International Journal of Advanced Research in Mathematics
ISSN: 2297-6213, Vol. 4, pp 1-7
doi:10.18052/www.scipress.com/IJARM.4.1

Submitted: 2016-01-13
Accepted: 2016-02-10
Online: 2016-02-15

On the one extremal problem on the Riemann sphere

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Keywords: Inner radius of domain, quadratic differential, piecewise-separating transformation, Green function, radial systems of points, logarithmic capacity, variational formula.

Abstract. Sharp estimates of product of inner radii for pairwise disjoint domains are obtained. In particular, we solve an extremal problem in the case of arbitrary finite number of rays containing arbitrary even number of free poles.

Introduction

This paper belongs to the theory of extremal problems on classes of non-overlapping domain, which is a separate direction in geometric theory of functions of a complex variable. The begin of these investigations associated with the paper of M. A. Lavrent'ev [1] in 1934. He found the maximum of some functional with respect to two simply connected domains with two fixed points. We note that this result was needed him for applying to some aerodynamics problems. In 1947, G. M. Goluzin solved a similar problem for three fixed points on the complex plane [2]. Then the topic began to evolve rapidly. In this connection we may recall papers of many authors, including Y. E. Alenitsina, M. A. Lebedev, J. Jenkins, P. M. Tamrazov, P. P. Kufareva and others. Using the idea of P. M. Tamrazov, in 1975 G. P. Bakhtin solved first the problem with so-called "free poles" on the unit circle, see, e.g., [3].

An important step for the development of this topic was papers of V. N. Dubinin. He developed a new method of research that is method of piecewise-separating transformation. He also first solved numerous of extremal problems for an arbitrary but fixed multi connected non-overlapping domains (see, e.g., [4], [5], [6]). Now this type of extremal problems is used for investigations in holomorphic dynamics.

In the last decade actively used Bakhtin's method of "managing functional". He managed to solve a series of extremal problems for so-called "radial systems of points" (see, e.g., [4], [7], [8], [9], [10]). In the present paper we use the mentioned about Bakhtin's method.

Theory

Let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ be the sets of natural, real and complex numbers respectively. Let $\overline{\mathbb{C}} := \mathbb{C} \bigcup \{\infty\}$ be the Riemannian sphere and $\mathbb{R}^+ := (0, \infty)$.

We fix a number $n \in \mathbb{N}$. A system of points $A_n = \{a_k \in \mathbb{C} : k = \overline{1, n}\}$, we will called *the n-equiangular system of points on rays*, if for all $k = \overline{1, n}$, the following relations are satisfied:

$$\arg a_k = \frac{2\pi}{n}(k-1). \tag{1}$$

We introduce the following "managing" functionals for an arbitrary n-equiangular system of points on rays

$$\mathcal{L}^{(\alpha)}\left(A_n\right)=\left[\prod_{k=1}^{n}\chi\left(\left|\frac{a_k}{a_{k+1}}\right|^{\frac{n}{4}}\right)\right]^{1-\frac{2\alpha}{n^2}}\cdot\left[\prod_{k=1}^{n}|a_k|\right]^{1+\frac{\alpha}{n}},$$

where $\chi(t)=\frac{1}{2}(t+\frac{1}{t})$, $t\in\mathbb{R}^{+}$.

Let us consider a system of angular domains:

$$P_k:=\{w\in\mathbb{C}:\frac{2\pi}{n}(k-1)<\arg w<\frac{2\pi}{n}k\},\quad k=\overline{1,n}.$$

Let $\{B_k,B_\infty\}$ be an arbitrary non-overlapping domains such that

$$a_k\in B_k,\infty\in B_\infty,\qquad B_k,B_\infty\subset\overline{\mathbb{C}},\quad k=\overline{1,n}.$$

Let

$$g_B\left(B,a\right)=h_{B,a}(z)+\log\frac{1}{|z-a|}$$

generalized Green's function of domains B with respect to a point $a\in B$. If $a=\infty$, then

$$g_B\left(B,\infty\right)=h_{B,\infty}(z)+\log\frac{1}{|z|}.$$

The value of

$$r(B,a):=\exp\left(h_{B,a}(z)\right)$$

the define of inner radius domain $B\subset\overline{\mathbb{C}}$ with respect to a point $a\in B$ (see [4], [5], [6], [11], [12], [13]).

