The Joint Approximate Point Spectrum of Elements of C^*- Algebras

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Abstract

Let A be a normal operator on a complex Hilbert space, and let $\sigma(A)$ and $\sigma_{ap}(A)$ denote the spectrum and the approximate point spectrum of A respectively. Then, $\sigma(A) = \sigma_{ap}(A)$, and the Gelfand- Naimark Theorem proves that there exists a 1-1 correspondence between the set of characters on $C^*(A)$ and $\sigma(A)$ $(= \sigma_{ap}(A))$. The approximate point spectrum turns out to be the relevant part of the spectrum in the study of characters on the C^*- algebra generated by an arbitrary operator a : If φ is a character on $C^*(a)$ then $\varphi(a) \in \sigma_{ap}(a)$ [9]. Chapter I of the present thesis is devoted to the definition and study of the approximate point spectrum of arbitrary elements of C^*- algebras.

For the study of characters on C^*- algebras generated by more than one operator, the appropriate generalization of the approximate point spectrum turns out to be the concept of the joint approximate point spectrum. In chapter II we study the latter concept.

In chapter III the results of the two previous chapters are used to prove a rational functional calculus for the joint approximate point spectrum of a commuting family of operators.

Chapter IV is concerned mainly with the applications of the methods developed in the previous chapters to the theory of characters and finite operators.

Finally, applications to various topics (such as Rosenblum's theorem) are scattered throughout the present work.

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General Theory of the Approximate Point Spectrum

§1. Definition and elementary properties of the approximate point spectrum.

1.1. Introduction- In this section we recall certain results and establish certain others which will be repeatedly used in our later work. In order to avoid unnecessary repetition we state here that, unless otherwise stated, every C^*-algebra under consideration in this work will be assumed to contain an identity . In particular, any C^*-subalgebra of a given unital C^*-algebra will be assumed to contain the identity element of the given C^*-algebra.

If $\mathcal{A}$ is a (unital) C^*-algebra, and $(a_\lambda)_{\lambda\in\Lambda}$ is a family of elements of $\mathcal{A}$, the (unital) C^*-subalgebra of $\mathcal{A}$ generated by the family $(a_\lambda)_{\lambda\in\Lambda}$ will be denoted by $C^*((a_\lambda)_{\lambda\in\Lambda})$; when Λ is finite, say $\Lambda=\{1,\cdots,n\}$, we shall also use the notation $C^*(a_1,\cdots,a_n)$.

The state space of a C^*-algebra $\mathcal{A}$ will be denoted by $E(\mathcal{A})$; it is a w^*-compact convex subset of the dual space of $\mathcal{A}$. The set of extreme points of $E(\mathcal{A})$ (i.e. the set of pure states of $\mathcal{A}$) is denoted by $\mathcal{P}(\mathcal{A})$. Since by our convention $\mathcal{A}$ is implicitly assumed to be unital, the Krein-Millman theorem shows that every state of $\mathcal{A}$ is a w^*-limit of finite convex combinations of pure states of $\mathcal{A}$.

If ρ is a pure state of $\mathcal{A}$, the set

$$\mathcal{M}=\{y\in\mathcal{A} : \rho(y^*y)=0\}$$

is a maximal left ideal of $\mathcal{A}$, and conversely, every maximal left ideal of $\mathcal{A}$ is of this form for some pure state of $\mathcal{A}$ [22; Th. 2.9.5.]. We note incidentally that by the Cauchy-Schwartz inequality for positive linear functionals, we also have

$$\mathcal{M}=\{y\in\mathcal{A} : \rho(xy)=0\ (\forall x\in\mathcal{A})\}.$$

1.2. The Approximate Point Spectrum- Let A be a bounded linear operator on a complex Hilbert space $\mathcal{H}$. The approximate point spectrum of A, which we shall denote by $\sigma_{ap}(A)$, is defined to be the set of complex numbers λ with the following property:

$$\forall \epsilon > 0, \ \exists x \text{ s.t. } x \in \mathcal{H}, \ x \neq 0, \ \& \ \| Ax - \lambda x \| \leq \epsilon \|x\|.$$

Concerning the approximate point spectrum, the following proposition is proved in [9]:

1.2.1. Proposition- For $A \in \mathcal{BL}(\mathcal{H})$, we have

$$\sigma_{ap}(A) = \{ \lambda \in \sigma(A) : C^*(A)(A - \lambda I) \neq C^*(A) \}$$
$$= \{ \lambda \in \sigma(A) : \mathcal{BL}(\mathcal{H})(A - \lambda I) \neq \mathcal{BL}(\mathcal{H}) \},$$

where $\sigma(A)$ denotes the spectrum of A (which is the same relative to $\mathcal{BL}(\mathcal{H})$ and $C^*(A)$).

We remark that the condition $\lambda \in \sigma(A)$ is superfluous, since the condition $C^*(A)(A - \lambda I) \neq C^*(A)$ automatically implies that $\lambda \in \sigma(A)$.

Using proposition 1.2.1., we may now define the approximate point spectrum of elements of arbitrary C^*-algebras as follows:

1.2.1.1. Definition- Let $\mathcal{A}$ be a (unital) C^*-algebra, and let $a \in \mathcal{A}$. The approximate point spectrum of a is the set

$$\{ \lambda \in \mathbb{C} : \mathcal{A}(a - \lambda 1) \neq \mathcal{A} \}.$$

We then have the following analogue of proposition 1.2.1. :

1.2.2. Proposition- Let $\mathcal{A}$ be a C^*-algebra, and let a be an element of $\mathcal{A}$. Then

$$\sigma_{ap}(a) = \{ \lambda \in \mathbb{C} : C^*(a)(a - \lambda 1) \neq C^*(a) \}$$
$$(= \{ \lambda \in \sigma(a) : C^*(a)(a - \lambda 1) \neq C^*(a) \}).$$

Remark- Note that since by our general convention, $C^*(a)$ is assumed to contain the identity element of $\mathcal{A}$, the symbol $\sigma(a)$ may be used to denote the spectrum of a relative to $C^*(a)$ or $\mathcal{A}$ without ambiguity.

Proof- Let $\lambda \in \sigma_{op}(a)$ and suppose on the contrary, that $C^*(a)(a-\lambda 1) = C^*(a)$. Then $1 = b(a-\lambda 1)$ for some $b \in C^*(a)$, so that $1 \in \mathcal{A}(a-\lambda 1)$; but $\mathcal{A}(a-\lambda 1)$ is clearly a left ideal of $\mathcal{A}$, hence $\mathcal{A}(a-\lambda 1) = \mathcal{A}$ which is a contradiction. Thus

$$\sigma_{ap}(a) \subseteq \{\lambda \in \underset{\sim}{\mathbb{C}} : C^*(a)(a-\lambda 1) \neq C^*(a)\}. \qquad (1)$$

Conversely, let $\lambda \in$ R.H.S.(1) and suppose on the contrary that $\mathcal{A}(a-\lambda 1) = \mathcal{A}$. Since $C^*(a)(a-\lambda 1)$ is a proper left ideal in $C^*(a)$, there exists a maximal left ideal $\mathcal{M}$ of $C^*(a)$ such that $\mathcal{M} \supseteq C^*(a)(a-\lambda 1)$; then, there exists a pure state ρ of $C^*(a)$ such that

$$\mathcal{M} = \{x \in C^*(a) : \rho(x^*x) = 0\}.$$

Extend ρ to a pure state $\tilde{\rho}$ of $\mathcal{A}$. Since $\mathcal{A}(a-\lambda 1) = \mathcal{A}$ there exists an element b of $\mathcal{A}$ such that $1 = b(a-\lambda 1)$. Then, applying the Cauchy-Schwartz inequality to $\tilde{\rho}$ we have

$$\begin{aligned} 1 = |\tilde{\rho}(1)|^2 &= |\tilde{\rho}(b(a-\lambda))|^2 \\ &\leq \tilde{\rho}(bb^*)\,\tilde{\rho}((a-\lambda)^*(a-\lambda)) \\ &= \tilde{\rho}(bb^*)\,\rho((a-\lambda)^*(a-\lambda)) \\ &= 0 \end{aligned}$$

since $a-\lambda \in \mathcal{M}$.

The contradiction shows that

$$\sigma_{ap}(a) \supseteq \{\lambda \in \underset{\sim}{\mathbb{C}} : C^*(a)(a-\lambda 1) \neq C^*(a)\}. \quad \blacktriangle$$

1.2.2.1. Remarks- (a) Suppose that $\mathcal{B}$ is any C^*-algebra containing $C^*(a)$ (we assume as usual that the identity element of $C^*(a)$ is the same as that of $\mathcal{B}$). Then essentially the same argument as that used above shows that

$$\sigma_{ap}(a) = \{\lambda \in \underset{\sim}{\mathbb{C}} : \mathcal{B}(a-\lambda 1) \neq \mathcal{B}\};$$

in other words, the approximate point spectrum of an operator is independent of the C^*-algebra containing it.

(b)- Later on [Theorem 2.2.] we shall give other equivalent characterizations of the approximate point spectrum which will show, amongst other things, that when $\mathcal{A} = \mathcal{BL}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, the present definition coïncides with the usual definition of the approximate point spectrum as was given in 1.2. •

1.2.2.2. Corollary- Let $\mathcal{A}$ be a C^*-algebra and let $a \in \mathcal{A}$. Then, $\lambda \in \sigma_{ap}(a)$ if and only if there exists a pure state ρ of $C^*(a)$ such that $\Pi_\rho(a)\xi_\rho = \lambda \xi_\rho$ where Π_ρ and ξ_ρ are the associated irreducible representation and cyclic vector, respectively.

Proof- Necessity: If $\lambda \in \sigma_{ap}(a)$, then $C^*(a)(a-\lambda 1) \neq C^*(a)$, so the left ideal $C^*(a)(a-\lambda 1)$ is contained in a maximal left ideal $\mathcal{M}$ of $C^*(a)$ given by

$$\mathcal{M} = \{ \gamma \in C^*(a) \; : \; \rho(\gamma^*\gamma) = 0 \}$$

where $\rho \in \mathcal{P}(C^*(a))$. Hence

$$\begin{aligned} \| \Pi_\rho(a)\xi_\rho - \lambda \xi_\rho \|^2 &= \langle \Pi_\rho(a-\lambda)\xi_\rho \, , \, \Pi_\rho(a-\lambda)\xi_\rho \rangle \\ &= \langle \Pi_\rho((a-\lambda)^*(a-\lambda))\xi_\rho, \xi_\rho \rangle \\ &= \rho((a-\lambda)^*(a-\lambda)) \\ &= 0 \end{aligned}$$

since $a-\lambda \in \mathcal{M}$; hence $\Pi_\rho(a)\xi_\rho = \lambda \xi_\rho$.

Sufficiency: If $\lambda \notin \sigma_{ap}(a)$, there exists $b \in C^*(a)$ such that $b(a-\lambda 1) = 1$; if now Π is any irreducible representation of $C^*(a)$, we have

$$1 = \Pi(1) = \Pi(b)\,(\Pi(a) - \lambda 1)$$

hence $\lambda \notin \sigma_{ap}(\Pi(a))$. In particular, $\Pi(a)\xi \neq \lambda \xi$ for any non-zero ξ.

For Hilbert space operators the necessity part of the above corollary may be found in [9].

Later on [c.f. §3] we shall give a more general treatment of the behaviour of $\sigma_{ap}(a)$ under representations.

We close this section with the following proposition which relates the approximate point spectrum to the theory of characters.

1.2.3. Proposition- Let $\mathcal{A}$ be a C^*-algebra, and let $a \in \mathcal{A}$. If φ is a character on $C^*(a)$, then $\varphi(a) \in \sigma_{ap}(a)$.

Proof- The proof in [9, proposition 8] may be adapted to the more general present case, using proposition 1.2.2. . We omit the details.

A more general result will be proved in chapter II, proposition 1.7.

Let now $\mathcal{A}$ be a C^*-algebra, and let a be an arbitrary element of $\mathcal{A}$. The above proposition suggests that in studying all the characters on $C^*(a)$, we should begin by choosing a point λ in $\sigma_{ap}(a)$ and determine whether there exists a character φ on $C^*(a)$ such that $\varphi(a)=\lambda$. In certain special cases, corollary 1.2.2.2. may be used to reduce the original choice of λ to a point in the point spectrum of a (i.e. the set of eigenvalues of a).

To be more precise, suppose a is an operator such that for every irreducible representation Π of $C^*(a)$ (or of $\mathcal{A}$) , $\Pi(a)$ is an operator of the same class; examples of such operators include the class of all hyponormal operators ($a^*a \geq aa^*$), the class of all paranormal operators ($a^{*2}a^2 - 2\mu a^*a + \mu^2 \geq 0$ for $\forall \mu \geq 0$), etc. . To fix the ideas, let a be hyponormal. For any irreducible representation Π of $C^*(a)$ on a Hilbert space $\mathcal{H}$, we have

$$\Pi(a)^* \Pi(a) - \Pi(a) \Pi(a)^* = \Pi(a^*a - aa^*) \geq 0$$

since $a^*a \geq aa^*$ and Π is order preserving.

Hence $\Pi(a)$ is a hyponormal operator on $\mathcal{H}$.

Let now $\lambda \in \sigma_{ap}(a)$; by corollary 1.2.2.2., there exists a pure state ρ of $C^*(a)$,with the corresponding irreducible representation Π_ρ and cyclic vector ξ_ρ such that

$$\Pi_\rho(a)\xi_\rho = \lambda \xi_\rho ,$$

so that λ is an eigenvalue of the hyponormal operator $\Pi_\rho(a)$.

Suppose now that there exists a character Ψ on $C^*(\pi_\rho(a))$ such that

$$\Psi(\pi_\rho(a)) = \lambda \ ;$$

then, defining the positive linear functional φ on $C^*(a)$ by

$$\varphi(b) = \Psi(\pi_\rho(b)) \qquad (\forall b \in C^*(a))$$

we have

$$\varphi(1) = \Psi(\pi_\rho(1)) = \Psi(1) = 1$$

$$\varphi(a^*a) = \Psi(\pi_\rho(a^*a)) = \Psi(\pi_\rho(a)^*\pi_\rho(a)) = |\lambda|^2 \qquad \text{(i)}$$

$$\varphi(aa^*) = \Psi(\pi_\rho(aa^*)) = \Psi(\pi_\rho(a)\pi_\rho(a)^*) = |\lambda|^2 \qquad \text{(ii)}$$

$$|\varphi(a)|^2 = |\Psi(\pi_\rho(a))|^2 = |\lambda|^2. \qquad \text{(iii)}$$

Hence, for each $b \in C^*(a)$, we have

$$|\varphi(ab) - \varphi(a)\varphi(b)|^2 = |\varphi((a-\lambda 1)b)|^2$$

$$\leq \underbrace{\varphi((a-\lambda)(a-\lambda)^*)}_{=0}\varphi(b^*b)$$

by (i), (ii), and (iii) above. Hence,

$$\varphi(ab) = \varphi(a)\varphi(b) \qquad (\forall b \in C^*(a))$$

Similarly, we have

$$\varphi(a^*b) = \varphi(a^*)\varphi(b) \qquad (\forall b \in C^*(a))$$

Thus, φ is a character on $C^*(a)$.

Thus, we have shown that, if every point in the point spectrum of an arbitrary hyponormal operator gives rise to a character, then the same is true of every point in the approximate point spectrum of an arbitrary hyponormal operator.

The above method, as well as alternative methods, will be used in chapter IV in the study of characters on C^*-algebras generated by certain classes of operators.

§2. Equivalent Characterizations of the Approximate Point Spectrum

The purpose of this section is to obtain certain characterizations of the approximate point spectrum, and to show, as a result, that the two definitions given in 1.2. and 1.2.1.1. coïncide in the case of bounded linear operators on a Hilbert space.

We shall begin by recalling certain definitions.

Let $\mathcal{K}$ be a topological vector space, and let $\mathcal{L}$ be a non-empty subset of $\mathcal{K}$.

We say that $\mathcal{L}$ is absolutely convex if whenever $\{l_1, \ldots, l_n\}$ is a finite set of elements in $\mathcal{L}$,and $\{\lambda_1, \ldots, \lambda_n\}$ is a finite set of complex numbers with $\sum_{j=1}^{n} |\lambda_j| \leq 1$, we have $\sum_{j=1}^{n} \lambda_j l_j \in \mathcal{L}$.

The linear subspace of $\mathcal{K}$ spanned by $\mathcal{L}$ will be denoted by $\mathrm{Span}(\mathcal{L})$; the closed linear subspace of $\mathcal{K}$ spanned by $\mathcal{L}$ is denoted by $[\mathcal{L}]$.

Let $\mathcal{A}$ be a C^*-algebra, and let $a \in \mathcal{A}$. A bounded linear functional f on $\mathcal{A}$ is said to be left-multiplicative with respect to a if the following condition is satiafied:

$$f(xa) = f(x)f(a) \qquad (\forall x \in \mathcal{A})$$

Similarly, f is said to be right-multiplicative with respect to a if the following condition is satisfied:

$$f(ax) = f(x)f(a) \qquad (\forall x \in \mathcal{A})$$

Similar definitions apply if f is a bounded linear functional on $C^*(a)$.

2.1. Lemma- Let $\mathcal{L}$ be an absolutely convex subset of a Hilbert space $\mathcal{K}$ such that $\mathcal{K} = [\mathcal{L}]$,and let $A \in \mathcal{BL}(\mathcal{K})$. Then, A is positive if and only if

$$\langle Al, l \rangle \geqslant 0 \qquad (\forall l \in \mathcal{L}) \qquad \text{(i)}$$

Proof- Necessity is obvious.

Sufficiency: Suppose (i) above is satisfied, and let $\sum_{j=1}^{n} \lambda_j l_j$ be an arbitrary finite linear combination of elements of $\mathcal{L}$ where

$$\lambda_j \in \underset{\sim}{\mathbb{C}}, \quad l_j \in \mathcal{L} \qquad (j = 1, 2, \cdots, n).$$

Assuming, without real loss of generality, that $\sum_{j=1}^{n} |\lambda_j| > 0$, we have

$$\sum_{j=1}^{n} \lambda_j l_j = \sum_{j=1}^{n} |\lambda_j| \left(\sum_{k=1}^{n} \frac{\lambda_k}{\sum_{j=1}^{n} |\lambda_j|} l_k \right)$$

$$= \sum_{j=1}^{n} |\lambda_j| \left(\sum_{k=1}^{n} \mu_k l_k \right),$$

where

$$\mu_k = \frac{\lambda_k}{\sum_{j=1}^{n} |\lambda_j|} \qquad (k = 1, 2, \cdots, n).$$

Since $\sum_{k=1}^{n} |\mu_k| = 1$ and $\mathcal{L}$ is absolutely convex, it follows that $\sum_{k=1}^{n} \mu_k l_k \in \mathcal{L}$; let $l = \sum_{k=1}^{n} \mu_k l_k$.

Then

$$\left\langle A\left(\sum_{j=1}^{n} \lambda_j l_j \right), \sum_{j=1}^{n} \lambda_j l_j \right\rangle =$$
$$\left\langle A\left(\sum_{j=1}^{n} |\lambda_j| l \right), \sum_{j=1}^{n} |\lambda_j| l \right\rangle =$$
$$\left(\sum_{j=1}^{n} |\lambda_j| \right)^2 \langle Al, l \rangle \geqslant$$
$$0,$$

by (i). Hence

$$\langle Al, l \rangle \geqslant 0 \qquad (\forall l \in \mathrm{Span}(\mathcal{L})) \qquad (ii).$$

Finally, let x be an arbitrary element of $\mathcal{K}$. There exists a sequence l_n of elements of $\mathrm{Span}(\mathcal{L})$ such that $x = \lim_{n\to\infty} l_n$.

Hence

$$\langle Ax, x \rangle = \lim_{n\to\infty} \langle Al_n, l_n \rangle \geqslant 0,$$

Therefore A is positive.

2.1.1. Corollary- With the notations of lemma 2.1., let λ be a complex number such that

$$\mathcal{BL}(\mathcal{H})(A-\lambda I) \neq \mathcal{BL}(\mathcal{H}) \qquad \text{(i)}$$

(i.e., $\lambda \in \sigma_{ap}(A)$ in the sense of definition1.2.1.1.).

