A Threelogy in Two Parts

3-Algebras in BLG Models and a Study of TMG solutions

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(Patricia Ritter)

Abstract

This thesis is a review of research done over the course of the past 4 years, divided into two unrelated parts.

The first is set in the context of Bagger-Lambert-Gustavsson models, based on 3-Lie algebras. In particular I will describe theories with metric 3-algebras of indefinite signature: these present fields with negative kinetic terms. The problem can be solved by gaugeing away the non-physical degrees of freedom, to obtain other well understood theories. I will show how this procedure can be easily applied for 3-algebra metrics of any indefinite signature.

Part II of this thesis focuses on solutions of topologically massive gravity (TMG): particular attention is devoted to warped AdS_3 black holes, which are discussed in great detail. I will present a novel analysis of the near horizon geometries of these solutions. I further propose an approach for searching for new solutions to 3-dimensional gravity based on conformal symmetry. This approach is able to yield most of the known axisymmetric stationary TMG backgrounds.

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"E quando dottore lo fui finalmente, non volli tradire il bambino per l'uomo [...] E allora capii, fui costretto a capire, che fare il dottore è soltanto un mestiere, che la scienza non puoi regalarla alla gente..."

[F. De André]

To strong, interesting women everywhere in space-time:

Meiner Auntie Eri und meiner Lustigen Oma;

Meiner weisen, unabhängigen Mami.

Contents

Part I

3-Lie Algebras

Introduction

The first part of this thesis lays out some of the work done in collaboration with J. Figueroa-O'Farrill, P. de Medeiros and E. Méndez-Escobar on the study of the Bagger-Lambert-Gustavsson (BLG) model for coincident M2 branes. This model introduced the idea of 3-algebras into the world of gauge theories and thus attracted a lot of attention to the study of such triple systems. The original idea ([1], [2]) was that of using 3-algebras, i.e. a structure on the space of symmetry generators that involves a bracket with three slots, in lieu of Lie algebras in a 3-dimensional Chern-Simons theory coupled to matter, thus generating all the properties expected of a description of a stack of M2 branes.

Let us quickly recall some of the reasons why this proposal was of such great significance to the community. In the context of M-theory, the strong coupling limit of type IIA string theory, it had thus far seemed impossible to write a consistent worldsheet action for a stack of M-branes. Basu and Harvey had considered a stack of M2-branes ending on an M5-brane, and proposed a solution satisfying a generalised Nahm equation [3]. They could further write out a corresponding bosonic theory on the M2-brane worldvolume, but it was not clear how to interpret their results from first principles. Attempts to build up a consistent action from geometric and supersymmetric properties of M-branes always incurred in the same difficulties:

- M2 branes are the strong coupling limit of D2 branes, and as such are expected to yield a theory that is the infrared fixed point of a maximally supersymmetric 3-dimensional super-Yang-Mills theory. Such a theory, however, seems impossible to construct: the only interacting gauge theory with these properties in three dimensions is maximally supersymmetric Yang-Mills containing one vector field and seven scalars with $SO(7)$ symmetry. M2-branes need one more degree of freedom, i.e. eight scalars with $SO(8)$ symmetry. While this problem can be solved for the case of a single brane (by dualising the vector field from the D2 brane), there is no generalisation to the case of $N \neq 1$.

- The near horizon limit of a stack of N M2-branes is dual to a 3-dimensional conformal field theory with $N^{\frac{3}{2}}$ degrees of freedom.

- There is no free parameter in M-theory, so there is no obvious way in which to obtain a weakly coupled limit that would correspond to perturbative quantization of a classical Lagrangian.

The key problem that Bagger, Lamber and Gustavsson tackled, was that of constructing a supersymmetric scale invariant theory with manifest $SO(8)$ symmetry. They set up a Lagrangian containing eight scalar fields X^I, $I = 1, \ldots, 8$ taking values in a non-associative algebra. To discuss the supersymmetry transformations, they needed a term containing a triple product of these fields, satisfying certain symmetry requirements. It is precisely this structure that was found to be a 3-Lie algebra[1]. The introduction of this 3-bracket also in the equations of motion allows the supersymmetry algebra to close on world-sheet translations and a set of bosonic transformations involving the triple product. The new structure therefore makes its appearance in the Lagrangian as an interaction term, sextic in the scalar fields X^I. One can further gauge the theory by introducing a field $\mathcal{A}_\mu{}^a{}_b$, valued in the adjoint of representation of the algebra, which couples to the theory via a Chern-Simons like term.

The BLG idea generated a flurry of activity in the community, and theories for various types of 3-algebras were soon written. Indeed one can relax some of the original conditions (e.g. the total antisymmetry of the 3-bracket) to construct more general 3-structures resulting in different gauge theories.

My contribution to this topic is encapsulated in two papers: the first focused on finding the Lie-algebraic origin of 3-algebras [4] and the second on the analysis of BLG-style models for 3-algebras of indefinite signatures [5]. It is the latter work that is exposed in detail in this thesis, as we will see after the following quick review of the results of the former.

In [4] we studied how some types of metric 3-algebras can be deconstructed and reconstructed from pairs consisting of a metric real Lie algebra and a faithful unitary representation, $(\mathfrak{g}, V)$. This begins to explain, in algebraic terms, how superconformal Chern-Simons (SCCS) theories which were originally formulated

[1]If the scalar fields had been Lie-algebra valued, this term in the supersymmetry transformations would vanish, in which case one could not re-obtain the Basu-Harvey equation from this construction.

in terms of 3-algebras, can be rewritten using only Lie algebraic data. This (de)construction procedure was inspired by a general algebraic construction of pairs due to Faulkner [6]: the examples of three algebras that have been used in the literature on the BLG model correspond to the special cases of Faulkner's construction where the representation V is real orthogonal or complex unitary. The real case is shown to correspond to the generalised metric Lie 3-algebras of [7], appearing in $N = 2$ theories, while the complex case relates to the hermitian 3-algebras of [8], which appear in the $N = 6$ theories (the Faulkner construction also relates pairs where V is quaternionic unitary and 3-algebras generalising those which arise in $N = 5$ SCCS theory). In both cases we show how one can obtain the pair $(\mathfrak{g}, V)$ from the 3-algebra, and conversely how, starting from the pair, we can reconstruct a corresponding 3-algebra. In the real orthogonal case, this is in the same class of generalised metric 3-algebras of [7], so that we establish a one-to-one correspondence between isomorphism classes of such 3-algebras and classes of $(\mathfrak{g}, V)$ pairs. Therefore the problem of classification of generalised metric 3-algebras reduces to that of classifying metric Lie subalgebras of $\mathfrak{so}(V)$. For complex unitary V, the reconstructed 3-algebra is generally in a class which includes those of [8] as special cases. We showed how these are in one-to-one correspondence with a class of metric Lie superalgebras.

Another natural question to ask in the context of BLG-models concerns the signature of the 3-algebra used for the construction. Just like with Lie algebras, the metric associated to a 3-algebra need not be positive definite. In the BLG context, where matter fields are valued in the 3-algebra, this of course has consequences for the unitarity of the theory, and a non-zero index of the metric is expected to cause problems like negative energy states. This issue was addressed in our second paper [5]and the following two chapters follow precisely the work presented therein.

Metric 3-Lie Algebras for Unitary Bagger-Lambert Theories

The fundamental ingredient in the Bagger–Lambert–Gustavsson (BLG) model [1,2,9], proposed as the low-energy effective field theory on a stack of coincident M2-branes, is a metric 3-Lie algebra V on which the matter fields take values. This means that V is a real vector space with a symmetric inner product $\langle -, - \rangle$ and a trilinear, alternating 3-bracket $[-,-,-] : V \times V \times V \to V$ obeying the fundamental identity [10]

$$[x, y, [z_1, z_2, z_3]] = [[x, y, z_1], z_2, z_3] + [z_1, [x, y, z_2], z_3] + [z_1, z_2, [x, y, z_3]] \ , \qquad (1)$$

and the metricity condition

$$\langle [x, y, z_1], z_2 \rangle = - \langle z_1, [x, y, z_2] \rangle \ , \qquad (2)$$

for all $x, y, z_i \in V$. We say that V is indecomposable if it is not isomorphic to an orthogonal direct sum of nontrivial metric 3-Lie algebras. Every indecomposable metric 3-Lie algebra gives rise to a BLG model and this motivates their classification. It is natural to attempt this classification in increasing index — the index of an inner product being the dimension of the maximum negative-definite subspace. In other words, index 0 inner products are positive-definite (called euclidean here), index 1 are Lorentzian, et cetera. To this date there is a classification up to index 2, which we now review.

It was conjectured in [11] and proved in [12] (see also [13,14]) that there exists a unique nonabelian indecomposable metric 3-Lie algebra of index 0. It is the simple 3-Lie algebra [10] S_4 with underlying vector space $\mathbb{R}^4$, orthonormal basis

e_1, e_2, e_3, e_4, and 3-bracket

$$[e_i, e_j, e_k] = \sum_{\ell=1}^{4} \varepsilon_{ijk\ell} e_\ell \ , \tag{3}$$

where $\varepsilon = e_1 \wedge e_2 \wedge e_3 \wedge e_4$. Nonabelian indecomposable 3-Lie algebras of index 1 were classified in [15] and are given either by

- the simple lorentzian 3-Lie algebra $S_{3,1}$ with underlying vector space $\mathbb{R}^4$, orthonormal basis e_0, e_1, e_2, e_3 with e_0 timelike, and 3-bracket

$$[e_\mu, e_\nu, e_\rho] = \sum_{\sigma=0}^{3} \varepsilon_{\mu\nu\rho\sigma} s_\sigma e_\sigma \ , \tag{4}$$

  where $s_0 = -1$ and $s_i = 1$ for $i = 1, 2, 3$; or

- $W(\mathfrak{g})$, with underlying vector space $\mathfrak{g} \oplus \mathbb{R}u \oplus \mathbb{R}v$, where $\mathfrak{g}$ is a semisimple Lie algebra with a choice of positive-definite invariant inner product, extended to $W(\mathfrak{g})$ by declaring $u, v \perp \mathfrak{g}$ and $\langle u, u\rangle = \langle v, v\rangle = 0$ and $\langle u, v\rangle = 1$, and with 3-brackets

$$[u, x, y] = [x, y] \qquad \text{and} \qquad [x, y, z] = -\langle [x, y], z\rangle\, v \ , \tag{5}$$

  for all $x, y, z \in \mathfrak{g}$.

The latter metric 3-Lie algebras were discovered independently in [16, 17, 18] in the context of the BLG model. The index 2 classification is presented in [19]. There two classes of solutions were found, termed Ia and IIIb. The former class is of the form $W(\mathfrak{g})$, but where $\mathfrak{g}$ is now a lorentzian semisimple Lie algebra, whereas the latter class will be recovered as a special case of the results in the following two chapters and hence will be described in more detail below.

Let us now discuss the BLG model from a 3-algebraic perspective. The V-valued matter fields in the BLG model [1, 2, 9] comprise eight bosonic scalars X and eight fermionic Majorana spinors Ψ in three-dimensional Minkowski space $\mathbb{R}^{1,2}$. Triality allows one to take the scalars X and fermions Ψ to transform respectively in the vector and chiral spinor representations of the $\mathfrak{so}(8)$ R-symmetry. These matter fields are coupled to a nondynamical gauge field A which is valued in $\Lambda^2 V$ and described by a so-called twisted Chern–Simons term in the Bagger–Lambert Lagrangian [1,9]. The inner product $\langle -, -\rangle$ on V is used to describe the kinetic terms for the matter fields X and Ψ in the Bagger–Lambert lagrangian. Therefore if the index of V is positive (i.e. not euclidean signature) then the associated BLG model is not unitary as a quantum field theory, having 'wrong' signs for the kinetic terms for those matter fields in the negative-definite directions on V, thus carrying negative energy.

Indeed, for the BLG model based on the index-1 3-Lie algebra $W(\mathfrak{g})$, one encounters just this problem. Remarkably though, as noted in the pioneering works [16,17,18], here the matter field components X^v and Ψ^v along precisely one of the two null directions (u, v) in $W(\mathfrak{g})$ never appear in any of the interaction terms in the Bagger–Lambert Lagrangian. Since the interactions are governed only by the structure constants of the 3-Lie algebra then this property simply follows from the absence of v on the left hand side of any of the 3-brackets in (5). Indeed the one null direction v spans the centre of $W(\mathfrak{g})$ and the linear equations of motion for the matter fields along v force the components X^u and Ψ^u in the other null direction u to take constant values (preservation of maximal supersymmetry in fact requires $\Psi^u = 0$). By expanding around this maximally supersymmetric and gauge-invariant vacuum defined by the constant expectation value of X^u, one can obtain a unitary quantum field theory. Use of this strategy in [18] gave the first indication that the resulting theory is nothing but $N = 8$ super Yang–Mills theory on $\mathbb{R}^{1,2}$ with the euclidean semi-simple gauge algebra $\mathfrak{g}$. The super Yang–Mills theory gauge coupling here being identified with the $SO(8)$-norm of the constant X^u. This procedure is somewhat reminiscent of the novel Higgs mechanism introduced in [20] in the context of the Bagger–Lambert theory based on the euclidean Lie 3-algebra S_4. In that case an $N = 8$ super Yang-Mills theory with $\mathfrak{su}(2)$ gauge algebra is obtained, but with an infinite set of higher order corrections suppressed by inverse powers of the gauge coupling. As found in [18], the crucial difference is that there are no such corrections present in the lorentzian case.

Of course, one must be wary of naively integrating out the free matter fields X^v and Ψ^v in this way since their absence in any interaction terms in the Bagger–Lambert lagrangian gives rise to an enhanced global symmetry that is generated by shifting them by constant values. To account for this degeneracy in the action functional, in order to correctly evaluate the partition function, one must gauge the shift symmetry and perform a BRST quantisation of the resulting theory. Fixing this gauged shift symmetry allows one to set X^v and Ψ^v equal to zero while the equations of motion for the new gauge fields sets X^u constant and $\Psi^u = 0$.

Indeed this more rigorous treatment has been carried out in [21, 22] whereby the perturbative equivalence between the Bagger–Lambert theory based on $W(\mathfrak{g})$ and maximally supersymmetric Yang–Mills theory with euclidean gauge algebra $\mathfrak{g}$ was established (see also [23]). Thus the introduction of manifest unitarity in the quantum field theory has come at the expense of realising an explicit maximal superconformal symmetry in the BLG model for $W(\mathfrak{g})$, i.e. scale-invariance is broken by a nonzero vacuum expectation value for X^u. It is perhaps worth pointing out that the super Yang–Mills description seems to have not captured the intricate structure of a particular 'degenerate' branch of the classical maximally supersymmetric moduli space in the BLG model for $W(\mathfrak{g})$ found in [15]. The occurrence of this branch can be understood to arise from a degenerate limit of the theory wherein the scale $X^u = 0$ and maximal superconformal symmetry is restored. However, as found in [21, 22], the maximally superconformal unitary theory obtained by expanding around $X^u = 0$ describes a rather trivial free theory for eight scalars and fermions, whose moduli space does not describe said degenerate branch of the original moduli space.

Consider now a general indecomposable metric 3-Lie algebra with index r of the form $V = \bigoplus_{i=1}^{r}(\mathbb{R}u_i \oplus \mathbb{R}v_i) \oplus W$, where $\langle u_i, u_j\rangle = 0 = \langle v_i, v_j\rangle$, $\langle u_i, v_j\rangle = \delta_{ij}$ and W is a euclidean vector space. As explained in section 2.4 of [19], one can ensure that none of the null components X^{v_i} and Ψ^{v_i} of the matter fields appear in any of the interactions in the associated Bagger–Lambert Lagrangian provided that no v_i appear on the left hand side of any of the 3-brackets on V. This guarantees one has an extra shift symmetry for each of these null components suggesting that all the associated negative-norm states in the spectrum of this theory can be consistently decoupled after gauging all the shift symmetries and following BRST quantisation of the gauged theory. A more invariant way of stating the aforementioned criterion is that V should admit a maximally isotropic centre: that is, a subspace $Z \subset V$ of dimension equal to the index of the inner product on V, on which the inner product vanishes identically and which is central, so that $[Z, V, V] = 0$ in the obvious notation. The null directions v_i defined above along which we require the extra shift symmetries are thus taken to provide a basis for Z. In [19] all indecomposable metric 3-Lie algebras of index 2 with a maximally isotropic centre were classified. There are nine families of such 3-Lie algebras, which were termed type IIIb in that paper. In chapter 1 we will prove a structure theorem for general metric 3-Lie algebras which admit a maximally isotropic centre, thus characterising them fully. Although the structure theorem falls short of a classification, we will argue that it is the best possible result for this problem. The bosonic contributions to the Bagger–Lambert lagrangians for such 3-Lie algebras will be computed but we will not perform a rigorous analysis of the physical theory in the sense of gauging the shift symmetries and BRST quantisation. We will limit ourselves to expanding the theory around a suitable maximally supersymmetric and gauge-invariant vacuum defined by a constant expectation value for X^{u_i} (with $\Psi^{u_i} = 0$). This is the obvious generalisation of the procedure used in [18] for the lorentzian theory and coincides with that used more recently in [24] for more general 3-Lie algebras. We will comment explicitly on how all the finite-dimensional examples considered in section 4 of [24] can be recovered from our formalism.

As explained in sections 2.5 and 2.6 of [19], two more algebraic conditions are necessary in order to interpret the BLG model based on a general metric 3-Lie algebra with maximally isotropic centre as an M2-brane effective field theory. Firstly, the 3-Lie algebra should admit a (nonisometric) conformal automorphism that can be used to absorb the formal coupling dependence in the BLG model. In [19] it is shown that precisely four of the nine IIIb families of index 2 3-Lie algebras with maximally isotropic centre satisfy this condition. Secondly, parity invariance of the BLG model requires the 3-Lie algebra to admit an isometric antiautomorphism. This symmetry is expected of an M2-brane effective field theory based on the assumption that it should arise as an IR superconformal fixed point of $N = 8$ super Yang–Mills theory. In [19] one can see explicitly that each of the four IIIb families of index 2 3-Lie algebras admitting said conformal automorphism also admit an isometric antiautomorphism.

It is worth emphasising that the motivation for the two conditions above is distinct from that which led us to demand a maximally isotropic centre. The first two are required only for an M-theoretic interpretation while the latter is a basic physical consistency condition to ensure that the resulting quantum field theory is unitary. Moreover, even given a BLG model based on a 3-Lie algebra satisfying all three of these conditions, it is plain to see that the procedure we shall follow must generically break the initial conformal symmetry since it has introduced scales into the problem corresponding to the vacuum expectation values of X^{u_i}. It is inevitable that this breaking of scale-invariance will also be a feature resulting from a more rigorous treatment in terms of gauging shift symmetries and BRST.

Thus we shall concentrate just on the unitarity condition and, for the purposes of this exposition, we will say that a metric 3-Lie algebra is **(physically) admis-**

sible if it is indecomposable and admits a maximally isotropic centre. Chapter 1 will be devoted in essence to characterising finite-dimensional admissible 3-Lie algebras. Chapter 2 will describe the general structure of the gauge theories which result from expanding the BLG model based on these physically admissible 3-Lie algebras around a given vacuum expectation value for X^{u_i}. Particular attention will be paid to explaining how the 3-Lie algebraic data translates into physical parameters of the resulting gauge theories.

Part I of this thesis is therefore organised as follows. Chapter 1 is concerned with the proof of Theorem 1.2.7, which is outlined in section 1.2.6. The theorem may be paraphrased as stating that every finite-dimensional admissible 3-Lie algebra of index $r > 0$ is constructed via the following procedure. We start with the set of data:

- for each $\alpha = 1, \ldots, N$, a nonzero vector $0 \neq \kappa^\alpha \in \mathbb{R}^r$ with components κ_i^α, a positive real number $\lambda_\alpha > 0$ and a compact simple Lie algebra $\mathfrak{g}_\alpha$;
- for each $\pi = 1, \ldots, M$, a two-dimensional euclidean vector space E_π with a complex structure H_π, and two linearly independent vectors $\eta^\pi, \zeta^\pi \in \mathbb{R}^r$;
- a euclidean vector space E_0 and $K \in \Lambda^3\mathbb{R}^r \otimes E_0$ obeying the quadratic equations

$$\langle K_{ijn}, K_{k\ell m}\rangle - \langle K_{ijm}, K_{nk\ell}\rangle + \langle K_{ij\ell}, K_{mnk}\rangle - \langle K_{ijk}, K_{\ell mn}\rangle = 0,$$

  where $\langle -, -\rangle$ is the inner product on E_0;
- and $L \in \Lambda^4\mathbb{R}^r$.

On the vector space

$$V = \bigoplus_{i=1}^{r} (\mathbb{R}u_i \oplus \mathbb{R}v_i) \oplus \bigoplus_{\alpha=1}^{N} \mathfrak{g}_\alpha \oplus \bigoplus_{\pi=1}^{M} E_\pi \oplus E_0,$$

we define the following inner product extending the inner product on E_π and E_0:

- $\langle u_i, v_j\rangle = \delta_{ij}$, $\langle u_i, u_j\rangle = 0$, $\langle v_i, v_j\rangle = 0$ and u_i, v_j are orthogonal to the $\mathfrak{g}_\alpha$, E_π and E_0; and
- on each $\mathfrak{g}_\alpha$ we take $-\lambda_\alpha$ times the Killing form.

This makes V above into an inner product space of index r. On V we define the following 3-brackets, with the tacit assumption that any 3-bracket not listed here is meant to vanish:

$$\begin{aligned}
[u_i, u_j, u_k] &= K_{ijk} + \sum_{\ell=1}^{r} L_{ijk\ell} v_\ell \\
[u_i, u_j, x_0] &= -\sum_{k=1}^{r} \langle K_{ijk}, x_0 \rangle v_k \\
[u_i, u_j, x_\pi] &= (\eta_i^\pi \zeta_j^\pi - \eta_j^\pi \zeta_i^\pi) H_\pi x_\pi \\
[u_i, x_\pi, y_\pi] &= \langle H_\pi x_\pi, y_\pi \rangle \sum_{j=1}^{r} (\eta_i^\pi \zeta_j^\pi - \eta_j^\pi \zeta_i^\pi) v_j \\
[u_i, x_\alpha, y_\alpha] &= \kappa_i^\alpha [x_\alpha, y_\alpha] \\
[x_\alpha, y_\alpha, z_\alpha] &= -\langle [x_\alpha, y_\alpha], z_\alpha \rangle \sum_{i=1}^{r} \kappa_i^\alpha v_i,
\end{aligned} \tag{6}$$

for all $x_0 \in E_0$, $x_\pi, y_\pi \in E_\pi$, and $x_\alpha, y_\alpha, z_\alpha \in \mathfrak{g}_\alpha$. The resulting metric 3-Lie algebra has a maximally isotropic centre spanned by the v_i. It is indecomposable provided that there is no $x_0 \in E_0$ which is perpendicular to all the K_{ijk}, whence in particular $\dim E_0 \leq \binom{r}{3}$. The only non-explicit datum in the above construction are the K_{ijk} since they are subject to certain quadratic equations. However we will see that these equations are trivially satisfied for $r < 5$. Hence the above results constitutes, in principle, a classification for indices 3 and 4, extending the classification of index 2 in [19].

Using this structure theorem, in chapter 2 we are able to calculate the Lagrangian for the BLG model associated with a general physically admissible 3-Lie algebra. For the sake of clarity, we shall focus on just the bosonic contributions since the resulting theories will have a canonical maximally supersymmetric completion. Upon expanding this theory around the maximally supersymmetric vacuum defined by constant expectation values X^{u_i} (with all the other fields set to zero) we will obtain standard $N = 8$ supersymmetric (but nonconformal) gauge theories with moduli parametrised by particular combinations of the data appearing in Theorem 1.2.7 and the vacuum expectation values X^{u_i}. It will be useful to think of the vacuum expectation values X^{u_i} as defining a linear map, also denoted $X^{u_i} : \mathbb{R}^r \to \mathbb{R}^8$, sending $\xi \mapsto X^\xi := \sum_{i=1}^r \xi_i X^{u_i}$. Indeed it will be found that the physical gauge theory parameters are naturally expressed in terms of components in the image of this map. That is, in general, we find that neither the data in Theorem 1.2.7 nor the vacuum expectation values X^{u_i} on their own appear as physical parameters which instead arise from certain projections of the components of the data in Theorem 1.2.7 onto X^{u_i} in $\mathbb{R}^8$.

The resulting Bagger–Lambert Lagrangian will be found to factorise into a sum of decoupled maximally supersymmetric gauge theories on each of the euclidean components $\mathfrak{g}_\alpha$, E_π and E_0. The physical content and moduli on each component can be summarised as follows:

- On each $\mathfrak{g}_\alpha$ one has an $N = 8$ super Yang–Mills theory. The gauge symmetry is based on the simple Lie algebra $\mathfrak{g}_\alpha$. The coupling constant is given by $\|X^{\kappa^\alpha}\|$, which denotes the $SO(8)$-norm of the image of $\kappa^\alpha \in \mathbb{R}^r$ under the linear map X^{u_i}. The seven scalar fields take values in the hyperplane $\mathbb{R}^7 \subset \mathbb{R}^8$ which is orthogonal to the direction defined by X^{κ^α}. (If $X^{\kappa^\alpha} = 0$, for a given value of α, one obtains a degenerate limit corresponding to a maximally superconformal free theory for eight scalar fields and eight fermions valued in $\mathfrak{g}_\alpha$.)

- On each plane E_π one has a pair of identical free abelian $N = 8$ massive vector supermultiplets. The bosonic fields in each such supermultiplet comprise a massive vector and six massive scalars. The mass parameter is given by $\|X^{\eta^\pi} \wedge X^{\zeta^\pi}\|$, which corresponds to the area of the parallelogram in $\mathbb{R}^8$ defined by the vectors X^{η^π} and X^{ζ^π} in the image of the map X^{u_i}. The six scalar fields inhabit the $\mathbb{R}^6 \subset \mathbb{R}^8$ which is orthogonal to the plane spanned by X^{η^π} and X^{ζ^π}. (If $\|X^{\eta^\pi} \wedge X^{\zeta^\pi}\| = 0$, for a given value of π, one obtains a degenerate massless limit where the vector is dualised to a scalar, again corresponding to a maximally superconformal free theory for eight scalar fields and eight fermions valued in E_π.) Before gauge-fixing, this theory can be understood as an $N = 8$ super Yang–Mills theory with gauge symmetry based on the four-dimensional Nappi–Witten Lie algebra $\mathfrak{d}(E_\pi, \mathbb{R})$. Moreover we explain how it can be obtained from a particular truncation of an $N = 8$ super Yang-Mills theory with gauge symmetry based on any euclidean semisimple Lie algebra with rank 2, which may provide a more natural D-brane interpretation.

- On E_0 one has a decoupled $N = 8$ supersymmetric theory involving eight free scalar fields and an abelian Chern–Simons term. Since none of the matter fields are charged under the gauge field in this Chern–Simons term then its overall contribution is essentially trivial on $\mathbb{R}^{1,2}$.

Note on simultaneous literature

Contemporarily to our work in this topic, the paper [24] appeared whose results have noticeable overlap with those described here. In particular, they also describe the physical properties of BLG models based on certain finite-dimensional 3-Lie algebras with index greater than 1 admitting a maximally isotropic centre. The structure theorem we prove here for such 3-Lie algebras allows us to extend some of their results and make general conclusions about the nature of those unitary gauge theories which arise from BLG models based on physically admissible 3-Lie algebras. In terms of our data in Theorem 1.2.7, the explicit finite-dimensional examples considered in section 4 of [24] all have $K_{ijk} = 0 = L_{ijkl}$ with only one J_{ij} nonzero. This is tantamount to taking the index $r = 2$. The example in sections 4.1 and 4.2 of [24] has $\kappa^\alpha = 0$ (i.e. no $\mathfrak{g}_\alpha$ part) while the example in section 4.3 has $\kappa^\alpha = (1, 0)^t$. These are isomorphic to two of the four physically admissible IIIb families of index 2 3-Lie algebras found in [19].

Divide et impera!

Chapter 1

Towards a classification of admissible metric 3-Lie algebras

In this chapter we will prove a structure theorem for finite-dimensional indecomposable metric 3-Lie algebras admitting a maximally isotropic centre. We think it is of pedagogical value to first rederive the similar structure theorem for metric Lie algebras using a method similar to the one we will employ in the more involved case of metric 3-Lie algebras.

1.1 Metric Lie algebras with maximally isotropic centre

Recall that a Lie algebra $\mathfrak{g}$ is said to be metric, if it possesses an ad-invariant scalar product. It is said to be indecomposable if it is not isomorphic to an orthogonal direct sum of metric Lie algebras (of positive dimension). Equivalently, it is indecomposable if there are no proper ideals on which the scalar product restricts nondegenerately. A metric Lie algebra $\mathfrak{g}$ is said to have index r, if the ad-invariant scalar product has index r, which is the same as saying that the maximally negative-definite subspace of $\mathfrak{g}$ is r-dimensional. In this section we will prove a structure theorem for finite-dimensional indecomposable metric Lie algebras admitting a maximally isotropic centre, a result originally due to Kath and Olbrich [25].

1.1.1 Preliminary form of the Lie algebra

Let $\mathfrak{g}$ be a finite-dimensional indecomposable metric Lie algebra of index $r > 0$ admitting a maximally isotropic centre. Let v_i, $i = 1, \dots, r$, denote a basis for the centre. The inner product is such that $\langle v_i, v_j \rangle = 0$. Since the inner product on $\mathfrak{g}$ is nondegenerate, there exist u_i, $i = 1, \dots, r$, which obey $\langle u_i, v_j \rangle = \delta_{ij}$. It is always possible to choose the u_i such that $\langle u_i, u_j \rangle = 0$. Indeed, if the u_i do not span a maximally isotropic subspace, then redefine them by $u_i \mapsto u_i - \frac{1}{2} \sum_{j=1}^{r} \langle u_i, u_j \rangle v_j$ so that they do. The perpendicular complement to the $2r$-dimensional subspace spanned by the u_i and the v_j is then positive-definite. In summary, $\mathfrak{g}$ admits the following vector space decomposition

$$\mathfrak{g} = \bigoplus_{i=1}^{r} (\mathbb{R}u_i \oplus \mathbb{R}v_i) \oplus \mathfrak{r}, \tag{1.1}$$

where $\mathfrak{r}$ is the positive-definite subspace of $\mathfrak{g}$ perpendicular to all the u_i and v_j.

Metricity then implies that the most general Lie brackets on $\mathfrak{g}$ are of the form

$$\begin{aligned} [u_i, u_j] &= K_{ij} + \sum_{k=1}^{r} L_{ijk} v_k \\ [u_i, x] &= J_i x - \sum_{j=1}^{r} \langle K_{ij}, x \rangle v_j \\ [x, y] &= [x, y]_{\mathfrak{r}} - \sum_{i=1}^{r} \langle x, J_i y \rangle v_i, \end{aligned} \tag{1.2}$$

where $K_{ij} = -K_{ji} \in \mathfrak{r}$, $L_{ijk} \in \mathbb{R}$ is totally skewsymmetric in the indices, $J_i \in \mathfrak{so}(\mathfrak{r})$ and $[-,-]_{\mathfrak{r}} : \mathfrak{r} \times \mathfrak{r} \to \mathfrak{r}$ is bilinear and skewsymmetric. Metricity and the fact that the v_i are central, means that no u_i can appear on the right-hand side of a bracket. Finally, metricity also implies that

$$\langle [x, y]_{\mathfrak{r}}, z \rangle = \langle x, [y, z]_{\mathfrak{r}} \rangle, \tag{1.3}$$

for all $x, y, z \in \mathfrak{r}$.

It is not hard to demonstrate that the Jacobi identity for $\mathfrak{g}$ is equivalent to the following identities on $[-,-]_{\mathfrak{r}}$, J_i and K_{ij}, whereas L_{ijk} is unconstrained:

$$[x,[y,z]_{\mathfrak{r}}]_{\mathfrak{r}} - [[x,y]_{\mathfrak{r}},z]_{\mathfrak{r}} - [y,[x,z]_{\mathfrak{r}}]_{\mathfrak{r}} = 0 \tag{1.4a}$$

$$J_i[x,y]_{\mathfrak{r}} - [J_i x, y]_{\mathfrak{r}} - [x, J_i y]_{\mathfrak{r}} = 0 \tag{1.4b}$$

$$J_i J_j x - J_j J_i x - [K_{ij}, x]_{\mathfrak{r}} = 0 \tag{1.4c}$$

$$J_i K_{jk} + J_j K_{ki} + J_k K_{ij} = 0 \tag{1.4d}$$

$$\langle K_{\ell i}, K_{jk}\rangle + \langle K_{\ell j}, K_{ki}\rangle + \langle K_{\ell k}, K_{ij}\rangle = 0, \tag{1.4e}$$

for all $x, y, z \in \mathfrak{r}$.

1.1.2 $\mathfrak{r}$ is abelian

Equation (1.4a) says that $\mathfrak{r}$ is a Lie algebra under $[-,-]_{\mathfrak{r}}$, which because of equation (1.3) is metric. Being positive-definite, it is reductive, whence an orthogonal direct sum $\mathfrak{r} = \mathfrak{s} \oplus \mathfrak{a}$, where $\mathfrak{s}$ is semisimple and $\mathfrak{a}$ is abelian. We will show that for an indecomposable $\mathfrak{g}$, we are forced to take $\mathfrak{s} = 0$, by showing that $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{s}^\perp$ as a metric Lie algebra.

Equation (1.4b) says that J_i is a derivation of $\mathfrak{r}$, which we know to be skewsymmetric. The Lie algebra of skewsymmetric derivations of $\mathfrak{r}$ is given by $\operatorname{ad}\mathfrak{s} \oplus \mathfrak{so}(\mathfrak{a})$. Therefore under this decomposition, we may write $J_i = \operatorname{ad} z_i + J_i^{\mathfrak{a}}$, for some unique $z_i \in \mathfrak{s}$ and $J_i^{\mathfrak{a}} \in \mathfrak{so}(\mathfrak{a})$.

Decompose $K_{ij} = K_{ij}^{\mathfrak{s}} + K_{ij}^{\mathfrak{a}}$, with $K_{ij}^{\mathfrak{s}} \in \mathfrak{s}$ and $K_{ij}^{\mathfrak{a}} \in \mathfrak{a}$. Then equation (1.4c) becomes the following two conditions

$$[z_i, z_j]_{\mathfrak{r}} = K_{ij}^{\mathfrak{s}} \tag{1.5}$$

and

$$[J_i^{\mathfrak{a}}, J_j^{\mathfrak{a}}] = 0. \tag{1.6}$$

One can now check that the $\mathfrak{s}$-component of the Jacobi identity for $\mathfrak{g}$ is automatically satisfied, whereas the $\mathfrak{a}$-component gives rise to the two equations

$$J_i^{\mathfrak{a}} K_{jk}^{\mathfrak{a}} + J_j^{\mathfrak{a}} K_{ki}^{\mathfrak{a}} + J_k^{\mathfrak{a}} K_{ij}^{\mathfrak{a}} = 0 \tag{1.7}$$

and

$$\left\langle K^{\mathfrak{a}}_{\ell i}, K^{\mathfrak{a}}_{jk}\right\rangle + \left\langle K^{\mathfrak{a}}_{\ell j}, K^{\mathfrak{a}}_{ki}\right\rangle + \left\langle K^{\mathfrak{a}}_{\ell k}, K^{\mathfrak{a}}_{ij}\right\rangle = 0. \tag{1.8}$$

We will now show that $\mathfrak{g} \cong \mathfrak{s} \oplus \mathfrak{s}^\perp$, which violates the indecomposability of $\mathfrak{g}$ unless $\mathfrak{s} = 0$. Consider the isometry φ of the vector space $\mathfrak{g}$ defined by

$$\begin{aligned}
\varphi(u_i) &= u_i - z_i - \frac{1}{2}\sum_{j=1}^{r} \langle z_i, z_j\rangle\, v_j \\
\varphi(v_i) &= v_i \\
\varphi(x) &= x + \sum_{i=1}^{r} \langle z_i, x\rangle\, v_i,
\end{aligned} \tag{1.9}$$

for all $x \in \mathfrak{r}$. Notice that if $x \in \mathfrak{a}$, then $\varphi(x) = x$. It is a simple calculation to see that for all $x, y \in \mathfrak{s}$,

$$[\varphi(u_i), \varphi(x)] = 0 \qquad \text{and} \qquad [\varphi(x), \varphi(y)] = \varphi([x,y]_{\mathfrak{r}}). \tag{1.10}$$

In other words, the image of $\mathfrak{s}$ under φ is a Lie subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{s}$ and commuting with its perpendicular complement in $\mathfrak{g}$. In other words, as a metric Lie algebra $\mathfrak{g} \cong \mathfrak{s} \oplus \mathfrak{s}^\perp$, violating the decomposability of $\mathfrak{g}$ unless $\mathfrak{s} = 0$.

In summary, we have proved the following

Lemma 1.1.1. Let $\mathfrak{g}$ be a finite-dimensional indecomposable metric Lie algebra with index $r > 0$ and admitting a maximally isotropic centre. Then as a vector space

$$\mathfrak{g} = \bigoplus_{i=1}^{r} (\mathbb{R}u_i \oplus \mathbb{R}v_i) \oplus E, \tag{1.11}$$

where E is a euclidean space, $u_i, v_i \perp E$ and $\langle u_i, v_j\rangle = \delta_{ij}$, $\langle u_i, u_j\rangle = \langle v_i, v_j\rangle = 0$. Moreover the Lie bracket is given by

$$\begin{aligned}
[u_i, u_j] &= K_{ij} + \sum_{k=1}^{r} L_{ijk} v_k \\
[u_i, x] &= J_i x - \sum_{j=1}^{r} \langle K_{ij}, x\rangle\, v_j \\
[x, y] &= -\sum_{i=1}^{r} \langle x, J_i y\rangle\, v_i,
\end{aligned} \tag{1.12}$$

where $K_{ij} = -K_{ji} \in E$, $L_{ijk} \in \mathbb{R}$ is totally skewsymmetric in its indices, $J_i \in \mathfrak{so}(E)$ and in addition obey the following conditions:

$$J_i J_j - J_j J_i = 0 \tag{1.13a}$$

$$J_i K_{jk} + J_j K_{ki} + J_k K_{ij} = 0 \tag{1.13b}$$

$$\langle K_{\ell i}, K_{jk}\rangle + \langle K_{\ell j}, K_{ki}\rangle + \langle K_{\ell k}, K_{ij}\rangle = 0. \tag{1.13c}$$

The analysis of the above equations will take the rest of this section, until we arrive at the desired structure theorem.

1.1.3 Solving for the J_i

Equation (1.13a) says that the $J_i \in \mathfrak{so}(E)$ are mutually commuting, whence they span an abelian subalgebra $\mathfrak{h} \subset \mathfrak{so}(E)$. Since E is positive-definite, E decomposes as the following orthogonal direct sum as a representation of $\mathfrak{h}$:

$$E = \bigoplus_{\pi=1}^{s} E_\pi \oplus E_0, \tag{1.14}$$

where

$$E_0 = \{x \in E | J_i x = 0 \ \forall i\} \tag{1.15}$$

and each E_π is a two-dimensional real irreducible representation of $\mathfrak{h}$ with certain nonzero weight. Let (H_π) denote the basis for $\mathfrak{h}$ where

$$H_\pi H_\varrho = \begin{cases} 0 & \text{if } \pi \neq \varrho, \\ -\Pi_\pi & \text{if } \pi = \varrho, \end{cases} \tag{1.16}$$

where $\Pi_\pi \in \operatorname{End}(E)$ is the orthogonal projector onto E_π. Relative to this basis we can then write $J_i = \sum_\pi J_i^\pi H_\pi$, for some real numbers J_i^π.

1.1.4 Solving for the K_{ij}

Since $K_{ij} \in E$, we may decompose according to (1.14) as

$$K_{ij} = \sum_{\pi=1}^{s} K_{ij}^\pi + K_{ij}^0. \tag{1.17}$$

We may identify each E_π with a complex line where H_π acts by multiplication by i. This turns the complex number K^π_{ij} into one component of a complex bivector $K^\pi \in \Lambda^2\mathbb{C}^r$. Equation (1.13b) splits into one equation for each K^π and that equation says that

$$J^\pi_i K^\pi_{jk} + J^\pi_j K^\pi_{ki} + J^\pi_k K^\pi_{ij} = 0, \tag{1.18}$$

or equivalently that $J^\pi \wedge K^\pi = 0$, which has as unique solution $K^\pi = J^\pi \wedge t^\pi$, for some $t^\pi \in \mathbb{R}^r$. In other words,

$$K^\pi_{ij} = J^\pi_i t^\pi_j - J^\pi_j t^\pi_i. \tag{1.19}$$

Now consider the following vector space isometry $\varphi : \mathfrak{g} \to \mathfrak{g}$, defined by

$$\begin{aligned} \varphi(u_i) &= u_i - t_i - \frac{1}{2}\sum_{j=1}^{r} \langle t_i, t_j \rangle \, v_j \\ \varphi(v_i) &= v_i \\ \varphi(x) &= x + \sum_{i=1}^{r} \langle t_i, x \rangle \, v_i, \end{aligned} \tag{1.20}$$

for all $x \in E$, where $t_i \in E$ and hence $t_i = \sum_{\pi=1}^{s} t^\pi_i + t^0_i$. Under this isometry the form of the Lie algebra remains invariant, but K_{ij} changes as

$$K_{ij} \mapsto K_{ij} - J_i t_j + J_j t_i \tag{1.21}$$

and L_{ijk} changes in a manner which need not concern us here. Therefore we see that K^π_{ij} has been put to zero via this transformation, whereas K^0_{ij} remains unchanged. In other words, we can assume without loss of generality that $K_{ij} \in E_0$, so that $J_i K_{kl} = 0$, while still being subject to the quadratic equation (1.13c).