We use the concept of a quadratic differential. Recall that a quadratic differential on a Riemann surface S is a map

$$\varphi:TS\to\mathbb{C}$$

satisfying

$$\varphi(\lambda v)=\lambda^2\varphi(v)$$

for all $v\in TS$ and all $\lambda\in\mathbb{C}$. If $z\in U\to\mathbb{C}$, is a chart defined on some open set $U\subset S$ then φ is equal on U to

$$\varphi_U(z)dz^2$$

for some function φ_U defined on $z(U)$.

Suppose that two charts $z:U\to\mathbb{C}$ and $w:V\to\mathbb{C}$ on S overlap, and let

$$h:=w\circ z^{-1}$$

be the transition function. If φ is represented both as $\varphi_U(z)dz^2$ and $\varphi_V(w)dw^2$ on $U\cap V$, then we have

$$\varphi_V\left(h(z)\right)\left(h'(z)\right)^2=\varphi_U(z).$$

One way to say this is that quadratic differentials transform under pull-backs by the square of the derivative. As the main results associated with it can be found in [14].

Results

In this paper we investigate the following problems.

Problem. Let $n \in \mathbb{N}$, $n \geq 2$, $\alpha \leq n^2$, $\alpha \in \mathbb{R}^+$. We intend to find a maximum of the functional

$$r^{\alpha}\left(B_{\infty}, \infty\right) \cdot \prod_{k=1}^{n} r\left(B_k, a_k\right),$$

and to describe all its extremals, if A_n be an arbitrary n-equiangular system of points on rays satisfying the condition (1), and $\{B_k, B_{\infty}\}$ be an arbitrary set of non-overlapping domains, $a_k \in B_k, \infty \in B_{\infty}$, $B_k, B_{\infty} \subset \overline{\mathbb{C}}$, $k = \overline{1, n}$.

Theorem. *Let $n \geq 2, n \in \mathbb{N}$, $\alpha \in \mathbb{R}^+$, $\alpha \leq n^2$. Let also $A_n = \{a_k\}_{k=1}^{n}$ be an n-equiangular system of points on rays, and $B_{\infty}, B_k \subset \overline{\mathbb{C}}, k = \overline{1, n}$, $a_k \in B_k, \infty \in B_{\infty}$ be an arbitrary set of non-overlapping domains. Then we have the inequality*

$$r^{\alpha}\left(B_{\infty}, \infty\right) \prod_{k=1}^{n} r(B_k, a_k) \leq r^{\alpha}\left(B_{\infty}^{(0)}, \infty\right) \prod_{k=1}^{n} r\left(B_k^{(0)}, a_k^{(0)}\right).$$

The equality sign holds, if points $\{a_k\}$ and domains B_k, B_{∞} are the poles and the circular domains of the quadratic differential

$$Q(w)dw^2 = -w^{n-2}\frac{n^2 + (w^n - 1)\,\alpha}{(w^n - 1)^2}\, dw^2, \tag{2}$$

where $\mathcal{L}^{(\alpha)}\left(A_n\right) = 1$.

When $\alpha = n^2$ we obtain the result follows.

