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ON THE CALCULATION OF UNIL$_*$

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Abstract. Cappell's codimension 1 splitting obstruction surgery group UNil_n is a direct summand of the Wall surgery obstruction group of an amalgamated free product. For any ring with involution R we use the quadratic Poincaré cobordism formulation of the L-groups to prove that

$$L_n(R[x]) \ = \ L_n(R) \oplus \mathrm{UNil}_n(R;R,R) \ .$$

We combine this with M. Weiss' universal chain bundle theory to produce almost complete calculations of $\mathrm{UNil}_*(\mathbb{Z};\mathbb{Z},\mathbb{Z})$ and the Wall surgery obstruction groups $L_*(\mathbb{Z}[D_\infty])$ of the infinite dihedral group $D_\infty = \mathbb{Z}_2 * \mathbb{Z}_2$. Our main results are stated in 0.2.

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1. Introduction

The nilpotent K- and L-groups of rings are a rich source of algebraic invariants for geometric topology, giving results of two types: if the groups are zero it is possible to solve the associated splitting and classification problems, while if they are non-zero the groups are infinitely generated and the solutions to the problems are definitely obstructed. See Bass [2], Farrell [9], [10], Farrell and Hsiang [11], Cappell [4],[6], Ranicki [16], Connolly and Koźniewski [8].

The unitary nilpotent L-groups UNil_* arise as follows. Suppose given a closed n-dimensional manifold X which is expressed as a union of codimension 0 submanifolds $X_1, X_{-1} \subset X$

$$X \ = \ X_1 \cup X_{-1}$$

with

$$X_0 \ = \ X_1 \cap X_{-1} \ = \ \partial X_{-1} \ = \ \partial X_1 \subset X$$

a codimension 1 submanifold. Assume X, X_{-1}, X_0, X_1 are connected, and that the maps $\pi_1(X_0) \to \pi_1(X_{\pm 1})$ are injective, so that by the van Kampen theorem the fundamental group of X is an amalgamated free product

$$\pi_1(X) \ = \ \pi_1(X_1) *_{\pi_1(X_0)} \pi_1(X_{-1})$$

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with $\pi_1(X_i) \to \pi_1(X)$ $(i = -1, 0, 1)$ injective. Given another closed n-dimensional manifold M and a simple homotopy equivalence $f : M \to X$ there is a single obstruction

$$s(f) \in \mathrm{UNil}_{n+1}(R; \mathcal{B}_1, \mathcal{B}_{-1})$$

to deforming f by an h-cobordism of domains to a homotopy equivalence of the form

$$f_1 \cup f_{-1} \ : \ M_1 \cup M_{-1} \ \to \ X_1 \cup X_{-1}$$

with $f_{\pm 1} : (M_{\pm 1}, \partial M_{\pm 1}) \to (X_{\pm 1}, \partial X_{\pm 1})$ homotopy equivalences of manifolds with boundary, and

$$R \ = \ \mathbb{Z}[\pi_1(X_0)] \ , \ \mathcal{B}_{\pm 1} \ = \ \mathbb{Z}[\pi_1(X_{\pm 1}) \backslash \pi_1(X_0)] \ .$$

Cappell [4],[6] proved geometrically that the free Wall [21] surgery obstruction groups $L_* = L^h_*$ of the fundamental group ring

$$\Lambda \ = \ \mathbb{Z}[\pi_1(X)] \ = \ \mathbb{Z}[\pi_1(X_1)] *_{\mathbb{Z}[\pi_1(X_0)]} \mathbb{Z}[\pi_1(X_{-1})]$$

have direct sum decompositions

$$L_*(\Lambda) \ = \ \mathrm{UNil}_*(R; \mathcal{B}_1, \mathcal{B}_{-1}) \oplus L'_*(\mathbb{Z}[\pi_1(X_0)] \to \mathbb{Z}[\pi_1(X_1)] \times \mathbb{Z}[\pi_1(X_{-1})])$$

with L'_* appropriately decorated intermediate relative L-groups. The split monomorphism

$$\mathrm{UNil}_{n+1}(R; \mathcal{B}_1, \mathcal{B}_{-1}) \ \to \ L_{n+1}(\Lambda) \ ; \ s(f) \mapsto \sigma(g)$$

sends the splitting obstruction $s(f)$ to the surgery obstruction $\sigma(g)$ of an $(n+1)$-dimensional normal map g between f and a split homotopy equivalence, as given by the unitary nilpotent cobordism construction of [6]. The 4-periodicity $L_*(\Lambda) = L_{*+4}(\Lambda)$ extends to a 4-periodicity

$$\mathrm{UNil}_*(R; \mathcal{B}_1, \mathcal{B}_{-1}) \ = \ \mathrm{UNil}_{*+4}(R; \mathcal{B}_1, \mathcal{B}_{-1}) \ .$$

Farrell [10] obtained a remarkable factorization

$$\mathrm{UNil}_{n+1}(R; \mathcal{B}_1, \mathcal{B}_{-1}) \ \to \ \mathrm{UNil}_{n+1}(\Lambda; \Lambda, \Lambda) \ \to \ L_{n+1}(\Lambda)$$

For this reason (and some others too) the groups $\mathrm{UNil}_*(R; R, R)$ for any ring with involution R are of especial significance to us, and we introduce the abbreviation:

$$\mathrm{UNil}_n(R) \ = \ \mathrm{UNil}_n(R; R, R) \ .$$

But even the groups $\mathrm{UNil}_*(\mathbb{Z})$ have remained opaque for the last 30 years. Cappell [3],[4],[5] proved that $\mathrm{UNil}_{4k}(\mathbb{Z}) = 0$ and that $\mathrm{UNil}_{4k+2}(\mathbb{Z})$ is infinitely generated. The UNil-groups $\mathrm{UNil}_*(R; \mathcal{B}_1, \mathcal{B}_{-1})$ are 2-primary torsion groups. Farrell [10] proved that $4\mathrm{UNil}_*(R) = 0$, for any ring R. Connolly and Koźniewski [8] obtained an isomorphism

$$\mathrm{UNil}_{4k+2}(\mathbb{Z}) \ \cong \ \bigoplus_1^\infty \mathbb{F}_2 \ ,$$

together with information on $\mathrm{UNil}_{4k+2}(R)$ for various Dedekind domains and division rings. But that is nearly all that is known.

The infinite dihedral group is a free product of two copies of the cyclic group $\mathbb{Z}_2$ of order 2

$$D_\infty \ = \ \mathbb{Z}_2 * \mathbb{Z}_2 \ .$$

Since the surgery obstruction groups $L_*(R[D_\infty])$ are hard to compute directly, the split monomorphisms $\mathrm{UNil}_*(R) \to L_*(R[D_\infty])$ are more useful in computing $L_*(R[D_\infty])$ from $\mathrm{UNil}_*(R)$ than the other way round. Connolly and Koźniewski [8] expressed $\mathrm{UNil}_*(R;\mathcal{B}_1,\mathcal{B}_{-1})$ as the L-groups $L_*(\mathbb{A}_\alpha[x])$ of an *additive category* with involution $\mathbb{A}_\alpha[x]$. Although this expression did give new computations of $\mathrm{UNil}_*(R)$, the L-theory of additive categories with involution (Ranicki [17]) is not in general very computable.

The first goal of this paper therefore, is to provide a new description for $\mathrm{UNil}_n(R)$ in terms of L-groups, which can be used to computational advantage. Cappell and Farrell observed that the infinite dihedral group $D_\infty = \mathbb{Z}_2 * \mathbb{Z}_2$ can also be viewed as an extension of $\mathbb{Z}$ by $\mathbb{Z}_2$

$$\{1\} \to \mathbb{Z} \to D_\infty \to \mathbb{Z}_2 \to \{1\} \ ,$$

so that the classifying space can be viewed both as a one-point union

$$K(D_\infty,1) \ = \ K(\mathbb{Z}_2,1) \vee K(\mathbb{Z}_2,1) \ = \ \mathbb{RP}^\infty \vee \mathbb{RP}^\infty$$

and as the total space of a fibration

$$K(\mathbb{Z},1) \ = \ S^1 \to K(D_\infty,1) \to K(\mathbb{Z}_2,1) \ = \ \mathbb{RP}^\infty \ ,$$

and that this should have implications for codimension 1 surgery obstruction theory with $\pi_1 = D_\infty$. This observation was used in Ranicki [16] (pp. 737–745) to prove geometrically that for the group ring $R = \mathbb{Z}[\pi]$ of a finitely presented group π

$$\mathrm{UNil}_*(R) \ = \ NL_*(R) \ = \ \ker(L_*(R[x]) \to L_*(R))$$

with the involution on R extended to $R[x]$ by $\overline{x} = x$, and $R[x] \to R; x \mapsto 0$ the augmentation map. The NL-groups are L-theoretic analogues of the nilpotent K-group

$$NK_1(R) \ = \ \ker(K_1(R[x]) \to K_1(R)) \ = \ \widetilde{\mathrm{Nil}}_0(R)$$

of Chapter XII of Bass [2], which is such that

$$K_1(R[x]) \ = \ K_1(R) \oplus NK_1(R) \ .$$

Theorem A. *For any ring with involution R*

$$\mathrm{UNil}_*(R) \ = \ NL_*(R)$$

so that

$$L_n(R[x]) \ = \ L_n(R) \oplus \mathrm{UNil}_n(R) \ .$$

We develop a new method for calculating $\mathrm{UNil}_*(R)$, adopting the following strategy. The symmetric L-groups $L^*(R)$ of a ring R with an involution $R \to R; x \mapsto \overline{x}$ were defined by Mishchenko [12] and Ranicki [15] to be the cobordism groups of symmetric Poincaré complexes over R. The quadratic L-groups $L_*(R)$ were expressed in [15] as the cobordism groups of quadratic

Poincaré complexes over R, and the two types of L-groups were related by an exact sequence

$$\cdots \to L_n(R) \to L^n(R) \to \widehat{L}^n(R) \to L_{n-1}(R) \to \dots$$

with the hyperquadratic L-groups $\widehat{L}^*(R)$ the cobordism groups of (symmetric,quadratic) Poincaré pairs. The symmetric and hyperquadratic L-groups are not 4-periodic in general, but there are defined natural maps

$$L^n(R) \to L^{n+4}(R)\ ,\ \widehat{L}^n(R) \to \widehat{L}^{n+4}(R)$$

(which are isomorphisms for certain R, e.g. a Dedekind ring or the polynomial extension of a Dedekind ring). The 4-periodic versions of the symmetric and hyperquadratic L-groups

$$L^{n+4*}(R) \ = \ \lim_{k\to\infty} L^{n+4k}(R)\ ,\ \widehat{L}^{n+4*}(R) \ = \ \lim_{k\to\infty} \widehat{L}^{n+4k}(R)$$

are related by an exact sequence

$$\cdots \to L_n(R) \to L^{n+4*}(R) \to \widehat{L}^{n+4*}(R) \to L_{n-1}(R) \to \dots\ .$$

The theory of Weiss [22] identified $\widehat{L}^{n+4*}(R)$ with the 'twisted Q-group' $Q_n(B^R,\beta^R)$ of the 'universal chain bundle' (B^R,β^R) over R, which can be computed (more or less effectively) from the Tate $\mathbb{Z}_2$-cohomology groups of the involution on R

$$\begin{aligned} H_n(B^R) \ &= \ \widehat{H}^n(\mathbb{Z}_2;R) \\ &= \ \{a\in R\,|\,\overline{a}=(-1)^n a\}/\{b+(-1)^n\overline{b}\,|\,b\in R\}\ . \end{aligned}$$

In Proposition 2.9 we show that for a Dedekind ring with involution R

$$L^n(R[x]) \ = \ L^n(R)\ ,\ NL^n(R) \ = \ 0$$

making the UNil-groups

$$\mathrm{UNil}_n(R) \ = \ \ker(Q_n(B^{R[x]},\beta^{R[x]}) \to Q_n(B^R,\beta^R))$$

accessible to computation.

Theorem B. *For the ring $\mathbb{Z}$, we have:*

$$\mathrm{UNil}_0(\mathbb{Z}) \ = \ 0\ ,\ \mathrm{UNil}_1(\mathbb{Z}) \ = \ 0$$

and there is an exact sequence:

$$0 \to \mathbb{F}_2[x]/\mathbb{F}_2 \xrightarrow{\psi^2-1} \mathbb{F}_2[x]/\mathbb{F}_2 \to \mathrm{UNil}_2(\mathbb{Z}) \to 0$$

with

$$\psi^2 \ :\ \mathbb{F}_2[x] \to \mathbb{F}_2[x]\ ;\ a \mapsto a^2$$

the Frobenius map. $\mathrm{UNil}_3(\mathbb{Z})$ *is not finitely generated, with* $4\mathrm{UNil}_3(\mathbb{Z})=0$.

We now give an outline of the rest of this paper.

In §1 we define the groups $\mathrm{UNil}_n(R)$ and the map $c:\mathrm{UNil}_n(R)\to L_n(R[x])$, as well as the various other morphisms and groups with which we will be working.

In §2 we prove Theorem A.

In §3 we relate $\mathrm{UNil}_n(R)$ for Dedekind R to the group of symmetric structures on the universal chain bundle of Weiss. We then make the calculations necessary to prove Theorem B.

2. Fundamental Concepts. The proof of Theorem A.

2.1. **Algebraic L-groups.** Throughout this paper R denotes a ring with an involution

$$R \to R \ ; \ r \mapsto \overline{r} \ .$$

Given a left R-module P let P^t be the right R-module with the same additive group and

$$P^t \times R \to P^t \ ; \ (x,r) \mapsto \overline{r}x \ .$$

The dual of a left R-module P is the left R-module

$$\begin{aligned} P^* &= \mathrm{Hom}_R(P,R) \ , \\ R \times P^* &\to P^* \ ; \ (r,f) \mapsto (x \mapsto f(x)\overline{r}) \ . \end{aligned}$$

Write the evaluation pairing as

$$\langle \ , \ \rangle \ : \ P^* \times P \to R \ ; \ (f,x) \mapsto \langle f,x \rangle \ = \ f(x) \ .$$

An element $\phi \in \mathrm{Hom}_R(P,P^*)$ determines a sesquilinear form on P

$$\langle \ , \ \rangle_\phi \ : \ P \times P \to R \ ; \ (x,y) \mapsto \langle \phi(x), y \rangle \ ,$$

and we identify $\mathrm{Hom}_R(P,P^*)$ with the additive group of such forms. The dual of a f.g. (= finitely generated) projective R-module P is a f.g. projective R-module P^*, and the morphism

$$P \to P^{**} \ ; \ x \mapsto (f \mapsto \overline{f(x)})$$

is an isomorphism, which we shall use to identify

$$P^{**} \ = \ P \ ,$$

and to define the ϵ-duality involution

$$T_\epsilon \ : \ \mathrm{Hom}_R(P,P^*) \to \mathrm{Hom}_R(P,P^*) \ ; \ \phi \mapsto \epsilon\phi^* \ , \ \langle x,y \rangle_{\phi^*} \ = \ \overline{\langle y,x \rangle_\phi} \ .$$

For $\epsilon = \pm 1$, any chain complex C of left R-modules and any $\mathbb{Z}[\mathbb{Z}_2]$-module chain complex X define the $\mathbb{Z}$-module chain complexes

$$\begin{aligned} X^{\%}(C,\epsilon) &= \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(X, C^t \otimes_R C) \ , \\ X_{\%}(C,\epsilon) &= X \otimes_{\mathbb{Z}[\mathbb{Z}_2]} (C^t \otimes_R C) \end{aligned}$$

with $T \in \mathbb{Z}_2$ acting on $C^t \otimes_R C$ by

$$T_\epsilon \ : \ (C_p)^t \otimes_R C_q \to (C_q)^t \otimes_R C_p \ ; \ x \otimes y \mapsto (-1)^{pq}\epsilon y \otimes x \ .$$

