

SOME ASPECTS OF CURRENT ALGEBRAS AND DISPERSION SUM RULES

Thesis

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CHAPTER I

INTRODUCTION

1. General Introductory Remarks

Symmetries with their impelling manifestations have always fascinated the enquiring mind. By their very innate nature of relating the structures existing in the universe they have been invaluable in the formulation of natural laws and in their generalizations. However, it is also obvious that nature prefers beauty at the cost of perfect symmetry. The subtle way that nature breaks the symmetries to generate perfect beauty is, perhaps, also one of the hardest things to comprehend, and this makes the formulation of natural laws more difficult and their generalizations even harder.

The symmetries of the world, as realized by a high-energy physicist, may be summed up in the following words of Sidney Coleman⁽¹⁾,

"The symmetries of the world form a group of unitary transformations that turn one particle states into one particle states, transform many particle states as if they were tensor products, and commute with the S-matrix, and further the connected part of the group is locally isomorphic to the direct product of the connected part of the Poincaré group and the group of internal symmetries." The seemingly exact and universal nature of Gell-Mann - Okubo mass formulae and the initial successes of non-relativistic $U(6)$ symmetry group naturally dawned as a cheerful prospect of realizing

the internal and space-time symmetries in a unified and non-trivial way. However, it soon became clear that any symmetry group that satisfied all the above criteria and contained within it the internal symmetry group and the Poincaré group in a non-trivial way was infested invariably with internal inconsistencies. A way out of this impasse was suggested by Dashen and Gell-Mann which, after repeated re-interpretations and refinements, has been the most successful theory of the present day high energy physics. In the next sections we shall briefly outline some of the developments of this theory and its ramifications going by the name of current algebras.

2. CVC, PCAC and the Adler-Weisberger Sum Rule

Current algebras use extensively the results of the S-matrix approach formulated in the Heisenberg picture of quantum field theory. The current operators used are local operators with well-defined matrix elements between physical states. Interactions between particles are conveniently expressed as products of currents.

Thus, in the Fermi theory of nuclear β -decay, the interaction Lagrangian is the product of the weak currents of the hadrons and the leptons. In the more recent formulation of this theory (universal V-A theory⁽²⁾), each of these currents is composed of a vector and an axial-vector part. If we work to lowest order in weak and electromagnetic interactions and to all orders in strong interactions, the leptonic weak current $J_{W,\ell}^\mu$ has a general representation in terms of electron, muon and neutrino fields as follows,

$$J_{W,e}^{\mu}(\alpha) = [\psi_e(\alpha)]^{\dagger} \gamma^0 \gamma^{\mu} (1 + \gamma^5) \psi_e(\alpha) + [\psi_{\mu}(\alpha)]^{\dagger} \gamma^0 \gamma^{\mu} (1 + \gamma^5) \psi_{\nu\mu}(\alpha), \quad (1.1)$$

but the hadronic weak current J_W^{μ} has a very complex structure in terms of all the strongly interacting particles, such that its matrix element has the form (in the case of $N \rightarrow P + e^{-} + \bar{\nu}_e$)

$$\begin{aligned} \langle P(p^f) | J_W^{\mu}(0) | N(p^i) \rangle = & \bar{u}_P(p^f) \left\{ F_{1,PN}^W(q^2) \gamma^{\mu} + F_{2,PN}^W(q^2) \sigma^{\mu\nu} (p^i - p^f)_{\nu} \right. \\ & + F_{3,PN}^W(q^2) (p^i - p^f)^{\mu} + G_{1,PN}^W(q^2) \gamma^5 \gamma^{\mu} \\ & \left. + G_{2,PN}^W(q^2) \gamma^5 \sigma^{\mu\nu} (p^i - p^f)_{\nu} + G_{3,PN}^W(q^2) \gamma^5 (p^i - p^f)^{\mu} \right\} u_N(p^i) \end{aligned} \quad (1.2)$$

where $q^2 = (p^i - p^f)^2$, and the form-factors take care of the meson clouds of the nucleons. If G_{μ} , G_V , G_A are respectively the weak coupling constants for μ -decay and for the hadronic vector and axial-vector parts of nuclear β -decay, i.e.,

$$\mathcal{L}(\mu^{-} \rightarrow e^{-} + \bar{\nu}_e + \nu_{\mu}) = \frac{G_{\mu}}{\sqrt{2}} J_{W,e}^{\mu} J_{W,\ell\mu}^{\dagger}, \quad (1.3)$$

and

$$\mathcal{L}(N \rightarrow P + e^{-} + \bar{\nu}_e) = \frac{1}{\sqrt{2}} J_{W,e}^{\mu} \left[G_V J_{W,\nu\mu}^{\dagger} + G_A J_{W,A\mu}^{\dagger} \right], \quad (1.4)$$

then, experimentally,

$$G_{\mu} \approx G_V \quad \text{and} \quad |G_V| \neq |G_A|$$

The corrections due to electromagnetic interactions (e.g., radiative corrections) increased the discrepancy between G_V and G_{μ} but only by a small amount. The near equality of G_V and G_{μ} (the fact that G_V is indeed not exactly equal to G_{μ} gave rise to the Cabibbo version of universality⁽³⁾ to be discussed later in this section), in spite of the meson cloud of the nucleon, encouraged Gershtein and Zeldovitch⁽²⁾ and Feynman and Gell-Mann⁽²⁾ to suggest the conserved vector current (CVC) hypothesis. According to this hypothesis the hadronic weak vector current, its hermitian conjugate and the isovector part of the electromagnetic current constitute an isotriplet of currents and the corresponding charges defined by the space integrals of their time components are just the generators of the isospin group.

Symbolically, in an hermitian basis,

$$I_i = \int J_{W,Vi}^0(x) d^3x, \quad (1.5)$$

$$[I_i, I_j] = i \epsilon_{ijk} I_k, \quad (1.6)$$

$$[I_i, J_{W,Vj}^{\mu}(x)] = i \epsilon_{ijk} J_{W,Vk}^{\mu}(x), \quad (1.7)$$

where $i, j, k = 1, 2, 3$.

This hypothesis then implies that $G_V = G_\mu$. The reason is that, for strong interactions, isospin is a good quantum number, and hence the currents are conserved. This would imply that G_V should not be affected by the presence of meson clouds (it is unrenormalized), and is, therefore, just equal to G_μ . This is analogous to the case of electric charge, the electron charge being equal to the proton charge (up to a sign), which is due to the conservation of the electromagnetic current, and, therefore, due to gauge invariance of the theory. Thus isotopic spin may be visualized in two different ways: (i) As a conventional symmetry group of transformations such that its generators obey the usual Lie algebra commutation relations

$$[I_i, I_j] = i\epsilon_{ijk} I_k \quad (1.6)$$

and the strong interactions are invariant under these transformations. Strong interactions are, therefore, characterised by a conserved quantum number, the isotopic spin, which corresponds to an invariant of the group. (ii) Alternatively, we may identify the space integrals of the time-components of the hadronic weak currents and the isovector part of the electromagnetic current with the generators of the isospin group so that these charges satisfy the usual commutation relations of SU(2) algebra and the currents transform as an isovector under this algebra as implied in eqs. (1.5), (1.6) and (1.7) given above. This identification is possible whether isospin is a good quantum number or not. The fact that isospin is a good quantum number, strong interactions being invariant under isospin transformations, implies that the

charges are time independent and the hadronic currents are conserved. In general, however, we may always postulate commutation relations at equal times between time dependent charge operators constructed out of nonconserved currents. If an algebra is closed in this way it will not correspond to a symmetry of the strong interactions unless further dynamical assumptions are made. In this way a host of algebraic relations between physical currents and charges may be obtained, and the symmetry (or partial symmetry) aspects of these relations are to be inserted as further dynamical assumptions (such as the saturation hypothesis used in Chapter II). This is essentially the principle involved in the theory of current algebras.

An important consequence of CVC theory is that it relates the electromagnetic and the weak form-factors of the hadronic vector current. The matrix element of the electromagnetic current between nucleon states is given by, ^{*}

$$\langle n(p^f) | j_{\mu}^{elm.}(x) | n(p^i) \rangle = e^{-i(p^f - p^i) \cdot x} \bar{u}(p^f) \left[i\gamma_{\mu} F_1^{elm.}(q^2) + i\sigma_{\mu\nu} (p^i - p^f)^{\nu} F_2^{elm.}(q^2) \right] u(p^i),$$

(1.8)

where $q^2 = (p^f - p^i)^2$, and $F_1^{elm.}$ and $F_2^{elm.}$ are Dirac-Pauli form-factors normalized such that $F_1^{elm.}(0)$ is the total charge of the nucleon (we have set $e = 1$) and $F_2^{elm.}(0)$ is the anomalous magnetic moment μ^A of the nucleon (in units of $\frac{e}{2m_p}$). CVC theory implies that $F_1, \text{PN}^W(0) = 1$, $F_2, \text{PN}^W(0) = \mu_P^A - \mu_N^A$ and

* total electromagnetic current (isoscalar + isovector).

$F_3, \frac{W}{PN}(q^2) = 0$ (the last one being true also by G-parity invariance).

The axial-vector current is not conserved, and therefore G_A is different from G_μ . The corresponding charges defined by,

$$\bar{I}_i(x^0) = \int \mathcal{J}_{W,A}^0(x) d^3x, \quad (1.9)$$

are not constants of motion, but are time-dependent. However, we may write equal-time commutation relations for them, e.g.,

$$[\bar{I}_i(t), \bar{I}_j(t)] = i\epsilon_{ijk} \bar{I}_k(t), \quad (1.10)$$

$$[\bar{I}_i(t), I_j] = i\epsilon_{ijk} \bar{I}_k(t). \quad (1.11)$$

These equations may be written from analogy with leptonic currents and they also follow from a quark model. We may consider them to be postulates of current algebra (the second of these equations, e.g. eq. (1.11) is, of course, always true, the axial-vector charges being isovectors) to be verified later by experiment. The first equation (eq. (1.10)), being nonlinear in \bar{I}_i , is useful for determining the axial-vector renormalization constant; it fixes along with eqs. (1.5), (1.6) and (1.7) the scale of the weak current. We shall presently see how eq. (1.10) leads to the Adler-Weisberger relation for the axial-vector renormalization constant ($-G_A/G_V$). As stated before, these commutation relations do not give any new information regarding the symmetry of the strong interactions apart

from the one already contained in SU(2) invariance due to conservation of isotopic spin. Instead of considering the vector and axial-vector charges separately we may construct the operators

$$x_i^\pm = I_i \pm \bar{I}_i \quad , \quad (1.12)$$

each of which generates an SU(2) algebra,

$$[x_i^\pm, x_j^\pm] = i\epsilon_{ijk} x_k^\pm \quad , \quad (1.13)$$

and commutes with the other,

$$[x_i^+, x_j^-] = 0 \quad . \quad (1.14)$$

Therefore, x_i^+ and x_i^- generate the chiral algebra $SU(2) \otimes SU(2)$ whose representations may be labelled by $[0, \frac{1}{2}]$, $[\frac{1}{2}, \frac{1}{2}]$ etc. However these representations are not invariant under parity transformations, since

$$P x_i^\pm P^{-1} = x_i^\mp \quad . \quad (1.15)$$

Only the representations of the algebra $SU(2) \otimes SU(2) \otimes P$, e.g. $[\frac{1}{2}, \frac{1}{2}]$, $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 0]$, etc. are parity invariant. One may postulate invariance of strong interactions under such chiral transformations. Such a symmetry group and its various generalizations have been discussed in the literature⁽⁴⁾. As in any other symmetry scheme the hadrons are assigned to definite irreducible

representations of the group in question. However, such classification schemes have not been very successful because of rather large symmetry breakings involved. We shall not pursue this point any further. A somewhat related approach is to make certain saturation assumptions and obtain information from matrix elements of the commutators. This technique is the subject of the discussion given in Chapter II, and is the way in which dynamical assumptions are imposed upon current algebra to obtain certain symmetry results.

At the present state of our knowledge of strong and weak interactions, eq. (1.10) cannot by itself give us an expression for $-G_A/G_V$ that can be checked experimentally. To do so we need the assumption of PCAC, which we shall discuss next. Since the divergence of the axial-vector current has the same quantum numbers as the pion, it is physically meaningful to use it as an interpolating field for the pion, i.e.,

$$\langle 0 | \int d^3x f_{\underline{k}}^*(x) \bar{D}^i(x) | \pi^j(k') \rangle = \text{const. } \delta^{ij} \delta^3(\underline{k} - \underline{k}') \quad (1.16)$$

where $f_{\underline{k}}(x)$ is a wave-packet with momentum centred around \underline{k} and which satisfies

$$[m_\pi^2 - \square_x] f_{\underline{k}}(x) = 0 \quad (1.17)$$

The constant in eq. (1.16) is related to the pion decay constant, f_π , defined by

$$\langle 0 | A_\mu^i(x) | \pi^j(k) \rangle = i f_\pi \delta^{ij} k_\mu e^{i k \cdot x} \quad (1.18)$$

so that

$$\langle 0 | \bar{D}^i(x) | \pi^j(k) \rangle = f_\pi m_\pi^2 \delta^{ij} \quad (1.19)$$

The decay constant, f_π , is found to be ≈ 94 Mev from the π_{l2} decays. The matrix elements of \bar{D} between single-particle states are analytic functions of the momentum transfer squared variable, t , except for a pole at the pion mass and cuts starting at different branch points on the real axis of t corresponding to physical thresholds for many particle channels (e.g., 3 pions, 5 pions, etc.). The assumption of PCAC (or PDDAC) is that for t in the neighbourhood of the pion pole the matrix elements of \bar{D} are dominated by the pion pole, all other contributions from higher singularities being negligible*. In the derivation of the Adler-Weisberger sum rule, it is assumed that the pion pole dominance assumption is valid down to $t = 0$, and both the pions in the (πN) scattering cross-sections that are relevant there are considered in the limit of vanishing four-momentum (the soft pion limit). Apparently such an assumption is reasonable in view of the fact that the next important singularity of \bar{D} after the pion pole starts at $t = 9m_\pi^2$ (the threshold for the 3π -cut contribution) which is relatively far from the origin, and, therefore, may have a negligibly small effect there. This seems to be borne out by experiments. However, recent investigations by Brown and West⁽⁵⁾ seems to suggest that one might require subtraction constants besides the pion pole in the matrix elements of \bar{D} for $t \approx m_\pi^2$. This point will be discussed further in Section 5 of this chapter.

We now illustrate Fubini's covariant method of doing current

* Actually, the PCAC assumption requires that, in the limit $m_\pi^2 \rightarrow 0$, not only $\langle 0 | \bar{D} | \pi \rangle \rightarrow 0$ but also \bar{D} itself tends to zero (Ref. 46).

algebra calculations by outlining the derivation of Adler-Weisberger sum rule for the axial-vector renormalization constant $g_A (= -G_A/G_V)$. Consider the integrals

$$T_{\nu\mu} = i \int d^4x e^{-ik_2 \cdot x} \langle P_2 | \theta(x^0) [\bar{J}_\nu^j(x), \bar{J}_\mu^i(0)] | P_1 \rangle, \quad (1.20a)$$

$$U_\mu = i \int d^4x e^{-ik_2 \cdot x} \langle P_2 | \theta(x^0) [\bar{D}^j(x), \bar{J}_\mu^i(0)] | P_1 \rangle, \quad (1.20b)$$

$$U'_\nu = i \int d^4x e^{-ik_2 \cdot x} \langle P_2 | \theta(x^0) [\bar{J}_\nu^j(x), \bar{D}^i(0)] | P_1 \rangle, \quad (1.20c)$$

$$V = i \int d^4x e^{-ik_2 \cdot x} \langle P_2 | \theta(x^0) [\bar{D}^j(x), \bar{D}^i(0)] | P_1 \rangle, \quad (1.20d)$$

$$t_{\nu\mu} = \frac{1}{2} \int d^4x e^{-ik_2 \cdot x} \langle P_2 | [\bar{J}_\nu^j(x), \bar{J}_\mu^i(0)] | P_1 \rangle, \quad (1.20e)$$

$$U_\mu = \frac{1}{2} \int d^4x e^{-ik_2 \cdot x} \langle P_2 | [\bar{D}^j(x), \bar{J}_\mu^i(0)] | P_1 \rangle, \quad (1.20f)$$

$$U'_\nu = \frac{1}{2} \int d^4x e^{-ik_2 \cdot x} \langle P_2 | [\bar{J}_\nu^j(x), \bar{D}^i(0)] | P_1 \rangle, \quad (1.20g)$$

$$V = \frac{1}{2} \int d^4x e^{-ik_2 \cdot x} \langle P_2 | [\bar{D}^j(x), \bar{D}^i(0)] | P_1 \rangle, \quad (1.20h)$$

where \bar{J} 's are axial-vector currents, and \bar{D} 's their divergences

$\langle p_2 |$ and $| p_1 \rangle$ are spin averaged nucleon states with momenta p_2 and p_1 respectively. We use the commutation relations for the currents,

$$[\bar{j}_0^j(x), \bar{j}_\mu^i(0)]_{x^0=0} = i \epsilon^{jik} j_\mu^k(0) \delta^3(\underline{x}) \quad (1.21)$$

$$[\bar{j}_0^j(x), \bar{D}^i(0)]_{x^0=0} \stackrel{def}{=} i s^{ji} \delta^3(\underline{x}) \quad (1.22)$$

where we have ignored gradient terms. The gradient terms in eqs. (1.21) and (1.22) and the covariance difficulties of eqs. (1.20a) to (1.20d) can be taken into account correctly⁽⁶⁾, and the following results are free from ambiguities arising from them. Partial integrations of eqs. (1.20a) to (1.20h) give

$$i k_2^\nu T_{\nu\mu} - U_\mu - \epsilon_{jik} \langle p_2 | j_\mu^k(0) | p_1 \rangle = 0 \quad , \quad (1.23a)$$

$$i k_2^\nu U'_\nu - V - \langle p_2 | s^{ji}(0) | p_1 \rangle = 0 \quad , \quad (1.23b)$$

$$i k_1^\mu T_{\nu\mu} + U'_\nu - \epsilon_{jik} \langle p_2 | j_\nu^k(0) | p_1 \rangle = 0 \quad , \quad (1.23c)$$

$$i k_1^\mu U_\mu + V + \langle p_2 | s^{ji}(0) | p_1 \rangle = 0 \quad , \quad (1.23d)$$

* We define $k_1 = p_2 + k_2 - p_1$.

$$i k_2^\nu t_{\nu\mu} - u_\mu = 0 \quad (1.23e)$$

$$i k_2^\nu u_\nu - v = 0 \quad (1.23f)$$

$$i k_1^\mu t_{\nu\mu} + u_\nu = 0 \quad (1.23g)$$

$$i k_1^\mu u_\mu + v = 0 \quad (1.23h)$$

We next expand $T_{\nu\mu}$ etc. in terms of invariant "amplitudes",

$$\begin{aligned} T_{\nu\mu} = & A P_\nu P_\mu + B^{(1)} P_\nu k_{1\mu} + B^{(2)} P_\nu k_{2\mu} + B^{(3)} k_{1\nu} P_\mu + B^{(4)} k_{2\nu} P_\mu \\ & + C^{(1)} k_{1\nu} k_{1\mu} + C^{(2)} k_{2\nu} k_{2\mu} + C^{(3)} k_{1\nu} k_{2\mu} + C^{(4)} k_{2\nu} k_{1\mu} + C^{(5)} g_{\nu\mu}, \end{aligned} \quad (1.24a)$$

$$t_{\nu\mu} = a P_\nu P_\mu + \dots \quad (1.24b)$$

$$u_\nu = L' P_\nu + M' k_{1\nu} + N' k_{2\nu} \quad (1.24c)$$

$$U_\mu = L P_\mu + \dots \quad (1.24d)$$

$$u'_\nu = \ell' P_\nu + \dots \quad (1.24e)$$

$$u_\mu = \ell P_\mu + \dots \quad (1.24f)$$

The amplitudes $A, B^{(1)}, \dots$ are functions of the invariants ν, t, k_1^2, k_2^2 defined as follows

$$P_\mu = \frac{1}{2} (p_1 + p_2)_\mu, \quad K_\mu = \frac{1}{2} (k_1 + k_2)_\mu, \quad \Delta_\mu = (p_2 - p_1)_\mu = (k_1 - k_2)_\mu,$$

$$t = -\Delta^2, \quad \nu = P \cdot K = P \cdot k_1 + \frac{1}{4} (-m_1^2 + m_2^2) \\ = P \cdot k_2 + \frac{1}{4} (m_1^2 - m_2^2). \quad (1.25)$$

Using the expansions given by eqs. (1.24a) to (1.24f) in eqs. (1.23a) to (1.23h) and comparing coefficients, we obtain,

Coefficient of P_μ :

$$i (k_2 \cdot P A + k_2 \cdot k_1 B^{(3)} + k_2^2 B^{(4)}) - L - 2 \epsilon_{jik} F_1^k(t) = 0, \quad (1.26a)$$

Coefficient of $k_{1\mu}$:

$$i (k_2 \cdot P B^{(1)} + k_2 \cdot k_1 C^{(1)} + k_2^2 C^{(4)}) - M - \epsilon_{jik} F_2^k(t) = 0, \quad (1.26b)$$

Coefficient of P_ν :

$$i (k_1 \cdot P A + k_1^2 B^{(1)} + k_1 \cdot k_2 B^{(2)}) + L' - 2 \epsilon_{jik} F_1^k(t) = 0, \quad (1.26c)$$

Coefficient of $k_{2\mu}$:

$$i (K_2 \cdot P B^{(2)} + K_2^2 C^{(2)} + K_2 \cdot K_1 C^{(3)}) - N + \epsilon_{jik} F_2^R(t) + iC^{(5)} = 0, \quad (1.26d)$$

Coefficient of $k_{1\nu}$:

$$i (K_1 \cdot P B^{(3)} + K_1^2 C^{(1)} + K_1 \cdot K_2 C^{(3)}) + M' - \epsilon_{jik} F_2^R(t) + iC^{(5)} = 0, \quad (1.26e)$$

Coefficient of $k_{2\nu}$:

$$i (K_1 \cdot P B^{(4)} + K_1 \cdot K_2 C^{(2)} + K_1^2 C^{(4)}) + \epsilon_{jik} F_2^R(t) + N' = 0, \quad (1.26f)$$

Also,

$$i L' K_2 \cdot P + iM' K_2 \cdot K_1 + iN' K_2^2 - V - \langle P_2 | S^{\beta\alpha} | P_1 \rangle = 0, \quad (1.26g)$$

$$i L K_1 \cdot P + iM K_1^2 + iN K_1 \cdot K_2 + V + \langle P_2 | S^{\beta\alpha} | P_1 \rangle = 0. \quad (1.26h)$$

We have used above the form-factors defined by

$$\langle P_2 | j_\mu^R(0) | P_1 \rangle = (P_2 + P_1)_\mu F_1^R(t) + (P_2 - P_1)_\mu F_2^R(t). \quad (1.27)$$

Actually, if the external states are spin averaged nucleon states of equal momenta and if j_μ^k is a vector current as in the Adler-Weisberger case, we have

$$\langle P | j_\mu^3(0) | P \rangle = 2p_\mu. \quad (1.28)$$

Equations similar to (1.26a) etc. without the form-factor terms

are obtained for $a, b^{(1)}$, etc. Assuming unsubtracted dispersion relations for $A, B^{(1)}, \dots$ at fixed space-like t, k_1^2, k_2^2 , we obtain,

$$\frac{1}{\pi} \int_{-\infty}^{\infty} a^{ji}(\nu', t, k_1^2, k_2^2) d\nu' = 2i \epsilon_{jik} F_1^R(t) \quad , \quad (1.29a)$$

$$\frac{1}{\pi} \int_{-\infty}^{\infty} b^{(1)ji}(\nu', t, k_1^2, k_2^2) d\nu' = i \epsilon_{jik} F_2^R(t) \quad , \quad (1.29b)$$

$$\frac{1}{\pi} \int_{-\infty}^{\infty} b^{(2)ji}(\nu', t, k_1^2, k_2^2) d\nu' = -i \epsilon_{jik} F_2^R(t) \quad , \quad (1.29c)$$

$$\frac{1}{\pi} \int_{-\infty}^{\infty} b^{(3)ji}(\nu', t, k_1^2, k_2^2) d\nu' = i \epsilon_{jik} F_2^R(t) \quad , \quad (1.29d)$$

$$\frac{1}{\pi} \int_{-\infty}^{\infty} b^{(4)ji}(\nu', t, k_1^2, k_2^2) d\nu' = -i \epsilon_{jik} F_2^R(t) \quad , \quad (1.29e)$$

and also,

$$\frac{1}{\pi} \int l_{ji}'(\nu', t, k_1^2, k_2^2) d\nu' = i \langle P_2 | S^{ji}(0) | P_1 \rangle \quad , \quad (1.29f)$$

$$\frac{1}{\pi} \int l_{ji}(\nu', t, k_1^2, k_2^2) d\nu' = -i \langle P_2 | S^{ji}(0) | P_1 \rangle \quad (1.29g)$$

It is not clear whether the assumption of unsubtracted dispersion relations is justified. We assume that this is the case. Not all the above sum rules are on the same footing, some of the integrals being more convergent than the others. The first of these sum rules

is the most reliable and yields the Adler-Weisberger relation

$$g_A^2 + \frac{1}{2\pi} \int_{\text{cont.}} \frac{\psi(\nu', 0, 0, 0) d\nu'}{\nu'^2} = 1, \quad (1.30)$$

where the first term comes from the nucleon contribution with g_A defined by

$$\langle p | \bar{f}_V^+(0) | n \rangle = \bar{u}(p) [i g_A G(q^2) \gamma_V \gamma_5 + R_V \gamma_5 \beta(q^2)] u(n) \quad (1.31)$$

$[G(0) = 1, k = p - n = q]$. The integral represents the rest of the contribution coming from the continuum where

$$\psi(\nu, 0, 0, 0) = \frac{1}{2} \int e^{-iK \cdot x} \langle p | [\bar{D}^+(x), \bar{D}^-(0)] | p \rangle d^4x, \quad (1.32)$$

$(K \rightarrow 0)$

In order to express eq. (1.29a) in terms of divergences as in (1.30) it is necessary to go to the limit $t = 0, k_1^2 = 0, k_2^2 = 0$ as will be seen from eqs. (1.23e), (1.23h) and (1.24b).

We now use PCAC assumption in the form

$$\lim_{q^2 = (p'-p)^2 \rightarrow 0} \langle p' | \bar{D}^+ | p \rangle = \frac{i\sqrt{2} f_\pi g_{\pi N} G_{\pi N}(q^2)}{m_\pi^2} \bar{u}_{p'} \gamma_5 u_p, \quad (1.33)$$

where

$$f_\pi = \frac{\sqrt{2} m_\pi^2 m g_A}{g_{\pi N} K_{\pi N}(0)} \quad (1.34)$$

The final form of the sum rule is

$$1 - \frac{1}{g_A^2} = \frac{4M_N^2}{g_r^2 K_{N\pi(0)}^2} \frac{1}{\pi} \int_{M_N + m_\pi} \frac{W dW}{W^2 - M_N^2} [\sigma_0^+(W) - \sigma_0^-(W)] \quad (1.35)$$

Here,

g_A : axial vector renormalization constant in nuclear β -decay;

M_N : nucleon mass;

g_r : renormalized (πN) coupling constant;

$K_{N\pi}$: pionic form-factor of the nucleon-normalized such that
 $K_{N\pi}^2(-m_K^2) = 1$. In the spirit of PCAC we take
 $K_{N\pi}^2(0) \approx 1$;

W : C.M.S. energy of the (πN) system;

$\sigma_0^\pm(W)$: total cross-sections for the scattering of zero mass pions (π^\pm) off the proton.

Since experimental cross-sections referring to physical pions are not involved in eq. (1.35) Adler used a model to estimate the unphysical cross-sections from the physical ones. He found that in his model the difference between the physical and unphysical cross-sections is small. Weisberger, on the other hand, ignored this correction. Their results compared well with experiment.

$$|g_A^{\text{Adler}}| \cong 1.24 \pm 0.03, \quad (1.36)$$

$$|g_A^{\text{Weisberger}}| \cong 1.15, \quad (1.37)$$

$$|g^{\text{Expt.}}| = 1.18 \pm .02, \quad (1.38)$$

It is to be noted that inclusion of N and N^* does not give a good saturation of the sum rule ($|g_A| = 5/3$). However, inclusion of all the known higher resonances does give a reasonable saturation of this sum rule. This would mean in symmetry language that there is a mixing between the various representations of $SU(6)$.

In order to estimate the contributions of higher resonances, we shall use the formulae:

$$\left. \begin{aligned} \sigma \pi^+ &= \frac{4\pi}{9} g_m f^{3/2} \\ \sigma \pi^- &= \frac{4\pi}{39} \left\{ g_m f^{1/2} + 2 g_m f^{3/2} \right\} \\ \sigma \pi^+ - \sigma \pi^- &= \frac{8\pi}{39} \left\{ g_m f^{3/2} - g_m f^{1/2} \right\} \end{aligned} \right\} (1.39)$$

We parametrize the phase-shifts as follows

$$\cot \delta_\ell = \left(\frac{s}{v} \right)^{\ell+1/2} \frac{W - W_R}{\gamma} \quad (1.40)$$

where δ_ℓ is the phase-shift for the ℓ -th partial wave, s is the square of the c.m. energy, ($s = W^2$), $W_R (= M_R)$ is the mass of the resonance, v is the square of the c.m. momentum and γ is related to the width. Then we have

$$\sigma_J^{\ell, I} = \frac{8\pi}{3v} \frac{\eta_R \gamma^2 (v/s)^{2\ell+1}}{(W_R - W)^2 + \gamma^2 (v/s)^{2\ell+1}} (-1)^{I+1/2}; \quad \gamma = \frac{\Gamma}{2} \left(\frac{s_R}{v_R} \right)^{\ell+1/2}, \quad (1.41)$$

where $\eta_R = \Gamma_{e\ell} / \Gamma_{\text{tot}}$ is the elasticity of the resonance, J is

the spin and I is the isospin of the resonance; Γ is the width of the resonance. For $l \gg 1$ this formula has the right threshold behaviour. For $l = 0$ the phase-shift goes to zero at threshold and the cross-section goes to a constant which is found to be small compared to its value at the resonance position. In the narrow width approximation these formulae take the form

$$\sigma_J^{l,I} = \frac{4\pi^2}{3v_R} \Gamma \gamma_R (J + \frac{1}{2}) (-1)^{I + \frac{1}{2}} \delta(W - W_R) \quad (1.42)$$

The results are given in Table IA. We give both finite width and narrow width results. They are seen to be in reasonable agreement with each other. The sum rule is reasonably convergent and the higher resonances are found to contribute by only a few percent to the sum rule. We have included the (πN) resonances up to 2.2 GeV as given in ref. (7). The saturation of the sum rule is good; it yields a value of $|g_A| = 1.30$ to be compared with the calculated values of Adler and Weisberger and also the experimental value as given in eqs. (1.36), (1.37) and (1.38). The inclusion of the higher resonances as given in ref. (45) alters g_A by less than 5%.

Besides the isovector axial and vector currents of hadrons which conserve strangeness, the hadronic weak current should also contain a part which changes strangeness. Therefore, we may write

$$J_W^\mu = J_W^{\mu(0)} + J_W^{\mu(1)} \quad (1.43)$$

where

$$\left. \begin{array}{l} J_W^{\mu(0)} : \Delta S = 0, \quad \Delta I_3 = +1, \quad \Delta I = 1, \\ J_W^{\mu(1)} : \Delta S = 1, \quad \Delta I_3 = +\frac{1}{2}, \quad \Delta I = \frac{1}{2}. \end{array} \right\} \quad (1.44)$$

TABLE IA

Resonance Saturation of Adler-Weisberger Sum Rule^{*}

Resonance	Mass (Mev)	Total Width (Mev)	Elasticity $\eta = \frac{\Gamma_{el}}{\Gamma_{tot}}$	Contribution to r.h.s. of eq. (1.35) narrow width	Contribution to r.h.s. of eq. (1.35) finite width
P ₃₃	1235.8	125.1	1	+0.7289	+0.669
P ₁₁	1466	211	0.658	-0.0747	-0.067
D ₁₃	1541	149	0.509	-0.0570	-0.058
S ₁₁	1591	268	0.696	-0.0567	-0.058
S ₃₁	1635	177	0.284	+0.0129	+0.013
D ₁₅	1678	173	0.391	-0.0444	-0.045
F ₁₅	1687	177	0.560	-0.0630	-0.063
P ₃₃	1688	281	0.098	+0.0116	+0.010
D ₃₃	1691	269	0.137	+0.0154	+0.013
P ₁₁	1751	327	0.320	-0.0179	-0.016
P ₁₃	1863	296	0.207	-0.0151	-0.014
F ₃₅	1913	350	0.163	+0.0184	+0.016
P ₃₁	1934	339	0.299	+0.0103	+0.009
F ₃₇	1946	221	0.386	+0.0338	+0.032
D ₃₅	1954	311	0.154	+0.0139	+0.013
F ₁₇	1983	225	0.128	-0.0104	-0.016
D ₁₃	2057	293	0.260	-0.0116	-0.010
G ₁₇	2265	298	0.349	-0.0206	-0.018

$$\epsilon_A = 1.30$$

* I am indebted to R. Kirsopp for a discussion on the (πN) resonances.

Now, if $SU(3)$ were an exact symmetry, there would exist an octet of currents j_{μ}^i ($i = 1, \dots, 8$) which are conserved and whose time components when integrated over all space give the generators of the group

$$\int j_0^i(x) d^3x = F^i \quad (1.45)$$

obeying the commutation relations

$$[F^i, F^j] = if^{ijk} F^k \quad (1.46)$$

The vector part of J_W^{μ} is built from this octet of currents, as follows,

$$J_{W,V}^{\mu} = \cos \theta (j^{\mu 1} + i j^{\mu 2}) + \sin \theta (j^{\mu 4} + i j^{\mu 5}) \quad (1.47)$$

We may likewise postulate an octet of axial-vector currents g_{μ}^i , whose space integrated time components satisfy similar commutation relations at equal times. These currents are not conserved. The axial-vector part of J_W^{μ} is built from these currents, in an analogous way,

$$J_{W,A}^{\mu} = \cos \theta (g^{\mu 1} + i g^{\mu 2}) + \sin \theta (g^{\mu 4} + i g^{\mu 5}) \quad (1.48)$$

Thus θ appears as a new universal constant (Cabibbo's angle), which governs the sharing of the weak interactions between strangeness conserving and strangeness violating processes.

Experimentally $\theta \approx 12^{\circ}$ and in this new form of universality of weak interactions $G_V = G_{\mu} \cos \theta$. The vector and the axial-vector charges together generate a chiral $SU(3) \otimes SU(3)$ algebra. If we consider all the components of the currents and integrate

them over all space, we obtain a chiral $U(6) \otimes U(6)$ algebra. A further extension of this algebra to $U(12)$ may be obtained by including also the integrals of scalar, pseudoscalar and tensor components. In order to get dynamical information from this $U(12)$ algebra Dashen and Gell-Mann make the assumption that its positive parity subgroup, isomorphic to $U(6) \otimes U(6)$ and generated by all the charge operators whose Dirac matrices commute with β , transforms one-particle states at rest into one-particle states at rest. We shall, however, not pursue this point further.

3. Superconvergence and Strong Interaction Sum Rules

In the previous section we saw how to arrive at sum rules of the type

$$\frac{1}{\pi} \int a(\nu, u_1, u_2, t) d\nu = F(t) \quad , \quad (1.49)$$

where, $u_1 = -q_1^2$, $u_2 = -q_2^2$ and $t = -(q_1 - q_2)^2$, and where q_1 , q_2 and $q_1 - q_2$ are kept fixed at some space-like values. As usual $F(t)$ is some current form-factor and a is defined through the expansion of the tensor

$$t_{\mu\nu}^{\alpha\beta} = \frac{1}{2} \int d^4x e^{-i q_2 \cdot x} \langle p_2 | [j_\mu^\alpha(x), j_\nu^\beta(0)] | p_1 \rangle \quad , \quad (1.50)$$

in terms of invariant amplitudes

$$t_{\mu\nu}^{\alpha\beta} = a P_\mu P_\nu + b_1 P_\mu q_{1\nu} + \dots \quad (1.51)$$

Here the isospin indices have been suppressed and we have defined

$P = \frac{1}{2}(p_1 + p_2)$, $t = -(p_2 - p_1)^2$, $v = p \cdot q_1 = p \cdot q_2$. For simplicity we have chosen the currents to be isovector vector currents and the states to be pions. Diagrammatically (Fig. IA) $t_{\mu\nu}$ is the absorptive part of the amplitude describing the process involving two pions and two currents.

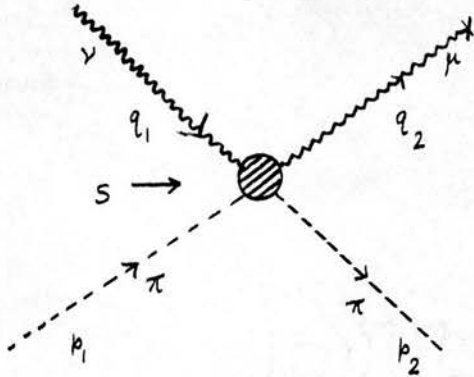


Fig. I.A.