Then, given a positive number ϵ there exists a non zero element l of $\mathcal{L}$ such that

$$\|(A-\lambda I)l\|^2 \leq \epsilon \|l\|^2 \qquad \text{(ii)}$$

Proof- Let λ satisfy (i), and suppose, on the contrary that (ii) does not hold. Then there exists a positive number ϵ such that

$$\|(A-\lambda I)l\|^2 \geqslant \epsilon \|l\|^2 \qquad (\forall l \in \mathcal{L})$$

i.e.,

$$\langle ((A^*-\bar{\lambda}I)(A-\lambda I)-\epsilon I)l, l \rangle \geqslant 0 \qquad (\forall l \in \mathcal{L}).$$

By lemma 2.1., this implies that

$$(A^*-\bar{\lambda}I)(A-\lambda I) \geqslant \epsilon I,$$

so that the positive operator $(A^*-\bar{\lambda})(A-\lambda)$ is bounded below, hence is invertible; let B be the inverse. Then

$$\mathcal{BL}(\mathcal{H})(A-\lambda I) \supseteq \mathcal{BL}(\mathcal{H})(B(A^*-\bar{\lambda}I)(A-\lambda I)) = \mathcal{BL}(\mathcal{H}),$$

contradicting (i).

This completes the proof.

2.2. Theorem- Let $\mathcal{A}$ be a C^*-algebra, let $a \in \mathcal{A}$, and let $\lambda \in \mathbb{C}$.

(a)- The following conditions are equivalent:

(i) $\mathcal{A}(a-\lambda 1) \neq \mathcal{A}$ (i.e., $\lambda \in \sigma_{ap}(a)$ in the sense of definition 1.2.1.1).

(ii) There exists a pure state ρ of $\mathcal{A}$ such that

$$\rho(a) = \lambda \ \ \& \ \ \rho(a^*a) = |\lambda|^2.$$

(iii) There exists a state f of $\mathcal{A}$ such that

$$f(a) = \lambda \ \ \& \ \ f(a^*a) = |\lambda|^2.$$

(iv) There exists a state f of $\mathcal{A}$ such that f is left multiplicative with respect to a with $f(a)=\lambda$.

(v) There exists a pure state ρ of $\mathcal{A}$ such that ρ is left multiplicative with respect to a with $\rho(a)=\lambda$.

(b) - Let A be a bounded linear operator on a Hilbert space $\mathcal{H}$, let $\mathcal{L}$ be an absolutely convex subset of $\mathcal{H}$ with $\mathcal{H}=[\mathcal{L}]$, and let $\lambda \in \mathbb{C}$.

The following conditions are equivalent:

(i) $\lambda \in \sigma_{ap}(A)$ in the sense of i.2., i.e.,

$$\forall \epsilon > 0 \quad \exists x \quad s.t. \quad x \in \mathcal{H}, \ x \neq 0, \ \& \quad \|(A-\lambda I)x\| \leq \epsilon \|x\|$$

(ii)

$$\forall \epsilon > 0 \quad \exists y \quad s.t. \quad y \in \mathcal{L}, \ y \neq 0, \ \& \quad \|(A-\lambda I)y\| \leq \epsilon \|y\|$$

(c)- With the above notations, suppose $\mathcal{A}=\mathcal{BL}(\mathcal{H})$, and $a=A$. Then, the conditions a(i)- a(v), b(i), and b(ii) are all equivalent.

We defer the proof of the above theorem to chapter II, where a more general result will be proved; c.f. theorem 1.4.

2.2.1. Corollary- Let $\mathcal{A}$ be a C^*-algebra, and let $a \in \mathcal{A}$. Then, the approximate point spectrum of a is a compact subset of $\mathbb{C}$.

Proof- It is well known that $\sigma_{ap}(a) \neq \emptyset$ [27; problem 63].

In order to show that $\sigma_{ap}(a)$ is closed, let λ_n be a sequence of elements of $\sigma_{ap}(a)$, and suppose that $\lambda_n \to \lambda$; by 2.2.a(iv), there exists a corresponding sequence $f_n \in E(\mathcal{A})$ such that

$$f_n(xa) = \lambda_n f_n(x) \qquad (\forall x \in \mathcal{A}, \ \forall n);$$

by the w^*-compactness of $E(\mathcal{A})$ there exists a subnet f_m of f_n and a state f of $\mathcal{A}$ such that

$$f_m \longrightarrow f \qquad (w^*\text{- limit}).$$

It is then easily verified that

$$f(xa) = f(x)f(a) = \lambda f(x) \qquad (\forall x \in \mathcal{A})$$

hence, $\lambda \in \sigma_{ap}(a)$ by theorem 2.2. .

We remark that for bounded linear operators on a Hilbert space, an elementary proof of the above corollary may be given; c.f. [27, problem 62].

The question now arises as to whether the existence of a self-adjoint linear functional f which is left multiplicative with respect to a with $f(a)=\lambda$ implies the existence of a state g with the same property. In some special cases, the answer is in the affirmative. For example, if a is an isometry, and if a self-adjoint linear functional f is left-multiplicative with respect to a then, an easy argument shows that $f|_{C^*(a)}$ is a character on $C^*(a)$. However, we are unable to provide an answer in the general case.

We close this section with the following remarks concerning theorem 2.2. .

(i)- Throughout the statement of theorem 2.2., the triple $(\mathcal{A}, E(\mathcal{A}), \mathcal{P}(\mathcal{A}))$ may be replaced by the triple $(C^*(a), E(C^*(a)), \mathcal{P}(C^*(a)))$ without changing any of the conclusions; further, each condition in the resulting theorem will be equivalent to the corresponding condition in theorem 2.2.; e.g., the existence of an $f \in E(C^*(a))$ with

$$f(xa) = f(x)f(a) = \lambda f(x) \qquad (\forall x \in C^*(a))$$

is equivalent to the existence of a $g \in E(\mathcal{A})$ with

$$g(xa) = g(x)g(a) = \lambda g(x) \qquad (\forall x \in \mathcal{A});$$

c.f., the proof of 1.2.2. .

(ii)- Let $\lambda \in \sigma_{ap}(a)$; by theorem 2.2., there exists $f \in E(\mathcal{A})$ such that

$$f(a) = \lambda \quad \& \quad f(a^*a) = |\lambda|^2.$$

Since

$$E(\mathcal{A}) = \overline{Co(\mathcal{P}(\mathcal{A}))}^{\,w^*\text{- closure}}$$

there exists a net f_α in $E(\mathcal{A})$ such that each f_α is a convex combination of pure states:

$$f_\alpha = \sum_{j=1}^{n} \lambda_j f_{\alpha_j} \qquad \left(f_{\alpha_j} \in \mathcal{P}(\mathcal{A}),\ \lambda_j \geq 0,\ \sum_{j=1}^{n} \lambda_j = 1 \right),$$

and such that

$$f_\alpha \to f \qquad (w^*\text{-limit})$$

It is natural to ask whether the pure states f_{α_j} may be chosen so as to satisfy (i) above. This question will be answered in §5..

§3. Representations-

In this section we examine the behaviour of the approximate point spectrum under representations of the underlying C^*-algebra. In particular, we obtain an extension of corollary 1.2.2.2. .

3.1. Proposition- Let $\mathcal{A}$ be a C^*-algebra, and let $a \in \mathcal{A}$, and $\lambda \in \underset{\sim}{\mathbb{C}}$. Suppose that there exists a non degenerate representation Π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ such that $\lambda \in \sigma_{ap}(\Pi(a))$.

Then $\lambda \in \sigma_{ap}(a)$.

Proof- Let $\lambda \in \sigma_{ap}(\Pi(a))$. By 2.2.a(iv), there exists a state f of $\mathcal{BL}(\mathcal{H})$ such that

$$f(\Pi(a)) = \lambda \quad ,$$

and

$$f(A\,\Pi(a)) = \lambda f(A) \qquad (\forall A \in \mathcal{BL}(\mathcal{H})).$$

Define a linear functional g on $\mathcal{A}$ by

$$g(x) = (f \circ \Pi)(x) \qquad (\forall x \in \mathcal{A}).$$

Since f is positive and Π is a $*$-homomorphism, it follows that g is positive. Further since Π is non degenerate, we have $\Pi(1) = I$, so that

$$g(1) = f(\Pi(1)) = f(1) = 1 \quad ,$$

hence $g \in E(\mathcal{A})$.

Now, for each x in $\mathcal{A}$, we have

$$\begin{aligned} g(xa) &= f(\Pi(xa)) \\ &= f(\Pi(x)\Pi(a)) \\ &= \lambda f(\Pi(x)) \\ &= \lambda g(x), \end{aligned}$$

and

$$g(a) = f(\Pi(a)) = \lambda;$$

hence by 2.2.a(iv), we have $\lambda \in \sigma_{ap}(a)$.

3.1.1. Remark- With the above notations, suppose f is also right-multiplicative with respect to $\Pi(a)$. Then, g is also right-multiplicative with respect to a.

The following example shows that if $\lambda \in \sigma_{ap}(a)$, then it is not necessarily the case that $\lambda \in \sigma_{ap}(\Pi(a))$ for any arbitrary representation of $\mathcal{A}$.

Let $\mathcal{H}$ be an infinite dimentional Hilbert space, and let $\mathcal{KL}(\mathcal{H})$ be the two-sided ideal of all compact operators on $\mathcal{H}$. Let $\mathcal{C}alk(\mathcal{H})$ denote the corresponding Calkin algebra:

$$\mathcal{C}alk(\mathcal{H}) = \mathcal{BL}(\mathcal{H}) / \mathcal{KL}(\mathcal{H}) .$$

Finally, let Π be the natural homomorphism of $\mathcal{BL}(\mathcal{H})$ onto $\mathcal{C}alk(\mathcal{H})$ defined by

$$\Pi(A) = A + \mathcal{KL}(\mathcal{H}) \qquad (A \in \mathcal{BL}(\mathcal{H})).$$

Then, for any compact operator A in $\mathcal{BL}(\mathcal{H})$ we have $\Pi(A) = 0$, so that $\sigma_{ap}(\Pi(A)) = \{0\}$.

Thus, it is sufficient to take any compact operator with non zero approximate point spectrum to obtain the desired example.

On the other hand, no such phenomenon can occur if Π injective.

We need the following lemma:

3.2. Lemma- Let $a \in \mathcal{A}$, and let Π be a non degenerate representation of $\mathcal{A}$. then,

$$C^*(\Pi(a)) = \Pi(C^*(a)).$$

Proof- For each polynomial $p(a, a^*)$ in a and a^* we have

$$\Pi(p(a,a^*)) = p(\Pi(a), \Pi(a)^*),$$

hence, by continuity,

$$\Pi(C^*(a)) \subseteq C^*(\Pi(a)).$$

On the other hand, $\Pi(C^*(a))$ is a C^*-algebra [22; Ch. I, Cor. 1.8.3.] and contains $\Pi(a)$, hence

$$\Pi(C^*(a)) \supseteq C^*(\Pi(a)).$$

This completes the proof.

3.3. Proposition- Let $\mathcal{A}$ be a C^*-algebra, and let $a \in \mathcal{A}$, and $\lambda \in \sigma_{ap}(a)$. Suppose Π is a non degenerate representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ such that

$$C^*(a) \cap \ker \Pi = \{0\}.$$

Then, $\lambda \in \sigma_{ap}(\Pi(a))$.

Proof- Suppose not; then, there exists an operator B in $C^*(\Pi(a))$ such that

$$B(\Pi(a) - \lambda 1) = I \quad ;$$

by lemma 3.2., $B = \Pi(b)$ for some $b \in C^*(a)$. Hence

$$\Pi(b)(\Pi(a) - \lambda 1) = I,$$

i.e.,

$$\Pi(b(a-\lambda 1)) = I.$$

Since $b(a-\lambda 1) \in C^*(a)$, this implies that

$$b(a-\lambda 1) = 1$$

contradicting $\lambda \in \sigma_{ap}(a)$.

This completes the proof.

3.3.1. Corollary- With the notations of proposition 3.3., suppose π is a faithful representation of $\mathcal{A}$. Then $\lambda \in \sigma_{ap}(\pi(a))$.

Proof- Since π is faithful, the condition

$$C^*(a) \cap \ker \pi = \{0\}$$

is trivially satisfied.

3.3.2. Corollary- Let $\mathcal{A}$ be a simple C*-algebra, let $a \in \mathcal{A}$, and $\lambda \in \mathbb{C}$. Suppose that π is a non degenerate representation of $\mathcal{A}$. Then

$$\lambda \in \sigma_{ap}(a) \Longleftrightarrow \lambda \in \sigma_{ap}(\pi(a)) .$$

Proof- Since $\mathcal{A}$ is simple, every representation of $\mathcal{A}$ is necessarily injective. The result now follows from proposition 3.1., and corollary 3.3.1. .

3.3.3. Corollary- Let π be the universal representation of a C*-algebra $\mathcal{A}$ on a Hilbert space $\mathcal{H}$, and let $a \in \mathcal{A}$, and $\lambda \in \mathbb{C}$.

A necessary and sufficient condition that $\lambda \in \sigma_{ap}(a)$ is that λ be an eigenvalue for $\pi(a)$.

Proof- Let $\lambda \in \sigma_{ap}(a)$. Since the universal representation of $\mathcal{A}$ is faithful, corollary 3.3.1. implies that $\lambda \in \sigma_{ap}(\pi(a))$. Hence, by 2.2.a(iii), there exists a state f of $\pi(\mathcal{A})$ such that

$$f(\pi(a)) = \lambda , \qquad (i)$$

and

$$f(\pi(a)^*\pi(a)) = |\lambda|^2 \qquad (ii) .$$

On the other hand, every state of $\pi(\mathcal{A})$ is a vector state [35; lemma 4.2.] ; hence, there exists a vector η in $\mathcal{H}$

with $\|\eta\|=1$ such that $f = \omega_\eta$. Then,

$$\|(\pi(a)-\lambda I)\eta\|^2 = \langle (\pi(a)^* - \bar{\lambda} I)(\pi(a)-\lambda I)\eta, \eta \rangle$$

$$= f((\pi(a)^* - \bar{\lambda} I)(\pi(a)-\lambda I))$$

$$= 0$$

by (i) and (ii).

Hence, λ is an eigenvalue for $\pi(a)$.

Conversely, if λ is an eigenvalue for $\pi(a)$, then in particular, $\lambda \in \sigma_{ap}(\pi(a))$; hence, by proposition 3.1., $\lambda \in \sigma_{ap}(a)$.

This completes the proof.

3.3.3.1. Remarks- (i) All the results of this section remain valid if $\mathcal{A}$ is replaced by $C^*(a)$, and π is replaced by a non-degenerate representation of $C^*(a)$; c.f., the remark at the end of §2. .

(ii)- The above corollary is the generalization of corollary 1.2.2.2. promised at the beginning of this section.

(iii)- The first explicit example of a representation of a C^*-algebra for which the conclusion of corollary 3.3.3. holds was given, in another context, by S.K.Berberian as follows:

Let $\mathcal{H}$ be a separable Hilbert space; there exists a Hilbert space $\mathcal{K}$ and a faithful representation π of $\mathcal{BL}(\mathcal{H})$ into $\mathcal{BL}(\mathcal{K})$ such that, for each $A \in \mathcal{BL}(\mathcal{H})$,

$$\sigma_{ap}(A) = \sigma_{ap}(\pi(A)) = \sigma_p(\pi(A)) ;$$

c.f.[5; §3].

It may be proved that, if A is normal, quasinormal, subnormal hyponormal, paranormal, or normaloid, then $\pi(A)$ belongs to the same classes of operators, respectively [34; theorem 1]. Thus, for Berberian's representation, the content of the remark at the end of §1. applies to these classes of operators.

§4. The ultraweak closure of $\mathcal{A}A$.

Let $\mathcal{A}$ be a von Neumann algebra(V.N. algebra) , and let $A \in \mathcal{A}$, and $\lambda \in \sigma_{ap}(A)$. Since the left ideal $\mathcal{A}(A-\lambda I)$ is proper, its norm-closure is again a proper left ideal. In this section we examine the corresponding property of $\mathcal{A}(A-\lambda I)$ with norm-closure replaced by the closure in any one of the weak, ultraweak, strong, and ultrastrong operator topologies. As a result, we give necessary and sufficient conditions for the existence of eigenvalues for bounded linear operators on a Hilbert space.

We begin with fixing some notations and recalling certain results.

The weak, ultraweak, strong, and ultrastrong operator topologies will be denoted by τ_ω , $\tau_{\sigma\omega}$, τ_s , and $\tau_{\sigma s}$, respectively. For the properties of the above topologies, we refer to [21; Ch. I, §3]. The norm topology will be denoted by τ_n .

If $\mathcal{N}$ is a subset of $\mathcal{BL}(\mathcal{H})$, the closure of $\mathcal{N}$ in the topology τ_j will be denoted by $\overline{\mathcal{N}}^{\tau_j}$.

For $A \in \mathcal{BL}(\mathcal{H})$, let

$$\mathcal{L} = \ker A = \{ \eta \in \mathcal{H} : A\eta = 0 \},$$

and,

$$\mathcal{K} = \mathcal{L}^{\perp} = \{ \xi \in \mathcal{H} : \langle l, \xi \rangle = 0 \quad \forall l \in \mathcal{L} \}.$$

Then

$$\mathcal{L}^{\perp} = \{ \text{range } A^* \}^{-\tau_n} .$$

The right support of A is defined to be the projection Q onto $\mathcal{K}$. Q is the smallest projection with the property $A = AQ$.

The right support of A is denoted by $\operatorname{Supp} A$.

4.1. Lemma- Let $\mathcal{A}$ be a V.N. algebra acting on a Hilbert space $\mathcal{H}$, let $A \in \mathcal{A}$, and let $\mathcal{B}$ be the V.N. algebra generated by A .

Then $\operatorname{supp} A \in \mathcal{B}$.

Proof- Let $Q = \operatorname{supp} A$, let T be an arbitrary element of $\mathcal{B}'$ where, $\mathcal{B}'$ is the commutant of $\mathcal{B}$, and let η be an arbitrary point in range A^*; thus,

$$\eta = A^* \xi \qquad \text{for some } \xi \in \mathcal{H}.$$

Then

$$T\eta = TA^*\xi = A^*T\xi,$$

since $T \in \mathcal{B}'$; hence

$$T(\text{range } A^*) \subseteq \text{range } A^*,$$

so, by continuity,

$$T(\text{range } Q) \subseteq \text{range } Q,$$

i.e. T is invariant under the range of Q.

On the other hand, since $T \in \mathcal{B}'$ implies $T^* \in \mathcal{B}'$, the same argument shows that T^* is also invariant under the range of Q.

It follows that T commutes with Q. Since T was arbitrary, it follows that Q commutes with $\mathcal{B}'$, i.e., $Q \in \mathcal{B}'' = \mathcal{B}$.

This completes the proof.

4.2. Lemma- Let A and B be bounded linear operators on a Hilbert space $\mathcal{H}$.

(a)- The following conditions are equivalent:

(i)-

$$\text{range } A \subseteq \text{range } B;$$

(ii)- There exists a bounded linear operator C on $\mathcal{H}$ such that

$$A = BC.$$

(b)- Suppose A and B are positive, and $A \leq B$. Then, there exists $D \in \mathcal{BL}(\mathcal{H})$ such that

(i) $A^{\frac{1}{2}} = DB^{\frac{1}{2}}$, and

(ii) $\operatorname{Supp} D \subseteq \operatorname{Supp} B$

If A and B belong to a V.N. algebra, then so does D.

For the proof we refer to [23; Th. 1] and [21; Ch. I, §1, lemma 2].

The following example shows that the $\tau_{\sigma\omega}$ -closure of $\mathcal{A}A$ need not be proper, in general.

4.3. A Counter Example- Let $\mathcal{A} = \mathcal{BL}(\mathcal{H})$, where $\mathcal{H}$ is an infinite dimensional Hilbert space, and let A be a non invertible positive operator with dense range. Since for a positive operator the spectrum, and the approximate point spectrum coincide, we have

$$\mathcal{A}A \neq \mathcal{A}.$$

Let

$$\mathcal{M} = \overline{\mathcal{A}A}^{\tau_{\sigma\omega}}.$$

Then, $\mathcal{M}$ is an ultraweakly closed left ideal of $\mathcal{A}$; hence [21; Ch.I, §3] there exists a projection P in $\mathcal{A}$ such that

$$\mathcal{M} = \mathcal{A}P.$$

Since $\mathcal{A}A \subseteq \mathcal{A}P$, there exists an operator B in $\mathcal{A}$ such that $A = BP$; since A is positive, this implies $A = PB^*$. Hence, we have

$$\text{range } A \subseteq \text{range } P,$$

so that, since A has dense range, and the range of P is closed, we get

$$\text{range } P = \mathcal{H},$$

i.e., $P = I$. Therefore

$$\overline{\mathcal{A}A}^{\tau_{\sigma\omega}} = \mathcal{A}.$$

We remark that, for the purpose of the above example, it is essential that $\mathcal{H}$ be infinite-dimensional; c.f. 4.5.2.3.1(ii).

Later on, we shall construct a whole clsss of operators for which the above phenomenon occurs; c.f. corollary 4.5.2.2. .