As in Ranicki [15] the group of n-dimensional ϵ-symmetric (resp. ϵ-hyperquadratic, resp. ϵ-quadratic) structures on C is defined by:

$$Q^n(C,\epsilon) \ = \ H_n(W^{\%}C) \ , \ \widehat{Q}^n(C,\epsilon) \ = \ H_n(\widehat{W}^{\%}C) \ ,$$

$$Q_n(C,\epsilon) \ = \ H_n(W_{\%}C) \ = \ H_n((W^{-*})^{\%}C)$$

where W (resp. $\widehat{W}$) denotes the standard free $\mathbb{Z}[\mathbb{Z}_2]$-module resolution of $\mathbb{Z}$ (resp. complete resolution) and

$$W^{-*} \ = \ \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W,\mathbb{Z}[\mathbb{Z}_2]) \ .$$

If $S^{-1}W^{-*}$ denotes the desuspension of W^{-*}, the short exact sequence

$$0 \to S^{-1}W^{-*} \to \widehat{W} \to W \to 0$$

induces the exact sequence:

$$\dots \to Q_n(C,\epsilon) \to Q^n(C,\epsilon) \xrightarrow{J} \widehat{Q}^n(C,\epsilon) \to Q_{n-1}(C,\epsilon) \to \dots \ . \tag{2.1}$$

Definition 2.1. An *ϵ-symmetric form* (P,ϕ) (resp. an *ϵ-quadratic form* (P,ψ)) over R is a f.g. projective R-module P together with an element

$$\phi \in Q^0(C,\epsilon) \ = \ Q^{\epsilon}(P) \ = \ \ker(1-T_\epsilon : \mathrm{Hom}_R(P,P^*) \to \mathrm{Hom}_R(P,P^*)) \ ,$$

$$\psi \in Q_0(C,\epsilon) \ = \ Q_{\epsilon}(P) \ = \ \mathrm{coker}(1-T_\epsilon : \mathrm{Hom}_R(P,P^*) \to \mathrm{Hom}_R(P,P^*))$$

with C the 0-dimensional R-module chain complex

$$C \ : \ \dots \to 0 \to C_0 = P^* \to 0 \to \dots \ .$$

The form is *nonsingular* if the R-module morphism

$$\phi : P \to P^* \ (\text{resp. } N_\epsilon(\psi) = (1+T_\epsilon)\psi : P \to P^*)$$

is an isomorphism. □

We refer to Ranicki [15],[16],[19] for various accounts of the construction of the free ϵ-symmetric (resp. quadratic) L-groups $L^n(R,\epsilon)$ (resp. $L_n(R,\epsilon)$) as the cobordism groups of n-dimensional ϵ-symmetric (resp. ϵ-quadratic) Poincaré complexes over R $(C,\phi \in Q^n(C,\epsilon))$ (resp. $(C,\psi \in Q_n(C,\epsilon))$) with

$$C \ : \ \dots \to 0 \to C_n \to C_{n-1} \to \dots \to C_0 \to 0 \to \dots$$

an n-dimensional f.g. free R-module chain complex. The projective L-groups $L^*_p(R,\epsilon)$ (resp. $L^p_*(R,\epsilon)$) are constructed in the same way, using f.g. projective C.

The suspension of an R-module chain complex C is the R-module chain complex SC with

$$d_{SC} \ = \ d_C \ : \ (SC)_{r+1} \ = \ C_r \to (SC)_r \ = \ C_{r-1} \ .$$

As in Ranicki [15] (p. 105) use the natural $\mathbb{Z}$-module isomorphisms

$$S^2(W^{\%}(C,\epsilon)) \ \cong \ W^{\%}(SC,-\epsilon) \ , \ S^2(W_{\%}(C,\epsilon)) \ \cong \ W_{\%}(SC,-\epsilon)$$

to identify

$$Q^n(C,\epsilon) \;=\; Q^{n+2}(SC,-\epsilon)\ ,\ Q_n(C,\epsilon) \;=\; Q_{n+2}(SC,-\epsilon)$$

and to define the skew-suspension maps

$$\overline{S}^n \;:\; L^n(R,\epsilon) \to L^{n+2}(R,-\epsilon)\ ;\ (C,\phi) \mapsto (SC,\phi)\ ,$$

$$\overline{S}_n \;:\; L_n(R,\epsilon) \to L_{n+2}(R,-\epsilon)\ ;\ (C,\psi) \mapsto (SC,\psi)\ .$$

Definition 2.2. A ring R is 0-*dimensional* if it is hereditary and noetherian, or equivalently if every submodule of a f.g. projective R-module is f.g. projective.

□

In particular, Dedekind rings are 0-dimensional.

Proposition 2.3. ([15])
(i) *For every ring with involution R the $\pm\epsilon$-quadratic skew-suspension maps $\overline{S}_n$ are isomorphisms, so that*

$$L_n(R,\epsilon) \;=\; L_{n+2}(R,-\epsilon) \;=\; L_{n+4}(R,\epsilon)\ ,$$

with $L_{2n}(R,\epsilon) = L_0(R,(-1)^n\epsilon)$ the Witt group of stable isometry classes of nonsingular $(-1)^n\epsilon$-quadratic forms over R.
(ii) *If R is 0-dimensional then the $\pm\epsilon$-symmetric skew-suspension maps $\overline{S}^n$ are isomorphisms, so that*

$$L^n(R,\epsilon) \;=\; L^{n+2}(R,-\epsilon) \;=\; L^{n+4}(R,\epsilon)\ ,$$

with $L^{2n}(R,\epsilon) = L^0(R,(-1)^n\epsilon)$ the Witt group of stable isometry classes of nonsingular $(-1)^n\epsilon$-symmetric forms over R.

Proof. By algebraic surgery below the middle dimension, given by Proposition I.4.3 of [15] for (i), and Proposition I.4.5 of [15] for (ii). □

For $\epsilon = 1$ we write

$$X^{\%}(C,1) \;=\; X^{\%}C\ ,\ X_{\%}(C,1) \;=\; X_{\%}C\ ,$$

$$Q^*(C,1) \;=\; Q^*(C)\ ,\ \widehat{Q}^*(C,1) \;=\; \widehat{Q}^*(C)\ ,\ Q_*(C,1) \;=\; Q_*(C)\ ,$$

$$L^*(R,1) \;=\; L^*(R)\ ,\ L_*(R,1) \;=\; L_*(R)\ .$$

The hyperquadratic Q-groups $\widehat{Q}^*(C)$ are used in Section 3 to define chain bundles.

2.2. **The nilpotent L-groups $L\mathrm{Nil}$, $L\widetilde{\mathrm{Nil}}$.** Theorem A identifies the unitary nilpotent L-groups $\mathrm{UNil}_*(R)$ with the nilpotent L-groups $L\widetilde{\mathrm{Nil}}_*(R)$, whose definition we now recall.

We start with nilpotent K-theory.

Definition 2.4. (i) An *R-nilmodule* (P,ν) is a f.g. projective R-module P together with a nilpotent endomorphism $\nu : P \to P$, so that

$$\nu^N \;=\; 0 \;:\; P \to P$$

for some $N \geqslant 1$.
(ii) A *morphism* of R-nilmodules $f : (P,\nu) \to (P',\nu')$ is an R-module morphism $f : P \to P'$ such that $\nu' f = f\nu : P \to P'$.
(iii) The *nilpotent K-groups* of R are defined to be the K-groups

$$\mathrm{Nil}_*(R) \;=\; K_*(\mathrm{Nil}(R))$$

of the exact category $\mathrm{Nil}(R)$ be of R-nilmodules. The *reduced nilpotent K-groups*

$$\widetilde{\mathrm{Nil}}_*(R) \;=\; \ker(\mathrm{Nil}_*(R) \to K_*(R))$$

are such that

$$\mathrm{Nil}_*(R) \;=\; K_*(R) \oplus \widetilde{\mathrm{Nil}}_*(R) \;.$$

(iv) The *NK-groups* of R are defined by

$$NK_*(R) \;=\; \ker(K_*(R[x]) \to K_*(R)) \;,$$

so that

$$K_*(R[x]) \;=\; K_*(R) \oplus NK_*(R) \;. \qquad \square$$

Proposition 2.5. (Bass [2])
(i) *There is a natural identification*

$$NK_1(R) \;=\; \widetilde{\mathrm{Nil}}_0(R)$$

using the split injection

$$\widetilde{\mathrm{Nil}}_0(R) \to K_1(R[x]) \;;\; (P,\nu) \mapsto \tau(1 + x\nu : P[x] \to P[x]) \;.$$

(ii) *If R is 0-dimensional then*

$$\widetilde{\mathrm{Nil}}_0(R) \;=\; 0 \;.$$

Proof. (i) See Chapter XII of [2].
(ii) Given a nilmodule (P,ν) with $\nu^N = 0 : P \to P$ for some $N \geqslant 1$ define the nilmodules

$$(P',\nu') \;=\; (\ker(\nu), 0) \;,\; (P'',\nu'') \;=\; (\mathrm{im}(\nu), \nu|) \;,$$

using the 0-dimensionality of R to ensure that the R-modules $\ker(\nu), \mathrm{im}(\nu) \subseteq P$ are f.g. projective. It follows from the exact sequence

$$0 \to (P',\nu') \to (P,\nu) \to (P'',\nu'') \to 0$$

that

$$[P,\nu] \;=\; [P',\nu'] + [P'',\nu''] \in \mathrm{Nil}_0(R)\ .$$

Now $\nu' = 0$, $(\nu'')^{N-1} = 0$, so proceeding inductively we obtain

$$[P,\nu] \;=\; \sum_{i=1}^{N} [\ker(\nu^i)/\ker(\nu^{i-1}), 0] \in K_0(R) \subseteq \mathrm{Nil}_0(R)$$

and hence that $\widetilde{\mathrm{Nil}}_0(R) = 0$. □

Definition 2.6. An n-dimensional *R-nilcomplex* (C,ν) is a n-dimensional f.g. projective R-module chain complex

$$C \;:\; \cdots \to 0 \to C_n \to C_{n-1} \to \cdots \to C_1 \to C_0$$

together with a chain map $\nu : C \to C$ which is chain homotopy nilpotent, i.e. such that $\nu^N \simeq 0 : C \to C$ for some integer $N \geqslant 1$. □

Proposition 2.7. *The chain equivalence classes of the following types of chain complexes are in one-one correspondence:*

(i) *n-dimensional chain complexes of R-nilmodules*

$$(C,\nu) \;:\; \cdots \to 0 \to (C_n,\nu) \to (C_{n-1},\nu) \to \cdots \to (C_1,\nu) \to (C_0,\nu)\ ,$$

(ii) *n-dimensional R-nilcomplexes (C,ν),*
(iii) *$(n+1)$-dimensional f.g. projective $R[x]$-module chain complexes*

$$D \;:\; \cdots \to 0 \to D_{n+1} \to D_n \to \cdots \to D_1 \to D_0$$

such that

$$H_*(R[x,x^{-1}] \otimes_{R[x]} D) \;=\; 0\ .$$

Proof. (i) $\Longrightarrow$ (ii) An n-dimensional chain complex of R-nilmodules is an n-dimensional R-nilcomplex.
(ii) $\Longrightarrow$ (iii) Given an n-dimensional R-nilcomplex (C,ν) define the $(n+1)$-dimensional f.g. projective $R[x]$-module chain complexes

$$D \;=\; C(x-\nu : C[x] \to C[x])$$

such that

$$H_*(R[x,x^{-1}] \otimes_{R[x]} D) \;=\; 0\ ,$$

$$x \;=\; \nu \;:\; H_*(D) \;=\; H_*(C) \to H_*(D) \;=\; H_*(C)\ .$$

(i) $\Longleftrightarrow$ (iii) See Proposition 3.1.2 of Ranicki [16]. □

Now for nilpotent L-theory.

Definition 2.8. (Ranicki [16], p. 440, [18] p. 470)
(i) The *ϵ-symmetric* QNil*-groups* $Q\mathrm{Nil}^*(C,\nu,\epsilon)$ of an R-nilcomplex (C,ν) are the relative Q-groups in the exact sequence

$$\cdots \to Q^{n+1}(C,-\epsilon) \to Q\mathrm{Nil}^n(C,\nu,\epsilon) \to Q^n(C,\epsilon) \xrightarrow{\Gamma_\nu} Q^n(C,-\epsilon) \to \dots$$

with

$$\Gamma_\nu \ : \ W^{\%}(C,\epsilon) \to W^{\%}(C,-\epsilon) \ ; \ \phi \mapsto (1\otimes\nu)\phi - \phi(\nu\otimes 1) \ .$$

Similarly for the *ϵ-quadratic* QNil*-groups* $Q\mathrm{Nil}_*(C,\nu,\epsilon)$, with an exact sequence

$$\cdots \to Q_{n+1}(C,-\epsilon) \to Q\mathrm{Nil}_n(C,\nu,\epsilon) \to Q_n(C,\epsilon) \xrightarrow{\Gamma_\nu} Q_n(C,-\epsilon) \to \dots \ .$$

(ii) An *n-dimensional ϵ-symmetric Poincaré nilcomplex over R* $(C,\nu,\delta\phi,\phi)$ is an n-dimensional R-nilcomplex (C,ν) together with an element

$$(\delta\phi,\phi) \in Q\mathrm{Nil}^n(C,\nu,\epsilon)$$

such that $(C,\phi\in Q^n(C,\epsilon))$ is an n-dimensional ϵ-symmetric Poincaré complex over R. The *ϵ-symmetric* LNil*-group* $L\mathrm{Nil}^n(R,\epsilon)$ is the cobordism group of n-dimensional ϵ-symmetric Poincaré nilcomplexes over R. Similarly in the ϵ-quadratic case, with $L\mathrm{Nil}_n(R,\epsilon)$.
(iii) The *reduced ϵ-symmetric* LNil*-groups* are defined by

$$L\widetilde{\mathrm{Nil}}^*(R,\epsilon) \ = \ \ker(L\mathrm{Nil}^*(R,\epsilon) \to L^*_p(R,\epsilon)) \ ,$$

with

$$L\mathrm{Nil}^*(R,\epsilon) \ = \ L^*_p(R,\epsilon) \oplus L\widetilde{\mathrm{Nil}}^*(R,\epsilon) \ .$$

Similarly in the ϵ-quadratic case, with $L\widetilde{\mathrm{Nil}}_*(R,\epsilon)$.
(iv) Extend the involution to $R[x]$ by $\overline{x}=x$. Use the augmentation map

$$R[x] \to R \ ; \ x \mapsto 0$$

to define the *nilpotent ϵ-symmetric L-groups of R*

$$NL^*(R,\epsilon) \ = \ \ker(L^*(R[x],\epsilon) \to L^*(R,\epsilon))$$

with

$$L^*(R[x],\epsilon) \ = \ L^*(R,\epsilon) \oplus NL^*(R,\epsilon) \ .$$

Similarly for the *nilpotent ϵ-quadratic L-groups* $NL_*(R,\epsilon)$. □

Proposition 2.9. (i) *The* QNil*-groups of an R-nilcomplex (C,ν) are the Q-groups of the f.g. projective $R[x]$-module chain complex*

$$D \ = \ C(x-\nu : C[x] \to C[x])$$

with

$$x \ = \ \nu \ : \ H_*(D) \ = \ H_*(C) \to H_*(D) \ = \ H_*(C) \ ,$$

$$Q\mathrm{Nil}^n(C,\nu,\epsilon) \ = \ Q^{n+1}(D,-\epsilon) \ ,$$

$$Q\mathrm{Nil}_n(C,\nu,\epsilon) \ = \ Q_{n+1}(D,-\epsilon) \ .$$

(ii) *The morphisms*

$$L\mathrm{Nil}^n(R,\epsilon) \to L^n_{\widetilde{K}_0(R)}(R[x],\epsilon) \ ; \ (C,\nu,\delta\phi,\phi) \mapsto (C[x],\widetilde{\Phi})$$
$$(\widetilde{\Phi}_s = (1-x\nu)\phi_s - x\delta\phi_{s-1} \ , \ s \geqslant 0 \ , \ \delta\phi_{-1} = 0) \ ,$$
$$L\mathrm{Nil}_n(R,\epsilon) \to L_n^{\widetilde{K}_0(R)}(R[x],\epsilon) \ ; \ (C,\nu,\delta\psi,\psi) \mapsto (C[x],\widetilde{\Psi})$$
$$(\widetilde{\Psi}_s = (1-x\nu)\psi_s - x\delta\psi_{s+1} \ , \ s \geqslant 0)$$

are isomorphisms.
(iii) *The nilpotent ϵ-symmetric L-groups of a ring with involution R fit into split exact sequences:*

$$0 \to L\widetilde{\mathrm{Nil}}^n(R,\epsilon) \to L^n(R[x],\epsilon) \to L^n(R,\epsilon) \to 0 \ ,$$
$$0 \to L^n(R[x],\epsilon) \to L^n(R[x,x^{-1}],\epsilon) \to L\mathrm{Nil}^n(R,\epsilon) \to 0$$

with an identification

$$NL^n(R,\epsilon) \ = \ L\widetilde{\mathrm{Nil}}^n(R,\epsilon) \ .$$

Similarly in the ϵ-quadratic case.