In summary, we have proved the following theorem, originally due to Kath and Olbrich [25]:

Theorem 1.1.2. Let $\mathfrak{g}$ be a finite-dimensional indecomposable metric Lie algebra of index $r > 0$ admitting a maximally isotropic centre. Then as a vector space

$$\mathfrak{g} = \bigoplus_{i=1}^{r} (\mathbb{R}u_i \oplus \mathbb{R}v_i) \oplus \bigoplus_{\pi=1}^{s} E_\pi \oplus E_0, \tag{1.22}$$

where all direct sums but the one between $\mathbb{R}u_i$ and $\mathbb{R}v_i$ are orthogonal and the inner product is as in Lemma 1.1.1. Let $0 \neq J^\pi \in \mathbb{R}^r$, $K_{ij} \in E_0$ and $L_{ijk} \in \mathbb{R}$

and assume that the K_{ij} obey the following quadratic relation

$$\langle K_{\ell i}, K_{jk}\rangle + \langle K_{\ell j}, K_{ki}\rangle + \langle K_{\ell k}, K_{ij}\rangle . = 0. \tag{1.23}$$

Then the Lie bracket of $\mathfrak{g}$ is given by

$$\begin{aligned} [u_i, u_j] &= K_{ij} + \sum_{k=1}^{r} L_{ijk} v_k \\ [u_i, x] &= J_i^\pi H_\pi x \\ [u_i, z] &= -\sum_{j=1}^{r} \langle K_{ij}, z\rangle v_j \\ [x, y] &= -\sum_{i=1}^{r} \langle x, J_i^\pi H_\pi y\rangle v_i, \end{aligned} \tag{1.24}$$

where $x, y \in E_\pi$ and $z \in E_0$. Furthermore, indecomposability forces the K_{ij} to span all of E_0, whence $\dim E_0 \leq \binom{r}{2}$.

It should be remarked that the L_{ijk} are only defined up to the following transformation

$$L_{ijk} \mapsto L_{ijk} + \langle K_{ij}, t_k\rangle + \langle K_{ki}, t_j\rangle + \langle K_{jk}, t_i\rangle , \tag{1.25}$$

for some $t_i \in E_0$.

It should also be remarked that the quadratic relation (1.23) is automatically satisfied for index $r \leq 3$, whereas for index $r \geq 4$ it defines an algebraic variety. In that sense, the classification problem for indecomposable metric Lie algebras admitting a maximally isotropic centre is not tame for index $r > 3$.

1.2 Metric 3-Lie algebras with maximally isotropic centre

After the above warm-up exercise, we may now tackle the problem of interest, namely the classification of finite-dimensional indecomposable metric 3-Lie algebras with maximally isotropic centre. The proof is not dissimilar to that of Theorem 1.1.2, but somewhat more involved and requires new ideas. Let us summarise the main steps in the proof.

1. In section 1.2.1 we write down the most general form of a metric 3-Lie algebra V consistent with the existence of a maximally isotropic centre Z. As a vector space, $V = Z \oplus Z^* \oplus W$, where Z and Z^* are nondegenerately paired and W is positive-definite. Because Z is central, the 4-form $F(x, y, z, w) := \langle [x, y, z], w \rangle$ on V defines an element in $\Lambda^4(W \oplus Z)$. The decomposition

$$\Lambda^4(W \oplus Z) = \Lambda^4 W \oplus \left(\Lambda^3 W \otimes Z\right) \oplus \left(\Lambda^2 W \otimes \Lambda^2 Z\right) \oplus \left(W \otimes \Lambda^3 Z\right) \oplus \Lambda^4 Z \tag{1.26}$$

induces a decomposition of $F = \sum_{a=0}^{4} F_a$, where $F_a \in \Lambda^{4-a} W \otimes \Lambda^a Z$, where the component F_4 is unconstrained.

2. The component F_0 defines the structure of a metric 3-Lie algebra on W which, if V is indecomposable, must be abelian, as shown in section 1.2.2.

3. The component F_1 defines a compatible family $[-, -]_i$ of reductive Lie algebras on W. In section 1.2.3 we show that they all are proportional to a reductive Lie algebra structure $\mathfrak{g} \oplus \mathfrak{z}$ on W, where $\mathfrak{g}$ is semisimple and $\mathfrak{z}$ is abelian.

4. In section 1.2.4 we show that the component F_2 defines a family J_{ij} of commuting endomorphisms spanning an abelian Lie subalgebra $\mathfrak{a} < \mathfrak{so}(\mathfrak{z})$. Under the action of $\mathfrak{a}$, $\mathfrak{z}$ breaks up into a direct sum of irreducible 2-planes E_π and a euclidean vector space E_0 on which the J_{ij} act trivially.

5. In section 1.2.5 we show that the component F_3 defines elements $K_{ijk} \in E_0$ which are subject to a quadratic equation.

1.2.1 Preliminary form of the 3-algebra

Let V be a finite-dimensional metric 3-Lie algebra with index $r > 0$ and admitting a maximally isotropic centre. Let v_i, $i = 1, \ldots, r$, denote a basis for the centre. Since the centre is (maximally) isotropic, $\langle v_i, v_j \rangle = 0$, and since the inner product on V is nondegenerate, there exists u_i, $i = 1, \ldots, r$ satisfying $\langle u_i, v_j \rangle = \delta_{ij}$. Furthermore, it is possible to choose the u_i such that $\langle u_i, u_j \rangle = 0$. The perpendicular complement W of the $2r$-dimensional subspace spanned by the u_i and v_i is therefore positive definite. In other words, V admits a vector space decomposition

$$V = \bigoplus_{i=1}^{r} (\mathbb{R}u_i \oplus \mathbb{R}v_i) \oplus W. \tag{1.27}$$

Since the v_i are central, metricity of V implies that the u_i cannot appear in the right-hand side of any 3-bracket. The most general form for the 3-bracket for V consistent with V being a metric 3-Lie algebra is given for all $x, y, z \in W$ by

$$\begin{aligned}
[u_i, u_j, u_k] &= K_{ijk} + \sum_{\ell=1}^{r} L_{ijk\ell} v_\ell \\
[u_i, u_j, x] &= J_{ij} x - \sum_{k=1}^{r} \langle K_{ijk}, x \rangle v_k \\
[u_i, x, y] &= [x, y]_i - \sum_{j=1}^{r} \langle x, J_{ij} y \rangle v_j \\
[x, y, z] &= [x, y, z]_W - \sum_{i=1}^{r} \langle [x, y]_i, z \rangle v_i,
\end{aligned} \tag{1.28}$$

where $J_{ij} \in \mathfrak{so}(W)$, $K_{ijk} \in W$ and $L_{ijk\ell} \in \mathbb{R}$ are skewsymmetric in their indices, $[-,-]_i : W \times W \to W$ is an alternating bilinear map which in addition obeys

$$\langle [x, y]_i, z \rangle = \langle x, [y, z]_i \rangle , \tag{1.29}$$

and $[-,-,-]_W : W \times W \times W \to W$ is an alternating trilinear map which obeys

$$\langle [x, y, z]_W, w \rangle = - \langle [x, y, w]_W, z \rangle . \tag{1.30}$$

The following lemma is the result of a straightforward, if somewhat lengthy, calculation.

Lemma 1.2.1. The fundamental identity (1) of the 3-Lie algebra V defined by

(1.28) is equivalent to the following conditions, for all $t, w, x, y, z \in W$:

$$\begin{aligned}
[t,w,[x,y,z]_W]_W &= [[t,w,x]_W,y,z]_W + [x,[t,w,y]_W,z]_W \\
&\quad + [x,y,[t,w,z]_W]_W && \text{(1.31a)}\\
[w,[x,y,z]_W]_i &= [[w,x]_i,y,z]_W + [x,[w,y]_i,z]_W \\
&\quad + [x,y,[w,z]_i]_W && \text{(1.31b)}\\
[x,y,[z,t]_i]_W &= [z,t,[x,y]_i]_W + [[x,y,z]_W,t]_i \\
&\quad + [z,[x,y,t]_W]_i && \text{(1.31c)}\\
J_{ij}[x,y,z]_W &= [J_{ij}x,y,z]_W + [x,J_{ij}y,z]_W + [x,y,J_{ij}z]_W && \text{(1.31d)}\\
J_{ij}[x,y,z]_W - [x,y,J_{ij}z]_W &= [[x,y]_i,z]_j - [[x,y]_j,z]_i && \text{(1.31e)}\\
[x,y,K_{ijk}]_W &= J_{jk}[x,y]_i + J_{ki}[x,y]_j + J_{ij}[x,y]_k && \text{(1.31f)}\\
[J_{ij}x,y,z]_W &= [[x,y]_i,z]_j + [[y,z]_j,x]_i + [[z,x]_i,y]_j && \text{(1.31g)}\\
J_{ij}[x,y,z]_W &= [z,[x,y]_j]_i + [x,[y,z]_j]_i + [y,[z,x]_j]_i && \text{(1.31h)}\\
[x,y,K_{ijk}]_W &= J_{ij}[x,y]_k - [J_{ij}x,y]_k - [x,J_{ij}y]_k && \text{(1.31i)}\\
J_{ik}[x,y]_j - J_{ij}[x,y]_k &= [J_{jk}x,y]_i + [x,J_{jk}y]_i && \text{(1.31j)}\\
[x,J_{jk}y]_i &= [J_{ij}x,y]_k + [J_{ki}x,y]_j + J_{jk}[x,y]_i && \text{(1.31k)}\\
[K_{ijk},x]_\ell &= [K_{\ell ij},x]_k + [K_{\ell jk},x]_i + [K_{\ell ki},x]_j && \text{(1.31l)}\\
[K_{ijk},x]_\ell - [K_{ij\ell},x]_k &= (J_{ij}J_{k\ell} - J_{k\ell}J_{ij})\,x && \text{(1.31m)}\\
[x,K_{jk\ell}]_i &= (J_{jk}J_{i\ell} + J_{k\ell}J_{ij} + J_{j\ell}J_{ki})\,x && \text{(1.31n)}\\
J_{im}K_{jk\ell} &= J_{ij}K_{k\ell m} + J_{ik}K_{\ell mj} + J_{i\ell}K_{jkm} && \text{(1.31o)}\\
J_{ij}K_{k\ell m} &= J_{\ell m}K_{ijk} + J_{mk}K_{ij\ell} + J_{k\ell}K_{ijm} && \text{(1.31p)}\\
\langle K_{ijm},K_{nk\ell}\rangle + \langle K_{ijk},K_{\ell mn}\rangle &= \langle K_{ijn},K_{k\ell m}\rangle + \langle K_{ij\ell},K_{mnk}\rangle\,. && \text{(1.31q)}
\end{aligned}$$

Of course, not all of these equations are independent, but we will not attempt to select a minimal set here, since we will be able to dispense with some of the equations easily.

1.2.2 W is abelian

Equation (1.31a) says that W becomes a 3-Lie algebra under $[-,-,-]_W$ which is metric by (1.30). Since W is positive-definite, it is reductive [12,13,14,15], whence isomorphic to an orthogonal direct sum $W = S \oplus A$, where S is semisimple and A is abelian. Furthermore, S is an orthogonal direct sum of several copies of the unique positive-definite simple 3-Lie algebra S_4 [10, 26]. We will show that as metric 3-Lie algebras $V = S \oplus S^{\perp}$, whence if V is indecomposable then $S = 0$ and $W = A$ is abelian as a 3-Lie algebra. This is an extension of the result in [15] by which semisimple 3-Lie algebras S factorise out of one-dimensional double extensions, and we will, in fact, follow a similar method to the one in [15] by which we perform an isometry on V which manifestly exhibits a nondegenerate ideal isomorphic to S as a 3-Lie algebra.

Consider then the isometry $\varphi : V \to V$, defined by

$$\varphi(v_i) = v_i \qquad \varphi(u_i) = u_i - s_i - \frac{1}{2}\sum_{j=1}^{r} \langle s_i, s_j \rangle v_j \qquad \varphi(x) = x + \sum_{i=1}^{r} \langle s_i, x \rangle v_i, \tag{1.32}$$

for $x \in W$ and for some $s_i \in W$. (This is obtained by extending the linear map $v_i \to v_i$ and $u_i \mapsto u_i - s_i$ to an isometry of V.) Under φ the 3-brackets (1.28) take the following form

$$\begin{aligned}
[\varphi(u_i), \varphi(u_j), \varphi(u_k)] &= \varphi(K^{\varphi}_{ijk}) + \sum_{\ell=1}^{r} L^{\varphi}_{ijk\ell} v_\ell \\
[\varphi(u_i), \varphi(u_j), \varphi(x)] &= \varphi(J^{\varphi}_{ij} x) - \sum_{k=1}^{r} \left\langle K^{\varphi}_{ijk}, x \right\rangle v_k \\
[\varphi(u_i), \varphi(x), \varphi(y)] &= \varphi([x, y]^{\varphi}_i) - \sum_{j=1}^{r} \left\langle x, J^{\varphi}_{ij} y \right\rangle v_j \\
[\varphi(x), \varphi(y), \varphi(z)] &= \varphi([x, y, z]_W) - \sum_{i=1}^{r} \langle [x, y]^{\varphi}_i, z \rangle v_i,
\end{aligned} \tag{1.33}$$

where

$$\begin{aligned}
[x, y]^{\varphi}_i &= [x, y]_i + [s_i, x, y]_W \\
J^{\varphi}_{ij} x &= J_{ij} x + [s_i, x]_j - [s_j, x]_i + [s_i, s_j, x]_W \\
K^{\varphi}_{ijk} &= K_{ijk} - J_{ij} s_k - J_{jk} s_i - J_{ki} s_j + [s_i, s_j]_k + [s_j, s_k]_i + [s_k, s_i]_j - [s_i, s_j, s_k]_W \\
L^{\varphi}_{ijk\ell} &= L_{ijk\ell} + \langle K_{jk\ell}, s_i \rangle - \langle K_{k\ell i}, s_j \rangle + \langle K_{\ell ij}, s_k \rangle - \langle K_{ijk}, s_\ell \rangle \\
&\quad - \langle s_i, J_{k\ell} s_j \rangle - \langle s_k, J_{j\ell} s_i \rangle - \langle s_j, J_{i\ell} s_k \rangle + \langle s_\ell, J_{jk} s_i \rangle + \langle s_\ell, J_{ki} s_j \rangle \\
&\quad + \langle s_\ell, J_{ij} s_k \rangle + \langle [s_i, s_j]_\ell, s_k \rangle - \langle [s_i, s_j]_k, s_\ell \rangle - \langle [s_k, s_i]_j, s_\ell \rangle \\
&\quad - \langle [s_j, s_k]_i, s_\ell \rangle + \langle [s_i, s_j, s_k]_W, s_\ell \rangle .
\end{aligned} \tag{1.34}$$

Lemma 1.2.2. There exists $s_i \in S$ such that the following conditions are met for all $x \in S$:

$$[x,-]_i^\varphi = 0 \qquad J_{ij}^\varphi x = 0 \qquad \left\langle K_{ijk}^\varphi, x \right\rangle = 0. \tag{1.35}$$

Assuming for a moment that this is the case, the only nonzero 3-brackets involving elements in $\varphi(S)$ are

$$[\varphi(x), \varphi(y), \varphi(z)] = \varphi([x,y,z]_W), \tag{1.36}$$

and this means that $\varphi(S)$ is a nondegenerate ideal of V, whence $V = \varphi(S) \oplus \varphi(S)^\perp$. But this violates the indecomposability of V, unless $S = 0$.

Proof of the lemma. To show the existence of the s_i, let us decompose $S = S_4^{(1)} \oplus \cdots \oplus S_4^{(m)}$ into m copies of the unique simple positive-definite 3-Lie algebra S_4. As shown in [15, §3.2], since J_{ij} and $[x,-]_i$ define skewsymmetric derivations of W, they preserve the decomposition of W into $S \oplus A$ and that of S into its simple factors. One consequence of this fact is that $J_{ij}x \in S$ for all $x \in S$ and $[x,y]_i \in S$ for all $x, y \in S$, and similarly if we substitute S for any of its simple factors in the previous statement. Notice in addition that putting $i = j$ in equation (1.31g), $[-,-]_i$ obeys the Jacobi identity. Hence on any one of the simple factors of S — let's call it generically S_4 — the bracket $[-,-]_i$ defines the structure of a four-dimensional Lie algebra. This Lie algebra is metric by equation (1.29) and positive definite. There are (up to isomorphism) precisely two four-dimensional positive-definite metric Lie algebras: the abelian Lie algebra and $\mathfrak{so}(3) \oplus \mathbb{R}$. In either case, as shown in [15, §3.2], there exists a unique $s_i \in S_4$ such that $[s_i, x, y]_W = [x,y]_i$ for $x, y \in S_4$. (In the former case, $s_i = 0$.) Since this is true for all simple factors, we conclude that there exists $s_i \in S$ such that $[s_i, x, y]_W = [x,y]_i$ for $x, y \in S$ and for all i.

Now equation (1.31g) says that for all $x, y, z \in S$,

$$\begin{aligned}
[J_{ij}x, y, z]_W &= [[x,y]_i, z]_j + [[y,z]_j, x]_i + [[z,x]_i, y]_j \\
&= [s_j, [s_i, x, y]_W, z]_W + [s_i, [s_j, y, z]_W, x]_W + [s_j, [s_i, z, x]_W, y]_W \\
&= [[s_i, s_j, x]_W, y, x]_W, \qquad \text{using (1.31a)}
\end{aligned}$$

which implies that $J_{ij}x - [s_i, s_j, x]_W$ centralises S, and thus is in A. However, for $x \in S$, both $J_{ij}x \in S$ and $[s_i, s_j, x]_W \in S$, so that $J_{ij}x = [s_i, s_j, x]_W$. Similarly, equation (1.31i) says that for all $x, y \in S$,

$$
\begin{aligned}
[x, y, K_{ijk}]_W &= J_{ij}[x, y]_k - [J_{ij}x, y]_k - [x, J_{ij}y]_k \\
&= [s_i, s_j, [s_k, x, y]_W]_W - [s_k, [s_i, s_j, x]_W, y]_W - [s_k, x, [s_i, s_j y]_W]_W \\
&= [[s_i, s_j, s_k]_W, x, y]_W, \qquad \text{using (1.31a)}
\end{aligned}
$$

which implies that $K_{ijk} - [s_i, s_j, s_k]_W$ centralises S, whence $K_{ijk} - [s_i, s_j, s_k]_W = K^A_{ijk} \in A$. Finally, using the explicit formulae for J^φ_{ij} and K^φ_{ijk} in equation (1.34), we see that for all all $x \in S$,

$$
\begin{aligned}
J^\varphi_{ij}x &= J_{ij}x + [s_i, x]_j - [s_j, x]_i + [s_i, s_j, x]_W \\
&= [s_i, s_j, x]_W + [s_j, s_i, x]_W - [s_i, s_j, x]_W + [s_i, s_j, x]_W = 0
\end{aligned}
$$

and

$$
\begin{aligned}
K^\varphi_{ijk} &= K_{ijk} - J_{ij}s_k - J_{jk}s_i - J_{ki}s_j + [s_i, s_j]_k + [s_j, s_k]_i + [s_k, s_i]_j - [s_i, s_j, s_k]_W \\
&= K^A_{ijk} + [s_i, s_j, s_k]_W - [s_i, s_j, s_k]_W - [s_j, s_k, s_i]_W - [s_k, s_i, s_j]_W \\
&\quad + [s_k, s_i, s_j]_W + [s_i, s_j, s_k]_W + [s_j, s_k, s_i]_W - [s_i, s_j, s_k]_W = K^A_{ijk},
\end{aligned}
$$

whence $\left\langle K^\varphi_{ijk}, x \right\rangle = 0$ for all $x \in S$. □

We may summarise the above discussion as follows.

Lemma 1.2.3. Let V be a finite-dimensional indecomposable metric 3-Lie algebra of index $r > 0$ with a maximally isotropic centre. Then as a vector space

$$
V = \bigoplus_{i=1}^{r} (\mathbb{R}u_i \oplus \mathbb{R}v_i) \oplus W, \tag{1.37}
$$

where W is positive-definite, $u_i, v_i \perp W$, $\langle u_i, u_j \rangle = 0$, $\langle v_i, v_j \rangle = 0$ and $\langle u_i, v_j \rangle = \delta_{ij}$. The v_i span the maximally isotropic centre. The nonzero 3-brackets are given by

$$
\begin{aligned}
[u_i, u_j, u_k] &= K_{ijk} + \sum_{\ell=1}^{r} L_{ijk\ell} v_\ell \\
[u_i, u_j, x] &= J_{ij} x - \sum_{k=1}^{r} \langle K_{ijk}, x \rangle v_k \\
[u_i, x, y] &= [x, y]_i - \sum_{j=1}^{r} \langle x, J_{ij} y \rangle v_j \\
[x, y, z] &= - \sum_{i=1}^{r} \langle [x, y]_i, z \rangle v_i,
\end{aligned}
\tag{1.38}
$$

for all $x, y, z \in W$ and for some $L_{ijk\ell} \in \mathbb{R}$, $K_{ijk} \in W$, $J_{ij} \in \mathfrak{so}(W)$, all of which are totally skewsymmetric in their indices, and bilinear alternating brackets $[-,-]_i : W \times W \to W$ satisfying equation (1.29). Furthermore, the fundamental identity of the 3-brackets (1.38) is equivalent to the following conditions on K_{ijk}, J_{ij} and $[-,-]_i$:

$$[x, [y, z]_i]_j = [[x, y]_j, z]_i + [y, [x, z]_j]_i \tag{1.39a}$$

$$[[x, y]_i, z]_j = [[x, y]_j, z]_i \tag{1.39b}$$

$$J_{ij}[x, y]_k = [J_{ij} x, y]_k + [x, J_{ij} y]_k \tag{1.39c}$$

$$0 = J_{j\ell}[x, y]_i + J_{\ell i}[x, y]_j + J_{ij}[x, y]_\ell \tag{1.39d}$$

$$[K_{ijk}, x]_\ell - [K_{ij\ell}, x]_k = (J_{ij} J_{k\ell} - J_{k\ell} J_{ij})\, x \tag{1.39e}$$

$$[x, K_{jk\ell}]_i = (J_{jk} J_{i\ell} + J_{k\ell} J_{ij} + J_{j\ell} J_{ki})\, x \tag{1.39f}$$

$$J_{ij} K_{k\ell m} = J_{\ell m} K_{ijk} + J_{mk} K_{ij\ell} + J_{k\ell} K_{ijm} \tag{1.39g}$$

$$
\begin{aligned}
0 &= \langle K_{ijn}, K_{k\ell m} \rangle + \langle K_{ij\ell}, K_{mnk} \rangle \\
&\quad - \langle K_{ijm}, K_{nk\ell} \rangle - \langle K_{ijk}, K_{\ell mn} \rangle .
\end{aligned}
\tag{1.39h}
$$

There are less equations in (1.39) than are obtained from (1.31) by simply making W abelian. It is not hard to show that the equations in (1.39) imply the rest. The study of equations (1.39) will take us until the end of this section. The analysis of these conditions will break naturally into several steps. In the first step we will solve equations (1.39a) and (1.39b) for the $[-,-]_i$. We will then solve equations (1.39c) and (1.39d), which will turn allow us to solve equations (1.39e) and (1.39f) for the J_{ij}. Finally we will solve equation (1.39g). We will not solve equation (1.39h). In fact, this equation defines an algebraic variety (an intersection of conics) which parametrises these 3-algebras.

1.2.3 Solving for the $[-,-]_i$

Condition (1.39a) for $i = j$ says that $[-,-]_i$ defines a Lie algebra structure on W, denoted $\mathfrak{g}_i$. By equation (1.29), $\mathfrak{g}_i$ is a metric Lie algebra. Since the inner product on W is positive-definite, $\mathfrak{g}_i$ is reductive, whence $\mathfrak{g}_i = [\mathfrak{g}_i, \mathfrak{g}_i] \oplus \mathfrak{z}_i$, where $\mathfrak{s}_i := [\mathfrak{g}_i, \mathfrak{g}_i]$ is the semisimple derived ideal of $\mathfrak{g}_i$ and $\mathfrak{z}_i$ is the centre of $\mathfrak{g}_i$. The following lemma will prove useful.

Lemma 1.2.4. Let $\mathfrak{g}_i$, $i = 1, \ldots, r$, be a family of reductive Lie algebras sharing the same underlying vector space W and let $[-,-]_i$ denote the Lie bracket of $\mathfrak{g}_i$. Suppose that they satisfy equations (1.39a) and (1.39b) and in addition that one of these Lie algebras, $\mathfrak{g}_1$ say, is simple. Then for all $x, y \in W$,

$$[x, y]_i = \kappa_i [x, y]_1, \tag{1.40}$$

where $\kappa_i \in \mathbb{R}$.

Proof. Equation (1.39a) says that for all $x \in W$, $\operatorname{ad}_i x := [x, -]_i$ is a derivation of $\mathfrak{g}_j$, for all i, j. In particular, $\operatorname{ad}_1 x$ is a derivation of $\mathfrak{g}_i$. Since derivations preserve the centre, $\operatorname{ad}_1 x : \mathfrak{z}_i \to \mathfrak{z}_i$, whence the subspace $\mathfrak{z}_i$ is an ideal of $\mathfrak{g}_1$. Since by hypothesis, $\mathfrak{g}_1$ is simple, we must have that either $\mathfrak{z}_i = W$, in which case $\mathfrak{g}_i$ is abelian and the lemma holds with $\kappa_i = 0$, or else $\mathfrak{z}_i = 0$, in which case $\mathfrak{g}_i$ is semisimple. It remains therefore to study this case.

Equation (1.39a) again says that $\operatorname{ad}_i x$ is a derivation of $\mathfrak{g}_1$. Since all derivations of $\mathfrak{g}_1$ are inner, this means that there is some element y such that $\operatorname{ad}_i x = \operatorname{ad}_1 y$. This element is moreover unique because ad_1 has trivial kernel. In other words, this defines a linear map

$$\psi_i : \mathfrak{g}_i \to \mathfrak{g}_1 \qquad \text{by} \qquad \operatorname{ad}_i x = \operatorname{ad}_1 \psi_i x \qquad \forall x \in W. \tag{1.41}$$

This linear map is a vector space isomorphism since $\ker \psi_i \subset \ker \operatorname{ad}_i = 0$, for $\mathfrak{g}_i$ semisimple. Now suppose that $I \lhd \mathfrak{g}_i$ is an ideal, whence $\operatorname{ad}_i(x) I \subset I$ for all $x \in \mathfrak{g}_i$. This means that $\operatorname{ad}_1(y) I \subset I$ for all $y \in \mathfrak{g}_1$, whence I is also an ideal of $\mathfrak{g}_1$. Since $\mathfrak{g}_1$ is simple, this means that $I = 0$ or else $I = W$; in other words, $\mathfrak{g}_i$ is simple.

Now for all $x, y, z \in W$, we have

$$
\begin{aligned}
[\psi_i[x,y]_i, z]_1 &= [[x,y]_i, z]_i && \text{by equation (1.41)}\\
&= [x,[y,z]_i]_i - [y,[x,z]_i]_i && \text{by the Jacobi identity of } \mathfrak{g}_i\\
&= [\psi_i x, [\psi_i y, z]_1]_1 - [\psi_i y, [\psi_i x, z]_1]_1 && \text{by equation (1.41)}\\
&= [[\psi_i x, \psi_i y]_1, z]_1 && \text{by the Jacobi identity of } \mathfrak{g}_1
\end{aligned}
$$

and since $\mathfrak{g}_1$ has trivial centre, we conclude that

$$\psi_i[x,y]_i = [\psi_i x, \psi_i y]_1,$$

whence $\psi_i : \mathfrak{g}_i \to \mathfrak{g}_1$ is a Lie algebra isomorphism.

Next, condition (1.39b) says that $\mathrm{ad}_1[x,y]_i = \mathrm{ad}_i[x,y]_1$, whence using equation (1.41), we find that $\mathrm{ad}_1[x,y]_i = \mathrm{ad}_1 \psi_i[x,y]_1$, and since ad_1 has trivial kernel, $[x,y]_i = \psi_i[x,y]_1$. We may rewrite this equation as $\mathrm{ad}_i x = \psi_i \mathrm{ad}_1 x$ for all x, which again by virtue of (1.41), becomes $\mathrm{ad}_1 \psi_i x = \psi_i \mathrm{ad}_1 x$, whence ψ_i commutes with the adjoint representation of $\mathfrak{g}_1$. Since $\mathfrak{g}_1$ is simple, Schur's Lemma says that ψ_i must be a multiple, κ_i say, of the identity. In other words, $\mathrm{ad}_i x = \kappa_i \mathrm{ad}_1 x$, which proves the lemma. □

Let us now consider the general case when none of the $\mathfrak{g}_i$ are simple. Let us focus on two reductive Lie algebras, $\mathfrak{g}_i = \mathfrak{z}_i \oplus \mathfrak{s}_i$, for $i = 1, 2$ say, sharing the same underlying vector space W. We will further decompose $\mathfrak{s}_i$ into its simple ideals

$$\mathfrak{s}_i = \bigoplus_{\alpha=1}^{N_i} \mathfrak{s}_i^\alpha . \tag{1.42}$$

For every $x \in W$, $\mathrm{ad}_1 x$ is a derivation of $\mathfrak{g}_2$, whence it preserves the centre $\mathfrak{z}_2$ and each simple ideal $\mathfrak{s}_2^\beta$. This means that $\mathfrak{z}_2$ and $\mathfrak{s}_2^\beta$ are themselves ideals of $\mathfrak{g}_1$, whence

$$\mathfrak{z}_2 = E_0 \oplus \bigoplus_{\alpha \in I_0} \mathfrak{s}_1^\alpha \qquad \text{and} \qquad \mathfrak{s}_2^\beta = E_\beta \oplus \bigoplus_{\alpha \in I_\beta} \mathfrak{s}_1^\alpha \qquad \forall \beta \in \{1, 2, \ldots, N_2\}\,, \tag{1.43}$$

and where the index sets $I_0, I_1, \ldots, I_{N_2}$ define a partition of $\{1, \ldots, N_1\}$, and

$$\mathfrak{z}_1 = E_0 \oplus E_1 \oplus \cdots \oplus E_{N_2} \tag{1.44}$$

is an orthogonal decomposition of $\mathfrak{z}_1$. But now notice that the restriction of $\mathfrak{g}_1$ to $E_\beta \oplus \bigoplus_{\alpha\in I_\beta} \mathfrak{s}_1^\alpha$ is reductive, whence we may apply Lemma 1.2.4 to each simple $\mathfrak{s}_2^\beta$ in turn. This allows us to conclude that for each β, either $\mathfrak{s}_2^\beta = E_\beta$ or else $\mathfrak{s}_2^\beta = \mathfrak{s}_1^\alpha$, for some $\alpha \in \{1, 2, \ldots, N_1\}$ which depends on β, and in this latter case, $[x, y]_{\mathfrak{s}_2^\beta} = \kappa[x, y]_{\mathfrak{s}_1^\alpha}$, for some nonzero constant κ.

This means that, given any one Lie algebra $\mathfrak{g}_i$, any other Lie algebra $\mathfrak{g}_j$ in the same family is obtained by multiplying its simple factors by some constants (which may be different in each factor and may also be zero) and maybe promoting part of its centre to be semisimple.

The metric Lie algebras $\mathfrak{g}_i$ induce the following orthogonal decomposition of the underlying vector space W. We let $W_0 = \bigcap_{i=1}^r \mathfrak{z}_i$ be the intersection of all the centres of the reductive Lie algebras $\mathfrak{g}_i$. Then we have the following orthogonal direct sum $W = W_0 \oplus \bigoplus_{\alpha=1}^N W_\alpha$, where restricted to each $W_{\alpha>0}$ at least one of the Lie algebras, $\mathfrak{g}_i$ say, is simple and hence all other Lie algebras $\mathfrak{g}_{j\neq i}$ are such that for all $x, y \in W_\alpha$,

$$[x, y]_j = \kappa_{ij}^\alpha [x, y]_i \qquad \exists \kappa_{ij}^\alpha \in \mathbb{R}. \tag{1.45}$$

To simplify the notation, we define a semisimple Lie algebra structure $\mathfrak{g}$ on the perpendicular complement of W_0, whose Lie bracket $[-,-]$ is defined in such a way that for all $x, y \in W_\alpha$, $[x, y] := [x, y]_i$, where $i \in \{1, 2, \ldots, r\}$ is the smallest such integer for which the restriction of $\mathfrak{g}_i$ to W_α is simple. (That such an integer i exists follows from the definition of W_0 and of the W_α.) It then follows that the restriction to W_α of every other $\mathfrak{g}_{j\neq i}$ is a (possibly zero) multiple of $\mathfrak{g}$.

We summarise this discussion in the following lemma, which summarises the solution of equations (1.39a) and (1.39b).

Lemma 1.2.5. Let $\mathfrak{g}_i$, $i = 1, \ldots, r$, be a family of metric Lie algebras sharing the same underlying euclidean vector space W and let $[-,-]_i$ denote the Lie bracket of $\mathfrak{g}_i$. Suppose that they satisfy equations (1.39a) and (1.39b). Then there is an orthogonal decomposition

$$W = W_0 \oplus \bigoplus_{\alpha=1}^N W_\alpha, \tag{1.46}$$

where

$$[x, y]_i = \begin{cases} 0 & \text{if } x, y \in W_0; \\ \kappa_i^\alpha [x, y] & \text{if } x, y \in W_\alpha, \end{cases} \tag{1.47}$$

for some $\kappa_i^\alpha \in \mathbb{R}$ and where $[-,-]$ are the Lie brackets of a semisimple Lie algebra $\mathfrak{g}$ with underlying vector space $\bigoplus_{\alpha=1}^N W_\alpha$.

1.2.4 Solving for the J_{ij}

Next we study the equations (1.39c) and (1.39d), which involve only J_{ij}. Equation (1.39c) says that each J_{ij} is a derivation over the $\mathfrak{g}_k$ for all i,j,k. Since derivations preserve the centre, every J_{ij} preserves the centre of every $\mathfrak{g}_k$ and hence it preserves their intersection W_0. Since J_{ij} preserves the inner product, it also preserves the perpendicular complement of W_0 in W, which is the underlying vector space of the semisimple Lie algebra $\mathfrak{g}$ of the previous lemma. Equation (1.39c) does not constrain the component of J_{ij} acting on W_0 since all the $[-,-]_k$ vanish there, but it does constrain the components of J_{ij} acting on $\bigoplus_{\alpha=1}^N W_\alpha$. Fix some α and let $x, y \in W_\alpha$. Then by virtue of equation (1.47), equation (1.39c) says that

$$\kappa_k^\alpha \left(J_{ij}[x,y] - [J_{ij}x, y] - [x, J_{ij}y] \right) = 0. \tag{1.48}$$

Since, given any α there will be at least some k for which $\kappa_k^\alpha \neq 0$, we see that J_{ij} is a derivation of $\mathfrak{g}$. Since $\mathfrak{g}$ is semisimple, this derivation is inner, where there exists a unique $z_{ij} \in \mathfrak{g}$, such that $J_{ij}y = [z_{ij}, y]$ for all $y \in \mathfrak{g}$. Since the simple ideals of $\mathfrak{g}$ are submodules under the adjoint representation, J_{ij} preserves each of the simple ideals and hence it preserves the decomposition (1.46). Let z_{ij}^α denote the component of z_{ij} along W_α. Equation (1.39d) can now be rewritten for $x, y \in W_\alpha$ as

$$\kappa_i^\alpha [z_{j\ell}^\alpha, [x,y]] + \kappa_j^\alpha [z_{\ell i}^\alpha, [x,y]] + \kappa_\ell^\alpha [z_{ij}^\alpha, [x,y]] = 0. \tag{1.49}$$

Since $\mathfrak{g}$ has trivial centre, this is equivalent to

$$\kappa_i^\alpha z_{j\ell}^\alpha + \kappa_j^\alpha z_{\ell i}^\alpha + \kappa_\ell^\alpha z_{ij}^\alpha = 0, \tag{1.50}$$

which can be written more suggestively as $\kappa^\alpha \wedge z^\alpha = 0$, where $\kappa^\alpha \in \mathbb{R}^r$ and $z^\alpha \in \Lambda^2 \mathbb{R}^r \otimes W_\alpha$. This equation has as unique solution $z^\alpha = \kappa^\alpha \wedge s^\alpha$, for some $s^\alpha \in \mathbb{R}^r \otimes W_\alpha$, or in indices

$$z_{ij}^\alpha = \kappa_i^\alpha s_j^\alpha - \kappa_j^\alpha s_i^\alpha \qquad \exists s_i^\alpha \in W_\alpha. \tag{1.51}$$

Let $s_i = \sum_\alpha s_i^\alpha \in \mathfrak{g}$ and consider now the isometry $\varphi : V \to V$ defined by

$$\begin{aligned} \varphi(v_i) &= v_i \\ \varphi(z) &= z \\ \varphi(u_i) &= u_i - s_i - \frac{1}{2}\sum_j \langle s_i, s_j\rangle\, v_j \\ \varphi(x) &= x + \sum_i \langle s_i, x\rangle\, v_i, \end{aligned} \tag{1.52}$$

for all $z \in W_0$ and all $x \in \bigoplus_{\alpha=1}^N W_\alpha$. The effect of such a transformation on the 3-brackets (1.38) is an uninteresting modification of K_{ijk} and $L_{ijk\ell}$ and the more interesting disappearance of J_{ij} from the 3-brackets involving elements in W_α. Indeed, for all $x \in W_\alpha$, we have

$$\begin{aligned} [\varphi(u_i), \varphi(u_j), \varphi(x)] &= [u_i - s_i, u_j - s_j, x] \\ &= [u_i, u_j, x] + [u_j, s_i, x] - [u_i, s_j, x] + [s_i, s_j, x] \\ &= J_{ij}x + [s_i, x]_j - [s_j, x]_i + \text{central terms} \\ &= [z_{ij}^\alpha, x] + \kappa_j^\alpha [s_i^\alpha, x] - \kappa_i^\alpha [s_j^\alpha, x] + \text{central terms} \\ &= [z_{ij}^\alpha + \kappa_j^\alpha s_i^\alpha - \kappa_i^\alpha s_j^\alpha, x] + \text{central terms} \\ &= 0 + \text{central terms}, \end{aligned}$$

where we have used equation (1.51).

This means that without loss of generality we may assume that $J_{ij}x = 0$ for all $x \in W_\alpha$ for any α. Now consider equation (1.39f) for $x \in \bigoplus_{\alpha=1}^N W_\alpha$. The right-hand side vanishes, whence $[K_{ijk}, x]_\ell = 0$. Also if $x \in W_0$, then $[K_{ijk}, x]_\ell = 0$ because x is central with respect to all $\mathfrak{g}_\ell$. Therefore we see that K_{ijk} is central with respect to all $\mathfrak{g}_\ell$, and hence $K_{ijk} \in W_0$.

In other words, we have proved the following

Lemma 1.2.6. In the notation of Lemma 1.2.5, the nonzero 3-brackets for V

may be brought to the form

$$\begin{aligned}
[u_i, u_j, u_k] &= K_{ijk} + \sum_{\ell=1}^{r} L_{ijk\ell} v_\ell \\
[u_i, u_j, x_0] &= J_{ij} x_0 - \sum_{k=1}^{r} \langle K_{ijk}, x_0 \rangle v_k \\
[u_i, x_0, y_0] &= - \sum_{j=1}^{r} \langle x_0, J_{ij} y_0 \rangle v_j \\
[u_i, x_\alpha, y_\alpha] &= \kappa_i^\alpha [x, y] \\
[x_\alpha, y_\alpha, z_\alpha] &= - \langle [x_\alpha, y_\alpha], z_\alpha \rangle \sum_{i=1}^{r} \kappa_i^\alpha v_i,
\end{aligned} \tag{1.53}$$

for all $x_\alpha, y_\alpha, z_\alpha \in W_\alpha$, $x_0, y_0 \in W_0$ and for some $L_{ijk\ell} \in \mathbb{R}$, $K_{ijk} \in W_0$ and $J_{ij} \in \mathfrak{so}(W_0)$, all of which are totally skewsymmetric in their indices.

Since their left-hand sides vanish, equations (1.39e) and (1.39f) become conditions on $J_{ij} \in \mathfrak{so}(W_0)$:

$$J_{ij} J_{k\ell} - J_{k\ell} J_{ij} = 0, \tag{1.54}$$

$$J_{jk} J_{i\ell} + J_{k\ell} J_{ij} + J_{j\ell} J_{ki} = 0. \tag{1.55}$$

The first condition says that the J_{ij} commute, whence since the inner product on W_0 is positive-definite, they must belong to the same Cartan subalgebra $\mathfrak{h} \subset \mathfrak{so}(W_0)$. Let H_π, for $\pi = 1, \dots, \lfloor \frac{\dim W_0}{2} \rfloor$, denote a basis for $\mathfrak{h}$, with each H_π corresponding to the generator of infinitesimal rotations in mutually orthogonal 2-planes in W_0. In particular, this means that $H_\pi H_\varrho = 0$ for $\pi \neq \varrho$ and that $H_\pi^2 = -\Pi_\pi$, with Π_π the orthogonal projector onto the 2-plane labelled by π. We write $J_{ij}^\pi \in \mathbb{R}$ for the component of J_{ij} along H_π. Fixing π we may think of J_{ij}^π as the components of $J^\pi \in \Lambda^2 \mathbb{R}^r$. Using the relations obeyed by the H_π, equation (1.55) separates into $\lfloor \frac{\dim W_0}{2} \rfloor$ equations, one for each value of π, which in terms of J^π can be written simply as $J^\pi \wedge J^\pi = 0$. This is a special case of a Plücker relation and says that J^π is decomposable; that is, $J^\pi = \eta^\pi \wedge \zeta^\pi$ for some $\eta^\pi, \zeta^\pi \in \mathbb{R}^r$. In other words, the solution of equations (1.54) and (1.55) is

$$J_{ij} = \sum_\pi \left(\eta_i^\pi \zeta_j^\pi - \eta_j^\pi \zeta_i^\pi \right) H_\pi \tag{1.56}$$

living in a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{so}(W_0)$.