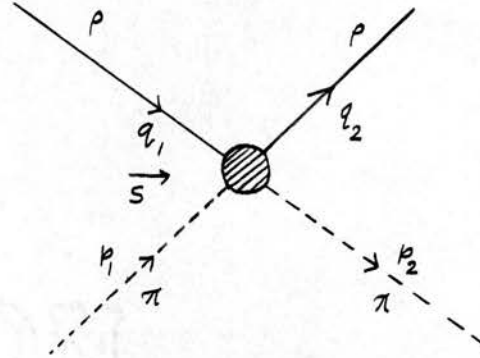


Fig. I.B.

We assume that eq. (1.49) can be continued analytically into a region where u_1 and u_2 are time-like. We note that $a(v, u_1, u_2, t)$ is a function of the external masses u_1 and u_2 associated with the currents. In particular, it has poles at $u_1 = m_\rho^2$ and at $u_2 = m_\rho^2$. But the right hand side of eq. (1.49) is a function of t only. Thus the effect of integrating over v should be such that the dependence on u_1 and u_2 compensate each other. Multiplying eq. (1.49) through by $(u_1 - m_\rho^2)(u_2 - m_\rho^2)$ and going to the limit $u_1 \rightarrow m_\rho^2$, $u_2 \rightarrow m_\rho^2$, we get

$$\lim_{\substack{u_1 \rightarrow m_\rho^2 \\ u_2 \rightarrow m_\rho^2}} \int a(v, u_1, u_2, t) (u_1 - m_\rho^2)(u_2 - m_\rho^2) dv = 0$$

But, $\lim_{\substack{u_1 \rightarrow m_\rho^2 \\ u_2 \rightarrow m_\rho^2}} a(v, u_1, u_2, t) (u_1 - m_\rho^2)(u_2 - m_\rho^2)$ is just the residue of the

poles in a at $u_1 = m_\rho^2$, $u_2 = m_\rho^2$ and is, therefore, the absorptive part of the invariant amplitude A in the physical $\rho\pi$ scattering process (Fig. I.8),

$$T = A \epsilon_2 \cdot P \epsilon_1 \cdot P + B (\epsilon_2 \cdot P \epsilon_1 \cdot Q + \epsilon_2 \cdot Q \epsilon_1 \cdot P) + C \epsilon_2 \cdot Q \epsilon_1 \cdot Q + D \epsilon_2 \cdot \epsilon_1 \quad (1.52)$$

where, $Q = \frac{1}{2}(q_1 + q_2)$. Hence, eq. (1.49) becomes

$$\int \text{Im} A(\nu, t) d\nu = 0, \quad \text{at fixed } t. \quad (1.53)$$

An important distinction between eq. (1.49) and eq. (1.53) is that whereas the former contains information about the weak and the electromagnetic structure of the pion, the latter deals with the strong interaction between ρ and π . Thus, in going from eq. (1.49) to eq. (1.53) we have lost the current algebra characteristic scaling of the form-factor $F(t)$. The derivation of eq. (1.53), therefore, should not depend upon the actual nature of the current algebra used. Indeed, all we need is that the commutator of two currents contains $\delta(\underline{x} - \underline{y})$ or its derivatives, which follows from the locality of the currents, anyway. Hence, eq. (1.53) has nothing to do with current algebra. In fact, it may be derived directly from the requirement of analyticity, unitarity and appropriate high energy bounds for the scattering amplitudes. For example, an analytic function $f(\nu)$ satisfies a dispersion relation

$$f(\nu) = \frac{1}{\pi} \int \frac{\text{Im} f(\nu') d\nu'}{\nu' - \nu}$$

if
$$\left. \begin{array}{l} |f(\nu)| \sim \nu^\beta \\ \beta < 0 \end{array} \right\} \text{ for } \nu \rightarrow \infty$$

and it satisfies a superconvergence relation

$$\int \text{Im} f(\nu') d\nu' = 0$$

$$\text{if } \left. \begin{array}{l} |f(\nu)| \sim \nu^\beta \\ \beta < -1 \end{array} \right\} \text{for } \nu \rightarrow \infty .$$

Sum rules that follow from eq. (1.53) involve only parameters like strong coupling constants and masses and are, therefore, referred to as strong interaction sum rules. When all the particles involved in a scattering process have no spin there is just one amplitude, and it behaves asymptotically as $\nu^{\alpha(t)}$ (apart from a factor of some power of $\ln \nu$) with $1 > \alpha(0) > 0$. It cannot, therefore, satisfy a superconvergence relation. If, however, one or more of these particles have spin, there is, in general, more than one amplitude, and these amplitudes may have different asymptotic behaviour, such as of the form $\nu^{\alpha(t)-n}$ where n varies, in general, from one amplitude to the other and depends upon the number of units of helicity flip in the t -channel that is associated with the amplitude in question. In some cases $(\alpha - n)$ may become less than -1 , and the corresponding amplitude will then satisfy a superconvergence relation. A simple example would be to consider $\rho\pi$ scattering as discussed by de Alfaro, Fubini, Furlan and Rossetti⁽⁸⁾. We shall instead consider a slightly more complicated case of $\rho + \pi \rightarrow K^* + \bar{K}$.

We first find the asymptotic behaviour of the various invariant amplitudes, A, B, B', C and D that appear in the expansion,

$$T = A \varepsilon_2 \cdot P \varepsilon_1 \cdot P + \frac{B}{2} (\varepsilon_2 \cdot P \varepsilon_1 \cdot Q + \varepsilon_2 \cdot Q \varepsilon_1 \cdot P) + \frac{B'}{2} (\varepsilon_2 \cdot P \varepsilon_1 \cdot Q - \varepsilon_2 \cdot Q \varepsilon_1 \cdot P) \\ + C \varepsilon_2 \cdot Q \varepsilon_1 \cdot Q + D \varepsilon_2 \cdot \varepsilon_1 , \quad (1.54)$$

by using the heuristic method of the above-mentioned authors.

In order to do so it is convenient to introduce another set of amplitudes such that the invariants are orthogonal to each other,

$$T = \alpha I_\alpha + \beta I_\beta + \gamma I_\gamma + \delta I_\delta + \epsilon I_\epsilon , \quad (1.55)$$

where ,

$$\left. \begin{aligned} I_\alpha &= \varepsilon_2 \cdot P' \varepsilon_1 \cdot P' , \\ I_\beta &= \varepsilon_2 \cdot P' \varepsilon_1 \cdot q_2 \frac{m_P}{2} + \varepsilon_2 \cdot q_1 \varepsilon_1 \cdot P' \frac{m_K^*}{2} , \\ I_\gamma &= \varepsilon_2 \cdot P' \varepsilon_1 \cdot q_2 \frac{m_P}{2} - \varepsilon_2 \cdot q_1 \varepsilon_1 \cdot P' \frac{m_K^*}{2} , \\ I_\delta &= \varepsilon_2 \cdot q_1 \varepsilon_1 \cdot q_2 \\ I_\epsilon &= \varepsilon_2 \cdot N' \varepsilon_1 \cdot N' . \end{aligned} \right\} (1.56)$$

with

$$P' = p_1 + a q_1 + b q_2 ,$$

$$N_\mu = \epsilon_{\mu\alpha\beta\gamma} P'_\alpha q_{1\beta} q_{2\gamma} ,$$

$$a = - \frac{2(-s+\pi+p)k^* + (-s-t+p+k)(t-p-k^*)}{(t-p-k^*)^2 - 4pk^*} , \quad (1.57)$$

$$b = \frac{(-s-t+p+k) + a(t-p-k^*)}{2k^*} ,$$

and $m_p^2 \equiv p$, $m_k^2 \equiv k$, etc .

It is easy to see that

$$P' \cdot q_1 = 0 , \quad P' \cdot q_2 = 0 , \quad P' \cdot N = 0$$

$$\sum_{\text{Pol. sum}} |I_\alpha I_\beta| = 0 , \quad \text{and so on.}$$

Also

$$\sum_{\text{Pol.}} |I_\alpha|^2 \sim s^4 \quad \text{as } s \rightarrow \infty \text{ and for fixed } t , \quad (1.58a)$$

$$\sum_{\text{Pol.}} |I_\beta|^2 \sim s^2 \quad \text{as } s \rightarrow \infty \text{ and for fixed } t , \quad (1.58b)$$

$$\sum_{\text{Pol.}} |I_\gamma|^2 \sim s^2 \quad \text{as } s \rightarrow \infty \text{ and for fixed } t , \quad (1.58c)$$

$$\sum_{\text{Pol.}} |I_\delta|^2 \sim \text{const.} \quad \text{as } s \rightarrow \infty \text{ and for fixed } t , \quad (1.58d)$$

$$\sum_{\text{Pol.}} |I_\epsilon|^2 \sim s^4 \quad \text{as } s \rightarrow \infty \text{ and for fixed } t , \quad (1.58e)$$

The orthogonal set of amplitudes $\alpha, \beta, \gamma, \delta, \epsilon$ are related to the perturbative set A, B, B', C, D as follows,

$$\begin{aligned}
 A &= \alpha + \epsilon [(q_1 q_2)^2 - q_1^2 q_2^2] \\
 B &= 2 \left\{ \alpha(b+a) + \frac{\beta}{2} (m_\rho + m_{K^*}) + \frac{\gamma}{2} (m_\rho - m_{K^*}) \right. \\
 &\quad \left. + \epsilon (P \cdot q_2 q_1^2 - P \cdot q_1 q_1 \cdot q_2 + q_2^2 P \cdot q_1 - P \cdot q_2 q_1 \cdot q_2) \right\} \\
 B' &= 2 \left\{ \alpha(b-a) + \frac{\beta}{2} (m_\rho - m_{K^*}) + \frac{\gamma}{2} (m_\rho + m_{K^*}) \right. \\
 &\quad \left. + \epsilon (P \cdot q_2 q_1^2 - P \cdot q_1 q_1 \cdot q_2 - q_2^2 P \cdot q_1 + P \cdot q_2 q_1 \cdot q_2) \right\} \\
 C &= 4ab\alpha + 2\beta(am_\rho + bm_{K^*}) + 2\gamma(am_\rho - bm_{K^*}) + 4\delta \\
 &\quad + 4\epsilon (P^2 q_1 \cdot q_2 - P \cdot q_1 P \cdot q_2) \\
 D &= \epsilon \{ P^2 q_1^2 q_2^2 - P^2 (q_1 q_2)^2 - (P \cdot q_1)^2 q_2^2 + 2P \cdot q_1 P \cdot q_2 q_1 \cdot q_2 - q_1^2 (P \cdot q_2)^2 \}.
 \end{aligned} \tag{1.59}$$

Using the optical theorem and assuming that the total cross-section is larger than the cross-section due to each of the amplitudes α, β, \dots , we get for large s

$$\int |\alpha(s,t)|^2 |I_\alpha|^2 dt < \text{const. } s^2 \sigma_{\text{tot.}}, \text{ etc.}, \tag{1.60}$$

since I_α, I_β, \dots , are orthogonal to each other. Making the constant shape assumption that $\alpha(s, t) = f(t) \alpha(s, 0)$, etc., we obtain

$$\begin{aligned}
 |\alpha| &\sim \text{const. } s^{-1} && \text{as } s \rightarrow \infty, \\
 |\beta| &\sim \text{const.} && \text{as } s \rightarrow \infty, \\
 |\gamma| &\sim \text{const.} && \text{as } s \rightarrow \infty, \\
 |\delta| &\sim \text{const. } s && \text{as } s \rightarrow \infty, \\
 |\varepsilon| &\sim \text{const. } s^{-1} && \text{as } s \rightarrow \infty;
 \end{aligned}
 \tag{1.61}$$

and, hence, for the original amplitudes,

$$\begin{aligned}
 |A| &\sim \text{const. } s^{-1} && \text{as } s \rightarrow \infty; \\
 |B| &\sim \text{const.} && \text{as } s \rightarrow \infty; \\
 |B'| &\sim \text{const.} && \text{as } s \rightarrow \infty; \\
 |C| &\sim \text{const. } s && \text{as } s \rightarrow \infty; \\
 |D| &\sim \text{const. } s && \text{as } s \rightarrow \infty;
 \end{aligned}
 \tag{1.62}$$

If we apply the above analysis to the case of spinless particles, we would obtain for the corresponding amplitude a behaviour $\text{const. } s$, asymptotically. Regge-pole theory would give instead a behaviour s^α . (The additional factors of $\ln s$ can be obtained by relaxing the constant shape assumption.) Therefore, we obtain the following asymptotic behaviour for the amplitudes

$$\begin{aligned}
 |A| &\sim s^{\alpha-2}, && , \\
 |B| &\sim s^{\alpha-1}, && , \\
 |B'| &\sim s^{\alpha-1}, && , \\
 |C| &\sim |D| \sim s^\alpha && .
 \end{aligned}
 \tag{1.63}$$

The value of α will depend upon the leading trajectory exchanged in the t -channel that will dominate the process under consideration.

It would appear that the use of the amplitudes α, β, \dots , will have the advantage of having two amplitudes which could go as $s^{\alpha-2}$. However, these amplitudes contain kinematic singularities in the variable s as eq. (1.59), relating the two sets of amplitudes, shows. Hence one cannot write superconvergence relations for these amplitudes. The amplitudes A, B, \dots , are free from such kinematic singularities and zeroes. A method of constructing such amplitudes free from kinematic singularities and useful for the discussion of superconvergence relations has been given by de Alfaro, Fubini, Furlan and Rossetti⁽⁹⁾. However, a generalization of their method to processes involving higher spin particles does not seem to be straightforward. For example, if one considers the amplitude with two vector currents and two nucleons there are 32 independent perturbative invariants. (At first sight it appears as if there are 34 of them, but as Gerstein⁽¹⁰⁾ has shown there are two constraint equations and only 32 of them are independent. He also gives a set such that the corresponding amplitudes are free from kinematic singularities in the variable s). If we follow the method of de Alfaro et al., there are only three vectors on the current side (two momenta associated with the currents and a derivative with respect to one of these momenta). From these we may construct ten tensors. On the nucleon side there are four vectors (γ matrix, two momenta and a derivative), but only two nontrivial invariants, e.g. 1 and $\gamma_{\lambda}^T \frac{\partial}{\partial p_{2\lambda}}$ can be constructed (p_2 is the momentum of one of the nucleons and γ_{λ}^T is defined in the above reference) after using

the equation of motion for the nucleons. Altogether, therefore, one can construct only twenty invariants. Since we need more we should introduce the symbol $\epsilon_{\mu\nu\lambda\rho}$. However then one can construct 38 tensors and it is not obvious as to which of the six have to be discarded such that the amplitudes have no kinematic singularities and zeroes in the variable s . If k_1, k_1' are the momenta associated with the currents, in the notation of the above reference, the 38 tensors are

$$\begin{aligned}
 & a_\mu a_\nu \left(1 + \gamma_\lambda^T \frac{\partial}{\partial p_{2\lambda}}\right) ; & \epsilon_{\mu\nu\rho\sigma} \gamma_5 k_{1\rho} k'_{1\sigma} \left(1 + \gamma_\lambda^T \frac{\partial}{\partial p_{2\lambda}}\right) ; \\
 & a_\mu k_{1\nu} \left(1 + \gamma_\lambda^T \frac{\partial}{\partial p_{2\lambda}}\right) ; & \epsilon_{\mu\nu\rho\sigma} \gamma_5 k_{1\rho} a_\sigma \left(1 + \gamma_\lambda^T \frac{\partial}{\partial p_{2\lambda}}\right) ; \\
 & a_\mu k'_{1\nu} \left(1 + \gamma_\lambda^T \frac{\partial}{\partial p_{2\lambda}}\right) ; & \epsilon_{\mu\nu\rho\sigma} \gamma_5 k'_{1\rho} a_\sigma \left(1 + \gamma_\lambda^T \frac{\partial}{\partial p_{2\lambda}}\right) ; \\
 & a_\nu k_{1\mu} \left(1 + \gamma_\lambda^T \frac{\partial}{\partial p_{2\lambda}}\right) ; & \epsilon_{\mu\lambda\rho\sigma} \gamma_5 k_{1\lambda} k'_{1\rho} a_\sigma k_{1\nu} \left(1 + \gamma_\alpha^T \frac{\partial}{\partial p_{2\alpha}}\right); \\
 & a_\nu k'_{1\mu} \left(1 + \gamma_\lambda^T \frac{\partial}{\partial p_{2\lambda}}\right) ; & \epsilon_{\mu\lambda\rho\sigma} \gamma_5 k_{1\lambda} k'_{1\rho} a_\sigma a_\nu \left(1 + \gamma_\alpha^T \frac{\partial}{\partial p_{2\alpha}}\right); \\
 & k'_{1\nu} k'_{1\mu} \left(1 + \gamma_\lambda^T \frac{\partial}{\partial p_{2\lambda}}\right) ; & \epsilon_{\nu\lambda\rho\sigma} \gamma_5 k_{1\lambda} k'_{1\rho} a_\sigma k_{1\mu} \left(1 + \gamma_\alpha^T \frac{\partial}{\partial p_{2\alpha}}\right); \\
 & k_{1\nu} k_{1\mu} \left(1 + \gamma_\lambda^T \frac{\partial}{\partial p_{2\lambda}}\right) ; & \epsilon_{\nu\lambda\rho\sigma} \gamma_5 k_{1\lambda} k'_{1\rho} a_\sigma k'_{1\mu} \left(1 + \gamma_\alpha^T \frac{\partial}{\partial p_{2\alpha}}\right); \\
 & k'_{1\nu} k_{1\mu} \left(1 + \gamma_\lambda^T \frac{\partial}{\partial p_{2\lambda}}\right) ; & \epsilon_{\nu\lambda\rho\sigma} \gamma_5 k_{1\lambda} k'_{1\rho} a_\sigma a_\mu \left(1 + \gamma_\alpha^T \frac{\partial}{\partial p_{2\alpha}}\right); \\
 & k_{1\nu} k'_{1\mu} \left(1 + \gamma_\lambda^T \frac{\partial}{\partial p_{2\lambda}}\right) ; & \epsilon_{\mu\lambda\rho\sigma} \gamma_5 k_{1\lambda} k'_{1\rho} a_\sigma k'_{1\nu} \left(1 + \gamma_\alpha^T \frac{\partial}{\partial p_{2\alpha}}\right); \\
 & \delta_{\mu\nu} \left(1 + \gamma_\lambda^T \frac{\partial}{\partial p_{2\lambda}}\right) ; &
 \end{aligned}
 \tag{1.64}$$

Furthermore, it is not clear whether the method would work when there are mass-less particles involved, especially when there are odd numbers of them, for then the problem would be very tricky⁽¹¹⁾. However, in simpler cases like $\rho\pi$ scattering or πN scattering, the method is seen to work well.

Another method of constructing kinematic singularity free amplitudes is to use helicity amplitudes⁽¹²⁾. Trueman⁽¹³⁾ has given a method of determining the superconvergent helicity amplitudes. As the method is well-explained in the literature⁽¹⁴⁾ we shall not pursue this question further.

The asymptotic behaviour of the amplitudes are seen to depend upon the value of $\alpha(t)$ corresponding to the dominant trajectory exchanged in the t channel and hence upon their isospin. Experimentally,

$$\alpha_{1/2}(t) < 1 ,$$

and if we further make the reasonable assumption

$$\alpha_{3/2}(t) < 0$$

for $t \leq 0$ and also over a small range of positive values of t then we may write, for t fixed in this region ,

$$\left. \begin{aligned}
 \int_{-\infty}^{+\infty} \text{Im} A^{3/2}(\nu', t) d\nu' &= 0, \\
 \int_{-\infty}^{\infty} \text{Im} A^{1/2}(\nu', t) d\nu' &= 0, \\
 \int_{-\infty}^{\infty} \text{Im} B^{3/2}(\nu', t) d\nu' &= 0, \\
 \int_{-\infty}^{\infty} \text{Im} B'^{3/2}(\nu', t) d\nu' &= 0, \\
 \int_{-\infty}^{\infty} \nu' \text{Im} A^{3/2}(\nu', t) d\nu' &= 0,
 \end{aligned} \right\} (1.65)$$

where the superscript refers to the isospin of the dominant Regge-pole exchanged in the t-channel.

Since these sum rules are true for arbitrary t we may in fact propose further sum rules by considering higher order terms in t. The reason is that if an amplitude has the behaviour

$$f \sim s^{\alpha(t) - n} \quad \text{as } s \rightarrow \infty \text{ for fixed } t$$

then,

$$\frac{\partial f}{\partial t} \sim \ln s \cdot s^{\alpha(t) - n} \cdot \alpha'(t) \quad \text{as } s \rightarrow \infty \text{ for fixed } t$$

and, in general

$$\frac{\partial^p f}{\partial t^p} \sim (\ln s \cdot \alpha'(t))^p s^{\alpha(t) - n} \quad \text{as } s \rightarrow \infty \text{ for fixed } t \text{ and any } p,$$

so that, if $f(s, t)$ is superconvergent over a range of values of t, so are its derivatives of any order with respect to t over the same range. Hence we may write (specialising to the forward direction),

$$\int_{-\infty}^{\infty} \text{Im} A^{3/2}(\nu', 0) d\nu' = 0, \quad \text{etc.},$$

and

$$\int_{-\infty}^{\infty} \frac{\partial}{\partial t} (\text{Im} A^{3/2}(\nu', 0)) d\nu' = 0, \quad \text{etc.},$$

and so on.

The practical applications of these sum rules to find physical information about strong coupling constants and mass relations are rather difficult. There have been various attempts to obtain such information from these sum rules by putting in a few resonances in the intermediate states. Even though these attempts have been partially successful in obtaining consistency among these sum rules and predicting reasonable experimental results, there does not seem to be any real justification for this truncation procedure. Phenomenologically, we see that it is reasonable to assume that the low mass and spin states will dominate the process under consideration and furthermore the sum rules obtained by taking higher derivatives with respect to t are expected to be less reliable than those for the amplitude itself because of additional powers of $(\ln s)$ contained in their asymptotic bounds. There is no simple group theoretical meaning to such a procedure, however. Since there are in fact an infinite number of sum rules that one can write down once the superconvergence criterion is satisfied, it appears that a consistent solution would not be possible by just putting in a few intermediate states. However, it should be noted that the t -dependence of the sum rule comes from the polarization sum for the intermediate states inserted. For example, a spin zero

intermediate state gives no t dependent term, with spin one intermediate states terms linear in t are obtained, and so on. Hence, by restricting to particles of spin up to J , say, we shall be retaining all powers of t up to t^J . If the amplitude is superconvergent, so are its derivatives with respect to t and, therefore, there will be $J+1$ superconvergence sum rules for a given amplitude at $t = 0$. (We have assumed that the amplitude has not got even faster convergence, that is we have assumed $-2 < \alpha(t) < -1$, but the generalization to the case $\alpha(t) - n < -2$ is straightforward. We have further restricted to a particular isospin exchange. Also we consider, for simplicity, only those cases when the external particles have integral spin.) Each sum rule will be a linear equation in variables coming from direct terms and from crossed terms. Some of these may vanish since they may not be allowed by conservation of quantum numbers. Each variable will be a product of two coupling constants, one for each vertex joining the intermediate particle with external lines (which are assumed to be all different). It is not obvious whether one can get a consistent solution for any J and even if we have such a solution it may not correspond to physics. Since the coupling constants appear only as products, we shall not be able to find the value of each coupling constant separately. If, however, the external particles are identical we shall be able to find the magnitude of the coupling constants but their sign will remain undetermined. We do not attempt to consider such a consistent saturation of all the sum rules for the process $\rho + \pi \rightarrow K^* + \bar{K}$, as the unsymmetrical nature of the problem makes it even more

difficult. In Chapter III a simple case of scattering of vector and pseudoscalar mesons to zeroth order terms in t and including up to particles of spin two in the intermediate state, is considered in the $SU(3)$ symmetric limit.

4. Pion Scattering Lengths

A specific process that has been under extensive investigation using current algebra and the PCAC hypothesis is that of low-energy $(\pi-\pi)$ scattering. The interest in this application ~~was~~ was started by Adler⁽¹⁵⁾ when he found that a sum rule of the Adler-Weisberger type, relating the axial-vector renormalization constant for nuclear β -decay to the integral over the difference of total cross-sections for $(\pi^+ \pi^-)$ and $(\pi^+ \pi^+)$ scattering could not be saturated by just including the contributions of ρ and f resonances, but required a rather large contribution from some other partial-wave which when assumed to be pure s-wave with isospin zero, yielded a rather large scattering length ($a_0^0 > 1.3$ or $< -.85$). Under the assumption that the commutator of the axial vector and the axial divergence corresponding to double charge exchange vanishes (i.e. $[\bar{T}^+, \bar{D}^+] = 0$) Furlan and Rosetti⁽¹⁶⁾ derived another sum rule for $(\pi-\pi)$ scattering. Inserting ρ , f and a conjectured σ they were able to get a saturation of both of these sum rules for σ -parameters (mass ≈ 400 Mev and width ≈ 100 Mev) in close agreement with those given by Brown and Singer⁽¹⁷⁾. However more recent experimental and phenomenological developments⁽¹⁸⁾ seem to disagree with these values. Our calculation, presented in Chapter IV, is an attempt to generalize their calculation in a more realistic way, taking into account the structure of the

resonances involved. Since the derivation of the Adler-Weisberger sum rule⁽¹⁹⁾ outlined in Section 2 of this chapter may easily be generalized to the case of $(\pi-\pi)$ scattering, we shall not consider this sum rule further. Another result of current algebra used in Chapter IV is Weinberg's result for $2a_0^0 - 5a_0^2$ for the s-wave $(\pi-\pi)$ scattering lengths⁽²⁰⁾ (where the superscript refers to the isospin channel evolved). We outline below Weinberg's original derivation of the scattering lengths.

We start off with the L.S.Z. reduction formula for the s-matrix for $(\pi-\pi)$ scattering,

$$\begin{aligned}
 S(\delta p_4, \gamma p_3; \beta p_2, \alpha p_1) &= I^{\delta\gamma; \beta\alpha} + \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 \frac{1}{(m_\pi^2)^4 F_\pi^4} e^{-ip_4x_4 - ip_3x_3 + ip_2x_2 + ip_1x_1} \\
 &\quad \times (m_\pi^2 - \square_{x_1})(m_\pi^2 - \square_{x_2})(m_\pi^2 - \square_{x_3})(m_\pi^2 - \square_{x_4}) \langle T \{ D^\delta(x_4) D^\gamma(x_3) D^\beta(x_2) D^\alpha(x_1) \} \rangle_0 \\
 &= I^{\delta\gamma; \beta\alpha} + \frac{4}{i=1} \frac{(p_i^2 + m_\pi^2)}{(m_\pi^2)^4 F_\pi^4} (2\pi)^4 \delta^4(p_4 + p_3 - p_2 - p_1) \int e^{-ip_4x_4 - ip_3x_3 + ip_2x_2} \\
 &\quad \langle T \{ D^\delta(x_4) D^\gamma(x_3) D^\beta(x_2) D^\alpha(0) \} \rangle_0 d^4x_4 d^4x_3 d^4x_2,
 \end{aligned}
 \tag{1.66}$$

where we have used $D^\gamma(x) = \partial^\mu A_\mu^\gamma(x)$, the divergence of the axial vector current as an interpolating field for the pion, and F_π is the pion decay constant defined by

$$\langle 0 | A_\mu^\alpha(x) | \pi^\beta(k) \rangle = i e^{ik \cdot x} k_\mu F_\pi \delta^{\alpha\beta}
 \tag{1.67}$$

We define the T-matrix by

$$S(\delta p_4, \gamma p_3; \beta p_2, \alpha p_1) = I^{\delta\gamma; \beta\alpha} - i(2\pi)^4 \delta^4(p_4 + p_3 - p_2 - p_1) T(\delta p_4, \gamma p_3; \beta p_2, \alpha p_1) \quad (1.68)$$

so that

$$T(\delta p_4, \gamma p_3; \beta p_2, \alpha p_1) = i \prod_{i=1}^4 \frac{p_i^2 + m_\pi^2}{m_\pi^8 F_\pi^2} \int e^{-ip_4 x_4 - ip_3 x_3 + ip_2 x_2} \times \\ \times \langle 0 | T(D^\delta(x_4) D^\gamma(x_3) D^\beta(x_2) D^\alpha(0)) | 0 \rangle d^4 x_4 d^4 x_3 d^4 x_2. \quad (1.69)$$

In the limit when $p_i^2 \rightarrow -m_\pi^2$ this is an exact identity. For p_i^2 different from $-m_\pi^2$, this defines the off-shell T-matrix, with all the four pions off the mass-shell. We interpret the PCAC hypothesis to imply that the off-shell amplitude so defined has a smooth extrapolation from physical values of p_i (i.e., $p_i^2 = -m_\pi^2$) to $p_i \rightarrow 0$.

In the spirit of PCAC we assume that we can expand the T-matrix in terms of the momenta and approximate it by retaining only up to quadratic terms in them. For the purpose of calculating the scattering lengths this expansion has to be valid at least in the region $-m_\pi^2 \leq p_i^2 \leq 0$ and $0 \leq s, t, u \leq 4m_\pi^2$, where we have defined $s = -(p_1 + p_2)^2$, $u = -(p_1 - p_4)^2$, $t = -(p_1 - p_3)^2$. The requirement of Bose statistics, crossing symmetry, isospin invariance and four-momentum conservation ($s + t + u = -\sum_1^4 p_i^2$) restricts the expansion to the following form,

$$T(\delta p_4, \gamma p_3; \beta p_2, \alpha p_1) = (A + B(s+t) + cu) \delta_{\beta\gamma} \delta_{\alpha\delta} + (A + B(s+u) + Ct) \delta_{\beta\delta} \delta_{\alpha\gamma} \\ + (A + B(u+t) + Cs) \delta_{\gamma\delta} \delta_{\alpha\beta} + \text{higher order terms} \\ \text{in momentum, which we neglect.} \quad (1.70)$$

(Actually, this expansion cannot be valid at and above threshold ($s = 4m_\pi^2$, $t = u = 0$) since it does not have the right cut structure and, therefore, violates unitarity. Our assumption that it can be used at the physical threshold and even slightly beyond it corresponds to assuming that the threshold singularity is weak and the violation of unitarity is negligibly small. This indirectly requires that the scattering lengths should be small.) Eq. (1.70) allows us to write the scattering lengths in terms of the constants A, B, C.

$$a_0^0 = \frac{1}{32\pi m_\pi} (5A + 8B m_\pi^2 + 12C m_\pi^2), \quad (1.71)$$

$$a_0^2 = \frac{1}{32\pi m_\pi} (2A + 8B m_\pi^2) \quad . \quad (1.72)$$

The next step is to exploit the current algebra commutators to evaluate A, B, C. We shall need the following commutation relations

$$[A_0^\alpha(\underline{x}, t), A_i^\beta(\underline{y}, t)] = i \epsilon^{\alpha\beta\gamma} V_i^\gamma(\underline{x}, t) \delta^3(\underline{x}-\underline{y}) - i d^{\alpha\beta} \partial_i \delta^3(\underline{x}-\underline{y}), \quad (1.73)$$

where $d^{\alpha\beta}$ is a c-number, and

$$[Q_5^\alpha(t), D^\beta(\underline{x}, t)] = i \sigma^{\alpha\beta}(\underline{x}, t) \quad . \quad (1.74)$$

Since the c-number Schwinger terms do not contribute to the connected part of the S-matrix, on integrating eq. (1.69) and

letting $p_1, p_3 \rightarrow 0$, we obtain

$$T(\delta p, \gamma_0, \beta p, \alpha_0) = -\frac{2}{F_\pi^2} p_1 \cdot p \left\{ \delta^{\alpha\beta} \delta^{\gamma\delta} - \delta^{\alpha\delta} \delta^{\gamma\beta} \right\} + \frac{1}{F_\pi^2} \langle \pi^\delta(p) | \sigma^{\alpha\gamma}(0) | \pi^\beta(p) \rangle + O(p_1^2) / F_\pi^2 \quad (1.75)$$

where we have defined $p_2 = p_4 = p$, $p_1 = p_3 \rightarrow 0$ so that $s \rightarrow m_\pi^2 - 2p_1 \cdot p$, $t \rightarrow 0$ and $u \rightarrow m_\pi^2 + 2p_1 \cdot p$. Comparing eqs. (1.70) and (1.75), we get

$$B - C = -\frac{1}{F_\pi^2}$$

and, therefore,

$$2a_0^0 - 5a_0^2 = \frac{3}{4\pi m_\pi F_\pi^2} \approx 0.69 m_\pi^{-1} \quad (1.76)$$

Using the Adler consistency condition,

$$\lim_{p_4 \rightarrow 0} T(\delta p_4, \gamma p_3, \beta p_2, \alpha p_1) = 0 \quad (1.77)$$

and the assumption

$$\sigma^{\alpha\gamma}(0) = \delta^{\alpha\gamma} \sigma(0)$$

(which follows from the quark model and is also used in the σ -model), it is easy to obtain another condition

$$2a_0^0 + 7a_0^2 = 0 \quad (1.78)$$

This yields the result $a_0^0 = 0.2 m_\pi^{-1}$ and $a_0^2 = -0.06 m_\pi^{-1}$.

These values are small compared with those obtained from the analysis of $(\pi-N)$ scattering data⁽²¹⁾. Since eq. (1.70), upon which this result is based, is rather dubious, Khuri⁽²²⁾ made the calculation including fourth order terms in momenta. He found that, under certain reasonable assumptions, Weinberg's values for the scattering lengths remain unchanged. If, however, one imposes the unitarity condition at the threshold, in addition to obtaining Weinberg's solution, other solutions with larger scattering lengths are also obtained⁽²³⁾. More recently there has been much work in this direction⁽²⁴⁾, but there does not seem to be one opinion as to what should be the correct values of the scattering lengths. Even though most people agree that a_0^0 should be positive and a_0^2 negative, certain calculations using dispersion relations seem to indicate that both of them are positive⁽²⁵⁾. On the other hand, it would be nice to have both of them negative as this would provide a natural solution of ghost problems in S-matrix theory⁽²⁶⁾.

A recent hard pion calculation due to Arnowitt, Friedman, Nath and Suitor⁽²⁷⁾ has shown that Weinberg's result is consistent with the experimental results for $(\pi-\pi)$ scattering due to Walker et al.⁽²⁸⁾ and also with the analysis of $(\pi N \rightarrow \pi\pi N)$ data by Malamud and Schlein⁽²⁹⁾. It may therefore appear that Weinberg's results are after all reasonable and any discrepancy in our results of Chapter IV may be due to some other source,

5. Ambiguities in Current Algebra Calculations and their Resolutions

Having outlined some of the successful applications of current algebras and PCAC let us now discuss certain paradoxical results of standard current algebra calculations and their resolutions through recent techniques and prescriptions. We do not intend to go into formal difficulties that arise in trying to set up a rigorous mathematical basis for current algebras, using axiomatic field theory⁽³⁰⁾. Such problems have been carefully tackled in the literature⁽³¹⁾. Nor do we intend to consider the more familiar difficulties⁽³²⁾ of gradient terms and of non-covariance aspects of retarded commutators of currents. Instead we shall consider certain technical difficulties that are present in the usual methods of doing current algebra calculations.

The first ambiguity that we want to mention arises in calculations in which more than one pion is extrapolated off its mass-shell. It was discussed first in connection with non-leptonic (2π) -decay modes of K-mesons⁽³³⁾. If one does the extrapolation of the two pions to zero mass sequentially, i.e., first takes one pion off its mass-shell, goes to the limit of zero four-momentum of the pion and uses PCAC smoothness assumption and then repeats the whole procedure for the other pion, then one obtains the result that $K \rightarrow 2\pi$ decay obeys $\Delta I = \frac{1}{2}$ rule⁽³⁴⁾. Thus it was believed that current algebras and PCAC implied the $\Delta I = \frac{1}{2}$ rule. However, if one extrapolates both the pions simultaneously, i.e., takes both the pions off their mass-shell and considers the limit when both the pion four-momenta vanish and assumes PCAC then one obtains additional, so called, σ terms arising from the commutator of the axial charge and the axial divergence at equal-times. Since the

σ -operator may have $I = 2$ as well as $I = 0$ components, (the former allowing $\Delta I = 3/2$ transitions) the $\Delta I = 1/2$ rule is no longer a natural consequence of Current algebra and PCAC. In fact, it is possible to explain the experimentally observed departures from $\Delta I = 1/2$ rule by allowing a reasonable admixture of the $I = 2$ and $I = 0$ σ -components⁽³³⁾. Even though there does not seem to exist any argument, which in principle would favour one extrapolation procedure against another, the general consensus is that the second procedure is the correct one and the σ -terms (which are physically meaningful) should be present. It is this method which leads to Weinberg's scattering lengths for pions, the first method allowing no such solution. Other processes where this method has been successful in re-estimating current algebra predictions are η -decay⁽³⁵⁾, intermediate vector boson mass⁽³⁶⁾, etc.

The next ambiguity that we want to consider is the derivation of the sum rule

$$1 - 2F_\pi^2 \delta_{\rho\pi\pi}^2 / m_\rho^2 \cong 0 \quad (1.79)$$

due to Kawarabayashi and Suzuki⁽³⁷⁾ using current algebra, PCAC and ρ -dominance assumptions. Here

$$\langle 0 | A_\mu^i | \pi^j, q \rangle = i \delta^{ij} F_\pi q_\mu \quad (1.80)$$

and

$$\Gamma_{\rho \rightarrow \pi\pi} = \frac{1}{6\pi} \left(\frac{\delta_{\rho\pi\pi}^2}{m_\rho^2} \right) p_\pi^3 \quad (1.81)$$

This sum rule is in good agreement with experiment. There have been many other derivations of this sum rule which, however,

require additional assumptions (38). Geffen (39) has argued that the original derivation of Kawarabayashi and Suzuki is ambiguous, and there does not seem to exist any derivation based on Current algebra and pole-dominance assumptions only. His argument is as follows.