Proof. (i)+(ii) Ranicki [18], Propositions 34.5, 34.8.
(iii) The ϵ-symmetric L-theory localization exact sequence of Chapter 3 of Ranicki [16]

$$\dots \to L^n(A,\epsilon) \to L^n(S^{-1}A,\epsilon) \to L^n(A,S,\epsilon) \to L^{n-1}(A,\epsilon) \to \dots$$

is defined for any ring with involution A and any central multiplicative subset $S \subset A$ of nonzero divisors, with $L_n(A,S,\epsilon)$ the cobordism group of $(n-1)$-dimensional ϵ-symmetric Poincaré complexes (C,ψ) over A such that

$$S^{-1}A \otimes_A C \ \simeq \ 0 \ .$$

Propositions 5.1.3, 5.1.4 of [16] give that for

$$(A,S) \ = \ (R[x],\{x^k | k \geqslant 0\}) \ , \ S^{-1}A \ = \ R[x,x^{-1}]$$

the localization exact sequence breaks up into split exact sequences

$$0 \to L^n(A,\epsilon) \to L^n(S^{-1}A,\epsilon) \to L^n(A,S,\epsilon) \to 0$$

and identifies

$$L^*(A,S,\epsilon) \ = \ L^*_{\widetilde{K}_0(R)}(A,\epsilon) \ = \ L\mathrm{Nil}^*(R,\epsilon) \ .$$

Similarly in the ϵ-quadratic case. □

In the applications of the nilpotent L-groups to the unitary nilpotent L-groups we shall be particularly concerned with the Witt groups of 'nilforms' over R.

Definition 2.10. ([16], p.452)
(i) A *nonsingular ϵ-symmetric nilform over R* is a triple (P, ν, ϕ) such that

(a) (P, ν) is an R-nilmodule,
(b) $\phi \in Q^{\epsilon}(P)$ with $\phi : P \to P^*$ an isomorphism and
$$\nu^*\phi \;=\; \phi\nu \;:\; P \to P^* \;.$$

Thus (P, ϕ) is a nonsingular ϵ-symmetric form over R, $\phi \in Q\mathrm{Nil}^0(C, \nu^*, \epsilon)$ with C the 0-dimensional R-module chain complex
$$C \;:\; \cdots \to 0 \to C_0 = P^* \to 0 \to \ldots \;,$$
and there is defined an isomorphism of R-nilmodules
$$\phi \;:\; (P, \nu) \to (P^*, \nu^*) \;.$$
A *lagrangian* for (P, ν, ϕ) is a direct summand $L \subset P$ such that

(c) $\nu(L) \subset L$,
(d) the sequence
$$0 \to L \xrightarrow{i} P \xrightarrow{i^*\phi} L^* \to 0$$
is exact, with $i : L \to P$ the inclusion.

In particular, L is a lagrangian for the nonsingular ϵ-symmetric form (P, ϕ).
(ii) A *nonsingular ϵ-quadratic nilform over R* is a quadruple $(P, \nu, \delta\psi, \psi)$ such that

(a) (P, ν) is an R-nilmodule,
(b) $\psi \in Q_{\epsilon}(P)$ with $N_{\epsilon}(\psi) = \psi + \epsilon\psi^* : P \to P^*$ an isomorphism,
(c) $\delta\psi \in Q_{-\epsilon}(P)$ with
$$N_{-\epsilon}(\delta\psi) \;=\; \nu^*\psi - \psi\nu \in Q^{-\epsilon}(P) \;,$$
so that
$$N_{\epsilon}(\psi)\nu \;=\; \nu^* N_{\epsilon}(\psi) \;:\; P \to P^* \;.$$

Thus (P, ψ) is a nonsingular ϵ-quadratic form over R, $(\delta\psi, \psi) \in Q\mathrm{Nil}_0(C, \nu^*, \epsilon)$ with C as in (i), and there is defined an isomorphism of R-nilmodules
$$N_{\epsilon}(\psi) \;:\; (P, \nu) \to (P^*, \nu^*) \;.$$
A *lagrangian* for $(P, \nu, \delta\psi, \psi)$ is a direct summand $L \subset P$ such that

(d) $\nu(L) \subset L$,
(e) the sequence
$$0 \to L \xrightarrow{i} P \xrightarrow{i^* N_{\epsilon}(\psi)} L^* \to 0$$
is exact, with $i : L \to P$ the inclusion,
(f) $[i^*\psi i] = 0 \in Q_{\epsilon}(L)$,
(g) $[i^*(\delta\psi)i] = 0 \in Q_{-\epsilon}(L)$.

In particular, L is a lagrangian for the nonsingular ϵ-quadratic form (P, ψ).
$\square$

The notion of stable isometry of nilforms is now defined in the usual way using lagrangians and orthogonal direct sums, and $L\mathrm{Nil}^0(R,\epsilon)$ (resp. $L\mathrm{Nil}_0(R,\epsilon)$) is the Witt group of nonsingular ϵ-symmetric (resp. ϵ-quadratic) nilforms over R. See Ranicki [16] (pp. 456-457) for the identification of $L\mathrm{Nil}^1(R,\epsilon)$ (resp. $L\mathrm{Nil}_1(R,\epsilon)$) with the Witt group of nonsingular ϵ-symmetric (resp. ϵ-quadratic) nilformations over R.

Proposition 2.11. (Ranicki [18], Proposition 41.3)
(i) *For any ring with involution R the skew-suspension maps in the nilpotent $\pm\epsilon$-quadratic L-groups are isomorphisms, so that*

$$L\mathrm{Nil}_n(R,\epsilon) \;=\; L\mathrm{Nil}_{n+2}(R,-\epsilon) \;=\; L\mathrm{Nil}_{n+4}(R,\epsilon)\ ,$$

with $L\mathrm{Nil}_{2n}(R,\epsilon)=L\mathrm{Nil}_0(R,(-1)^n\epsilon)$ the Witt group of nonsingular $(-1)^n\epsilon$-quadratic nilforms over R. Similarly for $L\widetilde{\mathrm{Nil}}_*(R,\epsilon)$.
(ii) *If R is a Dedekind ring with involution then*

$$L\mathrm{Nil}^n(R,\epsilon) \;=\; L\mathrm{Nil}^{n+2}(R,-\epsilon) \;=\; L\mathrm{Nil}^{n+4}(R,\epsilon)\ ,$$

$$L\mathrm{Nil}^n(R,\epsilon) \;=\; L^n_p(R,\epsilon)\ ,\ L\widetilde{\mathrm{Nil}}^n(R,\epsilon) \;=\; 0\ (n\geqslant 0)\ .$$

Proof. (i) In order to establish the 4-periodicity use algebraic surgery below the middle dimension, as for the ordinary ϵ-quadratic L-groups $L_n(R,\epsilon)$ in Proposition I.4.3 of [15] (cf. Proposition 2.2 above).
(ii) The explicit proof in the case $n=0$ ([18], p. 588) extends to the general case as follows. Let $(C,\nu,\delta\phi,\phi)$ be an n-dimensional ϵ-symmetric Poincaré nilcomplex over R, representing an element of $L\mathrm{Nil}^n(R,\epsilon)$, with

$$\nu^N \;=\; 0\ :\ C\to C$$

for some $N\geqslant 1$. We reduce to the case $N=1$ using the structure theory of f.g. modules over the Dedekind ring R : every f.g. R-module M fits into a split exact sequence

$$0\to T(M)\to M\to M/T(M)\to 0$$

with

$$T(M) \;=\; \{x\in M\,|\,ax=0\in M \text{ for some } a\neq 0\in R\}$$

the torsion R-submodule and the quotient torsion-free R-module $M/T(M)$ is f.g. projective. In particular, for any R-nilmodule (P,ν) with

$$\nu^N \;=\; 0\ :\ P\to P$$

the R-submodule of P defined by

$$T_N(P,\nu) \;=\; \{x\in P\,|\,ax\in\nu^{N-1}(P) \text{ for some } a\neq 0\in R\}$$

is such that

$$T_N(P,\nu)/\nu^{N-1}(P) \;=\; T(P/\nu^{N-1}(P))\ .$$

The torsion-free quotient R-module

$$(P/\nu^{N-1}(P))/T(P/\nu^{N-1}(P)) \;=\; P/T_N(P,\nu)$$

is f.g. projective, so that $T_N(P,\nu)$ is a direct summand of P. The inclusion defines a morphism of R-nilmodules

$$i \ : \ (T_N(P,\nu),0) \to (P,\nu)$$

Moreover, if (P',ν') is another R-nilmodule with $\nu'^N = 0$ and

$$\theta \ : \ (P,\nu) \to (P',\nu')^* \ = \ (P'^*,\nu'^*)$$

is a morphism of R-nilmodules then

$$i'^*\theta i \ = \ 0 \ : \ T_N(P,\nu) \to T_N(P',\nu')^*$$

since for any $x \in T_N(P,\nu)$, $x' \in T_N(P',\nu')$ there exist $a,a' \neq 0 \in R$, $y \in P$, $y' \in P'$ with

$$ax \ = \ \nu^{N-1}(y) \in P \ , \ a'x' \ = \ \nu'^{N-1}(x') \in P'$$

and

$$\begin{aligned} a'\theta(x)(x')\overline{a} \ &= \ \theta(ax)(a'x') \\ &= \ \theta(\nu^{N-1}(y))(\nu'^{N-1}(y')) \\ &= \ \theta(\nu^{2N-2}(y))(y') \\ &= \ 0 \in R \text{ (since } 2N-2 \geqslant N) \end{aligned}$$

so that

$$\theta(x)(x') \ = \ 0 \in R \ .$$

Returning to the n-dimensional ϵ-symmetric Poincaré nilcomplex $(C,\nu,\delta\phi,\phi)$ with $\nu^N = 0 : C \to C$, let $i : (B,0) \to (C^{n-*},\nu^*)$ be the inclusion of the subcomplex defined by

$$B_r \ = \ T_N(C^{n-r},\nu^*) \ .$$

The chain map of R-nilmodule chain complexes defined by

$$f \ = \ i^* \ : \ (C,\nu) \to (D,0) \ = \ (B^{n-*},0)$$

is such that

$$f^*(\delta\phi,\phi) \ = \ 0 \in Q\mathrm{Nil}^n(D,0,\epsilon) \ .$$

Algebraic surgery on $(C,\nu,\delta\phi,\phi)$ using the $(n+1)$-dimensional ϵ-symmetric nilpair $(f : (C,\nu) \to (D,0),(0,(\delta\phi,\phi)))$ over R results in a cobordant n-dimensional ϵ-symmetric Poincaré nilcomplex $(C',\nu',\delta\phi',\phi')$ over R with

$$\nu' \ \simeq \ 0 \ : \ C' \to C' \ .$$

$\square$

2.3. **The unitary nilpotent L-groups** UNil**.** Let R be any ring. An *involution* on an R-R bimodule $\mathcal{A}$ is a homomorphism

$$\mathcal{A} \to \mathcal{A} \; ; \; a \mapsto \bar{a}$$

which satisfies

$$\bar{\bar{a}} \;=\; a \;,\; \overline{ras} \;=\; \bar{s}\bar{a}\bar{r} \text{for all } a \in \mathcal{A},\; r, s \in R \;.$$

For any left R-module P there is defined a left R-module

$$\mathcal{A}P \;=\; \mathcal{A} \otimes_R P \;.$$

As in the special case $\mathcal{A} = R$ write the evaluation pairing as

$$\langle\;,\;\rangle \;:\; \mathcal{A}P^* \times P \to \mathcal{A} \;;\; (a \otimes f, x) \mapsto \langle a \otimes f, x\rangle \;=\; af(x) \;.$$

An element $\phi \in \mathrm{Hom}_R(P, \mathcal{A}P^*)$ determines a $\mathcal{A}$-valued sesquilinear form on P

$$\langle\;,\;\rangle_\phi \;:\; P \times P \to \mathcal{A} \;;\; (x, y) \mapsto \langle \phi(x), y\rangle \;,$$

and we identify $\mathrm{Hom}_R(P, \mathcal{A}P^*)$ with the additive group of such forms. For $\epsilon = \pm 1$ and a f.g. projective P define an involution

$$T_\epsilon \;:\; \mathrm{Hom}_R(P, \mathcal{A}P^*) \to \mathrm{Hom}_R(P, \mathcal{A}P^*) \;;\; \phi \mapsto \epsilon\phi^* \;,\; \langle x, y\rangle_{\phi^*} \;=\; \overline{\langle y, x\rangle_\phi} \;.$$

One then defines a map

$$N_\epsilon \;=\; 1 + T_\epsilon \;:\; \mathrm{Hom}_R(P, \mathcal{A}P^*) \to \mathrm{Hom}_R(P, \mathcal{A}P^*) \;; \phi \mapsto \phi + \epsilon\phi^t \tag{2.2}$$

with

$$\langle x, y\rangle_{N_\epsilon(\phi)} \;=\; \langle x, y\rangle_\phi + \epsilon\overline{\langle y, x\rangle_\phi} \;.$$

An $\mathcal{A}$-valued ϵ-symmetric form (P, λ) (resp. ϵ-quadratic form (P, μ)) over R is a f.g. projective R-module P together with an element of the group

$$\lambda \in Q^\epsilon(P, \mathcal{A}) \;=\; \ker(1 - T_\epsilon : \mathrm{Hom}_R(P, \mathcal{A}P^*) \to \mathrm{Hom}_R(P, \mathcal{A}P^*)) \;,$$

$$\mu \in Q_\epsilon(P, \mathcal{A}) \;=\; \mathrm{coker}(1 - T_\epsilon : \mathrm{Hom}_R(P, \mathcal{A}P^*) \to \mathrm{Hom}_R(P, \mathcal{A}P^*)) \;.$$

As usual, for $\lambda \in Q^\epsilon(P, \mathcal{A})$ we write

$$\lambda(x, y) \;=\; \langle \lambda(x), y\rangle \in \mathcal{A}$$

and for $\mu \in Q_\epsilon(P, \mathcal{A})$ we write

$$\mu(x) \;=\; \langle \mu(x), x\rangle \in \mathcal{A}/\{a - \epsilon\bar{a} | a \in \mathcal{A}\} \;.$$

The map N_ϵ induces a well defined map:

$$N_\epsilon \;:\; Q_\epsilon(P, \mathcal{A}) \to Q^\epsilon(P, \mathcal{A}) \;;\; [\mu] \mapsto \mu + \epsilon\mu^t \;. \tag{2.3}$$

Definition 2.12. (Cappell [4])
(i) Let $\mathcal{B}_1, \mathcal{B}_{-1}$ be R-bimodules with involution. Assume $\mathcal{B}_1, \mathcal{B}_{-1}$ are free as right R-modules. A *nonsingular ϵ-quadratic unilform over* $(R; \mathcal{B}_1, \mathcal{B}_{-1})$ is a quadruple

$$(P_1, P_{-1}, \mu_1, \mu_{-1})$$

where, for $\delta = \pm 1$, we require:

(a) (P_δ, μ_δ) is a stably f.g. free $\mathcal{B}_\delta$-valued ϵ-quadratic form over R,

(b) $P_\delta = P^*_{-\delta}$; we then identify $(P^*_\delta)^* = P_\delta$ in the usual way, and write the evaluation pairing as

$$\langle\ ,\ \rangle\ :\ P_1 \times P_{-1} \to R\ ;\ (x,f) \mapsto f(x)\ .$$

(c) If $\lambda_\delta = N_\epsilon(\mu_\delta)$ is the associated ϵ-symmetric form to μ_δ, then the composite

$$P_1 \xrightarrow{\lambda_1} \mathcal{B}_1 P_{-1} \xrightarrow{\lambda_{-1}\otimes 1} \mathcal{B}_{-1}\mathcal{B}_1 P_1 \xrightarrow{\lambda_1\otimes 1} \mathcal{B}_1\mathcal{B}_{-1}\mathcal{B}_1 P_{-1} \xrightarrow{\lambda_{-1}\otimes 1} \dots$$

is eventually zero. (See [4] for the precise meaning of 'eventually zero', which involves choices of filtration on P_1 and P_{-1}.)