4.4. Proposition- Let $\mathcal{A}$ be a V.N. algebra acting on a Hilbert space $\mathcal{H}$, and let $A \in \mathcal{A}$. Then

$$\overline{\mathcal{A}A}^{\tau_\omega} = \overline{\mathcal{A}A}^{\tau_{\sigma\omega}} = \overline{\mathcal{A}A}^{\tau_s} = \overline{\mathcal{A}A}^{\tau_{\sigma s}}.$$

Proof- Since $\overline{\mathcal{A}A}^{\tau_{\sigma\omega}}$ is an ultraweakly closed left ideal, it is weakly closed[21; Ch. I, §3, Cor. 3]. Since, on the other hand, $\overline{\mathcal{A}A}^{\tau_\omega}$ is the smallest weakly closed left ideal which contains $\mathcal{A}A$, we have

$$\overline{\mathcal{A}A}^{\tau_{\sigma\omega}} \supseteq \overline{\mathcal{A}A}^{\tau_\omega} \qquad \text{(i)}.$$

Also, $\tau_{\sigma\omega}$- convergence of operators implies τ_ω-convergence, so we have the reverse inclusion in (i). Hence

$$\overline{\mathcal{A}A}^{\tau_{\sigma\omega}} = \overline{\mathcal{A}A}^{\tau_\omega} \qquad \text{(ii)}.$$

Next, for convex subsets of $\mathcal{A}$, the τ_σ (resp. $\tau_{\sigma\omega}$)-closure is the same as the τ_s (resp. $\tau_{\sigma s}$)-closure [21; Ch. I, §3]; since $\mathcal{A}A$ is obviously convex, it follows that

$$\overline{\mathcal{A}A}^{\tau_s} = \overline{\mathcal{A}A}^{\tau_\omega} \qquad \text{(iii)}$$

and,

$$\overline{\mathcal{A}A}^{\tau_{\sigma s}} = \overline{\mathcal{A}A}^{\tau_{\sigma\omega}} \qquad \text{(iv)}.$$

The result now follows from (ii), (iii), and (iv).

The following result is probably well-known, but we can find no reference for it, and therefore include a complete proof.

4.5. Proposition- Let $\mathcal{A}$ be a V.N. algebra acting on a Hilbert space $\mathcal{H}$, and let $A \in \mathcal{A}$. Then

$$\overline{\mathcal{A}A}^{\tau_j} = \mathcal{A}Q \qquad (j = \omega, \sigma\omega, s, \sigma s)$$

where, $Q = \text{supp}\, A$.

Proof- By proposition 4.4., it is sufficient to prove the result for the case $\tau_j = \tau_{\sigma\omega}$.

Since $\overline{\mathcal{A}A}^{\tau_{\sigma\omega}}$ is an ultraweakly closed left ideal, there exists a projection P in $\mathcal{A}$ such that

$$\overline{\mathcal{A}A}^{\tau_{\sigma\omega}} = \mathcal{A}P.$$

In particular, $\mathcal{A}A \subseteq \mathcal{A}P$, so, there exists an element B in $\mathcal{A}$ such that $A = BP$, i.e., $A^* = PB^*$; hence

$$\text{range } A^* \subseteq \text{range } P,$$

so that

$$\text{range } Q \subseteq \text{range } P,$$

i.e., $Q \leq P$.

On the other hand, since $A = AQ$, we have

$$\mathcal{A}A = \mathcal{A}AQ \subseteq \mathcal{A}Q;$$

since $\mathcal{A}P$ is the smallest $\tau_{\sigma\omega}$- closed left ideal which contains $\mathcal{A}A$, and since $\mathcal{A}Q$ is a $\tau_{\sigma\omega}$- closed left ideal, it follows that

$$\mathcal{A}P \subseteq \mathcal{A}Q.$$

Hence

$$P = BQ$$

for some $B \in \mathcal{A}$, i.e., $P \leq Q$.

This completes the proof.

4.5.1. Corollary- With the notations of proposition 4.5., let $\mathcal{B}$ be the V.N. algebra generated by A. Then

$$\overline{\mathcal{B}A}^{\tau_j} = \mathcal{B}Q \qquad (j = \omega, \sigma\omega, s, \sigma s).$$

Proof- By lemma 4.1., we have $Q \in \mathcal{B}$. A similar reasoning as that of the above proposition now gives the result.

4.5.2. Theorem- Let $\mathcal{A}$ be a V.N. algebra, let $A \in \mathcal{A}$ and let $Q = \operatorname{supp} A$.

A necessary and sufficient condition that either of the two conditions (i) and (ii) below hold is that Q be a non- trivial projection (i.e., $Q \neq I$) :

(i)
$$\overline{\mathcal{A}A}^{\tau_j} \neq \mathcal{A} \qquad (j = \sigma, \sigma\omega, s, \sigma s) .$$

(ii)
$$\overline{\mathcal{B}A}^{\tau_j} \neq \mathcal{B} \qquad (j = \sigma, \sigma\omega, s, \sigma s) .$$

In particular, (i) and (ii) are equivalent.

Proof- (i)- The necessity follows from proposition 4.5. .

To prove the sufficiency, suppose $Q \neq I$, and assume, on the contrary, that
$$\overline{\mathcal{A}A}^{\tau_{\sigma\omega}} = \mathcal{A} ;$$
then, there exists a net of operators B_α in $\mathcal{A}$ such that
$$B_\alpha A \longrightarrow I \qquad \text{ultraweakly.}$$
Since Q is non-trivial, there exists $\eta \in \mathcal{H}$ such that $\eta \neq 0$, and $Q\eta = 0$. Then
$$\langle B_\alpha A\eta, \eta\rangle \rightarrow \langle \eta, \eta\rangle \qquad (*) ;$$
but
$$A\eta = AQ\eta = 0 ,$$
so (*) cannot hold.

The contradiction establishes the result.

(ii)- Using corollary 4.5.1., part (ii) may be established by a similar reasoning.

Finally, (i) and (ii) are equivalent, since each one of them is equivalent to the condition that Q be non-trivial.

4.5.2.1. Remark- With the notations of theorem 4.5.2., let $\mathcal{C}$ be any V.N. algebra containing $\mathcal{A}$. Then each one of the conditions (i), and (ii) is equivalent to the following condition:

$$\overline{\mathcal{C}\mathcal{A}}^{\tau_j} \neq \mathcal{C} \qquad (j = \omega, \sigma\omega, s, \sigma\omega);$$

compare proposition 1.2.2. .

For the next result, we shall need certain elementary properties of weighted shift operators.

Let $\mathcal{H}$ be an infinite- dimensional Hilbert space with an orthonormal basis $\{e_n : n = 0, 1, 2, \dots\}$. A (unilateral) weighted shift is an operator W which satisfies the relation

$$We_n = \alpha_n e_{n+1} \qquad (n = 0, 1, 2, \dots)$$

for some bounded sequence of complex numbers $\{\alpha_n\}_{n=0}^{\infty}$.

We shall need the following two results:

(a)- Suppose $\{\alpha_n\}_{n=0}^{\infty}$ is a bounded sequence of complex numbers such that $\alpha_n \neq 0$ $(\forall n)$. Then W has no eigenvalues.

(b)- Suppose $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence of positive numbers such that $\alpha_n \to 0$. Then, $\sigma(W) = \{0\}$.

For the proof of the above results, we refer to [27; No. 75].

4.5.2.2. Corollary- Let $\mathcal{A} = \mathcal{BL}(\mathcal{H})$, where $\mathcal{H}$ is an infinite dimensional Hilbert space with an orthonormal basis $\{e_n\}_{n=0}^{\infty}$, and let $\{\alpha_n\}_{n=0}^{\infty}$ be a sequence of positive numbers such that $\alpha_n \to 0$.

Then, the weighted shift W defined by the sequence $\{\alpha_n\}_{n=0}^{\infty}$ satisfies

$$\mathcal{A}W \neq \mathcal{A},$$

and

$$\overline{\mathcal{A}W}^{\tau_j} = \mathcal{A} \qquad (j = \omega, \sigma\omega, s, \sigma\omega).$$

Proof- Since $\sigma(W) = \{0\}$, and $\sigma_{ap}(W) \neq \emptyset$, we have $\sigma_{ap}(W) = \{0\}$, i.e. $\mathcal{A}W \neq \mathcal{A}$.

On the other hand, let $Q = \operatorname{supp} W$, and suppose $Q \neq I$. Then, there exists a non- zero element η of $\mathcal{H}$ such that $Q\eta = 0$. So,

$$W\eta = WQ\eta = 0 .$$

This is impossible, since W has no eigenvalues.

Hence, $Q = I$,and the result follows from theorem 4.5.2. .

4.5.2.3. Theorem- Let $\mathcal{A} = \mathcal{BL}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, and let $\lambda \in \mathbb{C}$, and $A \in \mathcal{A}$.

(a)- Each one of the following conditions is necessary and sufficient for λ to be an eigenvalue for A :

(i)

$$\overline{\mathcal{A}(A-\lambda)}^{\tau_j} \neq \mathcal{A} \qquad (j = \omega, \sigma\omega, s, \sigma s).$$

(ii) There exists a τ_j- continuous state f such that f is left-multiplicative with respect to A and $f(A) = \lambda$.

(b)- Let $\lambda \in \sigma_{ap}(A)$, and let $f \in E(\mathcal{A})$ be left-multiplicative with respect to A with $f(A) = \lambda$.

A necessary and sufficient condition that $\lambda \in \sigma_p(A)$ is that f be τ_j-continuous ($j = \omega, \sigma\omega, s, \sigma s$) .

Proof- a(i). Suppose $\overline{\mathcal{A}(A-\lambda)}^{\tau_j} \neq \mathcal{A}$. By theorem 4.5.2., there exists a non zero vector η in $\mathcal{H}$ such that $Q\eta = 0$; hence

$$(A-\lambda)\eta = (A-\lambda)Q\eta = 0 ,$$

i.e., λ is an eigenvalue for A .

Conversely, let λ be an eigenvalue for A , with corresponding eigenvector ξ , and suppose, on the contrary, that

$$\overline{\mathcal{A}(A-\lambda)}^{\tau_{\sigma\omega}} = \mathcal{A}.$$

Then, there exists a net of operators T_α in $\mathcal{A}$ such that

$$T_\alpha(A-\lambda) \longrightarrow I \qquad \text{ultraweakly.}$$

In particular,

$$\langle T_\alpha(A-\lambda)\xi, \xi \rangle \longrightarrow \langle \xi, \xi \rangle ;$$

but this is impossible, since $(A-\lambda)\xi = 0$, and $\xi \neq 0$.

This proves a(i).

a(ii)- Let $\lambda \in \sigma_p(A)$; there exists a vector η in $\mathcal{H}$ such that $\|\eta\|=1$ and $(A-\lambda)\eta=0$. Let $f = \omega_\eta$ be the vector state defined by η. An elementary calculation shows that

$$\omega_\eta(BA) = \omega_\eta(B)\,\omega_\eta(A) = \lambda\,\omega_\eta(B) \qquad (\forall B \in \mathcal{A}).$$

Hence, since ω_η is a τ_j-continuous state ($j = \omega, \sigma\omega, s, \sigma s$) it follows that a(ii) is necessary.

Conversely, fix j_0 in $\{\omega, \sigma\omega, s, \sigma s\}$, and let f be a τ_{j_0}-continuous state with

$$f(BA) = f(B)f(A) = \lambda f(B) \qquad (\forall B \in \mathcal{A}).\ (*)$$

Let

$$\mathcal{M} = \{B \in \mathcal{A} : f(B^*B) = 0\}.$$

Then $\mathcal{M}$ is a proper left ideal of $\mathcal{A}$, and is τ_{j_0}-closed, since f is τ_{j_0}- continuous. Further, using (*), an easy calculation shows that

$$\mathcal{A}(A-\lambda) \subseteq \mathcal{M};$$

hence, (since $\mathcal{M}$ is τ_{j_0}-closed) we have

$$\overline{\mathcal{A}(A-\lambda)}^{\tau_{j_0}} \subseteq \mathcal{M};$$

hence, by proposition 4.4.,

$$\overline{\mathcal{A}(A-\lambda)}^{\tau_j} \subseteq \mathcal{M} \qquad \{j = \omega, \sigma\omega, s, \sigma s\}.$$

Therefore, $\lambda \in \sigma_p(A)$ by a(i).

Part (b) follows from part (a).

This completes the proof.

4.5.2.3.1. Remarks- (i) Every ultraweakly continuous state of $\mathcal{A}$ has the form

$$f = \sum_{i=1}^{\infty} \omega_{\eta_i} \qquad (\eta_i \in \mathcal{H},\ \|\eta_i\|=1\ ;\ i=1,2,\cdots)$$

[21; Ch. I, §3, Th. 1.] . Theorem 4.5.2.3. then implies that if f is left-multiplicative with respect to A , then, there exists a single vector η in $\mathcal{H}$ with $\|\eta\|=1$, such that the corresponding vector state ω_η is left-multiplicative with respect to A ; moreover, $\omega_\eta(A) = f(A)$.

(ii)- Let $\mathcal{A}$ be a V.N. algebra acting on a finite- dimensional Hilbert space $\mathcal{H}$. Then , for each λ in $\sigma_{ap}(A)$, the τ_{δ} - closure of $\mathcal{A}(A-\lambda)$ is again a proper left ideal of $\mathcal{A}$. This is because for a finite-dimensional Hilbert space, the point spectrum and the approximate point spectrum coïncide (proof: τ_n-compactness of the unit ball) . The assertion now follows from theorem 4.5.2.3. .

§5. A geometric Characterization of $\sigma_{ap}(A)$.

Let a be an element of a C^*- algebra $\mathcal{A}$. By corollary 3.3.3., the investigation of the approximate point spectrum of a is equivalent to that of the approximate point spectrum of $\Pi(a)$, where Π is the universal representation of $\mathcal{A}$. The purpose of this section is to give a geometric characterization of $\sigma_{ap}(\Pi(a))$ in terms of certain faces of $E(\Pi(\mathcal{A}))$, and, as a result, to resolve the problem raised at the end of §2.

Let $\mathcal{K}$ be a convex subset of a real linear space. A subset F of $\mathcal{K}$ is called a face of $\mathcal{K}$ provided that the following two conditions are satisfied:

(a)- F is convex;

(b)- whenever $\rho, \sigma \in \mathcal{K}$, and, $0<\alpha<1$, then

$$\alpha\rho + (1-\alpha)\sigma \in F \text{ implies } \rho, \sigma \in F$$

Let $\mathcal{A}$ be a C^*-algebra acting in its universal representation on a Hilbert space $\mathcal{H}$. There exists an inclusion- reversing bijection between the norm-closed left ideals of $\mathcal{A}$ and the w^*-closed faces of $E(\mathcal{A})$. If the ideal $\mathcal{J}$ corresponds to the face $F(\mathcal{J})$, then (c.f., [35; theorem 5.14], and [41; theorem 5.11]) the following hold:

$$F(\mathcal{J}) = \{ f \in E(\mathcal{A}) : f(A^*A) = 0 \quad (\forall A \in \mathcal{J}) \},$$

and

$$\mathcal{J} = \{ A \in \mathcal{A} : f(A^*A) = 0 \quad (\forall f \in F(\mathcal{J})) \}.$$

For the rest of this section, it will be assumed, unless a statement is made to the contrary, that the C^*-algebra $\mathcal{A}$ acts in its universal representation on a Hilbert space $\mathcal{H}$.

Let $A \in \mathcal{A}$, let E be a subset of $E(\mathcal{A})$, and let $\lambda \in \mathbb{C}$. We say that E has property ($P_{A,\lambda}$) provided that

$$f(BA) = f(B)f(A) = \lambda f(B) \qquad (\forall B \in \mathcal{A}, \forall f \in E).$$

5.1. Lemma- Let $A \in \mathcal{A}$, and let $\mathcal{J}$ be a norm-closed left ideal of $\mathcal{A}$ such that

$$\mathcal{A}(A-\lambda) \subseteq \mathcal{J} \qquad (1).$$

Then, the corresponding face of $\mathcal{J}$ defined by

$$F(\mathcal{J}) = \{ f \in E(\mathcal{A}) : f(X^*X) = 0 \quad (\forall X \in \mathcal{J}) \}$$

has property ($P_{A,\lambda}$).

Proof- By (1), we have $(A-\lambda) \in \mathcal{J}$; hence, by the definition of $F(\mathcal{J})$ we have

$$f((A^*-\bar{\lambda})(A-\lambda)) = 0 \qquad (\forall f \in F(\mathcal{J})).$$

Let B be an arbitrary element of $\mathcal{A}$, and let $f \in F(\mathcal{J})$. Then

$$\begin{aligned} |f(BA) - \lambda f(B)|^2 &= |f(B(A-\lambda))|^2 \\ &\leq f(BB^*)\, f((A^*-\bar{\lambda})(A-\lambda)) \\ &= 0; \end{aligned}$$

hence,

$$f(BA) = \lambda f(B) \qquad (\forall B \in \mathcal{A}, \forall f \in F(\mathcal{J})).$$

This completes the proof.

5.2. Theorem- Let A be an element of $\mathcal{A}$, and let $\lambda \in \sigma_{op}(A)$. Define a subset $F_{A,\lambda}$ of $E(\mathcal{A})$ by

$$F_{A,\lambda} = \{ f \in E(\mathcal{A}) : f \text{ has property } (P_{A,\lambda}) \}.$$

Then

(i)- $F_{A,\lambda}$ is a non-empty w^*-compact face of $E(\mathcal{A})$, and the corresponding norm-closed left ideal $\mathcal{J}(F_{A,\lambda})$ defined by

$$\mathcal{J}(F_{A,\lambda}) = \{ X \in \mathcal{A} : f(X^*X) = 0 \quad (\forall f \in F_{A,\lambda}) \}$$

contains $\mathcal{A}(A-\lambda)$.

(ii)- If E is a subset of $E(\mathcal{A})$ with property $(P_{A,\lambda})$, then $E \subseteq F_{A,\lambda}$.

(iii)- Let $\mathcal{Y}$ be the intersection of all norm-closed left ideals of $\mathcal{A}$ containing $\mathcal{A}(A-\lambda)$. Then

$$\mathcal{J}(F_{A,\lambda}) = \mathcal{Y}.$$

Proof- (i). By theorem 2.2.a(iv), the set $F_{A,\lambda}$ is non-empty. It is easily verified that $F_{A,\lambda}$ is convex. To show that $F_{A,\lambda}$ is w^*-compact, let $\{f_\alpha\}$ be a net of elements of $F_{A,\lambda}$ and suppose that

$$f_\alpha \longrightarrow f \qquad (w^*\text{- topology})$$

Then

$$f_\alpha((A^*-\bar{\lambda})(A-\lambda)) \longrightarrow f((A^*-\bar{\lambda})(A-\lambda)).$$

since $f_\alpha \in F_{A,\lambda}$ $(\forall \alpha)$, we have

$$f_\alpha((A^*-\bar{\lambda})(A-\lambda)) = 0$$

hence

$$f((A^*-\bar{\lambda})(A-\lambda)) = 0$$

so (as in the proof of lemma 5.1.), we have $f \in F_{A,\lambda}$. It follows that $F_{A,\lambda}$ is w^*-closed, hence, being a w^*-closed subset of the w^*-compact set $E(\mathcal{A})$, it is w^*-compact.

Next, we show that $F_{A,\lambda}$ is a face of $E(\mathcal{A})$. Let

$$f = \alpha\rho + (1-\alpha)\sigma \in F_{A,\lambda} \qquad (\rho, \sigma \in E(\mathcal{A});\ 0<\alpha<1).$$

Then, since $f \in F_{A,\lambda}$, we have

$$\alpha\, \rho((A^*-\bar{\lambda})(A-\lambda)) + (1-\alpha)\sigma((A^*-\bar{\lambda})(A-\lambda)) = 0;$$

hence

$$\rho((A^*-\bar{\lambda})(A-\lambda)) = \sigma((A^*-\bar{\lambda})(A-\lambda)) = 0,$$

i.e., $\rho, \sigma \in F_{A,\lambda}$. Thus, $F_{A,\lambda}$ is a face of $E(\mathcal{A})$.

Finally, we show that

$$\mathcal{A}(A-\lambda) \subseteq \mathcal{J}(F_{A,\lambda}).$$

Let $X(A-\lambda)$ be an arbitrary element of $\mathcal{A}(A-\lambda)$, and let f be an element of $F_{A,\lambda}$; then

$$\begin{aligned} f((X(A-\lambda))^*(X(A-\lambda))) &= f((A-\lambda)^* X^* X (A-\lambda)) \\ &= 0 \end{aligned}$$

since f has property ($P_{A,\lambda}$). The result now follows from the definition of $\mathcal{J}(F_{A,\lambda})$.

This completes the proof of part (i).

(ii)- This obviously follows from the definition of $F_{A,\lambda}$.