The $(\rho \rightarrow 2\pi)$ decay amplitude may be extrapolated to zero four-momenta of all the three particles in three different ways:
 (i) Extrapolate two pions as usual to zero four-momentum so that ρ must also have zero four-momentum (by momentum conservation); this is the method used by Kawarabayashi and Suzuki. (ii) Extrapolate one of the pions and the ρ using for the latter the interpolating field V^μ/f_ρ where f_ρ is defined by,

$$\langle 0 | V_\mu^i | \rho^j, p \rangle = \delta^{ij} f_\rho \epsilon_\mu \quad (1.82)$$

(iii) Extrapolate all the three particles explicitly off their mass-shell. Only the last procedure is unambiguous as it shows how the extrapolation to zero four-momentum is done for all the three particles. However this method does not yield the Kawarabayashi-Suzuki sum rule, but instead,

$$\gamma_{\rho\pi\pi}(0,0,0) = (F_\pi(0)/F_\pi) m_\rho^2/f_\rho \quad (1.83)$$

This is consistent with PCAC and the ρ -dominance assumption for the electromagnetic form-factor of the pion. The second extrapolation procedure again yields the result (1.83). The first gives

$$2\gamma_{\rho\pi\pi}(k^2, q^2, p^2) \rightarrow f_\rho(p^2)/F_\pi^2, \quad k, q \rightarrow 0, \quad (1.84)$$

where $f_\rho(p^2)$ is the extrapolation of $f_\rho = f_\rho(-m_\rho^2)$. It is argued in ref. (39) that the estimation of $f_\rho(0)$ from ρ -dominance assumption is ambiguous because of the non-covariance of the vector current propagator that appears in $f_\rho(p^2)$.

Another ambiguity we discuss is the calculation of A_1 -width due to Renner⁽⁴⁰⁾ and Geffen⁽⁴⁰⁾ based on current algebra and meson pole dominance assumptions. The standard current algebra technique gave a width of the A_1 meson that was too large (~ 650 Mev) to be acceptable (the present experimental estimate is about $30 \sim 130$ Mev). The original explanation given was that A_1 was not a pure resonance and was a kinematic effect (Deck effect) and the actual 1^+ resonance was, as yet, unseen. Since then, however, several authors have produced more sensible values for the A_1 -width ($30 \sim 200$ Mev). Such calculations are usually done by considering both ($A_1 \rightarrow \rho\pi$) and ($\rho \rightarrow \pi\pi$) decay modes simultaneously. The first successful attempt was due to Schwinger⁽⁴¹⁾ in his phenomenological Lagrangian theory (chiral dynamics) which gave

$$\Gamma_{\rho \rightarrow \pi\pi} \simeq 78 \text{ Mev} , \quad (1.85)$$

$$\Gamma_{A_1 \rightarrow \rho\pi} \simeq 185 \text{ Mev} ,$$

with all the particles on their mass-shells. This method has been discussed in greater detail by Wess and Zumino⁽⁴²⁾ who have obtained additional possible solutions. Another calculation based on generalized Ward identities derived from current commutation relations and on meson dominance assumptions at finite non-zero energies has been done by Schnitzer and Weinberg⁽⁴³⁾. They obtain a one-parameter dependent set of solutions for $\Gamma_{\rho \rightarrow \pi\pi}$ and $\Gamma_{A_1 \rightarrow \rho\pi}$

which embody Schwinger's result and have a typical solution consistent with the presently accepted experimental widths

$$\Gamma_{\rho \rightarrow \pi\pi} \approx 128 \pm 20 \text{ Mev} \quad (1.86)$$

$$\Gamma_{A_1 \rightarrow \rho\pi} \approx 30 \sim 130 \text{ Mev} .$$

A calculation more in keeping with the original derivations⁽⁴⁰⁾ has been given by Brown and West⁽⁴⁴⁾. They point out that in the original derivation of the A_1 -width unsubtracted dispersion relations for matrix elements of the retarded commutators (rather, their Fourier transforms) were assumed in one of the two independent variables, while the other was kept fixed (preferably at zero). As a result, certain important pole contributions were missed out. In the derivation of the A_1 -width as given in ref. (40), the π -pole in the variable that was kept fixed ($= 0$) was lost. Brown and West attribute the discrepancy between experiment and theory to this unsatisfactory way of doing the calculation. By assuming an unsubtracted dispersion relation in one of the variables with a suitably defined variable (e.g., a linear combination of the two original variables with a free parameter) fixed such that no pole contributions are lost, they are able to obtain a consistent set of solutions for the $(A_1\rho\pi)$ system which are in agreement with the results of Schnitzer and Weinberg. They also show that the assumption of unsubtracted dispersion relations for all the current form-factors can lead to inconsistencies and some of these form-factors may, therefore, need subtractions. The assumption of meson-pole-dominance would accordingly need constant terms in addition to the pole terms. This is equally true of the matrix elements of the

divergence of the axial vector current. Therefore PCAC in its conventional form of pion-pole-dominance may need some modification in certain cases by requiring additional constant terms besides the pion poles.

We conclude this section by presenting a calculation for the scalar vertex $\langle \pi | \sigma | \pi \rangle$ based on the technique of Brown and West, mentioned above, and illustrate how the standard techniques used in current algebra calculations may be obtained as limiting cases of this method when one or the other variable is fixed. We find that as long as $m_\sigma^2 \gg m_\pi^2$ we get identical results (which may not be a general feature, but a peculiarity of our simple case) immaterial of whether we fix one variable or the other, or any linear combination of them, and this result is found to be the same as that obtained by combining eqs. (5.11) and (5.12) derived in Chapter V by a more general method.

We start with the equal-time commutator

$$[\int A_0^i(\underline{x}) d^3\underline{x}, \sigma(0)] = -i \partial^\mu A_\mu^i(0) \quad (1.87)$$

and consider its matrix element between a pion and vacuum

$$\begin{aligned} \langle 0 | [\int A_0^i(\underline{x}) d^3\underline{x}, \sigma(0)] | \pi^j(p) \rangle &= -i \langle 0 | \partial^\mu A_\mu^i(0) | \pi^j(p) \rangle \\ &= -i m_\pi^2 F_\pi \delta^{ij} \end{aligned} \quad (1.88)$$

($F_\pi \simeq 94$ Mev).

Define

$$W(p, q) = \int d^4x \theta(x^0) e^{-iq \cdot x} \langle 0 | [D^i(x), \sigma(0)] | \pi^j(p) \rangle \quad , \quad (1.89)$$

$$\omega(p, q) = \frac{1}{2i} \int d^4x e^{-iq \cdot x} \langle 0 | [D^i(x), \sigma(0)] | \pi^j(p) \rangle \quad . \quad (1.90)$$

Then, we may write eq. (1.88) as

$$\lim_{q \rightarrow 0} W(p, q) = -i F_\pi m_\pi^2 \delta^{ij} \quad (1.91)$$

Inserting intermediate states in eq. (1.90), we get

$$\begin{aligned} \omega(p, q) = \frac{1}{2i} \int d^4x \left\{ e^{i(p_n - q) \cdot x} \frac{\langle 0 | D^i(0) | \pi^j(p_n) \rangle \langle \pi^j(p_n) | \sigma(0) | \pi^j(p) \rangle d^3p_n}{(2\pi)^3 2p_n^0} \right. \\ \left. - e^{i(-q - p_n + p) \cdot x} \frac{\langle 0 | \sigma(0) | \sigma(p_n) \rangle \langle \sigma(p_n) | D^i(0) | \pi^j(p) \rangle d^3p_n}{(2\pi)^3 2p_n^0} \right\} \end{aligned} \quad (1.92)$$

$$\begin{aligned} = i\pi \delta(q^2 + m_\pi^2) \frac{F_\pi m_\pi^2 g_\sigma G_{\sigma\pi\pi} \delta^{ij}}{m_\sigma^2 + \Delta^2} \\ - i\pi \delta(\Delta^2 + m_\sigma^2) \frac{F_\pi m_\pi^2 g_\sigma G_{\sigma\pi\pi} \delta^{ij}}{m_\pi^2 + q^2} \end{aligned}$$

$$(1.93)$$

where $\Delta = p - q$.

We now keep $\mu = \alpha q^2 + (1-\alpha) \Delta^2$ fixed and write an UDR (unsubtracted dispersion relation) for W ,

$$W = \frac{1}{\pi} \int \frac{\omega(q''^2, \mu) dq''^2}{q''^2 - q^2} \quad \left(= \frac{1}{\pi} \int \frac{\omega(\Delta''^2, \mu) d\Delta''^2}{\Delta''^2 - \Delta^2} \right) \quad (1.94)$$

$$= i \delta^{ij} F_\pi m_\pi^2 g_\sigma G_{\sigma\pi\pi} \left\{ -\frac{1}{m_\pi^2 + q^2} \frac{1}{m_\sigma^2 + \Delta'^2} - \frac{1}{m_\sigma^2 - \Delta^2} \frac{1}{m_\pi^2 + q'^2} \right\} \quad (1.95)$$

where

$$\begin{aligned} \mu &= -\alpha m_\pi^2 + (1-\alpha) \Delta'^2 & ; \\ &= \alpha q'^2 - (1-\alpha) m_\sigma^2 & ; \end{aligned} \quad (1.96)$$

so that

$$\begin{aligned} \Delta'^2 &= \frac{\mu + \alpha m_\pi^2}{1-\alpha} & , \\ q'^2 &= \frac{\mu + (1-\alpha) m_\sigma^2}{\alpha} & . \end{aligned} \quad (1.97)$$

α is a free parameter. The equality within the parentheses in eq. (1.94) follows since $dq''^2 (q''^2 - q^2)^{-1} = d\Delta''^2 (\Delta''^2 - \Delta^2)^{-1}$

Therefore,

$$\begin{aligned} \lim_{q \rightarrow 0} W &= i \delta^{ij} F_\pi m_\pi^2 g_\sigma G_{\sigma\pi\pi} \left\{ \frac{\alpha - 1}{m_\pi^2 [(1-\alpha) m_\sigma^2 + \alpha m_\pi^2 + \mu]} \right. \\ &\quad \left. + \frac{\alpha}{(m_\sigma^2 - m_\pi^2)(\alpha m_\pi^2 + \mu + (1-\alpha) m_\sigma^2)} \right\} \quad (1.98) \end{aligned}$$

Case (i) $\alpha = 0$ (fixed Δ^2 UDR):

$$m_\pi^2 (m_\sigma^2 - m_\pi^2) = g_\sigma G_{\sigma\pi\pi} \quad (1.99)$$

Case (ii) $\alpha = 1$ (fixed q^2 UDR):

$$-m_\pi^2 (m_\sigma^2 - m_\pi^2) = g_\sigma G_{\sigma\pi\pi} \quad (1.100)$$

Even though these two cases cannot be correct simultaneously, the result

$$g_\sigma^2 G_{\sigma\pi\pi}^2 \simeq m_\pi^4 (m_\sigma^2 - m_\pi^2)^2, \quad (1.101)$$

is consistent with both of them. Eq. (1.101) is also obtained by combining eqs. (5.11) and (5.12) derived in Chapter V. According to Brown and West none of the above two cases are reasonable. We, therefore, allow α to be a free parameter.

Since

$$\mu = \alpha q^2 + (1-\alpha) (p-q)^2$$

we get

$$\lim_{q \rightarrow 0} \mu = - (1-\alpha) m_\pi^2 \quad (1.102)$$

Therefore, combining eqs. (1.91), (1.98) and (1.102), we get



$$-(2\alpha-1)m_{\pi}^2 + (\alpha-1)m_{\sigma}^2 = \frac{g_{\sigma} G_{\sigma\pi\pi}}{m_{\pi}^2 (m_{\sigma}^2 - m_{\pi}^2)} \left\{ (\alpha-1)(m_{\sigma}^2 - m_{\pi}^2) + \alpha m_{\pi}^2 \right\} .$$

(1.103)

Since this equation is true for all values of α (except perhaps at $\alpha = 0$ and at $\alpha = 1$), comparing the coefficients of various powers of α on both sides of the equation we obtain

$$-2m_{\pi}^2 + m_{\sigma}^2 = \frac{g_{\sigma} G_{\sigma\pi\pi} m_{\sigma}^2}{m_{\pi}^2 (m_{\sigma}^2 - m_{\pi}^2)} , \quad (1.104)$$

and

$$m_{\pi}^2 - m_{\sigma}^2 = - \frac{g_{\sigma} G_{\sigma\pi\pi}}{m_{\pi}^2} . \quad (1.105)$$

These solutions consistently imply (if $m_{\sigma}^2 \gg m_{\pi}^2$)

$$g_{\sigma} G_{\sigma\pi\pi} \cong m_{\pi}^2 (m_{\sigma}^2 - m_{\pi}^2) ,$$

or

$$g_{\sigma}^2 G_{\sigma\pi\pi}^2 \cong m_{\pi}^4 (m_{\sigma}^2 - m_{\pi}^2)^2 .$$

As shown in Chapter V this relation is not unreasonable.

CHAPTER II

MESON COUPLINGS AND MAGNETIC MOMENTS

FROM LEE-DASHEN-GELL-MANN METHOD

1. It was shown by Lee⁽¹⁾ and, independently, by Dashen and Gell-Mann⁽²⁾ that many useful results of SU(6) symmetry⁽³⁾ (e.g., the axial-vector renormalization constant in nuclear β -decay⁽⁴⁾, the ratio of proton and neutron total magnetic moments⁽⁵⁾, etc.) could be obtained without requiring such a symmetry if the U(6) algebra generated by the hadronic vector charges and spatial components of axial 'charges' was used along with SU(3) invariance and a saturation hypothesis. Their method was to take the matrix elements of the commutators between states of zero spatial momentum (e.g., one nucleon states at rest) and to insert a complete set of intermediate states between the two operators of the commutators. Even though conservation laws restricted the number of allowed states considerably, some kind of approximation was unavoidable in order that experimentally verifiable results could be arrived at. Hence they retained only certain single particle intermediate states (bound states or resonances), namely the octet of baryons and the decuplet of baryon-meson resonances at rest. In this way they obtained the SU(6) results for the axial-vector renormalization constant and the magnetic moment ratio for the nucleons. The assumption that the octet of baryons and the decuplet of baryon-meson resonances saturated the sum rule was equivalent to assuming that these states formed a basis of an irreducible representation of the U(6) algebra generated by the vector and appropriate axial-

vector charges. This algebra, however, was not assumed to correspond to any symmetry group. In particular, the Hamiltonian of the strong interactions under consideration was not assumed to commute with these charges (at least, not with all of them). Hence the Hamiltonian was not assumed to be invariant under the transformations of the group locally isomorphic to the algebra generated by these charges. The same technique was later applied to the case of mesons by Fayyazuddin, Riazuddin and Razmi⁽⁶⁾, and, independently, by Schnitzer⁽⁷⁾, again obtaining the SU(6) predicted values^{(8),(9)} for the meson couplings and magnetic moments (and also mean square radii). The case of nucleons was carefully examined by Ryan⁽¹⁰⁾ in the context of SU(4) algebra assuming only isospin invariance and keeping only a few intermediate states. Among many interesting results he reproduced the results of Lee and of Dashen and Gell-Mann except for an ambiguity in sign of the axial-vector renormalization constant which he attributed to the equivalence between two conjugate representations (which transform into each other under G-conjugation) at the SU(2) level in contrast to their difference at the SU(3) level. His analysis (restricted to nonstrange particles only) was much simpler as no complicated Clebsch-Gordan Coefficients were needed. He also demonstrated how the values of the axial-vector renormalization constant and the magnetic moment ratio depended upon the saturation assumption, and how important it was to take matrix elements between states at rest rather than between states of finite momentum.

In the following a similar calculation in the context of SU(4) algebra is presented for the case of nonstrange mesons. If there

were a $SU(4)$ symmetry (a higher symmetry, and not just an internal symmetry) the vector mesons (ρ, ω) and the pseudo-scalar mesons (π, η) would together belong to a sixteen ($1 \oplus 15$) dimensional representation of $SU(4)$. In the following analysis only these particles are involved (in particular, ϕ and η' are ignored as they do not appear in the representation of $SU(4)$ to which ρ, π, ω and η belong). However, no $SU(4)$ symmetry is assumed. Only isospin invariance is required to be valid. Matrix elements of the various commutators are taken between all possible pairs of these mesons at rest and the intermediate sum over states approximated by these very mesons. The calculations and results are given in the following sections. In section 2 commutation relations between the various charge operators closing an $SU(4)$ algebra are written down. They are shown to follow from a free quark model. Commutation relations involving magnetic dipole moment operators are then derived in the same model, first assuming that these moment operators have only $L = 1$ terms and next including an additional $L = 0$ term. (Here L refers to the orbital excitations of the quarks; in the free quark model that we consider there are no other basic particles except the quarks and L is just the orbital angular momentum of the quarks⁽¹¹⁾). In analogy with the $SU(3)$ situation^(12, 13) an algorithm of C-parity conservation is developed and used for writing the matrix elements of the various operators of interest to us. This automatically ensures conservation of G-parity and isospin. In section 3 the calculations are described and the results given. These are then discussed in section 4. PCAC constraints in the form of generalized Goldberger-Treiman relations are then applied to convert these results into

relations between strong meson couplings. Finally the implications of our results for magnetic moments and radiative transitions of the mesons are discussed. Table IIA summarizes the results of the calculation.

2. In a model based on a fundamental isotopic doublet field $q(\underline{x}, t)$, the vector and axial-vector currents are given by

$$v_i^\mu(x) = -\frac{1}{2} q^\dagger(x) \gamma^0 \gamma^\mu \tau_i q(x) \quad , \quad (2.1a)$$

$$a_i^\mu(x) = -\frac{1}{2} q^\dagger(x) \gamma^0 \gamma^5 \gamma^\mu \tau_i q(x) \quad , \quad (2.1b)$$

$$v_0^\mu(x) = -\frac{1}{2} q^\dagger(x) \gamma^0 \gamma^\mu q(x) \quad , \quad (2.1c)$$

$$a_0^\mu(x) = -\frac{1}{2} q^\dagger(x) \gamma^0 \gamma^5 \gamma^\mu q(x) \quad , \quad (2.1d)$$

where the Lorentz index $\mu = 0, 1, 2, 3$ and the isospin index $i = 1, 2, 3$. The corresponding charges are obtained by integrating them over all space. Thus,

$$V_i^0 = \frac{1}{2} \int q^\dagger(x) \tau_i q(x) d^3 \underline{x} \quad , \quad (2.2a)$$

$$A_i^a = \frac{1}{2} \int q^\dagger(x) \tau_i \sigma^a q(x) d^3 \underline{x} \quad , \quad (2.2b)$$

$$A_0^a = \frac{1}{2} \int q^\dagger(x) \sigma^a q(x) d^3 \underline{x} \quad , \quad (2.2c)$$

where $a = 1, 2, 3$ are space indices and $\sigma^a = \gamma^5 \gamma^0 \gamma^a$. For our notations and metric see Appendix (I). The fifteen operators satisfy the following commutation relations (this follows if we use anti-commutation relations for the quark fields), and thus close an SU(4) algebra,

$$[A_i^a, A_j^b] = i \epsilon_{ijk} \delta^{ab} V_k^0 + i \epsilon^{abc} \delta_{ij} A_0^c, \quad (2.3a)$$

$$[A_i^a, A_0^b] = i \epsilon^{abc} A_i^c, \quad (2.3b)$$

$$[A_i^a, V_j^0] = i \epsilon_{ijk} A_k^a, \quad (2.3c)$$

$$[A_0^a, V_j^0] = 0, \quad (2.3d)$$

$$[V_i^0, V_j^0] = i \epsilon_{ijk} V_k^0, \quad (2.3e)$$

$$[A_0^a, A_0^b] = i \epsilon^{abc} A_0^c \quad (2.3f)$$

In our notation $[\tau_i, \tau_j] = 2i \epsilon_{ijk} \tau_k$, $\{\sigma^a, \sigma^b\} = 2\delta^{ab}$.

As the axial currents are not conserved, the corresponding charges are time dependent and the corresponding Eqs. (2.3) hold at equal times. No Schwinger terms are involved as we are dealing with integrated quantities.

We next write down the isovector and isoscalar dipole moment

operators in the free quark model, under the assumption that they are pure $L = 1$ operators,

$$m_i^a = \frac{1}{2} \int d^3x \epsilon^{afg} x^f q^\dagger(x) \gamma^5 \sigma^g \frac{\partial_i}{2} q(x) , \quad (2.4a)$$

$$m_0^a = \frac{1}{2} \int d^3x \epsilon^{afg} x^f q^\dagger(x) \gamma^5 \sigma^g \frac{1}{2} q(x) . \quad (2.4b)$$

We shall further need to define the following moment operators

$$Q_k^{ab} = \int d^3x (3x^a x^b - \delta^{ab} x^2) q^\dagger(x) \frac{\partial_k}{2} q(x) , \quad (2.5a)$$

$$R_k = \int d^3x x^2 q^\dagger(x) \frac{\partial_k}{2} q(x) , \quad (2.5b)$$

$$O_0^c = \int d^3x x^c x^d q^\dagger(x) \sigma^d \frac{1}{2} q(x) . \quad (2.5c)$$

Q_k^{ab} is the quadrupole moment operator and R_k corresponds to the mean square radius. The commutation relations between the magnetic moment operators are found to be,

$$[m_i^a, m_j^b] = i \epsilon_{ijk} \left[\delta^{ab} \frac{R_k}{6} - \frac{Q_k^{ab}}{12} \right] + i \delta_{ij} \epsilon^{abc} \frac{O_0^c}{4} , \quad (2.6a)$$

$$[m_0^a, m_0^b] = \frac{i}{4} \epsilon^{abc} O_0^c , \quad (2.6b)$$

$$[m_i^a, A_j^b] = -\frac{i}{2} \int d^3x \delta^{ab} \delta_{ij} x \cdot v_0 + \frac{i}{2} \int d^3x \delta_{ij} x^b v_0^a - \frac{i}{2} \int d^3x \epsilon^{afg} x^f \epsilon_{ijk} a_R^g . \quad (2.6c)$$

It is to be noted that the magnetic moment operators do not close any finite dimensional algebra, higher and higher moments being generated on repeated commutation.

As pointed out by Ryan⁽¹⁰⁾ (see also ref. (11)) the magnetic moment operators given by (2.4a) may be written in the form

$$m_i^a = -\frac{1}{2} \epsilon^{afb} \int d^3x \ x^f v_i^b(x) \quad (2.7)$$

where v_i^b are isovector current space components defined in (2.1a) whose $i = 3$ component is just the isovector part of the electromagnetic current of the fundamental doublet (quark) field. If this current does not contain any derivative term then m_i^a is a pure $L = 1$ operator, where L is the orbital angular momentum of the quarks. However, if the fundamental field has a Pauli moment coupling to the electromagnetic field, then m_i^a should be modified such that the magnetic moment operator contains an $L = 0$ part in it. We shall denote such modified operators by M_i^a and they will have the form

$$M_i^a = \frac{1}{2} \int d^3x \ \epsilon^{afb} \ x^f q^\dagger(x) \gamma^5 \sigma^b \frac{\tau_i}{2} q(x) + \frac{i}{2} \mu_q \int d^3x \ q^\dagger(x) \gamma^0 \sigma^a \frac{\tau_i}{2} q(x), \quad (2.8a)$$

$$M_0^a = \frac{1}{2} \int d^3x \ \epsilon^{afb} \ x^f q^\dagger(x) \gamma^5 \sigma^b q(x) + \frac{i}{2} \mu_q \int d^3x \ q^\dagger(x) \gamma^0 \sigma^a \frac{1}{2} q(x), \quad (2.8b)$$

where μ_q is the quark magnetic moment. They satisfy the following commutation relations,

$$[M_i^a, M_j^b] = U + V + W + X, \quad (2.9a)$$

$$[M_0^a, M_0^b] = u + v + w + x, \quad (2.9b)$$

$$[M_i^a, A_j^b] = p + q + r + s + t, \quad (2.9c)$$

where,

$$U = i \epsilon_{ijk} \left[\delta^{ab} \frac{R_k}{6} - \frac{Q_k^{ab}}{12} \right] + i \delta_{ij} \epsilon^{abc} \frac{\rho_0^c}{4},$$

$$V = \frac{i}{4} \mu_q^2 \left\{ \delta_{ij} \epsilon^{abc} A_0^c + \delta^{ab} \epsilon_{ijk} V_k^0 \right\},$$

$$W = \frac{i}{8} \mu_q \int d^3 \underline{x} \epsilon^{afg} x^f q^g(x) \left\{ \delta_{ij} \delta^{gl} \gamma^5 \gamma^0 - \epsilon_{ijk} \epsilon^{ghl} \gamma^5 \gamma^0 \sigma^k \tau_k \right\} q(x),$$

$$X = -\frac{i}{8} \mu_q \int d^3 \underline{x} \epsilon^{bfh} x^f q^h(x) \left\{ \delta_{ij} \delta^{ga} \gamma^5 \gamma^0 - \epsilon_{jik} \epsilon^{gah} \gamma^5 \gamma^0 \sigma^k \tau_k \right\} q(x),$$

$$u = \frac{i}{4} \epsilon^{abc} \rho_0^c,$$

$$v = \frac{i}{4} \mu_q^2 \epsilon^{abc} A_0^c,$$

$$w = \frac{i}{8} \mu_q \int d^3 \underline{x} \epsilon^{afh} x^f q^h(x) \gamma^5 \gamma^0 q(x),$$

$$x = \frac{i}{8} \mu_q \int d^3 \underline{x} \epsilon^{afb} x^f q^\dagger(x) \gamma^5 \gamma^0 q(x) ,$$

$$p = -\frac{i}{2} \int d^3 \underline{x} \delta^{ab} \delta_{ij} x^b v_0 ,$$

$$q = \frac{i}{2} \int d^3 \underline{x} \delta_{ij} x^b v_0^a ,$$

$$r = -\frac{i}{2} \int d^3 \underline{x} \epsilon^{afb} x^f \epsilon_{ijk} a_k^0 ,$$

$$s = -\frac{\mu_q}{4} \int d^3 \underline{x} \delta_{ij} \epsilon^{abc} q^\dagger \gamma^0 \sigma^c q(x) ,$$

$$t = -\frac{\mu_q}{4} \int d^3 \underline{x} \epsilon_{ijk} \delta^{ab} q^\dagger(x) \gamma^0 \tau_k q(x) .$$

In order to write down the matrix elements of these operators, we now develop the concept of C-parity conservation in the case of SU(2) symmetry. As in the case of SU(3) symmetry (12, 13) each of the isospin multiplets ρ , π will be assigned a definite C-parity, and the C-parity of the second component will carry an additional minus sign. If C is the charge conjugation operator, under C the mesons transform as follows,

$$C : V_a \longrightarrow -\epsilon_a V_a \quad (\text{no summation}) ,$$

$$C : P_a \longrightarrow +\epsilon_a P_a \quad (\text{no summation}) ,$$

where $\alpha = 1, 2, 3$ are the isospin indices. $\epsilon_1 = \epsilon_3 = +$, $\epsilon_2 = -$. The vector and axial-vector currents transform as follows

$$C : \quad v_\alpha^\mu \longrightarrow -\epsilon_\alpha v_\alpha^\mu \quad (\text{no summation}),$$

$$a_\alpha^\mu \longrightarrow +\epsilon_\alpha a_\alpha^\mu \quad (\text{no summation}).$$

For isoscalar mesons C-parity and G-parity are identical. Thus

$$C : \quad V_0 \longrightarrow -V_0 ,$$

$$P_0 \longrightarrow +P_0 .$$

Also,

$$C : \quad v_0^\mu \longrightarrow -v_0^\mu$$

$$a_0^\mu \longrightarrow +a_0^\mu$$

Using the charge conjugation properties of the Dirac bilinears, we find,

$$C : \quad m_i^a \longrightarrow -\epsilon_i m_i^a \quad , \quad \rho_0^c \longrightarrow +\rho_0^c .$$

$$m_0^a \longrightarrow -m_0^a \quad ,$$

$$Q_k^{ab} \longrightarrow -\epsilon_k Q_k^{ab} \quad ,$$

$$R_k \longrightarrow -\epsilon_k R_k \quad ,$$

$$p \longrightarrow -p \quad ,$$

$$q \longrightarrow -q \quad ,$$

$$r_k \longrightarrow \epsilon_k r_k \quad ,$$

$$s \longrightarrow -s \quad ,$$

$$t_k \longrightarrow \epsilon_k t_k \quad ,$$

Let us consider the matrix elements $\langle V_\gamma' | v_j^0 | V_\alpha \rangle$, $\langle V_\gamma' | a_j^0 | V_\alpha \rangle$ and $\langle V_0' | a_j^0 | V_\alpha \rangle$ to illustrate how C-parity conservation imposes restrictions on matrix elements. Since V_γ' , v_j^0 , V_α have C-parity $-\epsilon_\gamma$, $-\epsilon_j$, $-\epsilon_\alpha$, the product becomes $-\epsilon_\gamma \epsilon_j \epsilon_\alpha$. In order that this remains positive only odd number of indices can take on the value 2. This implies after requiring isospin conservation that the matrix element must be of the form $\epsilon_{j\alpha\gamma}$. The second matrix element would vanish since it has an additional factor of minus sign and, therefore, $\epsilon_{j\alpha\gamma}$ type of coupling (which alone conserves isospin) is not allowed by C-parity conservation. The third matrix element should be of the form $\delta_{\alpha j}$ for only then will both C-parity and isospin be conserved.

The various nonvanishing matrix elements of interest to us are given below,

$$\langle V_0' | a_i^a | V_\gamma \rangle = i \delta_{i\gamma} \epsilon^{dea} \epsilon^{d*} \epsilon^e g_A \quad , \quad (2.10a)$$

$$\langle V_\alpha' | a_0^b | V_\gamma \rangle = i \delta_{\alpha\gamma} \epsilon^{fgb} \epsilon^{f*} \epsilon^g f_A \quad , \quad (2.10b)$$

$$\langle V_\gamma' | v_j^0 | V_\alpha \rangle = i \epsilon_{j\alpha\gamma} \tilde{\epsilon}^* \tilde{\epsilon} \quad 2 m_p \quad , \quad (2.10c)$$

$$\langle V_0' | a_0^c | V_0 \rangle = i \epsilon^{fgc} \epsilon^{f*} \epsilon^g f_A^{2s} \quad , \quad (2.10d)$$

$$\langle V_\gamma' | a_j^b | P_\alpha \rangle = i \epsilon_{j\alpha\gamma} \epsilon^{b*} h_A \quad , \quad (2.10e)$$

$$\langle P_\gamma | v_j^0 | P_\alpha \rangle = i \epsilon_{j\alpha\gamma} \quad 2 m_\pi \quad , \quad (2.10f)$$

$$\langle V'_k | M_i^a | V_j \rangle = \mu \epsilon_{ijk} \epsilon^{dea} \epsilon^{d*} \epsilon^e (2\pi)^3 \delta^3(\underline{k}) \quad , \quad (2.10g)$$

$$\langle P_k | M_i^a | V_0 \rangle = \mu_T^{(0)} \epsilon^a \delta_{ik} (2\pi)^3 \delta^3(\underline{k}) \quad , \quad (2.10h)$$

$$\langle P_0 | M_i^a | V_j \rangle = \mu_T^{(0)'} \epsilon^a \delta_{ij} (2\pi)^3 \delta^3(\underline{k}) \quad , \quad (2.10i)$$

$$\langle V_i | M_0^a | P_j \rangle = \mu_{Tis} \epsilon^{a*} \delta_{ij} (2\pi)^3 \delta^3(\underline{k}) \quad , \quad (2.10j)$$

$$\langle P_0 | M_0^a | V_0 \rangle = \mu_{Tis}' \epsilon^a (2\pi)^3 \delta^3(\underline{k}) \quad , \quad (2.10k)$$

$$\langle V'_k | R_i | V_j \rangle = i \epsilon_{ijk} \underline{\epsilon}^* \cdot \underline{\epsilon} \langle r_\rho^2 \rangle (2\pi)^3 \delta^3(\underline{k}) \quad , \quad (2.10l)$$

$$\langle P'_k | R_i | P_j \rangle = i \epsilon_{ijk} \langle r_\pi^2 \rangle (2\pi)^3 \delta^3(\underline{k}) \quad , \quad (2.10m)$$

$$\langle V'_k | Q_i^{ab} | V_j \rangle = \frac{Q}{2\sqrt{10}} i \epsilon_{ijk} [2\delta^{ab} \underline{\epsilon}^* \cdot \underline{\epsilon} - 3(\epsilon^{a*} \epsilon^b + \epsilon^{b*} \epsilon^a)] (2\pi)^3 \delta^3(\underline{k}) \quad (2.10n)$$

$$\langle V'_\beta | \mathcal{O}_0^c | V_\alpha \rangle = i \epsilon^{gfc} \epsilon^{g*} \epsilon^f F_V (2\pi)^3 \delta^3(\underline{k}) \delta_{\beta\alpha} \quad , \quad (2.10o)$$

$$\langle V_0 | \mathcal{O}_0^c | V_0 \rangle = i \epsilon^{gfc} \epsilon^{g*} \epsilon^f F_V^{is} (2\pi)^3 \delta^3(\underline{k}) \quad , \quad (2.10p)$$

$$i \langle P_\beta | \int d^3x q^\dagger(x) \tau_R \gamma^0 q(x) | P_0 \rangle = \delta_{\beta R} H_{PP} (2\pi)^3 \delta^3(\underline{k}) \quad , \quad (2.10q)$$

$$i \langle V_\beta | \int d^3 \underline{x} \, q^\dagger(\underline{x}) \tau_k \gamma^0 q(\underline{x}) | V_0 \rangle = \delta_{\beta k} \epsilon^* \epsilon H_{VV} (2\pi)^3 \delta^3(\underline{k}) , \quad (2.10r)$$

$$i \langle V_\beta | \int d^3 \underline{x} \, q^\dagger(\underline{x}) \gamma^0 \sigma^c q(\underline{x}) | P_\alpha \rangle = \delta_{\beta \alpha} \epsilon^{c*} G_{VP} (2\pi)^3 \delta^3(\underline{k}) , \quad (2.10s)$$

$$i \langle P_0 | \int d^3 \underline{x} \, q^\dagger(\underline{x}) \gamma^0 \sigma^c q(\underline{x}) | V_0 \rangle = \epsilon^c G_{VP}^{is} (2\pi)^3 \delta^3(\underline{k}) , \quad (2.10t)$$

$$\langle V_\beta | \int d^3 \underline{x} \, x^f a_k^0 | P_\alpha \rangle = i \epsilon_{k\alpha\beta} \epsilon^f K_{VP} (2\pi)^3 \delta^3(\underline{k}) , \quad (2.10u)$$

$$\langle V_\beta | \int d^3 \underline{x} \, x^f a_k^0 | V_0 \rangle = \delta_{\beta k} i \epsilon^{gef} \epsilon^g \epsilon^e K_{VV} (2\pi)^3 \delta^3(\underline{k}) , \quad (2.10v)$$

$$\langle V_\beta | J^a | V_\alpha \rangle = i \epsilon^{acb} \epsilon^{b*} \epsilon^c \delta_{\beta \alpha} 2m_p (2\pi)^3 \delta^3(\underline{k}) , \quad (2.10w)$$

$$\langle V_0 | J^a | V_0 \rangle = i \epsilon^{acb} \epsilon^{b*} \epsilon^c (2\pi)^3 2m_\omega \delta^3(\underline{k}) . \quad (2.10x)$$

The scale of these matrix elements is fixed by Eq. (2.10c). Eqs. (2.10f), (2.10w), (2.10x) have been chosen to be consistent with Eq. (2.10c). The various constants appearing in these equations such as $g_A, \dots, \mu, \dots, \langle r_p^2 \rangle, \dots, Q$ etc. are weak coupling constants, magnetic moments, mean-square radii, quadrupole moment etc. and they are defined by the corresponding equations. \underline{k} is the momentum transferred between incoming and outgoing states and tends to zero. Covariant normalization $\langle \underline{p}' | \underline{p} \rangle = (2\pi)^3 2p^0 \delta^3(\underline{p}' - \underline{p})$ has been used, as always. ϵ^* and ϵ are

polarization vectors of outgoing and incoming vector mesons. We shall consider the magnetic moments to be defined as above, whether the states involved have $L = 0$ or 1 .

3. The calculation is done as described below. In order to investigate the consistency of the solution obtained every possible case has to be examined. We, therefore, group the single particle states into fifteen subsets, four consisting of only one type of particle each ($P_0, V_0, P_\alpha, V_\alpha$), six with two types of particles each ($P_0 P_\alpha, P_0 V_\alpha, P_\alpha V_0, V_\alpha P_\alpha, V_\alpha V_0, P_\alpha V_0$), four with three types of particles each, and one with all the four types of particles in it. We then consider the matrix elements of all the commutators (Eqs. (2.3), (2.6), (2.9)) between every possible pair of states formed out of the states belonging to each subset and saturate them by inserting as intermediate states every type of particle that belongs to the subset. In the following this is illustrated by considering the case when all four types of particles are involved. The results of all the calculations are summarized in Table IIA.