Corollary. *Let $n \geq 2, n \in \mathbb{N}$, $\alpha \in \mathbb{R}^+$, $\alpha \leq n^2$. Let also $A_n = \{a_k\}_{k=1}^{n}$ be an n-equiangular system of points on rays, and $B_{\infty}, B_k \subset \overline{\mathbb{C}}, k = \overline{1, n}$, $a_k \in B_k, \infty \in B_{\infty}$ be an arbitrary set of non-overlapping domains. Then we have the inequality*

$$r^{n^2}\left(B_{\infty}, \infty\right) \prod_{k=1}^{n} r(B_k, a_k) \leq r^{n^2}\left(B_{\infty}^{(0)}, \infty\right) \prod_{k=1}^{n} r\left(B_k^{(0)}, a_k^{(0)}\right).$$

The equality sign holds, if points $\{a_k\}$ and domains B_k, B_{∞} are the poles and the circular domains of the quadratic differential

$$Q(w)dw^2 = -\frac{w^{2n-2}}{(w^n - 1)^2}\, dw^2,$$

where $\mathcal{L}^{(\alpha)}\left(A_n\right) = 1$.

Proof of Theorem. The proof leans on a method of the piece-dividing transformation developed by V. Dubinin (see [4], [5], [6]).

The function

$$z_k(w) = -i w^{\frac{n}{2}} \tag{3}$$

realizes univalent and conformal transformations of domain P_k on the right half-plane Re$z > 0$ for all $k = \overline{1, n}$.

Let $\omega_k^{(1)} := z_k\left(a_k\right), \omega_{k-1}^{(2)} := z_{k-1}\left(a_k\right), a_{n+1} := a_1, \omega_0^{(2)} := \omega_n^{(2)}, z_0 := z_n \; (k = \overline{1, n})$.

The family of functions $\{z_k(w)\}_{k=1}^{n}$ is a piece-dividing transformation (see [4], [5], [6]) of the domains $\left\{B_k :\; k = \overline{1, n}\right\}$ with respect to the system of angles $\{P_k\}_{k=1}^{n}$. For any domain $\Delta \in \mathbb{C}$ we define $(\Delta)^* := \left\{w \in \overline{\mathbb{C}} :\; \frac{1}{w} \in \Delta\right\}$. We denote by $\Omega_k^{(1)}$ the connected component

$$z_k\left(B_k\bigcap\overline{P}_k\right)\bigcup\left(z_k\left(B_k\bigcap\overline{P}_k\right)\right)^*$$

containing a point $\omega_k^{(1)}$, and by $\Omega_{k-1}^{(2)}$ we denote the connected component

$$z_{k-1}\left(B_k\bigcap\overline{P}_{k-1}\right)\bigcup\left(z_{k-1}\left(B_k\bigcap\overline{P}_{k-1}\right)\right)^*$$

containing a point $\omega_{k-1}^{(2)}$, $k=\overline{1,n}$, $\overline{P}_0:=\overline{P}_n$, $\Omega_0^{(2)}:=\Omega_n^{(2)}$. Generally speaking, the domains $\Omega_k^{(s)}$ are multiconnected domains, $k=\overline{1,n}$, $s=1,2$. The pair of domains $\Omega_{k-1}^{(2)}$ and $\Omega_k^{(1)}$ is result of piece-dividing transformation of domains B_k concerning families $\{P_{k-1},P_k\}$, $\{z_{k-1},z_k\}$ at a point a_k, $k=\overline{1,n}$. We denote by, too, $\Omega_k^{(\infty)}$ the connected component

$$z_k\left(B_\infty\bigcap\overline{P}_k\right)\bigcup\left(z_k\left(B_\infty\bigcap\overline{P}_k\right)\right)^*$$

containing a point ∞. The system of domains $\Omega_k^{(\infty)}$ is result of piece-dividing transformation of domains B_∞ concerning families $\{P_k\}$, $\{z_k\}$ at a point ∞, $k=\overline{1,n}$.