(ii) A *sublagrangian* for $(P_1, P_{-1}, \mu_1, \mu_{-1})$ is a pair of stably f.g. free direct summands $V_1 \subset P_1$, $V_{-1} \subset P_{-1}$ such that, for $\delta = \pm 1$

(2.4) $$\langle V_1, V_{-1}\rangle\ =\ 0\ ,\ \lambda_\delta(V_\delta) \subset \mathcal{B}_\delta V_{-\delta}\ ,\ \mu_\delta(V_\delta)\ =\ 0\ .$$

We call (V_1, V_{-1}) a *lagrangian* if in addition:

(2.5) $$V_1\ =\ V_{-1}^\perp.$$

$\square$

One can form orthogonal direct sums of ϵ-quadratic unilforms over $(R; \mathcal{B}_1, \mathcal{B}_{-1})$ in a rather obvious way. Cappell [4] defined $\mathrm{UNil}_{2n}(R; \mathcal{B}_1, \mathcal{B}_{-1})$ to be the Witt group of stable isometry classes of nonsingular $(-1)^n$-quadratic unilforms over $(R; \mathcal{B}_1, \mathcal{B}_{-1})$ modulo those admitting lagrangians, and showed (geometrically) that if π_{-1}, π_0, π_1 are finitely presented groups with $\pi_0 \subseteq \pi_{-1}$, $\pi_0 \subseteq \pi_1$ and

$$\pi\ =\ \pi_{-1} *_{\pi_0} \pi_1\ ,\ R\ =\ \mathbb{Z}[\pi_0]\ ,\ \mathcal{B}_{\pm 1}\ =\ \mathbb{Z}[\pi_{\pm 1} \backslash \pi_0]$$

then the morphism defined by

$$\begin{aligned}&\mathrm{UNil}_{2n}(R; \mathcal{B}_1, \mathcal{B}_{-1}) \to L_{2n}(\mathbb{Z}[\pi])\ ;\\ &(P_1, P_{-1}, \mu_1, \mu_{-1}) \mapsto (\mathbb{Z}[\pi] \otimes_{\mathbb{Z}[\pi_0]} (P_1 \oplus P_{-1}), \begin{pmatrix} \mu_1 & 1 \\ 0 & \mu_{-1}\end{pmatrix})\end{aligned}$$

is a split monomorphism.

If an ϵ-quadratic unilform $u = (P_1, P_{-1}, \mu_1, \mu_{-1})$ has a sublagrangian (V_1, V_{-1}), then one can form a new ϵ-quadratic unilform (see Connolly and Koźniewski [8], 6.3 (f))

$$u'\ =\ (V_{-1}^\perp / V_1, V_1^\perp / V_{-1}, \mu_1', \mu_{-1}')\ ,$$

so that

$$[u]\ =\ [u'] \in \mathrm{UNil}_{2n}(R; \mathcal{B}_1, \mathcal{B}_{-1})\ .$$

2.4. **The proof of Theorem A in the even-dimensional case.** We begin by defining maps:

$$L\widetilde{\mathrm{Nil}}_{2n}(R) \xrightarrow{c} \mathrm{UNil}_{2n}(R;R,R) \xrightarrow{r} NL_{2n}(R) \subset L_{2n}(R[x]).$$

The proof will show that the maps c, r are both isomorphisms.

Let $\epsilon = (-1)^n$.

Definition 2.13. The map

$$r \ : \ \mathrm{UNil}_{2n}(R;R,R) \to NL_{2n}(R) \ ; \ u \mapsto r(u)$$

sends an ϵ-quadratic unilform $u = (P_1, P_{-1}, \mu_1, \mu_{-1})$ over $(R;R,R)$ to the ϵ-quadratic form $r(u)$ over $R[x]$ given by:

$$r(u) \ = \ (P_1[x] \oplus P_{-1}[x], \psi_0 + x\psi_1)$$

where

$$\psi_0 \ = \ \begin{pmatrix} 0 & 1 \\ 0 & \mu_{-1} \end{pmatrix} \ , \ \psi_1 \ = \ \begin{pmatrix} \mu_1 & 0 \\ 0 & 0 \end{pmatrix} \ .$$

□

Here, $\psi_i : (P_1 \oplus P_{-1})[x] \to (P_{-1} \oplus P_1)[x]$ $(i = 0, 1)$ is the $R[x]$-module morphism induced, using change of coefficients, from the R-module morphism of the same name

$$\psi_i \ : \ (P_1 \oplus P_{-1}) \to (P_1 \oplus P_{-1})^* \ = \ (P_{-1} \oplus P_1) \ .$$

In order to verify that r is well-defined, first notice that

$$N_\epsilon(\psi_0 + x\psi_1) \ = \ \begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix} (1+\nu) \ : \ (P_1 \oplus P_{-1})[x] \to (P_1^* \oplus P_{-1}^*)[x]$$

where

$$\nu \ = \ \begin{pmatrix} 0 & \epsilon\lambda_{-1} \\ x\lambda_1 & 0 \end{pmatrix} \ : \ (P_1 \oplus P_{-1})[x] \to (P_1 \oplus P_{-1})[x] \ , \quad \lambda_{\pm 1} := \ N_\epsilon(\mu_{\pm 1}) \ .$$

Because

$$\nu^2 \ = \ \begin{pmatrix} x\epsilon\lambda_{-1}\lambda_1 & 0 \\ 0 & x\lambda_1\lambda_{-1} \end{pmatrix}$$

Definition (2.12) shows that ν is obviously nilpotent. Therefore $N_\epsilon(\psi_0 + x\psi_1)$ is nonsingular.

To see that $[r(u)] \in NL_{2n}(R)$, notice that $\eta_*[r(u)] = [P_1 \oplus P_{-1}, \psi_0]$, and that $P_1 \oplus 0$ is a lagrangian for $(P_1 \oplus P_{-1}, \psi_0)$.

The rule $u \mapsto r(u)$ preserves orthogonal direct sums of forms. If (V_1, V_{-1}) is a lagrangian for u, then $V_1[x] \oplus V_{-1}[x]$ is a lagrangian for $r(u)$. We thus have a well-defined homomorphism:

$$r \ : \ \mathrm{UNil}_{2n}(R;R,R) \to NL_{2n}(R) \ .$$

Definition 2.14. The map

$$c \ : \ L\widetilde{\mathrm{Nil}}_{2n}(R) \to \mathrm{UNil}_{2n}(R;R,R) \ ; \ z \mapsto c(z)$$

sends a nonsingular ϵ-quadratic nilform $z = (P, \nu, \delta\psi, \psi)$ over R (see 2.10) to $c(z) = (P_1, P_{-1}, \mu_1, \mu_{-1})$, where

$$P_1 \ = \ P \ , \ P_{-1} \ = \ P^* \ , \ \mu_1 \ = \ \delta\psi - \nu^*\psi \ , \ \mu_{-1} \ = \ -\phi^{-1}\psi^*\phi^{-1}$$

with $\phi = N_\epsilon(\psi) : P \to P^*$ an isomorphism. □

Using (2.10) set

$$\lambda_1 \ = \ N_\epsilon(\mu_1) \ = \ -\phi\nu \ = \ -\nu^*\phi \ ,$$

noting that $N_{-\epsilon}N_\epsilon(\delta\psi) = 0$. Set also

$$\lambda_{-1} \ = \ N_\epsilon(\mu_{-1}) \ = \ \epsilon\phi^{-1} \ .$$

Because $\lambda_{-1}\lambda_1 = \epsilon\nu$, and ν is nilpotent, it follows that $c(z)$ is an ϵ-quadratic unilform over $(R; R, R)$. The rule $z \mapsto c(z)$ preserves orthogonal direct sums. Moreover, if N is a lagrangian for z, then $(N, N^\perp)$ is a lagrangian for $c(z)$. Therefore, Definition (2.14) defines a homomorphism:

$$c \ : \ L\widetilde{\mathrm{Nil}}_{2n}(R) \to \mathrm{UNil}_{2n}(R;R,R) \tag{2.6}$$

Definition 2.15. The morphism

$$j \ : \ L\widetilde{\mathrm{Nil}}_{2n}(R) \to NL_{2n}(R) \ ; \ y \mapsto j(y)$$

sends $y = [P, \nu, \delta\psi, \psi]$ to

$$j(y) \ = \ [P[x], \psi + x(\delta\psi - \nu^*\psi)] \ .$$ □

It was proved in Ranicki [16], p. 445 that j is in fact an isomorphism. See (2.16) for the precise matching up of the formula in (2.15) with the morphism defined there.

The right hand side in (2.15) gives a nonsingular form because:

$$N_\epsilon(\psi + x(\delta\psi - \nu^*\psi)) \ = \ N_\epsilon(\psi)(1 - x\nu),$$

an isomorphism by Definition 2.10. Moreover this right hand side is in $NL_{2n}(R)$, also by (2.10).

Remark 2.16. In order to obtain the formula in Definition 2.15 for $j(y)$ from the formula in [16], p. 445 one must make the following translation of the terminology there to our terminology:

$$A \ = \ R \ , \ C^0 \ = \ P \ , \ C^i \ = \ 0 \text{ for } i \neq 0 \ ,$$
$$\psi_0 \ = \ \psi \ , \ \delta\psi_1 \ = \ \delta\psi \ ,$$

noting that the x^{-1} is our x, and the ν^* there is our ν. In the following argument we shall use the group $L\mathrm{Nil}_{2n}(R)$ and the split injection

$$\Delta \ : \ L\mathrm{Nil}_{2n}(R) \to L_{2n}(R[x,x^{-1}]) \ ; \ [P,\nu,\delta\psi,\psi] \mapsto [P[x,x^{-1}],(x^{-1}-\nu^*)\psi+\delta\psi]$$

defined there, along with the splitting map

$$\partial \ : \ L_{2n}(R[x,x^{-1}]) \to L\mathrm{Nil}_{2n}(R)$$

and the natural inclusion and projection:

$$L\widetilde{\mathrm{Nil}}_{2n}(R) \xrightarrow{i} L\mathrm{Nil}_{2n}(R) \xrightarrow{p} L\widetilde{\mathrm{Nil}}_{2n}(R) \ .$$

Let $\tilde{E} \ : \ NL_{2n}(R) \to L_{2n}(R[x,x^{-1}])$ be the restriction of the natural monomorphism

$$E \ : \ L_{2n}(R[x]) \to L_{2n}(R[x,x^{-1}]) \ .$$

Also, set

$$\begin{aligned} \tilde{\partial} \ &= \ p\partial \ : \ L_{2n}(R[x,x^{-1}]) \to L\widetilde{\mathrm{Nil}}_{2n}(R) \ , \\ \tilde{\Delta} \ &= \ \Delta i \ : \ L\widetilde{\mathrm{Nil}}_{2n}(R) \to L_{2n}(R[x,x^{-1}]) \ . \end{aligned}$$

Because $\partial\Delta = 1$, we get $\tilde{\partial}\tilde{\Delta} = 1$. According to the braid on page 448 of [16], $\tilde{\partial}\tilde{E}$ is an isomorphism. The map j of Definition 2.15 is $j = (\tilde{\partial}\tilde{E})^{-1}$. To get the formula for j in Definition 2.15, note that the "devissage" map $\tilde{\partial}$ satisfies:

$$\tilde{\partial} \ = \ \tilde{\partial} M$$

with

$$M \ : \ L_n(R[x,x^{-1}]) \to L_n(R[x,x^{-1}]) \ ; \ (P,\psi) \mapsto (P,x\psi) \ .$$

Then from [16], p. 445, we translate and find:

$$\begin{aligned} \tilde{\Delta}(y) \ &= \ \tilde{\Delta}([P,\nu,\delta\psi,\psi]) \\ &= \ [P[x,x^{-1}], x^{-1}\{\psi + x(\delta\psi - \nu^*\psi)\}] \ . \end{aligned}$$

So

$$\begin{aligned} j(y) \ &= \ j(\tilde{\partial} M \tilde{\Delta}(y)) \\ &= \ \tilde{E}^{-1} M \tilde{\Delta}(y) \\ &= \ \tilde{E}^{-1}([P[x,x^{-1}], \psi + x(\delta\psi - \nu^*\psi)]) \\ &= \ [P[x], \psi + x(\delta\psi - \nu^*\psi)] \ , \end{aligned}$$

as in 2.15. □

As explained above, [16] proves that j is an isomorphism.

Remark 2.17. The inverse of j

$$k \ = \ j^{-1} \ : \ NL_{2n}(R) \to L\widetilde{\mathrm{Nil}}_{2n}(R) \tag{2.7}$$

can be computed via Higman linearization (see Connolly and Koźniewski [8], 3.6 (a)) in the following way. By Higman linearization, each element of $NL_{2n}(R)$ can be represented in the form $[P[x], \psi_0 + x\psi_1]$. In these terms, the formula for $k = j^{-1}$ is:

$$k[P[x], \psi_0 + x\psi_1] \ = \ [P, \nu, \psi, \delta\psi] \ ,$$

where

$$\psi \;=\; \psi_0\ ,\ \nu \;=\; (N_\epsilon(\psi_0))^{-1}N_\epsilon(\psi_1)\ ,\ \delta\psi \;=\; \nu^*\psi_0+\psi_1\ .$$

It is clear that $jk=1$. □

We now turn to the proof of Theorem A in even dimensions. We only have to show that:

$$\text{(2.8)}\qquad \text{(i) } ckr \;=\; 1\ ,\ \text{(ii) } rc \;=\; j\ .$$

The proof of 2.8 (i) is easiest: let $(P_1,P_{-1},\mu_1,\mu_{-1})$ be an ϵ-quadratic unilform over $(R;R,R)$. By Definitions 2.13,2.14 and 2.17, and direct calculation, we obtain:

$$\text{(2.9)}\qquad ckr[P_1,P_{-1},\mu_1,\mu_{-1}] \;=\; [P_1\oplus P_{-1},P_{-1}\oplus P_1,\tilde{\mu}_1,\tilde{\mu}_{-1}]$$

where

$$\tilde{\mu}_1 \;=\; \begin{pmatrix}\mu_1 & 0\\ 0 & 0\end{pmatrix},\tilde{\mu}_{-1} \;=\; \begin{pmatrix}\mu_{-1} & 1\\ 0 & 0\end{pmatrix}\ .$$

Perform a sublagrangian construction on the right hand side of 2.9, using the sublagrangian

$$V_1 \;=\; 0\oplus P_{-1}\ ,\ V_{-1} \;=\; 0\ .$$

This yields:

$$[P_1\oplus P_{-1},P_{-1}\oplus P_1,\tilde{\mu}_1,\tilde{\mu}_{-1}] \;=\; [P_1,P_{-1},\mu_1,\mu_{-1}].$$

Therefore $ckr=1$, proving 2.8 (i).

Next we prove 2.8 (ii).