(iii)- Let $F(\mathcal{Y})$ be the corresponding face of $\mathcal{Y}$. By lemma 5.1 $F(\mathcal{Y})$ has property $(P_{A,\lambda})$. Hence, by part (ii), we get

$$F(\mathcal{Y}) \subseteq F_{A,\lambda}.$$

Therefore,

$$\mathcal{Y} \supseteq \mathcal{J}(F_{A,\lambda}).$$

Conversely, since by part (i) $\mathcal{J}(F_{A,\lambda})$ is a closed left ideal of $\mathcal{A}$ containing $\mathcal{A}(A-\lambda)$, and since $\mathcal{Y}$ is the smallest closed left ideal of $\mathcal{A}$ which contains $\mathcal{A}(A-\lambda)$, it follows that

$$\mathcal{Y} \subseteq \mathcal{J}(F_{A,\lambda}).$$

This completes the proof of part (iii).

This completes the proof.

We are now in a position to answer the question which was raised at the end of § 2.

5.3. Theorem- Let $A \in \mathcal{A}$, let $\lambda \in \sigma_{ap}(A)$, and let $f \in E(\mathcal{A})$ be left-multiplicative with respect to A with $f(A) = \lambda$ (by theorem 2.2. such a state always exists). Then, there exists a net $\{f_\alpha\}$ in $E(\mathcal{A})$ such that :

(i). Each f_α is a finite convex combination of pure states f_{α_j} of $\mathcal{A}$ with

$$f_{\alpha_j}(BA) = f_{\alpha_j}(B)\, f_{\alpha_j}(A) = \lambda\, f_{\alpha_j}(B) \qquad (\forall B \in \mathcal{A}).$$

(ii). $f_\alpha \longrightarrow f$ (w^*- topology).

Proof- Let $F_{A,\lambda}$ be the corresponding face constructed as in theorem 5.2. . Then, $f \in F_{A,\lambda}$ by theorem 5.2.(ii).

On the other hand, since $F_{A,\lambda}$ is a w^*-compact convex set, we have, by the Krein- Millman theorem,

$$f \in \overline{Co\left(Ext\left(F_{A,\lambda}\right)\right)}^{w^*} \qquad (*)$$

where $Ext\,(F_{A,\lambda})$ denotes the set of extreme points of $F_{A,\lambda}$.

Since $F_{A,\lambda}$ is a face of $E(\mathcal{A})$, every extreme point of $F_{A,\lambda}$ is an extreme point of $E(\mathcal{A})$ and is therefore a pure state of $\mathcal{A}$. The result now follows from ($*$), and the fact that $F_{A,\lambda}$ has property ($P_{A,\lambda}$).

This completes the proof.

5.3.1. Remark-The conclusion of theorem 5.3. remains true if $\mathcal{A}$ is any C^*-algebra (not necessarily acting in its universal representation). In fact, let $a \in \mathcal{A}$, and suppose that f is a state of $\mathcal{A}$ which is left-multiplicative with respect to a with $f(a)=\lambda$. Define the property ($P_{A,\lambda}$),and the face $F_{a,\lambda}$ as before. Exactly the same reasoning as that used in the proof of theorem 5.2.(i) shows that $F_{a,\lambda}$ is a w^*-compact face of $E(\mathcal{A})$; also, $f \in F_{a,\lambda}$. The argument of theorem 5.3. now goes through without any change.

5.3.2. Proposition- Let $\mathcal{A}$ be a C^*-algebra acting in its universal representation on a Hilbert space $\mathcal{H}$, let $A \in \mathcal{A}$, $\lambda \in \sigma_{ap}(A)$, and let α be a positive number. Then, with

$$H = ((A^*-\bar{\lambda})(A-\lambda))^{\alpha}$$

we have

$$\overline{\mathcal{A}(A-\lambda)} = \overline{\mathcal{A}H}$$

where — denotes norm-closure.

Proof- Let f be a state of $\mathcal{A}$. An application of the Cauchy-Schwartz inequality, together with the functional calculus shows that the following two statements are equivalent:

f is left-multiplicative with respect to A and $f(A)=\lambda$. (1)

f is left-multiplicative with respect to H and $f(H)=0$. (2)

(The Cauchy- Schwartz inequality shows that, with H replaced with $(A^*-\bar{\lambda})(A-\lambda)$, (1) and (2) above are equivalent. If n is a non-negative integer, the conditions $f(K)=0$ and $f(K^{\frac{1}{2^n}})=0$ are equivalent for any positive operator K ; c.f. the remarks immediately preceding proposition 1.6. of chapter II. Hence, by the functional calculus, the conditions $f(K)=0$ and $f(K^{\alpha})=0$ are equivalent for any α with $0<\alpha\leq 1$. A similar reasoning then shows that the conditions $f(K)=0$ and $f(K^{\alpha})=0$ are equivalent for any positive number α. Since for a positive operator K the condition $f(K)=0$ automatically implies that f is left-multiplicative with respect to K, it follows that (1) and (2) above are equivalent).

In particular,

$$\lambda \in \sigma_{ap}(A) \Longleftrightarrow 0 \in \sigma_{ap}(H) .$$

Hence, with the notations of theorem 5.2., we have

$$F_{A,\lambda} = F_{H,0} ,$$

i.e.,

$$\mathcal{J}(F_{A,\lambda}) = \mathcal{J}(F_{H,0}) .$$

The result now follows from theorem 5.2.(iii).

We close this section with the following remark concerning the above proposition and theorem 5.2. . Let $A \in \mathcal{A}$, let $\lambda \in \sigma_{ap}(A)$, and let $K = (A^* - \bar{\lambda})(A - \lambda)$. By theorem 5.2. and the above proof, the norm-closed left ideal $\mathcal{J}(F_{K,0})$ is the smallest norm-closed left ideal such that the corresponding face $F_{K,0}$ has property $(P_{K,0})$. If $\mathcal{A}$ is a separable C^*-algebra, the following converse holds: Given a proper norm-closed left ideal $\mathcal{J}$ of $\mathcal{A}$, there exists a positive operator R such that

$$\mathcal{J} = \mathcal{J}(F_{R,0}) .$$

For, every norm-closed left ideal in a separable C^*-algebra is the norm-closure of a principal left ideal generated by a positive operator [41;page 26]. The result now follows from theorem 5.2.(iii).

Chapter II

The Joint Approximate Point Spectrum of Operators

§1. The Joint Approximate Point Spectrum of a Finite Number of Operators

1.1. Introduction- Let $A_1, \ldots, A_n$ be bounded linear operators on a Hilbert space $\mathcal{H}$. The joint approximate point spectrum of $A_1, \ldots, A_n$ is, by definition, the set of n-tuples $(\lambda_1, \ldots, \lambda_n)$ of complex numbers with the following property:

$$\forall \epsilon > 0 \quad \exists x \text{ s.t. } x \in \mathcal{H},\ x \neq 0,\ \|(A_j - \lambda_j)x\| \leq \epsilon \|x\| \quad (j=1,\ldots,n)$$

For n=1, the above definition reduces to that of the approximate point spectrum of a single operator.

In this section, the above definition is extended to arbitrary (unital) C*-algebras, and a characterization of the joint approximate point spectrum will be given which will showthat, for $\mathcal{BL}(\mathcal{H})$ the two definitions coincide. In particular, Theorem 2.2. will be shown to be a special case of Theorem 1.4. below.

1.2. Definition- Let $\mathcal{A}$ be a C*-algebra, and let $a_1, \ldots, a_n$ be elements of $\mathcal{A}$.

The joint approximate point spectrum of $a_1, \ldots, a_n$ is defined to be the set

$$\left\{ (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n : \sum_{j=1}^{n} \mathcal{A}(a_j - \lambda_j) \neq \mathcal{A} \right\}$$

where $\sum_{j=1}^{n} \mathcal{A}(a_j - \lambda_j)$ denotes the left ideal of $\mathcal{A}$ generated by $a_1 - \lambda_1, \ldots, a_n - \lambda_n$.

The joint approximate point spectrum of $a_1, \ldots, a_n$ will be denoted by $J\sigma_{ap}(a_1, \ldots, a_n)$.

1.2.1. Proposition- With the notations of Definition 1.2., let $\mathcal{B}$ be any C*-algebra containing $a_1, \ldots, a_n$. Then

$$J\sigma_{ap}(a_1, \ldots, a_n) = \left\{ (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n : \sum_{j=1}^{n} \mathcal{B}(a_j - \lambda_j) \neq \mathcal{B} \right\}.$$

Remark: Thus, the joint approximate point spectrum of $a_1, \dots, a_n$ is independent of the C^*-algebra containing $\{a_1, \dots, a_n\}$.

Proof- The proof is essentially the same as that of proposition 1.2.2. of chapter I. We omit the details.

Before presenting the main result of this section, we state the following lemma, which may be proved in the same way as lemma 2.1., and corollary 2.1.1. of chapter I .

1.3. Lemma- Let $A_1, \dots, A_n$ be bounded linear operators on a Hilbert space $\mathcal{H}$, let $\mathcal{L}$ be an absolutely convex subset of $\mathcal{H}$ such that $[\mathcal{L}] = \mathcal{H}$, and let $(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$. Suppose that

$$\sum_{j=1}^{n} \mathcal{BL}(\mathcal{H})(A_j - \lambda_j) \neq \mathcal{BL}(\mathcal{H}).$$

Then, given a positive number ϵ , there exists a non-zero element l of $\mathcal{L}$ such that

$$\|(A_j - \lambda_j) l\| \leq \epsilon \qquad (j = 1, 2, \dots, n).$$

1.4. Theorem- Let $\mathcal{A}$ be a C^*-algebra, let $a_1, \dots, a_n$ be elements of $\mathcal{A}$, and let $(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$.

(a)- The following conditions are equivalent:

(i)-

$$\sum_{j=1}^{n} \mathcal{A}(a_j - \lambda_j) \neq \mathcal{A} ,$$

i.e.,

$$(\lambda_1, \dots, \lambda_n) \in J\sigma_{ap}(a_1, \dots, a_n)$$

in the sense of definition 1.2. .

(ii)- There exists a pure state ρ of $\mathcal{A}$ such that

$$\rho(a_j) = \lambda_j \quad \& \quad \rho(a_j^* a_j) = |\lambda_j|^2 \qquad (j = 1, 2, \dots, n).$$

(iii)- There exists a state f of $\mathcal{A}$ such that

$$f(a_j) = \lambda_j \quad \& \quad f(a_j^* a_j) = |\lambda_j|^2 \qquad (j = 1, 2, \dots, n).$$

(iv)- There exists a state f of $\mathcal{A}$ such that, for each j, $(j=1,\ldots,n)$, f is left-multiplicative with respect to a_j and $f(a_j)=\lambda_j$.

(v)- There exists a pure state ρ of $\mathcal{A}$ such that for each j, $(j=1,\ldots,n)$, ρ is left- multiplicative with respect to a_j and $\rho(a_j)=\lambda_j$.

(b)- Let $A_1, \ldots, A_n$ be bounded linear operators on a Hilbert space $\mathcal{H}$, let $\mathcal{L}$ be an absolutely convex subset of $\mathcal{H}$ with $\mathcal{H} = [\mathcal{L}]$, and let $(\lambda_1,\ldots,\lambda_n)\in\mathbb{C}^n$.

The following conditions are equivalent:

(i)-

$$(\lambda_1, \ldots, \lambda_n) \in \mathcal{J}\sigma_{ap}(A_1,\ldots,A_n)$$

in the sense of definition 1.1., i.e.,

$$\forall \epsilon>0 \ \exists x \ \text{ s.t. } x\in\mathcal{H},\ x\neq 0,\ \|(A_j-\lambda_j)x\|\leq\epsilon\|x\| \quad (j=1,\ldots,n).$$

(ii)-

$$\forall \epsilon>0 \ \exists l \ \text{ s.t. } l\in\mathcal{L},\ l\neq 0,\ \|(A_j-\lambda_j)l\|\leq\epsilon\|l\| \quad (j=1,\ldots,n).$$

(c)- With the above notations, suppose $\mathcal{A} = \mathcal{BL}(\mathcal{H})$, and $A_j = a_j \ (j=1,\ldots,n)$. Then, the conditions a(i)-a(v), b(i), and b(ii) are all equivalent.

Proof: We first prove part (a).

a(i) $\Longrightarrow$ a(ii)- Suppose

$$(\lambda_1,\ldots,\lambda_n)\in\mathcal{J}\sigma_{ap}(a_1,\ldots,a_n).$$

Then, the left ideal $\sum_{j=1}^{n}\mathcal{A}(a_j-\lambda_j)$ is a proper left ideal of $\mathcal{A}$ so it is contained in a maximal left ideal $\mathcal{M}$ of $\mathcal{A}$ given by

$$\mathcal{M} = \{x\in\mathcal{A} : \rho(yx)=0 \quad (\forall y\in\mathcal{A})\}$$

for some pure state ρ of $\mathcal{A}$.

In particular,

$$\mathcal{A}(a_j - \lambda_j) \subseteq \mathcal{M} \qquad (j=1,\ldots,n).$$

For each fixed j $(j=1,\ldots,n)$, taking x to be $a_j - \lambda_j$, and y to be 1 and $(a_j-\lambda_j)^*$ respectively, we get

$$\rho(a_j-\lambda_j)=0 \;\&\; \rho((a_j-\lambda_j)^*(a_j-\lambda_j))=0.$$

Hence

$$\rho(a_j) = \lambda_j \;\&\; \rho(a_j^* a_j) = |\lambda_j|^2 \qquad (j=1,\ldots,n).$$

a(ii) $\Longrightarrow$ a(iii). This is trivial.

a(iii) $\Longrightarrow$ a(iv). Let f satisfy a(iii). Then, for each fixed j in $\{1,\ldots,n\}$, and each x in $\mathcal{A}$ we have, by Cauchy-Schwartz inequality,

$$\begin{aligned} |f(xa_j) - f(x)f(a_j)|^2 &= |f(x(a_j-\lambda_j))|^2 \\ &\leq f(x^*x)\, f((a_j-\lambda_j)^*(a_j-\lambda_j)) \\ &= 0 ; \end{aligned}$$

Hence

$$f(x a_j) = \lambda_j f(x) \qquad (j=1,\ldots,n).$$

a(iv) $\Longrightarrow$ a(v). Let f have the stated property in a(iv). Let

$$\mathcal{N} = \{x \in \mathcal{A} : f(yx)=0 \quad (\forall y \in \mathcal{A})\}.$$

It is easily seen that $\mathcal{N}$ is a proper closed left ideal of $\mathcal{A}$. Let

$$\sum_{j=1}^{n} x_j (a_j - \lambda_j)$$

be an arbitrary element of $\sum_{j=1}^{n} \mathcal{A}(a_j-\lambda_j)$. Then, for each y in $\mathcal{A}$, we have

$$f\left(y\left(\sum_{j=1}^{n} x_j(a_j-\lambda_j)\right)\right) = \sum_{j=1}^{n} f(y x_j a_j) - \sum_{j=1}^{n} \lambda_j f(y x_j) = 0,$$

since

$$f(y x_j a_j) = f(y x_j) f(a_j) = \lambda_j f(y x_j) \qquad (j=1,\dots,n).$$

Hence

$$\sum_{j=1}^{n} \mathcal{A}(a_j-\lambda_j) \subseteq \mathcal{N} \subsetneq \mathcal{A}$$

It follows that $\sum_{j=1}^{n} \mathcal{A}(a_j-\lambda_j)$ is contained in a maximal left ideal $\mathcal{M}$ of $\mathcal{A}$ given by some pure state ρ. It then follows (as in the proof of a(i) $\Longrightarrow$ a(ii)) that for each j $(j=1,\dots,n)$, ρ is left-multiplicative with respect to each a_j with $\rho(a_j)=\lambda_j$.

a(v) $\Longrightarrow$ a(i). Let ρ have the stated property in a(v); let

$$\mathcal{M} = \{x \in \mathcal{A} : \rho(x^*x)=0\}.$$

Then, $\mathcal{M} \neq \mathcal{A}$, and it is easily verified that

$$\sum_{j=1}^{n} \mathcal{A}(a_j-\lambda_j) \subseteq \mathcal{M} \subsetneq \mathcal{A}.$$

This completes the proof of part (a).

Next, we prove part (b).

b(i) $\Longrightarrow$ b(ii). Let b(i) be satisfied. By lemma 1.3., it is sufficient to show that

$$\sum_{j=1}^{n} \mathcal{BL}(\mathcal{H})(A_j-\lambda_j) \neq \mathcal{BL}(\mathcal{H}).$$

Suppose not; then, there exits operators $B_1, \dots, B_n$ in $\mathcal{BL}(\mathcal{H})$ such that

$$\sum_{j=1}^{n} B_j(A_j-\lambda_j) = I.$$

Hence, for each ξ in $\mathcal{H}$, we have

$$\|\xi\| = \|\sum_{j=1}^{n} B_j (A_j - \lambda_j)\xi\|$$
$$\leq \sum_{j=1}^{n} \|B_j\| \, \|(A_j - \lambda_j)\xi\|,$$

so that

$$\sum_{j=1}^{n} \|(A_j - \lambda_j)\xi\| \geq M^{-1} \|\xi\|,$$

where

$$M = \sup_{1 \leq j \leq n} \|B_j\|.$$

This contradicts the hypothesis in b(i).

The required implication thus follows.

b(ii) $\Longrightarrow$ b(i). This holds trivially.

This completes the proof of part (b).

(c)- We shall prove that a(i) is equivalent to b(ii).

a(i) $\Longrightarrow$ b(ii). This is lemma 1.3. .

b(ii) $\Longrightarrow$ a(i). Partially order the set of all finite subsets $\mathcal{F}$ of $\mathbb{N}^+$(the set of all positive integers) with respect to inclusion. For each element F of $\mathcal{F}$, with

$$F = \{ m_1, m_2, \ldots, m_k \}$$

say, let

$$m_F = \max \{ m_1, m_2, \ldots, m_k \}.$$

By the hypothesis, given $F \in \mathcal{F}$, there exists an element x_F of $\mathcal{X}$ such that $\|x_F\| = 1$, and such that the corresponding vector state satisfies

$$\omega_{x_F}((A_k - \lambda_j)^*(A_j - \lambda_j)) \leq \frac{1}{m_F} \quad (j = 1, \ldots, n).$$

Since $E(\mathcal{B}\mathcal{L}(\mathcal{H}))$ is w*-compact, the net $\{ \omega_{x_F} : F \in \mathcal{F} \}$

has a cluster point f in $E(\mathcal{BL}(\mathcal{H}))$. We claim that

$$f((A_j-\lambda_j)^*(A_j-\lambda_j))=0 \qquad (j=1,\dots,n).$$

For let ε be an arbitrary positive number; choose an integer m such that $\frac{1}{m}<\varepsilon$; let $F_0=\{m\}$. Then, for each $F\in\mathcal{F}$ with $F\supseteq F_0$,we have

$$\begin{aligned}&\omega_{x_F}((A_j-\lambda_j)^*(A_j-\lambda_j))\le\\&\omega_{x_{F_0}}((A_j-\lambda_j)^*(A_j-\lambda_j))\le\\&\frac{1}{m}\le\epsilon \qquad\qquad (j=1,\dots,n);\end{aligned}$$

hence

$$f((A_j-\lambda_j)^*(A_j-\lambda_j))\le\varepsilon \qquad (j=1,\dots,n).$$

Since ε was arbitrary, this establishes our claim.

Thus, b(ii) $\Rightarrow$ a(iii) $\Leftrightarrow$ a(i) .

This completes the proof of the theorem.

1.4.1. Remarks- (a). In the course of the proof we have shown that, if a state f satisfies one of the following three conditions then it satisfies all three:

(1)-

$$f(a_j)=\lambda_j \ \ \& \ \ f(a_j^*a_j)=|\lambda_j|^2 \qquad (j=1,\dots,n)$$

(2)-

$$f((a_j-\lambda_j)^*(a_j-\lambda_j))=0 \qquad (j=1,\dots,n)$$

(3)- f is left-multiplicative with respect to a_j $(j=1,\dots,n)$, and $f(a_j)=\lambda_j$.

(b)- The equivalence of a(iv) and a(v) is proved in [37; theorem 4] by a different method.

The following proposition may be used to reduce ~~the study of~~

the study of the joint approximate point spectrum of a finite family of operators to that of the spectrum of a single positive operator.