1.2.5 Solving for the K_{ijk}

It remains to solve equations (1.39g) and (1.39h) for K_{ijk}. We shall concentrate on the linear equation (1.39g). This is a linear equation on $K \in \Lambda^3\mathbb{R}^r \otimes W_0$ and says that it is in the kernel of a linear map

$$\Lambda^3\mathbb{R}^r \otimes W_0 \longrightarrow \Lambda^2\mathbb{R}^r \otimes \Lambda^3\mathbb{R}^r \otimes W_0 \tag{1.57}$$

defined by

$$K_{ijk} \mapsto J_{ij}K_{k\ell m} - J_{\ell m}K_{ijk} - J_{mk}K_{ij\ell} - J_{k\ell}K_{ijm}. \tag{1.58}$$

The expression in the right-hand side is manifestly skewsymmetric in ij and $k\ell m$ separately, whence it belongs to $\Lambda^2\mathbb{R}^r \otimes \Lambda^3\mathbb{R}^r \otimes W_0$ as stated above. For generic r (here $r \geq 5$) we may decompose

$$\Lambda^2\mathbb{R}^r \otimes \Lambda^3\mathbb{R}^r = Y^{\ydiagram{2,2,1}}\mathbb{R}^r \oplus Y^{\ydiagram{2,1,1,1}}\mathbb{R}^r \oplus \Lambda^5\mathbb{R}^r, \tag{1.59}$$

where $Y^{\text{Young tableau}}$ denotes the corresponding Young symmetriser representation. Then one can see that the right-hand side of (1.58) has no component in the first of the above summands and hence lives in the remaining two summands, which are isomorphic to $\mathbb{R}^r \otimes \Lambda^4\mathbb{R}^r$.

We now observe that via an isometry of V of the form

$$\begin{aligned}
\varphi(v_i) &= v_i \\
\varphi(x_\alpha) &= x_\alpha \\
\varphi(u_i) &= u_i + t_i - \frac{1}{2}\sum_j \langle t_i, t_j \rangle \, v_j \\
\varphi(x_0) &= x_0 - \sum_i \langle x_0, t_i \rangle \, v_i,
\end{aligned} \tag{1.60}$$

for $t_i \in W_0$, the form of the 3-brackets (1.53) remains invariant, but with K_{ijk} and $L_{ijk\ell}$ transforming by

$$K_{ijk} \mapsto K_{ijk} + J_{ij}t_k + J_{jk}t_i + J_{ki}t_j, \tag{1.61}$$

and

$$\begin{aligned} L_{ijk\ell} \mapsto L_{ijk\ell} &+ \langle K_{ijk}, t_\ell \rangle - \langle K_{\ell ij}, t_k \rangle + \langle K_{k\ell i}, t_j \rangle - \langle K_{jk\ell}, t_i \rangle \\ &+ \langle J_{ij} t_k, t_\ell \rangle + \langle J_{ki} t_j, t_\ell \rangle + \langle J_{jk} t_i, t_\ell \rangle + \langle J_{i\ell} t_j, t_k \rangle \\ &+ \langle J_{j\ell} t_k, t_i \rangle + \langle J_{k\ell} t_i, t_j \rangle \,, \end{aligned} \tag{1.62}$$

respectively. In particular, this means that there is an ambiguity in K_{ijk}, which can be thought of as shifting it by the image of the linear map

$$\mathbb{R}^r \otimes W_0 \longrightarrow \Lambda^3 \mathbb{R}^r \otimes W_0 \tag{1.63}$$

defined by

$$t_i \mapsto J_{ij} t_k + J_{jk} t_i + J_{ki} t_j. \tag{1.64}$$

The two maps (1.57) and (1.63) fit together in a complex

$$\mathbb{R}^r \otimes W_0 \longrightarrow \Lambda^3 \mathbb{R}^r \otimes W_0 \longrightarrow \mathbb{R}^r \otimes \Lambda^4 \mathbb{R}^r \otimes W_0, \tag{1.65}$$

where the composition vanishes precisely by virtue of equations (1.54) and (1.55). We will show that this complex is acyclic away from the kernel of J, which will mean that without loss of generality we can take K_{ijk} in the kernel of J subject to the final quadratic equation (1.39h).

Let us decompose W_0 into an orthogonal direct sum

$$W_0 = \begin{cases} \displaystyle\bigoplus_{\pi=1}^{(\dim W_0)/2} E_\pi, & \text{if } \dim W_0 \text{ is even, and} \\ \mathbb{R} w \oplus \displaystyle\bigoplus_{\pi=1}^{(\dim W_0 - 1)/2} E_\pi, & \text{if } \dim W_0 \text{ is odd,} \end{cases} \tag{1.66}$$

where E_π are mutually orthogonal 2-planes and, in the second case, w is a vector perpendicular to all of them. On E_π the Cartan generator H_π acts as a complex structure, and hence we may identify each E_π with a complex one-dimensional vector space and H_π with multiplication by i. This decomposition of W_π allows us to decompose $K_{ijk} = K^w_{ijk} + \sum_\pi K^\pi_{ijk}$, where the first term is there only in the odd-dimensional situation and the K^π_{ijk} are complex numbers. The complex (1.65) breaks up into $\lfloor \frac{\dim W_0}{2} \rfloor$ complexes, one for each value of π. If $J^\pi = 0$ then K^π_{ijk} is not constrained there, but if $J^\pi = \eta^\pi \wedge \zeta^\pi \neq 0$ the complex turns out to have no homology, as we now show.

Without loss of generality we may choose the vectors η^π and ζ^π to be the elementary vectors e_1 and e_2 in $\mathbb{R}^r$, so that J^π has a $J^\pi_{12} = 1$ and all other $J^\pi_{ij} = 0$. Take $i = 1$ and $j = 2$ in the cocycle condition (1.57), to obtain

$$K^\pi_{k\ell m} = J^\pi_{\ell m} K^\pi_{12k} + J^\pi_{mk} K^\pi_{12\ell} + J^\pi_{k\ell} K^\pi_{12m}. \tag{1.67}$$

It follows that if any two of $k, \ell, m > 2$, then $K^\pi_{k\ell m} = 0$. In particular $K^\pi_{1ij} = K^\pi_{2ij} = 0$ for all $i, j > 2$, whence only K^π_{12k} for $k > 2$ can be nonzero. However for $k > 2$, $K^\pi_{12k} = J^\pi_{12} e_k$, with e_k the kth elementary vector in $\mathbb{R}^r$, and hence K^π_{12k} is in the image of the map (1.63); that is, a coboundary. This shows that we may assume without loss of generality that $K^\pi_{ijk} = 0$. In summary, the only components of K_{ijk} which survive are those in the kernel of all the J_{ij}. It is therefore convenient to split W_0 into an orthogonal direct sum

$$W_0 = E_0 \oplus \bigoplus_\pi E_\pi, \tag{1.68}$$

where on each 2-plane E_π, $J^\pi = \eta^\pi \wedge \zeta^\pi \neq 0$, whereas $J_{ij} x = 0$ for all $x \in E_0$. Then we can take $K_{ijk} \in E_0$.

Finally it remains to study the quadratic equation (1.39h). First of all we mention that this equation is automatically satisfied for $r \leq 4$. To see this notice that the equation is skewsymmetric in k, ℓ, m, n, whence if $r < 4$ it is automatically zero. When $r = 4$, we have to take k, ℓ, m, n all different and hence the equation becomes

$$\langle K_{ij1}, K_{234}\rangle - \langle K_{ij2}, K_{341}\rangle + \langle K_{ij3}, K_{412}\rangle - \langle K_{ij4}, K_{123}\rangle = 0,$$

which is skewsymmetric in i, j. There are six possible choices for i, j but by symmetry any choice is equal to any other up to relabeling, so without loss of generality let us take $i = 1$ and $j = 2$, whence the first two terms are identically zero and the two remaining terms satisfy

$$\langle K_{123}, K_{412}\rangle - \langle K_{124}, K_{123}\rangle = 0,$$

which is identically true. This means that the cases of index 3 and 4 are classifiable using our results. By contrast, the case of index 5 and above seems not to be tame. An example should suffice. So let us take the case of $r = 5$ and $\dim E_0 = 1$, so that the K_{ijk} can be taken to be real numbers. The solutions to (1.39h) now describe the intersection of five quadrics in $\mathbb{R}^{10}$:

$$\begin{aligned}
K_{125}K_{134} - K_{124}K_{135} + K_{123}K_{145} &= 0 \\
K_{125}K_{234} - K_{124}K_{235} + K_{123}K_{245} &= 0 \\
K_{135}K_{234} - K_{134}K_{235} + K_{123}K_{345} &= 0 \\
K_{145}K_{234} - K_{134}K_{245} + K_{124}K_{345} &= 0 \\
K_{145}K_{235} - K_{135}K_{245} + K_{125}K_{345} &= 0,
\end{aligned}$$

whence the solutions define an algebraic variety. One possible branch is given by setting $K_{1ij} = 0$ for all i, j, which leaves undetermined K_{234}, K_{235}, K_{245} and K_{345}. There are other branches which are linearly related to this one: for instance, setting $K_{2ij} = 0$, et cetera, but there are also other branches which are not linearly related to it.

1.2.6 Summary and conclusions

Let us summarise the above results in terms of the following structure theorem.

Theorem 1.2.7. Let V be a finite-dimensional indecomposable metric 3-Lie algebra of index $r > 0$ with a maximally isotropic centre. Then V admits a vector space decomposition into $r + M + N + 1$ orthogonal subspaces

$$V = \bigoplus_{i=1}^{r} (\mathbb{R}u_i \oplus \mathbb{R}v_i) \oplus \bigoplus_{\alpha=1}^{N} W_\alpha \oplus \bigoplus_{\pi=1}^{M} E_\pi \oplus E_0, \tag{1.69}$$

where W_α, E_π and E_0 are positive-definite subspaces with the E_π being two-dimensional, and where $\langle u_i, u_j\rangle = \langle v_i, v_j\rangle = 0$ and $\langle u_i, v_j\rangle = \delta_{ij}$. The 3-Lie algebra is defined in terms of the following data:

- $0 \neq \eta^\pi \wedge \zeta^\pi \in \Lambda^2\mathbb{R}^r$ for each $\pi = 1, \ldots, M$,
- $0 \neq \kappa^\alpha \in \mathbb{R}^r$ for each $\alpha = 1, \ldots, N$,
- a metric simple Lie algebra structure $\mathfrak{g}_\alpha$ on each W_α,
- $L \in \Lambda^4\mathbb{R}^r$, and
- $K \in \Lambda^3\mathbb{R}^r \otimes E_0$ subject to the equation

$$\langle K_{ijn}, K_{k\ell m}\rangle + \langle K_{ij\ell}, K_{mnk}\rangle - \langle K_{ijm}, K_{nk\ell}\rangle - \langle K_{ijk}, K_{\ell mn}\rangle = 0,$$

by the following 3-brackets,[1]

$$
\begin{aligned}
[u_i, u_j, u_k] &= K_{ijk} + \sum_{\ell=1}^{r} L_{ijk\ell} v_\ell \\
[u_i, u_j, x_0] &= -\sum_{k=1}^{r} \langle K_{ijk}, x_0 \rangle v_k \\
[u_i, u_j, x_\pi] &= J_{ij}^\pi H_\pi x_\pi \\
[u_i, x_\pi, y_\pi] &= -\sum_{j=1}^{r} \left\langle x_\pi, J_{ij}^\pi H_\pi y_\pi \right\rangle v_j \\
[u_i, x_\alpha, y_\alpha] &= \kappa_i^\alpha [x_\alpha, y_\alpha] \\
[x_\alpha, y_\alpha, z_\alpha] &= -\langle [x_\alpha, y_\alpha], z_\alpha \rangle \sum_{i=1}^{r} \kappa_i^\alpha v_i,
\end{aligned}
\tag{1.70}
$$

for all $x_0 \in E_0$, $x_\pi, y_\pi \in E_\pi$ and $x_\alpha, y_\alpha, z_\alpha \in W_\alpha$, and where $J_{ij}^\pi = \eta_i^\pi \zeta_j^\pi - \eta_j^\pi \zeta_i^\pi$ and H_π a complex structure on each 2-plane E_π. The resulting 3-Lie algebra is indecomposable provided that there is no $x_0 \in E_0$ which is perpendicular to all the K_{ijk}, whence in particular $\dim E_0 \leq \binom{r}{3}$.

1.3 Examples for low index

Let us now show how to recover the known classifications in index ≤ 2 from Theorem 1.2.7.

Let us consider the case of minimal positive index $r = 1$. In that case, the indices i, j, k, l in Theorem 1.2.7 can only take the value 1 and therefore J_{ij}, K_{ijk} and L_{ijkl} are not present. Indecomposability of V forces $E_0 = 0$ and $E_\pi = 0$, whence letting $u = u_1$ and $v = v_1$, we have $V = \mathbb{R}u \oplus \mathbb{R}v \oplus \bigoplus_{\alpha=1}^N W_\alpha$ as a vector space, with $\langle u, u \rangle = \langle v, v \rangle = 0$, $\langle u, v \rangle = 1$ and $\bigoplus_{\alpha=1}^N W_\alpha$ euclidean. The 3-brackets are:

$$
\begin{aligned}
[u, x_\alpha, y_\alpha] &= [x_\alpha, y_\alpha] \\
[x_\alpha, y_\alpha, z_\alpha] &= -\langle [x_\alpha, y_\alpha], z_\alpha \rangle v,
\end{aligned}
\tag{1.71}
$$

for all $x_\alpha, y_\alpha, z_\alpha \in W_\alpha$ and where we have redefined $\kappa^\alpha [x_\alpha, y_\alpha] \to [x_\alpha, y_\alpha]$, which is a simple Lie algebra on each W_α. This agrees with the classification of lorentzian 3-Lie algebras in [15] which was reviewed in the introduction.

[1] We understand tacitly that if a 3-bracket is not listed here it vanishes. Also every summation is written explicitly, so the summation convention is not in force. In particular, there is no sum over π in the third and fourth brackets.

Let us now consider $r = 2$. According to Theorem 1.2.7, those with a maximally isotropic centre may now have a nonvanishing J_{12} while K_{ijk} and L_{ijkl} are still absent. Indecomposability of V forces $E_0 = 0$. Therefore $W_0 = \bigoplus_{\pi=1}^{M} E_\pi$ and, as a vector space, $V = \mathbb{R}u_1 \oplus \mathbb{R}v_1 \oplus \mathbb{R}u_2 \oplus \mathbb{R}v_2 \oplus W_0 \oplus \bigoplus_{\alpha=1}^{N} W_\alpha$ with $\langle u_i, u_j\rangle = \langle v_i, v_j\rangle = 0$, $\langle u_i, v_j\rangle = \delta_{ij}$, $\forall i, j = 1, 2$ and $W_0 \oplus \bigoplus_{\alpha=1}^{N} W_\alpha$ is euclidean. The 3-brackets are now:

$$\begin{aligned}
[u_1, u_2, x_\pi] &= J x_\pi \\
[u_1, x_\pi, y_\pi] &= -\langle x_\pi, J y_\pi\rangle\, v_2 \\
[u_2, x_\pi, y_\pi] &= \langle x_\pi, J y_\pi\rangle\, v_1 \\
[u_1, x_\alpha, y_\alpha] &= \kappa_1^\alpha [x_\alpha, y_\alpha] \\
[u_2, x_\alpha, y_\alpha] &= \kappa_2^\alpha [x_\alpha, y_\alpha] \\
[x_\alpha, y_\alpha, z_\alpha] &= -\langle [x_\alpha, y_\alpha], z_\alpha\rangle\, \kappa_1^\alpha v_1 - \langle [x_\alpha, y_\alpha], z_\alpha\rangle\, \kappa_2^\alpha v_2,
\end{aligned} \tag{1.72}$$

for all $x_\pi, y_\pi \in E_\pi$ and $x_\alpha, y_\alpha, z_\alpha \in W_\alpha$. This agrees with the classification in [19] of finite-dimensional indecomposable 3-Lie algebras of index 2 whose centre contains a maximally isotropic plane. In that paper such algebras were denoted $V_{\mathrm{IIIb}}(E, J, \mathfrak{l}, \mathfrak{h}, \mathfrak{g}, \psi)$ with underlying vector space $\mathbb{R}(u, v) \oplus \mathbb{R}(\boldsymbol{e}_+, \boldsymbol{e}_-) \oplus E \oplus \mathfrak{l} \oplus \mathfrak{h} \oplus \mathfrak{g}$ with $\langle u, u\rangle = \langle v, v\rangle = \langle \boldsymbol{e}_\pm, \boldsymbol{e}_\pm\rangle = 0$, $\langle u, v\rangle = 1 = \langle \boldsymbol{e}_+, \boldsymbol{e}_-\rangle$ and all $\oplus$ orthogonal. The nonzero Lie 3-brackets are given by

$$\begin{aligned}
[u, \boldsymbol{e}_-, x] &= Jx \\
[u, x, y] &= \langle Jx, y\rangle\, \boldsymbol{e}_+ \\
[\boldsymbol{e}_-, x, y] &= -\langle Jx, y\rangle\, v \\
[\boldsymbol{e}_-, h_1, h_2] &= [h_1, h_2]_{\mathfrak{h}} \\
[h_1, h_2, h_3] &= -\langle [h_1, h_2]_{\mathfrak{h}}, h_3\rangle\, \boldsymbol{e}_+
\end{aligned}
\qquad
\begin{aligned}
[u, g_1, g_2] &= [\psi g_1, g_2]_{\mathfrak{g}} \\
[\boldsymbol{e}_-, g_1, g_2] &= [g_1, g_2]_{\mathfrak{g}} \\
[g_1, g_2, g_3] &= -\langle [g_1, g_2]_{\mathfrak{g}}, g_3\rangle\, \boldsymbol{e}_+ \\
&\quad - \langle [\psi g_1, g_2]_{\mathfrak{g}}, g_3\rangle\, v \\
[u, \ell_1, \ell_2] &= [\ell_1, \ell_2]_{\mathfrak{l}} \\
[\ell_1, \ell_2, \ell_3] &= -\langle [\ell_1, \ell_2]_{\mathfrak{l}}, \ell_3\rangle\, v,
\end{aligned} \tag{1.73}$$

where $x, y \in E$, $h, h_i \in \mathfrak{h}$, $g_i \in \mathfrak{g}$ and $\ell_i \in \mathfrak{l}$.

To see that this family of 3-algebras is of the type (1.72) it is enough to identify

$$u_1 \leftrightarrow u \qquad v_1 \leftrightarrow v \qquad u_2 \leftrightarrow e_- \qquad v_2 \leftrightarrow e_+ \tag{1.74}$$

as well as

$$W_0 \leftrightarrow E \qquad \text{and} \qquad \bigoplus_{\alpha=1}^{N} W_\alpha \leftrightarrow \mathfrak{l} \oplus \mathfrak{h} \oplus \mathfrak{g}, \tag{1.75}$$

where the last identification is not only as vector spaces but also as Lie algebras, and set

$$\begin{aligned} \kappa_1|_{\mathfrak{h}} &= 0 & \kappa_2|_{\mathfrak{h}} &= 1 \\ \kappa_1|_{\mathfrak{l}} &= 1 & \kappa_2|_{\mathfrak{l}} &= 0 \\ \kappa_1|_{\mathfrak{g}_\alpha} &= \psi_\alpha & \kappa_2|_{\mathfrak{g}_\alpha} &= 1, \end{aligned} \tag{1.76}$$

to obtain the map between the two families. As shown in [19] there are 9 different types of such 3-Lie algebras, depending on which of the four ingredients (E, J), $\mathfrak{l}$, $\mathfrak{h}$ or $(\mathfrak{g}, \psi)$ are present.

The next case is that of index $r = 3$, where there are up to 3 nonvanishing J_{ij} and one $K_{123} := K$, while L_{ijkl} is still not present. Indecomposability of V forces $\dim E_0 \leq 1$. As a vector space, V splits up as

$$V = \bigoplus_{i=1}^{3} (\mathbb{R}u_i \oplus \mathbb{R}v_i) \oplus \bigoplus_{\alpha=1}^{N} W_\alpha \oplus \bigoplus_{\pi=1}^{M} E_\pi \oplus E_0, \tag{1.77}$$

where all $\oplus$ are orthogonal except the second one, W_α. E_0 and E_π are positive-definite subspaces, $\dim E_0 \leq 1$, E_π is two-dimensional, and $\langle u_i, u_j \rangle = \langle v_i, v_j \rangle = 0$ and $\langle u_i, v_j \rangle = \delta_{ij}$. The 3-brackets are given by

$$\begin{aligned} [u_1, u_2, u_3] &= K \\ [u_i, u_j, x_0] &= -\sum_{k=1}^{r} \langle K_{ijk}, x_0 \rangle \, v_k \\ [u_i, u_j, x_\pi] &= J_{ij}^\pi H_\pi x_\pi \\ [u_i, x_\pi, y_\pi] &= -\sum_{j=1}^{r} \left\langle x_\pi, J_{ij}^\pi H_\pi y_\pi \right\rangle v_j \\ [u_i, x_\alpha, y_\alpha] &= \kappa_i^\alpha [x_\alpha, y_\alpha] \\ [x_\alpha, y_\alpha, z_\alpha] &= -\langle [x_\alpha, y_\alpha], z_\alpha \rangle \sum_{i=1}^{r} \kappa_i^\alpha v_i, \end{aligned} \tag{1.78}$$

for all $x_0 \in E_0$, $x_\pi, y_\pi \in E_\pi$ and $x_\alpha, y_\alpha, z_\alpha \in W_\alpha$, and where $J_{ij}^\pi = \eta_i^\pi \zeta_j^\pi - \eta_j^\pi \zeta_i^\pi$ and H_π a complex structure on each 2-plane E_π.

Finally, let us remark that the family of admissible 3-Lie algebras found in [24] are included in Theorem 1.2.7. In that paper, a family of solutions to equations (1.31) was found by setting each of the Lie algebra structures $[-,-]_i$ to be nonzero in orthogonal subspaces of W. This corresponds, in the language used here, to the particular case of allowing precisely one κ_i^α to be nonvanishing in each W_α.

Notice that, as shown in (1.76), already in [19] there are examples of admissible 3-Lie algebras of index 2 which are not of this form as both κ_1 and κ_2 might be nonvanishing in the $\mathfrak{g}_\alpha$ factors.

To solve the rest of the equations, two Ansätze are proposed in [24]:

- the trivial solution with nonvanishing J, i.e. $\kappa_i^\alpha = 0$, $K_{ijk} = 0$ for all $i, j, k = 1, \ldots, r$ and for all α; and
- precisely one $\kappa_i^\alpha = 1$ for each α (and include those W_α's where all κ's are zero in W_0) and one $J_{ij} := J \neq 0$ assumed to be an outer derivation of the reference Lie algebra defined on W.

As pointed out in that paper, L_{ijkl} is not constrained by the fundamental identity, so it can in principle take any value, whereas the Ansatz provided for K_{ijk} is given in terms of solutions of an equation equivalent to (1.39h). In the Lagrangians considered, both L_{ijkl} and K_{ijk} are set to zero.

One thing to notice is that in all these theories there is certain redundancy concerning the index of the 3-Lie algebra. If the indices in the nonvanishing structures κ_i^α, J_{ij}, K_{ijk} and L_{ijkl} involve only numbers from 1 to r_0, then any 3-Lie algebra with such nonvanishing structures and index $r \geq r_0$ gives rise to the equivalent theories.

In this light, in the first Ansatz considered, one can always define the non vanishing J to be J_{12} and then the corresponding theory will be equivalent to one associated to the index-2 3-Lie algebras considered in [19].

In the second case, the fact that J is an outer derivation implies that it must live on the abelian part of W as a Lie algebra, since the semisimple part does not possess outer derivations. This coincides with what was shown above, i.e., that $J|_{W_\alpha} = 0$ for each α. Notice that each Lie algebra $[-,-]_i$ identically vanishes in W_0, therefore the structure constants of the 3-Lie algebra do not mix J and $[-,-]_i$. The theories in [24] corresponding to this Ansatz also have $K_{ijk} = 0$, whence again they are equivalent to the theory corresponding to the index-2 3-Lie algebra which was denoted $V(E, J, \mathfrak{h})$ in [19].

Yet what are all such gaieties to me whose
thoughts are full of indices and surds?
Lewis Carroll

Chapter 2

Indefinite Signature Bagger–Lambert Lagrangians

In this section we will consider the physical properties of the Bagger–Lambert theory based on the most general kind of admissible metric 3-Lie algebra, as described in Theorem 1.2.7.

In particular we will investigate the structure of the expansion of the corresponding Bagger–Lambert Lagrangians around a vacuum wherein the scalars in half of the null directions of the 3-Lie algebra take the constant values implied by the equations of motion for the scalars in the remaining null directions, spanning the maximally isotropic centre. This technique was also used in [24] and is somewhat reminiscent of the novel Higgs mechanism that was first introduced by Mukhi and Papageorgakis [20] in the context of the Bagger–Lambert theory based on the unique simple euclidean 3-Lie algebra S_4. Recall that precisely this strategy has already been employed in lorentzian signature in [18], for the class of Bagger–Lambert theories found in [16, 17, 18] based on the unique admissible lorentzian metric 3-Lie algebra $W(\mathfrak{g})$, where it was first appreciated that this theory is perturbatively equivalent to $N = 8$ super Yang–Mills theory on $\mathbb{R}^{1,2}$ with the euclidean semisimple gauge algebra $\mathfrak{g}$. That is, there are no higher order corrections to the super Yang–Mills Lagrangian here, in contrast with the infinite set of corrections (suppressed by inverse powers of the gauge coupling) found for the super Yang–Mills theory with $\mathfrak{su}(2)$ gauge algebra arising from higgsing the Bagger–Lambert theory based on S_4 in [20]. This perturbative equivalence between the Bagger–Lambert theory based on $W(\mathfrak{g})$ and maximally supersymmetric Yang–Mills theory with euclidean gauge algebra $\mathfrak{g}$ has since been shown more rigorously in [21, 22, 23].

We will show that there exists a similar relation with $N = 8$ super Yang–Mills theory after expanding around the aforementioned maximally supersymmetric vacuum the Bagger–Lambert theories based on the more general physically admissible metric 3-Lie algebras we have considered. However, the gauge symmetry in the super Yang–Mills theory is generally based on a particular indefinite signature metric Lie algebra here that will be identified in terms of the data appearing in Theorem 1.2.7. The physical properties of the these Bagger–Lambert theories will be shown to describe particular combinations of decoupled super Yang-Mills multiplets with euclidean gauge algebras and free maximally supersymmetric massive vector multiplets. We will identify precisely how the physical moduli relate to the algebraic data in Theorem 1.2.7. We will also note how the theories resulting from those finite-dimensional indefinite signature 3-Lie algebras considered in [24] are recovered.

2.1 Review of two gauge theories in indefinite signature

Before utilising the structural results of the previous section, let us briefly review some general properties of the maximal $N = 8$ supersymmetric Bagger–Lambert and Yang–Mills theories in three-dimensional Minkowski space that will be of interest to us, when the fields are valued in a vector space V equipped with a metric of indefinite signature. We shall denote this inner product by $\langle -, - \rangle$ and take it to have general indefinite signature $(r, r + n)$. We can then define a null basis $e_A = (u_i, v_i, e_a)$ for V, with $i = 1, \ldots, r$, $a = 1, \ldots, n$, such that $\langle u_i, v_j \rangle = \delta_{ij}$, $\langle u_i, u_j \rangle = 0 = \langle v_i, v_j \rangle$ and $\langle e_a, e_b \rangle = \delta_{ab}$.

For the sake of clarity in the forthcoming analysis, we will ignore the fermions in these theories. Needless to say that they both have a canonical maximally supersymmetric completion and none of the manipulations we will perform break any of the supersymmetries of the theories.

2.1.1 Bagger–Lambert theory

Let us begin by reviewing some details of the bosonic field content of the Bagger–Lambert theory based on the 3-bracket $[-, -, -]$ defining a metric 3-Lie algebra structure on V. The components of the canonical 4-form for the metric 3-Lie algebra are $F_{ABCD} := \langle [e_A, e_B, e_C], e_D \rangle$ (indices will be lowered and raised using the metric $\langle e_A, e_B \rangle$ and its inverse). The bosonic fields in the Bagger–Lambert theory have components X_I^A and $(\tilde{A}_\mu)^A{}_B = F^A{}_{BCD} A_\mu^{CD}$, corresponding respectively to the scalars ($I = 1, \ldots, 8$ in the vector of the $\mathfrak{so}(8)$ R-symmetry) and the gauge field ($\mu = 0, 1, 2$ on $\mathbb{R}^{1,2}$ Minkowski space). Although the supersymmetry transformations and equations of motion can be expressed in terms of $(\tilde{A}_\mu)^A_B$, the Lagrangian requires it to be expressed as above in terms of A_μ^{AB}.

The bosonic part of the Bagger–Lambert Lagrangian is given by

$$\mathscr{L} = -\frac{1}{2} \langle D_\mu X_I, D^\mu X_I \rangle + \mathscr{V}(X) + \mathscr{L}_{\text{CS}} \ , \tag{2.1}$$

where the scalar potential is

$$\mathscr{V}(X) = -\tfrac{1}{12} \langle [X_I, X_J, X_K], [X_I, X_J, X_K] \rangle \ , \tag{2.2}$$

the Chern–Simons term is

$$\mathscr{L}_{\text{CS}} = \frac{1}{2} \left(A^{AB} \wedge d\tilde{A}_{AB} + \tfrac{2}{3} A^{AB} \wedge \tilde{A}_{AC} \wedge \tilde{A}^C{}_B \right) \ , \tag{2.3}$$

and $D_\mu \phi^A = \partial_\mu \phi^A + (\tilde{A}_\mu)^A{}_B \phi^B$ defines the action on any field ϕ valued in V of the derivative D that is gauge-covariant with respect to $\tilde{A}^A{}_B$. The infinitesimal gauge transformations take the form $\delta \phi^A = -\tilde{\Lambda}^A{}_B \phi^B$ and $\delta (\tilde{A}_\mu)^A{}_B = \partial_\mu \tilde{\Lambda}^A{}_B + (\tilde{A}_\mu)^A{}_C \tilde{\Lambda}^C{}_B - \tilde{\Lambda}^A{}_C (\tilde{A}_\mu)^C{}_B$, where $\tilde{\Lambda}^A{}_B = F^A{}_{BCD} \Lambda^{CD}$ in terms of an arbitrary skewsymmetric parameter $\Lambda^{AB} = -\Lambda^{BA}$.

If we now assume that the indefinite signature metric 3-Lie algebra above admits a maximally isotropic centre which we can take to be spanned by the basis elements v_i then the 4-form components $F_{v_i ABC}$ must all vanish identically. There are two important physical consequences of this assumption. The first is that the covariant derivative $D_\mu X_I^{u_i} = \partial_\mu X_I^{u_i}$. The second is that the tensors F_{ABCD} and $F_{ABC}{}^G F_{DEFG} = F_{ABC}{}^g F_{DEFg}$ which govern all the interactions in the Bagger–Lambert Lagrangian contain no legs in the v_i directions. Therefore the components $A_\mu^{v_i A}$ of the gauge field do not appear at all in the Lagrangian while $X_I^{v_i}$ appear only in the free kinetic term $-D_\mu X_I^{u_i} \partial^\mu X_I^{v_i} = -\partial_\mu X_I^{u_i} \partial^\mu X_I^{v_i}$. Thus $X_I^{v_i}$ can be integrated out imposing that each $X_I^{u_i}$ be a harmonic function on $\mathbb{R}^{1,2}$ which must be a constant if the solution is to be nonsingular. (We will assume this to be the case henceforth but singular monopole-type solutions may also be worthy of investigation, as in [27].) It is perhaps just worth noting that, in addition to setting $X_I^{u_i}$ constant, one must also set the fermions in all the u_i directions to zero which is necessary and sufficient for the preservation of maximal supersymmetry here.

The upshot is that we now have $-\frac{1}{2}\langle D_\mu X_I, D^\mu X_I\rangle = -\frac{1}{2}D_\mu X_I^a D^\mu X_I^a$ (with contraction over only the euclidean directions of V) and each $X_I^{u_i}$ is taken to be constant in (2.1). Since both $X_I^{v_i}$ and $A_\mu^{v_i A}$ are now absent, it will be more economical to define $X_I^i := X_I^{u_i}$ and $A_\mu^{ia} := A_\mu^{u_i a}$ henceforth.

2.1.2 Super Yang–Mills theory

Let us now perform an analogous review for $N = 8$ super Yang–Mills theory, with gauge symmetry based on the Lie bracket $[-,-]$ defining a metric Lie algebra structure $\mathfrak{g}$ on V. The components of the canonical 3-form on $\mathfrak{g}$ are $f_{ABC} := \langle [e_A, e_B], e_C\rangle$. The bosonic fields in the theory consist of a gauge field A_μ^A and seven scalar fields X_I^A (where now $I = 1, \ldots, 7$ in the vector of the $\mathfrak{so}(7)$ R-symmetry) with all fields taking values in V. The field strength for the gauge field takes the canonical form $F_{\mu\nu} = [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ in terms of the gauge-covariant derivative $D_\mu = \partial_\mu + [A_\mu, -]$. This theory is not scale-invariant and has a dimensionful coupling constant κ.

The bosonic part of the super Yang-Mills Lagrangian is given by

$$\mathscr{L}^{SYM}(A^A, X_I^A, \kappa|\mathfrak{g}) = -\frac{1}{2}\langle D_\mu X_I, D^\mu X_I\rangle - \tfrac{1}{4\kappa^2}\langle F_{\mu\nu}, F^{\mu\nu}\rangle - \tfrac{\kappa^2}{4}\langle [X_I, X_J], [X_I, X_J]\rangle . \quad (2.4)$$

Noting explicitly the dependence on the data on the left hand side will be useful when we come to consider super Yang-Mills theories with a much more elaborate gauge structure.

Assuming now that $\mathfrak{g}$ admits a maximally isotropic centre, again spanned by the basis elements v_i, then the 3-form components $f_{v_i AB}$ must all vanish identically. This property implies $DX_I^{u_i} = dX_I^{u_i}$, $F^{u_i} = dA^{u_i}$ and that the tensors f_{ABC} and $f_{AB}{}^E f_{CDE} = f_{AB}{}^e f_{CDe}$ which govern all the interactions contain no legs in the v_i directions. Therefore $X_I^{v_i}$ and A^{v_i} only appear linearly in their respective free kinetic terms, allowing them to be integrated out imposing that $X_I^{u_i}$ is constant and A^{u_i} is exact. Setting the fermions in all the u_i directions to zero again ensures the preservation of maximal supersymmetry.

The resulting structure is that all the inner products using $\langle e_A, e_B\rangle$ in (2.4) are to be replaced with $\langle e_a, e_b\rangle$ while all $X_I^{u_i}$ are to be taken constant and $A^{u_i} = d\phi^{u_i}$, for some functions ϕ^{u_i}. With both $X_I^{v_i}$ and A^{v_i} now absent, it will be convenient to define $X_I^i := X_I^{u_i}$ and $\phi^i := \phi^{u_i}$ henceforth.

Let us close this review by looking in a bit more detail at the physical properties of a particular example of a super Yang–Mills theory in indefinite signature with maximally isotropic centre, whose relevance will become clear in the forthcoming sections. Four-dimensional Yang–Mills theories based on such gauge groups were studied in [28]. The gauge structure of interest is based on the lorentzian metric Lie algebra defined by the double extension $\mathfrak{d}(E,\mathbb{R})$ of an even-dimensional vector space E with euclidean inner product. Writing $V = \mathbb{R}u \oplus \mathbb{R}v \oplus E$ as a lorentzian vector space, the nonvanishing Lie brackets of $\mathfrak{d}(E,\mathbb{R})$ are given by

$$[u,x] = Jx \ , \qquad [x,y] = -\langle x, Jy\rangle\, v \ , \tag{2.5}$$

for all $x, y \in E$ where the skewsymmetric endomorphism $J \in \mathfrak{so}(E)$ is part of the data defining the double extension. The canonical 3-form for $\mathfrak{d}(E,\mathbb{R})$ therefore has only the components $f_{uab} = J_{ab}$ with respect to the euclidean basis e_a on E. It will be convenient to take J to be nondegenerate and so the eigenvalues of J^2 will be negative-definite.

We shall define the positive number $\mu^2 := X_I^u X_I^u$ as the $SO(7)$-norm-squared of the constant 7-vector X_I^u and the projection operator $P_{IJ}^u := \delta_{IJ} - \mu^{-2}\, X_I^u X_J^u$ onto the hyperplane $\mathbb{R}^6 \subset \mathbb{R}^7$ orthogonal to X_I^u. It will also be convenient to define $x^a := X_I^u X_I^a$ as the projection of the seventh super Yang–Mills scalar field along X_I^u and $\mathcal{D}\Phi := d\Phi - d\phi^u \wedge J\Phi$ where Φ can be any p-form on $\mathbb{R}^{1,2}$ taking values in E. In terms of this data, the super Yang–Mills Lagrangian $\mathscr{L}^{SYM}((d\phi^u, A^a), (X_I^u, X_I^a), \kappa|\mathfrak{d}(E,\mathbb{R}))$ can be more succinctly expressed as

$$\begin{aligned} -\frac{1}{2} P_{IJ}^u \mathcal{D}_\mu X_I^a \mathcal{D}^\mu X_J^a + \tfrac{\kappa^2\mu^2}{2} (J^2)_{ab} P_{IJ}^u X_I^a X_J^b - \tfrac{1}{4\kappa^2} (2\,\mathcal{D}_{[\mu} A_{\nu]}^a)(2\,\mathcal{D}^{[\mu} A^{\nu]\,a}) \\ - \tfrac{1}{2\mu^2} \left(\mathcal{D}_\mu x^a + \mu^2 J^{ab} A_\mu^b\right) \left(\mathcal{D}^\mu x^a + \mu^2 J^{ac} A^{\mu\, c}\right) \ . \end{aligned} \tag{2.6}$$

From the first line we see that the six scalar fields $P_{IJ}^u X_J^a$ are massive with mass-squared given by the eigenvalues of the matrix $-\kappa^2\mu^2(J^2)_{ab}$. All the fields couple to $d\phi^u$ through the covariant derivative $\mathcal{D}$, but the second line shows that only the seventh scalar x^a couples to the gauge field A^a. However, the gauge symmetry of (2.6) under the transformations $\delta A^a = \mathcal{D}\lambda^a$ and $\delta x^a = -\mu^2 J^{ab}\lambda^b$, for any parameter $\lambda^a \in E$, shows that x^a is in fact pure gauge and can be removed in (2.6) by fixing $\lambda^a = \mu^{-2}(J^{-1})^{ab}x^b$. The remaining gauge symmetry of (2.6) is generated by the transformations $\delta\phi^u = \alpha$ and $\delta\Phi = \alpha\, J\Phi$ for all fields $\Phi \in E$, where α is an arbitrary scalar parameter. This is obvious since $\mathcal{D} = \exp(\phi^u J) d \exp(-\phi^u J)$ and therefore, one can take $\mathcal{D} = d$ in (2.6) by fixing $\alpha = -\phi^u$.

Thus, in the gauge defined above, the Lagrangian becomes simply

$$\mathscr{L}^{SYM}((d\phi^u, A^a), (X^u_I, X^a_I), \kappa|\mathfrak{d}(E,\mathbb{R})) = -\frac{1}{2}P^u_{IJ}\partial_\mu X^a_I \partial^\mu X^a_J$$
$$+ \tfrac{\kappa^2\mu^2}{2}(J^2)_{ab}P^u_{IJ}X^a_I X^b_J - \tfrac{1}{4\kappa^2}(2\,\partial_{[\mu}A^a_{\nu]})(2\,\partial^{[\mu}A^{\nu]\,a}) + \tfrac{\mu^2}{2}(J^2)_{ab}A^a_\mu A^{\mu\, b}\ , \quad (2.7)$$

describing $\dim E$ decoupled free abelian $N = 8$ supersymmetric massive vector multiplets, each of which contains bosonic fields given by the respective gauge field $\frac{1}{\kappa}A^a_\mu$ plus six scalars $P^u_{IJ}X^a_I$, all with the same mass-squared equal to the respective eigenvalue of $-\kappa^2\mu^2(J^2)_{ab}$.