Suppose $a=[P,\nu,\delta\psi,\psi]\in L\widetilde{\mathrm{Nil}}_{2n}(R)$. By direct calculation and Definitions 2.13, 2.14, we have

$$\text{(2.10)}\qquad rc(a) \;=\; [P[x]\oplus P^*[x],\Psi_0+x\Psi_1]$$

where

$$\Psi_0 \;=\; \begin{pmatrix}0 & 1\\ 0 & -\phi^{-1}\psi^*\phi^{-1}\end{pmatrix}\ ,\ \Psi_1 \;=\; \begin{pmatrix}\delta\psi-\nu^*\psi & 0\\ 0 & 0\end{pmatrix}$$

with $\phi=N_\epsilon(\psi)$. By hypothesis (see Definition 2.10), (P,ψ) admits a lagrangian, say $N\subset P$. Let

$$V \;=\; (\phi N)[x]\subset P^*[x]\subset P[x]\oplus P^*[x]\ .$$

By 2.10 V is a sublagrangian for $\Psi_0+x\Psi_1$. In fact, setting $\Phi=N_\epsilon(\Psi_0+x\Psi_1)$, one readily computes that the Φ-orthogonal complement of V is

$$V_\Phi^\perp \;=\; \{(u,v)\in P[x]\oplus P^*[x]\,|\,\phi(u)-v\in V\}\ .$$

Therefore one obtains an isomorphism

$$g\ :\ P[x]\to V_\Phi^\perp/V\ ;\ u\mapsto(u,\phi(u))\ .$$

Let $(V_\Phi^\perp/V,\Psi')$ be the sublagrangian construction on $cr(a)$ using V. We claim that

$$\text{(2.11)}\qquad g\ :\ (P[x],\psi+x(\delta\psi-\nu^*\psi))\to(V_\Phi^\perp/V,\Psi')$$

is an isometry. Since the right hand side of 2.11 represents $rc(a)$, and the left hand side is $j(a)$, this claim (2.11) will prove (2.8) (ii).

We prove (2.11) using the duality pairing

$$\begin{aligned}\{\ ,\ \}\ :\ & (P^*[x]\oplus P[x])\times(P[x]\oplus P^*[x])\to R[x]\ ;\\ & ((\xi,\eta),(\eta',\xi'))\mapsto\{(\xi,\eta),(\eta',\xi')\}\ =\ \langle\xi,\eta'\rangle+\overline{\langle\xi',\eta\rangle}\ .\end{aligned}$$

(2.11) amounts to the identity:

$$(2.12)\quad \langle[\psi+x(\delta\psi-\nu^*\psi)](u),v\rangle\ =\ \{\Psi(u,\phi(u)),(v,\phi(v))\}\ (u,v\in P[x])\ ,$$

where $\Psi=\Psi_0+x\Psi_1$. The right hand side of 2.12 is computed from 2.10 as:

$$\begin{aligned}&\langle[\phi+x(\delta\psi-\nu^*\psi)](u),v\rangle+\overline{\langle\phi(v),-\phi'\psi^*(u)\rangle}\\ &\qquad=\ \langle[\phi+x(\delta\psi-\nu^*\psi](u),v\rangle+\langle-\phi^*\phi^{-1}\psi^*(u),v\rangle\\ &\qquad=\ \langle[(\phi-\epsilon\psi^*)+x(\delta\psi-\nu^*\psi)](u),v\rangle,\end{aligned}$$

which is the left hand side of 2.12. This proves 2.11 and therefore also 2.8 (ii). Therefore the proof of Theorem A, when n is even, is complete.

Remark 2.18. It seems appropriate to record here an explicit formula for the inverse isomorphism

$$c^{-1}\ :\ \mathrm{UNil}_{2n}(R;R,R)\to L\widetilde{\mathrm{Nil}}_{2n}(R)$$

which can be derived from 2.17,2.8 and Definition 2.13, as follows.

For $[P_1,P_{-1},\mu_1,\mu_{-1}]\in\mathrm{UNil}_{2n}(R;R,R)$ we have:

$$(2.13)\qquad c^{-1}([P_1,P_{-1},\mu_1,\mu_{-1}])\ =\ (P_1\oplus P_{-1},\nu,\delta\psi,\psi)$$

where

$$\begin{aligned}\psi\ &=\ \begin{pmatrix}0&1\\0&\mu_{-1}\end{pmatrix}:P_1\oplus P_{-1}\to P_{-1}\oplus P_1\ ,\ \psi_1\ =\ \begin{pmatrix}\mu_1&0\\0&0\end{pmatrix}\ ,\\ \delta\psi\ &=\ \nu^*\psi+\psi_1\ :\ P_1\oplus P_{-1}\to P_{-1}\oplus P_1\ ,\\ \nu\ &=\ -N_\epsilon(\psi)^{-1}N_\epsilon(\psi_1)\ =\ \begin{pmatrix}\epsilon\lambda_{-1}\lambda_1&0\\-\lambda_1&0\end{pmatrix}\end{aligned}$$

with $\lambda_{\pm1}=N_\epsilon(\mu_{\pm1})$. □

2.5. **The proof of Theorem A in the odd-dimensional case.** We begin by commenting that the "simple L-theory" version of Theorem A, in even dimensions, proceeds uneventfully, along the same lines as above. We explain this in some detail now.

$\mathrm{UNil}^s_{2n}(R;\mathcal{B}_1,\mathcal{B}_{-1})$ is defined in Cappell [4] (p.1118). Also,

$$L\mathrm{Nil}^s_n(R)\ =\ L^s_n(R)\oplus L\widetilde{\mathrm{Nil}}^s_n(R)$$

is defined in ([16], p. 466-468), where there are also constructed exact sequences:

$$0 \to L_n^{I_+}(R[x]) \xrightarrow{E^s} L_n^J(R[x,x^{-1}]) \longrightarrow L_n^p(R) \oplus L\widetilde{\mathrm{Nil}}_n^s(R) \to 0$$

$$0 \to L_n^{I_-}(R[x^{-1}]) \longrightarrow L_n^J(R[x,x^{-1}]) \xrightarrow{\partial^s} L_n^p(R) \oplus L\widetilde{\mathrm{Nil}}_n^s(R) \to 0$$

where

$$\begin{aligned} I_\pm &= \widetilde{K}_1(R) \subseteq \widetilde{K}_1(R[x^{\pm 1}]) , \\ J &= \widetilde{K}_1(R) \oplus K_0(R) \subseteq \widetilde{K}_1(R[x,x^{-1}]) . \end{aligned}$$

Define

$$NL_n^s(R) = \ker(\eta_* : L_n^{K_1(R)}(R[x]) \to L_n(R))$$

and let $\widetilde{\Delta}^s, \tilde{E}^s, \tilde{\partial}^s$ be as in Remark 2.16, concluding that $\tilde{\partial}^s\tilde{E}^s$ is an isomorphism. As before, define

$$j^s = (\tilde{\partial}^s\tilde{E}^s)^{-1} : L\widetilde{\mathrm{Nil}}_{2n}(R) \to NL_{2n}^s(R)$$

using formula 2.15. The maps

$$L\widetilde{\mathrm{Nil}}_{2n}^s(R) \xrightarrow{c^s} \mathrm{UNil}_{2n}^s(R;R,R) \xrightarrow{r^s} NL_{2n}^s(R), \text{and } k^s = (j^s)^{-1}$$

are now defined exactly as in (2.13)–(2.17), and the proof that these are isomorphisms can now be repeated without change. In summary, we have:

Proposition 2.19. *The maps* $L\widetilde{\mathrm{Nil}}_{2n}^s(R) \xrightarrow{c^s} \mathrm{UNil}_{2n}^s(R;R,R) \xrightarrow{r^s} NL_{2n}^s(R)$ *described in the paragraph above are isomorphisms. Moreover,* $j^s = r^s c^s$.

We now complete the proof of Theorem A in odd dimensions.

Let $S = R[z,z^{-1}]$, extending the involution on R to S by

$$\overline{z} = z^{-1} .$$

Let $i : R \to S$ be the inclusion. The split exact sequence of Shaneson [20] and Ranicki [13]

$$0 \to L_n^s(R) \xrightarrow{i_*} L_n^s(S) \to L_{n-1}(R) \to 0$$

yields the split exact sequence

$$0 \to NL_n^s(R) \xrightarrow{i_*} NL_n^s(S) \to NL_{n-1}(R) \to 0 .$$

Cappell [4] defined $\mathrm{UNil}_{2n-1}(R;R,R)$ as the cokernel in the split exact sequence:

$$0 \to \mathrm{UNil}_{2n}^s(R;R,R) \to \mathrm{UNil}_{2n}^s(S;S,S) \to \mathrm{UNil}_{2n-1}(R;R,R) \to 0.$$

The isomorphism r^s of Proposition 2.19, being functorial, therefore induces an isomorphism:

$$r : \mathrm{UNil}_{2n-1}(R;R,R) \to NL_{2n-1}(R) .$$

This proves Theorem A.

3. Chain bundles and the proof of Theorem B.

3.1. **Universal chain bundles.** We begin with a resumé of the results of Ranicki [15],[19] and Weiss [22] which we need. As in Section 2, A is a ring with involution.

A *chain bundle* (B,β) over R is an R-module chain complex B together with a 0-cycle

$$\beta \in (\widehat{W}^{\%}B^{-*})_0 \ .$$

A map of chain bundles $f:(C,\gamma)\to(B,\beta)$ is a chain map $f:C\to B$ such that

$$[\widehat{f}^{\%}(\beta)] \ = \ [\gamma] \in \widehat{Q}^0(C^{-*}) \ ,$$

with $\widehat{f}^{\%}:\widehat{W}^{\%}B^{-*}\to W^{\%}C^{-*}$ the chain map induced by f. Each chain bundle (B,β) determines a homomorphism

$$J_\beta \ : \ Q^n(B)\to\widehat{Q}^n(B) \ ; \ \phi\mapsto J(\phi)-\widehat{\phi}_0^{\%}(S^n\beta) \tag{3.1}$$

where J is as in 2.1, $\phi_0^{\%}$ is the map induced by $\phi_0:B^{n-*}\to B$, and $S^n:\widehat{W}^{\%}C\to\Sigma^{-n}\widehat{W}^{\%}\Sigma^n(C)$ is the natural isomorphism of chain complexes. The map J_β is not induced by a chain map.

The Tate $\mathbb{Z}_2$-cohomology group

$$\widehat{H}^r(\mathbb{Z}_2;R) \ = \ \{x\in R\,|\,\bar{x}=(-1)^r x\}/\{y+(-1)^r\bar{y}\,|\,y\in R\}$$

is a left R-module via

$$R\times\widehat{H}^r(\mathbb{Z}_2;R)\to\widehat{H}^r(\mathbb{Z}_2;R) \ ; \ (a,x)\mapsto ax\bar{a} \ .$$

The *Wu classes* of a chain bundle (B,β) are the R-module morphisms

$$v_r(\beta) \ : \ H_r(B)\to\widehat{H}^r(\mathbb{Z}_2;R) \ ; \ x\mapsto\langle\beta_{-2r},x\otimes x\rangle \ (r\in\mathbb{Z}) \ . \tag{3.2}$$

The *universal chain bundle* (B^R,β^R) exists for each R. It is the chain bundle (unique up to equivalence) characterized by the requirement that the map 3.2 is an isomorphism for each r. This implies the more general property that for each f.g. free chain complex C the map

$$k_C \ : \ H_n(C\otimes_R B^R)\to\widehat{Q}^n(C) \ ; \ f\mapsto S^{-n}f^{\%}(\beta) \tag{3.3}$$

is an isomorphism. A cycle $f\in(C\otimes_R B^R)_n$ is a chain map $f:(B^R)^{-*}\to S^{-n}C$, inducing a morphism

$$S^{-n}f^{\%} \ : \ \widehat{Q}^0((B^R)^{-*})\to\widehat{Q}^0(S^{-n}C) \ = \ \widehat{Q}^n(C) \ .$$

See Weiss [22] and Ranicki [19].

3.2. **The chain bundle exact sequence and the theorem of Weiss.** For each chain bundle (B,β), the map J_β above fits into an exact sequence:

$$\cdots \to \widehat{Q}^{n+1}(B) \xrightarrow{H} Q_n(B,\beta) \xrightarrow{N_\beta} Q^n(B) \xrightarrow{J_\beta} \widehat{Q}^n(B) \to \dots \tag{3.4}$$

where the group $Q_n(B,\beta)$ of "*twisted quadratic structures*" and the maps N_β and H are defined as follows.

$Q_n(B,\beta)$ is defined as the abelian group of equivalence classes of pairs (ϕ,θ) (called *symmetric structures on* (B,β)) where $\phi \in (W^{\%}B)_n$, $\theta \in (\widehat{W}^{\%}B)_{n+1}$ satisfy

$$d\phi \;=\; 0 \;,\; d\theta \;=\; J_\beta(\phi) \;.$$

The addition is defined by

$$(\phi,\theta)+(\phi',\theta') \;=\; (\phi+\phi',\theta+\theta'+\xi) \text{ where } \xi_s \;=\; \phi_0\beta_{s-n+1}\phi'_0 \;.$$

One says that (ϕ,θ) is equivalent to (ϕ',θ') if there exist $\zeta \in (W^{\%}B)_{n+1}$, $\eta \in (\widehat{W}^{\%}B)_{n+2}$ such that

$$d\zeta \;=\; \phi'-\phi \;,\; d\eta \;=\; \theta'-\theta+J(\zeta)+(\zeta_0,\phi_0,\phi'_0)^{\%}(S^n\beta) \;.$$

Here $(\zeta_0,\phi_0,\phi'_0)^{\%} : (\widehat{W}^{\%}B^{-*})_n \to (\widehat{W}^{\%}B)_{n+1}$ is the chain homotopy from $\phi_0^{\%}$ to $(\phi')_0^{\%}$ induced by ζ_0. (See Ranicki [19], section 3).

The map H is defined by: $H(\theta)=[0,\theta]$.

The map N_β is defined by: $N_\beta([\phi,\theta])=[\phi]$.

When $\beta=0$, then $Q_n(B,0)=Q_n(B)$ and 3.4 reduces to 2.1.

Recall now from ([16],p.19,p.39, p.137), the cobordism groups $L_n(R,\epsilon)$ (resp. $L^n(R,\epsilon),\widehat{L}^n(R,\epsilon)$) of free n-dimensional ϵ-quadratic (resp. symmetric, resp. hyperquadratic) Poincaré complexes over R, where $\epsilon=\pm 1$. These are related by a long exact sequence and a skew-suspension functor:

(3.5)

$$\begin{array}{ccccccc}
\widehat{L}^{n+1}(R,\epsilon) & \xrightarrow{H} & L_n(R,\epsilon) & \longrightarrow & L^n(R,\epsilon) & \longrightarrow & \widehat{L}^n(R,\epsilon) \\
\big\downarrow \widehat{\overline{S}}^{n+1} & & \big\downarrow \overline{S}_n & & \big\downarrow \overline{S}^n & & \big\downarrow \widehat{\overline{S}}^n \\
\widehat{L}^{n+3}(R,-\epsilon) & \longrightarrow & L_{n+2}(R,-\epsilon) & \longrightarrow & L^{n+2}(R,-\epsilon) & \longrightarrow & \widehat{L}^{n+2}(R,-\epsilon).
\end{array}$$

$\overline{S}_n$ is an isomorphism for all n, and $L_n(R,1)$ is the Wall surgery obstruction group, $L_n(R)$. But $\widehat{\overline{S}}^n$ and $\overline{S}^n$ are not isomorphisms in general. Instead, the main result of Weiss [22] (see also Ranicki [19]) identifies the limit of the maps $\widehat{\overline{S}}^n$ in terms of a functorial isomorphism:

$$\lim_{k\to\infty} \widehat{L}^{n+2k}(R,(-1)^k) \xrightarrow{\cong} Q_n(B^R,\beta^R). \tag{3.6}$$

The skew-suspension maps $\widehat{S}^n$, S^n are isomorphisms for 0-dimensional R.

3.3. UNil **and** 0**-dimensional rings.** Recall from Definition 2.2 that a ring R is said to be 0-dimensional if it is hereditary and noetherian.