1.5. Proposition- Let $a_1, \ldots, a_n$ be elements of a C^*-algebra $\mathcal{A}$, and let $(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$. Then

$$(\lambda_1, \ldots, \lambda_n) \in J\sigma_{ap}(a_1, \ldots, a_n) \tag{1}$$

if and only if

$$0 \in \sigma\left(\sum_{j=1}^{n} (a_j - \lambda_j)^*(a_j - \lambda_j) \right) \tag{2}$$

Proof- If

$$0 \notin \sigma\left(\sum_{j=1}^{n} (a_j - \lambda_j)^*(a_j - \lambda_j) \right)$$

there exists b in $\mathcal{A}$ such that

$$b\left(\sum_{j=1}^{n} (a_j - \lambda_j)^*(a_j - \lambda_j) \right) = 1,$$

hence

$$\sum_{j=1}^{n} \mathcal{A}(a_j - \lambda_j) = \mathcal{A},$$

i.e.,

$$(\lambda_1, \ldots, \lambda_n) \notin J\sigma_{ap}(a_1, \ldots, a_n).$$

Conversely, let (2) be satisfied. Then

$$\mathcal{A}\left(\sum_{j=1}^{n} (a_j - \lambda_j)^*(a_j - \lambda_j) \right) \neq \mathcal{A},$$

so, there exists a maximal left ideal $\mathcal{M}$ of $\mathcal{A}$, given by a corresponding pure state ρ such that

$$\mathcal{A}\left(\sum_{j=1}^{n} (a_j - \lambda_j)^*(a_j - \lambda_j) \right) \subseteq \mathcal{M}.$$

It follows that

$$\rho\left(\sum_{j=1}^{n} (a_j - \lambda_j)^*(a_j - \lambda_j) \right) = 0.$$

In particular,

$$\rho((a_j-\lambda_j)^*(a_j-\lambda_j))=0 \qquad (j=1,\dots,n).$$

The result now follows from Remark 1.4.1.(a) and theorem 1.4. ∎

Let h be a positive operator in a C^*-algebra $\mathcal{A}$, and let $f\in E(\mathcal{A})$. For each non-negative integer m, we have, by the Cauchy-Schwartz inequality,

$$f(h^{\frac{1}{2^m}}) \le (f(h))^{\frac{1}{2^m}}.$$

Hence, if $f(h)=0$ then

$$f(h^{\frac{1}{2^m}}) = 0 \qquad (m=0,1,\dots)$$

Conversely, suppose $f(h^{\frac{1}{2^m}})=0$ for some non-negative integer m. Then,

$$h = h^{\frac{1}{2^{m+1}}}x,$$

where

$$x = h^{\frac{1}{2^{m+1}}}.h^{\frac{1}{2^m}}.\dots.h^{\frac{1}{2}}.$$

Hence, by Cauchy-Schwartz inequality,

$$0\le f(h)^2 \le f(h^{\frac{1}{2^m}}).f(x^2)=0,$$

i.e., $f(h)=0$.

Using proposition 1.5. and the above result, we have the following characterization of the joint approximate point spectrum:

1.6. Proposition: Let $a_1,\dots,a_n$ be elements of a C^*-algebra $\mathcal{A}$, and let $(\lambda_1,\dots,\lambda_n)\in\mathbb{C}^n$.

The following conditions are equivalent:

(i)-

$$(\lambda_1,\dots,\lambda_n)\in J\sigma_{ap}(a_1,\dots,a_n).$$

(ii)- There exists a non-negative integer m such that:

$$(0,0,\ldots,0) \in J\sigma_{ap}\left(((a_1-\lambda_1)^*(a_1-\lambda_1))^{\frac{1}{2^m}}, \ldots, ((a_n-\lambda_n)^*(a_n-\lambda_n))^{\frac{1}{2^m}}\right).$$

(iii)- For each non-negative integer m, we have

$$(0,0,\ldots,0) \in J\sigma_{ap}\left(((a_1-\lambda_1)^*(a_1-\lambda_1))^{\frac{1}{2^m}}, \ldots, ((a_n-\lambda_n)^*(a_n-\lambda_n))^{\frac{1}{2^m}}\right).$$

Proof- Let (i) be satisfied. By proposition 1.5., we have

$$0 \in \sigma\left(\sum_{j=1}^{n}(a_j-\lambda_j)^*(a_j-\lambda_j)\right).$$

Hence, there exists a state f of $\mathcal{A}$ such that

$$f\left(\sum_{j=1}^{n}(a_j-\lambda_j)^*(a_j-\lambda_j)\right)=0 \qquad (j=1,\ldots,n)$$

In particular,

$$f((a_j-\lambda_j)^*(a_j-\lambda_j))=0 \qquad (j=1,\ldots,n).$$

Hence, (ii) holds with $m=0$.

Conversely, if (ii) holds for some non-negative integer m then, there exists a state f of $\mathcal{A}$ such that

$$f(h_j)=0 \qquad (j=1,\ldots,n),$$

where

$$h_j=((a_j-\lambda_j)^*(a_j-\lambda_j))^{\frac{1}{2^m}} \qquad (j=1,\ldots,n).$$

By the remarks preceding the present proposition, it follows that

$$f((a_j-\lambda_j)^*(a_j-\lambda_j))=0 \qquad (j=1,\ldots,n).$$

Hence

$$f\left(\sum_{j=1}^{n}(a_j-\lambda_j)^*(a_j-\lambda_j)\right)=0$$

Hence, by proposition 1.5., (i) holds.

Next, we show the equivalence of (ii) and (iii).

Clearly, (iii) implies (ii).

Let now (ii) be satisfied for some non-negative integer m_0. As before, there exists a state f of $\mathcal{A}$ such that

$$f(h_j) = 0 \qquad (j = 1, \dots, n),$$

where

$$h_j = \left((a_j - \lambda_j)^* (a_j - \lambda_j) \right)^{\frac{1}{2^{m_0}}} \qquad (j = 1, \dots, n).$$

Hence

$$f\left((a_j - \lambda_j)^* (a_j - \lambda_j) \right) = 0 \qquad (j = 1, \dots, n).$$

Therefore

$$f\left(\left((a_j - \lambda_j)^* (a_j - \lambda_j) \right)^{\frac{1}{2^m}} \right) = 0 \qquad (j = 1, \dots, n;\ m = 1, 2, \dots).$$

Hence (iii) holds.

This completes the proof.

We close this section with the following result which relates the concept of the joint approximate point spectrum to the theory of characters.

1.7. Proposition- Let $\mathcal{A}$ be a C^*-algebra, let $a_1, \dots, a_n$ be elements of $\mathcal{A}$, and suppose that φ is a character on $C^*(a_1, \dots, a_n)$. Then

$$(\varphi(a_1), \dots, \varphi(a_n)) \in J\sigma_{ap}(a_1, \dots, a_n).$$

Proof- Suppose, on the contrary that

$$(\varphi(a_1), \dots, \varphi(a_n)) \notin J\sigma_{ap}(a_1, \dots, a_n)$$

By proposition 1.2.1., we have

$$\sum_{j=1}^{n} C^*(a_1, \dots, a_n)\,(a_j - \varphi(a_j)) = C^*(a_1, \dots, a_n)$$

Hence there exists elements $x_1, \dots, x_n$ in $C^*(a_1, \dots, a_n)$ such that

$$\sum_{j=1}^{n} x_j\,(a_j - \varphi(a_j)) = 1$$

Then

$$1 = \varphi(1) = \varphi\Big(\sum_{j=1}^{n} x_j (a_j - \varphi(a_j))\Big)$$
$$= \sum_{j=1}^{n} \varphi(x_j)\, \varphi(a_j - \varphi(a_j))$$
$$= 0$$

which is absurd.

This completes the proof.

§2. The existence of $J\sigma_{ap}(a_1, \dots, a_n)$.

It was noted in Chapter I that the approximate point spectrum of a single operator is always non-empty. This is no longer the case for the joint approximate point spectrum in general.

In this section, we give examples of operators whose joint approximate point spectrum is empty, and then establish some conditions under which the joint approximate point spectrum is non-empty.

2.1. Examples-

(a)- Let $\mathcal{H}$ be an infinite-dimensional Hilbert space, and let $\mathcal{K} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where

$$\mathcal{H}_j = \mathcal{H} \qquad (j = 1, 2).$$

Let P_j be the projection of $\mathcal{K}$ onto $\mathcal{H}_j$ $(j=1,2)$. Since $\mathcal{K}$ and $\mathcal{H}_j$ $(j=1,2)$ have the same Hilbert space dimension, it follows that P_j is equivalent to $I_{\mathcal{K}}$ $(j=1,2)$. Hence, for each j $(j=1,2)$, there exists a partial isometry V_j such that

$$V_j^* V_j = I_{\mathcal{K}} \qquad (j=1,2)$$
$$V_j V_j^* = P_j \qquad (j=1,2)$$

(Thus, V_j is in fact an isometry for $j=1,2$).

Suppose now that $J\sigma_{ap}(V_1, V_2) \neq \emptyset$, and let

$$(\lambda_1, \lambda_2) \in J\sigma_{ap}(V_1, V_2).$$

By theorem 1.4., there exists a state f of $\mathcal{B}(\mathcal{K})$ such that

$$f(AV_j) = \lambda_j f(A) \qquad (\forall A \in \mathcal{B}(\mathcal{K})\ ; j=1,2).$$

In particular, with $A = V_j^*$, we have

$$|\lambda_j|^2 = 1 \qquad (j=1,2).$$

Hence, putting $A = V_j V_j^*$, we get

$$f(V_j V_j^* V_j) = \lambda_j f(V_j V_j^*) \qquad (j=1,2),$$

i.e.,

$$\lambda_j = \lambda_j f(V_j V_j^*) \qquad (j=1,2),$$

so that

$$f(V_j V_j^*) = 1 \qquad (j=1,2).$$

Therefore,

$$\begin{aligned} 2 &= f(V_1 V_1^*) + f(V_2 V_2^*) \\ &= f(V_1 V_1^* + V_2 V_2^*) \\ &= f(P_1 + P_2) \\ &= 1 \end{aligned}$$

which is absurd.

Thus, $J\sigma_{ap}(V_1, V_2) = \emptyset$.

We remark that there exist unitary operators U_1 and U_2 which satisfy

$$U_1^2 = 1 \ \& \ U_2^3 = 1$$

such that $C^*(U_1, U_2)$ is simple; c.f. [17].

Similar ~~The same~~ analysis to ~~as~~ that of the above example then shows that

$J\sigma_{ap}(U_1, U_2) = \emptyset$.

(b)- We shall give an example of two self-adjoint operators whose joint approximate point spectrum is empty.

Let $\mathcal{A} = M_2(\mathbb{C})$, the C^*-algebra of complex 2×2 matrices. Let A and B be elements of $\mathcal{A}$ defined by

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix},$$

and

$$B = \frac{1}{2i}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

It is easily verified that A and B are self-adjoint; suppose that the joint approximate point spectrum of A and B is non-empty, and let

$$(\lambda, \mu) \in J\sigma_{ap}(A, B). \qquad \text{(i)}$$

Then, in particular,

$$\lambda \in \sigma_{ap}(A) \ \& \ \mu \in \sigma_{ap}(B),$$

so that λ and μ are real numbers; further, by proposition 1.5., we have

$$0 \in \sigma((A-\lambda)^2) \ \& \ 0 \in \sigma((B-\mu)^2)$$

i.e.,

$$\det (A-\lambda)^2 = 0 \ \& \ \det (B-\mu)^2 = 0.$$

An elementary calculation shows that

$$(A-\lambda)^2 = \begin{pmatrix} 1+(2-\lambda)^2 & 6-2\lambda \\ 6-2\lambda & 1+(4-\lambda)^2 \end{pmatrix}$$

and

$$(B-\mu)^2 = \begin{pmatrix} \frac{1}{4}+\mu^2 & -\frac{\mu}{i} \\ \frac{\mu}{i} & \mu^2+\frac{1}{4} \end{pmatrix}$$

Hence, we must have

$$(1+(2-\lambda)^2)(1+(4-\lambda)^2) = (6-2\lambda)^2 \tag{ii}$$

$$(\mu^2+\tfrac{1}{4})^2 = \mu^2. \tag{iii}$$

On the other hand, by (i) and proposition 1.5., we must have

$$\det\left((A-\lambda)^2+(B-\mu)^2\right) = 0.$$

But,

$$\begin{aligned} &\det\left((A-\lambda)^2+(B-\mu)^2\right) = \\ &(1+(2-\lambda)^2+(\mu^2+\tfrac{1}{4}))(1+(4-\lambda)^2+(\mu^2+\tfrac{1}{4})) - \\ &((6-2\lambda)+\tfrac{\mu}{i})((6-2\lambda)-\tfrac{\mu}{i}) = \\ &(1+(2-\lambda)^2)(1+(4-\lambda)^2)+(\mu^2+\tfrac{1}{4})(2+(2-\lambda)^2+(4-\lambda)^2)+ \\ &(\mu^2+\tfrac{1}{4})^2-((6-2\lambda)^2+\mu^2) = \\ &(\mu^2+\tfrac{1}{4})(2+(2-\lambda)^2+(4-\lambda)^2), \end{aligned}$$

by (ii) and (iii). Since λ and μ are real numbers, the last expression is always positive; so

$$\det\left((A-\lambda)^2+(B-\mu)^2\right) \neq 0,$$

a contradiction.

Thus

$$J\sigma_{ap}(A,B) = \emptyset$$

We remark that the existence of self-adjoint (indeed positive) operators whose joint approximate point spectrum is empty, is a consequence of a general result in the theory of characters to be proved in chapter IV, theorem 2.4..

We now turn to the investigation of necessary and sufficient conditions for the non-emptiness of the joint approximate point spectrum of operators.

2.2. Proposition- Let $a_1, \ldots, a_n$ be elements of a C^*-algebra $\mathcal{A}$, let $(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$, and let π be a non-degenerate representation of $\mathcal{A}$ such that

$$C^*(a_1, \ldots, a_n) \cap \operatorname{Ker} \pi = \{0\} .$$

Then

$$(\lambda_1, \ldots, \lambda_n) \in J\sigma_{ap}(a_1, \ldots, a_n)$$

if and only if

$$(\lambda_1, \ldots, \lambda_n) \in J\sigma_{ap}(\pi(a_1), \ldots, \pi(a_n)) .$$

The proof is essentially the same as that of proposition 3.3. of Chapter I. We omit the details.

2.3. Theorem- Let $a_1, \ldots, a_n$ be elements of a C^*-algebra $\mathcal{A}$ let $(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$, and let π be the universal representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$.

A necessary and sufficient condition that

$$(\lambda_1, \ldots, \lambda_n) \in J\sigma_{ap}(a_1, \ldots, a_n) \tag{i}$$

is that

$$\bigvee_{j=1}^{n} Q_j \neq I_{\mathcal{H}} , \tag{ii}$$

where

$$Q_j = \text{supp } (\pi(a_j) - \lambda_j) \qquad (j=1,\dots,n).$$

Proof- Suppose (i) is satisfied; since π is faithful, proposition 2.2. implies that

$$(\lambda_1,\dots,\lambda_n) \in J\sigma_{ap}(\pi(a_1),\dots,\pi(a_n)) .$$

Hence, there exists a state f of $\pi(\mathcal{A})$ such that

$$f(B\,\pi(a_j)) = \lambda_j\, f(B) \qquad (\forall B \in \mathcal{BL}(\mathcal{H});\ j=1,\dots,n)$$

Since every state of $\pi(\mathcal{A})$ is a vector state, there exists a vector ξ such that $\|\xi\|=1$, and such that the corresponding vector state ω_ξ satisfies

$$\omega_\xi((\pi(a_j)-\lambda_j)^*(\pi(a_j)-\lambda_j)) = 0 \qquad (j=1,\dots,n).$$

For each j $(j=1,\dots,n)$, let

$$(\pi(a_j)^* - \bar{\lambda}_j)\,\eta$$

be an arbitrary element of the range of $\pi(a_j)^* - \bar{\lambda}_j$. Then

$$\begin{aligned} |\langle(\pi(a_j)^*-\bar{\lambda}_j)\eta, \xi\rangle|^2 &= |\langle(\pi(a_j)-\lambda_j)\xi, \eta\rangle|^2 \\ &\le \|\eta\|^2\langle(\pi(a_j)-\lambda_j)^*(\pi(a_j)-\lambda_j)\xi, \xi\rangle \\ &= \|\eta\|^2\,\omega_\xi((\pi(a_j)-\lambda_j)^*(\pi(a_j)-\lambda_j)) \\ &= 0 . \end{aligned}$$

Hence

$$\xi \perp \bigcup_{j=1}^{n} \{\text{range}(\pi(a_j)^* - \bar{\lambda}_j)\} .$$

It follows that (ii) holds.

Conversely, if $\bigvee_{j=1}^{n} Q_j \neq I_{\mathcal{H}}$, then there exists a vector ξ in $\mathcal{H}$ such that $\|\xi\|=1$ and

$$Q_j\,\xi = 0 \qquad (j=1,\dots,n).$$

Since for each j $(j=1,\dots,n)$,

$$(\Pi(a_j)-\lambda_j)\,Q_j = \Pi(a_j)-\lambda_j\,,$$

an easy calculation shows that the vector state ω_ξ defined by ξ is left-multiplicative with respect to each $\Pi(a_j)$ and satisfies

$$\omega_\xi\,(\Pi(a_j)) = \lambda_j \qquad (j=1,\dots,n).$$

Hence, by theorem 1.4.,

$$(\lambda_1,\dots,\lambda_n)\in J\sigma_{ap}\,(\Pi(a_1),\dots,\Pi(a_n)),$$

so that

$$(\lambda_1,\dots,\lambda_n)\in J\sigma_{ap}\,(a_1,\dots,a_n)$$

by proposition 2.2. .

This completes the proof.

The following lemma gives a sufficient condition for the joint approximate point spectrum to be non-empty.

2.4. Lemma- Let $A_1,\dots,A_n$ be bounded linear operators on a Hilbert space $\mathcal{H}$. Suppose that $\mathcal{K}$ is a non-zero closed subspace of $\mathcal{H}$ such that $\mathcal{K}$ is invariant under each A_j $(j=1,\dots,n)$; let $A'_j = A_j|_{\mathcal{K}}$ (the restriction of A_j to $\mathcal{K}$).

If

$$(\lambda_1,\dots,\lambda_n)\in J\sigma_{ap}\,(A'_1,\dots,A'_n)\ ,$$

then

$$(\lambda_1,\dots,\lambda_n)\in J\sigma_{ap}\,(A_1,\dots,A_n).$$

Proof- Suppose not ; then, there exists a finite set $\{B_1,\dots,B_n\}$ of elements of $\mathcal{BL}(\mathcal{H})$ such that

$$\sum_{j=1}^{n} B_j(A_j-\lambda_j) = I_{\mathcal{H}}$$

Let P be the projection of $\mathcal{H}$ onto $\mathcal{K}$, let $B'_j = B_j|_{\mathcal{K}}$ $(j=1,\dots,n)$, and let

$$D_j = P B'_j \qquad (j = 1, \dots, n),$$

so that

$$D_j \in \mathcal{BL}(\mathcal{K}) \qquad (j = 1, \dots, n).$$

A simple calculation then shows that

$$\sum_{k=1}^{n} D_j (A'_j - \lambda I_{\mathcal{K}}) = I_{\mathcal{K}},$$

contradicting the hypothesis(by theorem 1.4.(c)).

This completes the proof.

2.4.1. Remark- It is well- known that, if a is an element of a Banach algebra $\mathcal{A}$, and if a is a topological divisor of zero in $\mathcal{A}$, then a is a topological divisor of zero as an element of any Banach algebra containing $\mathcal{A}$ (thus, a is "permanently singular"). The above lemma has a similar interpretation.

We are now ready to state and prove the main result of this section.

2.5. Theorem- Let $\mathcal{A}$ be a C^*-algebra, let Π be the universal representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$, and let $a_1, \dots, a_n$ be elements of $\mathcal{A}$.

Suppose that

(i)

$$(\lambda_1, \dots, \lambda_{n-1}) \in J\sigma_{ap}(a_1, \dots, a_{n-1}),$$

and

(ii) the subspaces

$$(\Pi(a_j)^* - \bar{\lambda}_j)(\mathcal{H}) \qquad (j = 1, \dots, n-1)$$

are invariant under $\Pi(a_n)^*$.

Then, there exists a complex number λ_n such that

$$(\lambda_1, \dots, \lambda_n) \in J\sigma_{ap}(a_1, \dots, a_n).$$

Proof- Let

$$Q_j = \operatorname{supp}(\Pi(a_j) - \bar{\lambda}_j) \qquad (j = 1, \dots, n-1),$$

and put

$$Q = 1 - \left(\bigvee_{j=1}^{n} Q_j\right),$$

and

$$\mathcal{K} = Q(\mathcal{H}).$$

By (i) and theorem 2.3., $\mathcal{K}$ is a non-zero closed subspace of $\mathcal{H}$. By (ii), $(\bigvee_{j=1}^{n-1} Q_j)(\mathcal{H})$ is invariant under $\Pi(a_n)^*$, so $\mathcal{K}$ is invariant under $\Pi(a_n)$. Hence, there exists a complex number λ_n such that

$$\lambda_n \in \sigma_{ap}(\Pi(a_n)')$$

where $\Pi(a_n)'$ is the restriction of $\Pi(a_n)$ to $\mathcal{K}$.

