition of Data and Method	Data Conditioning and Manipulation	Analysis and Statistical Method	Interpret--ation and Discussion	Final Approval	Total % contribution
Song Wang	√	√		√	√	√	20%
Shuhua Zhang			√	√			10%

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