(i) Matrix elements between $\langle V_\beta |$ and $| V_\alpha \rangle$

As outlined above, we take the matrix elements of all the commutators and insert in each case in the intermediate state $P_0, P_\gamma, V_\gamma, V_0$. Except in the following cases, we get the trivially consistent solution: $L.H.S. = R.H.S.$

Eq. (2.3a)

The only nonvanishing contributions come from P_γ and V_0 , and we get

$$\begin{aligned}
 & \int \langle V_{\beta'} | A_i^a | P_{\gamma}, k \rangle \frac{d^3 k}{2k^0 (2\pi)^3} \langle P_{\gamma}, k | A_j^b | V_{\alpha} \rangle + \sum_{\gamma=1}^3 \int \langle V_{\beta'} | A_i^a | V_0^{\gamma}, k \rangle \frac{d^3 k}{2k^0 (2\pi)^3} \langle V_0^{\gamma}, k | A_j^b | V_{\alpha} \rangle \\
 & - \int \langle V_{\beta'} | A_j^b | P_{\gamma}, k \rangle \frac{d^3 k}{2k^0 (2\pi)^3} \langle P_{\gamma}, k | A_i^a | V_{\alpha} \rangle - \sum_{\gamma=1}^3 \int \langle V_{\beta'} | A_j^b | V_0^{\gamma}, k \rangle \frac{d^3 k}{2k^0 (2\pi)^3} \langle V_0^{\gamma}, k | A_i^a | V_{\alpha} \rangle \\
 & = i \epsilon_{ijk} \delta^{ab} \langle V_{\beta'} | V_{k^0} | V_{\alpha} \rangle + i \epsilon^{abc} \delta_{ij} \langle V_{\beta'} | A_0^c | V_{\alpha} \rangle ,
 \end{aligned}$$

where $\sum_{\mathbf{r}}$ implies sum over spin states of V_0 . Using eq. (2.10), we obtain,

$$\begin{aligned}
 & \frac{\hbar_A^2}{2m_{\pi}} \epsilon^{a*} \epsilon^b (\delta_{ij} \delta_{\alpha\beta} - \delta_{i\alpha} \delta_{j\beta}) + \frac{g_A^2}{2m_{\omega}} (\underline{\epsilon}^* \cdot \underline{\epsilon} \delta^{ab} - \epsilon^{b*} \epsilon^a) \delta_{j\alpha} \delta_{i\beta} \\
 & - \frac{\hbar_A^2}{2m_{\pi}} \epsilon^{b*} \epsilon^a (\delta_{ij} \delta_{\alpha\beta} - \delta_{j\alpha} \delta_{i\beta}) - \frac{g_A^2}{2m_{\omega}} (\underline{\epsilon}^* \cdot \underline{\epsilon} \delta^{ab} - \epsilon^{a*} \epsilon^b) \delta_{i\alpha} \delta_{j\beta} \\
 & = - 2m_{\rho} \delta^{ab} \underline{\epsilon}^* \cdot \underline{\epsilon} (\delta_{i\alpha} \delta_{j\beta} - \delta_{i\beta} \delta_{j\alpha}) - \delta_{ij} \delta_{\beta\alpha} (\epsilon^{a*} \epsilon^b - \epsilon^a \epsilon^{b*})
 \end{aligned}$$

Comparing the L.H.S. and R.H.S. of this equation, we get,

$$\frac{2\hbar_A^2}{2m_{\pi}} + \frac{g_A^2}{2m_{\omega}} = - 3 f_A ,$$

$$\frac{\hbar_A^2}{2m_{\pi}} + \frac{2g_A^2}{2m_{\omega}} = 6 m_{\rho} ,$$

$$\frac{\hbar_A^2}{2m_{\pi}} = - f_A ,$$

which gives uniquely

$$f_A = -2m_{\rho} , \quad g_A^2 = 4m_{\rho} m_{\omega} , \quad \hbar_A^2 = 4m_{\rho} m_{\pi} \quad (2.11)$$

Eq. (2.3f):

$$f_A = -2m_{\rho} \quad (2.12)$$

Eq. (2.6a):

Making use of eqs. (2.10), we get the following set of results,

$$\frac{\mu_T'^{(0)2}}{2m_{\eta}} + \frac{\mu^2}{m_{\rho}} = \frac{\langle \gamma_{\rho}^2 \rangle}{2} \quad ,$$

$$\frac{\mu_T'^{(0)2}}{2m_{\eta}} + \frac{\mu^2}{m_{\rho}} = -\frac{3F_V}{4} \quad ,$$

$$\frac{\mu_T'^{(0)2}}{2m_{\eta}} + \frac{Q}{4\sqrt{10}} + \frac{F_V}{4} = 0 \quad ,$$

$$-\frac{3\mu_T'^{(0)2}}{2m_{\eta}} + \frac{\mu^2}{m_{\rho}} + \frac{Q}{4\sqrt{10}} - \frac{F_V}{4} = 0 \quad ,$$

$$-\frac{\mu^2}{m_{\rho}} + \frac{\langle \gamma_{\rho}^2 \rangle}{3} - \frac{Q}{6\sqrt{10}} = 0$$

These equations are consistent with each other and imply uniquely

$$Q = 0 \quad ; \quad \frac{3\mu^2}{m_{\rho}} = \langle \gamma_{\rho}^2 \rangle = -\frac{3}{2}F_V = \frac{3\mu_T'^{(0)2}}{m_{\eta}} \quad (2.13)$$

Eq. (2.6b)

$$F_V = -\frac{2\mu_{T\pi}^2}{m_{\pi}} \quad (2.14)$$

Eq. (2.9a):

$$\begin{aligned} & \frac{\mu^2}{2m_p} \left\{ (\delta_{\beta j} \delta_{i\alpha} - \delta_{\beta\alpha} \delta_{ij}) (\epsilon^a \epsilon^{b*} - \delta^{ab} \underline{\epsilon}^* \underline{\epsilon}) - (\delta_{\beta i} \delta_{j\alpha} - \delta_{\beta\alpha} \delta_{ij}) (\epsilon^b \epsilon^{a*} - \delta^{ab} \underline{\epsilon}^* \underline{\epsilon}) \right\} \\ & + \frac{\mu_T'^{(0)2}}{2m_\eta} \left\{ \delta_{i\beta} \delta_{j\alpha} \epsilon^{a*} \epsilon^b - \delta_{i\alpha} \delta_{j\beta} \epsilon^{b*} \epsilon^a \right\} \\ & = - \frac{m_p \mu_q^2}{2} \delta^{ab} (\delta_{i\alpha} \delta_{j\beta} - \delta_{i\beta} \delta_{j\alpha}) \underline{\epsilon}^* \underline{\epsilon} - \frac{\mu_q^2}{4} \delta_{\beta\alpha} \delta_{ij} (\epsilon^{a*} \epsilon^b - \epsilon^a \epsilon^{b*}) f_A \end{aligned}$$

Comparing the L.H.S. and the R.H.S., we get

$$\frac{\mu^2}{m_p} + \frac{\mu_T'^{(0)2}}{2m_\eta} = \frac{3}{2} m_p \mu_q^2 \quad ,$$

$$\frac{\mu^2}{m_p} + \frac{1}{2} f_A \mu_q^2 = 0$$

Therefore, using $f_A = -2m_p$ (as obtained above),

$$\mu^2 = m_p^2 \mu_q^2 \quad ; \quad \mu_T'^{(0)2} = m_\eta m_p \mu_q^2 \quad (2.15)$$

Eq. (2.9b):

$$\mu_{\tau_{is}}^2 = m_\pi m_p \mu_q^2 \quad (2.16)$$

Whenever we use equs. (2.6a), (2.6b) and (2.6c) we always assume that the states have $L = 1$, so that the matrix elements of m_i^a and

M_i^a are the same. Also, while using equs. (2.9a), (2.9b) and (2.9c) we always assume that the states have $L = 0$ and the matrix elements of any $L = 1$ operator between these states are dropped without any comment.

(ii) Matrix elements between $\langle P_0 |$ and $| P_0 \rangle$

Only the trivial result $L.H.S. = R.H.S.$ is obtained.

(iii) Matrix elements between $\langle P_\beta |$ and $| P_\alpha \rangle$

Again $L.H.S. = R.H.S.$ except with the following commutators.

Eq. (2.3a):

Only V_γ has a nonvanishing contribution, and we get

$$\frac{\delta^{ab} \hbar_A^2}{2m_\rho} (\delta_{i\beta} \delta_{j\alpha} - \delta_{i\alpha} \delta_{j\beta}) = 2m_\pi (\delta_{i\beta} \delta_{j\alpha} - \delta_{i\alpha} \delta_{j\beta}) \delta^{ab},$$

with the result

$$\hbar_A^2 = 4m_\pi m_\rho \quad (2.17)$$

Eq. 2.6a):

Only V_0 contributes, and we get

$$\frac{\mu_T^{(0)2}}{2m_\omega} \delta^{ab} (\delta_{i\beta} \delta_{j\alpha} - \delta_{j\beta} \delta_{i\alpha}) = \frac{\delta^{ab}}{6} \langle r_\pi^2 \rangle (\delta_{i\beta} \delta_{j\alpha} - \delta_{i\alpha} \delta_{j\beta}),$$

with the result,

$$\mu_T^{(0)2} = \frac{m_\omega}{3} \langle r_\pi^2 \rangle \quad (2.18)$$

Eq. (2.9a):

As in the previous case, we get

$$\frac{\mu_T^{(0)2}}{2m_\omega} \delta^{ab} (\delta_{i\beta} \delta_{j\alpha} - \delta_{j\beta} \delta_{i\alpha}) = \frac{\mu_q^2}{2} \delta^{ab} (\delta_{i\beta} \delta_{j\alpha} - \delta_{j\beta} \delta_{i\alpha}),$$

so that

$$\mu_T^{(0)2} = m_\pi m_\omega \mu_q^2 \quad (2.19)$$

(iv) Matrix elements between $\langle V_0 |$ and $|V_0 \rangle$

The following are the nontrivial cases:

Eq. (2.3a):

$$\frac{g_A^2}{2m_p} = -f_A^{is} \quad (2.20)$$

Eq. (2.3f):

$$f_A^{is} = -2m_\omega \quad (2.21)$$

Eq. (2.6a):

$$\mu_T^{(0)2} = -\frac{m_\pi}{2} F_V^{is} \quad (2.22)$$

Eq. (2.6b):

$$\mu_{Tis}'^2 = -\frac{m_\eta}{2} F_V^{is} \quad (2.23)$$

Eq. (2.9a):

$$\mu_T^{(0)2} = m_\pi m_\omega \mu_q^2 \quad (2.24)$$

Eq. (2.9b):

$$\mu_{Tis}'^2 = m_\eta m_\omega \mu_q^2 \quad (2.25)$$

(v) Matrix elements between $\langle V_\beta |$ and $|P_\alpha\rangle$.

The nontrivial cases are

Eq. (2.3a):

$$f_A = -2m_p. \quad (2.26)$$

Eq. (2.6c):

$$\begin{aligned} \frac{i\mu \hbar_A}{2m_p} \epsilon^{dba} \epsilon^{d*} (\delta_{\beta j} \delta_{i\alpha} - \delta_{\beta\alpha} \delta_{ij}) + \frac{i\mu_T^{(0)}}{2m_\omega} \delta_{i\alpha} \delta_{j\beta} \epsilon^{dba} \epsilon^{d*} g_A \\ = -\frac{i}{2} \epsilon^{afk} \epsilon^{f*} \epsilon_{ijk} i \epsilon_{k\alpha\beta}^{KVP}, \end{aligned}$$

so that

$$\mu_T^{(0)} = \mu = K_{VP} = 0. \quad (2.27)$$

Eq. (2.9c):

$$\begin{aligned} \frac{i\mu \hbar_A}{2m_p} (\delta_{\beta j} \delta_{i\alpha} - \delta_{\beta\alpha} \delta_{ij}) \epsilon^{dba} \epsilon^{d*} + \frac{i\mu_T^{(0)} g_A}{2m_\omega} \delta_{i\alpha} \delta_{j\beta} \epsilon^{dba} \epsilon^{d*} \\ = i \delta_{\beta\alpha} \epsilon^{c*} G_{VP} \frac{\mu g}{4} \delta_{ij} \epsilon^{abc}, \end{aligned}$$

$$\therefore \frac{\mu \hbar_A}{m_p} = \frac{G_{VP} \mu g}{2}, \quad (2.28)$$

$$\frac{\mu_T^{(0)} g_A}{m_\omega} = -\frac{G_{VP} \mu g}{2}. \quad (2.29)$$

(vi) Matrix elements between $\langle P_\beta |$ and $|P_0\rangle$

The nontrivial cases are

Eq. (2.6c):

$$\mu_T^{(0)'} k_A = 0 \quad (2.30)$$

Eq. (2.9c):

$$-i \epsilon_{j\beta\gamma} \delta^{ab} \delta_{i\gamma} k_A \mu_T^{(0)'} = \frac{i\mu_q}{4} \delta^{ab} \epsilon_{ijk} H_{PP} \delta_{\beta k}$$

$$\frac{k_A \mu_T^{(0)'}}{2m_p} = - \frac{\mu_q H_{PP}}{4} \quad (2.31)$$

(vii) Matrix elements between $\langle P_\beta |$ and $|V_0\rangle$

All commutators give the trivial result L.H.S. = R.H.S.

(viii) Matrix elements between $\langle V_\beta |$ and $|P_0\rangle$

As above L.H.S. = R.H.S.

(ix) Matrix elements between $\langle V_\beta |$ and $|V_0\rangle$

The only nontrivial cases are

Eq. (2.3b):

$$\frac{f_A}{2m_p} = \frac{f_A^{is}}{2m_w} = -1 \quad (2.32)$$

Eq. (2.6c):

$$-\frac{\mu}{2m_p} \epsilon_{i\gamma\beta} \epsilon^{dea} \epsilon^{d*} i \delta_{j\gamma} \epsilon^{feb} \epsilon^{fg} k_A - i \epsilon_{j\gamma\beta} \epsilon^{d*} \frac{k_A \mu_T^{(0)'}}{2m_p} \epsilon^a \delta_{i\gamma}$$

$$= -\frac{i}{2} \epsilon^{afb} \epsilon_{ijk} \delta_{\beta k} i \epsilon^{gef} \epsilon^{g*} k_{VV}$$

Therefore,

$$\mu g_A = \mu_T^{(0)} r_A = K_{VV} = 0 \quad (2.33)$$

Eq. (2.9c):

$$-\frac{i\mu}{2m_p} \epsilon_{ij\beta} (\epsilon^* \epsilon \delta^{ab} - \epsilon^a \epsilon^* \epsilon^b) g_A + \frac{i\mu_A}{2m_\pi} \mu_T^{(0)} \epsilon_{ij\beta} \epsilon^b \epsilon^* \epsilon^a$$

so that,

$$= \frac{i\mu_q}{4} \epsilon_{ij\beta} \delta^{ab} H_{VV} \tilde{\epsilon}^* \tilde{\epsilon}$$

$$\frac{\mu_T^{(0)} r_A}{m_\pi} = \frac{\mu_q H_{VV}}{2} \quad (2.34)$$

$$\frac{\mu g_A}{m_p} = -\frac{1}{2} \mu_q H_{VV} \quad (2.35)$$

(x) Matrix elements between $\langle P_0 |$ and $|V_0 \rangle$

The only nontrivial cases are:

Eq. (2.6c):

$$\mu_T^{(0)'} g_A = 0 \quad (2.36)$$

Eq. (2.9c):

$$-\delta_{ij} \epsilon^{dab} \epsilon^d g_A \frac{\mu_T^{(0)'}}{2m_p} = \frac{i\mu_q}{4} G_{VP}^{is} \delta_{ij} \epsilon^{abc} \epsilon^c$$

$$\frac{g_A \mu_T^{(0)'}}{m_p} = -\frac{\mu_q G_{VP}^{is}}{2} \quad (2.37)$$

We note that the solutions for coupling constant as given by eqs. (2.11), (2.12), (2.17), (2.20), (2.21), (2.26) and (2.32) are all consistent with each other and imply uniquely the following result

$$-\frac{f_A}{2m_\rho} = -\frac{f_A^{is}}{2m_\omega} = \frac{h_A^2}{4m_\rho m_\pi} = \frac{g_A^2}{4m_\rho m_\omega} = 1 \quad (2.38)$$

With pure $L = 1$ magnetic moment operators, we obtain for the magnetic moments the results contained in the eqs. (2.13), (2.14), (2.18), (2.22), (2.23), (2.27), (2.30), (2.33) and (2.36). Using the solutions obtained above for the coupling constant, we find that the only consistent result is that all the magnetic moments vanish. This trivial solution is not physically acceptable, and is the direct consequence of having used a pure $L = 1$ form (eq. (2.7)) for the magnetic dipole moment operator. We note that our method of saturating the commutators with P_0 , V_0 , V_γ and P_γ automatically requires these states to belong to a $(1 \oplus 15, 1)$ dimensional representation of $SU_{1,8}(4) \otimes R_3(L_1)$, so that these states have $L = 0$. To see this, we consider the matrix elements of the orbital angular momentum operator between these states:

$$\begin{aligned} \langle v_\beta | L^a | v_\alpha \rangle &= \langle v_\beta | J^a | v_\alpha \rangle - \langle v_\beta | A_0^a | v_\alpha \rangle \\ &= i \epsilon^{acb} \epsilon^{c*} \epsilon_{\beta\alpha} 2m_\rho (2\pi)^3 \delta^3(\underline{k}) - i \delta_{\beta\alpha} \epsilon^{acb} \epsilon^{c*} \epsilon_{f_A} \chi_\rho (2\pi)^3 \delta^3(\underline{k}) \end{aligned}$$

Our solution $f_A = -2m_\rho$, then implies that

$$\langle v_\beta | L^a | v_\alpha \rangle = 0 \quad .$$

Therefore, the states $|V_\alpha\rangle$ have $L = 0$. Similar arguments show, after using our solutions, eqs. (2.38), that $|V_0\rangle$, $|P_\alpha\rangle$, $|P_0\rangle$ all have $L = 0$. This explains why all the magnetic moments vanished. In order that we may obtain non-vanishing solutions for the magnetic moments we modify the corresponding operators by allowing additional $L = 0$ terms as in eq. (2.8). On repeating this calculation with these operators, we obtain the results given by the eqs. (2.15), (2.16), (2.19), (2.24), (2.25), (2.28), (2.29), (2.31), (2.34), (2.35) and (2.37). These equations are found to be consistent with each other and uniquely imply the following solution,

$$\mu_T^{(0)2} = m_\pi m_\omega \mu_q^2 ; \mu^2 = m_p^2 \mu_q^2 ; \mu_T'^{(0)2} = m_\eta m_p \mu_q^2 ;$$

$$\mu_{Tis}^2 = m_\pi m_p \mu_q^2 ; \mu_{Tis}'^2 = m_\eta m_\omega \mu_q^2 ; G_{VP}^2 = 16 m_p m_\pi ; \quad (2.39)$$

$$H_{PP}^2 = 16 m_\pi m_\eta ; H_{VV}^2 = 16 m_\omega m_p ; G_{VP}^{is2} = 16 m_\omega m_\eta ;$$

and further

$$G_{VP} = \frac{2\mu h_A}{m_p \mu_q} = - \frac{2\mu_T^{(0)} g_A}{m_\omega \mu_q} ;$$

$$H_{PP} = - \frac{2h_A \mu_T^{(0)'}}{\mu_q m_p} ; \quad (2.39a)$$

$$H_{VV} = \frac{2\mu_T^{(0)} h_A}{\mu_q m_\pi} = - \frac{2\mu g_A}{\mu_q m_p} ;$$

$$G_{VP}^{is} = - \frac{2g_A \mu_T'^{(0)'}}{m_p \mu_q}$$

In arriving at eqs. (2.39) we have used eqs. (2.38).

As outlined earlier, the whole calculation is repeated with different sets of particles. The results are summarized in Table (IIA). We note that consistent solutions are obtained only when all the four types of particles $P_0, V_0, P_\alpha, V_\alpha$ are considered, and in this case they constitute a $(16, 1)$ dimensional reducible representation of $SU(4) \otimes R_3(L_1)$.

4. In order to compare our results with $SU(6)$ predicted values, we make use of generalized Goldberger Treiman relations for the weak couplings. We shall also use our results on magnetic moments to obtain information on radiative decays of mesons. But before we go on to consider these matters, a few remarks on our results are in order. One noticeable feature is that we do not find any ambiguity in the sign of the coupling constants in addition to the ones already existing in the $SU(6)$ calculations of refs. (6) and (7). This is in contrast to the case of baryons where the $SU(6)$ calculation predicts unambiguously the correct sign of the axial-vector renormalization constant, while the sign is left undetermined in the corresponding $SU(4)$ calculation⁽¹⁰⁾. The reason is, whereas for baryons the particles and antiparticles belong to two conjugate representations which are distinct at the level of $SU(3)$ but equivalent at that of $SU(2)$, in the case of mesons both particles and antiparticles belong to one and the same representation both at $SU(3)$ and at $SU(2)$ levels. Another interesting feature is that our results on magnetic moments obtained from eqs. (2.9) and eqs. (2.6) give the same relation

$$\frac{\mu^2}{m_p^2} = \frac{\mu_T^{(0)2}}{m_p m_\eta} = \frac{\mu_T^{(0)2}}{m_\pi m_\omega} = \frac{\mu_{Tis}^2}{m_p m_\pi} = \frac{\mu_{Tis}^2}{m_\omega m_\eta}, \quad (2.40)$$

except for the fact that this is trivially true in the latter case, since all the magnetic moments vanish there. This is analogous to Ryan's⁽¹⁰⁾ observation that the magnetic moment ratio of neutron and proton obtained after taking account of the orbital angular momentum of the states is the same (apart from a sign ambiguity which arises for some other reason) as given by Lee, in spite of the fact that Lee did not take account of this, and, in fact, should have obtained vanishing magnetic moments. This encourages us to believe in the correctness of our relation between the various magnetic moments. A final remark we want to make is that we have not investigated the effect of including scalar, axial and tensor mesons and also various mixed states⁽¹⁴⁾. Nor could we find any information on the mean-square-radii and quadrupole moments of the mesons. They would, of course, vanish if our definitions eqs. (2.5) are correct. Presumably, however, these equations should be modified by adding suitable $L = 0$ terms in them. We do not investigate these generalizations. Our intention is only to see whether the good results of SU(6) symmetry for the nonstrange vector and pseudoscalar mesons can be obtained by using only SU(4) algebra and to illustrate the simplicity of the calculation as compared to the corresponding calculation using SU(6) algebra as given in refs. (6) and (7). We have seen that this is indeed the case, and we obtain a unique and consistent set of solutions.

We now apply the PCAC assumption (which would imply Goldberger-Treiman relations, referred to above) to obtain relations between the phenomenological strong coupling constants of mesons. For states of arbitrary momentum (suppressing isospin indices) the matrix elements of the axial currents are defined as follows,

$$\langle V' | a^\mu(0) | V \rangle = i \epsilon^{\mu\nu\rho\sigma} \epsilon_\nu^* \epsilon_\rho p'_\sigma F_1(t) + i \epsilon^{\mu\nu\rho\sigma} \epsilon_\nu^* \epsilon_\rho p_\sigma F_2(t), \quad (2.41a)$$

(Note $\epsilon^{0123} = +1$)

$$\langle V' | a^\mu(0) | P \rangle = \epsilon^{*\mu} H_1(t) + \epsilon^* \cdot p p^\mu H_2(t) + \epsilon^* \cdot p p^\mu H_3(t), \quad (2.41b)$$

where $t = -(p' - p)^2$, in obvious notations. In the limit $p, p' \rightarrow 0$, we obtain

$$\langle V' | a^a | V \rangle = m_{V'} i \epsilon^{dea} \epsilon^{d*} \epsilon^e F_1(t) + m_V i \epsilon^{dea} \epsilon^{d*} \epsilon^e F_2(t), \quad (2.42a)$$

$$\langle V' | a^a | P \rangle = \epsilon^{*a} H_1(t), \quad (2.42b)$$

where $t = + (m_{V'} - m_V)^2$. Comparing this, with our definitions (2.10), we get

$$g_A = m_\omega F_1^{\omega p}(t) + m_\rho F_2^{\omega p}(t), \quad t = (m_\omega - m_\rho)^2 \quad (2.43a)$$

$$f_A = m_\rho F_1^{\rho p}(t) + m_\rho F_2^{\rho p}(t), \quad t = 0 \quad (2.43b)$$

$$f_A^{i5} = m_\omega F_1^{\omega\omega}(t) + m_\omega F_2^{\omega\omega}(t) , \quad t = 0 \quad ; \quad (2.43c)$$

$$h_A = H_1^{\rho\pi}(t) , \quad t = (m_\rho - m_\pi)^2 . \quad (2.43d)$$

From eq. (2.41a):

$$\langle V' | \partial_\mu a^\mu | V \rangle = \epsilon^{\mu\nu\rho\sigma} p'_\mu \epsilon'_\nu \epsilon_\rho p_\sigma (F_1(t) + F_2(t)) . \quad (2.44)$$

We define

$$\langle 0 | \partial_\mu a^\mu | P \rangle = f_P m_P^2 \quad [P = \pi^i, \eta] , \quad (2.45)$$

then

$$\langle V'_0 | \partial_\mu a^\mu_i | V_\gamma \rangle = \frac{m_\pi^2 f_\pi}{m_\pi^2 - t} \delta_{i\gamma} \langle V'_0 | J_\pi | V \rangle . \quad (2.46)$$

We do not know how to calculate $\langle \omega' | J_\pi | \rho \rangle$ for arbitrary t .

But for $t = m_\pi^2$

$$\langle \omega' | J_\pi | \rho \rangle = \epsilon^{\mu\nu\rho\sigma} p'_\mu \epsilon'_\nu \epsilon_\rho p_\sigma g_{\omega\rho\pi} , \quad (2.47)$$

where $g_{\omega\rho\pi}$ is the physical coupling constant at $(\rho - \omega - \pi)$ vertex.

In the spirit of pion-pole-dominance (PCAC) we assume that this is true also for $t \rightarrow 0$ (and also in the neighbourhood of $t = m_\pi^2$).

Then for $t \rightarrow 0$ (i.e. $m_\omega \approx m_\rho$), we get from (2.43a), (2.44),

(2.45), (2.46) and (2.47) the result,

$$f_\pi g_{\omega\rho\pi} = \frac{g_A}{m_\rho} = \pm 2 , \quad (m_\rho \approx m_\omega) \quad (2.48a)$$

Similarly,

$$f_{\eta} g_{\rho\rho\eta} = \frac{f_A}{m_{\rho}} = -2 \quad (2.48b)$$

Also

$$f_{\eta} g_{\omega\omega\eta} = \frac{f_A^{is}}{m_{\omega}} = -2 \quad (2.48c)$$

where $g_{\rho\rho\eta}$ and $g_{\omega\omega\eta}$ are defined as in eq. (2.47). Eqs. (2.48b) and (2.48c) are true without requiring $m_{\rho} = m_{\omega}$, and furthermore they imply

$$g_{\rho\rho\eta} = g_{\omega\omega\eta} \quad (2.49a)$$

The sign is also determined in this case. From eq. (2.43d), we get as $t \rightarrow 0$ (i.e., assuming $m_{\rho} = m_{\pi}$, which is far from true)

$$f_{\pi} g_{\rho\pi\pi} = \frac{R_A}{2} = \pm m_{\rho} \quad (2.48d)$$

where $g_{\rho\pi\pi}$ is defined by $J_{\rho \rightarrow \pi\pi} = g_{\rho\pi\pi} i\varepsilon.(p+p')$.

Hence

$$\left(\frac{g_{\rho\pi\pi}}{g_{\omega\rho\pi}} \right)^2 = \frac{m_{\rho}^2}{4} \quad (2.49b)$$

in the approximation, $m_{\pi} = m_{\omega} = m_{\rho}$. This is also obtained if one assumes SU(6) symmetry⁽⁸⁾ and agrees with the results of the Gell-Mann, Sharp, Wagner⁽¹⁵⁾ model.

We now consider the implications of our relation between the various magnetic moments (eq. (2.40)). The equality

$$\frac{\mu_T^{(0)2}}{m_\pi m_\omega} = \frac{\mu_T'^{(0)2}}{m_\rho m_\eta} ,$$

implies

$$\frac{\Gamma(\omega^0 \rightarrow \pi^0 \gamma)}{\Gamma(\rho^0 \rightarrow \eta \gamma)} = \frac{m_\pi m_\omega}{m_\eta m_\rho} , \quad (2.50)$$

i.e., the rates of radiative transitions $\omega \rightarrow \pi^0 \gamma$ and $\rho^0 \rightarrow \eta \gamma$ will be equal if $m_\rho = m_\omega$ and $m_\pi = m_\eta$. This is just one of the many results of SU(3) symmetry applied to weak and electromagnetic transitions, under the assumption that the symmetry breaking interaction is negligible⁽¹⁶⁾. Next we show that the equality

$$\frac{\mu_T^{(0)2}}{m_\pi m_\omega} = \frac{\mu^2}{m_\rho^2} ,$$

implies that the rate for the radiative transition $\omega \rightarrow \pi^0 \gamma = 1.2$ MeV, in agreement with experiment. This follows if we assume that the magnetic moment of ρ^+ and P are the same (up to a constant determined by our definition of these magnetic moments). For μ_P defined in the standard way and for μ defined by our eq. (2.10) we should have

$$2 m_\rho \mu_P = \mu , \quad (\mu_P = 2.8 \frac{e}{2M_P}) . \quad (2.51)$$

This assumption is equivalent to assuming that all the charge form factors are dominated by the ρ meson. (This eq. (2.51) is also true in a nonrelativistic quark model⁽¹⁷⁾, so that our μ_q is just $2\mu_P$). Defining the coupling constant $g(\omega\pi)$ covariantly as follows,

$$\mathcal{L}_{\omega\pi\gamma} = g(\omega\pi) \epsilon^{\mu\nu\rho\sigma} \partial_\mu \omega_\nu \partial_\rho A_\sigma \pi, \quad (2.52)$$

we get in the rest frame of ω

$$\mathcal{L}_{\omega\pi\gamma} = \frac{m_\omega}{i} g(\omega\pi) \underline{\omega} \cdot \underline{\mathcal{H}} \pi, \quad (2.53)$$

where $\underline{\mathcal{H}}$ is the magnetic field of the photon. This expression is just the transition moment which we have defined by

$$\frac{1}{i} \mu_T^{(0)} \underline{\omega} \cdot \underline{\mathcal{H}} \pi,$$

$$\mu_T^{(0)} = m_\omega g(\omega\pi) \quad (2.54)$$

Using eq. (2.52), the width of ω^0 corresponding to the radiative decay $\omega^0 \rightarrow \pi^0 \gamma$ is given by the expression

$$\Gamma(\omega^0 \rightarrow \pi^0 \gamma) = \frac{g^2(\omega\pi)}{12\pi} \left(\frac{m_\omega^2 - m_\pi^2}{2m_\omega} \right)^3 \quad (2.55)$$

Comparing eqs. (2.51) and (2.54), we get

$$g(\omega\pi) = \frac{\mu_T^{(0)}}{m_\omega} = 2\mu_P \sqrt{\frac{m_\pi}{m_\omega}}$$

If we assume $m_\pi \approx m_\omega$, then

$$g(\omega\pi) = 2\mu_P \quad (2.56)$$

This is then substituted in eq. (2.55).

TABLE IIA

All the results are shown only in those cases where consistent solutions are obtained. Whenever there is an inconsistency only a few examples are cited and the rest ignored.

SET 1 : P_0

No nontrivial result. $\langle P_0 | L^a | P_0 \rangle = 0$, therefore P_0 by itself belongs to (1, 1) representation of $SU(4) \otimes R_3(L_1)$, in our approximation.

SET 2 : V_0

Eq. (2.3a): $f_A^{1s} = 0$; Eq. (2.3f): $f_A^{1s} = -2m_\omega$. Inconsistent. $\langle V_0 | L^a | V_0 \rangle$ undetermined. $SU(4) \otimes R_3(L_1)$ representation to which V_0 by itself alone would belong does not exist.

SET 3 : P_α

Eq. (2.3a): $m_\pi = 0$. Impossible. P_α by itself does not belong to a $SU(4) \otimes R_3(L_1)$ representation.

SET 4 : V_α

Eq. (2.3a): $f_A = 0$, $m_\rho = 0$. Impossible. V_α by itself does not belong to a $SU(4) \otimes R_3(L_1)$ representation.

SET 5 : P_0, V_0

$\langle V_0 | \dots | V_0 \rangle$: Eq. (2.3a): $f_A^{1s} = 0$; Eq. (2.3f): $f_A^{1s} = -2m_\omega$. Inconsistent. L -value of the states not determined. Hence P_0, V_0 together by themselves (without other states) do not belong to any $SU(4) \otimes R_3(L_1)$ representation.

TABLE IIA (Contd.)

SET 6 : P_0, V_α

$\langle V_\beta | \dots | V_\alpha \rangle$: Eq. (2.3a) : $f_A = 0, m_\rho = 0$. Impossible.
 P_0 and V_α together by themselves cannot form any representation of $SU(4) \otimes R_3(L_1)$ as there is no consistent solution.

SET 7 : P_0, P_α

$\langle P_\beta | \dots | P_\alpha \rangle$: Eq. (2.3a) : $m_\pi = 0$. Impossible. Same remark as above.

SET 8 : V_0, V_α

$\langle V_\beta | \dots | V_\alpha \rangle$: Eq. (2.3a) : $f_A = 0, m_\rho = 0$. Impossible. Same remark as above.

SET 9 : P_α, V_α

$\langle V_\beta | \dots | V_\alpha \rangle$: Eq. (2.3a) : $f_A = 0, m_\rho = 0$. Impossible. Same remarks as above.

SET 10 : P_α, V_0

$\langle P_\beta | \dots | P_\alpha \rangle$: Eq. (2.3a) : $m_\pi = 0$. Impossible. Same remarks as above.

TABLE IIA (Continued)

SET 11 : P₀, V₀, P_α

$\langle P_\beta | \dots | P_\alpha \rangle$: Eq. (2.3a) : $m_\pi = 0$. Impossible. Same remarks as in the previous case.

SET 12 : P₀, V₀, V_α

$\langle V_\beta | \dots | V_\alpha \rangle$: Eq. (2.3a) : $f_A = 0, g_A = 0, m_\rho = 0$. Impossible. Same remarks as above.

SET 13 : P₀, P_α, V_α

$\langle V_\beta | \dots | V_\alpha \rangle$: Eq. (2.3a) : $f_A = 0, m_\rho = 0$. Impossible. Same remarks as above.

SET 14 : V₀, V_α, P_α

$\langle P_\beta | \dots | P_\alpha \rangle$: Eq. (2.3a) : $k_A^2 = 4 m_\pi m_\rho$;

Eq. (2.6a) : $\mu_T^{(0)2} = \frac{1}{3} m_\omega \langle \gamma_\pi^2 \rangle$;

Eq. (2.9a) : $\mu_T^{(0)2} = m_\pi m_\omega \mu_q^2$;

$\langle V_\beta | \dots | V_\alpha \rangle$: Eq. (2.3a) : $f_A = -2 m_\rho, g_A^2 = 4 m_\rho m_\omega, k_A^2 = 4 m_\rho m_\pi$;

Eq. (2.3f) : $f_A = -2 m_\rho$;

Eq. (2.6a) : $Q = 0, \mu^2 = \langle \gamma_p^2 \rangle = F_V = 0$;

Eq. (2.6b) : $F_V = - 2 \mu_{Tis}^2 / m_\pi$;

TABLE IIA (Continued)

	Eq. (2.9a) :	$\mu^2 = m_p^2 \mu_q^2, m_\eta m_p \mu_q^2 = 0;$	(Impossible!)
	Eq. (2.9b) :	$\mu_{T_{is}}^2 = m_\pi m_p \mu_q^2 ;$	
$\langle v_0 \dots v_0 \rangle :$	Eq. (2.3a) :	$g_A^2 = -2 m_p f_A^{is} ;$	
	Eq. (2.3f) :	$f_A^{is} = -2 m_\omega ;$	
	Eq. (2.6a) :	$\mu_T^{(0)2} = -\frac{m_\pi}{2} F_V^{is} ;$	
	Eq. (2.6b) :	$m_\eta F_V^{is} = 0 ;$	
	Eq. (2.9a) :	$\mu_T^{(0)2} = m_\pi m_\omega \mu_q^2 ;$	
	Eq. (2.9b) :	$m_\eta m_\omega \mu_q^2 = 0 ;$	(Impossible!)
$\langle v_\beta \dots p_\alpha \rangle :$	Eq. (2.3b) :	$f_A = -2 m_p ;$	
	Eq. (2.6c) :	$\mu_T^{(0)} = \mu = K_{VP} = 0 ;$	
	Eq. (2.9c) :	$G_{VP} \mu_q = 2\mu h_A / m_p = -2\mu_T^{(0)} g_A / m_\omega ;$	
$\langle p_\beta \dots v_0 \rangle :$	Trivial result L.H.S. \equiv R.H.S. ;		
$\langle v_\beta \dots v_0 \rangle :$	Eq. (2.3b) :	$f_A = -2 m_p, f_A^{is} = -2 m_\omega ;$	

TABLE IIA (Continued)

$$\langle v_\beta | \dots | v_0 \rangle: \text{Eq. (2.6c)} : \mu g_A = \mu_T^{(0)} r_A = K_{VV} = 0 ;$$

$$\text{Eq. (2.9c)} : \frac{\mu_T^{(0)} r_A}{m_\pi} = \frac{\mu_q H_{VV}}{2}, \quad \frac{\mu g_A}{m_p} = -\frac{1}{2} \mu_q H_{VV}.$$

Since m_η, m_ω, m_ρ do not vanish, we get $\mu_q = 0$. This result is not acceptable. Thus, with our definitions of the moment operators (eqs. (2.4) and (2.8)) we can not get acceptable solutions with nonvanishing magnetic moments by just considering V_0, V_α and P_α . It is to be noted that, if we confine ourselves only to the vector and axial charges we get a unique and consistent solution for the weak couplings, and this solution remains unchanged whether P_0 is included or not. The reason is that the relevant charge operators have no nonvanishing matrix elements when one of the states involved is P_0 (cf. eqs. (2.10)). Furthermore, considering the matrix elements of $L^a = J^a - A_0^a$ between these states we find that they have $L = 0$. Our expectation that with pure $L = 1$ moment operators (eqs. (2.4)) we ought to get zero magnetic moments is vindicated by requiring consistency between solutions obtained for eqs. (2.6) listed above. However, we do not get sensible results with the moment operators defined in eqs. (2.8) even though they possess an additional $L = 0$ term. We interpret this rather paradoxical result as follows:-

We maintain that eqs. (2.8) have the correct form. Our method of saturating the sum rules has forced P_α, V_α and V_0 states to form a (15, 1) dimensional irreducible representation of

TABLE IIA (Continued)

SU(4) \otimes R₃(L₁). In order to get a consistent result for the magnetic moments we should use a reducible representation (16 = 1 \oplus 15, 1) of SU(4) \otimes R₃ formed by P₀, V_a, P_a, V₀. We do find a consistency when all these particles are considered, as discussed in section 3, the results of which are again summarized below.