On the other hand, since

$$Q_j Q = Q Q_j = 0 \qquad (j = 1, \dots, n-1)$$

we have

$$Q_j' = 0 \qquad (j = 1, \dots, n-1)$$

where Q_j' is the restriction of Q_j to $\mathcal{K}$.

Hence

$$\left(\sum_{j=1}^{n-1} \mathcal{BL}(\mathcal{K}) Q_j'\right) + \mathcal{BL}(\mathcal{K})(\Pi(a_n)' - \lambda_n) \neq \mathcal{BL}(\mathcal{K})$$

Hence, by lemma 2.4., we have

$$\left(\sum_{j=1}^{n-1} \mathcal{BL}(\mathcal{H}) Q_j\right) + \mathcal{BL}(\mathcal{H})(\Pi(a_n) - \lambda_n) \neq \mathcal{BL}(\mathcal{H})$$

Since, for each j, $(j=1,\ldots,n-1)$

$$\mathcal{BL}(\mathcal{H})(\pi(a_j)-\lambda_j) = \mathcal{BL}(\mathcal{H})(\pi(a_j)-\lambda_j)\,Q_j$$
$$\subseteq \mathcal{BL}(\mathcal{H})\,Q_j$$

it follows that

$$\sum_{j=1}^{n} \mathcal{BL}(\mathcal{H})(\pi(a_j)-\lambda_j) \neq \mathcal{BL}(\mathcal{H}).$$

Hence

$$(\lambda_1,\ldots,\lambda_n) \in J\sigma_{ap}(\pi(a_1),\ldots,\pi(a_n)).$$

The result now follows from proposition 2.2. .

2.5.1. Corollary- Let $\{a_1,\ldots,a_n\}$ be a finite set of pair-wise commuting operators in a C^*-algebra $\mathcal{A}$. Then, given

$$(\lambda_1,\ldots,\lambda_{n-1}) \in J\sigma_{ap}(a_1,\ldots,a_{n-1}),$$

there exists a complex number λ_n such that

$$(\lambda_1,\ldots,\lambda_n) \in J\sigma_{ap}(a_1,\ldots,a_n).$$

In particular, the joint approximate point spectrum of $a_1,\ldots,a_n$ is non-empty.

Proof- Since a_n commutes with a_j $(j=1,\ldots,n-1)$, condition (ii) of theorem 2.5. is trivially satisfied. This proves the first part.

The second part follows, by induction, from the first part and the fact that the approximate point spectrum of a single operator is always non-empty.

This completes the proof.

The above corollary was first raised,as an open question,in [20; problem 2] ; it was first solved by J.Bunce [10; proposition 1] and, independently, by W. Zelazko [57; theorem] . Bunce's proof makes a similar use of lemma 2.4., while Zelazko uses the idea of the joint topological divisors of zero; c.f., [50] and [56].

2.5.2. Corollary- With the notations of theorem 2.5., suppose 2.5.(i) holds. Suppose, further, that there exists a set $\{ b_1, \ldots, b_{n-1} \}$ of elements of $\mathcal{A}$ such that

$$a_n = b_j (a_j - \lambda_j) \qquad (j = 1, \ldots, n-1).$$

Then, the conclusion of theorem 2.5. holds.

Proof- We have

$$\pi(a_n)^* = (\pi(a_j)^* - \bar{\lambda}_j) \pi(b_j)^* \qquad (j = 1, \ldots, n-1).$$

Hence

$$\text{range}(\pi(a_n)^*) \subseteq \text{range}(\pi(a_j)^* - \bar{\lambda}_j) \qquad (j = 1, \ldots, n).$$

In particular, $\pi(a_n)^*$ is invariant under the range of $\pi(a_j)^* - \bar{\lambda}_j$, $(j = 1, \ldots, n-1)$.

The result now follows from theorem 2.5. •

We close this section with two examples where the situation described in corollary 2.5.2. occurs naturally.

The following simple lemma will be needed below.

2.6. Lemma- Let a and b be positive elements of a C^*-algebra $\mathcal{A}$ such that $a \leq b$. Suppose that

$$(\lambda, \mu) \in J\sigma_{ap}(a, b).$$

Then $\lambda \leq \mu$.

Proof- There exists a state f of $\mathcal{A}$ such that

$$f(a) = \lambda \ \& \ f(b) = \mu .$$

Hence

$$\lambda = f(a) \leq f(b) = \Gamma$$

This completes the proof.

2.7. Examples-

(a)- Let $\mathcal{H}$ be a separable Hilbert space. Let W be the (unilateral) weighted shift defined by the sequence of weights

$$\{\alpha, \beta, 1, 1, 1, \ldots\},$$

where $0 < \alpha < \beta < 1$.

Let $x = (\xi_0, \xi_1, \ldots)$ be an arbitrary element of $\mathcal{H}$. An easy calculation shows that

$$Wx = (0, \alpha\xi_0, \beta\xi_1, \xi_2, \xi_3, \ldots),$$

and

$$W^*x = (\alpha\xi_1, \beta\xi_2, \xi_3, \xi_4, \ldots);$$

hence

$$\langle W^*Wx, x\rangle = \alpha^2|\xi_0|^2 + \beta^2|\xi_1|^2 + \sum_{n=2}^{\infty}|\xi_n|^2,$$

and

$$\langle WW^*x, x\rangle = \alpha^2|\xi_1|^2 + \beta^2|\xi_2|^2 + \sum_{n=3}^{\infty}|\xi_n|^2.$$

So

$$\langle (W^*W - WW^*)x, x\rangle = \alpha^2|\xi_0|^2 + (\beta^2-\alpha^2)|\xi_1|^2 + (1-\beta^2)|\xi_2|^2$$

$$\geq 0,$$

since $0 < \alpha < \beta < 1$.

Let now $\lambda \in \sigma_{ap}(W)$; then (proposition 1.6.)

$$0 \in \sigma((W-\lambda)^*(W-\lambda)^{\frac{1}{2}}).$$

On the other hand, since

$$(W-\lambda)(W-\lambda)^* \leq (W-\lambda)^*(W-\lambda),$$

there exists an operator B in $\mathcal{BL}(\mathcal{H})$ such that

$$\left((W-\lambda)(W-\lambda)^* \right)^{\frac{1}{2}} = B \left((W-\lambda)^*(W-\lambda) \right)^{\frac{1}{2}}$$

(Ch.I, lemma 4.2.(b)).

Hence, by corollary 2.5.2., there exists a complex number μ such that

$$(\mu, 0) \in J\sigma_{ap}\left((W-\lambda)(W-\lambda)^{*\frac{1}{2}}, \ (W-\lambda)^*(W-\lambda)^{\frac{1}{2}} \right)$$

Therefore,

$$(\lambda, \bar{\lambda}) \in J\sigma_{ap}(W, W^*)$$

by lemma 2.6. and proposition 1.6. .

We remark that corollary 2.5.1. is not applicable here, since W is not normal (in fact, W is an example of a hyponormal operator which is not subnormal; c.f. [27; solution 160].

(b)- Let W be the (unilateral) weighted shift defined by the sequence of weights

$$\{ 1, 2, 1, 1, 1, \ldots \}$$

It may be shown that there exists a positive number M such that

$$(W-\lambda)(W-\lambda)^* \leq M (W-\lambda)^*(W-\lambda)$$

for all complex numbers λ ; c.f.[54; Example].

As in example (a), an application of corollary 2.5.2. then shows that

$$J\sigma_{ap}(W, W^*) = \{ (\lambda, \bar{\lambda}) : \lambda \in \sigma_{ap}(W) \} .$$

We remark that, W is an example of an M-hyponormal operator which is not hyponormal (to show that W is not hyponormal, choose an $x = (\xi_0, \xi_1, \ldots)$ in $\mathcal{H}$ with $\xi_0 = 0$ and $0 < |\xi_1|^2 < |\xi_2|^2$; it is then easily verified that $\langle W^*W\tau,\tau\rangle < \langle WW^*\tau,\tau\rangle$).

§ 3. The joint approximate point spectrum of a finite family of commuting operators.

Let $\mathcal{A}$ be a Banach algebra, and let a and b be commuting elements of $\mathcal{A}$. It is well-known that

$$\sigma(a+b) \subseteq \sigma(a)+\sigma(b)$$

and that

$$\sigma(ab) \subseteq \sigma(a)\,\sigma(b)$$

c.f.[47; theorem 11.23].

In this section, we shall obtain analogous resuls for the joint approximate point spectrum of commuting operators, and extend the result to direct sums of operators.

3.1. Proposition- Let $\{a_1,\dots,a_n\}$ be a pair-wise commuting set of operators in a C^*-algebra $\mathcal{A}$. Then

(i) $$\sigma_{ap}\Big(\sum_{j=1}^{n} a_j\Big) = \Big\{\sum_{j=1}^{n}\lambda_j : (\lambda_1,\dots,\lambda_n)\in J\sigma_{ap}(a_1,\dots,a_n)\Big\}$$
$$\Big(\subseteq \sum_{j=1}^{n}\sigma_{ap}(a_j)\Big)$$

and

(ii) $$\sigma_{ap}\Big(\prod_{j=1}^{n} a_j\Big) = \Big\{\prod_{j=1}^{n}\lambda_j : (\lambda_1,\dots,\lambda_n)\in J\sigma_{ap}(a_1,\dots,a_n)\Big\}$$
$$\Big(\subseteq \prod\sigma_{ap}(a_j)\Big)$$

Remark- The expression $\sum_{j=1}^{n}\sigma_{ap}(a_j)$ denotes the set

$$\Big\{\sum_{j=1}^{n}\lambda_j : \lambda_j\in\sigma_{ap}(a_j) \quad (j=1,\dots,n)\Big\}$$

The expression $\prod_{j=1}^{n}\sigma_{ap}(a_j)$ is defined similarly.

Proof- Let $\lambda\in\sigma_{ap}\big(\sum_{j=1}^{n}a_j\big)$; since the set

$$\Big\{\sum_{j=1}^{n}a_j\ ,\ a_1, a_2,\dots,a_n\Big\}$$

is a pair-wise commuting set of elements of $\mathcal{A}$, theorem 2.5.

implies the existence of an n-tuple $(\lambda_1, \ldots, \lambda_n)$ of complex numbers such that

$$(\lambda, \lambda_1, \ldots, \lambda_n) \in J\sigma_{ap}\left(\left(\sum_{j=1}^{n} a_j\right), a_1, \ldots, a_n \right).$$

By theorem 1.4., there exists a state f of $\mathcal{A}$ such that

$$f\left(\sum_{j=1}^{n} a_j \right) = \lambda$$

and

$$f(x a_j) = \lambda_j f(x) \qquad (\forall x \in \mathcal{A};\ j = 1, \ldots, n).$$

In particular,

$$(\lambda_1, \ldots, \lambda_n) \in J\sigma_{ap}(a_1, \ldots, a_n);$$

further,

$$\sum_{j=1}^{n} \lambda_j = \sum_{j=1}^{n} f(a_j) = f\left(\sum_{j=1}^{n} a_j \right) = \lambda.$$

Hence

$$\sigma_{ap}\left(\sum_{j=1}^{n} a_j \right) \subseteq \left\{ \sum_{j=1}^{n} \lambda_j : (\lambda_1, \ldots, \lambda_n) \in J\sigma_{ap}(a_1, \ldots, a_n) \right\}$$

Conversely, suppose that

$$(\lambda_1, \ldots, \lambda_n) \in J\sigma_{ap}(a_1, \ldots, a_n).$$

then, there exists a state f of $\mathcal{A}$ such that f is left-multiplicative with respect to each a_j with $f(a_j) = \lambda_j$; hence, f is left-multiplicative with respect to $\sum_{j=1}^{n} a_j$ with $f\left(\sum_{j=1}^{n} a_j\right) = \sum_{j=1}^{n} \lambda_j$.

This completes the proof of (i).

(ii) may be proved in a similar way.

This completes the proof.

Let $\mathcal{A}_1, \ldots, \mathcal{A}_n$ be V.N.algebras acting on Hilbert spaces $\mathcal{H}_1, \ldots, \mathcal{H}_n$ respectively. Let

$$T_j \in \mathcal{A}_j \qquad (j = 1, \ldots, n).$$

Let $T = \overset{n}{\underset{j=1}{\oplus}} T_j$ be the direct sum of the operators $T_1, \ldots, T_n$, defined on the Hilbert space $\mathcal{H} = \overset{n}{\underset{j=1}{\oplus}} \mathcal{H}_j$ by

$$T x = \overset{n}{\underset{j=1}{\oplus}} (T_j x_j),$$

where

$$x = \overset{n}{\underset{j=1}{\oplus}} x_j \qquad (x_j \in \mathcal{H}_j \ , \ j=1,\ldots,n).$$

Let

$$\mathcal{A} = \left\{ T : \ T = \overset{n}{\underset{j=1}{\oplus}} T_j \qquad (T_j \in \mathcal{A}_j \ , \ j=1,\ldots,n) \right\}.$$

Then, $\mathcal{A}$ is a V.N.algebra acting on the Hilbert space $\mathcal{H} = \overset{n}{\underset{j=1}{\oplus}} \mathcal{H}_j$ [21; Ch. I , §2.] . We call $\mathcal{A}$ the direct sum of $\mathcal{A}_1, \ldots, \mathcal{A}_n$, and denote it by $\overset{n}{\underset{j=1}{\oplus}} \mathcal{A}_j$.

3.2. Proposition- For each j $(j=1,\ldots,n)$, let $\mathcal{A}_j$ be a V.N. algebra acting on a Hilbert space $\mathcal{H}_j$, let A_j and B_j be elements of $\mathcal{A}_j$, let $A = \overset{n}{\underset{j=1}{\oplus}} A_j$, $B = \overset{n}{\underset{j=1}{\oplus}} B_j$, and $\mathcal{A} = \overset{n}{\underset{j=1}{\oplus}} \mathcal{A}_j$.

Then

$$J\sigma_{ap}(A, B) \subseteq \bigcup_{j=1}^{n} J\sigma_{ap}(A_j, B_j) .$$

Proof- Let $(\lambda, \mu) \in \mathbb{C}^2$, and suppose that

$$(\lambda, \mu) \notin \bigcup_{j=1}^{n} J\sigma_{ap}(A_j, B_j) .$$

Then, for each j $(j=1,\ldots,n)$,there exists operators A_j' and B_j' in $\mathcal{A}_j$ such that

$$A_j'(A_j - \lambda I_{\mathcal{H}_j}) + B_j'(B_j - \mu I_{\mathcal{H}_j}) = I_{\mathcal{H}_j} \qquad (j=1,\ldots,n).$$

Define A' and B' on $\overset{n}{\underset{j=1}{\oplus}} \mathcal{H}_j$ by

$$A' = \overset{n}{\underset{j=1}{\oplus}} A_j' \quad \& \quad B' = \overset{n}{\underset{j=1}{\oplus}} B_j'$$

then, A' and B' are in $\mathcal{A}$. further, an easy calculation shows that

$$A'(A-\lambda I_{\mathcal{H}}) + B'(B-\mu I_{\mathcal{H}}) =$$
$$\bigoplus_{j=1}^{n} \left((A_j'A_j - \lambda I_{\mathcal{H}_j}) + (B_j'B_j - \mu I_{\mathcal{H}_j}) \right) =$$
$$\bigoplus_{j=1}^{n} I_{\mathcal{H}_j} = I_{\mathcal{H}} \quad ;$$

Hence

$$(\lambda, \mu) \notin J\sigma_{ap}(A, B).$$

This completes the proof.

3.2.1. Remarks-

(a)- Proposition 3.2. may easily be generalized to cover the case of any finite number of operators in $\mathcal{A}$.

(b)- Proposition 3.2. fails if finite direct sums are replaced with infinite direct sums.

To see this, define a sequence of operators A_n as follows:

$$A_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

etc. .

It is easily verified that, for each j $(j=1,2,\dots)$, we have

$$\sigma(A_j) = \sigma_{ap}(A_j) = \{0\} \quad ,$$

so that

$$\bigcup_{j=1}^{\infty} (\sigma_{ap} (A_j)) = \{ 0 \} .$$

On the other hand, $\bigoplus_{j=1}^{\infty} A_j$ is the unilateral weighted shift A defined by the sequence of weights

$$\{ \alpha_n \}_{n=0}^{\infty} = \{ 1, 0, 1, 1, 0, 1, 1, 1, 0, \dots \} .$$

The spectral radius, and the norm of $\bigoplus_{j=1}^{\infty} A_j$ are given by

$$r(A) = \lim_{k \to \infty} \left(\sup_{n \geq 0} \left(\prod_{i=0}^{k-1} \alpha_{n+i} \right)^{1/k} \right)$$

and

$$\| A \| = \sup_{n \geq 0} | \alpha_n |$$

respectively [27; solution 77].

Hence

$$r(A) = \| A \| = 1 .$$

It follows that there exists a complex number λ such that $|\lambda| = 1$ and $\lambda \in \sigma_{ap} (A)$ (c.f. chapter IV, § 4, theorem 4.8.).

Hence

$$\sigma_{ap} (\bigoplus_{j=1}^{\infty} A_j) \not\subseteq \bigcup_{j=1}^{\infty} \sigma_{ap} (A_j) .$$

§ 4. The joint spectrum in Banach algebras.

Our attention has so far been confined to the joint approximate point spectrum of operators in C^*-algebras. However, even for applications to bounded linear operators on a Hilbert space, it is necessary to consider the joint spectra of operators on a Banach space (c.f. corollary 4.6.1.). The present section is therefore devoted to developing the necessary tools for the latter purpose.

4.1. Definition- Let $\mathcal{X}$ be a Banach space, let $A_1, \ldots, A_n$ be bounded linear operators on $\mathcal{X}$, and let $(\lambda_1,\ldots,\lambda_n)\in\mathbb{C}^n$ and $\mathcal{B}=\mathcal{BL}(\mathcal{X})$.

(a)- The joint left spectrum of $A_1, \ldots, A_n$ is defined to be the set of n-tuples $(\lambda_1, \ldots, \lambda_n)$ of complex numbers such that

$$\sum_{j=1}^{n} \mathcal{B}(A_j - \lambda_j I_{\mathcal{X}}) \neq \mathcal{B}.$$

The joint left spectrum is denoted by $J\sigma_l(A_1,\ldots,A_n)$.

(b)-The joint approximate point spectrum of $A_1, \ldots, A_n$ is the set of n-tuples $(\lambda_1,\ldots,\lambda_n)$ of complex numbers such that

$$\forall \epsilon > 0 \ \exists x \text{ s.t. } x \in \mathcal{X},\ x \neq 0,\ \|(A_j - \lambda_j I_{\mathcal{X}})x\| \leq \epsilon \quad (j=1,\ldots,n).$$

The joint approximate point spectrum is denoted by $J\sigma_{ap}(A_1,\ldots,A_n)$.

(c)- The joint right spectrum of $A_1, \ldots, A_n$ is defined to be the set of n-tuples $(\lambda_1,\ldots,\lambda_n)$ of complex numbers such that

$$\sum_{j=1}^{n} (A_j - \lambda_j I_{\mathcal{X}})\mathcal{B} \neq \mathcal{B}.$$

The joint right spectrum is denoted by $J\sigma_r(A_1,\ldots,A_n)$.

(d)- The joint spectrum of $A_1, \ldots, A_n$, which we shall denote by $J\sigma(A_1,\ldots,A_n)$ is defined by

$$J\sigma(A_1,\ldots,A_n) = J\sigma_r(A_1,\ldots,A_n) \cup J\sigma_l(A_1,\ldots,A_n).$$

4.1.1. Remarks.

(1)- When n=1, the above definitions reduce to those of the left spectrum, approximate point spectrum, right spectrum, and spectrum of a single operator, respectively.

(2)- If $\mathcal{X} = \mathcal{H}$ for some Hilbert space $\mathcal{H}$, then definitions 4.1.(a) and 4.1.(b) are equivalent. However, for an arbitrary Banach space $\mathcal{X}$, the two conditions need not be equivalent, even if n=1; we always have

$$J\sigma_{ap}(A_1,\ldots,A_n) \subseteq J\sigma_l(A_1,\ldots,A_n),$$

but the reverse inclusion is false, in general; c.f. [51; proposition 1.7.].

(3)- Definition 4.1.(d) is one of several possible (in general distinct) definitions of the joint spectrum; c.f. [51; Introduction]. Our definition is the same as that given in [7; §2, definition 11] and used in [29].

4.2. Proposition- Let $A_1, \dots, A_n$ be bounded linear operators on a Banach space $\mathfrak{X}$, and let $\mathcal{B} = \mathcal{BL}(\mathfrak{X})$.