It is worth pointing out that one can also obtain precisely the theory above from a particular truncation of an $N = 8$ super Yang–Mills theory with euclidean semisimple Lie algebra $\mathfrak{g}$. If one introduces a projection operator P_{IJ} onto a hyperplane $\mathbb{R}^6 \subset \mathbb{R}^7$ then one can rewrite the seven scalar fields in this euclidean theory in terms of the six projected fields $P_{IJ}X^a_J$ living on the hyperplane plus the single scalar y^a in the complementary direction. Unlike in the lorentzian theory above however, this seventh scalar is not pure gauge. Indeed, if we expand the super Yang–Mills Lagrangian (2.4) for this euclidean theory around a vacuum where y^a is constant then this constant appears as a physical modulus of the effective field theory, namely it gives rise to mass terms for the gauge field A^a and the six projected scalars $P_{IJ}X^a_J$. If one then truncates the effective field theory to the Coulomb branch, such that the dynamical fields A and $P_{IJ}X_J$ take values in a Cartan subalgebra $\mathfrak{t} < \mathfrak{g}$ (while the constant vacuum expectation value $y \in \mathfrak{g}$), then the Lagrangian takes precisely the form (2.7) after making the following identifications. First one must take $E = \mathfrak{t}$ whereby the gauge field A^a and coupling κ are the the same for both theories. Second one must identify the six-dimensional hyperplanes occupied by the scalars X^a_I in both theories such that P^u_{IJ} in (2.7) is identified with P_{IJ} here. Finally, the mass matrix for the euclidean theory is $-\kappa^2[(\mathrm{ad}_y)^2]_{ab}$ which must be identified with $-\kappa^2\mu^2(J^2)_{ab}$ in (2.7). This last identification requires some words of explanation. We have defined $\mathrm{ad}_y\,\Phi := [y, \Phi]$

for all $\Phi \in \mathfrak{g}$, where $[-,-]$ denotes the Lie bracket on $\mathfrak{g}$. Since we have truncated the dynamical fields to the Cartan subalgebra $\mathfrak{t}$, only the corresponding legs of $(\mathrm{ad}_y)^2$ contribute to the mass matrix. However, clearly y must not also be contained in $\mathfrak{t}$ or else the resulting mass matrix would vanish identically. Indeed, without loss of generality, one can take y to live in the orthogonal complement $\mathfrak{t}^\perp \subset \mathfrak{g}$ since it is only these components which contribute to the mass matrix. Thus, although $(\mathrm{ad}_y)^2$ can be nonvanishing on $\mathfrak{t}$, ad_y cannot. Thus we cannot go further and equate ad_y with μJ, even though their squares agree on $\mathfrak{t}$. To summarise all this more succinctly, after the aforementioned gauge-fixing of the lorentzian theory and truncation of the euclidean theory, we have shown that

$$\begin{aligned} \mathscr{L}^{SYM}\left((d\phi^u, A|_E), (X_I^u, P_{IJ}^u X_J|_E, x|_E), \kappa|\mathfrak{d}(E,\mathbb{R})\right) = \\ = \mathscr{L}^{SYM}\left(A|_E, (P_{IJ}X_J|_E, y|_{E^\perp}), \kappa|\mathfrak{g}\right) \ , \quad (2.8) \end{aligned}$$

where $E = \mathfrak{t}$, $y \in \mathfrak{t}^\perp \subset \mathfrak{g}$ is constant and $(\mathrm{ad}_y)^2 = \mu^2 J^2$ on $\mathfrak{t}$. Of course, it is not obvious that one can always solve this last equation for y in terms of a given μ and J nor indeed whether this restricts ones choice of $\mathfrak{g}$. However, it is the particular case of $\dim E = 2$ that will be of interest to us in the context of the Bagger–Lambert theory in 2.2.2 where we shall describe a nontrivial solution for any rank-2 semisimple Lie algebra $\mathfrak{g}$. Obvious generalisations of this solution give strong evidence that the equation can in fact always be solved.

2.2 Bagger–Lambert theory for admissible metric 3-Lie algebras

We will now substitute the data appearing in Theorem 1.2.7 into the bosonic part of the Bagger–Lambert Lagrangian (2.1), that is after having integrated out $X_I^{v_i}$ to set all $X_I^i := X_I^{u_i}$ constant.

Since we will be dealing with components of the various tensors appearing in Theorem 1.2.7, we need to introduce some index notation for components of the euclidean subspace $\bigoplus_{\alpha=1}^N W_\alpha \oplus \bigoplus_{\pi=1}^M E_\pi \oplus E_0$. To this end we partition the basis $e_a = (e_{a_\alpha}, e_{a_\pi}, e_{a_0})$ on the euclidean part of the algebra, where subscripts denote a basis for the respective euclidean subspaces. For example, $a_\alpha = 1, \dots, \dim W_\alpha$ whose range can thus be different for each α. Similarly $a_0 = 1, \dots, \dim E_0$, while $a_\pi = 1, 2$ for each two-dimensional space E_π. Since the decomposition

$\bigoplus_{\alpha=1}^{N} W_\alpha \oplus \bigoplus_{\pi=1}^{M} E_\pi \oplus E_0$ is orthogonal with respect to the euclidean metric $\langle e_a, e_b \rangle = \delta_{ab}$, we can take only the components $\langle e_{a_\alpha}, e_{b_\alpha} \rangle = \delta_{a_\alpha b_\alpha}$, $\langle e_{a_\pi}, e_{b_\pi} \rangle = \delta_{a_\pi b_\pi}$ and $\langle e_{a_0}, e_{b_0} \rangle = \delta_{a_0 b_0}$ to be nonvanishing. Since these are all just unit metrics on the various euclidean factors then we will not need to be careful about raising and lowering repeated indices, which are to be contracted over the index range of a fixed value of α, π or 0. Summations of the labels α and π will be made explicit.

In terms of this notation, we may write the data from Theorem 1.2.7 in terms of the following nonvanishing components of the canonical 4-form F_{ABCD} of the algebra

$$\begin{aligned} F_{u_i a_\alpha b_\alpha c_\alpha} &= \kappa_i^\alpha f_{a_\alpha b_\alpha c_\alpha} \\ F_{u_i u_j a_\pi b_\pi} &= \left(\eta_i^\pi \zeta_j^\pi - \eta_j^\pi \zeta_i^\pi \right) \epsilon_{a_\pi b_\pi} \\ F_{u_i u_j u_k a_0} &= K_{ijka_0} \\ F_{u_i u_j u_k u_l} &= L_{ijkl} \ , \end{aligned} \tag{2.9}$$

where $f_{a_\alpha b_\alpha c_\alpha}$ denotes the canonical 3-form for the simple metric Lie algebra structure $\mathfrak{g}_\alpha$ on W_α and we have used the fact that the 2x2 matrix H_π has only components $\epsilon_{a_\pi b_\pi} = -\epsilon_{b_\pi a_\pi}$, with $\epsilon_{12} = -1$, on each 2-plane E_π.

A final point of notational convenience will be to define $Y^{AB} := X_I^A X_I^B$ and the projection $X_I^\xi := \xi_i X_I^i$ for any $\xi \in \mathbb{R}^r$. Combining these definitions allows us to write certain projections which often appear in the Lagrangian like $Y^{\xi\varsigma} := X_I^\xi X_I^\varsigma$ and $Y^{\xi a} := X_I^\xi X_I^a$ for any $\xi, \varsigma \in \mathbb{R}^r$. It will sometimes be useful to write $Y^{\xi\xi} \equiv \|X^\xi\|^2 \geq 0$ where $\|X^\xi\|$ denotes the $SO(8)$-norm of the vector X_I^ξ. A similar shorthand will be adopted for projections of components of the gauge field, so that $A_\mu^{\xi\varsigma} := \xi_i \varsigma_j A_\mu^{ij}$ and $A_\mu^{\xi a} := \xi_i A_\mu^{ia}$.

It will be useful to note that the euclidean components of the covariant derivative $D_\mu X_I^A = \partial_\mu X_I^A + (\tilde{A}_\mu)^A{}_B X_I^B$ from section 2.1.1 can be written

$$\begin{aligned} D_\mu X_I^{a_\alpha} &= \partial_\mu X_I^{a_\alpha} - \kappa_i^\alpha f^{a_\alpha b_\alpha c_\alpha} \left(2\, A_\mu^{i b_\alpha} X_I^{c_\alpha} + A_\mu^{b_\alpha c_\alpha} X_I^i \right) \\ &=: \mathscr{D}_\mu X_I^{a_\alpha} - 2\, B_\mu^{a_\alpha} X_I^{\kappa^\alpha} \\ D_\mu X_I^{a_\pi} &= \partial_\mu X_I^{a_\pi} + 2\, \eta_i^\pi \zeta_j^\pi \epsilon^{a_\pi b_\pi} \left(A_\mu^{ij} X_I^{b_\pi} - A_\mu^{i b_\pi} X_I^j + A_\mu^{j b_\pi} X_I^i \right) \\ &= \partial_\mu X_I^{a_\pi} + 2\, \epsilon^{a_\pi b_\pi} \left(A_\mu^{\eta^\pi \zeta^\pi} X_I^{b_\pi} - A_\mu^{\eta^\pi b_\pi} X_I^{\zeta^\pi} + A_\mu^{\zeta^\pi b_\pi} X_I^{\eta^\pi} \right) \\ D_\mu X_I^{a_0} &= \partial_\mu X_I^{a_0} - K_{ijk}{}^{a_0} A_\mu^{ij} X_I^k \ . \end{aligned} \tag{2.10}$$

The second line defines two new quantities on each W_α: $B_\mu^{a_\alpha} := \frac{1}{2} f^{a_\alpha b_\alpha c_\alpha} A_\mu^{b_\alpha c_\alpha}$ and the covariant derivative $\mathscr{D}_\mu X_I^{a_\alpha} := \partial_\mu X_I^{a_\alpha} - 2\, f^{a_\alpha b_\alpha c_\alpha} \kappa_i^\alpha A_\mu^{i b_\alpha} X_I^{c_\alpha}$. The latter object is just the canonical covariant derivative with respect to the projected gauge field $\mathscr{A}_\mu^{a_\alpha} := -2\, A_\mu^{\kappa^\alpha a_\alpha}$ on each W_α. The associated field strength $\mathscr{F}_{\mu\nu} = [\mathscr{D}_\mu, \mathscr{D}_\nu]$ has components

$$\mathscr{F}^{a_\alpha} = -2\, \kappa_i^\alpha \left(dA^{i a_\alpha} - \kappa_j^\alpha f^{a_\alpha b_\alpha c_\alpha} A^{i b_\alpha} \wedge A^{j c_\alpha} \right) \ . \tag{2.11}$$

Although somewhat involved, the nomenclature above will help us understand more clearly the structure of the Bagger–Lambert Lagrangian. Let us consider now the contributions to (2.1) coming from the scalar kinetic terms, the sextic potential and the Chern–Simons term in turn.

The kinetic terms for the scalar fields give

$$\begin{aligned} -\frac{1}{2}\langle D_\mu X_I, D^\mu X_I \rangle = &-\frac{1}{2} D_\mu X_I^{a_0} D^\mu X_I^{a_0} \\ &-\frac{1}{2}\sum_{\alpha=1}^{N} D_\mu X_I^{a_\alpha} D^\mu X_I^{a_\alpha} - \frac{1}{2}\sum_{\pi=1}^{M} D_\mu X_I^{a_\pi} D^\mu X_I^{a_\pi} \end{aligned} \tag{2.12}$$

which expands to

$$\begin{aligned} &\sum_{\alpha=1}^{N}\left[-\frac{1}{2}\mathscr{D}_\mu X_I^{a_\alpha} \mathscr{D}^\mu X_I^{a_\alpha} + 2\, X_I^{\kappa^\alpha} B_\mu^{a_\alpha} \mathscr{D}^\mu X_I^{a_\alpha} - 2\, Y^{\kappa^\alpha \kappa^\alpha} B_\mu^{a_\alpha} B^{\mu\, a_\alpha} \right] \\ &+ \sum_{\pi=1}^{M}\Big[-\frac{1}{2}\partial_\mu X_I^{a_\pi} \partial^\mu X_I^{a_\pi} - 2\, \partial^\mu X_I^{a_\pi} \epsilon^{a_\pi b_\pi} \left(A_\mu^{\eta^\pi \zeta^\pi} X_I^{b_\pi} - A_\mu^{\eta^\pi b_\pi} X_I^{\zeta^\pi} + A_\mu^{\zeta^\pi b_\pi} X_I^{\eta^\pi} \right) \\ &\qquad - 2 \left(A_\mu^{\eta^\pi \zeta^\pi} X_I^{a_\pi} - A_\mu^{\eta^\pi a_\pi} X_I^{\zeta^\pi} + A_\mu^{\zeta^\pi a_\pi} X_I^{\eta^\pi} \right) \\ &\qquad \times \left(A^{\mu\, \eta^\pi \zeta^\pi} X_I^{a_\pi} - A^{\mu\, \eta^\pi a_\pi} X_I^{\zeta^\pi} + A^{\mu\, \zeta^\pi a_\pi} X_I^{\eta^\pi} \right) \Big] \\ &- \frac{1}{2}\partial_\mu X_I^{a_0} \partial^\mu X_I^{a_0} + K_{ijk}{}^{a_0} A_\mu^{ij} \partial^\mu Y^{k a_0} - \frac{1}{2} K_{ijka_0} K_{lmna_0} Y^{kl} A_\mu^{ij} A^{\mu\, mn} \ . \end{aligned} \tag{2.13}$$

The scalar potential can be written $\mathscr{V}(X) = \mathscr{V}^W(X) + \mathscr{V}^E(X) + \mathscr{V}^{E_0}(X)$ where

$$\begin{aligned} \mathscr{V}^W(X) &= -\tfrac{1}{4}\sum_{\alpha=1}^{N} f^{a_\alpha b_\alpha e_\alpha} f^{c_\alpha d_\alpha e_\alpha} \left(Y^{\kappa^\alpha \kappa^\alpha} Y^{a_\alpha c_\alpha} - Y^{\kappa^\alpha a_\alpha} Y^{\kappa^\alpha c_\alpha} \right) Y^{b_\alpha d_\alpha} \\ \mathscr{V}^E(X) &= -\tfrac{1}{2}\sum_{\pi=1}^{M} \Big\{ Y^{a_\pi a_\pi} \left(Y^{\eta^\pi \eta^\pi} Y^{\zeta^\pi \zeta^\pi} - (Y^{\eta^\pi \zeta^\pi})^2 \right) + 2\, Y^{\eta^\pi a_\pi} Y^{\zeta^\pi a_\pi} Y^{\eta^\pi \zeta^\pi} \\ &\qquad - Y^{\eta^\pi a_\pi} Y^{\eta^\pi a_\pi} Y^{\zeta^\pi \zeta^\pi} - Y^{\zeta^\pi a_\pi} Y^{\zeta^\pi a_\pi} Y^{\eta^\pi \eta^\pi} \Big\} \\ \mathscr{V}^{E_0}(X) &= -\tfrac{1}{12} K_{ijka_0} K_{lmna_0} Y^{il} Y^{jm} Y^{kn} \ . \end{aligned} \tag{2.14}$$

Notice that $\mathscr{V}^{E_0}(X)$ is constant and will be ignored henceforth.

And finally, the Chern–Simons term can be written $\mathscr{L}_{\rm CS} = \mathscr{L}^W_{\rm CS} + \mathscr{L}^E_{\rm CS} + \mathscr{L}^{E_0}_{\rm CS}$ where

$$
\begin{aligned}
\mathscr{L}^W_{\rm CS} &= -2 \sum_{\alpha=1}^{N} B^{a_\alpha} \wedge \mathscr{F}^{a_\alpha} \\
\mathscr{L}^E_{\rm CS} &= -4 \sum_{\pi=1}^{M} \Big\{ \epsilon^{a_\pi b_\pi}\, A^{\eta^\pi a_\pi} \wedge A^{\zeta^\pi b_\pi} + 2\, A^{\eta^\pi \zeta^\pi} \wedge A^{\eta^\pi a_\pi} \wedge A^{\zeta^\pi a_\pi} \\
&\qquad - \frac{1}{2} \epsilon^{a_\pi b_\pi} A^{a_\pi b_\pi} \wedge dA^{\eta^\pi \zeta^\pi} \Big\} \\
\mathscr{L}^{E_0}_{\rm CS} &= 2\, K_{ijka_0} A^{ij} \wedge dA^{ka_0} - \tfrac{1}{3} K_{ikla_0} K_{jmna_0} A^{ij} \wedge A^{kl} \wedge A^{mn} + \frac{1}{2} L_{ijkl} A^{ij} \wedge dA^{kl} \ .
\end{aligned}
\tag{2.15}
$$

These expressions are valid only up to total derivative terms that will be discarded.

Clearly there is a certain degree of factorisation for the Bagger–Lambert Lagrangian into separate terms living on the different components of $\bigoplus_{\alpha=1}^N W_\alpha \oplus \bigoplus_{\pi=1}^M E_\pi \oplus E_0$. Indeed let us define accordingly $\mathscr{L}^W = -\frac{1}{2} \sum_{\alpha=1}^N D_\mu X_I^{a_\alpha} D^\mu X_I^{a_\alpha} + \mathscr{V}^W(X) + \mathscr{L}^W_{\rm CS}$ and likewise for E and E_0. This is mainly for notational convenience however and one must be wary of the fact that $\mathscr{L}^E$ and $\mathscr{L}^{E_0}$ could have some fields, namely components of A^{ij}, in common.

To relate the full Lagrangian $\mathscr{L}$ with a super Yang-Mills theory, one has first to identify and integrate out those fields which are auxiliary or appear linearly as Lagrange multipliers. This will be most easily done by considering $\mathscr{L}^W$, $\mathscr{L}^E$ and $\mathscr{L}^{E_0}$ in turn.

2.2.1 $\mathscr{L}^W$

The field B^{a_α} appears only algebraically as an auxiliary field in $\mathscr{L}^W$. Its equation of motion implies

$$
2\, Y^{\kappa^\alpha \kappa^\alpha} B^{a_\alpha} = X_I^{\kappa^\alpha} \mathscr{D} X_I^{a_\alpha} + *\mathscr{F}^{a_\alpha} \ , \tag{2.16}
$$

for each value of α. Substituting this back into $\mathscr{L}^W$ then gives

$$
\begin{aligned}
&- \frac{1}{2} \sum_{\alpha=1}^{N} D_\mu X_I^{a_\alpha} D^\mu X_I^{a_\alpha} + \mathscr{L}^W_{\rm CS} = \\
&\qquad \sum_{\alpha=1}^{N} \Big\{ -\frac{1}{2} P_{IJ}^{\kappa^\alpha} \mathscr{D}_\mu X_I^{a_\alpha} \mathscr{D}^\mu X_J^{a_\alpha} - \tfrac{1}{4 Y^{\kappa^\alpha \kappa^\alpha}} \mathscr{F}^{a_\alpha}_{\mu\nu} \mathscr{F}^{\mu\nu\, a_\alpha} \Big\} \ ,
\end{aligned}
\tag{2.17}
$$

where, for each α, $P^{\kappa^\alpha}_{IJ} := \delta_{IJ} - \frac{X_I^{\kappa^\alpha} X_J^{\kappa^\alpha}}{Y^{\kappa^\alpha \kappa^\alpha}}$ is the projection operator onto the hyperplane $\mathbb{R}^7 \subset \mathbb{R}^8$ which is orthogonal to the 8-vector $X_I^{\kappa^\alpha}$ that κ_i^α projects the constant X_I^i onto.

Furthermore, in terms of the Lie bracket $[-,-]_\alpha$ on $\mathfrak{g}_\alpha$, the scalar potential can be written

$$\mathscr{V}^W(X) = -\tfrac{1}{4} \sum_{\alpha=1}^{N} Y^{\kappa^\alpha \kappa^\alpha} \, P^{\kappa^\alpha}_{IK} P^{\kappa^\alpha}_{JL} \, [X_I, X_J]_\alpha^{a_\alpha} [X_K, X_L]_\alpha^{a_\alpha} \ . \tag{2.18}$$

In conclusion, we have shown that upon integrating out B^{a_α} one can identify

$$\mathscr{L}^W = \sum_{\alpha=1}^{N} \mathscr{L}^{SYM} \left(\mathscr{A}^{a_\alpha}, P^{\kappa^\alpha}_{IJ} X_J^{a_\alpha}, \|X^{\kappa^\alpha}\| | \mathfrak{g}_\alpha \right) \ . \tag{2.19}$$

The identification above with the Lagrangian in (2.4) has revealed a rather intricate relation between the data κ_i^α and $\mathfrak{g}_\alpha$ on W_α from Theorem 1.2.7 and the physical parameters in the super Yang–Mills theory. In particular, the coupling constant for the super Yang–Mills theory on W_α corresponds to the $SO(8)$-norm of $X_I^{\kappa^\alpha}$. Moreover, the direction of $X_I^{\kappa^\alpha}$ in $\mathbb{R}^8$ determines which hyperplane the seven scalar fields in the super Yang–Mills theory must occupy and thus may be different on each W_α. The gauge symmetry is based on the euclidean Lie algebra $\bigoplus_{\alpha=1}^N \mathfrak{g}_\alpha$.

The main point to emphasise is that it is the projections of the individual κ_i^α onto the vacuum described by constant X_I^i (rather than the vacuum expectation values themselves) which determine the physical moduli in the theory. For example, take $N = 1$ with only one simple Lie algebra structure $\mathfrak{g} = \mathfrak{su}(n)$ on W. The Lagrangian (2.19) then describes precisely the low-energy effective theory for n coincident D2-branes in type IIA string theory, irrespective of the index r of the initial 3-Lie algebra. The only difference is that the coupling $\|X^\kappa\|$, to be interpreted as the perimeter of the M-theory circle, is realised as a different projection for different values of r.

Thus, in general, we are assuming a suitably generic situation wherein none of the projections $X_I^{\kappa^\alpha}$ vanish identically. If $X_I^{\kappa^\alpha} = 0$ for a given value of α then the W_α part of the scalar potential (2.14) vanishes identically and the only occurrence of the corresponding B^{a_α} is in the Chern–Simons term (2.15). Thus, for this particular value of α, B^{a_α} has become a Lagrange multiplier imposing $\mathscr{F}^{a_\alpha} = 0$ and so $\mathscr{A}^{a_\alpha}$ is pure gauge. The resulting Lagrangian on this W_α therefore describes a free $N = 8$ supersymmetric theory for the eight scalar fields $X_I^{a_\alpha}$.

2.2.2 $\mathscr{L}^E$

The field $\epsilon^{a_\pi b_\pi} A^{a_\pi b_\pi}$ appears only linearly in one term in $\mathscr{L}^E_{\mathrm{CS}}$ and is therefore a Lagrange multiplier imposing the constraint $A^{\eta^\pi \zeta^\pi} = d\phi^{\eta^\pi \zeta^\pi}$, for some some scalar fields $\phi^{\eta^\pi \zeta^\pi}$, for each value of π. The number of distinct scalars $\phi^{\eta^\pi \zeta^\pi}$ will depend on the number of linearly independent 2-planes in $\mathbb{R}^r$ which the collection of all $\eta^\pi \wedge \zeta^\pi$ span for $\pi = 1, \ldots, M$. Let us henceforth call this number k, which is clearly bounded above by $\binom{r}{2}$.

Moreover, up to total derivatives, one has a choice of taking just one of the two gauge fields $A^{\eta^\pi a_\pi}$ and $A^{\zeta^\pi a_\pi}$ to be auxiliary in $\mathscr{L}^E$. These are linearly independent gauge fields by virtue of the fact that $\eta^\pi \wedge \zeta^\pi$ span a 2-plane in $\mathbb{R}^r$ for each value of π. Without loss of generality we can take $A^{\eta^\pi a_\pi}$ to be auxiliary and integrate it out in favour of $A^{\zeta^\pi a_\pi}$. After implementing the Lagrange multiplier constraint above, one finds that the equation of motion of $A^{\eta^\pi a_\pi}$ implies

$$\begin{aligned} 2\, Y^{\zeta^\pi \zeta^\pi} A^{\eta^\pi a_\pi} = -\epsilon^{a_\pi b_\pi} \Big\{ X_I^{\zeta^\pi} \left(dX_I^{b_\pi} + 2\, \epsilon^{b_\pi c_\pi} \left(X_I^{c_\pi} d\phi^{\eta^\pi \zeta^\pi} + X_I^{\eta^\pi} A^{\zeta^\pi c_\pi} \right) \right) \\ + 2 * \left(dA^{\zeta^\pi b_\pi} + 2\, \epsilon^{b_\pi c_\pi} d\phi^{\eta^\pi \zeta^\pi} \wedge A^{\zeta^\pi c_\pi} \right) \Big\} \ . \quad (2.20) \end{aligned}$$

Substituting this back into $\mathscr{L}^E$ then, following a rather lengthy but straightforward calculation, one finds that

$$\begin{aligned} -\frac{1}{2} \sum_{\pi=1}^{M} D_\mu X_I^{a_\pi} D^\mu X_I^{a_\pi} + \mathscr{L}^E_{\mathrm{CS}} = \\ -\frac{1}{2} \sum_{\pi=1}^{M} P_{IJ}^{\zeta^\pi} \left(\partial_\mu X_I^{a_\pi} + 2\, \epsilon^{a_\pi b_\pi} \left(X_I^{b_\pi} \partial_\mu \phi^{\eta^\pi \zeta^\pi} + X_I^{\eta^\pi} A_\mu^{\zeta^\pi b_\pi} \right) \right) \\ \times \left(\partial^\mu X_J^{a_\pi} + 2\, \epsilon^{a_\pi c_\pi} \left(X_J^{c_\pi} \partial^\mu \phi^{\eta^\pi \zeta^\pi} + X_J^{\eta^\pi} A^{\mu\, \zeta^\pi c_\pi} \right) \right) \\ - \sum_{\pi=1}^{M} \tfrac{4}{Y^{\zeta^\pi \zeta^\pi}} \left(\partial_{[\mu} A_{\nu]}^{\zeta^\pi a_\pi} + 2\, \epsilon^{a_\pi b_\pi} \partial_{[\mu} \phi^{\eta^\pi \zeta^\pi} A_{\nu]}^{\zeta^\pi b_\pi} \right) \\ \times \left(\partial^\mu A^{\nu\, \zeta^\pi a_\pi} + 2\, \epsilon^{a_\pi c_\pi} \partial^\mu \phi^{\eta^\pi \zeta^\pi} A^{\nu\, \zeta^\pi c_\pi} \right) \ , \quad (2.21) \end{aligned}$$

where, for each π, $P_{IJ}^{\zeta^\pi} := \delta_{IJ} - \frac{X_I^{\zeta^\pi} X_J^{\zeta^\pi}}{Y^{\zeta^\pi \zeta^\pi}}$ projects onto the hyperplane $\mathbb{R}^7 \subset \mathbb{R}^8$ orthogonal to the 8-vector $X_I^{\zeta^\pi}$ which ζ_i^π projects the constant X_I^i onto.

We have deliberately written (2.21) in a way that is suggestive of a super

Yang–Mills description for the fields on E however, in contrast with the preceding analysis for W, the gauge structure here is not quite so manifest. To make it more transparent, let us fix a particular value of π and consider a 4-dimensional lorentzian vector space of the form $\mathbb{R}e_+ \oplus \mathbb{R}e_- \oplus E_\pi$, where the particular basis (e_+, e_-) for the two null directions obeying $\langle e_+, e_- \rangle = 1$ and $\langle e_\pm, e_\pm \rangle = 0 = \langle e_\pm, e_{a_\pi} \rangle$ can of course depend on the choice of π (we will omit the π label here though). If we take E_π to be a euclidean 2-dimensional abelian Lie algebra then we can define a lorentzian metric Lie algebra structure on $\mathbb{R}e_+ \oplus \mathbb{R}e_- \oplus E_\pi$ given by the double extension $\mathfrak{d}(E_\pi, \mathbb{R})$. The nonvanishing Lie brackets of $\mathfrak{d}(E_\pi, \mathbb{R})$ are

$$[e_+, e_{a_\pi}] = -\epsilon_{a_\pi b_\pi} e_{b_\pi} \ , \qquad [e_{a_\pi}, e_{b_\pi}] = -\epsilon_{a_\pi b_\pi} e_- \ . \tag{2.22}$$

This double extension is precisely the Nappi–Witten Lie algebra.

For each value of π we can collect the following sets of scalars and gauge fields, $\mathsf{A}^\pi := (2\, d\phi^{\eta^\pi \zeta^\pi}, 0, -2\, A^{\zeta^\pi a_\pi})$ and $\mathsf{X}_I^\pi := (X_I^{\eta^\pi}, X_I^{\zeta^\pi}, X_I^{a_\pi})$ respectively, into elements of the aforementioned vector space $\mathbb{R}e_+ \oplus \mathbb{R}e_- \oplus E_\pi$. The virtue of doing so being that if $\mathsf{D} = d + [\mathsf{A}, -]$, for each value of π, is the canonical gauge-covariant derivative with respect to each $\mathfrak{d}(E_\pi, \mathbb{R})$ then $(\mathsf{D}\mathsf{X}_I)^{a_\pi} = dX_I^{a_\pi} + 2\, \epsilon^{a_\pi b_\pi} \left(X_I^{b_\pi} d\phi^{\eta^\pi \zeta^\pi} + X_I^{\eta^\pi} A^{\zeta^\pi b_\pi} \right)$ while the associated field strength $\mathsf{F}_{\mu\nu} = [\mathsf{D}_\mu, \mathsf{D}_\nu]$ has $\mathsf{F}^{a_\pi} = -2 \left(dA^{\zeta^\pi a_\pi} + 2\, \epsilon^{a_\pi b_\pi} d\phi^{\eta^\pi \zeta^\pi} \wedge A^{\zeta^\pi b_\pi} \right)$. These are exactly the components appearing in (2.21)!

Moreover, the scalar potential $\mathscr{V}^E(X)$ can be written

$$\mathscr{V}^E(X) = -\tfrac{1}{4} \sum_{\pi=1}^{M} Y^{\zeta^\pi \zeta^\pi} P_{IK}^{\zeta^\pi} P_{JL}^{\zeta^\pi} [\mathsf{X}_I, \mathsf{X}_J]^{a_\pi} [\mathsf{X}_K, \mathsf{X}_L]^{a_\pi} \ , \tag{2.23}$$

where $[-,-]$ denotes the Lie bracket on each $\mathfrak{d}(E_\pi, \mathbb{R})$ factor.

Thus it might appear that $\mathscr{L}^E$ is going to describe a super Yang–Mills theory whose gauge algebra is $\bigoplus_{\pi=1}^M \mathfrak{d}(E_\pi, \mathbb{R})$, which indeed has a maximally isotropic centre and so is of the form noted in section 2.1.2. However, this need not be the case in general since the functions $\phi^{\eta^\pi \zeta^\pi}$ appearing in the e_+ direction of each A^π must describe the same degree of freedom for different values of π precisely when the corresponding 2-planes in $\mathbb{R}^r$ spanned by $\eta^\pi \wedge \zeta^\pi$ are linearly dependent. Consequently we must identify the (e_+, e_-) directions in all those factors $\mathfrak{d}(E_\pi, \mathbb{R})$ for which the associated $\eta^\pi \wedge \zeta^\pi$ span the same 2-plane in $\mathbb{R}^r$. It is not hard to see that, with respect to a general basis on $\bigoplus_{\pi=1}^M E_\pi$, the resulting Lie algebra $\mathfrak{k}$ must take the form $\bigoplus_{[\pi]=1}^k \mathfrak{d}(E_{[\pi]}, \mathbb{R})$ of an orthogonal direct sum over the number of independent 2-planes k spanned by $\eta^{[\pi]} \wedge \zeta^{[\pi]}$ of a set of k double extensions $\mathfrak{d}(E_{[\pi]}, \mathbb{R})$ of even-dimensional vector spaces $E_{[\pi]}$, where $\bigoplus_{\pi=1}^{M} E_\pi = \bigoplus_{[\pi]=1}^{k} E_{[\pi]}$. That is each $[\pi]$ can be thought of as encompassing an equivalence class of π values for which the corresponding 2-forms $\eta^\pi \wedge \zeta^\pi$ are all proportional to each other. The data for $\mathfrak{k}$ therefore corresponds to a set of k nondegenerate elements $J_{[\pi]} \in \mathfrak{so}(E_{[\pi]})$ where, for a given value of $[\pi]$, the relative eigenvalues of $J_{[\pi]}$ are precisely the relative proportionality constants for the linearly dependent 2-forms $\eta^\pi \wedge \zeta^\pi$ in the equivalence class. Clearly $\mathfrak{k}$ therefore has index k, dimension $2\left(k + \left[\frac{\dim W_0}{2}\right]\right)$ and admits a maximally isotropic centre.

Putting all this together, we conclude that

$$\mathscr{L}^E = \sum_{[\pi]=1}^{k} \mathscr{L}^{SYM}\left(\mathsf{A}^{[\pi]}, P_{IJ}^{\zeta^{[\pi]}} \mathsf{X}_J^{[\pi]}, \|X^{\zeta^{[\pi]}}\| \Big| \mathfrak{d}(E_{[\pi]}, \mathbb{R})\right) \ . \tag{2.24}$$

One can check from (2.14) and (2.21) that the contributions to the Bagger–Lambert Lagrangian on E coming from different E_π factors, but with π values in the same equivalence class $[\pi]$, are precisely accounted for in the expression (2.24) by the definition above of the elements $J_{[\pi]}$ defining the double extensions.

The identification above again provides quite an intricate relation between the data on E_π from Theorem 1.2.7 and the physical super Yang–Mills parameters. However, we know from section 2.1.2 that the physical content of super Yang–Mills theories whose gauge symmetry is based on a lorentzian Lie algebra corresponding to a double extension is rather more simple, being described in terms of free massive vector supermultiplets. Let us therefore apply this preceding analysis to the theory above.

The description above of the Lagrangian on each factor E_π has involved projecting degrees of freedom onto the hyperplane $\mathbb{R}^7 \subset \mathbb{R}^8$ orthogonal to $X_I^{\zeta^\pi}$. The natural analogy here of the six-dimensional subspace occupied by the massive scalar fields in section 2.1.2 is obtained by projecting onto the subspace $\mathbb{R}^6 \subset \mathbb{R}^8$ which is orthogonal to the plane in $\mathbb{R}^8$ spanned by $X^{\eta^\pi} \wedge X^{\zeta^\pi}$, i.e. the image in $\Lambda^2\mathbb{R}^8$ of the 2-form $\eta^\pi \wedge \zeta^\pi$ under the map from $\mathbb{R}^r \to \mathbb{R}^8$ provided by the vacuum expectation values X_I^i. This projection operator can be written

$$P_{IJ}^{\eta^\pi \zeta^\pi} = \delta_{IJ} - X_I^{\eta^\pi} Q_J^{\eta^\pi} - X_I^{\zeta^\pi} Q_J^{\zeta^\pi} \ , \tag{2.25}$$

where

$$Q_I^{\eta^\pi} := \frac{1}{(\Delta_{\eta^\pi\zeta^\pi})^2}\left(Y^{\zeta^\pi\zeta^\pi}X_I^{\eta^\pi} - Y^{\eta^\pi\zeta^\pi}X_I^{\zeta^\pi}\right)$$
$$Q_I^{\zeta^\pi} := \frac{1}{(\Delta_{\eta^\pi\zeta^\pi})^2}\left(Y^{\eta^\pi\eta^\pi}X_I^{\zeta^\pi} - Y^{\eta^\pi\zeta^\pi}X_I^{\eta^\pi}\right) \ , \tag{2.26}$$

and

$$(\Delta_{\eta^\pi\zeta^\pi})^2 := \|X^{\eta^\pi}\wedge X^{\zeta^\pi}\|^2 \equiv Y^{\eta^\pi\eta^\pi}Y^{\zeta^\pi\zeta^\pi} - (Y^{\eta^\pi\zeta^\pi})^2 \ . \tag{2.27}$$

The quantities defined in (2.26) are the dual elements to $X_I^{\eta^\pi}$ and $X_I^{\zeta^\pi}$ such that $Q_I^{\eta^\pi}X_I^{\eta^\pi} = 1 = Q_I^{\zeta^\pi}X_I^{\zeta^\pi}$ and $Q_I^{\eta^\pi}X_I^{\zeta^\pi} = 0 = Q_I^{\zeta^\pi}X_I^{\eta^\pi}$. The expression (2.27) identifies $\Delta_{\eta^\pi\zeta^\pi}$ with the area in $\mathbb{R}^8$ spanned by $X^{\eta^\pi}\wedge X^{\zeta^\pi}$. From these definitions, it follows that $P_{IJ}^{\eta^\pi\zeta^\pi}$ in (2.25) indeed obeys $P_{IJ}^{\eta^\pi\zeta^\pi} = P_{JI}^{\eta^\pi\zeta^\pi}$, $P_{IK}^{\eta^\pi\zeta^\pi}P_{JK}^{\eta^\pi\zeta^\pi} = P_{IJ}^{\eta^\pi\zeta^\pi}$ and $P_{IJ}^{\eta^\pi\zeta^\pi}X_J^{\eta^\pi} = 0 = P_{IJ}^{\eta^\pi\zeta^\pi}X_J^{\zeta^\pi}$.

The scalar potential (2.23) on E has a natural expression in terms of the objects defined in (2.25) and (2.27) as

$$\mathscr{V}^E(X) = -\tfrac{1}{2}\sum_{\pi=1}^{M}(\Delta_{\eta^\pi\zeta^\pi})^2\, P_{IJ}^{\eta^\pi\zeta^\pi}X_I^{a_\pi}X_J^{a_\pi} \ . \tag{2.28}$$

Furthermore, using the identity

$$P_{IJ}^{\eta^\pi\zeta^\pi} \equiv P_{IJ}^{\zeta^\pi} - \frac{(\Delta_{\eta^\pi\zeta^\pi})^2}{Y^{\zeta^\pi\zeta^\pi}}Q_I^{\eta^\pi}Q_J^{\eta^\pi} \ , \tag{2.29}$$

allows one to reexpress the remaining terms

$$-\frac{1}{2}\sum_{\pi=1}^{M}D_\mu X_I^{a_\pi}D^\mu X_I^{a_\pi} + \mathscr{L}_{\rm CS}^E \tag{2.30}$$

in (2.21) as

$$\begin{aligned}\sum_{\pi=1}^{M} -\frac{1}{2}\,P_{IJ}^{\eta^\pi\zeta^\pi}\mathcal{D}_\mu X_I^{a_\pi}\mathcal{D}^\mu X_J^{a_\pi} - \tfrac{1}{Y^{\zeta^\pi\zeta^\pi}}\left(2\,\mathcal{D}_{[\mu}A_{\nu]}^{\zeta^\pi a_\pi}\right)\left(2\,\mathcal{D}^\mu A^{\nu\,\zeta^\pi a_\pi}\right)\\ -\frac{1}{2}\sum_{\pi=1}^{M}\frac{Y^{\zeta^\pi\zeta^\pi}}{(\Delta_{\eta^\pi\zeta^\pi})^2}\left(X_I^{\eta^\pi}P_{IJ}^{\zeta^\pi}\mathcal{D}_\mu X_J^{a_\pi} + 2\,\frac{(\Delta_{\eta^\pi\zeta^\pi})^2}{Y^{\zeta^\pi\zeta^\pi}}\epsilon^{a_\pi b_\pi}A_\mu^{\zeta^\pi b_\pi}\right)\\ \times\left(X_K^{\eta^\pi}P_{KL}^{\zeta^\pi}\mathcal{D}^\mu X_L^{a_\pi} + 2\,\frac{(\Delta_{\eta^\pi\zeta^\pi})^2}{Y^{\zeta^\pi\zeta^\pi}}\epsilon^{a_\pi c_\pi}A^{\mu\,\zeta^\pi c_\pi}\right) \ ,\end{aligned} \tag{2.31}$$

where we have introduced the covariant derivative $\mathcal{D}\Phi^{a_\pi} := d\Phi^{a_\pi} + 2\,\epsilon^{a_\pi b_\pi}\,d\phi^{\eta^\pi\zeta^\pi}\wedge\Phi^{b_\pi}$ for any differential form Φ^{a_π} on $\mathbb{R}^{1,2}$ taking values in E_π. Similar to what we saw in section 2.1.2, the six projected scalars $P_{IJ}^{\eta^\pi \zeta^\pi} X_J^{a_\pi}$ in the first line of (2.31) do not couple to the gauge field $A^{\zeta^\pi a_\pi}$ on each E_π. Moreover, the remaining scalar in the second line of (2.31) can be eliminated from the Lagrangian, for each E_π, using the gauge symmetry under which $\delta A^{i a_\pi} = \mathcal{D}\Lambda^{i a_\pi}$ for any parameter $\Lambda^{i a_\pi}$ to fix $\Lambda^{\zeta^\pi a_\pi} = -\frac{1}{2} \frac{Y^{\zeta^\pi \zeta^\pi}}{(\Delta_{\eta^\pi \zeta^\pi})^2} \epsilon^{a_\pi b_\pi} X_I^{\eta^\pi} P_{IJ}^{\zeta^\pi} X_J^{b_\pi}$. There is a remaining gauge symmetry under which $\delta\phi^{\eta^\pi \zeta^\pi} = \Lambda^{\eta^\pi \zeta^\pi}$ and $\delta\Phi^{a_\pi} = -2\, \Lambda^{\eta^\pi \zeta^\pi} \epsilon^{a_\pi b_\pi} \Phi^{b_\pi}$ where the gauge parameter $\Lambda^{\eta^\pi \zeta^\pi} = \eta_i^\pi \zeta_j^\pi \Lambda^{ij}$, under which the derivative $\mathcal{D}$ transforms covariantly. This can also be fixed to set $\mathcal{D} = d$ on each E_π. Notice that one has precisely the right number of these gauge symmetries to fix all the independent projections $\phi^{\eta^\pi \zeta^\pi}$ appearing in the covariant derivatives.

After doing this one combines (2.28) and (2.31) to write

$$\begin{aligned} \mathscr{L}^E = & \sum_{\pi=1}^{M} -\frac{1}{2} P_{IJ}^{\eta^\pi \zeta^\pi} \partial_\mu X_I^{a_\pi} \partial^\mu X_J^{a_\pi} - \frac{1}{2} (\Delta_{\eta^\pi \zeta^\pi})^2 P_{IJ}^{\eta^\pi \zeta^\pi} X_I^{a_\pi} X_J^{a_\pi} \\ & + \sum_{\pi=1}^{M} -\tfrac{1}{Y^{\zeta^\pi \zeta^\pi}} (2\, \partial_{[\mu} A_{\nu]}^{\zeta^\pi a_\pi})(2\, \partial^{[\mu} A^{\nu]\, \zeta^\pi a_\pi}) - \tfrac{2}{Y^{\zeta^\pi \zeta^\pi}} (\Delta_{\eta^\pi \zeta^\pi})^2 A_\mu^{\zeta^\pi a_\pi} A^{\mu\, \zeta^\pi a_\pi} \ , \end{aligned} \tag{2.32}$$

describing precisely the bosonic part of the Lagrangian for free decoupled abelian $N = 8$ massive vector supermultiplets on each E_π, whose bosonic fields comprise the six scalars $P_{IJ}^{\eta^\pi \zeta^\pi} X_J^{a_\pi}$ and gauge field $-2 \frac{1}{\|X^{\zeta^\pi}\|} A^{\zeta^\pi a_\pi}$, all with mass $\Delta_{\eta^\pi \zeta^\pi}$ on each E_π. It is worth stressing that we have presented (2.32) as a sum over all E_π just so that the masses $\Delta_{\eta^\pi \zeta^\pi}$ on each factor can be written more explicitly. We could equally well have presented things in terms of a sum over the equivalence classes $E_{[\pi]}$, as in (2.24), whereby the relative proportionality constants for the $\Delta_{\eta^\pi \zeta^\pi}$ within a given class $[\pi]$ would be absorbed into the definition of the corresponding $J_{[\pi]}$.