SET 15 : P₀, V₀, P_a, V_a

The following are the consistent solutions obtained for the various weak coupling and magnetic moment form-factors:

$$f_A = -2m_p \quad ; \quad f_A^{is} = -2m_\omega \quad ; \quad h_A^2 = 4m_p m_\pi \quad ; \quad g_A^2 = 4m_p m_\omega$$

$$\mu_T^{(0)2} = m_\pi m_\omega \mu_q^2 \quad ; \quad \mu^2 = m_p^2 \mu_q^2 \quad ; \quad \mu_T'^{(0)2} = m_\eta m_p \mu_q^2 \quad ; \quad \mu_{Tis}^2 = m_\pi m_p \mu_q^2$$

$$\mu_{Tis}'^2 = m_\eta m_\omega \mu_q^2 \quad ; \quad G_{VP}^2 = 16m_p m_\pi \quad ; \quad H_{PP}^2 = 16m_\pi m_\eta$$

$$H_{VV}^2 = 16m_\omega m_p \quad ; \quad G_{VP}^{is2} = 16m_\omega m_\eta$$

Also

$$G_{VP} = \frac{2\mu h_A}{m_p \mu_q} = - \frac{2\mu_T^{(0)} g_A}{m_\omega \mu_q} \quad ; \quad H_{PP} = - \frac{2h_A \mu_T^{(0)'}}{\mu_q m_p}$$

$$H_{VV} = \frac{2\mu_T^{(0)} h_A}{\mu_q m_\pi} = - \frac{2\mu g_A}{\mu_q m_p} \quad ; \quad G_{VP}^{is} = - \frac{2g_A \mu_T^{(0)'}}{\mu_q m_p}$$

CHAPTER III

SUM RULES FOR PSEUDOSCALAR AND VECTOR MESONS SCATTERING

1. Sum rules for $(\rho-\pi)$ scattering involving only strong interaction parameters like masses of mesons and their couplings with each other were first derived by de Alfaro, Fubini, Furlan and Rosetti⁽¹⁾ without using current algebras. They followed a dispersion theoretic approach. In order to write down unsubtracted dispersion relations one makes assumptions concerning the high energy bounds of the scattering amplitudes. The convergence properties of the relevant dispersion integrals are assumed to be given by a Regge-pole model. In this model, the high energy behaviour of each amplitude is determined by the leading Regge trajectory, which is allowed to be exchanged in the crossed channel of the process under consideration. The trajectory is characterized by its intercept $\alpha(t=0)$ corresponding to $t=0$ (and also by its slope). Whereas the amplitude for scattering of scalar particles has the Froissart⁽²⁾-Gribov⁽³⁾ high energy bound, for particles with spin certain amplitudes corresponding to strong helicity-flip in the t -channel are found to have a more convergent (superconvergent) asymptotic bound. This has been shown explicitly by Trueman⁽⁴⁾. The convergence of the dispersion integral, therefore, depends on $\alpha(0)$ corresponding to the leading trajectory exchanged. Assuming $\alpha^{I=2}(0)$ (corresponding to $I=2$ exchange trajectory) to be negative, de Alfaro et al. obtained two nontrivial superconvergence relations. Certain other superconvergence relations were found to be trivially satisfied by crossing. Estimating the dispersion integrals with just 0^- and 1^- meson single particle

intermediate states they deduced two reasonable sum rules. Low⁽⁵⁾ pointed out that, under the assumptions of ref. (1), many more superconvergence relations could be written down even to first order in t ; the number increases as we go to higher orders in t . Saturation with 0^- and 1^- mesons only led to a trivial solution, in which all the coupling constants vanished. Inserting, in addition, 1^+ and 2^+ meson states, a reasonable consistency was found by Frampton and Taylor⁽⁶⁾. In this chapter we present a generalization of the above problem to the case of $SU(3)$ symmetric vector meson (V)-pseudoscalar meson (P) scattering. In this case we get more superconvergence relations. The most convergent of these sum rules has already been considered by Matsuda⁽⁷⁾. We examine all the sum rules at $t = 0$. We find that our results are in agreement with ref. (6) and ref. (7), and by retaining only up to first order terms in t no new results are obtained. Higher order terms in t may give new information, but our saturation with particles up to spin two will be less satisfactory.

2. The $SU(3)$ symmetric (V-P) scattering matrix is decomposed into the following kinematic form:

$$T = A \epsilon_2 \cdot P \epsilon_1 \cdot P + \frac{B}{2} (\epsilon_2 \cdot P \epsilon_1 \cdot Q + \epsilon_2 \cdot Q \epsilon_1 \cdot P) + C_1 \epsilon_2 \cdot Q \epsilon_1 \cdot Q + C_2 \epsilon_2 \cdot \epsilon_1,$$

where the amplitudes A , B , C_1 and C_2 are functions of the invariants ν and t , defined below:

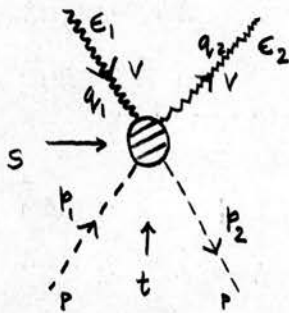


Fig. IIIA

$$P = \frac{1}{2}(p_1 + p_2) \quad S = - (p_1 + q_1)^2$$

$$v = - P \cdot Q$$

$$Q = \frac{1}{2}(q_1 + q_2) \quad t = - (p_1 - p_2)^2$$

$$\text{metric : } p^2 = - p_0^2 + \underline{p}^2 = -m^2.$$

ϵ_2 and ϵ_1 are polarization vectors of the vector mesons as shown. The various amplitudes behave asymptotically as follows⁽¹⁾:

$$A(v, t) \sim v^{\alpha(t)-2} \quad \text{as } v \rightarrow \infty ,$$

$$B(v, t) \sim v^{\alpha(t)-1} \quad \text{as } v \rightarrow \infty ,$$

$$C_1(v, t) \sim v^{\alpha(t)} \quad \text{as } v \rightarrow \infty ,$$

$$C_2(v, t) \sim v^{\alpha(t)} \quad \text{as } v \rightarrow \infty ,$$

where $\alpha(t)$ refers to the dominant Regge trajectory in the crossed, $P + \bar{P} \rightarrow V + \bar{V}$, t -channel. Experimental results suggest that the Pomeron trajectory and the trajectories associated with the nonet of vector mesons, all have intercepts such that $1 \gg \alpha(0) \gg 0$. Hence $A(v, t)$ is always superconvergent. If we further assume $\alpha^{(27)}(0)$ to be negative we can write superconvergence relations for $B^{(27)}(v, t)$ and $vA^{(27)}(v, t)$ also. The superscript 27 implies that it is the part of the amplitude corresponding to an exchange of a 27-plet in the t -channel. The assumption that $\alpha^{(10)}(0)$ is negative as made in ref. (7) yields no new relation, but it improves the convergence of the dispersion integrals and makes the saturation with single

particle states up to spin two more justifiable, especially for the most convergent case of $A^{(10)}$ and $A^{(10^*)}$. Crossing symmetry makes many of the superconvergence relations trivial, and the only nontrivial ones are:

$$\int_0^\infty \text{Im} A^{(8F)}(\nu, t) d\nu = 0, \quad (3.1a)$$

$$\int_0^\infty \nu \text{Im} A^{(27)}(\nu, t) d\nu = 0, \quad (3.1b)$$

$$\int_0^\infty \text{Im} A^{(10)}(\nu, t) d\nu = 0, \quad (3.1c)$$

$$\int_0^\infty \text{Im} B^{(27)}(\nu, t) d\nu = 0. \quad (3.1d)$$

A similar relation for $A^{(10^*)}$ is equivalent to that for $A^{(10)}$ because of charge conjugation invariance. Eqs. (3.1) are valid for fixed $t \leq 0$, and also over a small range of $t > 0$; we restrict ourselves only to the $t = 0$ case.

These relations are well represented if we consider the following specific processes:

$$\rho^- + \pi^+ \rightarrow \rho^+ + \pi^- \text{ as a representative of } \int_0^\infty \text{Im} B^{(27)}(\nu) d\nu = 0, \\ \text{and } \int_0^\infty \nu \text{Im} A^{(27)}(\nu) d\nu = 0,$$

$$\rho^+ + k^0 \rightarrow k^{*0} + \pi^+ \text{ as a representative of } \int_0^\infty \text{Im} A^{(10)}(\nu) d\nu = 0,$$

$$\text{and } \rho^- + \pi^+ \rightarrow \rho^- + \pi^+ \text{ as a representative of } \int_0^\infty \text{Im} A^{(8F)}(\nu) d\nu = 0,$$

where the particle symbols specify the corresponding SU(3) states.

3. In order to evaluate the single particle contributions to the dispersion integrals, we consider the following form for the absorptive part of the T-matrix,

$$E_{2\mu}(q_2) E_{1\nu}(q_1) t_{\mu\nu}^{\alpha\beta\gamma\delta} = i E_{2\mu} E_{1\nu} \int \langle p_2^\alpha | [g_\mu^\beta(x), g_\nu^\gamma(0)] | p_1^\delta \rangle e^{-iq_2 \cdot x} d^4x.$$

Inserting single particle meson states up to spin two into this equation, we get

$$\begin{aligned} & E_{2\mu}(q_2) E_{1\nu}(q_1) t_{\mu\nu}^{\alpha\beta\gamma\delta} \\ &= \pi E_{2\mu}(q_2) E_{1\nu}(q_1) \left\{ \frac{\delta(q_{20} + p_{20} - p_{n0})}{2E_n} \langle p^\alpha(p_2) | g_\mu^\beta | p^\gamma(p_n) \rangle \langle p^\gamma(p_n) | g_\nu^\delta | P^\epsilon(p_1) \rangle \right. \\ & \quad + \frac{\delta(q_{20} + p_{20} - p_{n0})}{2E_n} \langle p^\alpha(p_2) | g_\mu^\beta | v^\gamma(p_n) \rangle \langle v^\gamma(p_n) | g_\nu^\delta | P^\epsilon(p_1) \rangle \\ & \quad + \frac{\delta(q_{20} + p_{20} - p_{n0})}{2E_n} \langle p^\alpha(p_2) | g_\mu^\beta | v^0(p_n) \rangle \langle v^0(p_n) | g_\nu^\delta | P^\epsilon(p_1) \rangle \\ & \quad + \frac{\delta(q_{20} + p_{20} - p_{n0})}{2E_n} \langle p^\alpha(p_2) | g_\mu^\beta | A^\gamma(p_n) \rangle \langle A^\gamma(p_n) | g_\nu^\delta | P^\epsilon(p_1) \rangle \\ & \quad + \frac{\delta(q_{20} + p_{20} - p_{n0})}{2E_n} \langle p^\alpha(p_2) | g_\mu^\beta | T^\gamma(p_n) \rangle \langle T^\gamma(p_n) | g_\nu^\delta | P^\epsilon(p_1) \rangle \\ & \quad \left. - [\beta \leftrightarrow \delta, \mu \leftrightarrow \nu, p_n \leftrightarrow p'_n] \delta(p'_{n0} + q_{20} - p_{10}) \cdot (3.2) \right\} \end{aligned}$$

where summation over all possible polarization states of the intermediate state is understood. The polarization sum for the vector as well as the axial vector mesons is given by

$$\sum_{\gamma=1}^3 \epsilon_{\mu}^{(\gamma)}(p) \epsilon_{\nu}^{(\gamma)*}(p) = \delta_{\mu\nu} + \frac{p_{\mu} p_{\nu}}{m^2} = \theta_{\mu\nu}$$

and, for the spin two mesons it is given by

$$\sum_{\gamma=1}^5 \epsilon_{\mu\nu}^{(\gamma)}(p) \epsilon_{\rho\sigma}^{(\gamma)*}(p) = -\frac{1}{3} \theta_{\mu\nu} \theta_{\rho\sigma} + \frac{1}{2} \theta_{\mu\rho} \theta_{\nu\sigma} + \frac{1}{2} \theta_{\mu\sigma} \theta_{\nu\rho}$$

The conserved currents g_{μ} are sources of the phenomenological fields for the vector mesons and the values of their form-factors, when the momentum transfer is equal to the square of the mass of the vector meson, are just the corresponding phenomenological coupling constants. The various matrix elements of interest are:

$$\langle P_k(p_2) | g_{\mu}^i(0) | P_j(p_1) \rangle = i f^{ijk} V_{PP} (p_2 - p_1)^2 (p_2 + p_1)_{\mu} ,$$

$$\stackrel{\text{def.}}{=} V_{PP}^{ijk}(k^2) (p_1 + p_2)_{\mu} , \quad [k = p_2 - p_1]$$

$$\langle P_k(p_2) | g_{\mu}^i(0) | V_j(p_1) \rangle = d_{ijk} [i V_{PV} (p_2 - p_1)^2 \epsilon_{\mu\rho\lambda\sigma} p_{1\rho} \epsilon_{\lambda}^{\nu} p_{2\sigma}]$$

$$\stackrel{\text{def.}}{=} i V_{PV}^{ijk}(k^2) \epsilon_{\mu\rho\lambda\sigma} p_{1\rho} \epsilon_{\lambda}^{\nu} p_{2\sigma} .$$

$$\langle P_R(p_2) | g_\mu^i(0) | V_0(p_1) \rangle = d o i k [i V_{p\nu 0} (p_2 - p_1)^2 \epsilon_{\mu\rho\lambda\sigma} p_{1\rho} \epsilon_\lambda^\nu p_{2\sigma}] ,$$

$$\stackrel{\text{def.}}{=} i V_{p\nu 0}^{iok} (k^2) \epsilon_{\mu\rho\lambda\sigma} p_{1\rho} \epsilon_\lambda^\nu p_{2\sigma} .$$

$$\langle P_R(p_2) | g_\mu^i(0) | A_j(p_1) \rangle = i f^{ijk} [i V_{PA}^{(1)} (p_2 - p_1)^2 \epsilon_\mu^A(p_1) + i V_{PA}^{(2)} \epsilon^A \cdot p_2 p_{2\mu}] ,$$

$$\stackrel{\text{def.}}{=} i V_{PA}^{(1)ijk} (k^2) \epsilon_\mu^A(p_1) + i V_{PA}^{(2)ijk} (k^2) \epsilon^A \cdot p_2 p_{2\mu} .$$

$$\langle P_R(p_2) | g_\mu^i(0) | T_j(p_1) \rangle = i f^{ijk} [V_{PT} (p_2 - p_1)^2 \epsilon_{\mu\rho\lambda\sigma} p_{1\rho} \epsilon_{\lambda\nu} p_{2\nu} p_{2\sigma}]$$

$$\stackrel{\text{def.}}{=} V_{PT}^{ijk} (k^2) \epsilon_{\mu\rho\lambda\sigma} p_{1\rho} \epsilon_{\lambda\nu} p_{2\nu} p_{2\sigma} .$$

where $i, j, k = 1, 2, \dots, 8$ are $SU(3)$ indices, and $\mu, \nu = 0, 1, 2, 3$ are Lorentz indices. Only C-conserving couplings have been retained. The V's are the form-factors and $\epsilon_{\lambda\nu}$ is the polarization tensor describing the spin states of the tensor mesons, T. The A's refer to the axial-vector mesons. The coefficients of the various invariants $\epsilon_2 \cdot P, \epsilon_1 \cdot P, \epsilon_2 \cdot P, \epsilon_1 \cdot Q$ etc. appearing in eq. (32) give $\text{Im A}, \text{Im B}$ etc., the imaginary parts of the amplitudes. They are substituted in the appropriate superconvergence relations eqs. (3.1) and, after carrying out the integrations,

we arrive at the corresponding sum rules. The various terms in eq. (3.2) are found to be:

Contribution of P to the +ve part of the commutator:

$$\frac{\pi}{2} \delta(\nu + \nu_P) V_{PP}^{\alpha\beta\gamma} (-m_V^2) V_{PP}^{\gamma\delta\epsilon} (-m_V^2) 4 (\epsilon_2 \cdot P \epsilon_1 \cdot P + \epsilon_2 \cdot P \epsilon_1 \cdot Q + \epsilon_2 \cdot Q \epsilon_1 \cdot P + \dots)$$

Contribution of P to the -ve part of the commutator:

$$-\frac{\pi}{2} \delta(\nu - \nu_P) V_{PP}^{\alpha\delta\gamma'} (-m_V^2) V_{PP}^{\gamma'\beta\epsilon} (-m_V^2) 4 (\epsilon_2 \cdot P \epsilon_1 \cdot P - \epsilon_2 \cdot P \epsilon_1 \cdot Q - \epsilon_2 \cdot Q \epsilon_1 \cdot P + \dots)$$

Contribution of V to the +ve part of the commutator:

$$-\frac{\pi}{2} \delta(\nu + \nu_V) V_{PV}^{\alpha\beta\gamma} (-m_V^2) V_{PV}^{\gamma\delta\epsilon} (-m_V^2) \left\{ m_V^2 \epsilon_2 \cdot P \epsilon_1 \cdot P + (m_P^2 - m_V^2) \epsilon_2 \cdot P \epsilon_1 \cdot Q \right. \\ \left. + (m_P^2 - m_V^2) \epsilon_2 \cdot Q \epsilon_1 \cdot P + \dots \right\}$$

Contribution of V to the -ve part of the commutator:

$$\frac{\pi}{2} \delta(\nu - \nu_V) V_{PV}^{\alpha\delta\gamma'} (-m_V^2) V_{PV}^{\gamma'\beta\epsilon} (-m_V^2) \left\{ m_V^2 \epsilon_2 \cdot P \epsilon_1 \cdot P - (m_P^2 - m_V^2) \epsilon_2 \cdot P \epsilon_1 \cdot Q \right. \\ \left. + (m_V^2 - m_P^2) \epsilon_2 \cdot Q \epsilon_1 \cdot P + \dots \right\}$$

Contribution of V⁰ to the +ve part of the commutator:

$$-\frac{\pi}{2} \delta(\nu + \nu_{V^0}) V_{PV^0}^{\alpha\beta 0} (-m_V^2) V_{PV^0}^{0\delta\epsilon} (-m_V^2) \left\{ m_V^2 \epsilon_2 \cdot P \epsilon_1 \cdot P + (m_P^2 - m_{V^0}^2) \epsilon_2 \cdot P \epsilon_1 \cdot Q \right. \\ \left. + (m_P^2 - m_{V^0}^2) \epsilon_2 \cdot Q \epsilon_1 \cdot P + \dots \right\}$$

Contribution of V⁰ to the -ve part of the commutator:

$$\frac{\pi}{2} \delta(\nu - \nu_{V^0}) V_{PV^0}^{\alpha\delta 0} (-m_V^2) V_{PV^0}^{0\beta\epsilon} (-m_V^2) \left\{ m_V^2 \epsilon_2 \cdot P \epsilon_1 \cdot P - (m_P^2 - m_{V^0}^2) \epsilon_2 \cdot P \epsilon_1 \cdot Q \right. \\ \left. - (m_P^2 - m_{V^0}^2) \epsilon_2 \cdot Q \epsilon_1 \cdot P + \dots \right\}$$

Contribution of A to the +ve part of the commutator:

$$\frac{\pi}{2} \delta(v+v_A) V_{PA}^{\alpha\beta\gamma} (-m_V^2) V_{PA}^{\gamma\delta\epsilon} (-m_V^2) \times$$

$$\left\{ E_2 \cdot P E_1 \cdot P \left[\frac{1}{m_A^2} + \frac{c(-m_P^2 + m_V^2 + m_A^2)}{m_A^3 m_V} + \frac{c^2}{m_V^2 m_A^2} \left(-m_P^2 + \frac{(-m_P^2 + m_V^2 - m_A^2)^2}{4m_A^2} \right) \right] \right\}$$

$$+ E_2 \cdot P E_1 \cdot Q \left[\frac{1}{m_A^2} + \frac{c(-m_P^2 + m_V^2 - m_A^2)}{m_A^3 m_V} + \frac{c^2}{m_V^2 m_A^2} \left(-m_P^2 + \frac{(-m_P^2 + m_V^2 - m_A^2)^2}{4m_A^2} \right) \right]$$

$$+ E_2 \cdot Q E_1 \cdot P \left[\frac{1}{m_A^2} + \frac{c(-m_P^2 + m_V^2 - m_A^2)}{m_A^3 m_V} + \frac{c^2}{m_V^2 m_A^2} \left(-m_P^2 + \frac{(-m_P^2 + m_V^2 - m_A^2)^2}{4m_A^2} \right) \right]$$

+ ... }

Here, and in the following we have defined

$$V^2 = \frac{cV^1}{m_V m_A} \quad \text{and} \quad V^1 = V$$

Contribution of A to the -ve part of the commutator:

$$- \frac{\pi}{2} \delta(v-v_A) V_{PA}^{\alpha\delta\gamma'} (-m_V^2) V_{PA}^{\gamma'\beta\epsilon} (-m_V^2) \{ E_2 \cdot P E_1 \cdot P - E_2 \cdot P E_1 \cdot Q - E_2 \cdot Q E_1 \cdot P + \dots \} \times$$

$$\times \left[\frac{1}{m_A^2} + \frac{c(-m_P^2 + m_V^2 + m_A^2)}{m_A^3 m_V} + \frac{c^2}{m_V^2 m_A^2} \left(-m_P^2 + \frac{(-m_P^2 + m_V^2 - m_A^2)^2}{4m_A^2} \right) \right]$$

Contribution of T to +ve part of the commutator:

$$\begin{aligned}
 & - \frac{\pi}{4} \delta(\nu + \nu_T) V_{PT}^{\alpha\beta\gamma} (-m_V^2) V_{PT}^{\gamma\delta\epsilon} (-m_V^2) \\
 & \left\{ \epsilon_2 \cdot P \epsilon_1 \cdot P \left[(-m_P^2 + m_V^2 - m_T^2)^2 - 4m_P^2 m_T^2 \right] \frac{m_V^2}{4m_T^2} \right. \\
 & + \epsilon_2 \cdot P \epsilon_1 \cdot Q \left[(-m_P^2 + m_P m_T) + \left(\frac{m_P^2}{4m_T^2} - \frac{1}{4} \right) (-m_P^2 + m_V^2 - m_T^2)^2 \right] \\
 & + \epsilon_2 \cdot Q \epsilon_1 \cdot P \left[(-m_P^2 + m_P m_T) + \left(\frac{m_P^2}{4m_T^2} - \frac{1}{4} \right) (-m_P^2 + m_V^2 - m_T^2)^2 \right] \\
 & \left. + \dots \right\}
 \end{aligned}$$

Contribution of T to -ve part of the commutator:

$$\begin{aligned}
 & \frac{\pi}{4} \delta(\nu - \nu_T) V_{PT}^{\alpha\delta\gamma'} (-m_V^2) V_{PT}^{\gamma'\beta\epsilon} (-m_V^2) \times \\
 & \times \left\{ \epsilon_2 \cdot P \epsilon_1 \cdot P \left[-4m_P^2 m_T^2 + (-m_P^2 + m_V^2 - m_T^2)^2 \right] \frac{m_V^2}{4m_T^2} \right. \\
 & - \epsilon_2 \cdot P \epsilon_1 \cdot Q \left[(-m_P^2 + m_P m_T) + \left(\frac{m_P^2}{4m_T^2} - \frac{1}{4} \right) (-m_P^2 + m_V^2 - m_T^2)^2 \right] \\
 & - \epsilon_2 \cdot Q \epsilon_1 \cdot P \left[(-m_P^2 + m_P m_T) + \left(\frac{m_P^2}{4m_T^2} - \frac{1}{4} \right) (-m_P^2 + m_V^2 - m_T^2)^2 \right] \\
 & \left. + \dots \right\}
 \end{aligned}$$

In all of these equations $\nu_B = (m_P^2 + m_V^2 - m_B^2)/2$ where $B = P, V, V^0, A, T$ as the case may be. Substituting in eq. (1), we obtain the following sum rules. (In the following equation, unless

when explicitly stated otherwise, ρ^0 belongs to the singlet representation of SU(3) and ω^0 belongs to the octet.)

$$\underline{\rho^- + \pi^+ \rightarrow \rho^+ + \pi^- : \int \text{Im B}^{(27)}(v) dv = 0}$$

$$4 V_{\pi^+\rho^- \rightarrow \pi^0} V_{\pi^0 \rightarrow \rho^+\pi^-}$$

$$- (m_\rho^2 - m_V^2) V_{\pi^+\rho^- \rightarrow \omega^0} V_{\omega^0 \rightarrow \rho^+\pi^-}$$

$$- (m_\rho^2 - m_{V^0}^2) V_{\pi^+\rho^- \rightarrow \phi^0} V_{\phi^0 \rightarrow \rho^+\pi^-}$$

$$+ \left\{ \frac{1}{m_A^2} + \frac{c}{m_A^3 m_V} (-m_\rho^2 + m_V^2 - m_A^2) + \frac{c^2}{m_V^2 m_A^2} \left(-m_\rho^2 + \frac{(-m_\rho^2 + m_V^2 - m_A^2)^2}{4m_A^2} \right) \right\} V_{\pi^+\rho^- \rightarrow A_1^0} \times V_{A_1^0 \rightarrow \rho^+\pi^-}$$

$$- \frac{1}{2} \left\{ -m_\rho^4 + m_\rho^2 m_T^2 + \left(\frac{m_\rho^2}{4m_T^2} - \frac{1}{4} \right) (-m_\rho^2 + m_V^2 - m_T^2) \right\} V_{\pi^+\rho^- \rightarrow A_2^0} V_{A_2^0 \rightarrow \rho^+\pi^-}$$

= 0.

$$\underline{\rho^- \pi^+ \rightarrow \rho^+ \pi^- : \int v \text{Im A}^{(27)}(v) dv = 0}$$

$$4 v_P V_{\pi^+\rho^- \rightarrow \pi^0} V_{\pi^0 \rightarrow \rho^+\pi^-}$$

$$- m_V^2 v_V V_{\pi^+\rho^- \rightarrow \omega^0} V_{\omega^0 \rightarrow \rho^+\pi^-}$$

$$- m_V^2 v_{V^0} V_{\pi^+\rho^- \rightarrow \phi^0} V_{\phi^0 \rightarrow \rho^+\pi^-}$$

$$+ v_A \left\{ \frac{1}{m_A^2} + \frac{c(-m_\rho^2 + m_V^2 + m_A^2)}{m_A^3 m_V} + \frac{c^2}{m_V^2 m_A^2} \left(-m_\rho^2 + \frac{(-m_\rho^2 + m_V^2 - m_A^2)^2}{4m_A^2} \right) \right\} V_{\pi^+\rho^- \rightarrow A_1^0} \times V_{A_1^0 \rightarrow \rho^+\pi^-}$$

$$- \frac{v_T}{2} \left\{ \frac{m_V^2}{4m_T^2} \left((-m_P^2 + m_V^2 - m_T^2)^2 - 4m_P^2 m_T^2 \right) \right\} V_{\pi^+ \rho^-} \rightarrow A_2^0 V_{A_2^0} \rightarrow \rho^+ \pi^-$$

$$= 0$$

$$\underline{\rho^- \pi^+ \rightarrow \rho^- \pi^+ : \int \text{Im} A^{(8_F)}(v) dv = 0}$$

$$4 V_{\pi^+ \rho^-} \rightarrow \pi^0 V_{\pi^0} \rightarrow \rho^- \pi^+$$

$$- m_V^2 V_{\pi^+ \rho^-} \rightarrow \omega^0 V_{\omega^0} \rightarrow \rho^- \pi^+$$

$$- m_V^2 V_{\pi^+ \rho^-} \rightarrow \phi^0 V_{\phi^0} \rightarrow \rho^- \pi^+$$

$$+ \left\{ \frac{1}{m_A^2} + \frac{c(-m_P^2 + m_V^2 + m_A^2)}{m_A^3 m_V} + \frac{c^2}{m_V^2 m_A^2} \left(-m_P^2 + \frac{(-m_P^2 + m_V^2 - m_A^2)^2}{4m_A^2} \right) \right\} V_{\pi^+ \rho^-} \rightarrow A_1^0 V_{A_1^0} \rightarrow \rho^- \pi^+$$

$$- \frac{1}{2} \left\{ \frac{m_V^2}{4m_T^2} \left((-m_P^2 + m_V^2 - m_T^2)^2 - 4m_P^2 m_T^2 \right) \right\} V_{\pi^+ \rho^-} \rightarrow A_2^0 V_{A_2^0} \rightarrow \rho^- \pi^+$$

$$= 0$$

$$\underline{\rho^+ K^0 \rightarrow K^{*0} \pi^+ : \int \text{Im} A^{(10)}(v) dv = 0}$$

$$4 V_{K^0 \rho^+} \rightarrow K^+ V_{K^+} \rightarrow K^{*0} \pi^+$$

$$- m_V^2 V_{K^0 \rho^+} \rightarrow K^{*+} V_{K^{*+}} \rightarrow K^{*0} \pi^+$$

$$+ \left\{ \frac{1}{m_A^2} + \frac{c(-m_P^2 + m_V^2 + m_A^2)}{m_A^3 m_V} + \frac{c^2}{m_V^2 m_A^2} \left(-m_P^2 + \frac{(-m_P^2 + m_V^2 - m_A^2)^2}{4m_A^2} \right) \right\} \times$$

$$\times V_{K^0 \rho^+} \rightarrow K_a^+ V_{K_a^+} \rightarrow K^{*0} \pi^+$$

$$-\frac{1}{2} \left\{ \frac{m_V^2}{4m_T^2} (-4m_P^2 m_T^2 (-m_P^2 + m_V^2 - m_T^2)^2) \right\} V_{K^0 \rho^+ \rightarrow K^+} V_{K^+ \rightarrow K^{*0} \pi^+}$$

$$-4V_{K^0 \bar{K}^{*0} \rightarrow \pi^0} V_{\pi^0 \rightarrow \rho^- \pi^+}$$

$$+m_V^2 V_{K^0 \bar{K}^{*0} \rightarrow \omega^0} V_{\omega^0 \rightarrow \rho^- \pi^+}$$

$$+m_V^2 V_{K^0 \bar{K}^{*0} \rightarrow \phi^0} V_{\phi^0 \rightarrow \rho^- \pi^+}$$

$$-\left\{ \frac{1}{m_A^2} + \frac{c(-m_P^2 + m_V^2 + m_A^2)}{m_A^3 m_V} + \frac{c^2}{m_V^2 m_A^2} \left(-m_P^2 + \frac{(-m_P^2 + m_V^2 - m_A^2)^2}{4m_A^2} \right) \right\} V_{K^0 \bar{K}^{*0} \rightarrow A_1^0} V_{A_1^0 \rightarrow \rho^- \pi^+}$$

$$+\frac{1}{2} \left\{ -4m_P^2 m_T^2 + (-m_P^2 + m_V^2 - m_T^2)^2 \right\} \frac{m_V^2}{4m_T^2} V_{K^0 \bar{K}^{*0} \rightarrow A_2^0} V_{A_2^0 \rightarrow \rho^- \pi^+}$$

$$= 0$$

The various special meson couplings encountered above are related to SU(3) symmetric couplings as follows:- (Here, as above, ϕ^0 is SU(3) singlet and ω^0 is the isoscalar member of the octet).

$$V_{K^0 \rho^+ \rightarrow K^+} V_{K^+ \rightarrow K^{*0} \pi^+} = -\frac{1}{2} F_{VPP}^2 \quad ;$$

$$V_{K^0 \rho^+ \rightarrow K^{*+}} V_{K^{*+} \rightarrow K^{*0} \pi^+} = \frac{1}{2} D_{VVP}^2 \quad ;$$

$$V_{K^0 \rho^+ \rightarrow K_a^+} V_{K_a^+ \rightarrow K^{*0} \pi^+} = -\frac{1}{2} F_{AVP}^2 ;$$

$$V_{K^0 \rho^+ \rightarrow K_t^+} V_{K_t^+ \rightarrow K^{*0} \pi^+} = -\frac{1}{2} F_{TVP}^2 ;$$

$$V_{\pi^+ \rho^- \rightarrow \phi^0} V_{\phi^0 \rightarrow \rho^- \pi^+} = \frac{2}{3} D_{VV^0P}^2 ;$$

$$V_{\pi^+ \rho^- \rightarrow \phi^0} V_{\phi^0 \rightarrow \rho^+ \pi^-} = \frac{2}{3} D_{VV^0P}^2 ;$$

$$V_{\pi^+ \rho^- \rightarrow \pi^0} V_{\pi^0 \rightarrow \rho^+ \pi^-} = -F_{VPP}^2 ;$$

$$V_{\pi^+ \rho^- \rightarrow \pi^0} V_{\pi^0 \rightarrow \rho^- \pi^+} = F_{VPP}^2 ;$$

$$V_{\pi^+ \rho^- \rightarrow \omega^0} V_{\omega^0 \rightarrow \rho^- \pi^+} = \frac{1}{3} D_{VVP}^2 ;$$

$$V_{\pi^+ \rho^- \rightarrow A_1^0} V_{A_1^0 \rightarrow \rho^- \pi^+} = F_{VAP}^2 ;$$

$$V_{\pi^+ \rho^- \rightarrow A_2^0} V_{A_2^0 \rightarrow \rho^- \pi^+} = F_{VTP}^2 ;$$

$$V_{K^0 \bar{K}^{*0} \rightarrow \pi^0} V_{\pi^0 \rightarrow \rho^- \pi^+} = -\frac{1}{2} F_{VPP}^2 ;$$

$$V_{K^0 \bar{K}^{*0} \rightarrow A_1^0} V_{A_1^0 \rightarrow \rho^- \pi^+} = -\frac{1}{2} F_{AVP}^2 \quad ;$$

$$V_{K^0 \bar{K}^{*0} \rightarrow A_2^0} V_{A_2^0 \rightarrow \rho^- \pi^+} = -\frac{1}{2} F_{VTP}^2 \quad ;$$

$$V_{K^0 \bar{K}^{*0} \rightarrow \omega^0} V_{\omega^0 \rightarrow \rho^- \pi^+} = -\frac{1}{6} D_{VVP}^2 \quad ;$$

$$V_{K^0 \bar{K}^{*0} \rightarrow \phi^0} V_{\phi^0 \rightarrow \rho^- \pi^+} = \frac{2}{3} D_{VVP}^2 \quad .$$

Hence our sum rules take the following forms:

$$\underline{\int \text{Im } B^{(27)}(v) dv = 0:}$$

$$- 4F_{VPP}^2 - \frac{1}{3} (m_P^2 - m_V^2) D_{VVP}^2 - \frac{2}{3} (m_P^2 - m_{V0}^2) D_{VVP}^2$$

$$- F_{VPA}^2 \left(\frac{1}{m_A^2} + \frac{c(-m_P^2 + m_V^2 - m_A^2)}{m_A^3 m_V} + \frac{c^2}{m_V^2 m_A^2} \left(-m_P^2 + \frac{(-m_P^2 + m_V^2 - m_A^2)^2}{4m_A^2} \right) \right)$$

$$+ \frac{1}{2} F_{VPT}^2 \left(-m_P^4 + m_P^2 m_T^2 + (-m_P^2 + m_V^2 - m_T^2)^2 \frac{m_P^2 - m_T^2}{4m_T^2} \right)$$

$$= 0.$$

$\int_V \text{Im A}^{(27)}(v) dv = 0:$

$$-4v_P F_{VPP}^2 - \frac{1}{3} m_V^2 v_V D_{VVP}^2 - \frac{2}{3} m_V^2 v_{V0} D_{VVOP}^2$$

$$-v_A F_{VAP}^2 \left(\frac{1}{m_A^2} + \frac{c(-m_P^2 + m_V^2 + m_A^2)}{m_A^3 m_V} + \frac{c^2}{m_V^2 m_A^2} \left(-m_P^2 + \frac{(-m_P^2 + m_V^2 - m_A^2)^2}{4m_A^2} \right) \right)$$

$$+ \frac{1}{2} v_T F_{VTP}^2 \left(-m_P^2 m_V^2 + (-m_P^2 + m_V^2 - m_T^2)^2 \frac{m_V^2}{4m_T^2} \right)$$

= 0

(3.4)

$\int \text{Im A}^{(8_F)}(v) dv = 0:$

$$4F_{VPP}^2 - \frac{1}{3} m_V^2 D_{VVP}^2 - \frac{2}{3} m_V^2 D_{VVOP}^2$$

$$+ F_{VAP}^2 \left\{ \frac{1}{m_A^2} + \frac{c(-m_P^2 + m_V^2 + m_A^2)}{m_A^3 m_V^2} + \frac{c^2}{m_V^2 m_A^2} \left(-m_P^2 + \frac{(-m_P^2 + m_V^2 - m_A^2)^2}{4m_A^2} \right) \right\}$$

$$- \frac{1}{2} F_{VTP}^2 \left\{ \frac{m_V^2}{4m_T^2} (-m_P^2 + m_V^2 - m_T^2)^2 - m_P^2 m_V^2 \right\}$$

= 0

(3.5)

$\int \text{Im A}^{(10)}(v) dv = 0:$

$$D_{VVP}^2 = D_{VVOP}^2$$

(3.6)

Sum rule (3.6) is essentially Matsuda's result⁽¹⁷⁾. It is consistent with Okubo's ansatz⁽⁸⁾. Okubo postulated that instead of considering the vector octet and the vector singlet separately all the nine vector mesons should be considered together. The nonet was represented by a non-traceless tensor G_{ν}^{μ} which was constructed out of the traceless tensor F_{ν}^{μ} representing the octet and a singlet ϕ such that

$$G_{\nu}^{\mu} = F_{\nu}^{\mu} + \frac{\delta_{\nu}^{\mu}}{\sqrt{3}} \phi$$

The ansatz is that ϕ should never occur by itself but should always be accompanied by F_{ν}^{μ} . This means that G_{λ}^{λ} ($= \sqrt{3} \phi$) should not appear in any mathematical expression used to describe the mesons. The immediate consequences are in good agreement with experiment. It implies (i) $m_{\omega}^2 = m_{\rho}^2$; $m_{\phi}^2 - m_K^2 = m_K^2 - m^2$; (ii) $g_{\phi \rho^+ \pi^-} = 0$; (iii) $\tan \theta = \sqrt{\frac{1}{2}}$; $\sin \theta = \sqrt{\frac{2}{3}}$; $\cos \theta = \sqrt{\frac{1}{3}}$ where θ is the mixing angle (this specific value of θ is called the ideal mixing angle) and relates the physical states ω and ϕ with the SU(3) octet member ω^0 and the singlet ϕ^0 as follows

$$\begin{aligned} \omega &= \sin \theta \omega^0 + \cos \theta \phi^0 \\ \phi &= \cos \theta \omega^0 - \sin \theta \phi^0 \end{aligned} .$$

It has other implications as well, which are, however, not relevant to us here. In quark model such an ideal mixing would mean that ω is made out of nonstrange quarks and ϕ is made of the strange quark. Our sum rule (3.6) with the assumption of ideal mixing reads as follows

$$g_{\omega\rho^+\pi^-}^2 + 2g_{\phi\rho^+\pi^-}^2 = g_{\omega\rho^+\pi^-}^2 - \frac{1}{2}g_{\phi\rho^+\pi^-}^2$$

which is identically true since $g_{\phi\rho^+\pi^-} = 0$.