(i)- Suppose that

$$(\lambda_1, \dots, \lambda_n) \in J\sigma(A_1, \dots, A_n).$$

Then

$$(\lambda_1, \dots, \lambda_j) \in J\sigma(A_1, \dots, A_j) \qquad (j = 1, \dots, n).$$

(ii)- Let $\{A_1, \dots, A_n\}$ be a commuting set of normal elements of a C^*-algebra $\mathcal{A}$, and let Φ be the set of characters on $C^*(A_1, \dots, A_n)$. Then

$$J\sigma(A_1, \dots, A_n) = J\sigma_{ap}(A_1, \dots, A_n)$$
$$= \{(\varphi(A_1), \dots, \varphi(A_n)) : \varphi \in \Phi\}.$$

Proof- (i). For each n-tuple $(\mu_1, \dots, \mu_n)$ of complex numbers, and each fixed j_0 in $\{1, \dots, n\}$, we have

$$\sum_{j=1}^{j_0} \mathcal{B}(A_j - \lambda_j) \subseteq \sum_{j=1}^{n} \mathcal{B}(A_j - \lambda_j)$$

and

$$\sum_{j=1}^{j_0} (A_j - \lambda_j)\mathcal{B} \subseteq \sum_{j=1}^{n} (A_j - \lambda_j)\mathcal{B}.$$

The result now follows form definition 4.1.(d).

(ii). Let

$$(\mu_1, \dots, \mu_n) \in J\sigma_{ap}(A_1^*, \dots, A_n^*).$$

Then, there exists a state f of $\mathcal{A}$ such that

$$f(xA_j^*) = \mu_j f(x) \qquad (j = 1, \dots, n).$$

For each j ($j = 1, \ldots, n$) we have

$$f((A_j^* - \Gamma_j)(A_j - \bar{\Gamma}_j)) = f((A_j - \bar{\Gamma}_j)(A_j^* - \Gamma_j)) = 0 .$$

Hence (c.f. remark 1.4.1.(a)), f is also left-multiplicative with respect to each A_j with $f(A_j) = \bar{\Gamma}_j$. Therefore

$$f\big|_{C^*(A_1, \ldots, A_n)} \in \Phi .$$

In particular,

$$(\bar{\Gamma}_1, \ldots, \bar{\Gamma}_n) \in \mathcal{J}\sigma_{ap}(A_1, \ldots, A_n)$$

by proposition 1.7. . Hence,

$$\mathcal{J}\sigma(A_1, \ldots, A_n) = \mathcal{J}\sigma_{ap}(A_1, \ldots, A_n)$$

by definition 4.1.(d).

Next, applying the above reasoning to the set $\{A_1, \ldots, A_n\}$ we see that, given

$$(\lambda_1, \ldots, \lambda_n) \in \mathcal{J}\sigma_{ap}(A_1, \ldots, A_n)$$

there exists $\varphi \in \Phi$ such that

$$\varphi(A_j) = \lambda_j \qquad (j = 1, \ldots, n) .$$

Hence, by proposition 1.7., we have

$$\mathcal{J}\sigma_{ap}(A_1, \ldots, A_n) = \{(\varphi(A_1), \ldots, \varphi(A_n)) : \varphi \in \Phi\} .$$

This proves (ii) and completes the proof of the theorem.

4.2.1. Remark- The property expressed in 4.2.(i) is called the projection property of the joint spectrum. The projection property does not hold, in general, for the commutant and the bicommutant definitions of the joint spectrum; for an example, we refer to [51; page 144].

Throughout the rest of this section, $\mathcal{H}$ denotes an infinite-dimensional Hilbert space, $\mathfrak{X} = \mathcal{BL}(\mathcal{H})$ is given the Banach space structure of $\mathcal{BL}(\mathcal{H})$, and $\mathcal{B}$ will denote the complex unital Banach algebra of bounded linear operators on $\mathcal{X}$.

4.3. Lemma- Let $\{A_1, \dots, A_n\}$ (respectively $\{B_1, \dots, B_n\}$) be a set of operators in $\mathcal{X}$. For each j ($j=1,\dots,n$) define the operators F_j' and F_j'' on $\mathcal{X}$ by

$$F_j'(X) = A_j X \qquad (j=1,\dots,n\,;\ X\in\mathcal{X})$$

and

$$F_j''(X) = X B_j \qquad (j=1,\dots,n\,;\ X\in\mathcal{X})$$

Then

$$F_j' \in \mathcal{B}\ ,\quad F_j'' \in \mathcal{B} \qquad (j=1,\dots,n)$$

and the following inclusions hold:

$$J\sigma_l(F_1',\dots,F_n') \subseteq J\sigma_{ap}(A_1,\dots,A_n) \tag{i}$$

$$J\sigma_r(F_1',\dots,F_n') \subseteq J\sigma_{ap}(A_1^*,\dots,A_n^*) \tag{ii}$$

$$J\sigma_l(F_1'',\dots,F_n'') \subseteq J\sigma_{ap}(B_1^*,\dots,B_n^*) \tag{iii}$$

$$J\sigma_r(F_1'',\dots,F_n'') \subseteq J\sigma_{ap}(B_1,\dots,B_n) \tag{iv}$$

Proof- We shall only prove (i), since the other inclusions may be proved in a similar way.

Let $(\lambda_1,\dots,\lambda_n)\in\mathbb{C}^n$, and suppose that

$$\sum_{j=1}^{n} \mathcal{BL}(\mathcal{H})(A_j-\lambda_j) = \mathcal{BL}(\mathcal{H}).$$

Then

$$\sum_{j=1}^{n} C_j(A_j-\lambda_j) = I_{\mathcal{H}}$$

for some $C_j \in \mathcal{BL}(\mathcal{H})$ $(j=1,\dots,n)$.

For each j $(j=1,\dots,n)$, define a bounded linear operator G_j on $\mathcal{H}$ by

$$G_j(X) = C_j X \qquad (j=1,\dots,n;\ X \in \mathcal{X})$$

Then, for each X in $\mathcal{X}$, we have

$$\Big(\sum_{j=1}^{n} G_j(F_j' - \lambda_j 1_{\mathcal{X}})\Big)(X) =$$

$$\sum_{j=1}^{n}\big(C_j A_j(X) - \lambda_j C_j(X)\big) =$$

$$\Big(\sum_{j=1}^{n} C_j(A_j - \lambda_j 1)\Big)(X) =$$

$$X.$$

Hence

$$\sum_{j=1}^{n} G_j(F_j' - \lambda_j 1_{\mathcal{X}}) = 1_{\mathcal{X}} \quad ,$$

i.e.,

$$(\lambda_1, \dots, \lambda_n) \notin J\sigma_l(F_1', \dots, F_n').$$

This completes the proof.

4.4. Lemma- Let $\{A_1, \dots, A_n\}$ (respectively $\{B_1, \dots, B_n\}$) be a mutually commuting set of elements of $\mathcal{X}$, and let F_j' and F_j'' $(j=1,\dots,n)$ be defined as in lemma 4.3. .

For each j $(j=1,\dots,n)$ define a bounded linear operator F_j on $\mathcal{X}$ by

$$F_j(X) = A_j X - X B_j \qquad (j=1,\dots,n;\ X \in \mathcal{X}).$$

Then, the set

$$\{F_1, \dots, F_n,\ F_1', \dots, F_n',\ F_1'', \dots, F_n''\}$$

is a mutually commuting set of elements of $\mathcal{B}$.

Proof- Let j and k be in $\{1, 2, \dots, n\}$, and let X be an arbitrary element of $\mathfrak{X}$. then

$$F_j' F_k'(X) = F_j'(A_k X)$$
$$= A_j A_k X$$
$$= A_k A_j X$$
$$= F_k' F_j'(X).$$

Hence, the set

$$\{F_1', \ldots, F_n'\}$$

is mutually commuting.

Similarly, the set

$$\{F_1'', \ldots, F_n''\}$$

is mutually commuting.

Next, for each arbitrary j and k in $\{1, 2, \cdots, n\}$ and each X in $\mathfrak{X}$ we have

$$F_j' F_k''(X) = F_j'(XB_k)$$
$$= A_j XB_k$$
$$= (F_j'(X)) B_k$$
$$= F_k'' F_j'(X).$$

Hence the set

$$\{F_1', \ldots, F_n', F_1'', \ldots, F_n''\}$$

is mutually commuting.

Finally, since for each j, we have

$$F_j = F_j' - F_j'' \qquad (j = 1, \cdots, n),$$

it follows that the set

$$\{F_1, \ldots, F_n, F_1', \ldots, F_n', F_1'', \ldots, F_n''\}$$

is mutually commuting.

This completes the proof.

The following theorem, which is similar to corollary 2.5.1., is proved in [29].

4.5. Theorem- Let $\{T_1, \ldots, T_n\}$ be a mutually commuting set of elements of $\mathcal{B}$, and suppose that

$$(\lambda_1, \ldots, \lambda_{n-1}) \in J\sigma_l\,(T_1, \ldots, T_{n-1}) .$$

Then, there exists a complex number λ_n such that

$$(\lambda_1, \ldots, \lambda_n) \in J\sigma_l\,(T_1, \ldots, T_n) .$$

4.5.1. Corollary- Let $\{T_1, \ldots, T_n\}$ be a mutually commuting set of elements of $\mathcal{B}$, and suppose that

$$(\lambda_1, \ldots, \lambda_{n-1}) \in J\sigma_r\,(T_1, \ldots, T_{n-1}) .$$

Then, there exists a complex number λ_n such thst

$$(\lambda_1, \ldots, \lambda_n) \in J\sigma_r\,(T_1, \ldots, T_n) .$$

Proof- By [51; lemma 2.10.] ,we have

$$J\sigma_r\,(T_1, \ldots, T_n) = J\sigma_l\,(T_1^*, \ldots, T_n^*)$$

where $T_1^*, \ldots, T_n^*$ are the conjugate operators defined on the conjugate Banach space $\mathcal{X}^*$.

The result now follows from theorem 4.5. •

The following convention will be adopted below:

Let S_1 and S_2 be subsets of $\mathbb{C}^n$. The expression $S_1 - S_2$ denotes the set of elements

$$\{((\lambda_1 - \mu_1), \ldots, (\lambda_n - \mu_n))\}$$

of $\mathbb{C}^n$ such that

$$(\lambda_1, \ldots, \lambda_n) \in S_1 \quad \& \quad (\mu_1, \ldots, \mu_n) \in S_2 .$$

We are now ready for the main result of this section.

4.6. Theorem- Let $\{A_1, \ldots, A_n\}$ (resp. $\{B_1, \ldots, B_n\}$) be a mutually commuting set of operators in $\mathcal{X}$. For each j, $(j=1, \ldots, n)$, define a bounded linear operator F_j on $\mathcal{X}$ by

$$F_j(X) = A_j X - X B_j \qquad (j=1,\ldots,n;\ X \in \mathcal{X}).$$

Then

(i)- The joint left spectrum of $F_1, \ldots, F_n$ is non-empty, and

$$J\sigma_l(F_1, \ldots, F_n) \subseteq J\sigma_{ap}(A_1, \ldots, A_n) - J\sigma_{ap}(B_1^*, \ldots, B_n^*) .$$

(ii)- The joint right spectrum of $F_1, \ldots, F_n$ is non-empty, and

$$J\sigma_r(F_1, \ldots, F_n) \subseteq J\sigma_{ap}(A_1^*, \ldots, A_n^*) - J\sigma_{ap}(B_1, \ldots, B_n) .$$

(iii)- The joint spectrum of $F_1, \ldots, F_n$ is non-empty, and

$$J\sigma(F_1, \ldots, F_n) \subseteq J\sigma(A_1, \ldots, A_n) - J\sigma(B_1, \ldots, B_n) .$$

Proof- (i). For each j $(j=1,\ldots,n)$ let F_j' and F_j'' be defined as in lemma 4.3.; since $F_1, \ldots, F_n$ mutually commute, their joint left spectrum is non-empty; let

$$(\lambda_1, \ldots, \lambda_n) \in J\sigma_l(F_1, \ldots, F_n) .$$

By lemma 4.4. and theorem 4.5., there exists a 2n-tuple

$$(\lambda_1', \ldots, \lambda_n', \lambda_1'', \ldots, \lambda_n'')$$

of complex numbers

$$(\lambda_1', \ldots, \lambda_n', \lambda_1'', \ldots, \lambda_n'')$$

such that

$$(\lambda_1, \ldots, \lambda_n, \lambda_1', \ldots, \lambda_n', \lambda_1'', \ldots, \lambda_n'') \in$$
$$J\sigma_l(F_1, \ldots, F_n, F_1', \ldots, F_n', F_1'', \ldots, F_n''). \qquad (1)$$

Let $\mathfrak{F}$ be the commutative unital Banach algebra generated by the set

$$\{F_1, \ldots, F_n, F_1', \ldots, F_n', F_1'', \ldots, F_n''\}$$

and the identity operator on $\mathfrak{X}$.

By (1) and definition 4.1.(a), we have

$$\sum_{j=1}^{n} \mathfrak{F}(F_j - \lambda_j) + \sum_{j=1}^{n} \mathfrak{F}(F_j' - \lambda_j') + \sum_{j=1}^{n} \mathfrak{F}(F_j'' - \lambda_j'') \neq \mathfrak{F}.$$

By the Gelfand theory of commutative Banach algebras [6; Ch.2], there exists a maximal ideal $\mathcal{M}$ of $\mathfrak{F}$, and a corresponding non-zero multiplicative linear functional φ on $\mathfrak{F}$, such that

$$\sum_{j=1}^{n} \mathfrak{F}(F_j - \lambda_j) + \sum_{j=1}^{n} \mathfrak{F}(F_j' - \lambda_j') + \sum_{j=1}^{n} \mathfrak{F}(F_j'' - \lambda_j'') \subseteq \mathcal{M}$$

and $\mathcal{M} = \ker \varphi$.

In particular,

$$\varphi(F_j) = \lambda_j \qquad (j = 1, \ldots, n),$$
$$\varphi(F_j') = \lambda_j' \qquad (j = 1, \ldots, n),$$
$$\varphi(F_j'') = \lambda_j'' \qquad (j = 1, \ldots, n),$$

so that

$$\begin{aligned} \lambda_j &= \varphi(F_j) \\ &= \varphi(F_j') - \varphi(F_j'') \\ &= \lambda_j' - \lambda_j'' \end{aligned} \qquad (j = 1, \ldots, n) \quad (2)$$

Moreover, since

$$\sum_{j=1}^{n} \mathfrak{F}(F_j' - \lambda_j') \subseteq \ker \varphi,$$

we have

$$(\lambda_1', \ldots, \lambda_n') \in \mathcal{J}\sigma_l(F_1', \ldots, F_n')$$
$$\subseteq \mathcal{J}\sigma_{ap}(A_1, \ldots, A_n), \tag{3}$$

by lemma 4.3.(i).

Similarly,

$$(\lambda_1'', \ldots, \lambda_n'') \in \mathcal{J}\sigma_{ap}(B_1^*, \ldots, B_n^*) \tag{4}$$

Part (i) now follows from (2), (3), and (4).

A similar analysis establishes part(ii).

Finally, by parts(i) and (ii), and definition 4.1.(d), we have

$$\begin{aligned}&\mathcal{J}\sigma(F_1, \ldots, F_n) \subseteq \\ &(\mathcal{J}\sigma_{ap}(A_1, \ldots, A_n) - \mathcal{J}\sigma_{ap}(B_1^*, \ldots, B_n^*)) \cup \\ &(\mathcal{J}\sigma_{ap}(A_1^*, \ldots, A_n^*) - \mathcal{J}\sigma_{ap}(B_1, \ldots, B_n)) \subseteq \\ &\mathcal{J}\sigma(A_1, \ldots, A_n) - \mathcal{J}\sigma(B_1, \ldots, B_n)\end{aligned}$$

since each one of the sets

$$\mathcal{J}\sigma_{ap}(A_1, \ldots, A_n) - \mathcal{J}\sigma_{ap}(B_1^*, \ldots, B_n^*)$$

and

$$\mathcal{J}\sigma_{ap}(A_1^*, \ldots, A_n^*) - \mathcal{J}\sigma_{ap}(B_1, \ldots, B_n)$$

is contained in

$$\mathcal{J}\sigma(A_1, \ldots, A_n) - \mathcal{J}\sigma(B_1, \ldots, B_n).$$

This completes the proof.

4.6.1. Corollary- (Rosenblum's theorem)- Let A and B be bounded linear operators defined on a Hilbert space $\mathcal{H}$, and let F be the operator defined on $\mathcal{BL}(\mathcal{H})$ by

$$F(X) = AX - XB \qquad (X \in \mathcal{BL}(\mathcal{H})).$$

Then

$$\sigma(F) \subseteq \sigma(A) - \sigma(B)$$

Proof- Take $j = 1$ in 4.6.(iii)

Corollary 4.6.1. was first proved by Rosenblum in[46] in the case where A and B are elements of a Banach algebra; his proof involves computing an integral formula for the resolvent of F. A more elementary proof is given in [42; corollary 0.13], where a number of applications of corollary 4.6.1. are also given.

4.6.2. Corollary- Let A and B be bounded linear operators on a Hilbert space $\mathcal{H}$,

(a)- If

$$\sigma_{ap}(A^*) \cap \sigma_{ap}(B) = \emptyset$$

then, given Y in $\mathcal{B}$, there exists X in $\mathcal{B}$ such that

$$AX - XB = Y$$

(b) (Rosenblum's corollary)- If

$$\sigma(A) \cap \sigma(B) = \emptyset$$

then, given Y in $\mathcal{B}$, there exists a unique X in $\mathcal{B}$ such that

$$AX - XB = Y$$

Proof- (a) follows from theorem 4.6. by taking $j = 1$ in 4.6.(ii) and (b) follows from corollary 4.6.1 (since F is now invertible).

This completes the proof.

§5. Compactness of $J\sigma_{ap}(a_1, \dots, a_n)$.

Let $a_1, \ldots, a_n$ be elements of a C^*-algebra $\mathcal{A}$. If the joint approximate point spectrum of $a_1, \ldots, a_n$ is non-empty (in particular, if $a_1, \ldots, a_n$ mutually commute) then, an argument similar to that used in the proof of corollary 2.2.1. of chapter I shows that the joint approximate point spectrum of $a_1, \ldots, a_n$ is, in fact, compact.

The purpose of this section is to give a characterization of the joint approximate point spectrum and to show, as a result, that $J\sigma_{ap}(a_1, \ldots, a_n)$ is homeomorphic to a space consisting of equivalence classes of a certain subset of $E(\mathcal{A})$. The result on the compactness of $J\sigma_{ap}(a_1, \ldots, a_n)$ will then follow as a corollary.

For the rest of this section, S will denote a fixed set of elements $\{a_1, \ldots, a_n\}$ of a C^*-algebra $\mathcal{A}$, whose joint approximate point spectrum will be assumed to be non-empty.

A subset E of $E(\mathcal{A})$ is said to have property $P(S)$ if and only if each element of E is left multiplicative with respect to a_j $(j=1, \ldots, n)$.

A state f is said to have property $P(S)$ if and only if the set $\{f\}$ has property $P(S)$.

Let

$$\mathcal{X} = \{ f \in E(\mathcal{A}) : f \text{ has property } P(S) \}$$

Define a mapping $\theta : \mathcal{X} \to J\sigma_{ap}(a_1, \ldots, a_n)$ by

$$\theta(f) = (f(a_1), \ldots, f(a_n))$$

By theorem 1.4.(iv), the map θ is surjective.

The sets $\mathcal{X}$ and $J\sigma_{ap}(a_1, \ldots, a_n)$ are given the relative w^*-topology and the usual topology of $\mathbb{C}^n$, respectively.

Recall that a mapping τ of a topological space $\mathfrak{X}$ into a topological space $\mathcal{Y}$ is said to be a closed map provided that τ maps closed sets of $\mathfrak{X}$ into closed sets of $\mathcal{Y}$. It is easily seen that τ is a closed map if and only if for each subset X of $\mathfrak{X}$,

$$\overline{\tau(X)} \subseteq \tau(\overline{X})$$

where $-$ denotes the respective closure operations.

5.1. Proposition- The map $\theta : \mathfrak{X} \to \mathcal{J}\sigma_{ap}(a_1,\dots,a_n)$ defined by

$$\theta(f) = (f(a_1),\dots,f(a_n))$$

is a continuous, closed, surjective map.

In particular, θ is a quotient map.

Remark- Given two topological spaces $\mathfrak{X}$ and $\mathcal{Y}$, and a mapping $\tau : \mathfrak{X} \to \mathcal{Y}$, we say that τ is a quotient map, provided that τ is surjective, and that a subset $\mathcal{U}$ of $\mathcal{Y}$ is open if and only if $\tau^{-1}(\mathcal{U})$ is open in $\mathfrak{X}$.