The Lagrangian on a given E_π in the sum (2.32) can also be obtained from the truncation of an $N = 8$ super Yang–Mills theory with euclidean gauge algebra $\mathfrak{g}$ via the procedure described at the end of section 2.1.2. In particular, let us identify a given E_π with the Cartan subalgebra of a semisimple Lie algebra $\mathfrak{g}$ of rank two. Then we require $-\|X^{\zeta^\pi}\|^2 (\mathrm{ad}_y)^2 = (\Delta_{\eta^\pi \zeta^\pi})^2\, \mathbf{1}_2$ on E_π for some constant $y \in E_\pi^\perp \subset \mathfrak{g}$. In this case $\mathfrak{g}$ must be either $\mathfrak{su}(3)$, $\mathfrak{so}(5)$, $\mathfrak{so}(4)$ or $\mathfrak{g}_2$ and $E_\pi^\perp$ is identified with the root space of $\mathfrak{g}$ whose dimension is 6, 8, 4 or 12 respectively. A solution in this case is to take y proportional to the vector with only +1/-1 entries along the positive/negative roots of $\mathfrak{g}$. The proportionality constant here being $\frac{\Delta_{\eta^\pi \zeta^\pi}}{\sqrt{h(\mathfrak{g})}\|X^{\zeta^\pi}\|}$ where $h(\mathfrak{g})$ is the dual Coxeter number of $\mathfrak{g}$ and equals 3, 3, 2 or 4 for $\mathfrak{su}(3)$, $\mathfrak{so}(5)$, $\mathfrak{so}(4)$ or $\mathfrak{g}_2$ respectively (it is assumed that the longest root has norm-squared equal to 2 with respect to the Killing form in each case).

Recall from [29] that several of these rank two Lie algebras are thought to correspond to the gauge algebras for $N = 8$ super Yang–Mills theories whose IR superconformal fixed points are described by the Bagger–Lambert theory based on S_4 for two M2-branes on $\mathbb{R}^8/\mathbb{Z}_2$ (with Lie algebras $\mathfrak{so}(4)$, $\mathfrak{so}(5)$ and $\mathfrak{g}_2$ corresponding to Chern–Simons levels $k = 1, 2, 3$). It would interesting to understand whether there is any relation with the aforementioned truncation beyond just numerology! The general mass formulae we have obtained are somewhat reminiscent of equation (26) in [29] for the BLG model based on S_4 which describes the mass in terms of the area of the triangle formed between the location of the two M2-branes and the orbifold fixed point on $\mathbb{R}^8/\mathbb{Z}_2$. More generally, it would be interesting to understand whether there is a specific D-brane configuration for which $\mathscr{L}^E$ is the low-energy effective Lagrangian?

2.2.3 $\mathscr{L}^{E_0}$

The field A^{ia_0} appears only linearly in one term in $\mathscr{L}^0_{\text{CS}}$ and is therefore a Lagrange multiplier imposing the constraint $K_{ijka_0} A^{jk} = d\gamma_{ia_0}$, where γ_{ia_0} is a scalar field on $\mathbb{R}^{1,2}$ taking values in $\mathbb{R}^r \otimes E_0$.

Substituting this condition into the Lagrangian allows us to write

$$
\begin{aligned}
-\frac{1}{2} D_\mu X_I^{a_0} D^\mu X_I^{a_0} + \mathscr{L}^{E_0}_{\text{CS}} = &- \frac{1}{2}\partial_\mu \left(X_I^{a_0} - \gamma_i{}^{a_0} X_I^i \right) \partial^\mu \left(X_I^{a_0} - \gamma_j{}^{a_0} X_I^j \right) \\
&- \tfrac{1}{3} A^{ij} \wedge d\gamma_{ia_0} \wedge d\gamma_{ja_0} + \frac{1}{2} L_{ijkl} A^{ij} \wedge dA^{kl} \ .
\end{aligned}
\tag{2.33}
$$

The first line shows that we can simply redefine the scalars $X_I^{a_0}$ such that they decouple and do not interact with any other fields in the theory.

Notice that none of the projections $A^{\eta^\pi \zeta^\pi} = d\phi^{\eta^\pi \zeta^\pi}$ of A^{ij} that appeared in $\mathscr{L}^E$ can appear in the second line of (2.33) since the corresponding terms would be total derivatives. Consequently, our indifference to $\mathscr{L}^{E_0}$ in the gauge-fixing that was described for $\mathscr{L}^E$, resulting in (2.32), was indeed legitimate. Furthermore, there can be no components of A^{ij} along the 2-planes in $\mathbb{R}^r$ spanned by the nonanishing components of K_{ijka_0} here for the same reason.

The contribution coming from the Chern–Simons term in the second line of (2.33) is therefore completely decoupled from all the other terms in the Lagrangian. It has a rather unusual-looking residual gauge symmetry, inherited from that in the original Bagger–Lambert theory, under which $\delta\gamma_{ia_0} = \sigma_{ia_0} := K_{ia_0kl}\Lambda^{kl}$ and $L_{ijkl}\left(\delta A^{kl} - d\Lambda^{kl}\right) = \sigma_{[i}{}^{a_0} d\gamma_{j]a_0}$ for any gauge parameter Λ^{ij}. In addition to the second line of (2.33) being invariant under this gauge transformation, one can easily check that so is the tensor $L_{ijkl}dA^{kl} - d\gamma_{ia_0} \wedge d\gamma_{ja_0}$. This is perhaps not surprising since the vanishing of this tensor is precisely the field equation resulting from varying A^{ij} in the second line of (2.33). The important point though is that this gauge-invariant tensor is exact and thus the field equations resulting from the second line of (2.33) are precisely equivalent to those obtained from an abelian Chern–Simons term for the gauge field $C_{ij} := L_{ijkl}A^{kl} - \gamma_{[i}{}^{a_0} \wedge d\gamma_{j]a_0}$ (where the $[ij]$ indices do not run over any 2-planes in $\mathbb{R}^r$ which are spanned by the nonvanishing components of $\eta^{\pi}_{[i}\zeta^{\pi}_{j]}$ and K_{ijka_0}).

In summary, up to the aforementioned field redefinitions, we have found that

$$\mathscr{L}^{E_0} = -\frac{1}{2}\partial_\mu X_I^{a_0}\partial^\mu X_I^{a_0} + \frac{1}{2}M^{ijkl}C_{ij} \wedge dC_{kl} \ , \tag{2.34}$$

for some constant tensor M^{ijkl}, which can be taken to obey $M^{ijkl} = M^{[ij][kl]} = M^{klij}$, that is generically a complicated function of the components L_{ijkl} and K_{ijka_0}. Clearly this redefined abelian Chern–Simons term is only well-defined in a path integral provided the components M^{ijkl} are quantised in suitable integer units. However, since none of the dynamical fields are charged under C_{ij} then we conclude that the contribution from $\mathscr{L}^{E_0}$ is essentially trivial.

2.3 Examples

Let us end by briefly describing an application of this formalism to describe the unitary gauge theory resulting from the Bagger–Lambert theory associated with two of the admissible index-2 3-Lie algebras in the IIIb family from [19] that were detailed in section 1.3.

2.3.1 $V_{\mathbf{IIIb}}(0, 0, 0, \mathfrak{h}, \mathfrak{g}, \psi)$

The data needed for this in Theorem 1.2.7 is $\kappa|_{\mathfrak{h}} = (0, 1)^t$, $\kappa|_{\mathfrak{g}_\alpha} = (\psi_\alpha, 1)^t$. The resulting Bagger–Lambert Lagrangian will only get a contribution from $\mathscr{L}^W$ and describes a sum of separate $N = 8$ super Yang–Mills Lagrangians on $\mathfrak{h}$ and on each factor $\mathfrak{g}_\alpha$, with the respective euclidean Lie algebra structures describing the gauge symmetry. The super Yang–Mills theory on $\mathfrak{h}$ has coupling $\|X^{u_2}\|$ and the seven scalar fields occupy the hyperplane orthogonal to X^{u_2} in $\mathbb{R}^8$. Similarly, the $N = 8$ theory on a given $\mathfrak{g}_\alpha$ has coupling $\|\psi_\alpha X^{u_1} + X^{u_2}\|$ with scalars in the hyperplane orthogonal to $\psi_\alpha X^{u_1} + X^{u_2}$. This is again generically a super Yang–Mills theory though it degenerates to a maximally supersymmetric free theory for all eight scalars if there are any values of α for which $\psi_\alpha X^{u_1} + X^{u_2} = 0$.

2.3.2 $V_{\mathbf{IIIb}}(E, J, 0, \mathfrak{h}, 0, 0)$

The data needed for this in Theorem 1.2.7 is $\kappa|_{\mathfrak{h}} = (0,1)^t$ and $J^\pi = \eta^\pi \wedge \zeta^\pi$ where η^π and ζ^π are 2-vectors spanning $\mathbb{R}^2$ for each value of π and $E = \bigoplus_{\pi=1}^{M} E_\pi$. The data comprising J^π can also be understood as a special case of a general admissible index r 3-Lie algebra having all $\eta^\pi \wedge \zeta^\pi$ spanning the same 2-plane in $\mathbb{R}^r$ (when $r = 2$ this is unavoidable, of course). The resulting Bagger–Lambert Lagrangian will get one contribution from $\mathscr{L}^W$, describing precisely the same $N = 8$ super Yang–Mills theory on $\mathfrak{h}$ we saw above, and one contribution from $\mathscr{L}^E$. The latter being the simplest case of the Lagrangian (2.24) where there is just one equivalence class of 2-planes spanned by all $\eta^\pi \wedge \zeta^\pi$ and the gauge symmetry is based on the lorentzian Lie algebra $\mathfrak{d}(E, \mathbb{R})$. The physical degrees of freedom describe free abelian $N = 8$ massive vector supermultiplets on each E_π with masses $\Delta_{\eta^\pi\zeta^\pi}$ as in (2.32). Mutatis mutandis, this example is equivalent to the Bagger–Lambert theory resulting from the most general finite-dimensional 3-Lie algebra example considered in section 4.3 of [24].

Summary

In this first part of the thesis we studied admissible metric 3-Lie algebras of indefinite signature and the corresponding BLG models. In these theories, the matter fields are valued in the 3-algebra and their kinetic terms, described via the inner product, can become negative if the signature of the metric is not positive definite, thus making the model seem non-unitary. This problem was tackled i.a. by [30], where a 3-algebra with an inner product with one negative eigenvalue was used in building a BLG model. It was shown there that the matter fields with values along one of two the null directions decouple (this is clear from the 3-algebra structure constants), while the fields along the other are forced to constant values by the equations of motion. If one then expands around the maximally supersymmetric gauge-invariant vacuum defined by this constant field, one obtains a unitary theory: an $N = 8$ super Yang-Mills theory, with coupling constants defined by the norm of the constant field. This procedure can be made more rigorous using BRST quantisation, gauging the constant-shift symmetry of the decoupling fields (i.a. [31]). In any case, the result is a maximally supersymmetric Yang-Mills theory based on a euclidean gauge algebra, having broken scale-invariance with respect to the BLG model based on the 3-algebra.

In these first two chapters we generalised this discussion to metric 3-Lie algebras with generic indefinite signature: in chapter 1 we first characterised physically admissible finite-dimensional 3-Lie algebras, i.e. those that are indecomposable and have a maximally isotropic centre. For these, the fields along the null directions can effectively decouple from the theory. We show how the vector space on which the 3-algebra is defined factorises into the part spanned by the null vectors times different types of euclidean sectors. We then use this characterisation, in chapter 2 to calculate the Lagrangian for the general associated BLG model. This was found to factorise into a sum of decoupled, maximally supersymmetric gauge theories on the different euclidean components of V.

3-algebras and even n-algebras still constitute an extremely active research direction, as more applications to M-theoretic computations are constantly being discovered.

Going back to the original motivation that lead to the discovery of 3-algebraic gauge theories, let us also note at this point that significant progress has been made in the search for model describing stacks of coincident M2 branes. The restrictions on positive definite signature 3-Lie algebras used in BLG models result in such a model being able to describe the world-volume theory at best of 2 M2 branes. Indeed, it is shown in [32] that BLG theory with the S_4 3-Lie algebra can be rewritten as a "traditional" Chern-Simons theory with Lie algebra $SU(2)\times SU(2)$, coupled to matter fields valued in the bifundamental representation and with two Chern-Simons terms of opposite sign preserving parity invariance. The moduli space of such a theory does not coincide with what is expected of a model describing a stack of M2 branes. Furthermore, not all the superconformal primary operators predicted by the AdS/CFT correspondence can be constructed from the field content of the BLG theory. Consequently, theories based on 3-Lie algebras could at best describe two coincident branes, the special case in which both these issues could be resolved.

The key idea in [32], however, inspired the work of [33], the much acclaimed ABJM theory. Here the $SU(2)\times SU(2)$ quiver proposed by Van Raamsdonk is generalised to a $U(N)\times U(N)$ and $SU(N)\times SU(N)$ structures, allowing explicit $\mathcal{N}=6$ superconformal symmetry. Again one writes $d=3$ Chern-Simons theories at levels k and $-k$, coupled to bifundamental matter - the theory is expected to describes the low energy limit of a stack of N M2 branes probing a $\mathbb{C}^4/\mathbb{Z}_k$ singularity. The large N limit would then be dual to M-theory on an $\mathrm{AdS}_4\times S^7/\mathbb{Z}_k$.

In the special case of $N=2$ in the $SU(N)\times SU(N)$ theory, one recovers precisely the BLG theory, as the superconformal symmetry is enhanced to $\mathcal{N}=8$. This type of construction obviously sheds some serious doubt on the necessity of introducing triple structures in the first place, since the same and more general results can be obtained using direct product Lie algebras. For general N, ABJM theory at levels $k=1$, 2 is still expected to have enhanced $\mathcal{N}=8$ supersymmetry, so that it describes a stack of M2 branes probing flat space and $\mathbb{R}^8/\mathbb{Z}_2$ respectively. Needless to say, a lot of literature focuses on examining the theory for generic k for different amounts of conserved supersymmetry.

More recently the more general case of ABJ theories has also been developed beyond the classical level. Such theories are based on Lie algebras of the type

$U(N)\times U(M)$, contain the usual CS factors at levels k and $-k$ and classically have explicit $\mathcal{N}=6$ superconformal symmetry. It is argued in [34] that for $k=\pm 2$ and $M=N+1$ a hidden $\mathcal{N}=8$ supersymmetry emerges at the quantum level[1]. ABJ theories for these special values of N, M and k have the same moduli space as $U(N)_k \times U(N)_{-k}$ ABJM theories, but are not isomorphic to them. BLG theories for low values of k are found to be isomorphic, at the quantum level, to one or the other such theories at $N=2$.

Finally let us mention one of the most important results following from the ABJM proposal for a world-sheet theory of coincident M2 branes. We recalled in the introduction that in the near horizon limit the dual 3-dimensional CFT has $N^{\frac{3}{2}}$ degrees of freedom. Such a result seemed for a long time very difficult to reproduce via any of the proposed models. The key element that finally allowed a corresponding calculation of this quantity in ABJM theory, was the realisation that the partition function and Wilson loop observables of the theory could be encoded via a zero dimensional super-matrix model (see e.g. [35]). Without going into the detail of such a model, let us just observe that it allowed for the calculation, in [36], of the planar free energy, matching at strong coupling that of classical supergravity action on $\mathrm{AdS}_4 \times \mathbb{CP}^3$. Furthermore it reproduces the correct $N^{\frac{3}{2}}$ scaling for the number of degrees of freedom of the M2 brane theory.

The successes and all the activity around ABJM theories over the last few years have made this a very fertile research ground, not to mention a more and more probable candidate for the M2 brane description. The techniques that have been developed around this topic are furthermore leading to interesting mathematical generalisations that will bring a clearer understanding of these theories. It is not excluded that a more transparent connection to BLG models and 3-algebras (including the more generic triple structures, beyond 3-Lie algebras) will be established.

[1]A hidden parity invariance is also argued to emerge at the quantum level, consistently with what is expected of $\mathcal{N}=8$ theories.

And now for something completely different!

Monty Python

Part II

3-dimensional Gravity

Introduction

The second part of this work is set in the context of topologically massive gravity (TMG). This is a cosmological gravity theory in 3 dimensions, containing an Einstein-Hilbert (EH) term with negative cosmological constant $-\frac{2}{\ell^2}$, but also a gravitational Chern-Simons term. Recall that pure Einstein gravity in 3 dimensions is “trivial”, in the sense that it does not contain any dynamic degrees of freedom in its linear expansion and its solutions are described solely by their global properties [37, 38]. It is therefore an extremely simple setting in which to analyse gravity, but of limited use in the quest for quantisation. The same simplicity, however, together with the hope for a better UV behaviour of perturbative theories in three dimensions, motivates us to look at generlisations of pure 3-dimensional gravity including terms of higher order in the derivatives of the metric. In fact, it turns out that there are two possibilities for such a generalisation that do yield a physical propagating field in the linearised theory: the addition of a gravitational Chern-Simons term, as in TMG (see e.g. [39, 40, 41, 42]); the inclusion of specific higher order derivative terms, going like $R_{\mu\nu}R^{\mu\nu}$ and R^2, giving rise to what is known as new massive gravity [43]. Both cases have drawbacks. In particular, since we are here focusing on TMG, let us point out that it is a parity violating model, and, even though it does feature a propagating graviton of helicity ± 2, the linearised theory is unitary only for the “wrong” sign of the EH term. Indeed, using the standard sign on the EH term causes non-unitary propagation of the spin 2 modes, implying non-unitarity of the boundary CFT [44]. On the other hand, the opposite sign implies negative mass for black holes in the bulk, which translates to a negative central charge for the boundary CFT. Nonetheless, the simplicity of the theory makes it a very useful toy model within which to explore quantum gravity, not least since it poses considerable interest in the context of holography, as we will elucidate in the next section.

In what follows, we will first motivate our interest in space-like warped backgrounds in TMG in a short introductory section, to then proceed into a detailed analysis of the locally warped AdS_3 solutions to the theory. That is, we will review warped AdS_3 and those quotients of it that result in causal singularities protected by horizons. Our aim in this first part is to present a very clear, geometrically obvious construction, that will clarify the relation of the emergence of closed time-like curves to particular choices of coordinates for locally warped AdS_3 spaces. We will proceed to studying these solutions in depth and present a detailed analysis of the near horizon limits that can be defined for warped AdS_3 black holes. Here we chose to emphasize how to take such limits from the quotient construction directly, leading us back to the local parametrizations introduced earlier. The whole process gives a systematic, geometrically motivated understanding of warped AdS in TMG.

Overall, this work was motivated by our ambition to understand holography for such backgrounds. Keeping this hope in mind, we propose in the final chapter a conformal approach to solving TMG via a Kaluza-Klein (KK) reduction. Again, the aim being some insight into holography in this context, we hoped to find solutions that interpolate between AdS and warped AdS space. Understanding such a background could lead to some insight as to what should be expected as a dual boundary gauge theory for the warped case. Our approach was found to require too much symmetry to yield such novel solutions, but it does reproduce most known stationary axisymmetric TMG backgrounds.

Chapters 3 and 4 therefore follow precisely [45]and [46][2] respectively.

[2]Currently under review for publication in “General Relativity and Gravitation”.

Let's do the space-warp again!

Chapter 3

TMG and warped AdS_3

3.1 Why warp?

The action of TMG contains an Einstein-Hilbert term with negative cosmological constant $-1/\ell^2$ plus a gravitational Chern-Simons term

$$16\pi G\, S[g] = \int d^3x\sqrt{-g}\left[\left(R+\frac{2}{\ell^2}\right)+\frac{\ell}{6\nu}\epsilon^{\lambda\mu\nu}\Gamma^r_{\lambda\sigma}\left(\partial_\mu\Gamma^\sigma_{r\nu}+\frac{2}{3}\Gamma^\sigma_{\mu\tau}\Gamma^\tau_{\nu\rho}\right)\right] .$$

In three dimensions, the gravitational constant G has dimension of length and ν is a dimensionless positive constant that we shall take $\nu > 1$. The equations of motion are

$$R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}-\frac{1}{\ell^2}g_{\mu\nu} = +\frac{\ell}{3\nu}\epsilon_\mu{}^{\rho\sigma}(R_{\nu\rho}-\frac{1}{4}g_{\nu\rho}R)_{;\sigma} \equiv -C_{\mu\nu} ,$$

where the Cotton tensor $C_{\mu\nu}$ is a measure of conformal flatness. A solution of TMG is given by a metric along with a preferred orientation of the Levi-Civita tensor $\epsilon_{\mu\nu\rho}$. It is clear from the above that AdS_3 space is always a solution of this theory, since it solves both sides of the equations of motion being set to zero separately.

The theory is worth some attention since its solution space is more relevant to four-dimensional physics than what one might expect from such a simplification. The near-horizon geometry of the extremal Kerr black hole [47], at fixed polar angle, is a particular solution of TMG, the self-dual warped AdS_3 space in Poincaré coordinates (see also [48]). The geometry of warped AdS_3 therefore plays a pivotal role in TMG.

The last couple of years have seen a flurry of activity in TMG, due to the conjecture that the black hole solutions obtained by quotients of spacelike warped AdS_3 are dual to a CFT with separate left and right central charges [49]. More recently, real-time correlators were obtained for the self-dual geometry in accelerating coordinates that were chiral [50]. This motivates us to take a tour in the quotient construction and obtain the self-dual geometry as a spacetime limit of the black hole quotients.

The next two sections can be read as a review of spacelike warped AdS_3 and the black hole quotients. In section 3.2 we describe warped AdS_3 as the universal cover of $SL(2,\mathbb{R})$ equipped with a "non-round" metric. We give three coordinate systems that will be of use: the (global) warped AdS_3 coordinates, accelerating coordinates and Poincaré-like coordinates. In section 4.1 we present the black hole quotient construction following [49] paying particular attention to the case when causal singularities do exist behind the Killing horizons. As customary, it is for this case that we shall call the quotient a 3d black hole [38]. We explicitly write a corresponding inequality on the ADT mass and angular momentum for the black hole quotients in two commonly used conventions, those of [49] and [51].

We find that the phase space is such that the ratio of left to right temperature T_L/T_R has a lower bound, and that there is a critical value of the ratio when the inner horizon coincides with the causal singularity. In section 4.2 we accordingly find that the causal diagrams fall into three different classes. These are similar to those of the non-extremal charged Reissner-Nordström 4d black hole (RN) for a generic ratio T_L/T_R, the extremal RN when $T_R = 0$, and the uncharged RN when the ratio is at its critical value.

In the last section we describe the various spacetime limits that one can take in the black hole phase space. We describe the regular[1] extremal limit, the near-horizon limit of the extremal black holes, a near-extremal limit $T_R \to 0$ for the non-extremal black holes, and the limit when both temperatures T_R and T_L go to zero while keeping the Hawking temperature fixed. The extremal and near-extremal limits give the self-dual warped AdS_3 geometry in coordinates that respect the nature of the horizon. The limit when both temperatures go to zero while keeping the Hawking temperature fixed gives the vacuum solution and is universal for all ratios T_L/T_R.

[1] regular in the sense of a continuous limit in the ADM metric form.

3.2 Spacelike warped AdS_3

In this section we review the geometry of spacelike warped AdS_3. This will prepare us for a clear understanding of the quotient construction in section 4.1. We describe the metric in warped, accelerating and Poincaré coordinates. In summary, the metric will be written in the form

$$g_{\ell,\nu} = \frac{\ell^2}{\nu^2+3}\left(-f(x)d\tau^2 + \frac{dx^2}{f(x)} + \frac{4\nu^2}{\nu^2+3}(du + x d\tau)^2\right), \tag{3.1}$$

where

$$f(x) = \begin{cases} x^2+1 & \text{for warped coordinates,} \\ x^2-1 & \text{for accelerating coordinates,} \\ x^2 & \text{for Poincaré coordinates.} \end{cases}$$

The metric (3.1) satisfies the TMG equation of motion with $\epsilon_{\tau x u} = +\sqrt{-g}$. We will use the same labels (τ, x, u) for accelerating and Poincaré coordinates, hoping this will not cause confusion. For the warped coordinates we will use instead the coordinate labels $(\tilde{t}, \sigma, \tilde{u})$, where we replace $x \to \sinh\sigma$, $u \to \tilde{u}$ and $\tau \to \tilde{t}$.

3.2.1 Warped coordinates

Let us start by expressing AdS_3 as the universal cover of the special linear group $\mathrm{SL}(2,\mathbb{R})$:

$$\mathrm{SL}(2,\mathbb{R}) = \left\{ \begin{pmatrix} T_1 + X_1 & T_2 + X_2 \\ X_2 - T_2 & T_1 - X_1 \end{pmatrix} : \quad T_1^2 + T_2^2 - X_1^2 - X_2^2 = 1 \right\} .$$

As a group, $\mathrm{SL}(2,\mathbb{R})$ acts on the left and right on the group manifold. We write the action as $\mathrm{SL}(2,\mathbb{R})_L \times \mathrm{SL}(2,\mathbb{R})_R$. We choose a basis of the right- and left-invariant vector fields, respectively, l_a and r_a:

$$\begin{aligned}
l_1\,(r_2) &= \frac{1}{2}\left(-X_2\frac{\partial}{\partial T_1} - T_1\frac{\partial}{\partial X_2} \pm T_2\frac{\partial}{\partial X_1} \pm X_1\frac{\partial}{\partial T_2}\right) \\
&= \frac{1}{2}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \in \mathfrak{sl}(2,\mathbb{R})_{L(R)}\ ; \\
l_0\,(r_0) &= \frac{1}{2}\left(-T_1\frac{\partial}{\partial T_2} + T_2\frac{\partial}{\partial T_1} \pm X_1\frac{\partial}{\partial X_2} \mp X_2\frac{\partial}{\partial X_1}\right) \\
&= \frac{1}{2}\begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix} \in \mathfrak{sl}(2,\mathbb{R})_{L(R)}\ ; \\
l_2\,(r_1) &= \frac{1}{2}\left(-X_1\frac{\partial}{\partial T_1} - T_1\frac{\partial}{\partial X_1} \mp X_2\frac{\partial}{\partial T_2} \mp T_2\frac{\partial}{\partial X_2}\right) \\
&= \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix} \in \mathfrak{sl}(2,\mathbb{R})_{L(R)}\ .
\end{aligned}$$

The non-zero commutators of the generators are $[l_a, l_b] = \epsilon_{ab}{}^c l_c$ and $[r_a, r_b] = \epsilon_{ab}{}^c r_c$, where the indices $a = 0, 1, 2$ are raised with a mostly-plus Lorentzian signature metric and $\epsilon_{012} = +1$. We associate to the bases l_a and r_a the dual left- and right-invariant one forms θ^a and $\bar{\theta}^a$, so that $\theta^a(l_b) = \delta^a_b$ and $\bar{\theta}^a(r_b) = \delta^a_b$. The Lie derivative therefore acts as $\mathscr{L}_{l_a}\theta^b = \epsilon_a{}^b{}_c\theta^c$ and $\mathscr{L}_{r_a}\bar{\theta}^b = \epsilon_a{}^b{}_c\bar{\theta}^c$. The left-invariant one-forms allow us to write metrics on $\mathrm{SL}(2,\mathbb{R})$ with symmetry of rank 3,4 and 6.

The Killing form, or "round" metric, is simply

$$g_\ell = \frac{\ell^2}{4}\left(-\theta^0\otimes\theta^0 + \theta^1\otimes\theta^1 + \theta^2\otimes\theta^2\right)\ .$$

Let us introduce the parametrization

$$\begin{aligned}
T_1 &= \cosh\tfrac{\sigma}{2}\cosh\tfrac{\tilde{u}}{2}\cos\tfrac{\tilde{t}}{2} + \sinh\tfrac{\sigma}{2}\sinh\tfrac{\tilde{u}}{2}\sin\tfrac{\tilde{t}}{2}, \\
T_2 &= \cosh\tfrac{\sigma}{2}\cosh\tfrac{\tilde{u}}{2}\sin\tfrac{\tilde{t}}{2} - \sinh\tfrac{\sigma}{2}\sinh\tfrac{\tilde{u}}{2}\cos\tfrac{\tilde{t}}{2}, \\
X_1 &= \cosh\tfrac{\sigma}{2}\sinh\tfrac{\tilde{u}}{2}\cos\tfrac{\tilde{t}}{2} + \sinh\tfrac{\sigma}{2}\cosh\tfrac{\tilde{u}}{2}\sin\tfrac{\tilde{t}}{2}, \\
X_2 &= \cosh\tfrac{\sigma}{2}\sinh\tfrac{\tilde{u}}{2}\sin\tfrac{\tilde{t}}{2} - \sinh\tfrac{\sigma}{2}\cosh\tfrac{\tilde{u}}{2}\cos\tfrac{\tilde{t}}{2},
\end{aligned} \tag{3.2}$$

which was shown in [52] to cover the whole of $\mathrm{SL}(2,\mathbb{R})$ with $\tilde{u}, \sigma \in \mathbb{R}$ and $\tilde{t} \sim$

$\tilde{t} + 4\pi$. These are the hyperbolic asymmetric coordinates of [53]. We use the conventions in (3.2), so that with the above parametrization the θ^a are

$$\theta^0 = -d\tilde{t}\cosh\tilde{u}\cosh\sigma + d\sigma\sinh\tilde{u}, \tag{3.3}$$

$$\theta^1 = -d\sigma\cosh\tilde{u} + d\tilde{t}\cosh\sigma\sinh\tilde{u}, \tag{3.4}$$

$$\theta^2 = d\tilde{u} + d\tilde{t}\sinh\sigma\ , \tag{3.5}$$

the left-invariant vectors are

$$r_0 = -\partial_{\tilde{t}}\ , \tag{3.6}$$

$$r_1 = \sin\tilde{t}\,\partial_\sigma + \cos\tilde{t}\tanh\sigma\,\partial_{\tilde{t}} + \cos\tilde{t}\,\mathrm{sech}\,\sigma\,\partial_{\tilde{u}}\ , \tag{3.7}$$

$$r_2 = -\cos\tilde{t}\,\partial_\sigma + \sin\tilde{t}\tanh\sigma\,\partial_{\tilde{t}} + \mathrm{sech}\,\sigma\sin\tilde{t}\,\partial_{\tilde{u}} \tag{3.8}$$

and the right-invariant vectors are

$$l_0 = -\sinh\tilde{u}\,\partial_\sigma - \cosh\tilde{u}\,\mathrm{sech}\,\sigma\,\partial_{\tilde{t}} + \cosh\tilde{u}\tanh\sigma\,\partial_{\tilde{u}}\ , \tag{3.9}$$

$$l_1 = -\cosh\tilde{u}\,\partial_\sigma - \mathrm{sech}\,\sigma\sinh\tilde{u}\,\partial_{\tilde{t}} + \sinh\tilde{u}\tanh\sigma\,\partial_{\tilde{u}}\ , \tag{3.10}$$

$$l_2 = \partial_{\tilde{u}}\ . \tag{3.11}$$

The round metric becomes

$$g_\ell = \frac{\ell^2}{4}\left[-\cosh^2\sigma d\tilde{t}^2 + d\sigma^2 + (d\tilde{u} + \sinh\sigma d\tilde{t})^2\right]\ . \tag{3.12}$$

The isometry group of $\mathrm{SL}(2,\mathbb{R})$ when endowed with the round metric is $\mathrm{SO}(2,2) = (\mathrm{SL}(2,\mathbb{R})_L \times \mathrm{SL}(2,\mathbb{R})_R)/\mathbb{Z}_2$, where we take into account that $-\mathbb{1}$ acts similarly on each side. Unwrapping $\tilde{t} \in \mathbb{R}$ gives the AdS_3 metric in warped coordinates [52], as a hyperbolic fibration over AdS_2. The isometry group becomes a diagonal universal cover of $(\mathrm{SL}(2,\mathbb{R})_L \times \mathrm{SL}(2,\mathbb{R})_R)/\mathbb{Z}_2$.

Keeping the time identification $\tilde{t} \sim \tilde{t} + 4\pi$, one observes in (3.12) that it covers twice a quadric base space. This is because the isometry generated by l_2 defines a non-trivial real-line fibration of $\mathrm{SL}(2,\mathbb{R})$ over the quadric

$$\tilde{T}_1^2 + \tilde{T}_2^2 - \tilde{X}^2 = 1. \tag{3.13}$$

Explicitly, the coordinates defined by

$$\begin{aligned}\tilde{T}_1 - \tilde{X} &= 2(X_1 + T_1)(X_2 - T_2),\\ \tilde{T}_1 + \tilde{X} &= 2(X_1 - T_1)(X_2 + T_2),\\ \tilde{T}_2 &= 2(T_2^2 - X_2^2) - 1\end{aligned} \tag{3.14}$$

are invariant under l_2 and satisfy (3.13). For every point $(\tilde{T}_1, \tilde{T}_2, \tilde{X}_1)$ that satisfies the quadric (3.13), there are two different orbits in $\mathrm{SL}(2,\mathbb{R})$ compatible with (3.14). Indeed, (3.14) can be solved depending on the value of $\tilde{T}_2$: if $\tilde{T}_2 < -1$ the solutions will cross $T_2 = 0$ and the two orbits are distinguished by the sign of X_2; similarly, if $\tilde{T}_2 > -1$ the same happens, but with T_2 and X_2 exchanged; if $\tilde{T}_2 = -1$ the two orbits are given by $T_2 = \pm X_2$. One can easily check that the action of the vector field r_0 induces on the quadric base space a rotation of period 2π:

$$\begin{aligned}\mathscr{L}_{r_0}(\tilde{T}_2) &= \tilde{T}_1\\ \mathscr{L}_{r_0}(\tilde{T}_1) &= -\tilde{T}_2\\ \mathscr{L}_{r_0}(\tilde{X}) &= 0,\end{aligned}$$

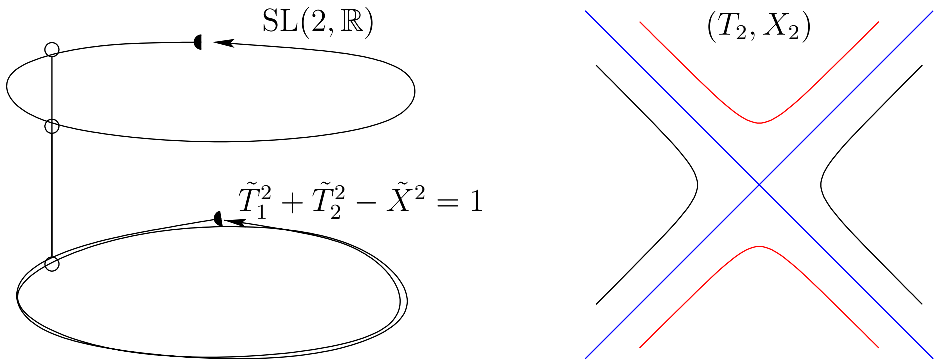
while, from (3.2), it has period 4π in $\mathrm{SL}(2,\mathbb{R})$. The double cover is depicted in figures 3.1(a) and 3.1(b). Note that this is slightly different from the Hopf fibration of the three-sphere, which covers the two-sphere once. If two complex numbers z_1, z_2 are used to describe the three-sphere as $|z_1|^2 + |z_2|^2 = 1$, then the projection is $\pi(z_1, z_2) = (2z_1 z_2^*, |z_1|^2 - |z_2|^2) \in S^2$. For every point in S^2 there is precisely one orbit given by $(z_1, z_2) \mapsto (e^{i\theta} z_1, e^{i\theta} z_2)$.

Along these lines we approach the spacelike warped metric

$$g_{\ell,\nu} = \frac{\ell^2}{\nu^2+3}\left(-\theta^0 \otimes \theta^0 + \theta^1 \otimes \theta^1 + \frac{4\nu^2}{\nu^2+3}\theta^2 \otimes \theta^2\right) , \tag{3.15}$$

so that for $\nu > 1$ or $\nu < 1$ we have a respectively stretching or squashing of the fiber in the direction of l_2 [54, 53, 55]. The isometry group is broken to that generated by the l_2 and the r_a. In the warped coordinates $(\tilde{t}, \sigma, \tilde{u})$, the warped metric is

$$g_{\ell,\nu} = \frac{\ell^2}{\nu^2+3}\left(-\cosh^2\sigma\, d\tilde{t}^2 + d\sigma^2 + \frac{4\nu^2}{\nu^2+3}\left(d\tilde{u} + \sinh\sigma\, d\tilde{t}\right)^2\right) , \tag{3.16}$$



(a) The flow of r_0 covers twice the quadric.

(b) The fiber is two distinct orbits of l_2.

Figure 3.1: Hyperbolic fibration

where the coordinate $\tilde{t}$ covers the quadric base space twice. As before, we unwrap the time coordinate to run over $\tilde{t} \in \mathbb{R}$. This is the warped AdS_3 geometry in the global warped coordinates, which was given in (3.1) for $f(x) = x^2 + 1$. The isometry group is the universal cover $\widetilde{\mathrm{SL}(2,\mathbb{R})} \times \mathbb{R}$.

If we compactify spacelike warped AdS_3 along l_2, that is $\tilde{u} \sim \tilde{u} + 2\pi\alpha$, we obtain the so-called self-dual solution of TMG. In warped coordinates, the metric is

$$g_{\ell,\nu} = \frac{\ell^2}{\nu^2+3}\left(-\cosh^2\sigma\, d\tilde{t}^2 + d\sigma^2 + \frac{4\nu^2}{\nu^2+3}\left(\alpha\, d\tilde{\phi} + \sinh\sigma\, d\tilde{t}\right)^2\right) ,$$

with $\tilde{t}, \sigma \in \mathbb{R}$ and $\tilde{\phi} \sim \tilde{\phi} + 2\pi$. The isometry group of the self-dual geometry becomes $\widetilde{\mathrm{SL}(2,\mathbb{R})} \times U(1)$.

3.2.2 Accelerating coordinates

Let us ask how we would write the warped AdS_3 metric in other coordinate systems (τ, x, u) where ∂_τ is a linear combination of the r_a and l_2. Since l_2 acts freely we can choose u to be such that $\partial_u = l_2$. The metric would have as a manifest symmetry the translations in τ and u. We still need to make an appropriate choice for the coordinate x, which should be invariant under ∂_τ and ∂_u. That is, we require $\partial_\tau x = \partial_u x = 0$. We choose $x = \frac{(\nu^2+3)^2}{4\nu^2\ell^2} g_{\ell,\nu}(\partial_u, \partial_\tau)$, which is indeed invariant because ∂_u and ∂_τ are commuting Killing vectors. The coordinate system (τ, x, u) is thus described by the surfaces (u, τ) generated by the flows of two Killing vectors, and a coordinate x which smoothly labels them.

Under an $\mathrm{SL}(2,\mathbb{R})_R$ rotation on the r_a and an $\mathrm{GL}(2,\mathbb{R})$ transformation on

(u, τ) we can bring ∂_τ to one of the following forms: r_0, $-r_2$, or $r_0 \pm r_2$. We always keep $\partial_u = l_2$ as before. The timelike case $\partial_\tau = -r_0$ corresponds to the warped coordinates, see (3.6) and (3.11). In this subsection we consider the second, spacelike case, $\partial_\tau = r_2$, and in the next subsection we will consider the null case. Here, we thus have a set of coordinates defined by the action of the Killing vectors r_2 and l_2 and their metric product. Using (3.16) and the present data, we can write the metric

$$g_{\ell,\nu} = \frac{\ell^2}{\nu^2+3}\left(-(x^2-1)d\tau^2 + \frac{dx^2}{x^2-1} + \frac{4\nu^2}{\nu^2+3}(du + x\, d\tau)^2\right) , \qquad (3.17)$$

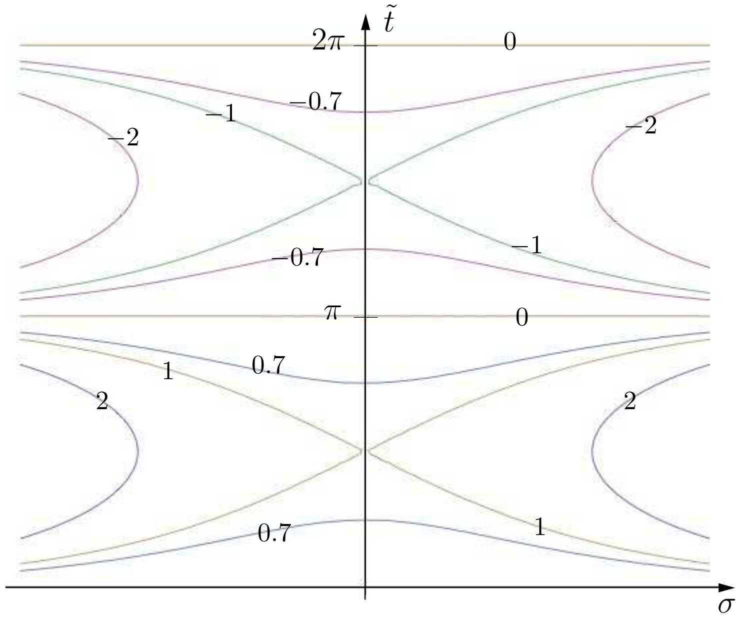
where we fixed dx to be orthogonal to the (u, τ) hypersurfaces. This is precisely the metric (3.1), with $f(x) = x^2 - 1$. The self-dual solution in accelerating coordinates is obtained by replacing $u = \alpha\phi$ in (3.17), with $\phi \sim \phi + 2\pi$.