From the formula (3) we obtain the following asymptotic expressions:

$$\Big|z_k(w)-z_k(a_m)\Big|\sim\frac{n}{2}\cdot|a_m|^{\frac{n}{2}-1}\cdot|w-a_m|,\quad w\to a_m,\ w\in\overline{P}_k,\quad m=k,k+1. \tag{4}$$

From the Theorem 1.9 [11] (see also [5], [6]) and the formulae (4), we have the inequalities

$$r\left(B_k,a_k\right)\leqslant\frac{2}{n}\cdot\left(\frac{r\left(\Omega_k^{(1)},\omega_k^{(1)}\right)\cdot r\left(\Omega_{k-1}^{(2)},\omega_{k-1}^{(2)}\right)}{\left(|a_k||a_{k+1}|\right)^{\frac{n-2}{2}}}\right)^{\frac{1}{2}},\quad k=\overline{1,n},$$

$$r\left(B_\infty,\infty\right)\leqslant\prod_{k=1}^{n}\left(r\left(\Omega_k^{(\infty)},\infty\right)\right)^{\frac{2}{n^2}}. \tag{5}$$

Using the inequality (5), we get the following relations:

$$r^{\alpha}\left(B_{\infty},\infty\right)\prod_{k=1}^{n} r(B_k,a_k)\leq$$

$$\leq\left(\frac{2}{n}\right)^{n}\cdot\prod_{k=1}^{n}\left(r^{\frac{4\alpha}{n^2}}\left(\Omega_k^{(\infty)},\infty\right)\cdot\frac{r\left(\Omega_k^{(1)},\omega_k^{(1)}\right)\cdot r\left(\Omega_{k-1}^{(2)},\omega_{k-1}^{(2)}\right)}{\left(|a_k||a_{k+1}|\right)^{\frac{n-2}{2}}}\right)^{\frac{1}{2}}=$$

$$=\left(\frac{2}{n}\right)^{n}\cdot\prod_{k=1}^{n}\frac{\left(|a_k|^{\frac{n}{2}}+|a_{k+1}|^{\frac{n}{2}}\right)^{1-\frac{2\alpha}{n^2}}\cdot\left(|a_k|^{\frac{n}{2}}\cdot|a_{k+1}|^{\frac{n}{2}}\right)^{\frac{2\alpha}{n^2}}}{\left(|a_k||a_{k+1}|\right)^{\frac{n}{4}}}|a_k|=$$

$$\times\prod_{k=1}^{n}\left(r^{\frac{4\alpha}{n^2}}\left(\Omega_k^{(\infty)},\infty\right)\cdot\frac{r\left(\Omega_k^{(1)},\omega_k^{(1)}\right)\cdot r\left(\Omega_{k-1}^{(2)},\omega_{k-1}^{(2)}\right)}{\left|\omega_k^{(1)}-\omega_k^{(2)}\right|^{2-\frac{4\alpha}{n^2}}\cdot\left(\left|\omega_k^{(1)}\right|\left|\omega_k^{(2)}\right|\right)^{\frac{4\alpha}{n^2}}}\right)^{\frac{1}{2}}=$$

$$=\left(\frac{2}{n}\right)^{n}\cdot\prod_{k=1}^{n}|a_k|^{1+\frac{\alpha}{n}}\cdot\prod_{k=1}^{n}\left(\frac{|a_k|^{\frac{n}{2}}+|a_{k+1}|^{\frac{n}{2}}}{\left(|a_k||a_{k+1}|\right)^{\frac{n}{4}}}\right)^{1-\frac{2\alpha}{n^2}}\times$$

$$\times\prod_{k=1}^{n}\left(\frac{r^{\frac{4\alpha}{n^2}}\left(\Omega_k^{(\infty)},\infty\right)\cdot r\left(\Omega_k^{(1)},\omega_k^{(1)}\right)\cdot r\left(\Omega_{k-1}^{(2)},\omega_{k-1}^{(2)}\right)}{\left|\omega_k^{(1)}-\omega_k^{(2)}\right|^{2-\frac{4\alpha}{n^2}}\cdot\left(\left|\omega_k^{(1)}\right|\left|\omega_k^{(2)}\right|\right)^{\frac{4\alpha}{n^2}}}\right)^{\frac{1}{2}}=$$

$$=\left(\frac{2}{n}\right)^{n}\cdot 2^{n-\frac{2\alpha}{n}}\cdot\mathcal{L}^{(\alpha)}\left(A_n\right)\times$$