The other sum rules are the SU(3) symmetric limits of the Frampton Taylor sum rules⁽⁶⁾ for the case $t = 0$. Perhaps one remark that is worth making is that whereas the neglect of pseudoscalar mass may be justifiable in ref. (6) because of the relatively low mass of π , this is no longer true in the SU(3)-symmetric case. Furthermore the ω^0 contribution to sum rule (3.4) is rather sensitive to the pseudoscalar mass and so the approximation of neglecting pseudoscalar mass would lead to unreliable results.

Our calculation therefore did not yield anything fundamentally different from the calculations given in ref. (6) and (7). The investigation of nonforward sum rules may turn out more informative, but the whole procedure of saturating by putting in a few low-lying states does not seem to be very useful except in a few cases where the integrals converge rapidly.

CHAPTER IV

PI - PI SUM RULES AND THEIR SATURATION

1. A forward unsubtracted dispersion relation is written down for the component of $(\pi-\pi)$ scattering amplitude dominated by the exchange in the t -channel of an $I = 2$ boson trajectory under the assumption that the corresponding intercept, $\alpha^{(I=2)}$ ($t = 0$) is negative. Its consistency with Weinberg's low-energy parameters for $(\pi-\pi)$ scattering and Adler's $(\pi-\pi)$ sum rule is examined, putting in all known resonances, and, using the most recent available data. The s -wave is parameterized in a resonant form. A reasonable saturation of Adler sum rule can be obtained for suitable s -wave parameters. However the unsubtracted dispersion relation cannot be saturated for realistic values of these parameters. Some remarks concerning the finite width formulae for s -, p -, and d -partial wave cross-sections are made.

2. Recently Gatto⁽¹⁾ obtained a reasonable estimate of a universal D/F - ratio for meson-baryon couplings from two assumptions.

(1) The Regge trajectory for the exchange of a 27-plet of $SU(3)$ in the t -channel has a negative intercept at $t = 0$, i.e. $\alpha^{\underline{27}}(t = 0) < 0$. Consequently one can write an unsubtracted dispersion relation for the forward scattering amplitude $A(\underline{27}, t)_{(s)}$ corresponding to 27-exchange in the t -channel.

(2) At the scattering threshold, there is no appreciable contribution from 27-exchange in the t -channel. This result is deduced from recent calculations of meson-baryon scattering lengths using

current algebras⁽²⁾, which are equivalent to describing low energy meson-baryon scattering by the exchange of a vector meson nonet⁽³⁾. With these assumptions, Gatto expresses $A^{(27,t)}(s = (m_B + m_M)^2) = 0$ in terms of an unsubtracted dispersion integral and saturates it with low-lying meson-baryon resonances, using an ideal value of $3/2$ for the D/F - ratio. A fair saturation of the sum rule was obtained.

Here we want to apply Gatto's ideas to $(\pi-\pi)$ scattering⁽⁴⁾, with some modifications.

(1) Only $SU(2)$ invariance is assumed and $SU(3)$ symmetry is not needed.

(2) A negative intercept is assumed for all kinds of isospin two exchange, irrespective of its nature (cut⁽⁵⁾ or trajectory). The corresponding amplitude $A^{(2,t)}(s)$ then satisfies an unsubtracted dispersion relation.

(3) The value of the amplitude $A^{(2,t)}(s)$ at the threshold is estimated using Weinberg's results for low-energy $(\pi-\pi)$ scattering⁽²⁾. We do not, however, confine ourselves to his solution for the scattering lengths (e.g. $a_0 = .2m_\pi^{-1}$, $a_2 = -.06m_\pi^{-1}$) but allow a_0 and a_2 to vary over a reasonable range such that $2a_0 - 5a_2 = .7m_\pi^{-1}$, which is one of his results (obtained prior to the assumption that σ^{ij} is pure isoscalar). We use Weinberg's results because of lack of better results.

(4) Since it turns out that the sum rule so obtained is less convergent than Adler's sum rule⁽⁹⁾ for $(\pi\pi)$ scattering, a simultaneous saturation of both of these sum rules is considered. The contributions of the resonances ρ , f and g ⁽⁶⁾ are estimated both in narrow width and finite width approximations. Since the

finite width expressions for the partial wave cross-sections as given by Balazs⁽⁷⁾ overestimate these contributions (owing to the presence of undesirable bumps at high energies), they are slightly modified such that the phase-shifts go asymptotically always to π .

(5) The remaining contribution is assumed to be due to the possible existence of an $I = 0$, s-wave (π - π) resonance, parametrized as in ref. (10). For the parameters fitted to K_{e4} decay⁽¹⁰⁾ a reasonable saturation of Adler's (π - π) sum rule is obtained. However, the other sum rule does not seem to be saturated for any reasonable values of these parameters.

Our analysis is similar in spirit to that of various authors⁽⁹⁾, in particular, Furlan and Rossetti who have tried to extract information on the s-wave (π - π) resonance in the isospin zero channel, but differs in detail from them.

3. The amplitude $A^{(2,t)}(s)$ is related to the various isospin amplitudes in the s-channel as follows

$$A^{(2,t)}(s) = \frac{1}{3} A^{(0,s)}(s) - \frac{1}{2} A^{(1,s)}(s) + \frac{1}{6} A^{(2,s)}. \quad (4.1)$$

With the assumption

$$a^{(2,t)}(t=0) < 0,$$

we may write an unsubtracted dispersion relation in the forward direction for the amplitude $A^{(2,t)}$,

$$A^{(2,t)}(s) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\text{Im} A^{(2,t)}(s') ds'}{s' - s - i\epsilon}$$

$$= \frac{1}{\pi} \int_{4m_{\pi}^2}^{\infty} ds' \text{Im} A^{(2,t)}(s') \left\{ \frac{1}{s' - s} + \frac{1}{s' + s - 4m_{\pi}^2} \right\}, \quad (4.2)$$

where the second equality follows from crossing.

Using the optical theorem and by Bose statistics, and putting $m_{\pi} = 1$, we have

$$A^{(2,t)}(s) = \frac{1}{\pi} \int_4^{\infty} ds' \left\{ \frac{1}{s' - s} + \frac{1}{s' + s - 4} \right\} \sqrt{s'(s' - 4)} \times$$

$$\times \left\{ \frac{1}{3} \sum_{\ell=0}^{\text{even}} \sigma_{\ell}^{(0,s)}(s') - \frac{1}{2} \sum_{\ell=1}^{\text{odd}} \sigma_{\ell}^{(1,s)}(s') + \frac{1}{6} \sum_{\ell=0}^{\text{even}} \sigma_{\ell}^{(2,s)}(s') \right\}.$$

Keeping only s-, p-, d- and f-wave terms, and neglecting $I = 2$ terms, we get,

$$A^{(2,t)}(s) = \frac{2}{\pi} \int_4^{\infty} ds' \frac{(s' - 2) \sqrt{s'(s' - 4)}}{(s' - s)(s' + s - 4)} \times$$

$$\times \left\{ \frac{1}{3} \sigma_0^{(0,s)}(s') + \frac{1}{3} \sigma_2^{(0,s)}(s') - \frac{1}{2} \sigma_1^{(1,s)}(s') - \frac{1}{2} \sigma_3^{(1,s)}(s') \right\}. \quad (4.3)$$

At the physical threshold (suppressing isospin labels),

$$A^{(2,t)}(s=4, u=t=0) = \frac{2}{\pi} \int_4^{\infty} \frac{ds' (s' - 2)}{\sqrt{s'(s' - 4)}} \left\{ \frac{1}{3} \sigma_0(s') + \frac{1}{3} \sigma_2(s') - \frac{1}{2} \sigma_1(s') - \frac{1}{2} \sigma_3(s') + \dots \right\},$$

i.e.,

$$32\pi \left(\frac{1}{3} a_0^0 + \frac{1}{6} a_0^2 \right) = \frac{2}{\pi} \int_4^\infty \frac{ds'(s'-2)}{\sqrt{s'(s'-4)}} \times$$

$$\times \left\{ \frac{1}{3} \sigma_0(s') + \frac{1}{3} \sigma_2(s') - \frac{1}{2} \sigma_1(s') - \frac{1}{2} \sigma_3(s') + \dots \right\} \quad (4.4)$$

In the same notation as above, Adler sum-rule may be written as

$$\frac{4M_N^2}{g_r^2} \frac{1}{2\pi} \int_4^\infty \frac{ds'}{s'-1} \left\{ \frac{1}{3} \frac{\sqrt{s'(s'-4)}}{(s'-1)} \sigma_0(s') + \frac{1}{3} \left(\frac{(s'-1)^2}{s'(s'-4)} \right)^{3/2} \sigma_2(s') \right.$$

$$\left. + \frac{1}{2} \left(\frac{(s'-1)^2}{s'(s'-4)} \right)^{1/2} \sigma_1(s') + \frac{1}{2} \left(\frac{(s'-1)^2}{s'(s'-4)} \right)^{5/2} \sigma_3(s') + \dots \right\} = \frac{2}{g_A^2}, \quad (4.5)$$

where $g_A (= 1.18)$ is the axial vector renormalization constant, $g_r^2/(4\pi) (= 14.6)$ is the rationalized renormalized (πN) coupling constant, M_N is the nucleon mass in units of m_π .

4. We now want to estimate the contribution of p-, d-, and f-waves. For this we need explicit forms⁽⁷⁾ for $\sigma_1(1, S)$, $\sigma_2(0, S)$ and $\sigma_3(1, S)$. We parametrize the phase shifts by

$$\left(\frac{\nu}{\nu+1} \right)^{l+1/2} \cot \delta_l = \frac{\nu_R - \nu}{\gamma_l}, \quad (4.6)$$

rather than by

$$\sqrt{\frac{\nu^{2l+1}}{\nu+1}} \cot \delta_\ell = \frac{\nu_R - \nu}{\gamma_\ell} \quad (4.6a)$$

The latter leads to Balaz's formulae for cross-sections. These formulae have undesirable high energy behaviour and as a consequence the contributions of the resonances are overestimated. The reason is that the phase-shifts do not go to π as energy tends to infinity but go to some constant which is different from $\pi/2$ in the case of p-wave and is equal to $\pi/2$ for all higher waves. No such difficulty arises if we use eq. (4.6). Corresponding to eq. (4.6) the ℓ -th partial-wave cross-section will be given by

$$\sigma_\ell^{(s)} = \frac{8\pi (2\ell+1) \nu^{2\ell} \gamma_\ell^2 / (\nu+1)^{2\ell+1}}{(\nu_R - \nu)^2 + \nu^{2\ell+1} \gamma_\ell^2 / (\nu+1)^{2\ell+1}} \quad ; \quad \gamma_\ell^2 = \frac{\Gamma_R^2 (\nu_R+1)^{2\ell+2}}{4\nu_R^{2\ell+1}} \quad (4.7)$$

where ℓ , Γ_R , ν_R are the spin, width and position of the resonance. In the above formulae ν is the square of the c.m. momentum of the pion ($s = 4(\nu+1)$). The narrow width form for eq. (4.7) becomes

$$\sigma_\ell^{(s)} = 32\pi^2 (2\ell_R+1) \Gamma_R m_R \delta(s-s_R) (m_R^2 - 4)^{-1} \quad (4.8)$$

where m_R is the mass of the resonance. We find that the contributions of the resonances to the sum rules calculated using the finite width and the narrow width formulae are reasonably close to each other.

For the $I = 0$, s-wave we use the parameterization given by Berends, Donnachie and Oades⁽⁸⁾

$$\sqrt{\frac{s-4}{4}} \cot \delta_0^0 = \frac{(s-4+\frac{\gamma}{a_0})(s-s_R)}{\gamma(4-s_R)} \quad (4.9)$$

where a_0 is the scattering length, γ is related to the width, δ_0^0 is the $I = 0$, s-wave phase-shift and s_R is the square of the resonance mass. The phase-shift goes to π as energy becomes very large. As the resonance is rather lopsided the relation between γ and the width is found by computing the derivative of δ_0^0 at the position of the resonance,

$$\frac{1}{\Gamma_R M_R} = \left. \frac{d\delta_0^0}{ds} \right|_{\delta_0^0 = \frac{\pi}{2}} = \frac{2(s_R-4+\frac{\gamma}{a_0})}{\gamma(s_R-4)^{3/2}} ; \quad \Gamma_R = \frac{\gamma(M_R^2-4)^{3/2}}{2M_R(M_R^2-4+\frac{\gamma}{a_0})} \quad (4.10)$$

5. The results are given in Table IVA. There we use the parameters $M_R = 700$ MeV, $M_R = 300$ MeV and $a_0 = 0.6 \text{ m}_\pi^{-1}$. The left hand side of eq. (4.4) is calculated using $(2a_0 - 5a_2) = 0.7 \text{ m}_\pi^{-1}$. In Table IVB we give the contribution of $I = 0$, s-wave resonance to the sum rules for a range of values for the s-wave parameters. It is seen from Table IVA that the contributions of ρ , f and g together are insufficient to saturate the sum rules. A substantial contribution must come either from the high energy region or from some s-wave resonance (or large phase shift), or from a combination of both. In view of

insufficient experimental results at very high energies and the necessity of some reasonably strong s-wave $I = 0$ ($\pi\text{-}\pi$) interaction as implied by other calculations⁽⁹⁾, we are tempted, as a first approximation, to forget about the high-energy region altogether. This makes our neglect of $I = 2$ contributions more reasonable since no $I = 2$ resonance has been observed. It is seen from Table IVA that the s-wave parameters fitted to K_{e4} decay⁽⁸⁾ give a reasonable saturation of Adler sum rule, but for the same parameters the other sum rule is not saturated. The discrepancy between the left and right hand sides of this sum rule is too large to be accounted for by varying the s-wave parameters. This is seen on inspecting Table IVB. None of the values of the parameters given there can saturate this sum rule. On the other hand the Adler sum rule can be saturated for various sets of parameters. In particular a fairly small value of a_0 ($\sim .35m_\pi^{-1}$) can make the saturation possible provided M_R is reasonably small (~ 400 MeV) and Γ_R is reasonably large (~ 100 MeV). The reason that the sum rule (4.4) remains unsaturated, even after including σ , ρ , f and g , lies in its poor convergence. It is seen that the g -contribution is larger than the f -contribution. Consequently this sum rule depends as much on high-energy contributions as on low-energy ones. In this case, therefore, the neglect of high energy contributions is not justified. For example, a spin four resonance of isospin zero will contribute appreciably to the sum rule if it has a reasonable mass and width. Also the contribution of $I = 2$ resonances may not be negligible.

TABLE IVA : Contribution of resonances to sum rules ($a_0^0 = .6m_\pi^{-1}$).

Resonance	Mass (MeV)	Width (MeV)	Sum rule (4.4)		Adler sum rule (4.5)		
			Narrow Width RHS	Finite Width RHS	Narrow Width LHS	Finite Width LHS	
$\sigma (0^+, 0^+)$	700	300		37.54		.566	
$\rho (1^-, 1^+)$	774	128	-57	-53	.51	.560	
$f (2^+, 0^+)$	1254	117	33	32	.109	.125	
$g (3^-, 1^+)$	1630	100	-44	-43	.085	.096	
Total				-26	22	1.347	1.43

TABLE IVB : Contribution of I = 0 s-wave (π - π) resonance to sum rules (4.4)

Mass = 420 MeV

Width (MeV) Contributions to Eq. (4.4) Contribution to Eq. (4.5)

Width (MeV)	Contributions to Eq. (4.4)					Contribution to Eq. (4.5)				
	$a_0 = 1$ $.2m_\pi$.4	.6	.8	1	.2	.4	.6	.8	1
50	14.24	16.04	18.62	21.32	24.12	.338	.382	.426	.462	.496
100		27.24	29.30	32.96	36.94		.624	.680	.744	.804
150			39.30	41.20	45.26			.866	.922	.992
200				50.62	52.30				1.076	1.128
250					61.34					1.258

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Mass = 560 MeV

100	15.28	18.44	21.82	25.30	28.86	.252	.324	.388	.446	.496
200		29.06	33.50	38.38	43.40		.502	.606	.704	.792
300			41.58	46.58	52.08			.734	.848	.954
400				53.14	58.40			.828	.942	1.056
500				60.12	63.94				1.018	1.133
600					69.86					1.19
700					78.98					1.252

TABLE IVB (Continued)

Width (MeV)	Mass = 700 MeV									
	Contribution to Eq. (4.4)					Contribution to Eq. (4.5)				
	$a_0 = \frac{1}{.2m_\pi}$.4	.6	.8	1	.2	.4	.6	.8	1
100	12.24	15.26	18.32	21.50	24.78	.152	.212	.266	.314	.360
200	20.62	25.08	29.90	34.76	39.66	.246	.352	.452	.542	.624
300		31.92	37.54	43.34	49.14		.440	.566	.682	.788
400		37.74	43.20	49.38	55.64		.502	.642	.774	.894
500		44.56	47.94	54.08	60.52		.552	.698	.838	.968
600			52.50	58.10	64.50			.740	.886	1.022
700			57.64	61.82	67.96			.778	.924	1.064

Mass = 840 MeV										
100	10.42	13.20	16.06	19.04	22.16	.104	.158	.206	.250	.294
200	18.00	22.76	27.46	32.16	36.90	.176	.274	.364	.448	.528
300	23.88	29.52	35.42	41.24	47.02	.224	.350	.470	.582	.686
400	30.24	34.66	41.22	47.76	54.18	.262	.406	.544	.674	.794
500		38.98	45.74	52.66	59.50		.444	.598	.740	.872
600		42.98	49.50	56.60	63.64		.476	.638	.790	.928
700		47.22	52.82	59.88	67.04		.502	.670	.826	.972
800		52.92	55.90	62.78	69.94		.526	.696	.856	1.006

CHAPTER V*

A MODEL FOR π - π S-WAVE PHASE SHIFTS FROM CURRENT ALGEBRA
AND PCAC

In this chapter a set of sum rules for the $(\pi-\pi)$ s-wave interaction in the isospin-zero channel is derived from the algebra of axial-vector charges and divergences, and using extensively the principle of pion-pole dominance. On approximating these sum rules by the σ -meson, treated as a single-particle state, a relation between its mass and width is obtained. The numerical results are fairly reasonable in view of the drastic assumptions involved. In the approximation of elastic unitarity the sum rules take the form of an integral equation for the vertex function $\langle \pi | \sigma | \pi \rangle$. This equation implies that the $(\pi-\pi)$ s-wave phase shift δ_0 in the isospin-zero channel satisfies the inequality $0 < \delta_0 < \pi$, quite generally. The approximate integral equation cannot be solved exactly, thus reflecting the drawback of our earlier PCAC assumption. On introducing an effective cut-off function as a correction factor, solutions are obtained. They give a scattering length slightly larger than Weinberg's result and a phase shift that has a broad maximum around 700 Mev., the height of the phase-shift being sensitive to the cut-off parameter.

1. Current algebra has been used mainly in deriving sum rules and low-energy theorems based on the soft-pion technique⁽¹⁾. In these applications current algebra makes definite predictions on off-mass-shell amplitudes (or vertex functions) in the limit of vanishing four-momentum for one or more of the pions involved in the process

* This chapter is based on work done in collaboration with Drs. A. Pagnamenta and B. Renner.

under consideration. Only after making PCAC smoothness assumption can one obtain information about on-mass-shell quantities. This extrapolation from off- to on-mass-shell is rather vague and not without ambiguities⁽⁹⁾. Recently there have been various attempts to do current algebra calculations without using the soft-pion technique. In particular, Schnitzer and Weinberg⁽²⁾ have developed a method which does not invoke the soft-pion limit. In this method current algebras are used to derive generalized Ward identities for proper vertices. These are then supplemented by crossing relations to determine the form of the proper vertices which, in the spirit of pole dominance, are assumed to be smooth functions of momenta. There does not, however, seem to exist an obvious generalization of this method to scalar vertices. Moreover one does not know the precise nature of the difficulties one would encounter when large extrapolations are involved. To clarify these problems we present a model for the $(\pi-\pi)$ s-wave phase shifts in the isospin-zero channel based on the algebra of axial-vector charges and divergences, and on an extensive use of pion-pole-dominance. Unlike earlier authors⁽³⁾ we try to assume as little as possible beforehand about the strength and energy dependence of the $(\pi-\pi)$ s-wave interactions, such as whether the (unknown) unitarity cut allows certain extrapolations from zero-energy to threshold or not, or whether there is a σ -resonance or not. To offset this lack of information we need to use the principle of pion-pole-dominance very extensively, far more than can be justified on the basis of the relative distances of singularities. We maintain that even this extreme use of pion-pole-dominance deserves exploration since the limits of its applicability are hardly known at present⁽⁴⁾.

In section 3 we derive a set of sum rules involving the off-mass-shell vertex $\langle \pi | \sigma | \pi \rangle$. As a preliminary test, we consider these

sum rules in a model of single-particle dominance and obtain results acceptable for the conjectured σ -meson⁽⁵⁾. In section 3 we abandon the single-particle model and, in the approximation of elastic unitarity for $(\pi-\pi)$ scattering, we convert the sum rule into an integral equation for the vertex $\langle \pi | \sigma | \pi \rangle$. This equation implies that the $(\pi-\pi)$ s-wave phase shift δ_0 in the isospin-zero channel satisfies the inequality $0 < \delta_0 < \pi$, quite generally. Exact solutions of the approximate integral equation, however, cannot be constructed. In section 4 we apply the N/D formalism and find that our PCAC approximate analysis of the vertex would lead to an N-function in the $(\pi-\pi)$ s-wave scattering amplitude without a left hand cut. To correct for this, we introduce, as a first step, an effective interaction pole which at the same time will serve as a cut-off function in the integral equation. The solutions for δ_0 give a scattering length somewhat larger than Weinberg's result⁽⁶⁾: $0.23 m_\pi^{-1} \sim 0.33 m_\pi^{-1}$, and a broad maximum around 700 MeV. Its height is sensitive to the cut-off parameter. With our phase shifts we can saturate the Adler-Weisberger relation for $(\pi-\pi)$ scattering with a reasonable cut-off value. We have not ruled out, however, the possibility of more complicated corrections. Some such investigations (section 5), however, seem to imply that the qualitative features of the above results are left unaltered.

2. In this section we derive the sum rules and consider their single-particle saturation.

We begin with some comments on the equal-time commutator for the axial-vector charges and the axial-vector divergences

$$\left[\int A_0^i(\underline{x}) d^3x, \partial^\mu A_\mu^j(0) \right] = i\sigma^{ij}(0) \quad (5.1)$$

where $i, j = 1, 2, 3$ are isospin indices, and $\mu = 0, 1, 2, 3$ are Lorentz indices. This commutator is encountered in deriving low-energy theorems whenever the emission or absorption of two or more pions is involved. We further assume Gell-Mann's commutator for the axial-vector charges,

$$\left[\int A_0^i(x) d^3x, \int A_0^j(y) d^3y \right]_{x^0=y^0} = i\epsilon^{ijk} \int v_0^k(y) d^3y \quad (5.2)$$

Since the vector current is conserved, we have

$$\left[\int \partial^\mu A_\mu^i(x) d^3x, \int A_0^j(y) d^3y \right]_{x^0=y^0} = \left[\int \partial^\mu A_\mu^j(x) d^3x, \int A_0^i(y) d^3y \right]_{x^0=y^0}$$

so that σ^{ij} is symmetric in its isospin labels, i.e.

$$\sigma^{ij}(0) = \sigma^{ji}(0)$$

Following the suggestions of the quark model and of the σ -model, it was conjectured⁽⁶⁾ that σ^{ij} is purely isoscalar, i.e.,

$$\sigma^{ij}(0) = \delta^{ij} \sigma(0) \quad (5.3)$$

Though eq. (5.3) has not been confirmed directly, its validity is assumed in the successful applications of eqs. (5.1) and (5.2) to low-energy $(\pi-\pi)$ scattering^(6,7), pion electromagnetic mass difference⁽⁸⁾ and to nonleptonic K-meson decays⁽⁹⁾.[‡] Using eqs. (5.1) and (5.2) in the Jacobi identity satisfied by the operators

$\int A_0^i(x) d^3x$, $\int A_0^j(y) d^3y$ and $\partial^\mu A_\mu^k(z)$ at equal times, we obtain

$$\left[\int A_0^i(x) d^3x, \sigma(0) \right] = -i\partial^\mu A_\mu^i(0) \quad (5.4)$$

Consider now, the off-shell-vertex $\langle \pi^j(q_2) | \sigma(0) | \pi^i(q_1) \rangle =$

‡- In ref. (9) σ^{ij} is allowed to have both $I = 0$ and $I = 2$ components.

$$\delta^{ij} f_{\sigma}(-q_1^2, -q_2^2; t) = - \frac{(m_{\pi}^2 + q_1^2)(m_{\pi}^2 + q_2^2)}{F_{\pi}^2 m_{\pi}^4} \iint e^{iq_1 \cdot x} e^{-iq_2 \cdot y} d^4x d^4y \times \langle T \{ \partial^{\mu} A_{\mu}^i(x) \partial^{\nu} A_{\nu}^j(y) \sigma(0) \} \rangle_0 \quad (5.5)$$

where $t = -(q_2 - q_1)^2$, and F_{π} is defined by

$$\langle 0 | A_{\mu}^i(0) | \pi^j(p) \rangle = i F_{\pi} p_{\mu} \delta^{ij} \quad ; \quad F_{\pi} \approx 90 \text{ MeV}$$

Pion-pole dominance for the integrand in eq. (5.5) asserts that for q_1^2 and q_2^2 not too far away from m_{π}^2 , the variation of f_{σ} with q_1^2 and q_2^2 may be neglected⁽⁷⁾,

$$f_{\sigma}(m_{\pi}^2, m_{\pi}^2; t) \approx f_{\sigma}(-q_1^2, -q_2^2; t) \quad ; \quad q_1^2, q_2^2 \approx -m_{\pi}^2$$

We write the off-mass-shell vertex $f_{\sigma}(-q_1^2, -q_2^2; t)$ as a product of the on-mass-shell vertex $F_{\sigma}(t) \equiv f_{\sigma}(m_{\pi}^2, m_{\pi}^2; t)$ and a correction factor,

$$f_{\sigma}(-q_1^2, -q_2^2; t) = F_{\sigma}(t) X(-q_1^2, -q_2^2; t) \quad (5.6)$$

By definition, we have $X(m_{\pi}^2, m_{\pi}^2; t) = 1$, and we know that $X(-q_1^2, -q_2^2; t) \approx 1$ for $q_1^2 \approx -m_{\pi}^2$ and $q_2^2 \approx -m_{\pi}^2$. However we do not know the extent of the region in which X may be approximated by unity. Conventional PCAC asserts that $X \approx 1$ is reasonable for $q_1^2 \rightarrow 0$, $q_2^2 \rightarrow 0$ at $t = 0(m_{\pi}^2)$. We shall try to keep $X \approx 1$ in as large a domain as is possible and is needed. This assumption defines the model. Eq. (5.6), supplemented by the assumption that X is a slowly varying function of its arguments, is related in spirit to the technique of Schnitzer and Weinberg⁽²⁾

in which the off-mass-shell vertex functions are factored into propagators and proper vertices, the latter being assumed to be slowly varying functions of momenta.

In eq. (5.5), we make the usual partial integration with respect to y , use eqs. (5.1) and (5.2) and let $q_{2\mu}$ tend to zero,

$$\begin{aligned} \delta^{ij} f_{\sigma}(t, 0; t) &= -i \frac{m_{\pi}^2 - t}{F_{\pi}^2 m_{\pi}^2} \int e^{iq_1 \cdot x} \left\{ \langle 0 | T(\partial^{\mu} A_{\mu}^i(x) \partial^{\nu} A_{\nu}^j(0)) | 0 \rangle \right. \\ &\quad \left. - \delta^{ij} \langle 0 | T(\sigma(x)\sigma(0)) | 0 \rangle \right\} d^4x \\ &= \delta^{ij} F_{\sigma}(t) \chi(t, 0; t) \stackrel{\text{def.}}{=} F_{\sigma}(t) \alpha(t) \delta^{ij} \end{aligned} \tag{5.7}$$

We introduce intermediate states into the propagators and in the spirit of pion-pole-dominance, we keep only the one-pion state in the pseudoscalar propagator and a yet unspecified continuum in the scalar propagator:

$$F_{\sigma}(t) \alpha(t) = -m_{\pi}^2 + \frac{(m_{\pi}^2 - t)}{m_{\pi}^2 F_{\pi}^2} \int_{4m_{\pi}^2}^{\infty} \frac{ds \rho_{\sigma}(s)}{s-t}, \tag{5.8}$$

where

$$\rho_{\sigma}(s) = \sum_n \delta^3(p_n - q_1) \delta(m_n^2 - s) |\langle 0 | \sigma | n \rangle|^2.$$

A consistent treatment of the pseudoscalar and scalar propagators would demand the inclusion of continuum contributions in the pseudo-scalar propagator as well (such as the three-pion-cut, etc.),

however, the success of pion-pole-dominance in most cases considered so far makes their neglect reasonable.

Let us examine eq. (5.8) in more detail. The factor $x(t)$ on its left-hand-side takes account of the difference between $f_{\sigma}(t, 0; t)$ and $f_{\sigma}(m_{\pi}^2, m_{\pi}^2; t)$. Near $t = m_{\pi}^2$ these two form-factors are approximately equal to each other (pion-pole-dominance) and $x(t) \approx 1$, but for general values of t , there is no way of finding $x(t)$ and its t -dependent structure is unknown to us. One might assume certain "smooth" forms for $x(t)$ in analogy with the work of Schnitzer and Weinberg⁽²⁾, but there does not seem to be an obvious way of determining $x(t)$. Crossing symmetry might be of some help here^{*}. In view of this lack of information on $x(t)$, we set as a first (admittedly crude) approximation $x(t) = 1$, i.e., the pion-pole-dominance assumption is true for any value of t so that $f_{\sigma}(t, 0; t) \cong f_{\sigma}(m_{\pi}^2, m_{\pi}^2; t)$. This is a drastic assumption and will need correcting as we shall see in due course. Likewise the integral term on the right-hand-side of eq. (5.8) represents the next important contribution to the off-mass-shell vertex after the pion-pole term. Therefore, in order to treat the corrections to pion-pole-dominance consistently one might suggest that if we are to set $x(t) = 1$, we should as well drop the integral term. However we shall not do so. The reason is that, unlike $x(t)$, we know the precise nature of the correction term on the right-hand-side. This integral introduces the elastic unitarity cut for $(\pi-\pi)$ scattering starting correctly at the threshold $4m_{\pi}^2$.

* This point needs further investigation.

which presumably represents the next important singularity of the off-mass-shell vertex after the pion poles^{*}. We do not want to relinquish this physically meaningful information. We emphasize that we have set $x(t) = 1$ simply due to our ignorance about $x(t)$ and only as a first approximation to a more realistic situation. Diagrammatically our model of setting $x(t) = 1$ for any t may be represented as in Fig. VA.

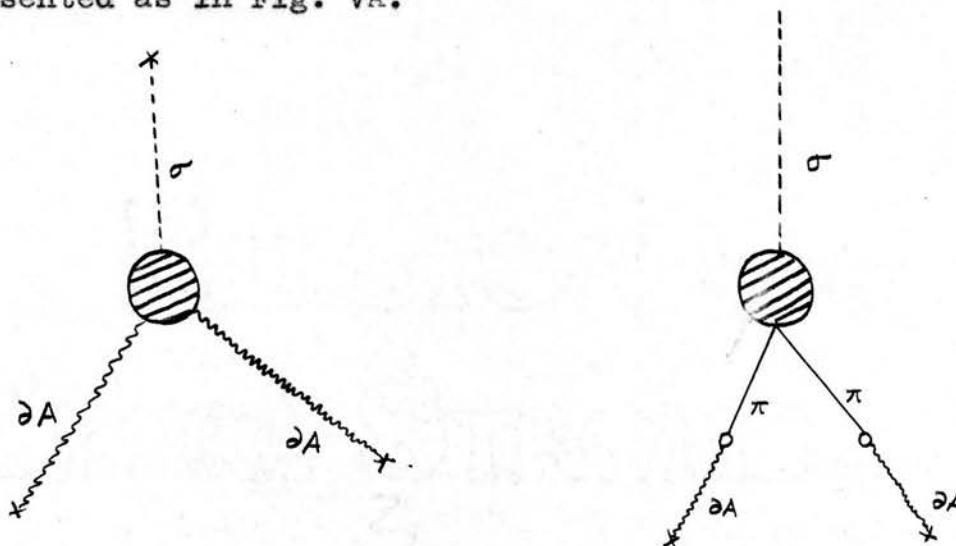


Fig. VA

In Fig. VA the pion-poles in ∂A correspond to our assumption that the pseudoscalar propagator is dominated by pion-poles and does not involve more complicated singularities like the three pion cut, etc. The assumption $x(t) = 1$ for any t implies that the t -dependent structure of the off-mass-shell vertex function is entirely due to the σ -channel. In eq. (5.8) this implies that only the second term on its right-hand-side determines the t -dependence of the vertex and not the pion-pole terms. This would mean that the pions cannot interact before they meet as shown in Fig. VA. The presence of any initial state interaction for the pions would invariably imply that

* This is the reason why we preferred the off-mass-shell extrapolation defined by eq. (5.5) to a successive reduction of two pions; the latter fails to produce the scalar propagator. For further details see ref. (9).

the t -dependent structure of the off-mass-shell vertex arises not only due to the integral term on the right-hand-side of eq. (5.8) but also due to the pion-pole-terms. This is the case only if $x(t) \neq 1$ and depends upon t . Such diagrams are illustrated in Fig. VB.