We call this set of coordinates "accelerating" as they have a lot in common with those of the Rindler spacetime. Accelerating coordinates are those of observers with proper velocity $v = \frac{\partial_\tau}{|\partial_\tau|}$, whose acceleration $\nabla_v v$ is position dependent. In contrast to Rindler coordinates though, where ∂_τ is a Lorentz boost in Minkowski space, here ∂_τ is never timelike with respect to the metric (recall that it is taken to be r_2). Nevertheless, note how the $\tau =$const. surfaces are spacelike. As expected for metrics expressed in Rindler-like coordinates, there are apparent Killing horizons appearing at $x = \pm 1$. Here the flow of r_2 takes us to a line where r_2 becomes collinear to l_2. Thus this coordinate system is valid only away from the Killing horizon. The warped AdS spacetime has an infinite number of such regions. The figure in 3.2(a) gives a visualisation of the situation[2]. The value of the level x tells us where we are with respect to the Killing horizons in each region, for each of which there is an appropriate isometric embedding of (τ, x, u) in warped AdS_3.

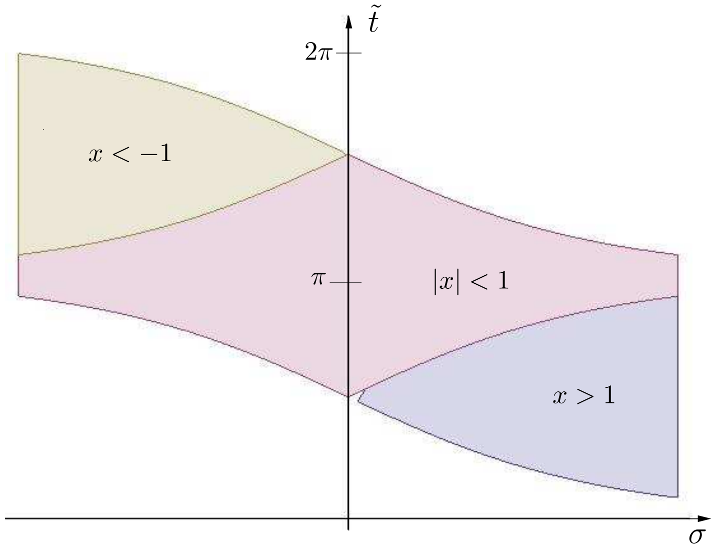
Let us present an explicit embedding as in figure 3.2(b). The region $x > 1$ with metric (3.17) isometrically embeds in warped AdS under

$$\begin{aligned} \sinh\sigma &= \sqrt{x^2-1}\cosh\tau \\ \cot\tilde{t} &= -\frac{\sqrt{x^2-1}}{x}\sinh\tau \\ \tilde{u} &= u + \tanh^{-1}\left(\frac{\tanh\tau}{x}\right) . \end{aligned} \qquad (3.18)$$

[2]in the figure we take $\tilde{u} =$const., which is possible because $\tilde{u}$ is defined globally.



(a) Integral curves of r_2 in warped AdS_3



(b) the regions of warped AdS_3 covered by (3.18) and (3.19)

Figure 3.2: The $(\sigma, \tilde{t})$ plane of warped AdS_3 at fixed $\tilde{u}$. Each line is the flow of ∂_τ and the level numbers are $x = \cosh\sigma \sin\tilde{t}$. At $\sigma = 0$, $\tilde{t} = \frac{\pi}{2} \mod \pi$ we have a fixed point $r_2 = 0$.

This covers $\tilde{u} \in \mathbb{R}$, $\sigma > 0$, and $\tilde{t} \in (0, \pi)$ with $\cosh\sigma \sin\tilde{t} > 1$. The inverse of (3.18) is

$$\begin{aligned} x &= \cosh\sigma \sin\tilde{t} \\ \tanh\tau &= -\coth\sigma\cos\tilde{t} \\ u &= \tilde{u} + \tanh^{-1}\frac{\cot\tilde{t}}{\sinh\sigma} \ , \end{aligned}$$

which is well-defined for $\sigma > 0$, $\tilde{t} \in (0, \pi)$ and

$$\left|\frac{\cot\tilde{t}}{\sinh\sigma}\right| < 1 \Leftrightarrow |\cosh\sigma\sin\tilde{t}| > 1 \ .$$

Similarly the region $|x| < 1$ can be embedded with

$$\begin{aligned} \sinh\sigma &= \sqrt{1-x^2}\sinh\tau \\ \tan\tilde{t} &= -\frac{x}{\sqrt{1-x^2}}\frac{1}{\cosh\tau} \\ \tilde{u} &= u + \tanh^{-1}(x\tanh\tau) \ , \end{aligned} \tag{3.19}$$

whose inverse is

$$\begin{aligned} x &= \cosh\sigma\sin\tilde{t} \\ \tanh\tau &= -\frac{\tanh\sigma}{\cos\tilde{t}} \\ u &= \tilde{u} + \tanh^{-1}(\sinh\sigma\tan\tilde{t}) \ . \end{aligned}$$

Here we cover $\sigma \in \mathbb{R}$, $\tilde{u} \in \mathbb{R}$ and

$$\left|\sinh\sigma\tan\tilde{t}\right| < 1 \Leftrightarrow |\cosh\sigma\sin\tilde{t}| < 1 \ .$$

3.2.3 Poincaré-like coordinates

We can go through the same construction as above, but this time choosing $\partial_\tau = -r_0 + r_2$. We define as before $\partial_u = l_2$ and $x = \frac{(\nu^2+3)^2}{4\nu^2\ell^2} g_{\ell,\nu}(\partial_u, \partial_\tau)$. We also use the freedom to make x hypersurface orthogonal. The metric is

$$g_{\ell,\nu} = \frac{\ell^2}{\nu^2+3}\left(-x^2 d\tau^2 + \frac{dx^2}{x^2} + \frac{4\nu^2}{\nu^2+3}(du + x\,d\tau)^2\right), \tag{3.20}$$

in what have been called Poincaré coordinates of warped AdS for obvious reasons. This is the metric in (3.1) with $f(x) = x^2$.

The case $\partial_\tau = r_0 + r_2$ is similar to the above, simply by the warped AdS

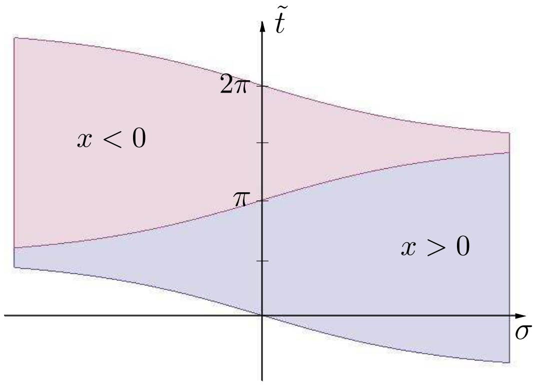


Figure 3.3: Isometric embedding in Poincaré coordinates.

discreet symmetry $(\tilde{t}, \tilde{u}) \mapsto (-\tilde{t}, -\tilde{u})$ that flips the sign of r_0 while preserving that of r_2. Note how rescaling $x \mapsto e^{\zeta} x$ and $\tau \mapsto e^{-\zeta}\tau$ is an isometry. This is because it is the action of $e^{\zeta r_1}$, and we can show that $r_1 = x\partial_x - \tau\partial_\tau$ by noting the following:

$$r_1(x) = \frac{(\nu^2+3)^2}{4\nu^2\ell^2}\mathscr{L}_{r_1}(g_{\ell,\nu}(\partial_u, \partial_\tau)) = \frac{(\nu^2+3)^2}{4\nu^2\ell^2} g_{\ell,\nu}(\partial_u, [r_1, \partial_\tau]) = x \; ; \tag{3.21}$$

$$\begin{aligned}
[r_1, \partial_\tau] = \partial_\tau \Rightarrow \partial_\tau(r_1(\tau)) = -1 \text{ and } \partial_\tau(r_1(u)) = 0 \; , \\
[r_1, \partial_u] = 0 \Rightarrow \partial_u(r_1(u)) = 0 \text{ and } \partial_u(r_1(\tau)) = 0 \; .
\end{aligned}$$

In (3.21) we used that r_1 is Killing and we have also used the commutation relations. Compactifying along l_2, that is $u \sim u + 2\pi\alpha$, results in the self-dual solution in Poincaré coordinates

$$g_{\ell,\nu} = \frac{\ell^2}{\nu^2+3}\left(-x^2 d\tau^2 + \frac{dx^2}{x^2} + \frac{4\nu^2}{\nu^2+3}\left(\alpha\, d\phi + x\, d\tau\right)^2\right) .$$

An explicit embedding for $x \lesseqgtr 0$ that covers the range $\sinh\sigma + \sin\tilde{t}\cosh\sigma \lesseqgtr 0$, as in figure 3.3, is given by

$$\begin{aligned}
x &= \sinh\sigma + \sin\tilde{t}\cosh\sigma \; , \\
x\tau &= -\cos\tilde{t}\cosh\sigma \; , \\
u &= \tilde{u} + \ln\left(\pm\frac{\cosh\sigma/2\cos t/2 + \sinh\sigma/2\sin t/2}{\cosh\sigma/2\sin t/2 + \sinh\sigma/2\cos t/2}\right) \; .
\end{aligned}$$

The first equation above is the definition of x, while the second follows from

$$\partial_t x = -\mathscr{L}_{r_0} x = \tfrac{(\nu^2+3)^2}{4\nu^2\ell^2} g_{\ell,\nu}(\partial_u, r_1) \ ,$$

where we used the Killing property of r_0 and its commutation relations, and we use the relation $r_1 = x\partial_x - \tau\partial_\tau$ described in the previous paragraph. The last equation relating $u - \tilde{u}$ is an integral of $\sinh\sigma d\tilde{t} - x d\tau$ so that x is hypersurface orthogonal. We easily confirm that $x \lesseqgtr 0$ is equivalent to

$$\frac{\cosh\sigma/2\cos t/2 + \sinh\sigma/2\sin t/2}{\cosh\sigma/2\sin t/2 + \sinh\sigma/2\cos t/2} \lesseqgtr 0 \ .$$

3.3 Warpping up

At the end of this chapter, we are now more than familiar with the warped AdS_3 geometry and we are able to describe it in three different sets of coordinates, according to how we choose our Killing vector ∂_τ:

- for $\partial_\tau \sim r_0$ timelike, using the parametrization (3.2), we obtain the set $(\tilde{t}, \sigma, \tilde{u})$, well defined on all of AdS_3, of **warped coordinates**;
- for $\partial_\tau \sim r_2$ spacelike, we obtain the set (τ, x, u), exhibiting apparent horizons and therefore only locally well defined, of **accelerating coordinates** ;
- for $\partial_\tau \sim r_2 + r_0$ null, similarly, we obtain the set (τ, xu), with apparent singularity at $x = 0$, of **Poincaré coordinates** .

We explicitly derived and listed the isometries that embed the patches of the local coordinates into the total AdS_3 space, for completeness, although the general form (3.1) visualising all these choices does give the clearest idea of the salient characteristics of each map. It is precisely this visualisation, together with the relation between explicit isometries and different coordinate maps, that we want to keep in mind for the next sections. We will be using these maps as starting point to construct non-extremal and extremal warped BTZ black holes, where we hope to clarify how solutions with actual causal singularities are very easily related to global AdS space via the appropriate coordinate patches.

***Farnsworth*:** Shizz, baby. So paradox free time-travel is possible after all.
***Bubblegum*:** Right on. But dig this multiplicand here.
***Farnsworth*:** The doom field? That must be what corrects the paradoxes.
***Curly Joe*:** When that momma rises exponentially, it could rupture the very fabric of causality!

Futurama

Chapter 4

Black hole quotients

4.1 The quotient construction

Here we will follow the construction of [49], and find the quotients of spacelike warped AdS that have causal singularities hidden behind Killing horizons. Up to an $\mathrm{SL}(2,\mathbb{R})_R$ rotation, we quotient spacelike warped AdS by $\exp(2\pi\partial_\theta)$ with ∂_θ given by

$$\partial_\theta = \begin{cases} 2\pi\ell\, T_R\, r_2 + 2\pi\ell\, T_L\, l_2 & \text{non-extremal black holes} \\ (r_2 \pm r_0) + 2\pi\ell\, T_L\, l_2 & \text{extremal black holes.} \end{cases} \tag{4.1}$$

The timelike case $\partial_\theta = A\, r_0 + B\, l_2$ yields naked closed timelike curves (CTCs). Up to an $\mathrm{SL}(2\mathbb{R})_R$ rotation, which is an isometry of warped AdS, these three cases cover all choices of ∂_θ.

We pay attention to two points of interest. The first is that singular regions of a non-extremal quotient can be hidden behind a Killing horizon only when T_L/T_R is bigger than a critical value. The second is that the Ansatz for T_L and T_R as a function of r_+ and r_- in [49] is not one-to-one for T_L/T_R smaller than a second (different) critical value.

The method we employ is to describe the quotient in accelerating or, for the case of extremal black holes, Poincaré coordinates. The reason is quite simple:

other than ∂_θ we would like a metric where the remaining isometry ∂_t is manifest. The coordinates (t, θ) should then be given by a $\mathrm{GL}(2,\mathbb{R})$ transformation on the accelerating, respectively Poincaré, coordinates (τ, u). The remaining radial coordinate r is then any function of x that labels the integral flows of $(\partial_\tau, \partial_u)$. The non-extremal black hole horizons are none other than the Killing horizons of warped AdS at $x = \pm 1$, while the extremal black hole horizon lies on the Poincaré horizon $x = 0$.

4.1.1 Non-extremal black holes

Assume the accelerating coordinates (τ, x, u) and the quotient defined by[1]

$$\begin{pmatrix} t \\ \theta \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tau \\ u \end{pmatrix} . \tag{4.2}$$

The periodicity $\theta \sim \theta + 2\pi$ is preserved under the coordinate transformation

$$\begin{pmatrix} t' \\ \theta' \end{pmatrix} = \begin{pmatrix} A & 0 \\ B & 1 \end{pmatrix}_{A \neq 0} \begin{pmatrix} t \\ \theta \end{pmatrix} . \tag{4.3}$$

That is, the quotient matrix in (4.2) is equivalent under

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \approx \begin{pmatrix} Aa & Ab \\ aB + c & Bb + d \end{pmatrix} .$$

When $b = 0$ we bring the matrix to the form

$$\begin{pmatrix} t \\ \theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} \tau \\ u \end{pmatrix} . \tag{4.4}$$

This quotient is the self-dual solution albeit in accelerating coordinates. When

[1] recall $l_2 = \partial_u$ and $r_2 = \partial_\tau$.

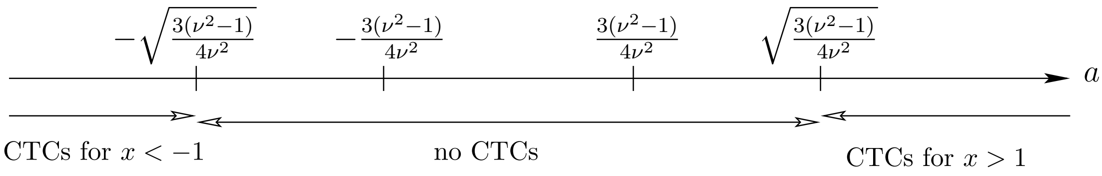


Figure 4.1: CTCs versus the parameter a.

$b \neq 0$ we bring the matrix to the form

$$\begin{pmatrix} t \\ \theta \end{pmatrix} = \frac{2\nu}{\nu^2+3} \begin{pmatrix} a & 1 \\ c & 0 \end{pmatrix} \begin{pmatrix} \tau \\ u \end{pmatrix} \tag{4.5}$$

$$\Leftrightarrow \begin{pmatrix} \tau \\ u \end{pmatrix} = \frac{\nu^2+3}{2\nu} \begin{pmatrix} 0 & 1/c \\ 1 & -a/c \end{pmatrix} \begin{pmatrix} t \\ \theta \end{pmatrix} \tag{4.6}$$

$$\Leftrightarrow \begin{pmatrix} \partial_t \\ \partial_\theta \end{pmatrix} = \frac{\nu^2+3}{2\nu} \begin{pmatrix} 0 & 1 \\ 1/c & -a/c \end{pmatrix} \begin{pmatrix} \partial_\tau \\ \partial_u \end{pmatrix} , \tag{4.7}$$

where our choice is to normalize the length $|\partial_t|^2 = \ell^2$. Note that $1/c \neq 0$ and so we cannot describe the extremal case $T_R = 0$ regularly.

We now ask when singular regions $|\partial_\theta|^2 \leq 0$ exist and whether they are hidden behind the Killing horizon $x = 1$. By reflecting $\theta \mapsto -\theta$ if necessary, we choose $c > 0$. Observe that we have not yet restricted the parameter a in (4.5). A simple calculation in accelerating coordinates reveals

$$c^2|\partial_\theta|^2 = \ell^2 \frac{\nu^2+3}{4\nu^2} \left(-(x^2-1) + \frac{4\nu^2}{\nu^2+3}(x-a)^2 \right) ,$$

with determinant

$$\Delta_x = \ell^4 \frac{\nu^2+3}{\nu^2} \left(a^2 - 3\frac{\nu^2-1}{4\nu^2} \right)$$

and

$$\partial_x(c^2|\partial_\theta|^2) = \ell^2 \frac{\nu^2+3}{2\nu^2} \left(3\frac{\nu^2-1}{\nu^2+3} x - \frac{4\nu^2}{\nu^2+3} a \right) .$$

It follows that for $|a| < \frac{\sqrt{3(\nu^2-1)}}{2\nu}$ there are no CTCs, for $a < -\frac{\sqrt{3(\nu^2-1)}}{2\nu}$ CTCs exist in $x < -1$ and for $a > \frac{\sqrt{3(\nu^2-1)}}{2\nu}$ there are CTCs after $x > 1$. This is summarized in figure 4.1.

The quotient in [49] is parametrized by

$$r = \frac{r_+ - r_-}{2} x + \frac{r_+ + r_-}{2} \ , \tag{4.8}$$

$$c = \frac{2}{\nu(r_+ - r_-)} \ , \tag{4.9}$$

$$a = -\frac{\nu(r_+ + r_-) - \sqrt{r_+ r_-(\nu^2+3)}}{\nu(r_+ - r_-)} \ , \tag{4.10}$$

so that the right and left temperatures in (4.1) are given by

$$T_R = \frac{(\nu^2+3)(r_+ - r_-)}{8\pi\ell} \ , \tag{4.11}$$

$$T_L = \frac{\nu^2+3}{8\pi\ell}\left(r_+ + r_- - \frac{\sqrt{r_+ r_-(\nu^2+3)}}{\nu}\right) \ . \tag{4.12}$$

The local coordinate transformation into the global warped coordinates is[2]

$$\tilde{t} = \tan^{-1}\left(\frac{2\sqrt{(r-r_+)(r-r_-)}}{2r - r_+ - r_+}\sinh\left(\frac{1}{4}(r_+ - r_-)(\nu^2+3)\theta\right)\right) \ ,$$

$$\sigma = \sinh^{-1}\left(\frac{2\sqrt{(r-r_+)(r-r_-)}}{r_+ - r_+}\cosh(\frac{1}{4}(r_+ - r_-)(\nu^2+3)\theta)\right) \ ,$$

$$\begin{aligned}\tilde{u} = &\frac{\nu^2+3}{4\nu}\left(2t + \left(\nu(r_+ + r_-) - \sqrt{r_+ r_-(\nu^2+3)}\right)\theta\right) \\ &+ \coth^{-1}\left(\frac{2r - r_+ - r_-}{r_+ - r_+}\coth\left(\frac{1}{4}(r_+ - r_-)(\nu^2+3)\theta\right)\right),\end{aligned}$$

and the Levi-Civita tensor transforms to $\epsilon_{tr\theta} = +\sqrt{-g}$. The coordinate transformation from the accelerating coordinates allows one to write the black hole metric in the ADM form

$$\begin{aligned} ds^2 &= \ell^2 dt^2 + \ell^2 R^2 d\theta(d\theta + 2N^\theta dt) + \frac{\ell^4 dr^2}{4R^2N^2} \\ &= N^2\left(-dt + \frac{\ell^2 dr}{2RN^2}\right)\left(dt + \frac{\ell^2 dr}{2RN^2}\right) + \ell^2 R^2 (d\theta + N_\theta dt)^2 \ , \end{aligned} \tag{4.13}$$

[2]the transformation in eqs.(5.3)-(5.5) of [49] are defined in $r_- < r < r_+$, whereas ours is in $r > r_+$. Note that we have translated $\tilde{t} \mapsto \tilde{t} + \frac{\pi}{2}$ with respect to (3.18).

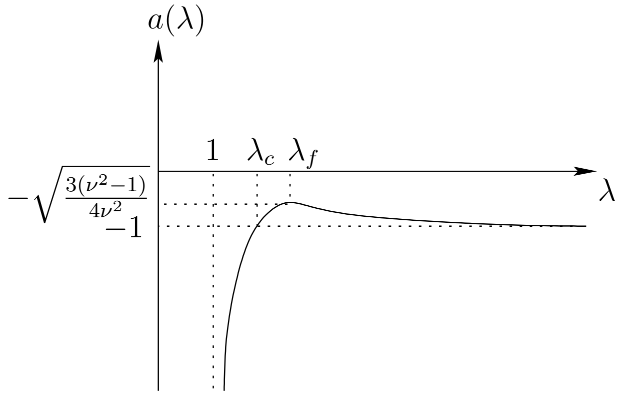


Figure 4.2: The parameter a versus the ration $\lambda = r_+/r_- > 1$.

where

$$R^2 = \frac{3(\nu^2-1)}{4} r(r-r_0) \tag{4.14}$$

$$N^2 = \frac{\ell^2(\nu^2+3)}{4R^2}(r-r_+)(r-r_-) = \frac{\ell^2(\nu^2+3)}{3(\nu^2-1)}\frac{(r-r_-)(r-r_+)}{r(r-r_0)} \tag{4.15}$$

$$N_\theta = \frac{2\nu r - \sqrt{r_+ r_-(\nu^2+3)}}{2R^2} \tag{4.16}$$

$$r_0 = \frac{4\nu\sqrt{r_+ r_-(\nu^2+3)} - (\nu^2+3)(r_+ + r_-)}{3(\nu^2-1)} \ . \tag{4.17}$$

It is instructive to draw the graph of the parameter a in (4.10) as a function of $\lambda \equiv r_+/r_- > 1$, see figure 4.2. By a suitable choice of $r_- > 0$, the parameter $1/c > 0$ is kept arbitrary. We find that a grows from minus infinity until the maximum at

$$\lambda_f = 1 + 6\frac{\nu^2-1}{\nu^2+3}\left(1 + \frac{\sqrt{3}}{3}\frac{2\nu}{\sqrt{\nu^2-1}}\right) \ , \tag{4.18}$$

for which value

$$a(\lambda_f) = -\sqrt{\frac{3(\nu^2-1)}{4\nu^2}} > -1 \ ,$$

and then asymptotes to -1. There is thus a hidden isometry between the pairs

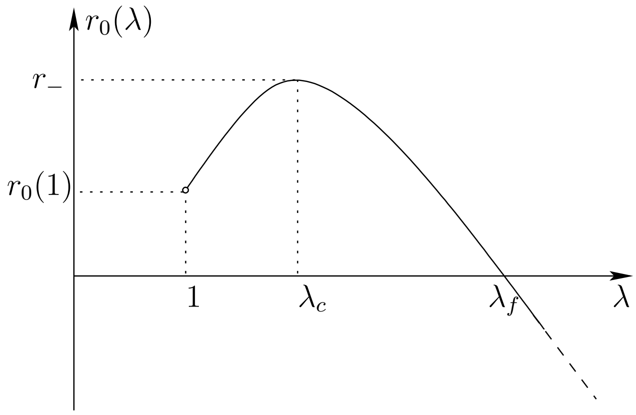


Figure 4.3: Graph of $r_0(r_+/r_-)$ for fixed (ν, r_-).

(r_+, r_-) in the two regions (λ_c, λ_f) and (λ_f, ∞), where

$$\lambda_c = \frac{4\nu^2}{\nu^2+3} \ .$$

The isometry relates black hole metrics with

$$\begin{aligned} r_+ &= \frac{\nu^2+3}{3(\nu^2-1)}\left(\sqrt{\tilde{r}_-} - \frac{2\nu}{\sqrt{\nu^2+3}}\sqrt{\tilde{r}_+}\right)^2 \\ r_- &= \frac{\nu^2+3}{3(\nu^2-1)}\left(\frac{2\nu}{\sqrt{\nu^2+3}}\sqrt{\tilde{r}_-} - \sqrt{\tilde{r}_+}\right)^2 , \end{aligned}$$

for the radial coordinate transform $r \mapsto \tilde{r}$ given by

$$\frac{2r - r_+ - r_-}{r_+ - r_-} = \frac{2\tilde{r} - \tilde{r}_+ - \tilde{r}_-}{\tilde{r}_+ - \tilde{r}_-} = x \ .$$

It is worth pointing out that r_0 in (4.17), as a function of the ratio $\lambda \equiv r_+/r_- \geq 1$ with r_- fixed, presents a maximum $r_0(\lambda_c) = r_-$ and then decreases monotonously, see figure 4.3. In particular, $r_0(\lambda_f) = 0$. As a result, the maximum root of $R(r)^2$, denoted $\bar{r}_0$ hereafter, is

$$\bar{r}_0 = \begin{cases} 0 & \text{if } r_0 < 0 \ \ i.e.\ \ \lambda > \lambda_f \\ r_0 & \text{if } r_0 \in [0, r_-] \ \ i.e.\ \ 1 \leq \lambda \leq \lambda_f, \end{cases}$$

and so $R(r)^2 > 0$ for $r > r_-$. The equality $R(r_-)^2 = 0$ holds only for $r_0(\lambda_c) = r_-$, that is when the inner horizon coincides with the singularity. For later use, let us define

$$R_{\pm}^2 \equiv R(r_{\pm})^2 = \frac{r_{\pm}}{4}\left(2\nu\sqrt{r_{\pm}} - \sqrt{(\nu^2+3)r_{\mp}}\right)^2 \geq 0.$$

Altogether, we have that for $r > \bar{r}_0$ the flow of ∂_θ is spacelike.

We should stress that we arrive at global results using accelerating coordinates. This is because ∂_θ in (4.1) is a global identification and one can choose to cover any of the infinite regions discussed in §3.2 using accelerating coordinates. In fact, the values $a > \frac{\sqrt{3(\nu^2-1)}}{2\nu}$ tell us that $x > 1$ is an accelerating patch where CTCs exist. One can then move by the discreet isometry $(x, u) \mapsto (-x, -u)$ to the outer region of the black hole. This essentially flips the sign of a, or equivalently we choose the region $x > 1$ to be the outer region as we did here. The lower bound in T_L/T_R was discussed in [49, §6.1.1]. Furthermore, the parametrization of T_L and T_R in terms of r_- and r_+ is such that the lower bound is satisfied. A subtle feature of the parametrization is the isometry in parameter space for $r_+/r_- \geq \lambda_c$. Let us also comment that, by the above analysis, the parameter $a = 0$ appears special and disconnected from the region $a < -\frac{\sqrt{3(\nu^2-1)}}{2\nu}$. We will nevertheless obtain it as the vacuum limit of the non-extremal black holes in section 4.3.

4.1.2 Extremal black holes

In the quotient given by the matrix in (4.7), observe that the parameter $1/c \sim T_R$ is always positive. One can thus never reach the extremal black holes from a regular quotient of that type. It is clear though that the non-extremal black holes have an extremal limit given by setting $r_+ = r_-$ in the non-extremal black hole metric in ADM form (4.13). We shall later recover this result as a limit of the non-extremal quotient (4.7).

The quotient that gives the extremal black holes in terms of the second Killing vector in (4.1) does not present any particular point of interest. We can repeat the previous derivation mutatis mutandis, where now the coordinates (τ, u) in (4.2) are the Poincaré coordinates of warped AdS. The case $b = 0$, see (4.4), is the self-dual solution in Poincaré coordinates. The case $b \neq 0$ gives the black hole solution in ADM form (4.13), when setting $r_+ = r_-$ in (4.12) and using $x = r - r_-$. The singular regions are behind $r < r_-$ for all values of $T_L \neq 0$, which can be chosen positive by reflecting θ if necessary. As explained beneath (4.1), we are free to rescale and normalize the factor in front of ∂_τ. We will use this later in order to obtain the near-horizon limit of the extremal black holes, which is the self-dual solution in Poincaré coordinates.

4.1.3 Thermodynamics

We would like to recall here the thermodynamic quantities that were computed for the spacelike warped black holes in [49]. It is also noteworthy to translate the condition in (4.18) and the region $r_+/r_- < \lambda_f$ into conventions used in the literature. However, let us first briefly comment on the ADM form. A general stationary, axisymmetric, asymptotically-flat black hole uniquely normalizes the Killing vector

$$\xi = \partial_t - \Omega \partial_\theta$$

that is null on its horizon, by using the asymptotically defined t and θ. For example, the surface gravity $\kappa_0 = 2\pi T_H$ on its horizon $\mathscr{H}$ is given unambiguously by

$$\nabla_\xi \xi = |_{\mathscr{H}} \kappa_0 \xi \ .$$

Were we to use a different time and angle

$$\begin{aligned} t' &= \Lambda\, t \\ \theta' &= \theta + b\, t \ , \end{aligned} \tag{4.19}$$

the Hawking temperature, angular velocity Ω, and ADT charges [56,57,58,59,60], here the mass M_{ADT} and angular momentum J_{ADT}, would transform as

$$\begin{aligned} T'_H &= \frac{1}{\Lambda} T_H \ , \\ \Omega' &= \frac{\Omega + b}{\Lambda} \ , \\ \delta M'_{\mathrm{ADT}} &= \frac{1}{\Lambda}\delta M_{\mathrm{ADT}} - \frac{b}{\Lambda}\delta J_{\mathrm{ADT}} \ , \\ \delta J'_{\mathrm{ADT}} &= \delta J_{\mathrm{ADT}} \ . \end{aligned}$$

On the other hand, the entropy variation in the first law, $\delta S = \frac{1}{T_H}(\delta M_{\mathrm{ADT}} - \Omega \delta J_{\mathrm{ADT}})$, is seen to be invariant under (4.19). The Wald formula for the entropy [61] as applied for TMG in [62] (see also [57, §4.2]) depends on the asymptotic orthonormal frame and its spin connection, and therefore is indeed invariant under (4.19).

We shall normalize the thermodynamic quantities with respect to the frame where

$$g(\partial_t, \partial_t) = \ell^2 \ .$$

This is compatible to the asymptotically warped AdS_3 conditions in [51]. In particular, it fixes both t and θ coordinates as in the ADM form (4.13). From [49], we have

$$
\begin{aligned}
T_H &= \frac{\nu^2+3}{4\pi\ell\nu}\frac{T_R}{T_L+T_R} , \\
\Omega &= -\frac{\nu^2+3}{4\pi\nu}\frac{1}{T_R+T_L} , \\
M_{\text{ADT}} &= \frac{\pi}{3G}\ell T_L , \\
J_{\text{ADT}} &= \frac{\nu\ell}{3(\nu^2+3)G}\left((2\pi\ell T_L)^2 - \frac{5\nu^2+3}{4\nu^2}(2\pi\ell T_R)^2\right) , \\
S &= \frac{\pi^2\ell}{3}\left(\frac{5\nu^2+3}{\nu(\nu^2+3)G}\ell T_R + \frac{4\nu}{(\nu^2+3)G}\ell T_L\right) .
\end{aligned}
$$

The CFT correspondence conjecture in [49] allows one to write the entropy in the form of Cardy's formula [63] with left/right central extension charges $c_R = \frac{5\nu^2+3}{\nu(\nu^2+3)G}\ell$ and $c_L = \frac{4\nu}{(\nu^2+3)G}\ell$. The bound in (4.18) and $T_R \geq 0$ become, respectively, the left-hand side and right-hand side of

$$
-\frac{8\nu\ell G}{\nu^2-1}M_{\text{ADT}}^2 \leq J_{\text{ADT}} \leq \frac{12\nu\ell G}{\nu^2+3}M_{\text{ADT}}^2 .
$$

There is yet another form of the black hole metrics[3] that is given in [64], [51] and [65]. The metric in [64] with parameters (ν', J', a', L') is related to the one in [51], which we write here

$$
\begin{aligned}
ds^2 = dT'^2 + (\frac{3}{\ell^2}(\nu^2-1)R'^2 - \frac{4\mathfrak{j}\ell}{\nu} + 12\mathfrak{m}R')d\theta^2 \\
- 4\frac{\nu}{\ell}R'dT'd\theta + \frac{dR'^2}{\frac{3+\nu^2}{\ell^2}R'^2 - 12\mathfrak{m}R' + \frac{4\mathfrak{j}\ell}{\nu}} ,
\end{aligned}
\tag{4.20}
$$

by $\mathfrak{j} = GJ'$, $6\mathfrak{m} = 4G\nu'$, $a' = -\nu/\ell$ and $L' = \sqrt{2\ell/(3-\nu^2)}$. The metric in (4.20) is related to (4.13) under the transformation $R' = \frac{\ell^2}{2}r - \frac{\ell^2}{4\nu}\sqrt{r_+r_-(\nu^2+3)}$ and $T' = \ell t$ with

$$
\begin{aligned}
6\mathfrak{m} &= \frac{\nu^2+3}{4}\left(r_+ + r_- - \frac{\sqrt{r_+r_-(\nu^2+3)}}{\nu}\right) = 2\pi\ell T_L , \\
4\mathfrak{j} &= \frac{5\nu^2+3}{16\nu}(\nu^2+3)\ell r_-r_+ - \frac{(\nu^2+3)^{\frac{3}{2}}}{8}\ell(r_+ + r_-)\sqrt{r_-r_+} .
\end{aligned}
$$

[3] the black hole metric was first found in [57] using the dimensionally reduced equations.

The condition of a Killing horizon is that $g_{R'R'}$ vanishes[4] for some R'. Its determinant $\Delta_{R'}$ with respect to R' is

$$\Delta_{R'} = (12\mathfrak{m})^2 - 16\mathfrak{j}\frac{\nu^2+3}{\nu\ell} = \left(\frac{\nu^2+3}{2}(r_+ - r_-)\right)^2 = (4\pi\ell T_R)^2 \geq 0 \ .$$

The condition that there are singularities hidden behind a Killing horizon, that is T_L/T_R is bounded from below, is that $g_{\theta\theta}$ vanishes somewhere. The positive determinant condition of $g_{\theta\theta}$, or equivalently the upper bound of r_+/r_- in (4.18), becomes

$$\mathfrak{j} \geq -3\mathfrak{m}^2\nu\ell/(\nu^2-1) \ .$$

For smaller values of $\mathfrak{j}$ for fixed $\mathfrak{m}$ we continue in the region where there are no CTCs.

4.2 Causal structure

In this section we will examine the causal structure of the spacelike warped black holes in a manner similar to [37]. Although these geometries are ideal (also referred to as "eternal", that is symmetric under $t \to -t$, so that they can be extended to regions including new singularities in the past), they are likely to appear as the end state of physical processes where chronology is protected. We will show that the Penrose-Carter diagram of a generic non-extremal or extremal black hole is similar to the 4d non-extremal, respectively extremal, Reissner-Nordström black hole. Recall that we uncovered a critical value $r_0 = r_-$ that is isometric to $r_- = 0$. We accordingly find that the $r_- = r_0$ black hole has a causal diagram similar to that of the Schwarzschild black hole, that is the uncharged Reissner-Nordström black hole.

In what follows we will work with the two-dimensional metric g_2

$$g = \underbrace{-N^2dt^2 + \frac{\ell^4 dr^2}{4R^2N^2}}_{g_2} + \ell^2R^2(d\theta + N_\theta dt)^2 \ .$$

If a curve $\gamma : [0,1] \to M$ has tangent vector $\dot\gamma \in \gamma^*TM$, then

$$g_2(\dot\gamma,\dot\gamma) > 0 \Longrightarrow g(\dot\gamma,\dot\gamma) > 0 \ ,$$

[4]recall that $g_{R'R'}$ is inverse proportional to the lapse function squared N^2.

thus a causal curve γ must be non-positive on g_2

$$g(\dot{\gamma},\dot{\gamma}) \leq 0 \Longrightarrow g_2(\dot{\gamma},\dot{\gamma}) \leq 0 \ .$$

On the other hand, any causal curve $g_2(\dot{\gamma},\dot{\gamma}) \leq 0$ can be lifted to a causal curve on g, e.g. by choosing the horizontal lift

$$\dot{\theta} + N_\theta \dot{t} = 0 \ . \tag{4.21}$$

Let us note that the metric g_2 does not capture the behaviour of causal geodesics, see e.g. [66]. However null curves on g such that (4.21) holds are geodesic on g_2. They correspond to zero angular momentum $p_\theta = g(\dot{\gamma}, \partial_\theta)$.

The metric g_2 then tells us about all causal relations by neglecting the angle θ. One might wonder why we do not take a $\theta =$ const. section. After disentangling the angle one can indeed find a Kruskal extension, as done generically in [67]. However, the angle is not defined globally on the different Kruskal patches, so our choice is simpler since the connection $d\theta + N_\theta dt$ is global. Furthermore, a local θ-section will not give us information on causal relations, nor can it be compatible with any geodesic. Indeed, observe that for large enough r no Killing vector $\partial_{t'}$ can be timelike, so the restriction of the metric on a constant angle will always be positive definite far away from the horizon.

The similarities with the RN black holes are not coincidental. Our method involves reducing the causal properties to the two-dimensional quotient space under the angular isometry ∂_θ. The difference to the Reissner-Nordström solution then, other than the dimensionality of the sphere, is a non-trivial connection one-form $d\theta + N_\theta dt$, compare e.g. with Carter's extension in [68].

We will first describe the future horizon ingoing coordinates. This is done so as to intermediately introduce the Regge-Wheeler tortoise coordinate r_*. We then write down the Kruskal-Szekeres extension in a straightforward way. We can finally conformally compactify and draw the causal diagrams. We shall also use the ingoing coordinates in section 4.3, in order to derive the near-horizon geometry of extremal black holes.

4.2.1 Ingoing Eddington-Finkelstein coordinates

To introduce Eddington-Finkelstein coordinates, one first solves for the Regge-Wheeler tortoise coordinate r_*, which in our case satisfies

$$\frac{dr_*}{dr} = \frac{\ell^2}{2RN^2} = \frac{\sqrt{3(\nu^2-1)}}{\nu^2+3}\frac{\sqrt{r(r-r_0)}}{(r-r_-)(r-r_+)} \ . \tag{4.22}$$

For $r > \bar{r}_0$ and $r_+ \neq r_-$, the solution is branched as follows

$$\begin{aligned} r_* = \frac{\sqrt{3(\nu^2-1)}}{\nu^2+3}\Bigg[& \frac{\sqrt{r_+(r_+-r_0)}}{r_+-r_-}\ln\left(\frac{|r-r_+|}{\left(\sqrt{r}\sqrt{r_+-r_0}+\sqrt{r-r_0}\sqrt{r_+}\right)^2}\right) \\ & - \frac{\sqrt{r_-(r_--r_0)}}{r_+-r_-}\ln\left(\frac{|r-r_-|}{\left(\sqrt{r}\sqrt{r_--r_0}+\sqrt{r-r_0}\sqrt{r_-}\right)^2}\right) \\ & + 2\ln\left(\sqrt{r}+\sqrt{r-r_0}\right)\Bigg] \ . \end{aligned} \tag{4.23}$$

For the critical value $r_+/r_- = 4\nu^2/(\nu^2+3)$, the solution (4.23) is also well-defined. For the extremal case $r_+ = r_-$, (4.22) becomes

$$\frac{dr_*}{dr} = \frac{\sqrt{3(\nu^2-1)}}{\nu^2+3}\frac{\sqrt{r(r-r_0)}}{(r-r_-)^2} \tag{4.24}$$

and its solution is branched as

$$\begin{aligned} r_* = \frac{\sqrt{3(\nu^2-1)}}{\nu^2+3}\Bigg(& -\frac{\sqrt{r(r-r_0)}}{r-r_-} + 2\ln(\sqrt{r}+\sqrt{r-r_0}) \\ & + \frac{1}{2}\frac{2r_--r_0}{\sqrt{r_-(-r_0+r_-)}}\ln\frac{|r-r_-|}{(\sqrt{r(r_--r_0)}+\sqrt{r_-(r-r_0)})^2}\Bigg) \ . \end{aligned} \tag{4.25}$$

The ingoing coordinate is defined as $u = t + r_*$.

The coordinates (u, r) are well-defined on and past the future horizon. In contrast, the angle θ is entangled, that is it diverges for geodesics that cross the horizon. For $r_+ \neq r_-$ and $r_+/r_- \neq 4\nu^2/(\nu^2+3)$ we define the angle

$$\theta_{in} = \theta$$
$$+ \frac{4\nu}{\nu^2+3}\frac{1}{\nu(r_+ - r_-)}\left(\frac{2\nu r_+ + \sqrt{r_+ r_-(\nu^2+3)}}{2\nu r_+ - \sqrt{r_+ r_-(\nu^2+3)}}\ln\left(\sqrt{r(r_+ - r_0)} + \sqrt{r_+(r-r_0)}\right)\right.$$

$$\left. - \frac{2\nu r_- + \sqrt{r_+ r_-(\nu^2+3)}}{|2\nu r_- - \sqrt{r_+ r_-(\nu^2+3)}|}\ln\left(\sqrt{r(r_- - r_0)} + \sqrt{r_-(r-r_0)}\right)\right) + N_\theta(r_+)u \ ,$$

while for $r_+/r_- = 4\nu^2/(\nu^2+3)$ we define

$$\theta_{in} = \theta - \frac{4}{r_- 3(\nu^2-1)}\ln\left(\sqrt{r} + \frac{2\nu}{\sqrt{3(\nu^2-1)}}\sqrt{r-r_-}\right) + N_\theta(r_+)u \ .$$

For the extremal black holes $r_+ = r_-$ we define

$$\theta_{in} = \theta + N_\theta(r_-)u + \frac{4\nu}{\sqrt{3(\nu^2-1)}(\nu^2+3)}\left(- \frac{\sqrt{r(r-r_0)}}{r_-(r-r_-)}\right.$$
$$\left. + \frac{r_0}{2r_-\sqrt{r_-(-r_0+r_-)}}\ln\frac{r-r_-}{(\sqrt{r(r_- - r_0)} + \sqrt{r_-(r-r_0)})^2}\right) .$$

These definitions are such that, in (u, r, θ_{in}) coordinates, in all cases the metric becomes

$$g = -N^2 du^2 + \frac{\ell^2}{R}dr du + \ell^2 R^2 (d\theta_{in} + N_{\theta_{in}} du)^2 \ , \tag{4.26}$$

with $N_{\theta_{in}}(r) = N_\theta(r) - N_\theta(r_+)$ being zero on the horizon. The coordinates (u, r, θ_{in}) are regular on the future horizon $r = r_+$ and valid until $r = r_-$. The Hamiltonian of a free-falling particle is

$$\mathcal{H} = \frac{2}{\ell^4}\left(\ell^2 R p_u p_r + N^2 R^2 p_r^2 + \frac{\ell^2}{4R^2}p_{\theta_{in}}^2 - \ell^2 R N_{\theta_{in}} p_r p_{\theta_{in}}\right) \ ,$$

where $p_{\theta_{in}}$, p_u are constants of motion. Null geodesics, $\mathcal{H} = 0$, satisfy

$$\dot{u} = \frac{2}{\ell^2}R p_r$$

and for $p_{\theta_{in}} = 0$ the ingoing rays are those with $p_r \equiv 0$.