$$\times\prod_{k=1}^{n}\left(\frac{r^{\frac{4\alpha}{n^2}}\left(\Omega_k^{(\infty)},\infty\right)\cdot r\left(\Omega_k^{(1)},\omega_k^{(1)}\right)\cdot r\left(\Omega_{k-1}^{(2)},\omega_{k-1}^{(2)}\right)}{\left|\omega_k^{(1)}-\omega_k^{(2)}\right|^{2-\frac{4\alpha}{n^2}}\cdot\left(\left|\omega_k^{(1)}\right|\left|\omega_k^{(2)}\right|\right)^{\frac{4\alpha}{n^2}}}\right)^{\frac{1}{2}}. \tag{6}$$

Functional

$$\frac{r^{\alpha_1}(B_1,a_1)\cdot r^{\alpha_2}(B_2,a_2)\cdot r^{\alpha_3}(B_3,a_3)}{|a_1-a_2|^{\alpha_1+\alpha_2-\alpha_3}\cdot|a_1-a_3|^{\alpha_1-\alpha_2+\alpha_3}\cdot|a_2-a_3|^{-\alpha_1+\alpha_2+\alpha_3}},$$

$a_k\in B_k\subset\overline{\mathbb{C}}$, $B_k\cap B_p$, $k\neq p$, $\alpha_k\in\mathbb{R}^{+}$, $k,p=1,2,3$, relatively invariant conformal automorphisms of the complex plane $\overline{\mathbb{C}}$ ([4], [15]).

Taking into account the ratio of the last assertion of from (6), we have:

$$r^{\alpha}\left(B_{\infty},\infty\right)\prod_{k=1}^{n} r(B_k,a_k)\leq\left(\frac{2}{n}\right)^{n}\cdot\mathcal{L}^{(\alpha)}\left(A_n\right)$$

$$\times\prod_{k=1}^{n}\left(r^{\frac{4\alpha}{n^2}}\left(G_k^{(\infty)},\infty\right)\cdot r\left(G_k^{(1)},-i\right)\cdot r\left(G_{k-1}^{(2)},i\right)\right)^{\frac{1}{2}}$$

$$=\left(\frac{2}{n}\right)^{n}\cdot\mathcal{L}^{(\alpha)}\left(A_n\right)\cdot\left(r^{\frac{4\alpha}{n^2}}\left(D_{\infty},\infty\right)\cdot r\left(D_1,-i\right)\cdot r\left(D_2,i\right)\right)^{\frac{n}{2}}, \tag{7}$$

where D_{∞},D_1,D_2 – the circular domains of the quadratic differential

$$Q(z)dz^2=\frac{n^2-\alpha(1+z^2)}{(1+z^2)^2}dz^2. \tag{8}$$

In the quadratic differentials (8) make the change from the formula (3).

$$Q(w)dw^2 = \frac{n^2 - \alpha(1 + \left(-iw^{\frac{n}{2}}\right)^2)}{(1 + \left(-iw^{\frac{n}{2}}\right)^2)^2} \left(-i\frac{n}{2}w^{\frac{n}{2}-1}\right)^2 dw^2 =$$

$$= \frac{n^2 + \alpha(w^n - 1)}{(w^n - 1)^2} \left(-\frac{n^2}{4}\right) w^{n-2} dw^2.$$

We obtain the differential (2).

Using the relation (7) we finally obtain

$$r^{\alpha}\left(B_{\infty}, \infty\right) \prod_{k=1}^{n} r(B_k, a_k) \leq r^{\alpha}\left(B_{\infty}^{(0)}, \infty\right) \prod_{k=1}^{n} r\left(B_k^{(0)}, a_k^{(0)}\right),$$

where points $\{a_k\}$ and domains B_k, B_∞ are the poles and the circular domains of the quadratic differential (2). **The theorem is proved**.

Acknowledgement. The author is grateful to Prof. Bakhtin for suggesting problems and useful discussions.

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