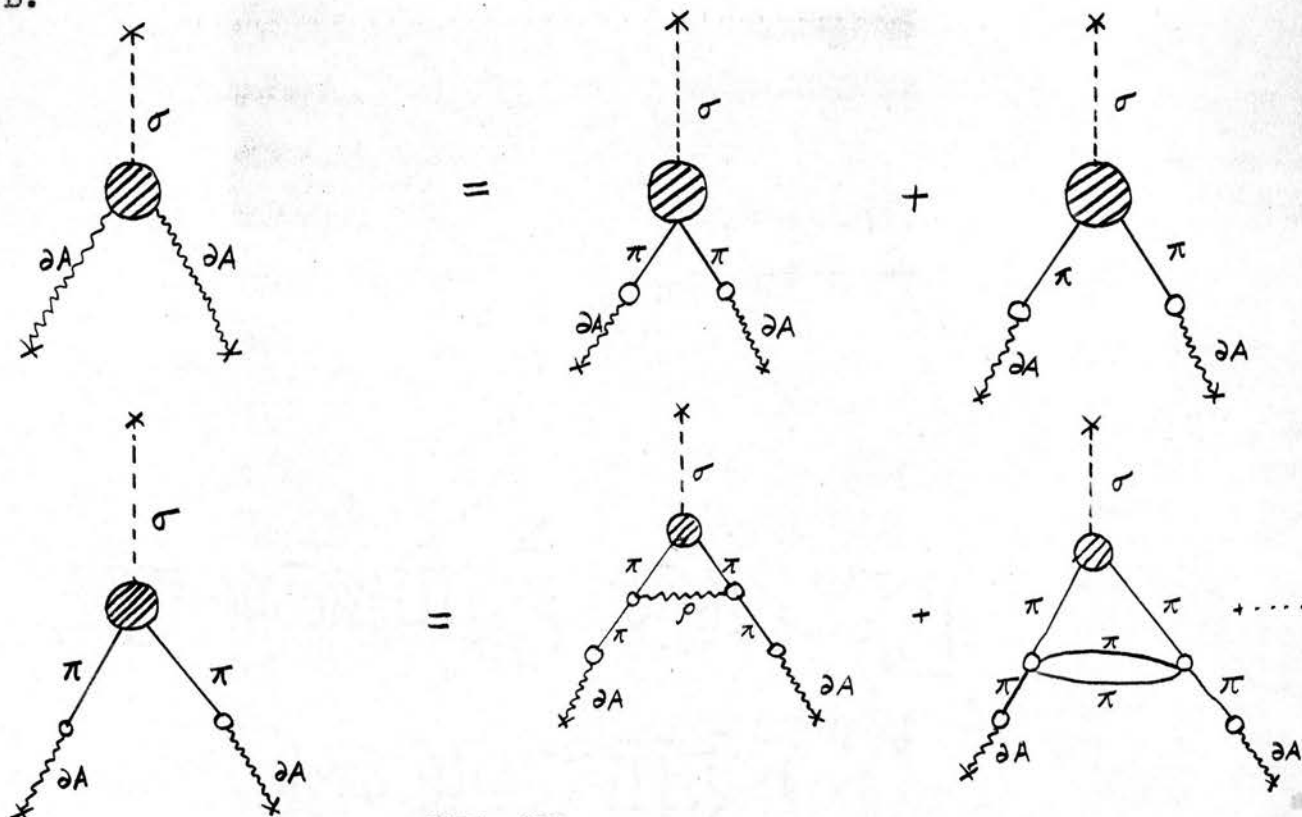


FIG. VB

In Fig. VB we have assumed that the only important singularities in the off-mass-shell vertex are the single pion-poles of ∂A and the two-pion contribution to σ . This is presumably a reasonable assumption as the respective three pion and four pion cut contributions are expected to be quite small. These diagrams correspond to $x(t) \neq 1$, in general. A particularly interesting case is that when $x(t) = \frac{m^2 + t}{m^2 + m_\pi^2}$, so as to restore the conventional requirement that $x(t) \approx 1$ for $t \approx m_\pi^2$. Such a t -dependent form of $x(t)$ would imply that the t -dependent structure is not only due to the σ -term but also due to the initial state interaction of the pion-

poles of the vertex. More precisely it corresponds to an exchange of some particle between the two pions in the initial state which would generate a crossed pole in the \bar{t} -channel. By setting $x(t)$ to this form we are allowing for all such exchanges responsible for the left-hand cut in $(\pi-\pi)$ scattering by an effective pole in the t -channel. This form of $x(t)$ will be used in Section 4 where we attempt to modify our assumptions on $x(t)$. For the rest of this section, however, we shall tacitly assume that $x(t) = 1$ for any t thereby retaining contributions to the off-shell vertex from diagrams of the type illustrated in Fig. VA only.

To see whether such an approximation is at all reasonable, we test eq. (5.8) in a single particle model, by introducing the conjectured σ -meson as a pole in $F_\sigma(t)$ ^{*} and as the dominant single particle state in $\rho_\sigma(s)$,

$$\rho_\sigma(s) \approx \delta(s - m_\sigma^2) | \langle 0 | \sigma(0) | \sigma \rangle |^2 \quad (5.9)$$

$$= \delta(s - m_\sigma^2) g_\sigma^2 ,$$

$$F_\sigma(t) \cong \frac{g_\sigma G_{\sigma\pi\pi}}{t - m_\sigma^2} \quad (5.10)$$

We introduce eqs. (5.9) and (5.10) into eq. (5.8) with $x(t) = 1$, compare coefficients in t and obtain

$$g_\sigma^2 \approx m_\pi^4 F_\pi^2 \quad (5.11)$$

$$F_\pi G_{\sigma\pi\pi} \approx (m_\sigma^2 - m_\pi^2) \quad (5.12)$$

* This is analogous to the pole dominance assumption used in connection with vector currents

Eq. (5.11) is equivalent to interpreting $-i[A_0^i(x) d^3 \underline{x}, \rho_\pi^j(y)] = \delta^{ij} \sigma(y) / (m_\pi^2 F_\pi)$ as the interpolating σ -field⁽¹⁰⁾. Eq. (5.12) can also be derived in the σ -model⁽¹¹⁾ in lowest order perturbation theory. The width of the σ -meson as predicted by eq. (5.12), as a function of its mass, is found to be

$$\Gamma_\sigma = \frac{3}{2} \frac{1}{8\pi} \left(\frac{G_{\sigma\pi\pi}}{m_\sigma} \right)^2 p_\pi \quad (5.13)$$

where p_π is the momentum of each pion in the rest frame of the σ . The numerical results are given in Table VA.

Table VA

m_σ (Mev)	350	400	450	500	550
Γ_σ (Mev)	70	130	220	330	470
Contribution to Adler sum rule	1.71	1.54	1.39	1.34	1.31

The σ -parameters so obtained are not inconsistent with the results of Brown and Singer⁽⁵⁾, recently supported by current algebra sum rules⁽¹²⁾, but considerably higher masses with broader widths are also allowed. In the narrow width approximation Adler's (π - π) scattering sum rule gets the following contributions from the various (π - π) resonances: σ -contribution + ρ -contribution (≈ 0.51) + f -contribution (≈ 0.11) + g -contribution (≈ 0.08) ≈ 1.43 (5.14) so that the σ -contribution ≈ 0.73 , i.e., it contributes about 50%.

The last row in Table VA shows that the σ -contribution is by $\sim 100\%$, the contribution decreasing slowly for higher masses. In view of our crude approximations the disagreement is not appreciable.

3. Since the single-particle model of Section 2 ignores unitarity corrections we cannot consider it to be an accurate solution of eq. (5.8). Its only purpose was to demonstrate that the approach is not misleading. Now we abandon the single-particle model and introduce a continuum of intermediate states into $\rho_\sigma(s)$. The two pion contribution is given by

$$\rho_\sigma^{(2\pi)}(s) = \frac{3}{32\pi^2} \frac{\sqrt{(s-4m_\pi^2)}}{s} |F_\sigma(s)|^2 \quad (5.15)$$

For $s > (4m_\pi)^2$ there are also inelastic contributions; we account for these by introducing a factor $R(s)$

$$\rho_\sigma(s) = R(s) \rho_\sigma^{(2\pi)}(s) \quad (5.16)$$

with $R(s) = 1$ for $s < (4m_\pi)^2$, and $R(s) > 1$ for $s > (4m_\pi)^2$. Eq. (5.8) then becomes

$$F_\sigma(t) \alpha(t) = -m_\pi^2 + \frac{(m_\pi^2 - t)}{m_\pi^2 F_\pi} \frac{3}{32\pi^2} \int_{4m_\pi^2}^{\infty} ds \frac{R(s)}{(s-t)} \frac{\sqrt{s-4m_\pi^2}}{s} |F_\sigma(s)|^2 \quad (5.17)$$

In order that the integral on the right of eq. (5.17) may converge,

we require that $\lim_{s \rightarrow \infty} F_{\sigma}(s) = 0^*$. Further information on $F_{\sigma}(t)$ can be obtained from the linearized unitarity relations for vertex functions, which allow us to represent $F_{\sigma}(t)$ as an Omnès function

$$F_{\sigma}(t \pm i\epsilon) = -m_{\pi}^2 \exp \left\{ \frac{t - m_{\pi}^2}{\pi} \int_{4m_{\pi}^2}^{\infty} \frac{ds \delta_{\sigma}(s)}{(s - t \mp i\epsilon)(s - m_{\pi}^2)} \right\} \quad (5.18)$$

$$= -|F_{\sigma}(t)| e^{\pm i\delta_{\sigma}(t)}$$

For $t < (4m_{\pi})^2$ the phase $\delta_{\sigma}(t)$ is equal to the $(\pi\pi)$ s-wave scattering phase $\delta_{\sigma}(t)$ in the isospin-zero channel, and this relation remains approximately true as long as inelasticity can be neglected. Comparing the discontinuities of $F_{\sigma}(t)$ according to eqs. (5.17) and (5.18) (assuming that $\text{disc} x(t)$ may be neglected), we get

$$2\pi i \frac{3}{32\pi^2} \frac{(m_{\pi}^2 - t)}{m_{\pi}^2 F_{\pi}^2} \sqrt{\frac{t - 4m_{\pi}^2}{t}} \frac{R(t)}{x(t)} |F_{\sigma}(t)|^2 = -2i \sin \delta_{\sigma}(t) F_{\sigma}(t). \quad (5.19)$$

This implies that as long as $x(t) > 0$

$$\sin \delta_{\sigma}(t) \approx \sin \delta_{\sigma}(t) > 0 \quad ; \quad 0 \lesssim \delta_{\sigma}(t) \lesssim \pi \quad (5.20)$$

We emphasize that this result requires only local validity of our assumptions; in particular the actual form of $x(t)$ does not matter

* If we assume that $\lim_{s \rightarrow \infty} x(s) F_{\sigma}(s) = 0$ then eq. (5.17) reduces to a sum rule which was derived by Woo⁽¹⁵⁾ assuming asymptotic chiral invariance. His failure to saturate this sum rule may be related to our difficulties in solving eq. (5.17).

so long as $x(t) > 0$. Hence it will not be impaired by the difficulties and ambiguities that arise in trying to solve eq. (5.17) by assuming some simple forms for $x(t)$.

To solve eq. (5.17) we define a new function $D_\sigma(t)$ by

$$F_\sigma(t) = - \frac{m_\pi^2}{D_\sigma(t)} \quad (5.21)$$

The discontinuity of $D_\sigma(t)$ across the cut $t \gg 4m_\pi^2$ is given by
 $\text{disc. } D_\sigma(t) = 2i \text{ Im } D_\sigma(t)$

$$= 2\pi i \frac{3}{32\pi^2} \frac{(m_\pi^2 - t)}{F_\pi^2} \sqrt{\frac{t - 4m_\pi^2}{t}} \frac{R(t)}{x(t)} \quad (5.22)$$

for $t > 4m_\pi^2$.

We next show that if $x(t) = 1$, then $F_\sigma(t)$ has no zeroes.

From eq. (5.17), we deduce

$$F_\sigma(\infty) = 0 = -m_\pi^2 + \frac{3}{32\pi^2 m_\pi^2 F_\pi^2} \int_{4m_\pi^2}^{\infty} ds \sqrt{\frac{s - 4m_\pi^2}{s}} R(s) |F_\sigma(s)|^2 \quad (5.23)$$

Subtracting eq. (5.23) from eq. (5.17), we get

$$F_\sigma(t) = \frac{3}{32\pi^2 m_\pi^2 F_\pi^2} \int_{4m_\pi^2}^{\infty} ds \sqrt{\frac{s - 4m_\pi^2}{s}} R(s) |F_\sigma(s)|^2 \left(\frac{m_\pi^2 - s}{s - t} \right)$$

For real $t < 4m_\pi^2$, $F_\sigma(t)$ is always negative since the integral is negative definite. For complex t

$$\text{Im } F_{\sigma}(t) = \frac{3}{32 \pi^2 m_{\pi}^2 F_{\pi}^2} \int_{4m_{\pi}^2}^{\infty} ds \sqrt{\frac{s - 4m_{\pi}^2}{s}} R(s) \frac{(m_{\pi}^2 - s) |F_{\sigma}(s)|^2 \text{Im } t}{(s - \text{Re } t)^2 + (\text{Im } t)^2}$$

since $\text{Im } t \neq 0$, $\text{Im } F_{\sigma}(t)$ cannot vanish.

Therefore, $F_{\sigma}(t) \neq 0$ for $x(t) = 1$. This means that there should not be any CDD pole in $D_{\sigma}(t)$. Using its discontinuity as found in eq. (5.22), we can represent $D_{\sigma}(t)$ as a dispersion integral subtracted at $t = m_{\pi}^2$ with $D_{\sigma}(m_{\pi}^2) = 1$. The addition of arbitrary subtraction polynomials is limited, because they would require superconvergence of $F_{\sigma}(t)$:

$$\lim_{t \rightarrow \infty} t F_{\sigma}(t) = 0 ;$$

this is not compatible with a negative definite $\text{Im } F_{\sigma}(t)$ (unless $x(t)$ changes sign).

Neglecting inelasticity and PCAC corrections (i.e. putting $R(t) = x(t) = 1$) and integrating $D_{\sigma}(t)$, we get

$$\begin{aligned} D_{\sigma}(t) &= 1 + a(m_{\pi}^2 - t) + (m_{\pi}^2 - t)^2 \frac{3}{32 \pi^2} \frac{1}{F_{\pi}^2} \int_{4m_{\pi}^2}^{\infty} \sqrt{\frac{s - 4m_{\pi}^2}{s}} \frac{ds}{(m_{\pi}^2 - s)(s - t)} \\ &= 1 + (m_{\pi}^2 - t) \left\{ a' + \frac{3}{32 \pi^2 F_{\pi}^2} \sqrt{\frac{t - 4m_{\pi}^2}{t}} \log \left(\frac{\sqrt{4m_{\pi}^2 - t} - \sqrt{-t}}{\sqrt{4m_{\pi}^2 - t} + \sqrt{-t}} \right) \right\} , \end{aligned}$$

for $t < 0$

$$\begin{aligned}
 &= 1 + (m_\pi^2 - t) \left\{ a' - \frac{\sqrt{4m_\pi^2 - t}}{t} \frac{3}{16\pi^2 F_\pi^2} \arctan \left(\sqrt{\frac{t}{4m_\pi^2 - t}} \right) \right. \\
 &\hspace{20em} \left. \text{for } 0 < t < 4m_\pi^2 \right. \\
 &= 1 + (m_\pi^2 - t) \left\{ a' + \frac{3}{32\pi^2 F_\pi^2} \sqrt{\frac{t - 4m_\pi^2}{t}} \left(\log \left(\frac{\sqrt{t} - \sqrt{t - 4m_\pi^2}}{\sqrt{t} + \sqrt{t - 4m_\pi^2}} \right) \pm i\pi \right) \right\} , \\
 &\hspace{10em} \text{for } t = t_R \pm i\epsilon, \quad t_R > 4m_\pi^2 \hspace{10em} (5.24)
 \end{aligned}$$

The unknown subtraction constant a' has been redefined from a in the course of the calculation. Since $D_0(t)$ has no CDD poles the solution (5.24) is unique. It is easy to see that none of these solutions is acceptable. At $t = m_\pi^2$ we have $D_0(m_\pi^2) = 1$, but at $t \rightarrow -\infty$, the last term dominates and we obtain $D_0(t) \rightarrow -\infty$. This means $D_0(t)$ passes through zero at some value of t . But a zero in $D_0(t)$ corresponds to a pole in $F_0(t)$, which is not contained in eq. (5.17). By keeping $R(s) \gg 1$, the negative term is only enhanced. So we conclude that eq. (5.17) has no solutions with $x(t) = 1$.

4. Being forced to introduce corrections to pion-pole dominance, it will be our aim to keep the model simple and to avoid having too many undetermined and unmotivated parameters. Comparing eq. (5.22) with the usual (N/D) equations of $(\pi-\pi)$ s-wave scattering we have

$$T_{l=0, I=0}(t) = 32\pi \sqrt{\frac{t}{t - 4m_\pi^2}} e^{i\delta_0(t)} \sin \delta_0(t) = \frac{N(t)}{D(t)} , \hspace{2em} (5.24)$$

and

$$\text{disc. } D(t) = 2i \left(-\frac{1}{32\pi} \right) \sqrt{\frac{t - 4m_\pi^2}{t}} N(t) \quad (5.25)$$

With $D(t)$ given by $D_\sigma(t)$, we would have a linear N-function

$$N_\sigma(t) = -3 \frac{(m_\pi^2 - t)}{F_\pi^2} \left(\frac{R(t)}{x(t)} \right), \quad (5.26)$$

up to inelasticity and corrections to PCAC. This may well be a good approximation for t in the neighbourhood of m_π^2 , but it fails at negative t , where $N(t)$ should have its left-hand cut. A linear rise of $N(t)$ at large t appears also unlikely. On these grounds it seems reasonable to correct $N(t)$ by introducing a factor $R(t)/x(t) = (m^2 + m_\pi^2)/(m^2 + t)$ (compare the explanation given for Fig. VB in Section 2) to simulate some effects of the omitted left-hand cut for large t in the integration region $t > 4m_\pi^2$. Not to distort the current algebra predictions for small t , we should choose $m^2 \gg m_\pi^2$. As to the size of m^2 , we have taken different values in the ρ -exchange region and above ($m = 600, 760, 1000, 1500, 2000$ Mev). We do not want to commit ourselves to a final statement on this point and, for the moment, prefer to regard m^2 as some cut-off parameter. The precise nature of the required PCAC correction needs further investigation.

The modified D-function can be integrated, and we get

$$D_{\sigma}^m(t) = 1 + (m_{\pi}^2 - t) \frac{3}{32\pi^2} \frac{1}{F_{\pi}^2} \int_{4m_{\pi}^2}^{\infty} \sqrt{\frac{s - 4m_{\pi}^2}{s}} \frac{m^2 + m_{\pi}^2}{m^2 + s} \frac{ds}{s - t}$$

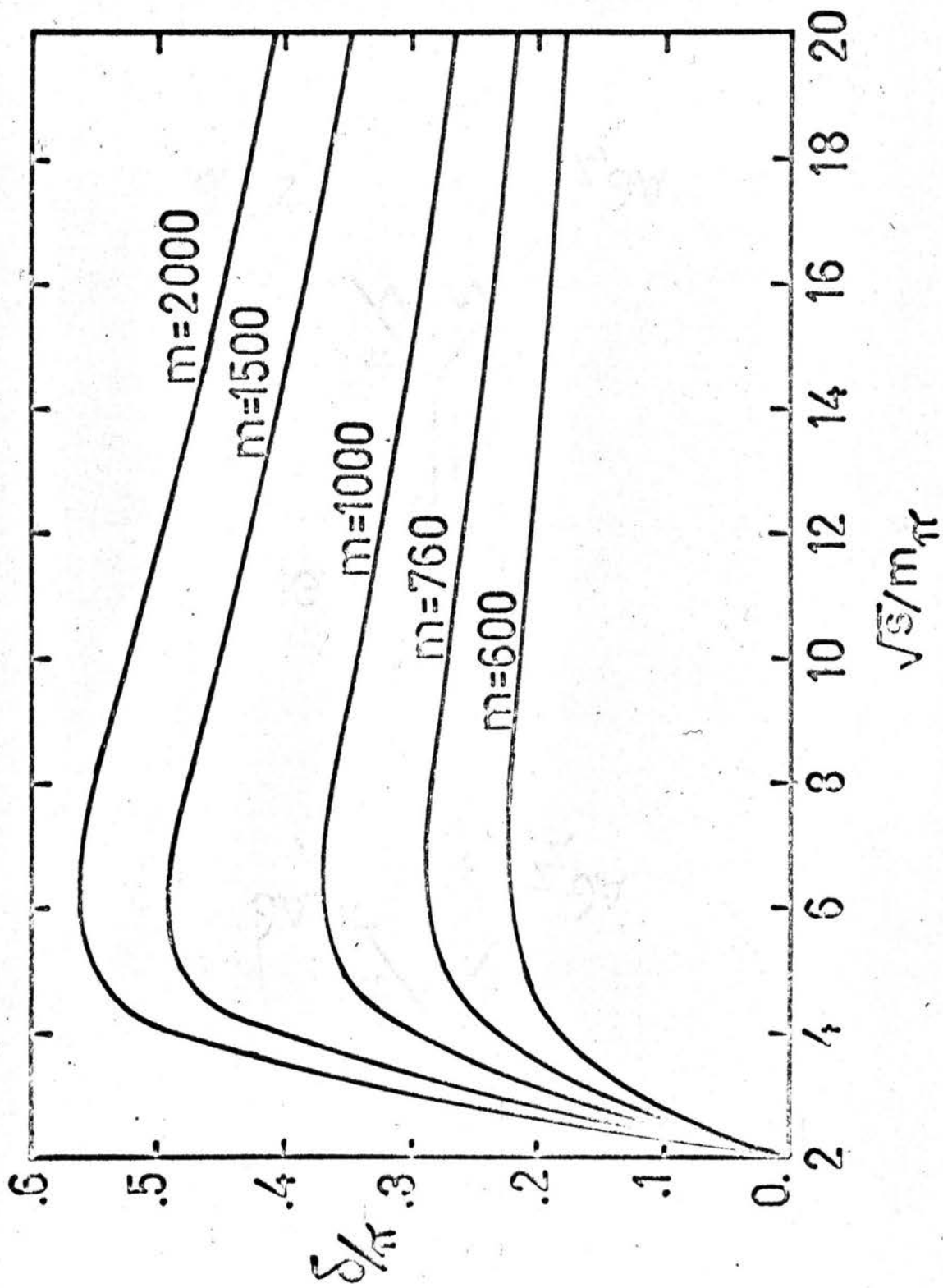
$$= 1 + \frac{3(m^2 + m_{\pi}^2)(m_{\pi}^2 - t)}{32\pi^2 F_{\pi}^2 (m^2 + t)} \left\{ \sqrt{\frac{4m_{\pi}^2 + m^2}{m^2}} \log \left(\frac{\sqrt{4m_{\pi}^2 + m^2} + m}{\sqrt{4m_{\pi}^2 + m^2} - m} \right) - \sqrt{\frac{t - 4m_{\pi}^2}{t}} \log \left(\frac{\sqrt{4m_{\pi}^2 - t} + \sqrt{-t}}{\sqrt{4m_{\pi}^2 - t} - \sqrt{-t}} \right) \right\} \text{ for } t < 0,$$

$$= 1 + \frac{3(m^2 + m_{\pi}^2)(m_{\pi}^2 - t)}{32\pi^2 F_{\pi}^2 (m^2 + t)} \left\{ \sqrt{\frac{4m_{\pi}^2 + m^2}{m^2}} \log \left(\frac{\sqrt{4m_{\pi}^2 + m^2} + m}{\sqrt{4m_{\pi}^2 + m^2} - m} \right) - 2 \sqrt{\frac{4m_{\pi}^2 - t}{t}} \operatorname{arc tan} \left(\sqrt{\frac{t}{4m_{\pi}^2 - t}} \right) \right\} \text{ for } 0 < t < 4m_{\pi}^2,$$

$$= 1 + \frac{3(m^2 + m_{\pi}^2)(m_{\pi}^2 - t)}{32\pi^2 F_{\pi}^2 (m^2 + t)} \left\{ \sqrt{\frac{4m_{\pi}^2 + m^2}{m^2}} \log \left(\frac{\sqrt{4m_{\pi}^2 + m^2} + m}{\sqrt{4m_{\pi}^2 + m^2} - m} \right) - \sqrt{\frac{t - 4m_{\pi}^2}{t}} \left(\log \left(\frac{\sqrt{t} + \sqrt{t - 4m_{\pi}^2}}{\sqrt{t} - \sqrt{t - 4m_{\pi}^2}} \right) \mp i\pi \right) \right\}$$

$$\text{for } t = t_R \pm i\epsilon, \quad t_R > 4m_{\pi}^2. \quad (5.27)$$

FIG. VC



For $t < 4m_\pi^2$, $D_\sigma^{(m)}(t)$ is seen to be real and has no zeroes (in the range of m considered). For $t > 4m_\pi^2$, neglecting inelasticity, its phase is the negative of the s -wave ($\pi-\pi$) phase shift δ_0 in the isospin zero channel.

$$\delta_0^{(m)}(t) = -\arg D_\sigma^{(m)}(t) \quad \text{for } t > 4m_\pi^2 \quad (5.28)$$

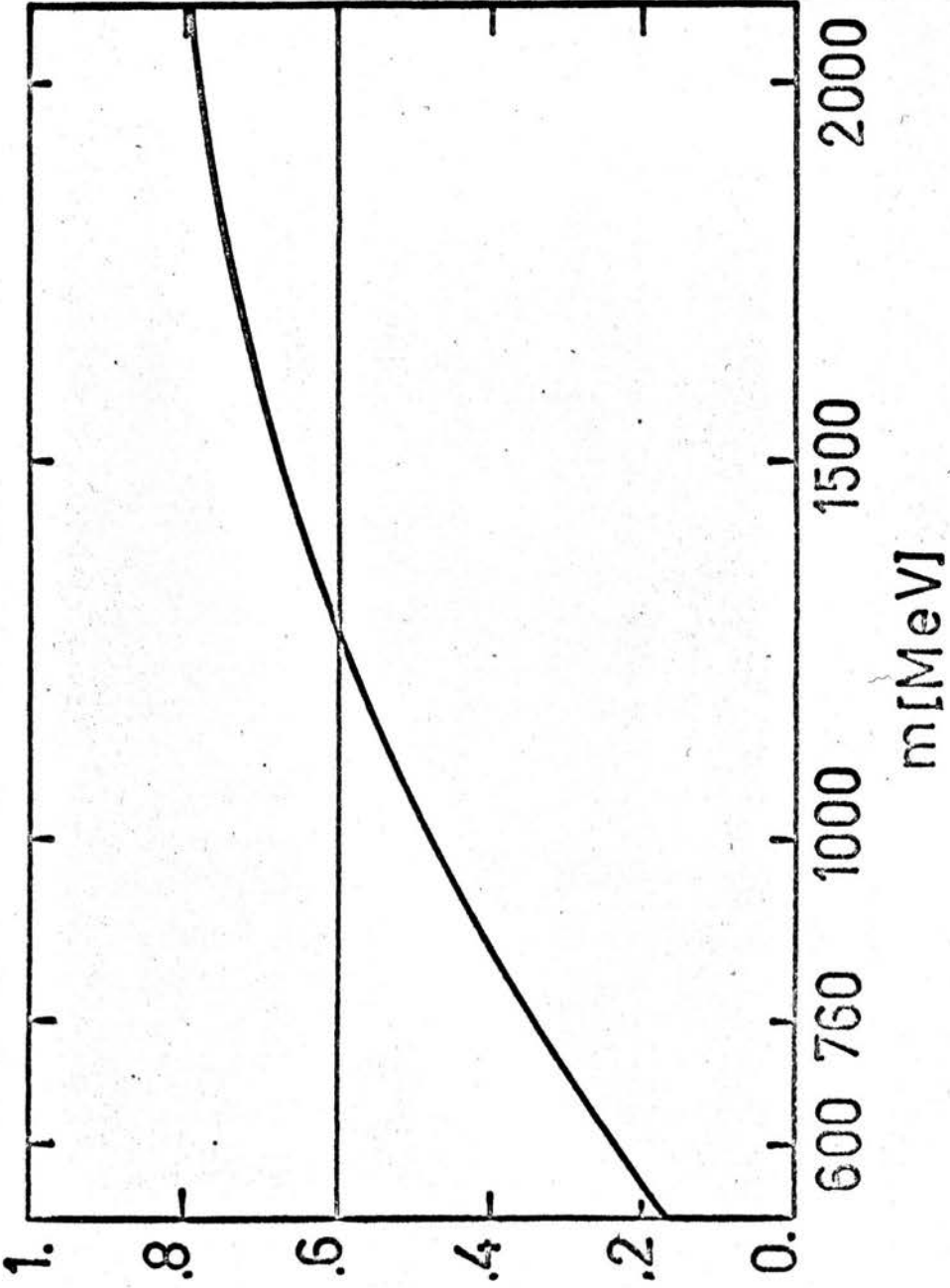
In Figure VC we have plotted $\delta_0^{(m)}(t)$ for different values of m . The resulting scattering lengths are given by

$$a_0(m) = \lim_{t \rightarrow 4m_\pi^2} \frac{(-2) \operatorname{Im} D_\sigma^{(m)}(t)}{\sqrt{t-4m_\pi^2} \operatorname{Re} D_\sigma^{(m)}(t)} \approx \frac{\pi/m_\pi}{\frac{32\pi^2 F_\pi^2}{9m_\pi^2} - 2 \log\left(\frac{m}{m_\pi}\right)} \quad \text{for } \left(\frac{m}{m_\pi}\right) \gg 1. \quad (5.29)$$

For a wide range of values of the cut-off parameter m , the scattering length varies only by a small amount: $0.23 m_\pi^{-1} \sim 0.33 m_\pi^{-1}$, which is slightly larger than Weinberg's result⁽⁶⁾. They are given in Table VB. The phase shift exhibits a broad maximum at about 700 Mev falling off very slowly at larger energies. This shape resembles qualitatively the results of Lovelace, Heinz and Donnachie⁽¹³⁾, but we prefer to reserve our opinion at present, because the height of the maximum is sensitive to the cut-off parameter m , and there is every reason to regard the tail at $t \gtrsim m^2$ as cut-off dependent (see Fig. VC).

Using the calculated values of $\delta_0^{(m)}(t)$ as a function of m we examine the saturation of the Adler-Weisberger relation for ($\pi-\pi$) scattering⁽¹⁴⁾ with the resonances ρ and f (contributing about

FIG. YD



44%) and the isospin-zero s-wave continuum. We require, therefore

$$0.56 \approx |F_\pi|^2 \int_{4m_\pi^2}^{\infty} \frac{32}{3} \frac{ds}{(s-m^2)^2} \sqrt{\frac{s}{s-4m_\pi^2}} \sin^2 \delta_0^{(\pi)}(s) \quad (5.30)$$

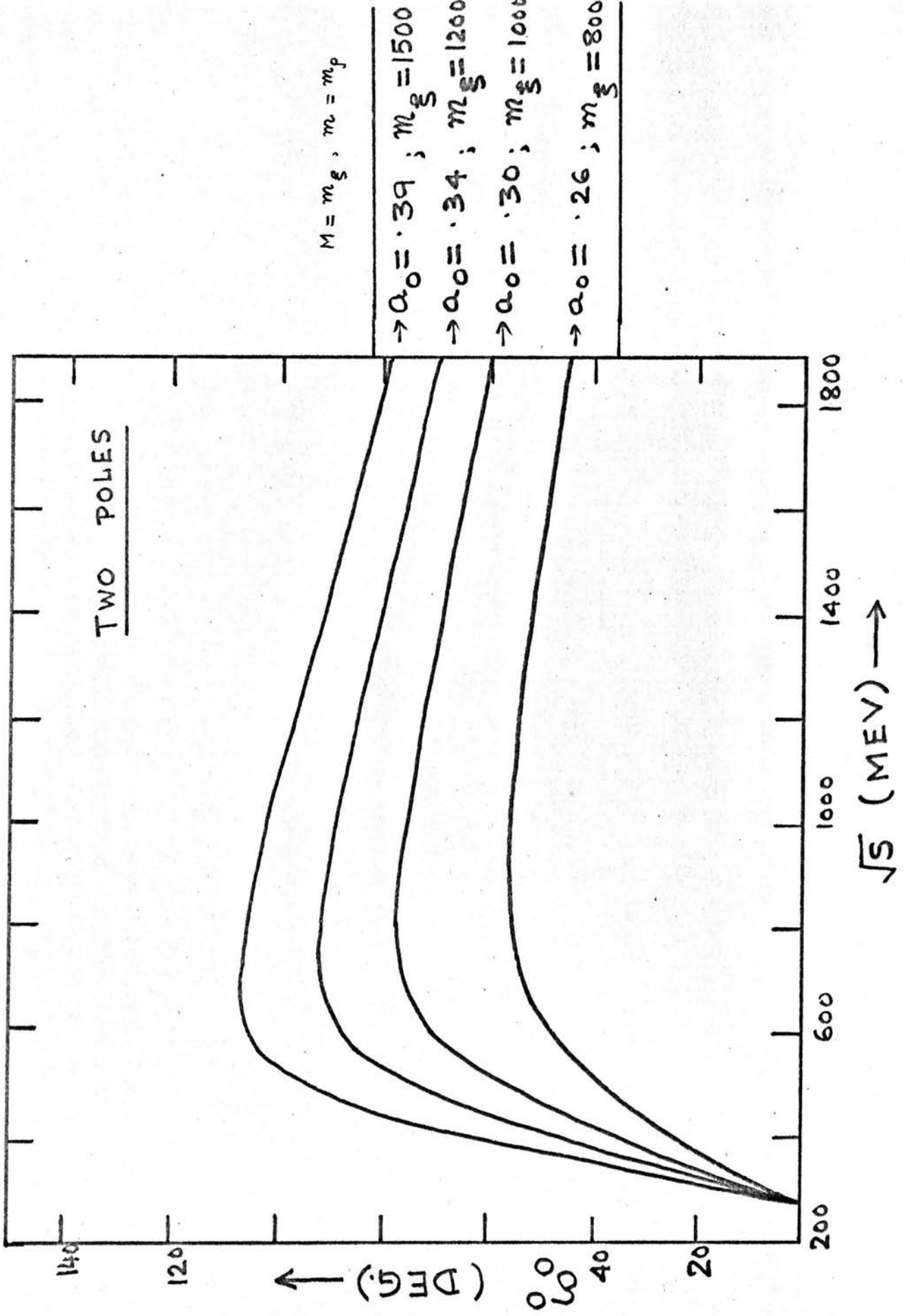
(We have neglected the soft-pion correction as given by Adler.) We find approximate saturation for $m \approx 1200$ Mev. Inclusion of the g-resonance would lower the s-wave contribution to about 51% which would correspond to a cut-off value of about $m \approx 1100$ Mev. The right-hand-side of eq. (5.30), plotted as a function of the cut-off parameter m , is given in Fig. VD.

Table VB

m (Mev)	$a_0 (m_\pi^{-1})$
600	0.231
760	0.252
1000	0.276
1500	0.307
2000	0.330

The s-wave scattering length a_0 as a function of the cut-off mass m .