Proposition 3.1. *For any* 0*-dimensional ring* R *with involution, and any* $n \geqslant 0$*, there is a short exact sequence:*

$$0 \to \mathrm{UNil}_n(R;R,R) \to Q_{n+1}(B^{R[x]},\beta^{R[x]}) \to Q_{n+1}(B^R,\beta^R) \to 0$$

Proof. Following 2.8 define

$$NQ_n(R) \;=\; \ker\{Q_n(B^{R[x]},\beta^{R[x]}) \to Q_n(B^R,\beta^R)\}. \tag{3.7}$$

By Proposition 2.9, $NL^n(R) = 0$ for all $n \geqslant 0$. So by 3.5 we get a square of isomorphisms, for all $n \geqslant 0$:

$$\begin{array}{ccc} N\widehat{L}^{n+1}(R,\epsilon) & \xrightarrow{\cong} & NL_n(R,\epsilon) \\ {\scriptstyle \widehat{S}^n}\Big\downarrow{\scriptstyle \cong} & & {\scriptstyle S_n}\Big\downarrow{\scriptstyle \cong} \\ N\widehat{L}^{n+3}(R,-\epsilon) & \xrightarrow{\cong} & NL_{n+2}(R,-\epsilon). \end{array} \tag{3.8}$$

By Theorem A,3.6,3.7, and 3.8, for all $n \geqslant 0$, we have:

$$\begin{aligned} \mathrm{UNil}_n(R;R,R) \;&\cong\; NL_n(R,1) \;\cong\; N\widehat{L}^{n+1}(R,1) \\ &\cong\; \lim_k\, N\widehat{L}^{n+1+2k}(R,(-1)^k) \;\cong\; NQ_{n+1}(R). \end{aligned}$$

This proves 3.1. □

3.4. **Rules for calculating** $Q_n(C,\gamma)$**.** Our goal, in the light of Proposition 3.1, is to compute $Q_n(B^A,\beta^A)$, especially when $A = \mathbb{Z}$. But first we explain three tools for computing $Q_n(C,\gamma)$ for any chain bundle (C,γ) over any ring with involution A.

A) Suppose (C,γ) is a chain bundle and $C \otimes_A C$ is n-connected. Then:

$$Q_i(C,\gamma) \;=\; 0 \text{ for } i \leqslant n-1 \quad \text{and } Q^{n+1}(C) \overset{J_\gamma^{n+1}}{\to} \widehat{Q}^{n+1}(C) \overset{H^{n+1}}{\to} Q_n(C,\gamma) \to 0$$

is exact. Moreover, for $i \leqslant n, J^i_\gamma = J^i : Q^i(C) \to \widehat{Q}^i(C)$, and J^i_γ is an isomorphism.

Proof of A): Use the spectral sequence:

$$E^2_{p,q} \;=\; H_p(\mathbb{Z}_2; H_q(C \otimes_A C)) \Rightarrow H_{p+q}((W^{-*})^{\%}C) \;=\; Q_{p+q}(C)\ .$$

This proves $Q_i(C) = 0$, for $i \leqslant n$. Next,

$$J^i_\gamma([\phi]) \;=\; J^i([\phi]) - \phi_0^{\%}([S^i\gamma])$$

for any $[\phi] \in Q^i(C)$. But if $i \leqslant n, \phi_0$ is null homotopic because $[\phi_0] = 0 \in H_i(C \otimes_A C)$. Consequently, $\phi_0^{\%} = 0$ and $J^i_\gamma = J^i$ for all $i \leqslant n$. But by the exact sequence 2.1, it follows that J^i_γ is an isomorphism for all $i \leqslant n$, and H^{n+1} is an epimorphism. This proves A).

B) Suppose (C,γ) is a chain bundle for which the chain complex C splits as:

$$C \;=\; \sum_{i=-\infty}^{\infty} C(i)\ .$$

Then

$$\gamma \;=\; \sum_{i=-\infty}^{\infty} \gamma(i)$$

where $\gamma(i)\in \widehat{Q}^0(C(i))$, and the inclusions $C(i)\to C$ induce a long exact sequence:

$$\begin{array}{ll}(3.9)\quad \cdots \to \displaystyle\sum_{i=-\infty}^{\infty} Q_n(C(i),\gamma(i)) \to Q_n(C,\gamma) & \\ \qquad\qquad \to \displaystyle\sum_{i<j} H_n(C(i)\otimes C(j)) \to \sum_{i=-\infty}^{\infty} Q_{n-1}(C(i),\gamma(i)) \to \dots\ . & \end{array}$$

Proof of B): On general principles

$$\widehat{Q}^n(C) \;=\; \sum_i \widehat{Q}^n(C(i))$$

and

$$Q^n(\sum_i C(i)) \;=\; \sum_i Q^n(C(i)) \oplus \sum_{i<j} H_n(C(i)\otimes C(j)).$$

Therefore, B) is a consequence of a diagram chase applied to the following map of exact sequences obtained from (3.4):

$$\begin{tikzcd}
\sum_i Q_n(C(i),\gamma(i)) \arrow[r] \arrow[d] & \sum_i Q^n(C(i)) \arrow[r, "\Sigma J_{\beta(i)}"] \arrow[d] & \sum_i \widehat{Q}^n(C(i)) \arrow[r] \arrow[d, "\cong"] & {} \\
Q_n(C,\gamma) \arrow[r] & Q^n(C) \arrow[r, "J_\beta"] & \widehat{Q}^n(C) \arrow[r] & {}\ .
\end{tikzcd}$$

C) Suppose the chain complex C is concentrated in degrees $\leqslant n$. Then $Q^k(C)=0$ if $k>2n$. If, in addition, $H_n(C)=0$, then $Q^{2n}(C)=0$ as well.

The proof of C. is straightforward from the definition of $W^{\%}C$.

3.5. $Q_n(B^A,\beta^A)$ **in the** 0**-dimensional case.** Throughout this section we suppose $2A=0$, the involution on A is trivial (and consequently A is commutative), and that $\widehat{H}^r(\mathbb{Z}_2;A)$ is a f.g. free A-module for each r. This occurs, for example, when $A=\mathbb{F}$ or $\mathbb{F}[x]$, where $\mathbb{F}$ is a perfect field of characteristic 2.

The Frobenius map

$$\psi^2\ :\ A\to A\ ;\ a\mapsto \psi^2(a)=a^2\ .$$

is a ring homomorphism which makes the target copy of A a module over the source copy of A. We denote this A-module as A', and note that there is an A-module isomorphism

$$\widehat{H}^r(\mathbb{Z}_2; A) \;\cong\; A' \;.$$

In this case on can easily construct the universal chain bundle (B^A, β^A) for A with

$$d \;=\; 0 \;:\; (B^A)_r \;=\; A' \to (B^A)_{r-1} \;=\; A' \;.$$

The class

$$\beta \;=\; \sum_r \beta^{-2r} \in \sum_r \widehat{Q}^0(B_r^{-*})$$

is obtained as follows. (Here and below we view B_r as a chain complex concentrated in degree r. Its dual chain complex is B_r^{-*}).

Let $x_1 \dots x_k$ be a basis of A' over A. Let $x^1 \dots x^k$ denote the dual basis. Then $x^i \otimes x^i \in \widehat{H}^0(\mathbb{Z}_2; B^r \otimes B^r) = \widehat{Q}^0(B_r^{-*})$. We set

$$\beta_{-2r} \;=\; \sum_{i=1}^{k} x_i(x^i \otimes x^i)$$

a symmetric bilinear form on the based module A'. The matrix of this symmetric bilinear form is diagonal:

$$\begin{bmatrix} x_1 & 0 & 0 & \dots \\ 0 & x_2 & 0 & \dots \\ 0 & 0 & x_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

It follows that $\widehat{v}_r : H_r(B) \to A'$ is the identity map. So (B, β) is universal. Inclusion induces a map of chain bundles, $(B_r, \beta_{-2r}) \xrightarrow{\iota_r} (B, \beta)$.

Lemma 3.2. *Assume $2A = 0$, the involution on A is trivial, and A' is free and finitely generated over A. With notation as above, the map $\iota_r : Q_*(B_r, \beta_{-2r}) \to Q_*(B, \beta)$, and the exact sequence 3.4 for (B_r, β_{-2r}), combine to give an exact sequence for each r:*

$$(3.10) \qquad 0 \to Q_{2r}(B, \beta) \to Q^{2r}(B_r) \xrightarrow{J_{\beta_{-2r}}} \widehat{Q}^{2r}(B_r) \to Q_{2r-1}(B, \beta) \to 0.$$

Proof. By (3.3) we have an isomorphism $B_r \otimes B_{n-r} \overset{k_{B_r}}{\cong} \widehat{Q}^n(B_r)$. By 3.4. A), we have $Q_n(B_s, \beta_{-2s}) \;=\; 0$ for $n < 2s - 1$. Therefore (3.9) can be written:

(3.11)

$$\sum_{s \leqslant r} \widehat{Q}^{2r+1}(B_s) \to \sum_{s \leqslant r} Q_{2r}(B_s, \beta_{-2s}) \to Q_r(B, \beta) \to \sum_{s<r} \widehat{Q}^{2r-1}(B_s) \to$$
$$\sum_{s \leqslant r} Q_{2r-1}(B_s, \beta_{-2s}) \to Q_{2r-1}(B, \beta) \to \sum_{s<r} \widehat{Q}^{2r}(B_s) \to Q_{2r-2}(B_s, \beta_{-2s})$$

Now, for dimensional reasons, if $n > 2s$, $Q^n(B_s) = 0$, and so $\widehat{Q}^{n+1}B_s) \xrightarrow{H} Q_n(B_x, \beta_{-2s})$ is an isomorphism. So (3.11) reduces to two pieces:

$$(3.12) \quad \widehat{Q}^{2r+1}(B_r) \xrightarrow{H} Q_{2r}(B_r, \beta_{-2r}) \to Q_{2r}(B, \beta) \to 0$$

$$Q_{2r-1}(B_r, \beta_{-2r}) \overset{\iota_r}{\cong} Q_{2r}(B, \beta).$$

Now apply the exact sequence (3.4) to B_r to get:

$$0 \to \operatorname{coker}(H_{\beta_{-2r}}) \longrightarrow Q^{2r}(B_r) \xrightarrow{J_{\beta_{-2r}}} \widehat{Q}^{2r}(B_r) \to Q_{2r-1}(B_r, \beta_{-2r}) \to 0,$$

which, together with (3.12) implies Lemma 3.2. $\square$

We now restrict ourselves to the case when $A = \mathbb{F}[x]$ where $\mathbb{F}$ is a perfect field of characteristic 2. Then A' is free of rank 2 over A, generated by 1 and x. Since $B_r = A'$ for all r, the abelian group $Q^{2r}(B_r)$ can be identified with the additive group, $\mathrm{Sym}_2(A)$, of 2×2 symmetric matrices over A. The A-module $\widehat{Q}^{2r}(B_r)$ can be identified with $\mathrm{Sym}_2(A)/\mathrm{Quad}_2(A)$ where $\mathrm{Quad}_2(A)$ denotes the matrices of the form $M + M^t$. The map $J_{\beta_{2r}} : \mathrm{Sym}_2(A) \to \mathrm{Sym}_2(A)/\mathrm{Quad}_2(A)$ then has the form:

$$\begin{aligned} J_\beta \begin{bmatrix} a & b \\ b & d \end{bmatrix} &= \begin{bmatrix} a & b \\ b & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & x \end{bmatrix} \begin{bmatrix} a & b \\ b & d \end{bmatrix} - \begin{bmatrix} a & b \\ b & d \end{bmatrix} \\ &= \begin{bmatrix} a^2 + a + xb^2 & * \\ * & b^2 + d + xd^2 \end{bmatrix} . \end{aligned}$$

We intend to show that the kernel and cokernel of J_β can be identified with the kernel and cokernel of the map $\psi^2 - 1 : A \to A$.

We have two inclusion maps $A \xrightarrow{\iota} \mathrm{Sym}_2(A)$, and $A \xrightarrow{\iota'} \mathrm{Sym}_2(A)/\mathrm{Quad}_2(A)$, both of the form:

$$a \to \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$$

Denote the images of these two maps as X, X'. Note that $(J_\beta)\iota = \iota'(\psi^2 - 1)$.

We use the following easily proved lemma:

Lemma 3.3. *Suppose X, X' are subgroups of two abelian groups Y, Y'. Suppose $j : Y \to Y'$ is a homomorphism such that $j(X) \subset X'$, and the induced map $\tilde{j} : Y/X \to Y'/X'$ is an isomorphism. Set $k = j|X : X \to X'$. Then* $\ker(k) = \ker(j)$, *and the inclusion $X' \to Y'$ induces an isomorphism*

$$\iota : \operatorname{coker}(k) \cong \operatorname{coker}(j) .$$

We want to apply this lemma when X, X' are as mentioned earlier and the role of $j : Y \to Y'$ is played by

$$J_\beta : \mathrm{Sym}_2(A) \to \mathrm{Sym}_2(A)/\mathrm{Quad}_2(A) .$$

This means we must first check that $\tilde{j}$ is an isomorphism. In other words, we must check that each element $p \in \mathbb{F}[x]$ can be written in one and only one way in the form $b^2 + d + xd^2$ where $b, d \in \mathbb{F}[x]$.

Write

$$p = \sum_{j=0}^{2n+1} a_j x^j, \quad b = \sum_i b_i x^i, \quad d = \sum_i d_i x^i .$$

Then:

$$b^2 + xd^2 + d = \sum_i (b_i^2 + d_{2i}) x^{2i} + \sum_i (d_i^2 + d_{2i+1}) x^{2i+1}.$$

Therefore the equation $p = b^2 + xd^2 + d$ reduces to equations,

$$d_i^2 + d_{2i+1} = a_{2i+1}; \quad b_i^2 + d_{2i} = a_{2i}.$$

One solves these recursively for d_i and b_i, working from higher to lower indices. Note that the first equation implies that $d_i = 0$ for all $i > n$. Therefore recursively, the equations

$$d_i^2 = d_{2i+1} + a_{2i+1}$$

specify d. Then the equations

$$b_i^2 = d_{2i} + a_{2i}$$

specify b. Here we use that $\mathbb{F}$ is perfect. Therefore $\tilde{j}$ is an isomorphism.

Applying the lemma, we conclude that if $A = \mathbb{F}[x]$ then

$$\ker(\psi^2 - 1) \overset{\iota}{\cong} \ker J_\beta \; ; \quad \operatorname{coker}(\psi^2 - 1) \overset{\tilde{\iota}}{\cong} \operatorname{coker}(J_\beta) \; . \tag{3.13}$$

The map $\tilde{\iota}$ is induced by $A \xrightarrow{\iota'} \mathrm{Sym}_2(A)/\mathrm{Quad}_2(A)$.

Note that if $A = \mathbb{F}_2[x]$, then $\ker(\psi^2 - 1) = \mathbb{F}_2$ and the cokernel of $A \xrightarrow{\psi^2-1} A$ can be identified with the vector space $\{\sum_i a_i x^i \mid a_{2i} = 0 \text{ for } i > 0\}$.

Summarizing, we have a confirmation of the calculation of Connolly and Koźniewski [8]:

Theorem 3.4. *: For all k, we have :*

$$\begin{aligned} \mathrm{UNil}_{2k+1}(\mathbb{F}_2; \mathbb{F}_2, \mathbb{F}_2) &= 0 \; , \\ \mathrm{UNil}_{2k}(\mathbb{F}_2; \mathbb{F}_2, \mathbb{F}_2) &\cong \operatorname{coker}(\mathbb{F}_2[x]/\mathbb{F}_2 \xrightarrow{\psi^2-1} \mathbb{F}_2[x]/\mathbb{F}_2) \\ &\cong \{\sum_i a_i x^i : a_{2i} = 0 \text{ for } i, a_i \in \mathbb{F}_2\} \end{aligned}$$

Proof. This is a consequence of Corollary 3.1, Lemma 3.2 and (3.13). □

3.6. **The one dimensional case.** In this section we deal with a ring A, whose universal chain bundle, (B^A, β^A), satisfies:

(3.14) For all i, $B^A_{2i} \xrightarrow{d} B^A_{2i-1}$ is zero; $B^A_{2i+1} \xrightarrow{d} B^A_{2i}$ is injective.

(We shall see that this holds for $A = \mathbb{Z}$ or $\mathbb{Z}[x]$. The point is that Corollary 3.1 reduces the calculation of $\mathrm{UNil}_*(\mathbb{Z};\mathbb{Z},\mathbb{Z})$ to that of $Q_*(B^A,\beta^A)$ for such rings A).