Observe that in the critical case, $r_0 = r_-$, $N_{\theta_{in}}$ simplifies considerably,

$$N_{\theta_{in}} = \frac{4}{3}\frac{\nu}{\nu^2-1}\left(\frac{1}{r}-\frac{1}{r_+}\right) \ .$$

and for $r_+ = r_-$ there is of-course a double root in N^2,

$$N^2 = \frac{\ell^2(\nu^2+3)}{3(\nu^2-1)}\frac{(r-r_-)^2}{r(r-r_0)} \ .$$

As said, we shall use these results later to obtain the near-horizon geometry. The tortoise coordinate we introduced is however also useful to maximally extend the spacetime.

4.2.2 Kruskal extension of non-extremal black holes

We first describe the Kruskal extension across $r = r_+$ for the case $r_+ \neq r_-$. With

$$b_+ = \frac{\nu^2+3}{4}\frac{r_+-r_-}{R_+} = \frac{1}{2}\frac{r_+-r_-}{\sqrt{r_+(r_+-r_0)}}\frac{\nu^2+3}{\sqrt{3(\nu^2-1)}}$$

and $\rho(r) = e^{b_+ r_*}$, define

$$\left.\begin{aligned} U &= \rho(r)e^{b_+ t} \\ V &= \rho(r)e^{-b_+ t} \\ \theta_+ &= \theta - \frac{N_\theta(r_+)}{2b_+}\ln\frac{U}{V} \end{aligned}\right\} \text{for } r > r_+ \quad \text{and}$$

$$\left.\begin{aligned} U &= \rho(r)e^{b_+ t} \\ V &= -\rho(r)e^{-b_+ t} \\ \theta_+ &= \theta + \frac{N_\theta(r_+)}{2b_+}\ln\frac{U}{V} \end{aligned}\right\} \text{for } r_- < r < r_+ \ .$$

The transformation in $r_- < r < r_+$ is given so that one can match the Kruskal patches using (4.13). In these coordinates, the metric becomes

$$ds^2 = \Omega_+^2 dUdV + \ell^2R^2\left(d\theta_+ + N_{UV}(VdU - UdV)\right)^2 \ , \tag{4.27}$$

where

$$\Omega_+^2 = \frac{4\ell^2}{\nu^2+3}\frac{r_+(r_+-r_0)}{(r_+-r_-)^2}\frac{(r-r_-)^{1+\sqrt{\frac{r_-(r_--r_0)}{r_+(r_+-r_0)}}}}{r(r-r_0)}(\sqrt{r}\sqrt{r_+-r_0}+\sqrt{r-r_0}\sqrt{r_+})^2$$
$$\times\left(\sqrt{r}\sqrt{r_--r_0}+\sqrt{r_-}\sqrt{r-r_0}\right)^{-2\sqrt{\frac{r_-(r_--r_0)}{r_+(r_+-r_0)}}}\left(\sqrt{r}+\sqrt{r-r_0}\right)^{-2\frac{r_+-r_-}{\sqrt{r_+(r_+-r_0)}}}$$

is everywhere positive and N_{UV} can be shown to be regular at $r = r_+$. The coordinate r is given implicitly by $UV = \rho^2(r)$, which is monotonous in $r > r_+$ and, separately, in $r_- < r \leq r_+$. We have the limits $\lim_{r\to+\infty} UV = +\infty$, $\lim_{r\to r_+} UV = 0$ and $\lim_{r\to r_-^+} UV = -\infty$. We can extend with the isometry $V \mapsto -V$ and $U \mapsto -U$, and the patch $K_+ = \{U, V \in \mathbb{R}\}$ is regular everywhere with a metric given by (4.27).

We now build an extension across r_- for $r_+ \neq r_-$ and $r_- \neq r_0$. With

$$b_- = -\frac{\nu^2+3}{4}\frac{r_+-r_-}{R_-} = -\frac{1}{2}\frac{r_+-r_-}{\sqrt{r_-(r_--r_0)}}\frac{\nu^2+3}{\sqrt{3(\nu^2-1)}}$$

and $\rho(r) = e^{b_- r_*}$, define

$$\left.\begin{aligned}\tilde{U} &= \tilde{\rho}(r)e^{b_- t}\\ \tilde{V} &= \tilde{\rho}(r)e^{-b_- t}\\ \theta_- &= \theta - \frac{N_\theta(r_-)}{2b_-}\ln\frac{\tilde{U}}{\tilde{V}}\end{aligned}\right\}\text{ for } \bar{r}_0 < r < r_- \quad \text{and}$$

$$\left.\begin{aligned}\tilde{U} &= -\tilde{\rho}(r)e^{b_- t}\\ \tilde{V} &= \tilde{\rho}(r)e^{-b_- t}\\ \theta_- &= \theta + \frac{N_\theta(r_-)}{2b_-}\ln\frac{\tilde{U}}{\tilde{V}}\end{aligned}\right\}\text{ for } r_- < r < r_+ \,.$$

The metric becomes

$$ds^2 = \Omega_-^2 d\tilde{U}d\tilde{V} + \ell^2 R^2(d\theta_- + N_{\tilde{U}\tilde{V}}(\tilde{V}d\tilde{U} - \tilde{U}d\tilde{V}))^2 \tag{4.28}$$

with

$$\Omega_-^2 = \frac{4\ell^2}{\nu^2+3}\frac{r_-(r_- - r_0)}{(r_+ - r_-)^2}\frac{(r_+ - r)^{1+\sqrt{\frac{r_+(r_+-r_0)}{r_-(r_--r_0)}}}}{r(r-r_0)}\left(\sqrt{r}\sqrt{r_- - r_0} + \sqrt{r-r_0}\sqrt{r_-}\right)^2$$
$$\times\left(\sqrt{r}\sqrt{r_+ - r_0} + \sqrt{r_+}\sqrt{r-r_0}\right)^{-2\sqrt{\frac{r_+(r_+-r_0)}{r_-(r_--r_0)}}}\left(\sqrt{r} + \sqrt{r-r_0}\right)^{2\frac{r_+-r_-}{\sqrt{r_-(r_--r_0)}}}$$

and r is given implicitly by $\tilde{U}\tilde{V}$, which is again monotonous in r. We have the limits $\lim_{r\to\bar{r}_0^+}\tilde{U}\tilde{V} = \rho_0^2 > 0$, $\lim_{r\to r_-}\tilde{U}\tilde{V} = 0$ and $\lim_{r\to r_+^-}\tilde{U}\tilde{V} = -\infty$.

We similarly extend the coordinate range with the isometry $U \mapsto -U$, $V \mapsto -V$. The patch $K_- = \{\tilde{U}, \tilde{V} \in \mathbb{R}\}$ is defined regularly throughout with the metric given in (4.28).

By transforming into the finite-range coordinates $\tan(u) = U$ and $\tan(v) = V$, and similarly $\tan(\tilde{u}) = \rho_0\,\tilde{U}$ and $\tan(\tilde{v}) = \rho_0\,\tilde{V}$, we draw in figure 4.4 the Carter-Penrose diagrams for the two patches. Note that the conformal factor multiplying the connection one-form in the metric blows up as

$$\frac{R^2}{U^2V^2\Omega_+^2} \sim \mathcal{O}\left(r^{\frac{r_+-r_-}{\sqrt{r_+(r_+-r_0)}}}\right)\ .$$

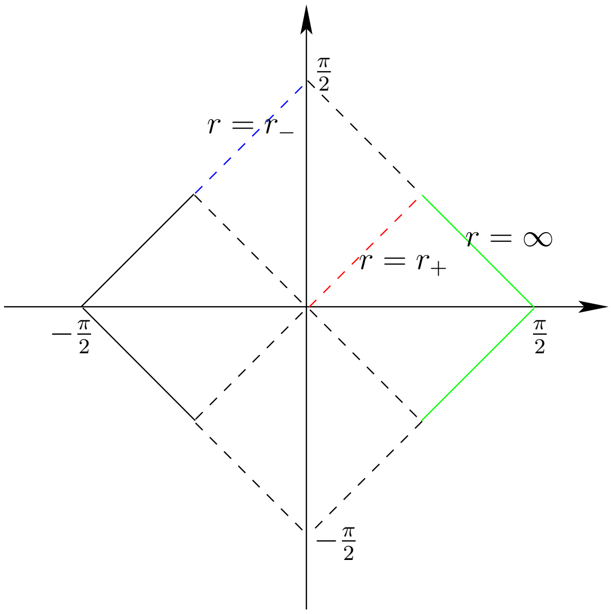
To circumvent any ambiguity, we compactify the manifold by using instead the coordinate system

$$\hat{U} = U^{z(U)+1}$$
$$\hat{V} = V^{z(V)+1}\ ,$$

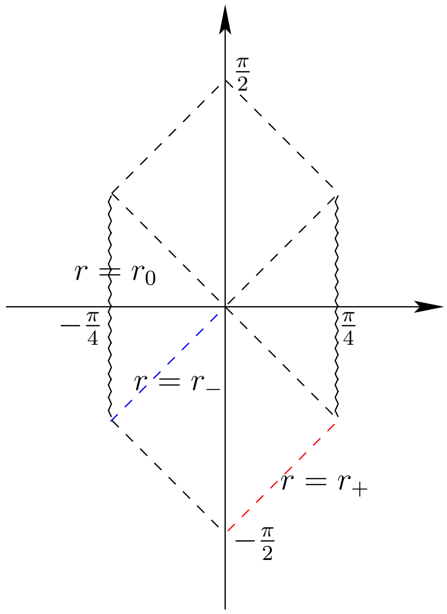
where the exponent $z(x)$ is a function that is zero for small but positive x and grows smoothly within a finite range up to the constant value of $\frac{r_+-r_-}{\sqrt{r_+(r_+-r_0)}}$. The factor multiplying the connection one-form then becomes finite and non-vanishing in the limit $r \to \infty$. The maximal extension is obtained by concutting K_+ after K_- ad infinitum, as in figure 4.5.

For the critical value $r_+/r_- = 4\nu^2/(\nu^2+3)$ we define the patch K_+ as before. With the special value

$$b_+ = \frac{\nu^2+3}{4\nu}\ ,$$



(a) K_+



(b) K_-

Figure 4.4: Penrose diagrams of Kruskal patches for $r_0 \neq r_-$ black holes.

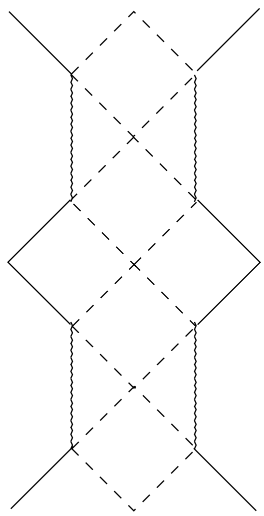


Figure 4.5: The Penrose diagram of maximally extended $r_0 \neq r_-$ black holes.

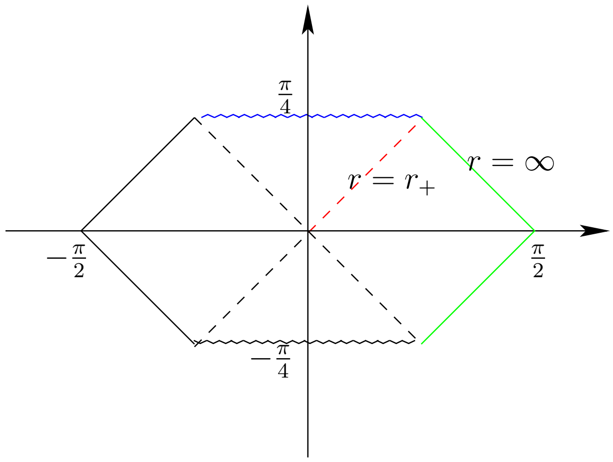


Figure 4.6: Penrose diagram for $r_0 = r_-$.

we find

$$\Omega_+^2 = \frac{4\ell^2}{3(\nu^2-1)}\frac{4\nu^2}{\nu^2+3}\frac{1}{r}\left(\sqrt{r}\sqrt{3(\nu^2-1)}+2\nu\sqrt{r-r_-}\right)^2 \times\left(\sqrt{r}+\sqrt{r-r_-}\right)^{-\frac{\sqrt{3(\nu^2-1)}}{\nu}} . \quad (4.29)$$

However, here we do not extend beyond the inner horizon r_- where $|\partial_\theta|^2 < 0$. The Kruskal coordinates have the limits $\lim_{r\to+\infty} UV = +\infty$, $\lim_{r\to r_+} UV = 0$ and

$$\lim_{r\to 0} UV = -\rho_0^2 = -\frac{1}{\nu^2+3} r_-^{\sqrt{\frac{3(\nu^2-1)}{2\nu}}} .$$

The Penrose diagram of the critical black hole is drawn in 4.6, where we use $U = \rho_0 \tan(u)$ and $V = \rho_0 \tan(v)$.

4.2.3 Kruskal extension of extremal black holes

Finally, we describe the extremal case. We present the conformal compactification at once, by using a transformation similar to the one for the extremal Reissner-Nordström in [68]. However, some care is needed to show that the connection one-form is also well-defined. Using the tortoise coordinate, define for $r > r_-$

$$\tan U = t + r_* , \quad (4.30)$$

$$\tan V = -t + r_* , \quad (4.31)$$

$$\theta_{UV} = \theta - N_\theta(r_-)t - C\left(2\tanh^{-1}\tan\frac{U}{2} - 2\tanh^{-1}\tan\frac{V}{2}\right) , \quad (4.32)$$

with the constant

$$C = -\frac{4\nu}{(\nu^2+3)\sqrt{3(\nu^2-1)}\sqrt{r_-(r_- - r_0)}} \ .$$

The metric takes the form

$$g = \Omega^2 dU dV + \ell^2 R^2 (d\theta_{UV} + \tilde{N}_{UV}(dU - dV))^2,$$

with

$$\Omega^2 = \frac{N^2}{\cos^2 U \cos^2 V}$$

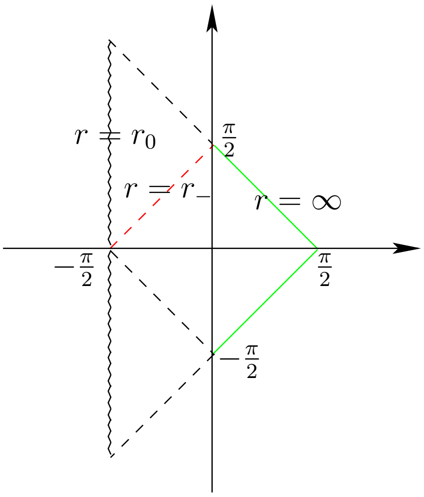
and $\tilde{N}_{UV}$ is zero on the horizon. We first observe that Ω^2 is non-zero on the future and past horizon. Indeed, the dangerous factor $\frac{(r-r_-)^2}{\cos^2 V}$ in the limit $V \to 0$ goes like

$$\left(\frac{1}{\cos V}\right)\left(\frac{1}{r-r_-}\right)^{-1} \longrightarrow \left(2\frac{\sin V}{\cos^2 V}\right)\left(-\frac{\frac{\partial r}{\partial V}}{(r-r_-)^2}\right)^{-1}$$
$$\longrightarrow 2\frac{\sqrt{3(\nu^2-1)}}{\nu^2+3}\sqrt{r_-(r_- - r_0)} \ ,$$

where the last equation uses the derivative of the tortoise coorindate in (4.24). It follows that $\lim_{V\to 0} \Omega^2$ is finite and non-vanishing on the future horizon, and similarly on the past horizon. We also defined θ_{UV} in (4.32) with the term linear in C so that a potential pole of $g(\partial_\theta, \partial_U) \sim \tilde{N}_{UV}$ in $r - r_-$ vanishes. Altogether, this means that we can use the same transformation on and behind the horizon but for a different domain of U, V, and by replacing $C \to -C$. The singular region is at $\tan U + \tan V = 2r_*$ which can be brought to zero by a suitable shift in r_*. The Penrose diagram of the extremal black hole is drawn in 4.7 and the maximal extension can be obtained with the isometry $U - V \mapsto U - V + 2\pi\mathbb{Z}$.

4.3 Spacetime limits

In the previous sections we explored the geometry of warped AdS, its black hole quotients and their causal properties. In particular, the extremal black holes are obtained from a different quotient than their non-extremal counterparts. At the same time, the extremal black holes are a regular limit of the non-extremal



(a) Kruskal Patch

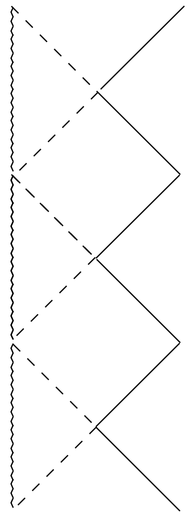

(b) Maximal Extension

Figure 4.7: Penrose diagrams of extremal black holes.

black holes, in the sense that we can set $r_- = r_+$ in the ADM form. In this section we explain this limit in more detail. We also want to ask what other classical[5] limits we can obtain from the warped AdS black holes. We will obtain the near-horizon geometry of extremal black holes and we will define several other spacetime limits, which give us the self-dual warped AdS, in either accelerating or Poincaré coordinates, and warped AdS with a proper time identification.

We find it helpful to recall Geroch's notion of a spacetime limit [69]. Here one collects a family of metric spacetimes (M_L, g_L), where $L > 0$, and constructs the augmented manifold $\mathscr{M} = \{(M_L, g_L, L)_L\}$. A spacetime limit, $L \to 0$, is invariantly defined on the boundary of $\mathscr{M}$. Spacetime limits are interesting for the properties of the family (M_L, g_L) that are inherited in the limit, a typical example being the rank of Killing vectors and Killing spinors [70]. Naturally, the spacetime limit (M_0, g_0) is of interest when its maximal extension is not included in the original phase space.

An instance of Geroch's notion is when there is a local isometry $f_L : M_L \to M_1$, for $L > 0$, between the metrics g_L and g_1. The limit can then be said to be of the metric itself g_1 rather than a limit in the family of metrics g_L. An example is the Penrose limit [70]. A metric limit typically involves blowing up a neighbourhood of the spacetime. Minkowski space is not only a spacetime limit of 4d black holes, where the mass $M \equiv L \to 0$, but can also be written in terms

[5]that is, we consider ℓ, G and ν fixed.

of a metric limit [69]. In the latter case, one is translating in the limit to the asymptotically flat region while keeping the mass M fixed. In this paper, we call a metric limit the near-horizon geometry of g_1 when the isometry f_L fixes the outer horizon.

In our case, the metrics are parametrized by (T_R, T_L) that we take as functions of $L > 0$. Each black hole in the phase space is given by the identification Killing vector ∂_θ as written in (4.1). Note, though, that the identification vector in (4.1) is unique up to $\mathrm{SL}(2,\mathbb{R})_R$ rotations. The question we ask is, what are the limits of the non-extremal black holes as $T_R \to 0$.

In order to simplify our discussion, we do not ask what happens in the limit behind the outer horizon. We thus take the M_L to cover only part of the maximally extended spacetime. In practice this means we can work with the accelerating, or Poincaré coordinates, and define the limits explicitly. The coordinates will thus depend explicitly on L. This description is complementary to the previous not only for practical reasons, but also because it describes the relation of the coordinate range of the limit manifold M_0 to that of $M_{L>0}$.

We first describe the near-horizon limit of the extremal black holes, using the coordinate description in the framework of [71, 72, 73]. We then consider spacetime limits of non-extremal black holes when $T_R \to 0$. There are two such limits. The first one gives the extremal black holes. The second gives us a geometry similar to the near-horizon geometry of the non-extremal ones, but in accelerating coordinates. We call the latter a near-extremal limit because of this similarity. We also describe the near-horizon geometry of extremal black holes in the invariant description. Finally, we consider the case when we send $T_R \to 0$ while keeping the Hawking temperature fixed.

4.3.1 Near-horizon limit

Let us erect Gaussian null coordinates on the future horizon of a spacelike warped black hole, as explained in [74]. The ingoing coordinates (u, r, θ_{in}) are such that θ_{in} is a well-defined angle on a spacelike section of the horizon and u is the group parameter of $\xi = \partial_u$. Recall that the metric in ingoing coordinates has the form (4.26):

$$g = -N^2 du^2 + \frac{\ell^2}{R} dr du + \ell^2 R^2 (d\theta_{in} + N_{\theta_{in}} du)^2 \ .$$

We are interested in defining a new coordinate $\bar{r}$ that is the affine parameter of a null geodesic congruence γ emanating from the horizon and parametrised by

(u, θ_{in}). We fix its velocity $\dot\gamma_0$ on the future horizon $\mathscr{H}^+$ to be the normalized null complement of ∂_u and $\partial_{\theta_{in}}$ with respect to the metric: $g(\dot\gamma_0, \partial_u)|_{\mathscr{H}^+} = 1/2$ and $g(\dot\gamma_0, \partial_{\theta_{in}})|_{\mathscr{H}^+} = 0$. The Hamiltonian of a free-falling particle and its geodesic equations are

$$\begin{aligned}
\mathcal{H} &= \frac{2}{\ell^4}\left(\ell^2 R p_u p_r + N^2 R^2 p_r^2 + \frac{\ell^2}{4R^2} p_{\theta_{in}}^2 - \ell^2 R N_{\theta_{in}} p_r p_{\theta_{in}}\right)\\
\dot r &= \frac{2}{\ell^4}\left(\ell^2 R p_u + 2N^2 R^2 p_r\right)\\
\dot\theta &= \frac{2}{\ell^2}\left(\frac{p_{\theta_{in}}}{2R^2} - R N_{\theta_{in}} p_r\right)\\
\dot u &= \frac{2}{\ell^2} R p_r \ .
\end{aligned}$$

The equations can easily be solved. The constraint $\mathcal{H} = 0$ implies $p_r|_{\mathscr{H}^+} = p_{\theta_{in}} = 0$ and with $p_u = \frac{1}{2}$ we find $p_r \equiv 0$, $\dot\theta = \dot u = 0$ and

$$\frac{dr}{d\bar r} = \frac{R}{\ell^2} \ , \tag{4.33}$$

where $\bar r$ is the affine parameter. This equation is solved generically by

$$r = r_0 \cosh^2\left(\frac{\sqrt{3(\nu^2-1)}}{4\ell^2}\bar r - c\right) , \tag{4.34}$$

where

$$\cosh c = \sqrt{\frac{r_+}{r_0}} \ \text{and} \ c > 0 \ . \tag{4.35}$$

The coordinate transformation (4.34) covers the region $r \in (r_0, +\infty)$, which corresponds to

$$\bar r \in \left(-\infty, \frac{4\ell^2}{\sqrt{3(\nu^2-1)}} c\right) .$$

The other coordinates remain $u \in \mathbb{R}$ and θ_{in} periodic.

For $r_+ \neq r_-$ the metric takes the form

$$g = -\bar r\, F(\bar r)\, du^2 + d\bar r du + \ell^2 R^2(r(\bar r))(d\theta_{in} + N_{\theta_{in}}(r(\bar r))\, du)^2 \ , \tag{4.36}$$

where $N^2 = \bar r\, F(\bar r)$ and $F(\bar r)$ is regular non-vanishing on the horizon $\bar r = 0$. It follows that the near-horizon limit cannot be defined for non-extremal black-holes.

Indeed, if we assume a diffeomorphism $\bar{r} \mapsto \bar{r}/L$ that zooms in on a neighbourhood of the horizon, then the component $g(\partial_u, \partial_{\bar{r}})$ dictates an appropriate rescaling $u \mapsto Lu$ so that $\lim_{L\to 0} g(\partial_u, \partial_{\bar{r}})$ remains finite. However, this blows up the component $g(\partial_u, \partial_u)$.

When $r_+ = r_-$, $F(\bar{r}) = \bar{r}\, H(\bar{r})$ where $H(\bar{r})$ is regular non-vanishing at $\bar{r} = 0$. Introducing the coordinate transformation

$$\begin{aligned} \bar{r}' &= \bar{r}/L \\ u' &= Lu, \end{aligned} \tag{4.37}$$

and sending $L \to 0$, gives the metric limit

$$g = \frac{\nu^2+3}{4\ell^2}\bar{r}'^2\, du'^2 + d\bar{r}'du' + \ell^2 R_-^2 \left(d\theta_{in} + \left.\frac{dN_{\theta_{in}}}{dr}\right|_{r_-} \frac{R_-}{\ell^2}\bar{r}'du' \right)^2 , \tag{4.38}$$

with

$$\left.\frac{dN_{\theta_{in}}}{dr}\right|_{r_-} = \frac{4}{r_-^2}\frac{\nu - 2\nu^2 r_- r_- + \nu\sqrt{\nu^2+3}r_-}{(2\nu - \sqrt{\nu^2+3})^2} \ .$$

Observe that as $L \to 0$, any point $\bar{r}$ close to $\bar{r} = 0$ is pushed away to infinity with respect to $\bar{r}'$. The metric in (4.38) is the self-dual solution with $\alpha = \frac{\nu^2+3}{2\nu}R_-$ in Poincaré coordinates. This can be verified by using the diffeomorphism

$$\begin{aligned} u' &= \tau - \frac{1}{x} \\ \bar{r}' &= \frac{2\ell^2}{\nu^2+3}x \\ \phi &= \theta + \frac{2\nu}{\nu^2+3}\frac{1}{R_-}\ln x \ . \end{aligned} \tag{4.39}$$

The above derivation zooms indefinitely into the future horizon of an extremal black hole along a geodesic congruence. Using the coordinate description we got the self-dual warped AdS in Poincaré coordinates. This result is universal. We would not have been able to arrive at the same geometry in, say, accelerating or warped coordinates. Since the horizon is non-bifurcate the same should be true for the limit spacetime. One could use equivalently the double null coordinates (u, v), where u is the ingoing and $v = -t + r_*$ is the outgoing coordinate. The description using (u, v) serves to show that we are zooming in on the whole of the horizon. Finally, we could have used the ADM coordinates (r, t). The limit is given by $r - r_- = r'\, L$ and $t = t'/L$. This description provides an equivalent explanation for why the limit is in Poincaré coordinates. This is the case because t is defined asymptotically by observers who wish to probe the horizon. As such, the near-horizon inherits a preferred time which is not related to the global warped time $\tilde{t}$.

We can already ask what properties are inherited in the limit. It is clear that one such property is the nature of the horizon. The size of the radius of θ on the horizon is also inherited, this being a consequence of definition (4.37) as an isometry that fixes the horizon. We will later describe the near-horizon geometry invariantly, using the identification vector ∂_θ, and see that this is related to the extremal black hole via $\alpha = 2\pi\ell\, T_L$.

4.3.2 Near-extremal limit

Although a non-extremal black hole does not admit a near-horizon limit, we can consider a limit in the black hole phase space (T_L, T_R) for $T_R \to 0$. This limit cannot be considered a metric limit because T_R is continuously varied. Furthermore, there is more than one way to take the limit. Here we will consider the case when the limit gives us the self-dual solution in accelerating coordinates. We call the limit the near-extremal near-horizon limit, or near-extremal limit for short, and we stress it is a spacetime limit in the phase space of non-extremal black holes.

A black hole is described by (T_R, T_L) that enter the definition (4.1) of the Killing vector ∂_θ,

$$\partial_\theta = 2\pi\ell\, T_R\, r_2 + 2\pi\ell\, T_L\, l_2 \ .$$

There are however two gauge freedoms that we can use in its description. The first is an active $\mathrm{SL}(2,R)_R$ rotation that isometrically maps the outer region as embedded in warped AdS to a new region. The rotation transforms $r_2 \mapsto Ar_2 \pm Br_0$, with $A^2 - B^2 = 1$, and we can use instead the vector

$$\partial_{\theta'} = 2\pi\ell\, T_R\, (A\, r_2 \pm B\, r_0) + 2\pi\ell\, T_L\, l_2 \ . \tag{4.40}$$

Note that we are considering an active transformation in warped AdS. That is, the rotation $\exp(\tanh^{-1}(\frac{B}{A})\, r_1)$ is not an isometry of the metric.

The second gauge freedom is how we describe time t. The $\mathrm{GL}(2,\mathbb{R})$ diffeomorphism in (4.3) keeps the identification vector ∂_θ invariant. However, we are redefining ∂_t and so the metric form in the new coordinate system does change.

It is this freedom that we shall use and fix here. Indeed, notice that if we simply take $T_R = 0$ in (4.7), that is send $1/c \to 0$ and keep a/c fixed in (4.7), we end up with ∂_t collinear with ∂_θ. The coordinates (t, θ) are thus ill-defined in the limit. We use the transformation

$$\begin{pmatrix} t' \\ \theta' \end{pmatrix} = \begin{pmatrix} -\frac{1}{b}\frac{\nu^2+3}{2\nu}\frac{T_R}{T_L} & 0 \\ \frac{\nu^2+3}{2\nu}\frac{1}{2\pi\ell T_L} & 1 \end{pmatrix} \begin{pmatrix} t \\ \theta \end{pmatrix}, \tag{4.41}$$

so that

$$\begin{pmatrix} \partial_{t'} \\ \partial_{\theta'} \end{pmatrix} = \begin{pmatrix} b & 0 \\ 2\pi\ell T_R & 2\pi\ell T_L \end{pmatrix} \begin{pmatrix} \partial_\tau \\ \partial_u \end{pmatrix} .$$

Here we have included an arbitrary $b > 0$ constant, which is equivalent to $b = 1$ by diffeomorphism invariance.

The near-extremal limit is now well-defined in coordinates t' and θ'. By simply setting $T_R = 0$ we get

$$\begin{aligned} \partial_{t'} &= b\,\partial_\tau \\ \partial_{\theta'} &= 2\pi\ell T_L \partial_u \ . \end{aligned}$$

This identification gives the self-dual geometry with $\alpha = 2\pi\ell T_L$ in accelerating coordinates. The identification with (3.17) is made by $\phi = \theta' = u/\alpha$ and $\tau = b\,t'$.

It is useful to describe the limit explicitly in coordinates. For this, we reuse the accelerating coordinate x, which is related to r via (4.8). Recall that x is given linearly by $g(\partial_\tau, \partial_u)$ and so it remains invariant under the transformation (4.41). We also use the coordinates (θ', t') from (4.41). Altogether we have

$$\begin{aligned} r &= \frac{r_+ - r_-}{2} x + \frac{r_+ + r_-}{2} \\ t &= -\frac{2\nu}{\nu^2+3} b \frac{T_L}{T_R} t' \\ \theta &= \phi + \frac{b}{2\pi\ell T_R} t'. \end{aligned} \tag{4.42}$$

The ADM metric at fixed T_L and $T_R > 0$ in (t', x, ϕ) coordinates is

$$g = -\frac{\ell^2}{\nu^2+3} b^2(x^2-1) \left(\frac{4\pi\nu\ell T_L}{R(r)(\nu^2+3)} \right)^2 dt'^2 \\ + \frac{\ell^2}{\nu^2+3} \frac{dx^2}{x^2-1} + \ell^2 R^2(r) \left(d\phi + N_{t'}(r)\, dt'\right)^2 , \quad (4.43)$$

with

$$N_{t'}(r) = \frac{b}{2\pi\ell T_R} \left(1 - \frac{2\nu}{\nu^2+3} 2\pi\ell T_L \frac{2\nu r - \sqrt{r_- r_+ (\nu^2+3)}}{2R^2(r)} \right) . \quad (4.44)$$

Note that in the limit $r_+ \to r_-$, $r(x) \to \frac{r_+ + r_-}{2}$. We also have that

$$R^2(\frac{r_+ + r_-}{2}) = \frac{(\nu^2+3)(2\pi\ell T_R)^2 + (4\pi\nu\ell T_L)^2}{(\nu^2+3)^2} \overset{r_+ \to r_-}{\longrightarrow} \left(\frac{4\pi\nu\ell T_L}{\nu^2+3} \right)^2 .$$

By using the above, and the equations for T_L and T_R in (4.11) and (4.12), one sees that the term in parentheses in (4.44) is zero as $r_+ \to r_-$. Therefore, it cancels the pole in T_R. In order to find the limit we expand the function

$$f\left(\frac{r_+ - r_-}{2} x + \frac{r_+ + r_-}{2} ; r_+, r_- \right) = \frac{2\nu r - \sqrt{r_- r_+ (\nu^2+3)}}{2R^2(r)} ,$$

which is symmetric in its last two arguments, in powers of L, with $r_\pm = r_e \pm L$ and keeping r_e and x fixed:

$$f(L\,x + r_e; r_e + L, r_- - L) = f(r_e; r_e, r_e) + f^{(1,0,0)}(r_e; r_e, r_e) L\, x \\ + f^{(0,1,0)}(r_e; r_e, r_e) L - f^{(0,0,1)}(r_e; r_e, r_e) L + \mathscr{O}(L^2)$$

$$= \partial_r \left(\frac{2\nu r - \sqrt{r_- r_+ (\nu^2+3)}}{2R^2(r)} \right) \Bigg|_{r=r_-=r_+} \cdot \frac{r_+ - r_-}{2} x + \mathscr{O}({T_R}^2) .$$

After some algebra, we find

$$\lim_{r \to r_+} N_{t'}(x) = \frac{1}{R_-} \frac{2\nu}{\nu^2+3} b\, x .$$

With $R_- = 4\pi\nu\ell T_L/(\nu^2+3)$, we confirm that the metric (4.43) becomes at $r_+ \to r_-$ the self-dual solution with $\alpha = 2\pi\ell T_L$ and $t' = b\,\tau$.

One might ask whether the transformation in (4.42) can be modified so as to describe a metric limit of a fixed geometry $T_R \neq 0$. An immediate guess $x \mapsto x/L$ and $t' \mapsto L\,t'$ in (4.43) keeping $r_- \neq r_+$ and sending $L \to 0$ gives us the self-dual solution in Poincaré coordinates. However, this limit commutes with taking the same limit after we send $r_+ \to r_-$.

Observe that the bifurcate nature of the horizon is inherited in accelerating coordinates. Although this is not a metric limit, in the sense that we have not fixed a black hole geometry, we intuitively understand (4.42) as zooming in close to the outer horizon of non-extremal black holes with $T_R \approx 0$. Finally note that, in taking $T_R \to 0$, we can keep T_L or some other combination of T_L and T_R fixed. The interpretation of the near-horizon limit in the context of the 4d Planck scale limit $L_p \to 0$ for Reissner-Nordström black holes has been discussed in [75, 76], see also [77].

4.3.3 Extremal limit

In the ADM form one can reach the extremal black holes by setting $r_+ = r_-$ in (4.13). We can describe this by combining the limit $T_R \to 0$ with an $\mathrm{SL}(2,\mathbb{R})_R$ transformation,

$$\partial_{\theta'} = 2\pi\ell\, T_R\,(A\, r_2 \pm B\, r_0) + 2\pi\ell\, T_L\, l_2, \tag{4.45}$$

where we set

$$A = \frac{1}{\ell\, T_R}\ , \quad B = \sqrt{\left(\frac{1}{\ell\, T_R}\right)^2 - 1}\ . \tag{4.46}$$

In the limit $T_R \to 0$ we have

$$\begin{aligned} \partial_t &= \frac{\nu^2+3}{2\nu}\partial_u \\ \partial_{\theta'} &= 2\pi(r_2 \pm r_0) + 2\pi\ell\, T_L l_2, \end{aligned}$$

which describe precisely the extremal black holes. Here we do not need to use a $\mathrm{GL}(2,\mathbb{R})$ transformation.

We claim that this limit is equivalent to setting $r_- = r_+$ in the ADM form. Indeed, in section 4.1 we only considered the case when ∂_θ is a linear combination of r_2 and l_2. Since $e^{\zeta r_1}$ is invertible, the identification along ∂_θ is equivalent to the identification along $\partial_{\theta'}$:

$$e^{2\pi\partial_\theta} p \sim p \Longleftrightarrow e^{2\pi\partial_{\theta'}} e^{\zeta r_1} p \sim e^{\zeta r_1} p$$

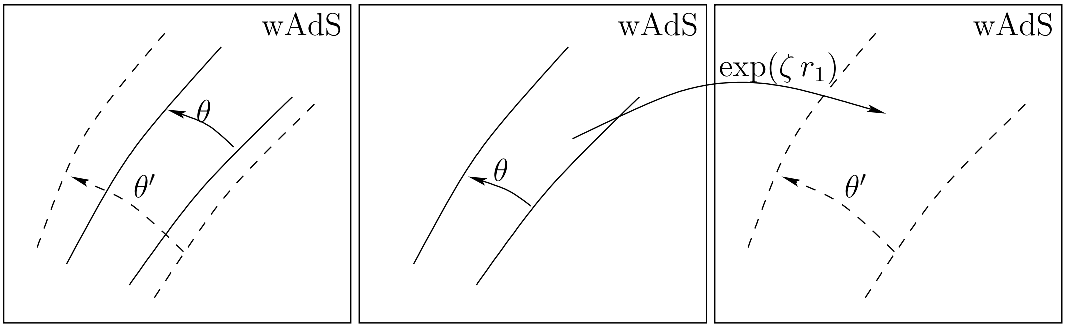


Figure 4.8: The field r_1 is not an isometry of the black hole metric, since it does not preserve the identification. However, the mapped region is by definition isometric to the black hole.

for every point p in warped AdS. We can define coordinates (r', t', θ') on the mapped region by using the (r, t, θ) coordinates of §4.1, with $r' = r$, $t' = t$, $\theta' = \theta$, see figure 4.8.

By using the invariant description of the identification vector, it is obvious that in sending $T_R \to 0$, and keeping T_L finite, non-extremal black holes can either limit to the near-extremal geometry with $\alpha = 2\pi\ell T_L$, or the extremal black hole with the same T_L. That is, we can either try to keep the term in ∂_θ that is multiplied by T_R (the extremal limit) or not (the near-extremal limit).

4.3.4 Near-horizon geometry, again

We are now able to describe the near-horizon geometry of the extremal black holes, which was given in §4.3.1, in an invariant way. Let us accordingly switch to Poincaré coordinates (x, τ, u). From (4.1) and by using an $\mathrm{SL}(2, \mathbb{R})_R$ rotation, the identification vector is

$$\partial_\theta = 2\pi\, L\,(r_2 + r_0) + 2\pi\ell\, T_L\, l_2 \quad \text{with } L > 0.$$

It is also necessary to use a matrix transformation as in §4.3.2, so that ∂_t is not collinear with ∂_θ in the limit $L \to 0$. We use a matrix transformation identical in form to (4.41), but replace T_R with L. In the limit $L \to 0$, we obtain the self-dual solution in Poincaré coordinates, with $\alpha = 2\pi\ell T_L$:

$$\begin{aligned} \partial_t &= b\partial_\tau \\ \partial_\theta &= 2\pi\ell\, T_L \partial_u\ . \end{aligned}$$

One can use coordinates to describe the above limit. In fact, the coordinate transformation follows closely §4.3.2, with some minor changes. In (4.42), the first equation should be replaced with $x = L\,(r - r_-)$, and T_R should be replaced with L in the other two equations. The metric in (r', t', ϕ') coordinates, (4.43), becomes

$$g = -\frac{\ell^2}{\nu^2+3} b^2\, x^2 \left(\frac{4\pi\nu\ell\, T_L}{R(r)(\nu^2+3)} \right)^2 dt'^2 \\ + \frac{\ell^2}{\nu^2+3} \frac{dx^2}{x^2} + \ell^2 R^2(r)\,(d\phi + N_{t'}(r)\,dt')^2 \ ,$$

and, in the limit $L \to 0$, the metric limits to the self-dual geometry in Poincaré coordinates, with $\alpha = 2\pi\ell\, T_L$ and $t' = b\,\tau$.

It might seem surprising that this is the same limit as in §4.3.1. Observe however that $\partial_{t'} - \partial_\phi$ is proportional to the Killing vector that is null on the horizon. In using the matrix transformation we are rescaling the ingoing coordinate as before. The radial coordinate is then rescaled appropriately so that the limit is finite.

4.3.5 Vacuum limit

We finally consider the limit $T_R, T_L \to 0$ with the ratio T_L/T_R kept constant. This is equivalent to keeping a fixed ratio r_+/r_- and sending $r_- \to 0$. In [49] this limit was called the vacuum solution. In order to keep ∂_θ finite, we use the $\mathrm{SL}(2,\mathbb{R})_R$ transformation in (4.45), with the same parameters (4.46), so that in the limit $T_R \to 0$ we obtain

$$\begin{aligned} \partial_t &= \frac{\nu^2+3}{2\nu}\partial_u \\ \partial_\theta &= 2\pi(r_2 \pm r_0)\ . \end{aligned} \tag{4.47}$$

Note that here we do not need the $\mathrm{GL}(2,\mathbb{R})$ transformation. Observe that the Killing vectors ∂_t and ∂_θ do not depend on r_+/r_-. The limit is thus universal.