FIG. VE



5. Finally to test the dependence of our result on the shape of the cut-off function we have repeated our calculation with a few different cut-off functions $(R(t)/x(t))$:

a. two poles : $\left(\frac{2M^2}{M^2+t} - \frac{m^2}{m^2+t} \right)$; $2M^2 > m^2$

b. a pole and a dipole : $\frac{1}{1-\beta} \left(\frac{M^2}{M^2+t} - \frac{\beta M^4}{(M^2+t)^2} \right)$; $0 < \beta < 1$

where we have imposed certain conditions on the parameters such that $D_{\sigma}^{(m)}(t)$ does not acquire a zero for $t < 4\frac{m^2}{\pi}$. The explicit solutions for $D_{\sigma}^{(m)}(t)$ in the two cases are listed below:

a. two poles:

$$D_{\sigma}^{(m)}(t)$$

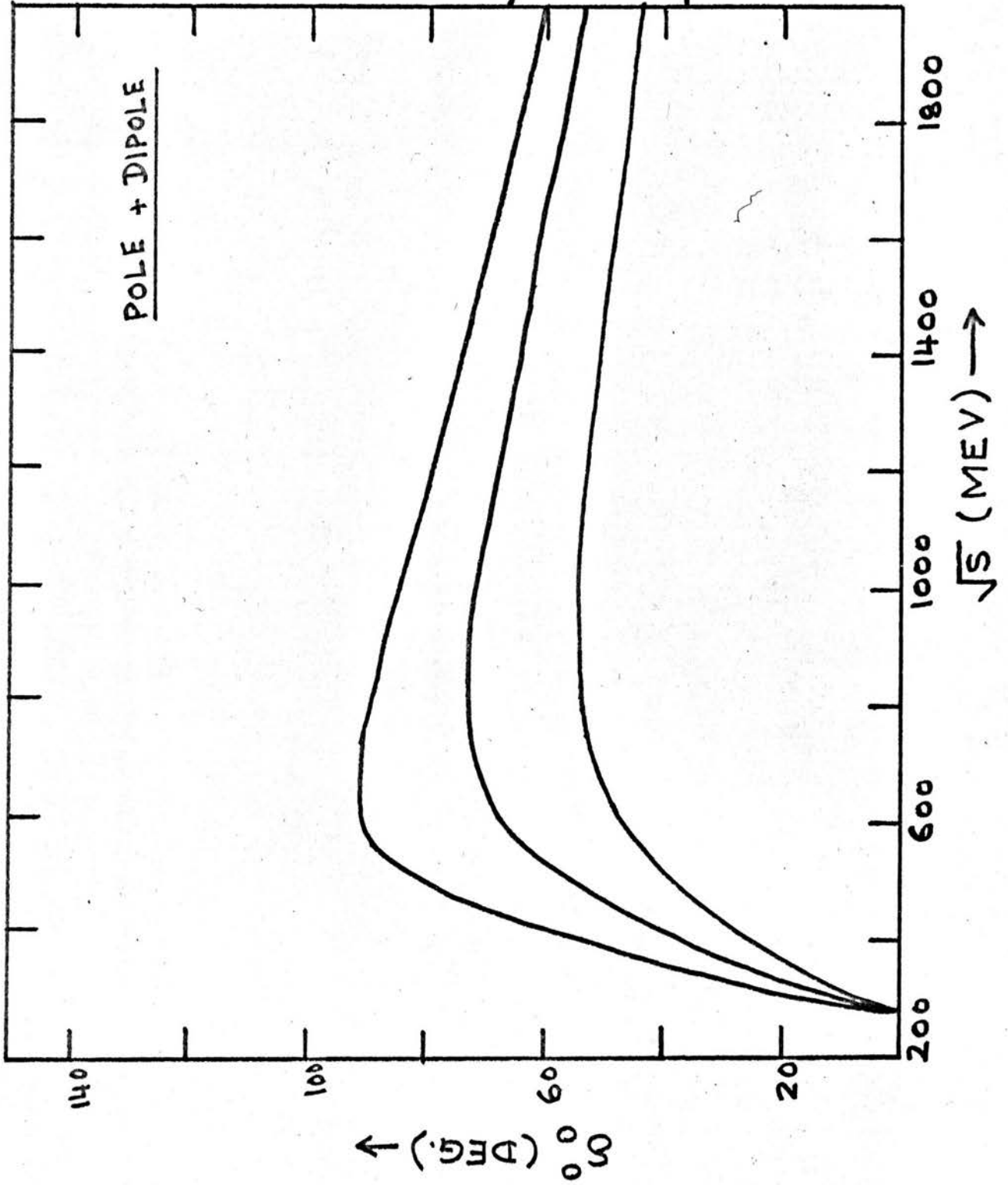
$$= 1 + \frac{8c(1-t)}{t(a+l')} \left\{ \sqrt{l'} \ln \left(\frac{\sqrt{l'}+1}{\sqrt{l'}-1} \right) - \sqrt{-a} \ln \left(\frac{\sqrt{-a}+1}{\sqrt{-a}-1} \right) \right\} - \frac{4c(1-t)}{t(a+l)} \left\{ \sqrt{l} \ln \left(\frac{\sqrt{l}+1}{\sqrt{l}-1} \right) - \sqrt{-a} \ln \left(\frac{\sqrt{-a}+1}{\sqrt{-a}-1} \right) \right\} \text{ for } t < 0,$$

$$= 1 + \frac{8c(1-t)}{t(a+l')} \left\{ \sqrt{l'} \ln \left(\frac{\sqrt{l'}+1}{\sqrt{l'}-1} \right) - 2\sqrt{a} \tan^{-1} \left(\frac{1}{\sqrt{a}} \right) \right\} - \frac{4c(1-t)}{t(a+l)} \left\{ \sqrt{l} \ln \left(\frac{\sqrt{l}+1}{\sqrt{l}-1} \right) - 2\sqrt{a} \tan^{-1} \left(\frac{1}{\sqrt{a}} \right) \right\} \text{ for } 0 < t < 4,$$

$$= 1 + \frac{8c(1-t)}{t(a+l')} \left\{ \sqrt{l'} \ln \left(\frac{\sqrt{l'}+1}{\sqrt{l'}-1} \right) - \sqrt{-a} \left(\ln \left(\frac{1+\sqrt{-a}}{1-\sqrt{-a}} \right) \mp i\pi \right) \right\} \quad (5.31)$$

$$- \frac{4c(1-t)}{t(a+l)} \left\{ \sqrt{l} \ln \left(\frac{\sqrt{l}+1}{\sqrt{l}-1} \right) - \sqrt{-a} \left(\ln \left(\frac{1+\sqrt{-a}}{1-\sqrt{-a}} \right) \mp i\pi \right) \right\} \text{ for } t = t_R \pm i\epsilon, \\ t_R > 4.$$

FIG. V F



b. a pole and a dipole:

$$\begin{aligned}
 D_{\sigma}^{(m)} &= 1 + \frac{4c(1-t)}{t(a+b)(1-\beta)} \left\{ \left[\sqrt{b} \ln \left(\frac{\sqrt{b}+1}{\sqrt{b}-1} \right) - \sqrt{-a} \ln \left(\frac{\sqrt{-a}+1}{\sqrt{-a}-1} \right) \right] \right. \\
 &\quad \left. + \beta \left[1 + \frac{b+b^2+3ab-a}{2\sqrt{b}(a+b)} \ln \left(\frac{\sqrt{b}-1}{\sqrt{b}+1} \right) + \sqrt{-a} \left(\frac{1+a}{a+b} \right) \ln \left(\frac{\sqrt{-a}+1}{\sqrt{-a}-1} \right) \right] \right\} \\
 &\hspace{25em} \text{for } t < 0, \\
 &= 1 + \frac{4c(1-t)}{t(a+b)(1-\beta)} \left\{ \left[\sqrt{b} \ln \left(\frac{\sqrt{b}+1}{\sqrt{b}-1} \right) - 2\sqrt{a} \tan^{-1} \frac{1}{\sqrt{a}} \right] \right. \\
 &\quad \left. + \beta \left[1 + \frac{b+b^2+3ab-a}{2\sqrt{b}(a+b)} \ln \left(\frac{\sqrt{b}-1}{\sqrt{b}+1} \right) + 2\sqrt{a} \left(\frac{1+a}{a+b} \right) \tan^{-1} \left(\frac{1}{\sqrt{a}} \right) \right] \right\} \\
 &\hspace{25em} \text{for } 0 < t < 4, \\
 &= 1 + \frac{4c(1-t)}{t(a+b)(1-\beta)} \left\{ \left[\sqrt{b} \ln \left(\frac{\sqrt{b}+1}{\sqrt{b}-1} \right) - \sqrt{-a} \left(\frac{1+a}{a+b} \right) \left(\ln \left(\frac{1+\sqrt{-a}}{1-\sqrt{-a}} \right) \mp i\pi \right) \right] \right\} \\
 &\hspace{25em} \text{for } t = t_R \pm i\epsilon, \quad t_R > 4.
 \end{aligned}$$

(5.32)

In the above solutions we have set $m_{\pi} = 1$ and have defined

$$a = \frac{4-t}{t}, \quad b = \frac{4+m^2}{m^2}, \quad b' = \frac{4+M^2}{M^2}, \quad c = \frac{3}{32\pi^2 F_{\pi}^2}$$

In Fig. VE and Fig. VF we have plotted the phase shifts for a few values of the parameters and for the two separate cases as indicated. Qualitatively the shapes of the phase shifts are the same as before. Again we find a maximum near 700 Mev, the height being dependent on the parameters.

APPENDIX I

NOTATIONS

1. Natural Units.

We set $c = \hbar = 1$, then every dimensional quantity will be expressed in units of some powers of mass, e.g., length will be expressed as inverse mass.

2. Relativistic Notations

Our metric is such that

$$k \cdot x \equiv k_{\mu} x^{\mu} = -k^0 x^0 + \underline{k} \cdot \underline{x} = -\omega t + \underline{k} \cdot \underline{x} \quad , \quad (\text{AI.1})$$

$$k^{\mu} \equiv (\omega, \underline{k}) \equiv (k^0, k^1, k^2, k^3) \quad , \quad (\text{AI.2})$$

$$x^{\mu} \equiv (t, \underline{x}) \equiv (x^0, x^1, x^2, x^3) \quad . \quad (\text{AI.3})$$

The scalar product as defined in eq. (AI.1) corresponds to the metric tensor,

$$g_{\mu\nu} \equiv g^{\mu\nu} = \left\{ \begin{array}{cccc} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{array} \right\} \quad . \quad (\text{AI.4})$$

Covariant and Contravariant tensors are defined by

$$\text{Contravariant vector: } x^{\mu} \equiv (x^0, x^1, x^2, x^3) = (t, \underline{x}) \quad (\text{AI.5})$$

$$\text{Covariant vector} \quad : \quad x_{\mu} \equiv (x_0, x_1, x_2, x_3) = (-t, \underline{x}) \quad . \quad (\text{AI.6})$$

Einstein summation convention is used. (Latin indices i, j, \dots run over 1,2,3, and Greek indices μ, ν, \dots over 0,1,2,3.) Further

$$g_{\mu\rho} g^{\rho\nu} = g_{\mu}^{\nu} = \delta_{\mu}^{\nu} = \left\{ \begin{matrix} 1 & & \\ & 1 & \\ & & 1 \end{matrix} \right\}, \quad (\text{AI.7})$$

$$dF = \frac{\partial F}{\partial x^{\mu}} dx^{\mu} = \frac{\partial F}{\partial x_{\mu}} dx_{\mu}, \quad (\text{AI.8})$$

$$\frac{\partial}{\partial x^{\mu}} \equiv \partial_{\mu} \equiv \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial \underline{x}} \right), \quad (\text{Covariant differentiation}) \quad (\text{AI.9})$$

$$\frac{\partial}{\partial x_{\mu}} \equiv \partial^{\mu} \equiv \left(-\frac{\partial}{\partial t}, \frac{\partial}{\partial \underline{x}} \right), \quad (\text{Contravariant differentiation}) \quad (\text{AI.10})$$

Sometimes $\frac{\partial}{\partial \underline{x}}$ is denoted by $\underline{\nabla}$ or $\underline{\partial}$ and $\partial_{\mu} \partial^{\mu}$ by \square .

3. Relativistic field equations

(a) Free scalar field $\phi(x)$ of mass m :

$$(m^2 - \square) \phi(x) = 0 \quad (\text{Klein-Gordon equation}), \quad (\text{AI.11})$$

(b) Free Dirac field $\psi(x)$ of mass m and spin half:

$$\frac{\partial \psi(x)}{\partial t} + \underline{\alpha} \cdot \underline{\nabla} \psi(x) + im_{\beta} \psi(x) = 0 \quad (\text{Dirac equation}), \quad (\text{AI.12})$$

$\underline{\alpha}$ and β are traceless, hermitian, 4×4 matrices which satisfy the anticommutation relations,

$$\left. \begin{aligned} \{ \alpha^i, \alpha^j \} &= 2 \delta^{ij}, \\ \{ \alpha^i, \beta \} &= 0, \end{aligned} \right\} \quad \begin{matrix} \\ \\ \end{matrix} \quad \begin{matrix} \\ \\ \end{matrix} \quad \left. \begin{matrix} \\ \\ \end{matrix} \right\} \quad (\text{AI.13})$$

also, $\alpha^1{}^2 = \alpha^2{}^2 = \alpha^3{}^2 = \beta^2 = 1$.

We choose the following representation for the matrices

$$\alpha^k = \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad k = 1, 2, 3 \quad (\text{AI.14})$$

where

$$\left. \begin{aligned} \sigma^1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \sigma^2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \\ \sigma^3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & 1 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned} \right\} \quad (\text{AI.15})$$

We define

$$\left. \begin{aligned} \gamma^i &= -i\beta\alpha^i = \begin{pmatrix} 0 & -i\sigma^i \\ i\sigma^i & 0 \end{pmatrix}, \\ \gamma^0 &= -i\beta = -i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ \gamma^5 &= i\gamma^1\gamma^2\gamma^3\gamma^0 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \end{aligned} \right\} \quad (\text{AI.16})$$

Then

$$\left. \begin{aligned} \gamma^{0\dagger} &= \gamma^0 & ; & \gamma^{0^2} &= -1 & ; \\ \gamma^{i\dagger} &= \gamma^i & ; & \gamma^{i^2} &= 1 & ; \\ \gamma^{5\dagger} &= \gamma^5 & ; & \gamma^{5^2} &= 1 & ; \\ \{\gamma^\mu, \gamma^\nu\} &= 2g^{\mu\nu} & ; & (\mu, \nu = 0, 1, \dots, 3, \\ \gamma^5 &= \frac{i}{4!} \epsilon_{\mu\nu\lambda\rho} \gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\rho & ; \\ \epsilon^{0123} &= -\epsilon_{0123} = +1 \end{aligned} \right\} \quad (\text{AI.17})$$

Covariant form for Dirac equation

$$(\not{\partial} + m) \psi(x) \equiv (\gamma_{\mu} \partial^{\mu} + m) \psi(x) = 0, \quad (\text{AI.18})$$

$$\gamma^{\mu} \equiv (\gamma^0, \gamma^1, \gamma^2, \gamma^3)$$

Define

$$\sigma^{\mu\nu} = \frac{1}{2i} [\gamma^{\mu}, \gamma^{\nu}] \quad (\text{AI.19})$$

Then

$$\sigma^{ij} = \frac{1}{2i} [\gamma^i, \gamma^j] = -i \gamma^i \gamma^j = \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} = \sigma^k, \quad (\text{AI.20})$$

if i, j, k are cyclic, i.e. ,

$$\sigma^{ij} = \epsilon^{ijk} \sigma^k \quad (\text{AI.21})$$

The adjoint spinor

$$\bar{\psi}(x) = i \psi^{\dagger}(x) \gamma^0, \quad (\text{AI.22})$$

satisfies the adjoint equation

$$\partial_{\mu} \bar{\psi}(x) \gamma^{\mu} - m \bar{\psi}(x) = 0. \quad (\text{AI.23})$$

Bilinear covariants (all hermitian):

$$\begin{aligned}
 \text{scalar} & : \bar{\psi} \psi, \\
 \text{vector} & : i \bar{\psi} \gamma^\mu \psi, \\
 \text{tensor} & : \bar{\psi} \sigma^{\mu\nu} \psi, \\
 \text{pseudovector} & : i \bar{\psi} \gamma^5 \gamma^\mu \psi, \\
 \text{pseudoscalar} & : i \bar{\psi} \gamma^5 \psi.
 \end{aligned} \tag{AI.24}$$

General free particle solution:

$$\psi^r(x) = \omega^r(\underline{p}) e^{i p \cdot x} \varepsilon^r, \quad r = 1, \dots, 4, \tag{AI.25}$$

where

$$\varepsilon^r = +1 \quad \text{for } r = 1, 2 \quad (\text{positive energy solutions}),$$

$$\varepsilon^r = -1 \quad \text{for } r = 3, 4 \quad (\text{negative energy solutions}),$$

and where

$$\begin{aligned}
 \omega^{(1)}(\underline{p}) &= \sqrt{E+m} \begin{pmatrix} 1 \\ 0 \\ \frac{p^3}{E+m} \\ \frac{p^1 + i p^2}{E+m} \end{pmatrix}; & \omega^{(2)}(\underline{p}) &= \sqrt{E+m} \begin{pmatrix} 0 \\ 1 \\ \frac{p^1 - i p^2}{E+m} \\ \frac{-p^3}{E+m} \end{pmatrix}; \\
 \omega^{(3)}(\underline{p}) &= \sqrt{E+m} \begin{pmatrix} \frac{p^3}{E+m} \\ \frac{p^1 + i p^2}{E+m} \\ 1 \\ 0 \end{pmatrix}; & \omega^{(4)}(\underline{p}) &= -\sqrt{E+m} \begin{pmatrix} \frac{p^1 - i p^2}{E+m} \\ \frac{-p^3}{E+m} \\ 0 \\ 1 \end{pmatrix} \tag{AI.26}
 \end{aligned}$$

$$(E \equiv p^0)$$

$\omega^{(r)}(\mathbf{p})$ satisfy

$$\left. \begin{aligned}
 (\not{k} - \varepsilon^r i m) \omega^{(r)}(\mathbf{p}) &= 0 & ; \\
 \bar{\omega}^{(r)}(\mathbf{p}) (\not{k} - \varepsilon^r i m) &= 0 & ; \\
 \omega^{(r)\dagger}(\mathbf{p}) \omega^r(\mathbf{p}) &= 2E\delta^{r4} & ; \\
 \bar{\omega}^{(r)}(\mathbf{p}) \omega^s(\mathbf{p}) &= 2m\delta^{rs} & ; \\
 \sum_{r=1}^4 \varepsilon^{(r)} \omega_{\alpha}^{(r)}(\mathbf{p}) \omega_{\beta}^{*(r)}(\mathbf{p}) &= 2m\delta_{\alpha\beta} & .
 \end{aligned} \right\} \quad (\text{AI.27})$$

Projection operator for positive energy solutions,

$$\Lambda_+(\mathbf{p}) = (m - i\not{k}) = \sum_{r=1}^2 \omega^{(r)}(\mathbf{p}) \bar{\omega}^{(r)}(\mathbf{p}) = \frac{[\Lambda_+(\mathbf{p})]^2}{2m} \quad (\text{AI.28})$$

Projection operator for negative energy solutions,

$$\Lambda_-(\mathbf{p}) = (m + i\not{k}) = - \sum_{r=3}^4 \omega^{(r)}(\mathbf{p}) \bar{\omega}^{(r)}(\mathbf{p}) = \frac{[\Lambda_-(\mathbf{p})]^2}{2m} \quad (\text{AI.29})$$

Equal-time commutator for Dirac fields,

$$[\psi_{\alpha}(x), \bar{\psi}_{\beta}(x')]_{t=t'} = i\gamma^0 \delta^3(\underline{x} - \underline{x}') \delta_{\alpha\beta} \quad ; \quad (\text{AI.30})$$

$$[\psi_{\alpha}(x), \psi_{\beta}^{\dagger}(x')]_{t=t'} = \delta^3(\underline{x} - \underline{x}') \delta_{\alpha\beta} \quad (\text{AI.31})$$

(c) Free massive vector field.

Field equation:

$$\left. \begin{aligned} (m^2 - \square) \phi^\mu(x) &= 0, \\ \partial_\mu \phi^\mu(x) &= 0 \end{aligned} \right\} ; \quad (\mu = 0, \dots, 3) \quad (\text{AI.32})$$

Plane wave solution,

$$\phi^\mu(x) = e^{ikx} \epsilon^\mu(k), \quad (\text{AI.33})$$

$$k_\mu \epsilon^\mu(k) = 0 \quad (\text{AI.34})$$

$$\sum_{\lambda=1}^3 \epsilon^\mu(k, \lambda) \epsilon_\nu(k, \lambda) = g^\mu{}_\nu + \frac{k^\mu k_\nu}{m^2} \quad (\text{AI.35})$$

Occasionally, this is loosely written as

$$\sum_{\tau=1}^3 \epsilon_\mu^{(\tau)}(k) \epsilon_\nu^{(\tau)}(k) = g_{\mu\nu} + \frac{k_\mu k_\nu}{m^2} \quad (\text{AI.36})$$

$$\epsilon^\mu(k, \lambda) \epsilon_\mu(k, \sigma) = \delta_{\lambda\sigma} \quad (\lambda, \sigma = 1, 2, 3). \quad (\text{AI.37})$$

(d) Field equation for free massive spin two particles.

Equation of motion:

$$\left. \begin{aligned}
 (m^2 - \square) \phi^{\mu\nu}(x) &= 0, \\
 \partial^\mu \phi_{\mu\nu}(x) &= 0, \\
 \phi_{\mu\nu}(x) &= \phi_{\nu\mu}(x), \\
 \phi_{\mu\nu} g^{\mu\nu} &= 0.
 \end{aligned} \right\} \quad (\text{AI.38})$$

Plane wave solution

$$\left. \begin{aligned}
 \phi^{\mu\nu}(x) &= \epsilon^{\mu\nu}(k) e^{ik \cdot x}, \\
 k_\mu \epsilon^{\mu\nu}(k) &= 0, \\
 \epsilon^{\mu\nu}(k) &= \epsilon^{\nu\mu}(k), \\
 \epsilon^{\mu\nu} g_{\mu\nu} &= 0.
 \end{aligned} \right\} \quad (\text{AI.39})$$

$\alpha, \beta = 1, \dots, 5$ are states of polarization (AI.40)

$$\epsilon^{\mu\nu}(k, \alpha) \epsilon_{\mu\nu}(k, \beta) = \delta_{\alpha\beta},$$

and

where
$$\sum_1^5 \epsilon^{\mu\nu}(k, \alpha) \epsilon_{\lambda\rho}(k, \alpha) = \theta^{\mu\nu}_{\lambda\rho},$$

$$\left. \begin{aligned}
 \theta^{\mu\nu}_{\lambda\rho} &= -\frac{1}{3} \theta^{\mu\nu} \theta_{\lambda\rho} + \frac{1}{2} \theta^\mu{}_\lambda \theta^\nu{}_\rho + \frac{1}{2} \theta^\mu{}_\rho \theta^\nu{}_\lambda, \\
 \theta^{\mu\nu} &= \left(g^{\mu\nu} - \frac{\partial^\mu \partial^\nu}{m^2} \right); \quad \theta^\mu{}_\nu = \left(g^\mu{}_\nu - \frac{\partial^\mu \partial^\nu}{m^2} \right).
 \end{aligned} \right\} \quad (\text{AI.41})$$

$e^{\mu\nu}_{\lambda\rho}$ satisfies the usual properties of projection operators,

$$\theta^{\mu\nu}_{\lambda\rho} \theta^{\lambda\rho}_{\sigma\chi} = \theta^{\mu\nu}_{\sigma\chi} \quad (\text{AI.42})$$

4. Normalization convention.

The states are normalized covariantly,

$$\langle p' | p \rangle = 2p^0 (2\pi)^3 \delta^3(p' - p) \quad (\text{AI.43})$$

Further,

$$\frac{\partial A}{\partial x_\mu} = -i [P^\mu, A(x)] \quad , \quad (\text{AI.44})$$

so that ,

$$e^{-iP \cdot a} A(x) e^{iP \cdot a} = A(x+a) \quad (\text{AI.45})$$

APPENDIX II

1. Notation for SU(2) Clebsch-Gordan Coefficients:

We use the Condon-Shortley phase convention.

2. Application to isotopic spin : Simple examples

$$[I_i, I_j] = i \epsilon_{ijk} I_k \quad ; \quad i, j, k = 1, 2, 3 \quad ; \quad (\text{AII.1})$$

$$I_{\pm} = I_1 \pm i I_2 \quad ; \quad (\text{AII.2})$$

$$[I_3, I_{\pm}] = \pm I_{\pm} \quad ; \quad (\text{AII.3})$$

$$[I^2, I_{\pm}] = 0 \quad ; \quad (\text{AII.4})$$

$$I_{\pm} |I, I_z\rangle = \sqrt{(I \mp I_z)(I \pm I_z + 1)} |I, I_z \pm 1\rangle \quad (\text{AII.5})$$

$$\frac{1 \otimes 1}{\sqrt{2}}$$

$$\langle \pi^i | \tau^j | \pi^k \rangle = i \epsilon_{ijk} \quad , \quad (\text{AII.6})$$

$$|\pi^+ \rangle = -\frac{1}{\sqrt{2}} |\pi^1 + i\pi^2 \rangle \quad , \quad (\text{AII.7})$$

$$|\pi^-\rangle = +\frac{1}{\sqrt{2}} |\pi^1 - i\pi^2\rangle \quad , \quad (\text{AII.8})$$

$$|\pi^0\rangle = |\pi^3\rangle \quad (\text{AII.9})$$

$$\begin{aligned}
 & |2,2\rangle = |\pi^+\rangle |\pi^+\rangle \\
 \langle \pi^+ | I^3 | \pi^+ \rangle &= 1 \quad ; \\
 \langle \pi^0 | I^3 | \pi^0 \rangle &= 0 \quad ; \\
 \langle \pi^- | I^3 | \pi^- \rangle &= -1 \quad ; \\
 \langle \pi^+ | I^+ | \pi^0 \rangle &= \sqrt{2} \quad ; \\
 \langle \pi^0 | I^+ | \pi^- \rangle &= \sqrt{2} \quad ; \\
 \langle \pi^0 | I^- | \pi^+ \rangle &= \sqrt{2} \quad ; \\
 \langle \pi^- | I^- | \pi^0 \rangle &= \sqrt{2} \quad ; \\
 & |2,1\rangle = \frac{1}{\sqrt{2}} \{ |\pi^0\rangle |\pi^+\rangle + |\pi^+\rangle |\pi^0\rangle \} \quad ; \\
 & |2,0\rangle = \frac{1}{\sqrt{6}} \{ |\pi^-\rangle |\pi^+\rangle + 2|\pi^0\rangle |\pi^0\rangle + |\pi^+\rangle |\pi^-\rangle \} \quad ; \\
 & |2,-1\rangle = \frac{1}{\sqrt{2}} \{ |\pi^-\rangle |\pi^0\rangle + |\pi^0\rangle |\pi^-\rangle \} \quad ; \\
 & |2,-2\rangle = |\pi^-\rangle |\pi^-\rangle \quad ; \\
 & |1,1\rangle = \frac{1}{\sqrt{2}} \{ |\pi^+\rangle |\pi^0\rangle - |\pi^0\rangle |\pi^+\rangle \} \quad ; \\
 & |1,0\rangle = \frac{1}{\sqrt{2}} \{ |\pi^+\rangle |\pi^-\rangle - |\pi^-\rangle |\pi^+\rangle \} \quad ; \\
 & |1,-1\rangle = \frac{1}{\sqrt{2}} \{ |\pi^0\rangle |\pi^-\rangle - |\pi^-\rangle |\pi^0\rangle \} \quad ; \\
 & |0,0\rangle = \frac{1}{\sqrt{3}} \{ |\pi^+\rangle |\pi^-\rangle - |\pi^0\rangle |\pi^0\rangle + |\pi^-\rangle |\pi^+\rangle \} .
 \end{aligned}
 \quad (\text{AII.10})$$

$\frac{1}{2} \otimes \frac{1}{2}$

$$\langle k^+ | I^3 | k^+ \rangle = +\frac{1}{2} ;$$

$$|1,1\rangle = |k^+\rangle |k^+\rangle ;$$

$$\langle k^0 | I^3 | k^0 \rangle = -\frac{1}{2} ;$$

$$|1,0\rangle = \frac{1}{\sqrt{2}} \{ |k^+\rangle |k^0\rangle + |k^0\rangle |k^+\rangle \} ; \quad (\text{AII.11})$$

$$\langle k^+ | I^+ | k^0 \rangle = 1 ;$$

$$|1,-1\rangle = |k^0\rangle |k^0\rangle ;$$

$$\langle k^0 | I^- | k^+ \rangle = 1 ;$$

$$|0,0\rangle = \frac{1}{\sqrt{2}} \{ |k^+\rangle |k^0\rangle - |k^0\rangle |k^+\rangle \} .$$

$1 \otimes \frac{1}{2}$

$$|3/2, 3/2\rangle = |\pi^+\rangle |k^+\rangle ;$$

$$|3/2, 1/2\rangle = \sqrt{\frac{2}{3}} |\pi^0\rangle |k^+\rangle + \sqrt{\frac{1}{3}} |\pi^+\rangle |k^0\rangle ;$$

$$|3/2, -1/2\rangle = \sqrt{\frac{1}{3}} |\pi^-\rangle |k^+\rangle + \sqrt{\frac{2}{3}} |\pi^0\rangle |k^0\rangle ;$$

$$|3/2, -3/2\rangle = |\pi^-\rangle |k^0\rangle ;$$

$$|1/2, 1/2\rangle = -\sqrt{\frac{1}{3}} |\pi^0\rangle |k^+\rangle + \sqrt{\frac{2}{3}} |\pi^+\rangle |k^0\rangle ;$$

$$|1/2, -1/2\rangle = \sqrt{\frac{1}{3}} |\pi^0\rangle |k^0\rangle - \sqrt{\frac{2}{3}} |\pi^-\rangle |k^+\rangle .$$

(AII.12)

3. Isospin crossing matrices:

$\pi\pi \rightarrow \pi\pi$

$$\begin{pmatrix} M_t^{(0)} \\ M_t^{(1)} \\ M_t^{(2)} \end{pmatrix} = \begin{pmatrix} 1/3 & 1 & 5/3 \\ 1/3 & 1/2 & -5/6 \\ 1/3 & -1/2 & 1/6 \end{pmatrix} \begin{pmatrix} M_s^{(0)} \\ M_s^{(1)} \\ M_s^{(2)} \end{pmatrix} \quad (\text{AII.13})$$

$M_t^{(0)}$, $M_t^{(1)}$, $M_t^{(2)}$ are t-channel isospin amplitudes, and $M_s^{(0)}$, $M_s^{(1)}$, $M_s^{(2)}$ are s-channel amplitudes.

$\pi N \rightarrow \pi N$

$$\begin{pmatrix} M_u^{(1/2)} \\ M_u^{(3/2)} \end{pmatrix} = \begin{pmatrix} -1/3 & 4/3 \\ 2/3 & 1/3 \end{pmatrix} \begin{pmatrix} M_s^{(1/2)} \\ M_s^{(3/2)} \end{pmatrix}, \quad (\text{AII.14})$$

and

$$\begin{pmatrix} M_t^{(0)} \\ M_t^{(1)} \end{pmatrix} = \begin{pmatrix} \sqrt{6}/3 & 2\sqrt{6}/3 \\ 2/3 & -2/3 \end{pmatrix} \begin{pmatrix} M_s^{(1/2)} \\ M_s^{(3/2)} \end{pmatrix} \quad (\text{AII.15})$$

4. P. C. T transformation properties of Dirac bilinears:

TABLE AIIA

	P	C	T
$\psi(x)$	$\beta \psi(-x, t)$	$\gamma^2 \tilde{\psi}^\dagger(x)$	$\gamma^3 \gamma^1 \tilde{\psi}^\dagger(x, -t)$
$\psi^\dagger(x)$	$\psi^\dagger(-x, t) \beta$	$\tilde{\psi} \gamma^2$	$\tilde{\psi} \gamma^1 \gamma^3$
$\bar{\psi}_a(x) \psi_b(x)$	$\bar{\psi}_a(-x, t) \psi_b(-x, t)$	$\bar{\psi}_a(x) \psi_a(x)$	$\bar{\psi}_a(x, -t) \psi_a(x, -t)$
$i \bar{\psi}_a(x) \gamma^\mu \psi_b(x)$	$\left\{ \begin{array}{l} -i \bar{\psi}_a(-x, t) \gamma^i \psi_b(-x, t) \\ i \bar{\psi}_a(-x, t) \gamma^0 \psi_b(-x, t) \end{array} \right\}$	$-i \bar{\psi}_a(x) \gamma^\mu \psi_a(x)$	$\left\{ \begin{array}{l} -i \bar{\psi}_a(x, -t) \gamma^i \psi_a(x, -t) \\ i \bar{\psi}_a(x, -t) \gamma^0 \psi_a(x, -t) \end{array} \right\}$
$\bar{\psi}_a(x) \sigma^{\mu\nu} \psi_b(x)$	$\left\{ \begin{array}{l} \bar{\psi}_a(-x, t) \sigma^{ij} \psi_b(-x, t) \\ -\bar{\psi}_a(-x, t) \sigma^{0i} \psi_b(-x, t) \end{array} \right\}$	$-\bar{\psi}_a(x) \sigma^{\mu\nu} \psi_a(x)$	$\left\{ \begin{array}{l} -\bar{\psi}_a(x, -t) \sigma^{ij} \psi_a(x, -t) \\ \bar{\psi}_a(x, -t) \sigma^{ij} \psi_a(x, -t) \end{array} \right\}$
$i \bar{\psi}_a(x) \gamma^5 \gamma^\mu \psi_b(x)$	$\left\{ \begin{array}{l} i \bar{\psi}_a(-x, t) \gamma^5 \gamma^i \psi_b(-x, t) \\ -i \bar{\psi}_a(-x, t) \gamma^5 \gamma^0 \psi_b(-x, t) \end{array} \right\}$	$i \bar{\psi}_a(x) \gamma^5 \gamma^\mu \psi_a(x)$	$\left\{ \begin{array}{l} -i \bar{\psi}_a(x, -t) \gamma^5 \gamma^i \psi_a(x, -t) \\ i \bar{\psi}_a(x, -t) \gamma^5 \gamma^0 \psi_a(x, -t) \end{array} \right\}$
$i \bar{\psi}_a(x) \gamma^5 \psi_b(x)$	$-i \bar{\psi}_a(-x) \gamma^5 \psi_b(-x)$	$i \bar{\psi}_a(x) \gamma^5 \psi_a(x)$	$-i \bar{\psi}_a(x, -t) \gamma^5 \psi_a(x, -t)$

APPENDIX III

1. Reduction Formulae.

(π - π) scattering:

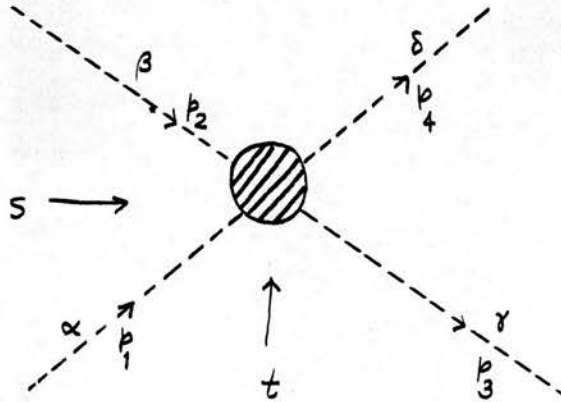


Fig. AIII

p_1, p_2, p_3, p_4 are four-momenta. $\alpha, \beta, \gamma, \delta = 1, 2, 3$ are charge indices.

$$s = - (p_1 + p_2)^2 = - (p_3 + p_4)^2 ;$$

$$t = - (p_1 - p_3)^2 = - (p_2 - p_4)^2 ;$$

$$u = - (p_1 - p_4)^2 = - (p_2 - p_3)^2 ;$$

$$s + t + u = - (p_1^2 + p_2^2 + p_3^2 + p_4^2) = 4m_\pi^2 .$$

(AIII.1)

S-matrix:

$$S^{\delta\gamma; \beta\alpha} = \left\langle \pi^{\delta}(p_4), \pi^{\gamma}(p_3) \mid \pi^{\beta}(p_2), \pi^{\alpha}(p_1) \right\rangle_{IN, OUT} \quad (\text{AIII.2})$$

$$= I^{\delta\gamma; \beta\alpha} + \left\{ \begin{matrix} -1 \\ -i \end{matrix} \right\} \int d^4x d^4y e^{-ip_4x + ip_2y} (m^2 - \square_x)(m^2 - \square_y) \left\langle \pi^{\gamma}(p_3) \mid \left\{ R^T \right\} \right. \\ \left. (\phi^{\delta}(x) \phi^{\beta}(y)) \mid \pi^{\alpha}(p_1) \right\rangle, \quad (\text{AIII.3})$$

$$= I^{\delta\gamma; \beta\alpha} + \int d^4x d^4y d^4z d^4v e^{-ip_4x - ip_3v + ip_2y + ip_1z} \\ (m^2 - \square_x)(m^2 - \square_y)(m^2 - \square_v)(m^2 - \square_z) \left\langle 0 \mid T(\phi^{\gamma}(v) \phi^{\delta}(x) \phi^{\beta}(y) \phi^{\alpha}(z)) \mid 0 \right\rangle; \quad (\text{AIII.4})$$

where :

$$T(\phi^{\gamma}(v) \phi^{\delta}(x) \phi^{\beta}(y) \phi^{\alpha}(z)) \\ = \phi^{\gamma}(v) \phi^{\delta}(x) \phi^{\beta}(y) \phi^{\alpha}(z), \quad \text{if } v^0 > x^0 > y^0 > z^0, \quad (\text{AIII.5})$$

$$= \phi^{\delta}(x) \phi^{\gamma}(v) \phi^{\beta}(y) \phi^{\alpha}(z), \quad \text{if } x^0 > v^0 > y^0 > z^0, \quad (\text{AIII.6})$$

and so on;

$$R(\phi^{\delta}(x) \phi^{\beta}(y)) = -i\theta(x^0 - y^0) [\phi^{\delta}(x), \phi^{\beta}(y)] \quad (\text{AIII.7})$$

2. Lehmann spectral representation for two point functions.

For scalar interpolating fields, the two point function

$$\Delta'_F(x) \equiv \langle 0 | T \{ \phi(x) \phi(0) \} | 0 \rangle, \quad (\text{AIII.8})$$

satisfies the spectral representation

$$\Delta'_F(x) = \int_0^\infty dM^2 \{ \delta(M^2 - m^2) + \sigma(M^2) \} \Delta_F(x|M^2), \quad (\text{AIII.9})$$

where Δ_F is the free-field propagator and $\sigma(M^2) \geq 0$.

In momentum space,

$$\Delta'_F(p) = \frac{1}{p^2 + m^2} + \int_{4m^2}^\infty \frac{\sigma(M^2) dM^2}{p^2 + M^2 - i\epsilon}. \quad (\text{AIII.10})$$

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