We clearly have:

$$(B^A,\beta^A) \;=\; \sum_{i=-\infty}^{\infty} (B^A(i),\beta^A(i)), \text{where } B^A(i) \text{ is: } \ldots 0 \to B^A_{2i+1} \xrightarrow{d} B^A_{2i} \to 0 \ldots .$$

We first relate $Q_n(B^A,\beta^A)$ to $Q_n(B^A(0),\beta^A(0))$, for $n = -1,0,1,2$, by analyzing the exact sequence (3.9), of the above direct sum splitting. By 3.4 A), we have:

$$\sum_{i=-\infty}^{\infty} Q_m(B^A(i),\beta^A(i)) \;=\; \sum_{i\leqslant\frac{m+1}{4}} Q_m(B^A(i),\beta^A(i)).$$

Next, because of (3.3), and dimensional reasons, we have

$$\begin{aligned}
\sum_{i<j} H_m(B^A(i)\otimes B^A(j)) \;&=\; \sum_{2i<[\frac{m}{2}]} H_m(B^A(i)\otimes B^A([\frac{m}{2}]-i)) \\
&=\; \sum_{2i<[\frac{m}{2}]} H_m(B^A(i)\otimes B^A) \\
&=\; \sum_{i<\frac{1}{2}[\frac{m}{2}]} \widehat{Q}^m(B^A(i)) \ .
\end{aligned}$$

But, by (3.4) and 3.4 C), the map $\widehat{Q}^{m+1}(B^A(i)) \to Q_m(B^A(i),\beta^A(i))$ is an isomorphism if $i \leqslant \dfrac{m-2}{4}$. Therefore, after we remove isomorphic direct summands from the exact sequence (3.9), it reduces to the much simpler long exact sequence:

$$\begin{aligned}
\cdots \to \sum_{\frac{m-2}{4}<i\leqslant\frac{m+1}{4}} Q_m(B^A(i),\beta^A(i)) \to Q_m(B^A,\beta^A) \\
\to \sum_{\frac{m-3}{4}<i<\frac{1}{2}[\frac{m}{2}]} \widehat{Q}^m(B^A(i)) \to \ldots .
\end{aligned}$$

So, we get:

(3.15) $Q_m(B^A,\beta^A) \xrightarrow{\cong} Q_m(B^A(0),\beta^A(0))$ for $m = -1,0,$ and:

$$\begin{aligned}
Q_1(B^A,\beta^A) \;&=\; \ker\{Q^1(B^A(0)) \xrightarrow{J^1_{\beta^A(0)}} \widehat{Q}^1(B^A(0))\} \\
Q_2(B^A,\beta^A) \;&=\; \mathrm{im}\{Q^2(B^A(0)) \xrightarrow{J^2_{\beta^A(0)}} \widehat{Q}^2(B^A(0))\} \;=\; 0 \text{ by } 3.4\,C),
\end{aligned}$$

whenever (B^A,β^A) is the universal chain bundle of A, and (B^A,β^A) satisfies (3.14).

Next we show that (3.14) holds when $A = \mathbb{Z}$ or $\mathbb{Z}[x]$.

3.6.1. *The construction of* (B^A, β^A) *for certain rings* A. Suppose A is a commutative ring with no elements of order 2, and trivial involution. Write

$$A_2 \ = \ A/2A \ .$$

Therefore $\widehat{H}^1(\mathbb{Z}_2; A) \ = \ 0$, and $\widehat{H}^0(\mathbb{Z}_2; A) \ = \ A_2'$, by which we mean the abelian group A_2, equipped with the A-module structure:

$$A \times A_2 \to A_2; \ (a, x) \mapsto (a^2 x).$$

Suppose further that there are elements $x_1, x_2, \ldots, x_r \in A, r > 0$, such that,

$$0 \to A^r \xrightarrow{\times 2} A^r \xrightarrow{j} A_2' \to 0$$

is exact, where

$$j \ : \ A^r \to A_2' \ ; \ (a_1, a_2, \ldots, a_r) \mapsto a_1^2 x_1 + a_2^2 x_2 + \cdots + a_r^2 x_r \ .$$

(For example, if $A = \mathbb{Z}$ then $r = 1, x_1 = 1$, while if $A = \mathbb{Z}[x]$ then $r = 2, x_1 = 1, x_2 = x$).

We show here how to construct the universal chain bundle (B, β) for A, so that (3.14) holds.

First we construct B. For all i, we define:

$$\begin{aligned} B_i \ &= \ A^r \\ B_{2i} &\xrightarrow{d=0} B_{2i-1} \\ B_{2i+1} \ = \ A^r &\xrightarrow{d \ = \ \times 2} A^r \ = \ B_{2i}. \end{aligned} \tag{3.16}$$

Next let $X \in M_r(A)$ be the diagonal matrix,

$$X \ = \ \begin{pmatrix} x_1 & 0 & \ldots & 0 \\ 0 & x_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & x_r \end{pmatrix}.$$

We define $\beta = \{\beta_{-i} \in (B^{-*} \otimes B^{-*})_i\}$ by:

$$\begin{aligned} \beta_{-4i} \ &= \ X \in M_r(A) \ = \ (B_{2i} \otimes B_{2i})^* \\ \beta_{-4i-1} \ &= \ (\delta \otimes 1)\beta_{-4i} \\ \beta_{-4i-2} \ &= \ -\frac{1}{2}(\delta \otimes \delta)\beta_{-4i} \text{for all } i. \end{aligned} \tag{3.17}$$

Here $\delta : B_0^{-*} \to B_{-1}^{-*}$ is the coboundary homomorphism.

As in 3.5, the map $\widehat{v}_{2i} : H_{2i}(B) \to A_2'$ is an isomorphism for all i, and so (B, β) is the universal chain bundle for A.

We can now apply the calculation (3.15) to the computation of $Q_n(B^A, \beta^A)$, when $A = \mathbb{Z}$ or $\mathbb{Z}[x]$. Specifically, (3.15) and Corollary 3.1 give us the split short exact sequence if $n = 0$ or -1:

(3.18)
$$0 \to \mathrm{UNil}_{n-1}(\mathbb{Z}; \mathbb{Z}, \mathbb{Z}) \to Q_n(B^{\mathbb{Z}[x]}(0), \beta^{\mathbb{Z}[x]}(0)) \xrightarrow{\eta_*} Q_n(B^{\mathbb{Z}}(0), \beta^{\mathbb{Z}}(0)) \to 0,$$

where $\eta : \mathbb{Z}[x] \to \mathbb{Z}$ is the augmentation map.

To simplify things further we define three families of groups, K_n, C_n, I_n, by the exactness of the following three split sequences:

$$0 \to K_n \to \ker(J^n_{\beta(0)}(\mathbb{Z}[x])) \xrightarrow{\eta_*} \ker(J^n_{\beta(0)}(\mathbb{Z})) \to 0$$
$$0 \to C_n \to \mathrm{coker}(J^{n+1}_{\beta(0)}(\mathbb{Z}[x])) \xrightarrow{\eta_*} \mathrm{coker}(J^{n+1}_{\beta(0)}(\mathbb{Z})) \to 0$$
$$0 \to I_{n+1} \to \mathrm{im}(J^{n+1}_{\beta(0)}(\mathbb{Z}[x])) \xrightarrow{\eta_*} \mathrm{im}(J^{n+1}_{\beta(0)}(\mathbb{Z})) \to 0$$

We next claim there is an isomorphism:

(3.19)
$$\mathrm{UNil}_{-2}(\mathbb{Z}; \mathbb{Z}, \mathbb{Z}) \cong C_{-1}.$$

To see this, note that $Q_n(B^A(0)) = 0$ for dimensional reasons if $n \leqslant -1$. Also, by 3.4 A),

$$J^{-1}_{\beta^A(0)} = J^{-1} : Q^{-1}(B^A(0)) \to \widehat{Q}^{-1}(B^A(0)),$$

which is a monomorphism by 2.1. This implies that

$$Q_{-1}(B^A(0), \beta^A(0)) \cong \mathrm{coker}(J^0_{\beta^A(0)}) .$$

Therefore (3.18) simplifies when $n = -1$, to (3.19).

Now 3.15, Corollary 3.1, and the exact sequence 3.4 for $(B^A(0), \beta^A(0))$ (when $A = \mathbb{Z}, \mathbb{Z}[x]$) yield the following calculations:

(3.20)
$$\begin{aligned} \mathrm{UNil}_0(\mathbb{Z}; \mathbb{Z}, \mathbb{Z}) &\cong K_1 \\ \mathrm{UNil}_1(\mathbb{Z}; \mathbb{Z}, \mathbb{Z}) &\cong I_2 \\ 0 \to C_0 \cong \mathrm{UNil}_{-1}(\mathbb{Z}; \mathbb{Z}, \mathbb{Z}) &\to K_0 \to 0 \\ C_{-1} &\cong \mathrm{UNil}_{-2}(\mathbb{Z}; \mathbb{Z}, \mathbb{Z}). \end{aligned}$$

Therefore our goal is to calculate C_0, C_{-1}, K_0, and K_1. This is done in the next two subsections.

3.6.2. *Calculation of* $Q^n(B^A(0))$ *and* $\widehat{Q}^n(B^A(0))$. Recall from Ranicki [15] that for any ring with involution A and for any A-module chain complex C an element $\phi \in (\widehat{W}^{\%}C)_n$ is specified by the sequence of elements $(\dots, \phi_{-1}, \phi_0, \phi_1, \dots)$ of $C \otimes_A C$ defined by

$$\phi_i = \phi(e_i) \in (C \otimes_A C)_{n+i} \ (i \in \mathbb{Z})$$

where $e_i \in \widehat{W}_i$ is the standard basis element. Likewise, an element $\phi \in (W^{\%}C)_n$ is specified by a sequence $(\phi_0, \phi_1, \dots)$, with $\phi_i = \phi(e_i)$.

For the rest of this section we assume A is a ring satisfying the hypotheses at the beginning of section 3.6.1.

Let $t : M_r(A) \to M_r(A)$ be the transpose map and define

$$\begin{aligned} \mathrm{Sym}_r(A) &= \ker(1 - T : M_r(A) \to M_r(A)) \\ \mathrm{Quad}_r(A) &= \mathrm{im}(1 + T : M_r(A) \to M_r(A)) \end{aligned}$$

Note that $B^A(0)$ is the algebraic mapping cone of the map $f : C \to D$, where $C = D = A^r$ is concentrated in degree 0, and $f = \times 2 : A^r \to A^r$. Therefore, for all m:

$$\widehat{Q}^{2m}(C) \;=\; \widehat{Q}^{2m}(D) \;\cong\; \mathrm{Sym}_r(A)/\mathrm{Quad}_r(A) : [\phi] \mapsto [\phi_{-2m}]$$

because $\phi_{-2m} \in A^r \otimes A^r = M_r(A)$ must be in the kernel of $1 - T$, for all $2m$-cycles $\phi \in (\widehat{W}^{\%} D)_{2m}$.

Also, $Q^{2m+1}(C) = \widehat{Q}^{2m+1}(D) = 0$ for all m. Since the induced map, $f^{\%} : \widehat{Q}^m(C) \to \widehat{Q}^m(D)$ is multiplication by 4, we see $f^{\%} = 0$. So the sequence:

$$0 \to \widehat{Q}^m(D) \to \widehat{Q}^m(B^A(0)) \to \widehat{Q}^m(\Sigma C) \to 0$$

is exact for all m.

If $m = 1$ the composite isomorphism,

$$\widehat{Q}^1(B^A(0)) \overset{\cong}{\to} \widehat{Q}^1(\Sigma C) \overset{\cong}{\to} \frac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)}$$

is written as

$$\widehat{Q}^1(B^A(0)) \overset{\beta^1}{\cong} \frac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)} : \quad \beta^1([(\phi_{-1}, \phi_0, \phi_1)]) \;=\; [\phi_1] \tag{3.21}$$

If $m = 0$ we write the inverse of the composite isomorphism

$$\frac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)} \;\cong\; \widehat{Q}^0(D) \;\cong\; \widehat{Q}^0(B^A(0))$$

as:

$$\widehat{Q}^0(B^A(0)) \overset{\beta^0}{\cong} \frac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)} : \quad \beta^0([(\phi_0, \phi_1, \phi_2)]) \;=\; [\phi_0] \;. \tag{3.22}$$

The calculation of $Q^m(B^A(0))$ requires more work.

Following Ranicki [15] we define $Q^m(f)$ as the m-th homology group of the mapping cone of $f^{\%}$:

$$Q^m(f) \;=\; H_m(f^{\%} : W^{\%} C \to W^{\%} D) \;,$$

for any chain map $f : C \to D$ of free A-module chain complexes. We also write $\mathcal{C}(f)$ for the mapping cone of such f, and we write $g : D \to \mathcal{C}(f)$ for the inclusion. The symmetrization map

$$H_m(C \otimes_A C) \to Q^m(C) \; ; \; \theta \mapsto \{\phi_s = \begin{cases} (1+T)\theta & \text{if } s = 0 \\ 0 & \text{if } s \geqslant 1 \end{cases}\}$$

fits into a natural transformation of exact sequences:

$$\begin{array}{ccccccc} H_m(C \otimes_A C) & \xrightarrow{f} & H_m(D \otimes C) & \xrightarrow{g} & H_m(\mathcal{C}(f) \otimes C) & \longrightarrow & H_{m-1}(C \otimes_A C) \\ \Big\downarrow{\scriptstyle (1+T)} & & \Big\downarrow{\scriptstyle (1+T)f} & & \Big\downarrow{\scriptstyle (1+T)f} & & \Big\downarrow{\scriptstyle (1+T)} \\ Q^m(C) & \xrightarrow{f^{\%}} & Q^m(D) & \longrightarrow & Q^m(f) & \longrightarrow & Q^{m-1}(C) \end{array}$$

This leads to a further exact sequence relating $Q^m(f)$ to $Q^m(\mathcal{C}(f))$:

$$\dots \to Q^{m+1}(\mathcal{C}(f)) \to H_m(\mathcal{C}(f) \otimes C) \xrightarrow{(1+T)f} Q^m(f) \to Q^m(\mathcal{C}(f)) \to \dots$$

Now in the case at hand (where $C = D = A^r$, and $\mathcal{C}(f) = B^A(0)$), we have

$$Q^m(C) \;=\; Q^m(D) \;=\; \begin{cases} \mathrm{Sym}_r(A), \text{ if } m = 0 \\ \dfrac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)}, \text{ if } m \text{ is even and } m < 0 \\ 0, \text{ in all other cases .} \end{cases}$$

But $f^{\%}$ is multiplication by 4. Thus

$$\begin{aligned} Q^0(f) &= \frac{\mathrm{Sym}_r(A)}{4\mathrm{Sym}_r(A)} \;, \\ Q^{2m}(f) &= \frac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)} \; (m < 0) \;, \\ Q^k(f) &= 0 \text{ for all other } k \;. \end{aligned}$$

So from the above exact sequence, we extract the following diagram with exact rows:

$$\begin{array}{ccccccc} H_0(\mathcal{C}(f) \otimes C) & \xrightarrow{(1+T)f} & Q^0(f) & \longrightarrow & Q^0(B^A(0)) & \longrightarrow & 0 \\ \Big\downarrow{\scriptstyle \cong} & & \Big\downarrow{\scriptstyle \cong} & & \Big\downarrow{\scriptstyle =} & & \\ \dfrac{M_r(A)}{2M_r(A)} & \xrightarrow{2(1+T)} & \dfrac{\mathrm{Sym}_r(A)}{4\mathrm{Sym}_r(A)} & \xrightarrow{\alpha} & Q^0(B^A(0)) & \longrightarrow & 0 \end{array}$$

Therefore α induces an isomorphism:

(3.23) $$\frac{\mathrm{Sym}_r(A)}{2\mathrm{Quad}_r(A)} \overset{\alpha^0}{\cong} Q^0(B^A(0)); \quad \alpha^0([M]) \;=\; [(M,0,0)],$$

where $(M,0,0)$ is a 0-cycle in $W^{\%}B^A(0)$, for any

$$M \in \mathrm{Sym}_r(A) \subset M_r(A) \;=\; A^r \otimes A^r \;=\; (B^A(0) \otimes B^A(0))_0 \;.$$

Now $Q^m(B^A(0)) = 0$ if $m \geqslant 2$ by 3.4 C). Also by 2.1, if $m \leqslant -1$, the map $Q^m(B^A(0)) \xrightarrow{J^m} \widehat{Q}^m(B^A(0))$ is an isomorphism.

Therefore, we are only left with the calculation of $Q^1(B^A(0))$. Instead of the above method (which would yield the result) we calculate this by hand both for its therapeutic value and for its greater explicitness. The bottom line will be 3.24.