The geometry we obtain is warped AdS in Poincaré coordinates with a periodic identification of the proper time τ. We can see this by using coordinates. As in the extremal limit, we use the metric in ADM form, and we send the parameters

r_- and r_+ to zero keeping r_+/r_- fixed. The metric becomes

$$\lim_{\substack{r_-\to 0\\ r_+\to 0}} g = -\ell^2 \frac{\nu^2+3}{3(\nu^2-1)}dt^2 + \frac{\ell^2}{\nu^2+3}\frac{dr^2}{r^2} + \ell^2 \frac{3(\nu^2-1)}{4} r^2 \left(d\theta + \frac{4\nu}{3(\nu^2-1)}\frac{1}{r}dt\right)^2$$

$$= -\ell^2 \frac{\nu^2+3}{4} r^2 d\theta^2 + \frac{\ell^2}{\nu^2+3}\frac{dr^2}{r^2} + \frac{4\nu^2\ell^2}{(\nu^2+3)^2}\left(\frac{\nu^2+3}{2\nu}dt + \frac{\nu^2+3}{2} r\, d\theta\right)^2 . \quad (4.48)$$

The identification with Poincaré coordinates can be made with $x = \frac{\nu^2+3}{2} r$.

The limit corresponds to sending M_{ADT} and J_{ADT} to zero while keeping the Hawking temperature fixed. One can also interpret the limit as a metric limit to the far-away region. That is, the metric in (4.48) corresponds to keeping the leading order components of the black hole metric when $r \gg r_+$.

4.4 Discussion

Having elaborated on the construction of warped AdS_3 from first principles in the previous chapter, we have now set up the quotient construction. We focused on the case when causal singularities do exist and are hidden behind a Killing horizon. The geometries are ideal, in the sense that they can be continued to regions that contain new singularities and new asymptotic regions. We found the causal structure and showed that the geometries fall into three classes that resemble the causal structure of the Reissner-Nordström black hole.

We pointed out two features that are usually suppressed in the literature. The first is that the black hole metric parametrized by r_+ and r_- presents a redundancy, in that for a certain region two sets of parameters (r_+, r_-) describe the same geometry. The second is that, the ratio of the left to right temperature is bounded from below, if the geometry is to describe a causal singularity that is hidden behind Killing horizons. In [51] care was taken to consistently define an asymptotically Killing algebra [78] that contains a centrally extended Virasoro algebra with generators $\mathscr{L}_m$, so that $\mathscr{L}_0$ has positive spectrum and a central extension that matches the AdS/CFT expectation [63]. The bound on the ratio of temperatures T_L/T_R would then imply an upper bound on $\mathscr{L}_0$. Indeed, in [51] $\mathscr{L}_0$ for a black hole goes like $c(\frac{3}{2}\mathfrak{m}^2 - \frac{2}{3c}\mathfrak{j})$ (for a specific, field-dependent, normalisation, which guarantees $\mathscr{L}_0 \geq 0$), for $\mathfrak{m}$ and $\mathfrak{j}$ as described in 4.1. We further saw in 4.1.3 that the lower bound on T_L/T_R corresponds to a lower bound on $\mathfrak{j}$, so that, for any given $\mathfrak{m}$, for too high values of $\mathscr{L}_0$ one does not have any causal singularities.

We also described various spacetime limits that one can take in the black hole phase space. We do this by studying the behaviour of the identification vector ∂_θ for different significant limits of the invariants T_R, T_L and their ratio. We chose this exposition for the clarity of the geometric interpretation of the limits, and also to avoid the ambiguities that could come from a coordinate description. In this description, it is easy to see that the possible limits using this method are again quotients of warped AdS. Furthermore, the spacetime limits inherit suitable coordinates that are not global. In particular, we get the self-dual solution in accelerating or Poincaré coordinates, and warped AdS in Poincaré coordinates under a proper time identification.

The spacelike stretched black holes are a subset of the general black holes of cosmological Einstein-Maxwell theory with gravitational and gauge Chern-Simons couplings, which were presented in [65], with $\mu_E/\mu_G = 2/3$ and $\beta^2 = (\nu^3+3)/(4\nu^2)$. There, the causal structure of the general black holes was also first reported. We have here presented an explicit Kruskal extension, which underlies the Penrose diagram of the maximal extension. Our derivation focuses on the metric g_2 that is defined on the two-dimensional quotient space of a black hole by the global isometry ∂_θ. One can successively remove detail from our presentation but retain the reduction on g_2, since this captures the essential causal relations of the 3d spacetime.

Also in [65], local coordinate transformations were given that relate the various black holes, the self-dual solution, and the vacuum. Here, we only write the local coordinate transformations between the black holes, or the self-dual solution, and spacelike warped AdS, which precisely define the first as discrete quotients of the latter. Furthermore, the vacuum and self-dual solution are obtained here as limits of the black holes. This was done invariantly using the identification vector, but also through well-defined coordinate transformations. We note this comparison so as to highlight the structure of this work. Let us also remark that the limits we consider are classical, that is ν, G and ℓ are kept fixed. This does not allow us to obtain, for instance, the black holes with vanishing cosmological constant [79].

Let us also compare to the construction to the Banados-Teitelboim-Zanelli black holes of Einstein gravity with a negative cosmological constant. The BTZ black holes are necessarily asymptotically locally AdS (AlAdS) and so their conformal boundary is always timelike. The causal diagrams of the BTZ black holes fall into two classes, depending on whether the geometry is extremal or not [37,38].

This is different from the present case, as warped AdS backgrounds are not AlAdS and exhibit fundamentally different behaviour.

One motivation for this work was to find a non-extremal spacetime limit where the acceleration coordinate ∂_τ would explicitly depend on a parameter b. This would imply that the limit inherits two parameters rather than the one in $u = 2\pi\ell\, T_L\, \phi$. Then one could approximate the chiral thermal Green functions of the near-extremal black holes with those computed in the self-dual warped AdS space in accelerating coordinates, see [50, 80]. It is for this reason that we introduced the constant b in (4.43). By diffeomorphism invariance though, we can set this constant equal to 1. We speculate on whether a suitable set of asymptotic conditions can break this freedom.

Topological massive gravity is expected to have a rich spectrum and we believe that the solution space will present new insight in generalisations of the AdS/CFT correspondence. Indeed AlAdS solutions have been seen to be dual to logarithmic CFT theories, while even the holography of the non-AlAdS case of null-warped AdS has been studied in detail. This sets up the motivation for the next chapter, wherein we investigate a possible Ansatz for the search of new solutions to TMG.

Wer nicht überzeugen kann, sollte wenigstens Verwirrung stiften.

Chapter 5

A kinky approach to 2d-Reduced TMG

We have often mentioned how our interest in TMG solutions, and especially the warped AdS space studied in Chapter 3, was motivated mainly by the conjecture of [81] regarding a CFT dual to spacelike warped AdS_3 black holes. As a matter of fact, for a critical value of the theory, the holography of null warped AdS_3 has already been studied extensively in [82], see also [83]. However, spacelike warped AdS_3 has different asymptotics to AdS_3 or the Schrödinger background and a similar analysis cannot be made. In particular it is not asymptotically locally AdS, so techniques such as a Fefferman-Graham expansion are not applicable.

The largest class of known solutions to TMG is the Kundt class [84], which includes the TMG wave [85] and spacelike warped AdS_3; the odd one out is time-like warped AdS_3, which is not a Kundt spacetime [81]. Various other solutions can be written up to identifications with one of the above [86]. One of our motivations here was the search for an "intermediate", or "interpolating" solution between AdS_3 and spacelike warped AdS_3, for generic values of the theory, which could be relevant to the warped-AdS/CFT correspondence.

Numerical solutions that are asymptotic to warped AdS_3 were found in [87], wherein the same question as ours is posed. Our ambition was further encouraged by [88], where an interesting solution was found for the purely gravitational Chern-Simons term that appears in the TMG action. These solutions can be related at a local level to kinks with interpolating behaviour, see also [89]. The hope was to generalise their approach to include the Einstein-Hilbert action, and search for a similar solution for the full model.

In [88], the authors used a Kaluza-Klein (KK) dimensional reduction on the three-dimensional theory, to obtain a system of differential equations in 2 dimensions. For the purely Chern-Simons part of the action, one of the equations of motion is actually a conformal Killing equation on the gradient of one of the reduced fields. It is the presence of this new symmetry that allows a simple solution to the problem.

We will see below that the approach of [88] does not generalise in a simple way for the full TMG action. Recalling the classification of Pope et al. [86] and the Kundt solutions to topologically massive gravity [84], we will show that our "kinky" approach only leads to a subset of these. The symmetries imposed by the Ansatz, i.e. an isometry along which to perform the KK-reduction and an exact conformal Killing symmetry generated by the dilaton, are too restrictive to yield new solutions. The approach does however yield locally most of the known stationary axisymmetric solutions of TMG as collected in [87].

Although these solutions are not gravitational kinks, we have retained use of the word since our method is influenced by [88]. In section 5.1 we set up our notation and introduce some helpful theorems to streamline our derivation. In section 5.2 we motivate our Ansatz and in 5.3 we identify the solutions it yields. We end with concluding remarks.

5.1 Setup and notation

In this first section we derive the equations of motion of the reduced action and set up some theorems that simplify the ensuing analysis.

5.1.1 2d reduced action

We write the full TMG action as

$$16\pi G\, S[g] = \int d^3x\sqrt{-g}\left(R + \frac{2}{\ell^2} + \frac{\ell}{6\nu}\epsilon^{\lambda\mu\nu}\Gamma^{r}_{\lambda\sigma}\left(\partial_\mu\Gamma^{\sigma}_{r\nu} + \frac{2}{3}\Gamma^{\sigma}_{\mu\tau}\Gamma^{\tau}_{\nu\rho}\right)\right) \ .$$

We follow the usual KK-reduction set-up, starting with a 3-dimensional metric

$$g^{(3)} = e^{2\alpha\phi}\bar{g} \pm e^{2\phi}\left(dz + A\right)^2 \ ,$$

where we assumed the isometry $z \mapsto z+\xi$. ϕ is a function and A a one-form on the remaining two coordinates. We raise/lower the 2-dimensional tensorial indices a and b with the metric $\bar{g}_{ab}$. The $\pm$ sign distinguishes spacelike and timelike reductions. We could absorb the α parameter above into $\bar{g}$, but we choose to leave this free for now. This freedom will allow us to find various solutions from one simple Ansatz.

By D_a we denote the 2-dimensional covariant derivative and write $D^2 = D_a D^a$. The field strength $F = dA = f\,\mathrm{dvol}_{\bar{g}}$ defines the scalar f in 2 dimensions by its Hodge dual. The 3-dimensional scalar curvature R written in terms of the 2-dimensional curvature $\bar{R}$ is given by

$$R = e^{-2\alpha\phi}\bar{R} - 2\,(\alpha+1)\,e^{-2\alpha\phi}D^2\phi - 2e^{-2\alpha\phi}|d\phi|^2 + \frac{1}{2}e^{-4\alpha\phi+2\phi}f^2 \tag{5.1}$$

and the Einstein-Hilbert part of the action is therefore

$$I_{EH} = \int \mathrm{dvol}_{\bar{g}} \left[e^{\phi}\bar{R} + 2\alpha e^{\phi}|d\phi|^2 + \frac{1}{2}e^{(-2\alpha+3)\phi}f^2 - 2(1+\alpha)D_a\left(e^{\phi}D^a\phi\right)\right]. \tag{5.2}$$

To KK-reduce the Chern-Simons-like terms in the action, we make use of the results of [88]. Schematically, in their set-up ($\alpha = 1$)

$$\left(\epsilon\Gamma(\frac{1}{2}\partial\Gamma + \frac{1}{3}\Gamma\Gamma)\right) \longrightarrow -\frac{1}{2}\sqrt{-g}(F\bar{R} + F^3)\,.$$

This schematic result has to be corrected by exponential factors for generic α. This can be easily done, since the metric of [88] is conformally related to our generic one by $g_{ab} = e^{2(\alpha-1)\phi}\bar{g}_{ab}$. We thus obtain the action

$$I_{CS} = \pm\frac{1}{2\mu}\int \mathrm{dvol}_{\bar{g}} \left(e^{(-2\alpha+2)\phi}f\bar{R} - 2(\alpha-1)e^{(-2\alpha+2)\phi}fD^2\phi + e^{(-4\alpha+4)\phi}f^3\right)\ , \tag{5.3}$$

where $\mu = 3\nu/\ell$. Both parts of the action are valid for either sign of the reduction.

5.1.2 Equations of motion

We can now either vary[1] the reduced action, or reduce the 3-dimensional equations. Either way one obtains the following set of equations of motion:

$$
\begin{aligned}
\mathbf{F}: \quad c &= \pm 2\mu\, e^{(-2\alpha+3\beta)\phi} f + e^{2(1-\alpha)\phi}\bar{R} - 2(\alpha-1)e^{2(1-\alpha)\phi}D^2\phi \\
&\quad +3e^{4(1-\alpha)\phi}f^2, \\
\mathbf{Dil:}\; -\tfrac{6}{\ell^2} &= e^{-2\alpha\phi}\bar{R} - 2(\alpha+1)e^{-2\alpha\phi}D^2\phi - 2e^{-2\alpha\phi}|d\phi|^2 + \tfrac{1}{2}e^{(-4\alpha+2\beta)\phi}f^2, \\
\mathbf{Kink:}\; 0 &= D^2 e^{\phi} + \tfrac{1}{2}e^{(-2\alpha+3\beta)\phi}f^2 - \tfrac{2}{\ell^2}e^{(2\alpha+1\beta)\phi} \pm \tfrac{1}{2\mu}\Big(D^2(e^{(-2\alpha+2\beta)\phi}f) \\
&\quad + e^{(-2\alpha+2\beta)\phi}f\bar{R} - 2(\alpha-1)e^{(-2\alpha+2\beta)\phi}fD^2\phi + 2e^{(-4\alpha+4\beta)\phi}f^3\Big), \\
\mathbf{CKV:}\; 0 &= e^{2\alpha\phi}D_{(a}\left(e^{-2\alpha\phi}D_{b)}e^{\phi}\right) \pm \tfrac{1}{2\mu}e^{2(\alpha-1)\phi}D_{(a}\left(e^{-2(\alpha-1)\phi}D_{b)}(e^{-2(\alpha-1)\phi}f)\right) \\
&\quad -\bar{g}_{ab}\Big[e^{2\alpha\phi}D^c\left(e^{-2\alpha\phi}D_c e^{\phi}\right) \\
&\qquad \pm\tfrac{1}{2\mu}e^{2(\alpha-1)\phi}D^c\left(e^{-2(\alpha-1)\phi}D_c(e^{-2(\alpha-1)\phi}f)\right)\Big].
\end{aligned}
$$

In the **F** equation c is a constant of integration that a solution will fix. The **CKV** equation is the traceless part of the Einstein equation and round brackets around indices indicate symmetrization of strength one. The trace of the Einstein equation is what we call the **Kink** equation. For the dilaton equation (**Dil.**) we can equivalently use (5.1) and the constant scalar curvature $-6/\ell^2$ of the 3d geometry. The two-dimensional equations have been examined before, e.g. in the conformal gauge in [90].

These equations exhibit two types of "symmetry": a scaling of $z \mapsto \xi\, z$ and a shift of $\alpha \mapsto \alpha + \tilde{\xi}$. The former rescales the fields as $e^{\phi} \mapsto \xi^2 e^{\phi}$, $\bar{g} \mapsto \xi^{-2\alpha}\bar{g}$ and $f \mapsto \xi^{2\alpha-1}f$, and can be used to normalize c. The latter transforms fields as $\bar{g} \mapsto e^{-2\tilde{\xi}\phi}\bar{g}$ and $f \mapsto e^{2\tilde{\xi}\phi}f$, whereas it leaves ϕ unchanged. Using this, α can be fixed from the onset but, as mentioned above, we keep this freedom and let it be fixed by a consistency requirement on our Ansatz.

[1] the variation of f is $\delta f = \frac{1}{2}f g_{\mu\nu}\delta[g^{\mu\nu}] \mp \epsilon^{\mu\nu}\partial_\mu \delta A_\nu$.

5.1.3 Conformal Killing vectors

Let us now focus on the last of the above equations of motion, labeled **CKV** because of its similarity to a conformal Killing equation. In fact it contains the conformal Killing equation of [88] for $D_a(e^{-2(\alpha-1)\phi}f)$, coming from the purely Chern-Simons part of the action, but is complicated by the similar equation for $D_a e^{\phi}$ coming from the Einstein-Hilbert term. Nevertheless, this equation motivates us to search for solutions where its content is that of a single conformal Killing vector equation. This is both for simplicity, but also in the hope of finding behaviour similar to [88]. Let us first list a set of propositions that will help us in the subsequent analysis.

The next proposition will be used to fix the metric.

Proposition 5.1.1. If $\bar{g}$ has a conformal Killing one-form $d\psi$ that is non-null and exact, then the metric can be written in some coordinate system as

$$\bar{g} = \frac{d\psi(x)}{dx}(dx^2 - dt^2) \ . \tag{5.4}$$

Proof. In a conformal gauge, the metric can be written as $g = \Lambda(x,t)dudv$, where conformal Killing vectors are of the form $X = g(v)\partial_v + h(u)\partial_u$. The condition that $\bar{g}(X)$ is non null ($g(v)h(u) \neq 0$), allows us to change coordinates

$$u \mapsto \int \frac{1}{h(u)},$$
$$v \mapsto \int \frac{1}{g(v)},$$

so that $g = \Lambda(x,t)(dx^2 - dt^2)$ with $X = \partial_x$, and $\bar{g}(X) = d\psi$ implies $\Lambda = \psi'(x)$. □

Lemma 5.1.2. If $\bar{g}$ has a Conformal Killing one-form $d\psi$ that is ***null*** and exact, then the metric can be written in some coordinate system as

$$\bar{g} = (dx^2 - dt^2) \ . \tag{5.5}$$

Proof. Conformal killing vectors X_ψ will have one of the coefficients ($h(u)$ or $g(v)$) equal to zero. Pick $X_\psi = h(u)\partial_u$, so that

$$\bar{g}(X, -) = \Lambda(u,v)h(u)dv = d\psi.$$

X_ψ is null, $g(X,X)=0$, so it has to be $X_\psi = h(u)\partial_u = \partial_x \pm \partial_t$ and

$$\bar{g} = \Lambda(u,v)du\,dv = \Lambda(x,t)(dx^2 - dt^2).$$

Therefore

$$\bar{g}(X,-) = d\psi = \Lambda(x,t)\,(dx \mp dt)$$

so that

$$\psi' = \mp\dot{\psi} = \Lambda(u,v).$$

Furthermore, since $d\psi = \Lambda(u,v)h(u)dv$, we have that $\psi = \psi(v)$, $\Lambda(u,v) = \Lambda(v)$ and $h(u) = \kappa$ =const. In other words we get

$$\bar{g} = \kappa\dot{\psi}(v)du\,dv \to \bar{g} = d\tilde{u}d\tilde{v} = d\tilde{x}^2 - d\tilde{t}^2\ .$$

□

The following proposition will be needed to complete our Ansatz.

Proposition 5.1.3. Assume two non-null conformal Killing one-forms, $F_1 dF_2$ and $d\psi$, with F_1 and F_2 functions of x in (5.4). They are necessarily related by

$$F_1\,dF_2 = \tilde{k}\,d\psi$$

for some constant $\tilde{k}$.

Proof. $F_1 dF_2$ is dual to a conformal Killing vector $X = g(v)\partial_v + h(u)\partial_u$, for some functions $g(v)$ and $h(u)$. Since the left hand side of

$$F_1 dF_2 = \frac{\psi'(x)}{2}\left((g(x+t)+h(x-t))\,dx + (g(x+t)-h(x-t))\,dt\right)\ .$$

is a function of x, we have $g(x+t) = h(x-t) =$ const. . □

Finally, we have

Proposition 5.1.4. Take $d\psi$ to be the metric dual to a conformal Killing vector as before. Then in the adapted coordinates we define

$$Z = \frac{1}{\psi'}\frac{d}{dx}$$

for which

$$ZD^2\psi = -\bar{R}\ . \tag{5.6}$$

Proof. For $g = e^{2\sigma(x)}(dx^2 - dt^2)$, the Laplacian is $D^2 = e^{-2\sigma(x)}(\partial_x^2 - \partial_t^2)$. At the same time, the curvature scalar is $\bar{R} = -2e^{-2\sigma(x)}\sigma''(x)$. We substitute $\psi'(x) = e^{2\sigma(x)}$. □

Equivalent statements for a null one-form $d\psi$ can also be written, however our method for the null case does not lead to any solutions. Proposition 5.1.4 will be used as in [88] to check for the consistency of a solution.

5.2 A general Ansatz

Before moving onto a general Ansatz involving functions generating conformal Killing vectors, we glance briefly at the simplest solution to the equations of motion.

5.2.1 Constant f or ϕ

From our Kaluza-Klein Ansatz, it is clear that we can obtain known solutions to TMG by simply setting f and ϕ to constant values $f = f_0$, $\phi = \phi_0$. For simplicity, let us set here $\alpha = 1$. From the dilaton (**Dil.**) equation of motion we obtain $\bar{R}$ in terms of these constants, while the **Kink** equation becomes

$$\frac{1}{2}(e^{\phi_0} \pm \frac{3}{2\mu}f_0)(f_0 - \frac{2}{\ell}e^{\phi_0})(f_0 + \frac{2}{\ell}e^{\phi_0}) = 0\,, \tag{5.7}$$

yielding AdS_3 or warped AdS_3, respectively for $f_0 = \pm\frac{2}{\ell}e^{\phi_0}$ and $e^{\phi_0} = \mp\frac{3}{2\mu}f_0$.

Along these lines, it is interesting to note that constancy for ϕ implies the same for f, and vice versa. This can be easily checked by setting one of the two functions to a constant value and studying the equations of motion for the other.

5.2.2 The Ansatz

Let us focus again on the **CKV** equation. If we view this as the sum of two conformal Killing equations coming separately from the Einstein part and Chern-Simons part, we can only obtain the AdS_3 solution. Trying to relax this idea, we can allow for a "mixing" of the functions appearing in the two gradients. For instance, focus on the first term

$$e^{2\alpha\phi}D_a(e^{-2\alpha\phi}D_b e^{\phi}) = e^{\phi}(D_aD_b\phi + (1-2\alpha)D_a\phi D_b\phi),$$

and write out the function f as

$$f = \pm 2\mu k e^{(2\alpha-1\beta)\phi} + e^{(2\alpha-2\beta)\phi}\tilde{f} \tag{5.8}$$

for a constant k. Inserting this into the second term of the **CKV** equation yields

$$\pm \frac{1}{2\mu} e^{(2\alpha-2\beta)\phi} D_{(a}\left(e^{(-2\alpha+2\beta)\phi} D_{b)}(e^{(-2\alpha+2\beta)\phi} f)\right) =$$

$$k\, e^{\phi}(D_a D_b \phi + (-2\alpha+3) D_a\phi D_b\phi) \pm \frac{1}{2\mu} e^{(2\alpha-2\beta)\phi} D_{(a}(e^{(-2\alpha+2\beta)\phi} D_{b)}\tilde{f}). \tag{5.9}$$

The most obvious approach is to impose that $\tilde{f}$ is zero so that we are left with the conformal Killing vector equation. For $k \neq \frac{1-2\alpha}{2\alpha-3}$ the left-hand side of the equation becomes

$$e^{\phi}(1+k)\left(D_a D_b\phi + \frac{(1-2\alpha)+k(-2\alpha+3)}{1+k} D_a\phi D_b\phi\right) = e^{(1-\epsilon)\phi}\frac{1+k}{\epsilon} D_a D_b e^{\epsilon\phi},$$

when

$$\epsilon = \frac{(1-2\alpha)+k(-2\alpha+3)}{1+k}$$

is well defined and non-zero. That is, $k \neq -1$ and $k \neq \frac{1-2\alpha}{2\alpha-3}$. It is then natural to take $d\psi = de^{\epsilon\phi}$ in proposition 5.1.1. If, on the other hand, we start by imposing $d\psi = de^{\epsilon\phi}$, then $\tilde{f}$ appears in the **CKV** equation that now takes the form of a conformal Killing vector equation, and so is fixed by using the **F** equation of motion to satisfy proposition 5.1.3. This way all fields are fixed and in particular

$$f = \pm 2\mu k e^{(2\alpha-1\beta)\phi} + \tilde{k} e^{(4\alpha-4+\epsilon)\phi} + \delta e^{(2\alpha-2\beta)\phi} \quad \text{when } \epsilon \neq 2-2\alpha \Leftrightarrow k \neq 1 \tag{5.10a}$$

$$f = \pm 2\mu k e^{(2\alpha-1\beta)\phi} + e^{(2\alpha-2\beta)\phi}(\tilde{k}\phi + \delta) \quad \text{when } \epsilon = 2-2\alpha \Leftrightarrow k = 1\ . \tag{5.10b}$$

The metric one obtains by choosing the conformal Killing generator to be $\psi = e^{\epsilon\phi}$ for some α is equivalent to the one obtained by the choice $\psi = \phi$ for $\alpha' = \alpha + \epsilon/2$. Our Ansatz is thus to assume $d\psi = d\phi$ is a conformal Killing one-form. We set

$$k = \frac{1-2\alpha}{2\alpha-3} \ ,$$

and by using proposition 5.1.3, which is satisfied by the **F** equation of motion, f is given by

$$f = \pm 2\mu k e^{(2\alpha-1\beta)\phi} + \tilde{k} e^{(4\alpha-4\beta)\phi} + \delta e^{(2\alpha-2\beta)\phi} \qquad \text{if } \alpha \neq \frac{3}{2}, 1 \qquad (5.11a)$$

$$f = \pm 2\mu e^{\phi} + \tilde{k}\phi + \delta \qquad \text{if } \alpha = 1 \ , \qquad (5.11b)$$

whereas the metric is given by (5.4) with $\psi = \phi$. This way the **CKV** equation is automatically satisfied and at the same time all fields are fixed. It remains to show that the other equations of motion are satisfied for suitable values of α, $\tilde{k}$, δ and c.

5.3 Solutions

In this section we check the consistency of our Ansatz, namely which functions f and ϕ related by our Ansatz satisfy the reduced TMG equations of motion. Starting with (5.11), we use the equations of section 5.1.2 to calculate the expressions for $|d\phi|^2$, $\bar{R}$ and $D^2\phi$ in terms of ϕ. We then use proposition 5.1.4 and compare $ZD^2\phi$, that is Z acting on the expression for $D^2\phi$, with the expression for $-\bar{R}$ obtained previously. When the two expressions match, the equation for $D^2\phi$ implies that of $\bar{R}$. Finally, the consistency of the equation for $|d\phi|^2 = \phi'$ is checked by the integral of the equation for $D^2\phi = \phi''/\phi'$.

The resulting conditions are in terms of long expressions involving exponentials of ϕ, schematically

$$\sum_{(m,n)\in S} e^{(m\alpha+n\beta)\phi} \ .$$

Recall the first consistency check is an equation of the type

$$ZD^2\phi + \bar{R} = 0 \ .$$

The simplest approach is to consider all the powers to be different, $m\alpha + n \neq m'\alpha + n'$, so that their coefficients have to vanish separately. We thus obtain three cases:

1. $\delta = 0$, $\alpha = 1/2$ and $\tilde{k}, c$ unconstrained;

2. $c - \delta^2 = \tilde{k} = \alpha = 0$;

3. $\delta = c = \tilde{k} = 0$ and $\mu^2\ell^2(2\alpha+1)^2 = (2\alpha-3)^2$.

One need also check the cases when the powers mentioned above are not all different. This happens when

$$\alpha = 0, 1/2, 3/4, 1, 9/8, 7/6, 5/4, 4/3, 11/8, 5/3, 7/4, 2, 5/2, 3.$$

For each of these values we simplify the result, but again find the same three possible solutions.

The final check is to verify that the expression for $|d\phi|^2$ is also satisfied. We therefore integrate the expression for $D^2\phi$

$$D^2\phi = H(\phi) \to \phi'' = H(\phi)\phi' \to |d\phi|^2 = \phi' = \int H d\phi + d \ ,$$

for a function $H(\phi)$ of ϕ, and compare with the expression for $|d\phi|^2$. One finds that for a suitable integration constant d, the two expressions always match for the three cases above.

We will now write down and identify the three classes of solutions that can be obtained via our Ansatz.

5.3.1 Case 1: $\delta = 0$, $\alpha = 1/2$

In this case our generalised Ansatz simply becomes

$$f = \tilde{k}e^{-2\phi} \ ,$$

so that $A = -\frac{\tilde{k}}{2}e^{-2\phi}dt$. Solving the equations of motion we get

$$\begin{aligned} D^2\phi &= \tfrac{1}{2}(c \mp 2\mu\tilde{k})e^{-\phi} + \tfrac{1}{\ell^2}e^{\phi} - \tfrac{3}{4}\tilde{k}^2 e^{-3\phi}, \\ \bar{R} &= \tfrac{1}{2}(c \mp 2\mu\tilde{k})e^{-\phi} - \tfrac{1}{\ell^2}e^{\phi} - \tfrac{9}{4}\tilde{k}^2 e^{-3\phi}. \end{aligned}$$

Integrating $D^2\phi = \phi''/\phi'$

$$|d\phi|^2 = \phi' = -\frac{1}{2}(c \mp 2\mu\tilde{k})e^{-\phi} + \frac{1}{\ell^2}e^{\phi} + \frac{1}{4}\tilde{k}^2 e^{-3\phi} + d$$

and inserting into the dilaton equation (along with $D^2\phi$) we find that $d = 0$.

The 3-dimensional metric can now be written in the ϕ coordinate as

$$g = e^{\phi}\left(\frac{d\phi^2}{\frac{\tilde{k}^2}{4}e^{-3\phi}+\frac{\gamma}{2}e^{-\phi}+\frac{1}{\ell^2}e^{\phi}} \mp (\frac{\tilde{k}^2}{4}e^{-3\phi}+\frac{\gamma}{2}e^{-\phi}+\frac{1}{\ell^2}e^{\phi})dt^2\right) \pm e^{2\phi}(dz-\frac{\tilde{k}}{2}e^{-2\phi}dt)^2. \quad (5.12)$$

Identifying this and the other metrics is particularly easy due to the classification of algebraically special solutions to TMG [86]. We suspect we are dealing with constant scalar invariant spaces (CSI), after evaluating the first three curvature invariants, in which case they are CSI Kundt, locally homogeneous, or both [91, 84]. Furthermore, the Ansatz we use implies two commuting symmetries ∂_t and ∂_z. To identify which particular Petrov-Segre class we are in, we study the Jordan normal form of the tensor

$$S_a{}^b = R_a{}^b - \frac{1}{3}R\delta_a{}^b.$$

For Case 1, the canonical $S_a{}^b$ turns out to be identically zero, i.e. the solution is of Petrov class O, corresponding to locally AdS_3.

5.3.2 Case 2: $c-\delta^2=\tilde{k}=\alpha=0$

The Ansatz here boils down to

$$f = \mp\frac{2\mu}{3}e^{-\phi}+\delta e^{-2\phi},$$

so that $A=\left(\pm\frac{2\mu}{3}e^{-\phi}-\frac{1}{2}\delta e^{-2\phi}\right)dt$. From the equations of motion we obtain that

$$\phi' = \mp\frac{2}{3}\mu\delta e^{-\phi}+\frac{1}{4}\delta^2e^{-2\phi}+d.$$

Furthermore we get that $d=\frac{3}{\ell^2}+\frac{1}{9}\mu^2$ and

$$\bar{R} = -\delta^2e^{-2\phi}\pm\frac{2}{3}\mu\delta e^{-\phi}.$$

The full 3-dimensional metric is then

$$g_3 = \frac{1}{\mp\frac{2}{3}\mu\delta e^{-\phi} + \frac{1}{4}\delta^2 e^{-2\phi} + d} d\phi^2 + \left(\pm\frac{2}{3}\mu\delta e^{-\phi} - \frac{1}{4}\delta^2 e^{-2\phi} - d\right) dt^2 + \left(e^{\phi} dz + \left(\pm\frac{2}{3}\mu - \frac{1}{2}\delta e^{-\phi}\right) dt\right)^2 . \tag{5.13}$$

For this solution, the canonical $S_a{}^b$ is given by

$$S_a{}^b = \begin{pmatrix} -\frac{2(\nu^2-1)}{\ell^2} & 0 & 0 \\ 0 & \frac{\nu^2-1}{\ell^2} & 0 \\ 0 & 0 & \frac{\nu^2-1}{\ell^2} \end{pmatrix},$$

placing it into Petrov class D, whence by the theorem in [86] it is locally spacelike or timelike warped AdS_3. In fact, case 2 covers both spacelike and timelike stretching. One can easily find a diffeomorphism that will bring the metric to one of the standard forms

$$g = \frac{\ell^2}{\nu^2+3}\left(\frac{dy^2}{y^2-\delta} \mp \left(y^2-\delta\right) du^2 \pm \frac{4\nu^2}{\nu^2+3}\left(d\tilde{t} + y du\right)^2\right) , \tag{5.14}$$

where the two values $\delta = 0, 1$ are isometric. The sign above distinguishes spacelike and timelike stretching and is the same as the one we used to distinguish between spacelike or timelike KK reduction.

5.3.3 Case 3: $\delta = c = \tilde{k} = 0$ and $\mu^2\ell^2(2\alpha+1)^2 = (2\alpha-3)^2$

The general Ansatz here is

$$f = \pm 2\mu \frac{1-2\alpha}{2\alpha-3} e^{(2\alpha-1\beta)\phi},$$

so that $A = \mp\frac{2\mu}{2\alpha-3} e^{(2\alpha-1\beta)\phi} dt$. The equations of motion here yield that

$$D^2\phi = 2\left(\frac{\alpha}{\ell^2} + \mu^2 \frac{(1-2\alpha)(2\alpha^2+3\alpha)}{(2\alpha-3)^2}\right) e^{2\alpha\phi} .$$

Consistency with the dilaton equation requires a $d = 0$ integration constant and

$$\phi' = \frac{4\mu^2}{(2\alpha-3)^2} e^{2\alpha\phi} .$$

The 3-dimensional metric is therefore given by

$$g = e^{2\alpha\phi}\left(\frac{d\phi^2}{\frac{4\mu^2}{(2\alpha-3)^2}e^{2\alpha\phi}} \mp \frac{4\mu^2}{(2\alpha-3)^2}e^{2\alpha\phi}dt^2\right) \pm e^{2\phi}\left(dz \mp \frac{2\mu}{2\alpha-3}e^{(2\alpha-1\beta)\phi}dt\right)^2 . \tag{5.15}$$

Again, to identify this solution we look for the Jordan normal form of the traceless Ricci tensor $S_a{}^b$, which in this case is

$$S_a{}^b = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} ,$$

corresponding to the Petrov class N. When $\nu \neq \pm 1/3$, a coordinate transformation can bring the metric to the form of an AdS pp-wave

$$g = \frac{\ell^2}{4}\frac{d\rho^2}{\rho^2} + s_1 \rho^{\frac{1}{2}(1-3\nu\, s_2)} dz^2 + \rho\, dz\, dt \ , \tag{5.16}$$

where s_1 and s_2 are uncorrelated signs. The sign s_1 keeps track of the sign of the KK reduction we used and the sign s_2 comes from the two possible solutions for α. When $\nu = \pm 1/3$, the solution for α is unique and our metric becomes that of AdS_3 in Poincaré coordinates.

The pp-wave (5.16) then corresponds to a TMG wave [85] with two commuting symmetries. It is locally isometric[2] to the Schrödinger sector solutions of [87, §4.2] for their $b = 0$, which were found and their causal structure analyzed in [92]. Our Ansatz is thus seen to reproduce locally all known stationary axisymmetric solutions to TMG [87] for generic values ℓ and ν, with the exception of the $b \neq 0$ in [87, §4.2].

5.4 Conclusion

The search for new solutions to TMG has lead us to exploring the power and range of the "kinky" approach to 3d-gravity as used in [88]. The idea of using an exact conformal Killing vector to simplify the reduced two-dimensional equations of motion seems very effective in leading to a whole range of possible solutions

[2] the diffeomorphism in [85] has an arbitrary function $f_1(z)$ that here should be a constant, see also the appendix in [83].

depending on a small set of parameters. However, the theorems we used impose strong restrictions on the Ansatz. A subset of valid parameter values is selected that corresponds to the already well-known and studied solutions of locally AdS_3, warped AdS_3 and the pp-wave.

Appealing as the Kinky Ansatz may look, it requires too much symmetry to yield any novel solutions. Nonetheless, this is a new, simplified way to obtain the most symmetric TMG backgrounds. We note how a simple and local Ansatz can reproduce a large class of the known stationary axisymmetric solutions in [87], without assumptions on the asymptotics. In this setting, the relationship between these is in terms of the functional dependence of the generator of a conformal isometry.

Our Case 3 corresponds to the special case of the family $W_1 = -2/\ell$ of type N CSI Kundt solutions where the $f_{01}(u)$ in [84] is constant. In this way, their general solution acquires an extra isometry, which is precisely what our Ansatz requires. One might wonder whether our Ansatz can be generalized to include other deformations of AdS_3 and warped AdS_3. Another natural question is whether the core idea behind this Ansatz, which was to automatically satisfy the traceless part of the Einstein equation, can be useful in studying other gravitational systems.

Summary and Outlook

This second part of the thesis was dedicated entirely to topologically massive gravity and solutions to it exhibiting isometry groups smaller than the maximal SO(2,2). In particular we dedicated one chapter (3) to a detailed study of warped AdS_3 space, to familiarise with the various possible ways of parametrisation. We then set up the quotient construction for warped AdS_3 black hole solutions to TMG in chapter 4: we saw how the identification procedure can lead to closed time-like curves that can be "naked", or hidden behind event horizons. We also explained in detail that the temperatures T_L and T_R, appearing as constant coefficients in the quotient construction, are unambiguous parameters by which to identify the black holes (in contrast with the horizon radii r_+ and r_-). We proceeded to a full layout of the causal structure of such solutions and gave a quick review of their thermodynamics.

Chapter 4 also gives an in-depth study of the near-horizon geometries of warped AdS_3 black holes. The analysis takes place in parameter space, where the near horizon behaviour is obtained by taking specific limits of the identifying parameters T_L and T_R. The salient point of these two chapters is the following: via an appropriate choice of parametrization, keeping a specific Killing symmetry "direction" ∂_τ manifest, and subsequently quotienting along a similar isometry (spacelike quotient for a spacelike ∂_τ etc.) yields the corresponding black hole solution. For example: choosing a spacelike ∂_τ, i.e. accelerating coordinates with two apparent horizons, and a spacelike quotient direction, one obtains the metric of a non-extremal black hole. Furthermore, zooming in towards the horizons gives us back a metric in the originally chosen coordinates, albeit a further identification yielding the "self-dual" version of these spaces. This is the merit of our construction and our setup. The analysis of locally warped AdS_3 solutions of TMG and its salient properties can be very easily laid out and understood from a careful geometrical discussion, toward the most obvious coordinate choices, via appropriate quotients, leading all the way to the near-horizon properties of warped black holes.

We further dedicated chapter 5 to our first attempt at finding new solutions to topologically massive gravity. We started with a KK dimensional reductions and hoped to be able to use a conformal isometry on the resulting 2-dimensional space to simplify the equations of motion. Ultimately we were aiming toward a “kink”-like solution, a metric that would interpolate in some way between the known locally AdS_3 solutions and warped AdS_3. As usual, we were motivated by our wish to formulate a holography conjecture for warped AdS_3 space. We recall this, so as to motivate the following, describing ideas for future related work.

One possible project emerging from this work, is to continue the above analysis for other, less symmetric backgrounds. Clearly this will not be straightforward, as one is dealing with non asymptotically locally AdS spaces, so finding the asymptotic boundary metric is a highly non-trivial problem. Since we are familiar with TMG and spacelike warped AdS solutions, one could start with this case. A lot of work has been done concerning the asymptotics of different TMG backgrounds (e.g. [93]), and in particular in [51] the authors give a set of consistent boundary conditions for space- and timelike warped AdS_3. Their analysis furthermore shows that the conserved charges are finite and integrable. One could also try to compute 1- and 2-point functions on the gravity side from the on-shell action, hoping to reproduce results from a recognizable gauge theory. This step is clearly far from obvious and probably such a gauge theory would have to be set up anew with the information that is known: the symmetry structure, the sources for the operators and the expected values of the 1- and 2-point correlators.

Within TMG, we are also interested in understanding some of the other known solutions, referred to as deformed AdS_3 and deformed warped AdS_3 spaces (see [86]). As we saw in section 5.3, these belong to the set of Kundt CSI solutions to TMG. One would first of all have to understand in what sense they are to be viewed as deformations of the more symmetric solutions, and subsequently try to understand their asymptotic structure. Here asymptotic boundary conditions have yet to be defined. Also, it has to be understood whether the most general conditions will yield finite charges, or if some sort of renormalisation has to take place. Preliminary results indicate that relaxed asymptotic conditions with finite charges do exist. Setting up the conditions will require a detailed analysis of all the symmetries and properties of these solutions, but the charge discussion will be a purely computational task. If there is anything conceptually well defined about the deformed adjective, then this could give some intuition about eventually deforming the known dual theories to TMG towards a gauge theory corresponding to these types of solutions. Considering the rather large class of Kundt CSI backgrounds in the bulk, one might be justified in expecting to find a corresponding "class" of dual theories. According to how many parameters are switched on or off, and how much symmetry is required, this class may split into subsectors: the most symmetric being CFTs, and the next in line being logarithmic CFT.

Summarising, my recent interests focus on gauge-gravity duality. I choose to approach the topic via 3-dimensional gravity theories to simplify the analyses without trivialising the problem, and Part II of this thesis is the result of my first steps in this direction.

List of Figures

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