For each $M \in M_r(A)$, define

$$\phi^M \;=\; (\phi_0^M, \phi_1^M) \in (W^{\%}B^A(0))_1$$

by:

$$\begin{aligned} \phi_1^M \;&=\; M \in M_r(A) \;=\; A^r \otimes A^r \;=\; B_1 \otimes B_1 \\ \phi_0^M \;&=\; M \oplus (-M) \in (B_1 \otimes B_0) \oplus (B_0 \otimes B_1). \end{aligned}$$

Lemma 3.5. *If $M \in \mathrm{Sym}_r(A)$, then ϕ^M is a 1-cycle in $W^{\%}B^A(0)$, and the rule $M \mapsto \phi^M$ induces an isomorphism:*

$$\alpha^1 \;:\; \frac{\mathrm{Sym}_r(A)}{2\mathrm{Sym}_r(A)} \;\cong\; Q^1(B^A(0)). \tag{3.24}$$

Proof. For any $\phi = (\phi_0, \phi_1) \in (W^{\%}B^A(0))_1$ we write

$$\phi_0 \;=\; \kappa_1 \oplus \kappa_2\;,$$

where $\kappa_1 \in B_1 \otimes B_0$, and $\kappa_2 \in B_0 \otimes B_1$. ϕ is a 1-cycle if and only if:

$$1)\ \partial\phi_0 \;=\; 0; \qquad 2)\ (T-1)\phi_0 \;=\; -\partial\phi_1; \qquad 3)\ (T+1)\phi_1 \;=\; 0\;,$$

where $T : B^A(0) \otimes B^A(0) \to B^A(0) \otimes B^A(0)$ is the twist chain map

$$T(x \otimes y) \;=\; (-1)^{|x||y|} y \otimes x\;.$$

These three conditions are equivalent to:

$$\kappa_2 \;=\; -\kappa_1; \qquad (1+T)\kappa_1 \;=\; 2\phi_1; \qquad t\phi_1 \;=\; \phi_1 \text{ in } A^r \otimes A^r \;=\; M_r(A)\;.$$

Here t denotes the transpose map in $M_r(A)$. Also a cycle ϕ as above is a boundary in $W^{\%}B^A(0)$ if and only if there is an element $\psi \in B_1 \otimes B_1$, such that $\kappa_1 = 2\psi$ in $A^r \otimes A^r = M_r(A)$. Therefore the map

$$Q^1(B^A(0)) \to \mathrm{Sym}_r(A)/2\mathrm{Sym}_r(A) \;:\; [\phi] \mapsto \kappa_1 \bmod (2A)$$

is an isomorphism.

The above discussion shows that if $M \in \mathrm{Sym}_r(A)$, then ϕ^M is a 1-cycle, and if $M \in 2\mathrm{Sym}_r(A)$, then ϕ^M is a boundary. Since the map $Q^1(B^A(0)) \to \mathrm{Sym}_r(A)/2\mathrm{Sym}_r(A)$ obviously sends ϕ^M to M, the proof is complete. $\square$

We summarize the calculations of this subsection as follows:

$$\begin{aligned}
\widehat{Q}^m(B^A(0)) &\cong \frac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)} \text{ for all } m,\\
Q^0(B^A(0)) &\cong \frac{\mathrm{Sym}_r(A)}{2\mathrm{Quad}_r(A)}\\
Q^1(B^A(0)) &\cong \frac{\mathrm{Sym}_r(A)}{2\mathrm{Sym}_r(A)}\\
Q^n(B^A(0)) &= 0 \text{ for } n \geqslant 2\\
Q^n(B^A(0)) &\overset{J^n}{\cong} \widehat{Q}^n(B^A(0)) \text{ if } n \leqslant -1.
\end{aligned} \tag{3.25}$$

3.6.3. *The maps $J^0_{\beta(0)}(A), J^1_{\beta(0)}(A)$ and the groups C_{-1}, C_0 and K_1, K_0.* We first analyze the map $J^0_{\beta(0)}(A) : Q^0(B^A(0)) \to \widehat{Q}^0(B^A(0))$, when $A = \mathbb{Z}$ or $\mathbb{Z}[x]$ using the isomorphisms of 3.21,3.22,3.23,3.24. By 3.1, $\beta^0 \circ J^0_{\beta(0)}(A) \circ \alpha^0$ sends a matrix $M \in \dfrac{\mathrm{Sym}_r(A)}{2\mathrm{Quad}_r(A)}$ to:

$$\beta^0(J^0([(M,0,0)])) - M^tXM \;=\; M - MXM \in \frac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)}.$$

In the case when $A = \mathbb{Z}$, so that $r = 1$, and $X = 1$, we have $\beta^0 J^0_{\beta(0)}(\mathbb{Z})\alpha^0$, sending $a \in \mathbb{Z}_4$ to $a - a^2\mathbb{Z}_4/2\mathbb{Z}_4 = \mathbb{Z}_2$. So $J^0_{\beta(0)}(\mathbb{Z}) = 0$. Therefore:

$$\ker J^0_{\beta(0)}(\mathbb{Z}) \;=\; Q^0(B^{\mathbb{Z}}(0)) \;\cong\; \mathbb{Z}_4; \quad \operatorname{coker} J^0_{\beta(0)}(\mathbb{Z}) \;=\; \widehat{Q}^0(B^{\mathbb{Z}}(0)) \;\cong\; \mathbb{Z}_2.$$

Now we let $A = \mathbb{Z}[x]$. Set

$$\mathcal{J}^0 \;=\; \beta^0 \circ J^0_{\beta(0)}(\mathbb{Z}[x]) \circ \alpha^0 : \mathrm{Sym}_2(\mathbb{Z}[x])/2\mathrm{Quad}_2(\mathbb{Z}[x]) \to \mathrm{Sym}_2(\mathbb{Z}[x])/\mathrm{Quad}_2(\mathbb{Z}[x])\,.$$

For any

$$\begin{pmatrix} a & b\\ b & d\end{pmatrix} \in \mathrm{Sym}_2(\mathbb{Z}[x])/2\mathrm{Quad}_2(\mathbb{Z}[x])$$

we compute from the above formula:

$$\mathcal{J}^0\begin{pmatrix} a & b\\ b & d\end{pmatrix} \;=\; \begin{pmatrix} a - a^2 - b^2x & b - ab - bdx\\ b - ab - bdx & d - b^2 - d^2x\end{pmatrix} \in \frac{\mathrm{Sym}_2(A)}{\mathrm{Quad}_2(A)} \tag{3.26}$$

We want to apply Lemma 3.3 again. Let $j = \mathcal{J}^0$, and:

$$Y \;=\; \frac{\mathrm{Sym}_r(\mathbb{Z}[x])}{2\mathrm{Quad}_r(\mathbb{Z}[x])},\; Y' \;=\; \frac{\mathrm{Sym}_r(\mathbb{Z}[x])}{\mathrm{Quad}_r(\mathbb{Z}[x])},\; X \;=\; (\mathbb{Z}_4[x]) \times (\mathbb{F}_2[x]),\; X' \;=\; (\mathbb{F}_2[x])$$

X and X' include into Y and Y' respectively by the rules: $(a,d) \mapsto \left(\begin{smallmatrix} a & 0\\ 0 & 2d\end{smallmatrix}\right)$, and $a \mapsto \left(\begin{smallmatrix} a & 0\\ 0 & 0\end{smallmatrix}\right)$. We first will have to show that $Y/X \to Y'/X'$ is an isomorphism. To this end, we note an isomorphism, $\mathbb{F}_2[x] \times \mathbb{F}_2[x] \;\cong\; Y/i(X)$, defined by: $(b,d) \mapsto \left(\begin{smallmatrix} 0 & b\\ b & d\end{smallmatrix}\right)$, and an isomorphism $\mathbb{F}_2[x] \;\cong\; Y'/i'(X')$, given by : $p \mapsto \left[\begin{smallmatrix} 0 & 0\\ 0 & p\end{smallmatrix}\right]$. Therefore the claim that j induces an isomorphism, $Y/X \to$

Y'/X', amounts to the statement that each $p \in \mathbb{F}_2[x]$ can be written uniquely in the form, $p = b^2 + d + xd^2$, for some $b, d \in \mathbb{F}_2[x]$. But this was proved already in section 3.5.

Define

$$k \ : \ \mathbb{Z}_4[x] \times \mathbb{F}_2[x] \to \mathbb{F}_2[x] \ ; \ (a,d) \mapsto a - a^2 \bmod 2 \ .$$

Clearly,

$$\begin{aligned} \ker(k) &= \{(a,d) \in \mathbb{Z}_4[x] \times \mathbb{F}_2[x] | a = a_0 + 2a_1, \text{for some } a_0 \in \mathbb{Z}_4, a_1 \in \mathbb{Z}_4[x]\} \ , \\ \operatorname{coker}(k) &= \operatorname{coker}(\psi^2 - 1) \ . \end{aligned}$$

Applying Lemma 3.3, we see that i and i' induce isomorphisms:

$$\ker(k) \overset{\iota}{\cong} \ker J^0_{\beta(0)}(\mathbb{Z}[x]); \quad \operatorname{coker}(\psi^2 - 1) \xrightarrow{\iota'} \operatorname{coker}(J^0_{\beta(0)}(\mathbb{Z}[x])).$$

Also, $\iota(a,d) = \alpha^0 \left[\begin{smallmatrix} a & 0 \\ 0 & 2d \end{smallmatrix}\right]$.

The augmentation map induced by η

$$Q^0(B^{\mathbb{Z}[x]}(0)) \xrightarrow{\eta_*} Q^0(B^{\mathbb{Z}}(0))$$

sends $\alpha^0 \left[\begin{smallmatrix} a & 0 \\ 0 & 2d \end{smallmatrix}\right]$ to $a_0 \in \mathbb{Z}_4$, the degree zero coefficient of a. The same formula holds as well for $\eta_* : Q^0(B^{\mathbb{Z}[x]}(0)) \to Q^0(B^{\mathbb{Z}}(0))$.

Restricting η_* to $\ker J^0_{\beta(0)}(\mathbb{Z}[x])$, we get a short exact sequence:

$$0 \to \mathbb{F}_2[x] \times \mathbb{F}_2[x] \xrightarrow{k_2} \ker(J^0_{\beta(0)}(\mathbb{Z}[x])) \xrightarrow{\eta_*} \ker(J^0_{\beta(0)}(\mathbb{Z})) \to 0$$

where k_2 is defined by:

$$k_2(a,d) = \begin{pmatrix} 2a & 0 \\ 0 & 2d \end{pmatrix} \ .$$

This yields isomorphisms:

(3.27)

$$\mathbb{F}_2[x] \times \mathbb{F}_2[x] \overset{k_2}{\cong} K_0, \quad \operatorname{coker}\{(\psi^2 - 1) : \mathbb{F}_2[x]/\mathbb{F}_2 \to \mathbb{F}_2[x]/\mathbb{F}_2\} \overset{k_2'}{\cong} C_{-1}.$$

Here $(\psi^2 - 1) : \mathbb{F}_2[x]/\mathbb{F}_2 \to \mathbb{F}_2[x]/\mathbb{F}_2$ is the map induced by $\psi^2 - 1 : \mathbb{F}_2[x] \to \mathbb{F}_2[x]$, and k_2' is induced by ι'.

Now we analyse $J^1_{\beta(0)}(A)$ similarly. Recall $B^A(0)$ is a chain complex concentrated in degrees 0 and 1: $B_0 = A^r$; $B_1 = A^r$, and its boundary map is $\partial = \times 2 : B_1 \to B_0$.

In order to understand the map $J^1_{\beta(0)}(A)$, we define, for any 1-cycle, $\phi \in (W^{\%} B^A(0))_1$, another 1-cycle

$$\gamma^\phi = \phi_0^{\%}(S^1(\beta(0))) \in (\widehat{W}^{\%} B^A(0))_1 \ .$$

We know $\gamma^\phi = (\gamma^\phi_{-1}, \gamma^\phi_0, \gamma^\phi_1)$, where

$$\gamma^\phi_i = \gamma_i = \tilde{\phi}_0 \otimes \tilde{\phi}_0(\beta(0)_{i-1}) \ .$$

Here $\tilde{\phi}_0 : B^A(0)^{1-*} \to B^A(0)$ is the chain map whose matrix is $\phi_0 \in (B^A(0) \otimes B^A(0))_1$.

We conclude:

$$\begin{aligned}\gamma_1 &= \tilde{\phi}_0 \otimes \tilde{\phi}_0(X) \in B_1 \otimes B_1,\\ \gamma_0 &= (1 \otimes \partial)\gamma_1 \in (B^A(0) \otimes B^A(0))_1,\\ \gamma_{-1} &= \frac{1}{2}(\partial \otimes \partial)\gamma_1 \in B_0 \otimes B_0.\end{aligned}$$

Therefore

$$J^1_{\beta(0)}(A) : Q^1(B^A(0)) \to \widehat{Q}^1(B^A(0)) \quad \text{is} : [\phi] \mapsto J^1([\phi]) - [\gamma^\phi].$$

Set

$$\mathcal{J}^1 \ = \ \beta^1 \circ J^1_{\beta(0)}(A) \circ \alpha^1 \ .$$

We get:

$$\begin{aligned}\mathcal{J}^1(M) &= \beta^1(J^1[\phi^M]) - [\gamma_1^{\phi^M}] \ = \ M - M^t X M\\ &= M - MXM \ \text{ mod } \mathrm{Quad}_r(A)\end{aligned}$$

for all $M \in \mathrm{Sym}_r(A)$. (The formulae for $\mathcal{J}^1$ and $\mathcal{J}^0$ are identical!). Therefore the formula 3.26 can also be used for $\mathcal{J}^1$. We therefore conclude at once that we have an isomorphism, induced by β^1:

(3.28) $$\mathrm{coker}\{(\psi^2 - 1) : \mathbb{F}_2[x]/\mathbb{F}_2 \to \mathbb{F}_2[x]/\mathbb{F}_2\} \ \cong \ C_0$$

To compute K_1, we note from 3.26 that the kernel of $J^1_{\beta(0)}(\mathbb{Z}[x]) \circ \alpha^1$ is:

$$\{\left[\begin{smallmatrix} a_0 & 0 \\ 0 & 0 \end{smallmatrix}\right] \in \mathrm{Sym}_2(\mathbb{F}_2[x]) : a_0 \in \mathbb{F}_2\}.$$

Since $\widehat{\eta}_* \left[\begin{smallmatrix} a_0 & 0 \\ 0 & 0 \end{smallmatrix}\right] \ = \ a_0 \in \mathbb{Z}_2$, we conclude at once that:

(3.29) $$K_1 \ = \ 0$$

3.7. **The calculation of** $\mathrm{UNil}_n(\mathbb{Z};\mathbb{Z},\mathbb{Z})$ **for all** n**.** The results of the last section allow us to prove Theorem B of the Introduction:

Theorem 3.6. *There are isomorphisms:*

$$\begin{aligned}\mathrm{UNil}_0(\mathbb{Z};\mathbb{Z},\mathbb{Z}) &= \ 0,\\ \mathrm{UNil}_1(\mathbb{Z};\mathbb{Z},\mathbb{Z}) &= \ 0,\\ \mathrm{UNil}_2(\mathbb{Z};\mathbb{Z},\mathbb{Z}) &\cong \ \mathrm{coker}\{ \ (\psi^2 - 1) : \mathbb{F}_2[x]/\mathbb{F}_2 \to \mathbb{F}_2[x]/\mathbb{F}_2 \ \},\end{aligned}$$

and an exact sequence:

$$0 \to \mathbb{F}_2[x]/\mathbb{F}_2 \xrightarrow{(\psi^2-1)} \mathbb{F}_2[x]/\mathbb{F}_2 \to \mathrm{UNil}_3(\mathbb{Z};\mathbb{Z},\mathbb{Z}) \to \mathbb{F}_2[x] \times \mathbb{F}_2[x] \to 0 \ .$$

Proof. Note $I_2 = 0$, by 3.15. Therefore 3.29 and 3.20 imply the first two equations at once. The third equation is immediate from 3.20 and 3.27. The final exact sequence is immediate from 3.20, 3.27, and 3.28. $\square$

See Banagl and Ranicki [1] and Connolly and Davis [7] for further computations.

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