Proof- Let

$$(\lambda_1,\dots,\lambda_n) \in \mathcal{J}\sigma_{ap}(a_1,\dots,a_n),$$

and, with ε an arbitrary positive number, let

$$V = \{(\Gamma_1,\dots,\Gamma_n) \in \mathcal{J}\sigma_{ap}(a_1,\dots,a_n) : |\Gamma_j - \lambda_j| \le \epsilon\}$$

be an arbitrary neighbourhood of $(\lambda_1,\dots,\lambda_n)$ in $\mathcal{J}\sigma_{ap}(a_1,\dots,a_n)$.

By theorem 1.4.(iv), there exists an element f_0 of $\mathfrak{X}$ such that

$$(\lambda_1,\dots,\lambda_n) = (f_0(a_1),\dots,f_0(a_n))$$

Let U_{f_0} be the neighbourhood of f_0 in $\mathfrak{X}$ defined by

$$U_{f_0} = \{f \in \mathfrak{X} : |f(a_j) - f_0(a_j)| \le \varepsilon \quad (j=1,\dots,n)\}$$

It is then easily verified that

$$f \in U_{f_0} \implies \theta(f) \in V.$$

Hence θ is continuous.

Next, we prove that θ is closed.

By the remark immediately preceding the proposition, it is sufficient to show that

$$\overline{\theta(X)} \subseteq \theta(\overline{X}^{\omega^*})$$

where X is an arbitrary subset of $\mathfrak{X}$.

Let

$$\{\underline{f}_k(\underline{a})\}_{k=1}^{\infty} = \{(f_k(a_1), \dots, f_k(a_n))\}_{k=1}^{\infty}$$

be a sequence of elements of $\theta(X)$, where

$$f_k \in X \qquad (k = 1, 2, \dots)$$

and suppose that

$$\underline{f}_k(\underline{a}) \longrightarrow (\lambda_1, \dots, \lambda_n)$$

as $k \to \infty$. Then

$$f_k(a_j) \longrightarrow \lambda_j \qquad (j = 1, \dots, n)$$

as $k \to \infty$.

Hence, since for each k, the state f_k is left-multiplicative with respect to a_j $(j=1,\dots,n)$, it follows that

$$f_k((a_j - \lambda_j)^*(a_j - \lambda_j)) \to 0 \qquad (j = 1, \dots, n) \tag{1}$$

as $k \to \infty$.

Partially order the set $\mathfrak{F}$ of all finite subsets of the set of all positive integers $\mathbb{N}^+$ with respect to inclusion; for each $F \in \mathfrak{F}$ let

$$m_F = \sup\{m : m \in F\}$$

By (1), corresponding to each element F of $\mathfrak{F}$, there exists an element f_{k_0} of $\{f_k\}_{k=1}^{\infty}$ such that

$$\left| f_{k_0}((a_j-\lambda_j)^*(a_j-\lambda_j)) \right| \leq \frac{1}{m_F} \qquad (j=1,\dots,n).$$

Thus, letting $f_F = f_{k_0}$,we get a well-defined net $\{ f_F : F \in \mathfrak{F} \}$ of elements of $\mathfrak{F}$. By the w^*-compactness of $E(\mathcal{A})$, the net $\{f_F : f \in \mathfrak{F}\}$ has a w^*-limit point f, say; then, $f \in \overline{X}^{\omega^*}$. Further, a similar calculation as that given in the proof of theorem 1.4.(c) shows that, for each j $(j=1,\dots,n)$, f is left-multiplicative with respect to a_j with $f(a_j)=\lambda_j$.

Hence

$$(\lambda_1,\dots,\lambda_n) = (f(a_1),\dots,f(a_n))$$
$$= \theta(f).$$

This proves (1) and completes the proof of the theorem.

Let $\mathfrak{X}$ be as before. Define a relation $\sim$ on $\mathfrak{X}$ as follows: Given f and g in $\mathfrak{X}$, let

$$f \sim g$$

if and only if

$$f(a_j) = g(a_j) \qquad (j=1,\dots,n).$$

It is easily seen that $\sim$ is an equivalence relation on $\mathfrak{X}$. Let $\tilde{\mathfrak{X}}$ denote the corresponding set of equivalence classes, and let $p: \mathfrak{X} \to \tilde{\mathfrak{X}}$ be the corresponding quotient map, which sends each element of $\mathfrak{X}$ to its equivalence class.

Clearly, we have

$$\tilde{\mathfrak{X}} = \{ \theta^{-1}(\{\underline{\lambda}\}) : \underline{\lambda} \in J\sigma_{ap}(a_1,\dots,a_n) \}.$$

5.2. Theorem- The map θ induces a homeomorphism

$$\tau : \tilde{\mathfrak{X}} \longrightarrow \mathcal{J}\sigma_{ap}(a_1, \dots, a_n)$$

where $\tilde{\mathfrak{X}}$ is given the quotient topology.

Remark- The situation may be described by the following commutative diagram:

$$\begin{tikzcd} \mathfrak{X} \arrow[d, "p"'] \arrow[dr, "\theta"] & \\ \tilde{\mathfrak{X}} \arrow[r, "\tau"'] & \mathcal{J}\sigma_{ap}(a_1, \dots, a_n) \end{tikzcd}$$

Proof- We define τ as follows: Let $\tilde{f}$ be an arbitrary element of $\tilde{\mathfrak{X}}$; then, the set

$$\theta(p^{-1}\{\tilde{f}\})$$

is a one point set in $\mathcal{J}\sigma_{ap}(a_1, \dots, a_n)$. If we let $\tau(\tilde{f})$ denote this point, then we have defined a map

$$\tau : \tilde{\mathfrak{X}} \longrightarrow \mathcal{J}\sigma_{ap}(a_1, \dots, a_n)$$

such that for each f in $\mathfrak{X}$, we have

$$\theta(f) = \tau(p(f)) .$$

This completes the definition of τ .

Since, by proposition 5.1., θ is a quotient map, a standard topological argument shows that τ is a homeomorphism (see [38; Ch. 2, Th. 11.2, for example).

This completes the proof.

5.3. Theorem- The topological spaces $\mathfrak{X}$, $\tilde{\mathfrak{X}}$, and $\mathcal{J}\sigma_{ap}(a_1, \dots, a_n)$ are compact.

Proof- To prove that $\mathfrak{X}$ is w^*-compact, it is sufficient to show that it is w^*-closed.

Let $\{f_\alpha\}$ be a net of elements of $\mathfrak{X}$, and suppose that

$$f_\alpha \to f \qquad (w^*\text{-topology})$$

For each fixed element x in $\mathcal{A}$, and each fixed j in $\{1, \ldots, n\}$, we have

$$f_\alpha(x a_j) \to f(x a_j),$$

and

$$f_\alpha(x) \to f(x),$$

and

$$f_\alpha(a_j) \to f(a_j).$$

Since each f_α is left-multiplicative with respect to a_j we have

$$f_\alpha(x a_j) = f_\alpha(x) f_\alpha(a_j).$$

Hence

$$f(x a) = f(x) f(a)$$

i.e., $f \in \mathfrak{X}$.

Thus, $\mathfrak{X}$ is w^*-compact.

Next, since θ is a closed map, we have

$$\begin{aligned} \overline{\theta(\mathfrak{X})} &\subseteq \theta(\overline{\mathfrak{X}}^{w^*}) \\ &= \theta(\mathfrak{X}) \\ &\subseteq \overline{\theta(\mathfrak{X})}. \end{aligned}$$

Hence, since

$$\theta(\mathfrak{X}) = J\sigma_{ap}(a_1, \ldots, a_n),$$

it follows that $J\sigma_{ap}(a_1, \ldots, a_n)$ is closed.

Hence, since $J\sigma_{ap}(a_1, \ldots, a_n)$ is a bounded subset of $\mathbb{C}^n$ (it is contained in the polydisc

$$\{(\mu_1, \ldots, \mu_n) \in \mathbb{C}^n : |\mu_j| \leq \|a_j\| \quad (j = 1, \ldots, n)\}$$

for example) it follows that $J\sigma_{ap}(a_1,\ldots,a_n)$ is compact.

Finally, since $\tilde{\mathfrak{X}}$ is homeomorphic to the compact space $J\sigma_{ap}(a_1,\cdots,a_n)$ it follows that $\tilde{\mathfrak{X}}$ is compact.

This completes the proof.

5.4. Remark- Let $S=\{a_1,\ldots,a_n\}$, and let $\underline{\lambda}=(\lambda_1,\ldots,\lambda_n)$ be a fixed point in $J\sigma_{ap}(a_1,\ldots,a_n)$; we say that a state f has property $P_{\underline{\lambda}}(S)$ provided that

$$f(xa_j)=f(x)f(a_j)=\lambda_j f(x) \qquad (x\in\mathcal{A};\ j=1,\ldots,n)$$

For each $\underline{\lambda}$ in $J\sigma_{ap}(a_1,\ldots,a_n)$, let

$$E_{\underline{\lambda}}=\{f\in E(\mathcal{A}) : f \text{ has property } P_{\underline{\lambda}}(S)\}.$$

It is then easily verified that

$$\tilde{\mathfrak{X}}=\{E_{\underline{\lambda}} : \underline{\lambda}\in J\sigma_{ap}(a_1,\cdots,a_n)\}$$

The sets $E_{\underline{\lambda}}$ correspond to generalized versions of the maximal faces $F_{A,\lambda}$ considered in §5 of chapter I .

§6- Finite V.N. algebras-

Let $\mathcal{A}$ be a V.N.algebra acting on a Hilbert space $\mathcal{H}$. If A is an element of $\mathcal{A}$ then, in general, $\sigma(A)\neq\sigma_{ap}(A)$. In the case where $\mathcal{A}$ is finite V.N.algebra, strong relations exist between the various spectra, which it is the purpose of this section to develop . We shall also consider the case of certain non-commutative C^*-algebras.

Let $\mathcal{A}$ be a V.N.algebra, and let P and Q be projections in $\mathcal{A}$

(i)- P is said to be equivalent to Q if and only if, there exists a partial isometry V in $\mathcal{A}$ such that

$$VV^*=P \quad \& \quad V^*V=Q$$

The equivalence of P and Q is denoted by $P \sim Q$.

(ii)- P is said to be weaker than Q if and only if, there exists a projection R in $\mathcal{A}$ such that

$$P \sim R \leq Q.$$

This is denoted by $P \precsim Q$.

A projection P in $\mathcal{A}$ is said to be finite if and only if, there does not exist a proper subprojection of P in $\mathcal{A}$ which is equivalent to P, i.e.,

$$\nexists Q \in \mathcal{A} \text{ s.t. } P \sim Q < P.$$

A V.N. algebra is said to be finite if and only if, the identity of $\mathcal{A}$ is a finite projection. This is equivalent to the following condition:

Whenever $A \in \mathcal{A}$ satiafies $A^*A = I$, then $AA^* = I$; c.f.[21 ; Ch.3, §8, Th.1.].

The following lemma must be well-known, but we can find no reference for it, and therefore include a complete proof.

6.1. Lemma- Let A be a bounded linear operator on a Hilbert space $\mathcal{H}$, and let $\lambda \in \sigma(A)$.

If $\lambda \notin \sigma_{ap}(A)$, then $\bar{\lambda} \in \sigma_p(A^*)$.

Proof- Suppose $\lambda \notin \sigma_{ap}(A)$; then, $A-\lambda$ is left invertible, hence it is bounded below. If $\bar{\lambda} \notin \sigma_p(A^*)$, then $A^*-\bar{\lambda}$ is injective so, since

$$\{\ker (A^*-\bar{\lambda})\}^{\perp} = \{\text{range } (A-\lambda)\}^{-},$$

it follows that $A-\lambda$ has dense range.

On the other hand, an operator which is bounded below and has dense range is invertible.

This contradicts $\lambda \in \sigma(A)$.

Hence $\bar{\lambda} \in \sigma_p(A^*)$.

6.2. Proposition- Let $\mathcal{A}$ be a finite V.N. algebra, let $A \in \mathcal{A}$, and let $\lambda \in \mathbb{C}$. Then

(i)- $\lambda \in \sigma_p(A)$ if and only if $\bar{\lambda} \in \sigma_p(A^*)$.

(ii)- $\lambda \in \sigma_{ap}(A)$ if and only if $\bar{\lambda} \in \sigma_{ap}(A^*)$.

(iii)- $\sigma(A) = \sigma_{ap}(A)$.

Proof- let

$$A-\lambda = U((A-\lambda)^*(A-\lambda))^{\frac{1}{2}}$$

be the polar decomposition of $A-\lambda$. Let

$$Q = UU^*, \quad P = U^*U.$$

It is well-known that

$$Q = \operatorname{supp}(A-\lambda),$$

and that

$$P = \operatorname{supp}(A-\lambda)^*.$$

Moreover

$$P \sim Q.$$

By theorems 4.5.2. and 4.5.2.3. of chapter I,

$$\lambda \in \sigma_p(A) \text{ if and only if } P \neq I$$

and

$$\lambda \in \sigma_p(A^*) \text{ if and only if } Q \neq I.$$

Since $\mathcal{A}$ is finite, we have

$$P \neq I \quad \text{if and only if } Q \neq I.$$

This proves (i).

To prove (ii), let $\lambda \in \sigma_{ap}(A)$; then, $\bar{\lambda} \in \sigma(A^*)$, hence

By lemma 6.1., either $\bar{\lambda} \in \sigma_{ap}(A^*)$ or $\lambda \in \sigma_p(A)$.

The result now follows from(i) .

Finally, let λ be an arbitrary point of $\sigma(A)$. By lemma 6.1., either $\lambda \in \sigma_{ap}(A)$ or $\bar{\lambda} \in \sigma_p(A^*)$; hence, by part (i), either $\lambda \in \sigma_{ap}(A)$, or $\lambda \in \sigma_p(A)$; hence

$$\sigma(A) \subseteq \sigma_{ap}(A) .$$

Since the reverse inclusion is obvious, this establishes (iii). This completes the proof.

6.2.1. Remark- Proposition 6.2.(iii) is also a consequence of the fact that in a finite V.N.algebra $\mathcal{A}$, the set of invertible operators is dense in $\mathcal{A}$; c.f. [15].

Suppose now that $\mathcal{A}$ is not finite; take an element A in $\mathcal{A}$ such that

$$A^*A = 1 \ \& \ AA^* < 1 .$$

It is easily seen that $0 \in \sigma(A)$ whereas $0 \notin \sigma_{ap}(A)$ since A is isometric Hence (c.f., [16; theorem 3]),

6.3. Theorem- A V.N. algebra $\mathcal{A}$ is finite if and only if

$$\sigma(A) = \sigma_{ap}(A)$$

for all A in $\mathcal{A}$.

Proof- Proposition 6.2.(iii) and the above example.

Let $\mathcal{A}$ be a commutative C^*-algebra. By proposition 4.2.(ii), we have

$$(\lambda_1, \dots, \lambda_n) \in J\sigma_{ap}(a_1, \dots, a_n)$$

if and only if

$$(\bar{\lambda}_1, \ldots, \bar{\lambda}_n) \in J\sigma_{ap}(A_1^*, \ldots, A_n^*)$$

The rest of this section is devoted to the study of the above result in more general (non-commutative) C^*-algebras.

Let $A_1, \ldots, A_n$ be bounded linear operators on a Hilbert space $\mathcal{H}$.

The joint point spectrum of $A_1, \ldots, A_n$ is defined to be the set of n-tuples of complex numbers $(\lambda_1, \ldots, \lambda_n)$ such that the following condition is satiafied:

$$\exists x \in \mathcal{H} \text{ s.t. } x \neq 0 \;\&\; A_j x = \lambda_j x \qquad (j=1,\ldots,n).$$

The joint point spectrum is denoted by $J\sigma_p(A_1, \ldots, A_n)$.

Clearly

$$J\sigma_p(A_1, \ldots, A_n) \subseteq J\sigma_{ap}(A_1, \ldots, A_n).$$

6.4. Proposition- Let $\mathcal{A}$ be a C^*-algebra acting on a finite-dimentional Hilbert space $\mathcal{H}$, and let $A_1, \ldots, A_n$ be elements of $\mathcal{A}$. Then

$$J\sigma_p(A_1, \ldots, A_n) = J\sigma_{ap}(A_1, \ldots, A_n)$$

Proof- It is sufficient to show that

$$J\sigma_p(A_1, \ldots, A_n) \supseteq J\sigma_{ap}(A_1, \ldots, A_n).$$

Let

$$(\lambda_1, \ldots, \lambda_n) \in J\sigma_{ap}(A_1, \ldots, A_n).$$

Let ε_n be a sequence of positive nubers decreasing to zero. By theorem 1.4.(c), there exists a sequence $\{x_k\}_{k=1}^{\infty}$ in $\mathcal{H}$ such that $\|x_k\|=1$ and

$$\|(A_j - \lambda_j)x_k\| \leq \varepsilon_k \qquad (j=1,\ldots,n;\; k=1,2,\ldots).$$

Since $\mathcal{H}$ is finite-dimensional, the unit ball of $\mathcal{H}$ is compact; hence, there exists a subsequence $\{x_{k_m}\}_{m=1}^{\infty}$ of $\{x_k\}$ and an element $x \in \mathcal{H}$ with $\|x\|=1$ such that

$$x_{k_m} \to x \qquad \text{as} \quad m \to \infty .$$

Hence

$$\|(A_j - \lambda_j)x\| = 0 \qquad\qquad (j=1,\ldots,n),$$

i.e.,

$$(\lambda_1, \ldots, \lambda_n) \in J\sigma_p(A_1, \ldots, A_n).$$

This completes the proof.

For the rest of this section, the following convention will be adopted:

If S is a subset of $\mathbb{C}^n$, the symbol $\bar{S}$ denotes the set

$$\{(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n : (\bar{\lambda}_1, \ldots, \bar{\lambda}_n) \in S\}.$$

6.5. Theorem- Let $\mathcal{A}$ be a C^*-algebra acting on a finite-dimensional Hilbert space, let $\{a_1, \ldots, a_n\}$ be a commuting set of elements of $\mathcal{A}$, and let $(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$. Then

$$(\lambda_1, \ldots, \lambda_n) \in J\sigma_{ap}(a_1, \ldots, a_n)$$

if and only if

$$(\bar{\lambda}_1, \ldots, \bar{\lambda}_n) \in J\sigma_{ap}(a_1^*, \ldots, a_n^*) .$$

We defer the proof to appendix I .

6.5.1. Remarks -

(a)- With the assumptions of theorem 6.5., we get

$$J\sigma(a_1, \ldots, a_n) = J\sigma_{ap}(a_1, \ldots, a_n)$$

(b)- The conclusion of theorem 6.5. is false if the assumption of commutativity is dropped.

To see this, let $\mathcal{A} = M_2(\mathbb{C})$, and let

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Then, P and Q are equivalent projections, so there exists a partial isometry V in $M_2(\mathbb{C})$ such that

$$V^*V = P \quad \& \quad VV^* = Q.$$

It is then easily verified that

$$\mathcal{A}V + \mathcal{A}Q = \mathcal{A}$$

whereas

$$V\mathcal{A} + Q\mathcal{A} \neq \mathcal{A}$$

i.e.,

$$(0,0) \notin J\sigma_{ap}(V, Q)$$

and

$$(0,0) \in J\sigma_{ap}(V^*, Q^*).$$

Let $\mathcal{A}$ be a C^*-algebra, and let π be an irreducible representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$. We say that π is finite-dimensional provided that $\mathcal{H}$ is a finite- dimensional Hilbert space.

6.6. Theorem- Let $\mathcal{A}$ be a C^*-algebra such that every irreducible representation of $\mathcal{A}$ is finite-dimentsional, and let $\{a_1, \dots, a_n\}$ be a mutually commuting set of elements in $\mathcal{A}$. Then

$$J\sigma_{ap}(a_1, \dots, a_n) = \overline{J\sigma_{ap}(a_1^*, \dots, a_n^*)}$$

In particular

$$J\sigma(a_1, \dots, a_n) = J\sigma_{ap}(a_1, \dots, a_n)$$

Proof- Let

$$(\lambda_1, \ldots, \lambda_n) \in J\sigma_{ap}(a_1, \ldots, a_n).$$

By theorem 1.4., there exists a pure state ρ such that

$$\rho((a_j-\lambda_j)^*(a_j-\lambda_j)) = 0 \qquad (j=1,\ldots,n). \quad (1)$$

Let Π_ρ, ξ_ρ ,and $\mathcal{H}_\rho$ be the associated irreducible representation, cyclic vector, and Hilbert space, respectively; then, for each j $(j=1,\ldots,n)$, we have

$$\begin{aligned} \| \Pi_\rho(a_j-\lambda_j)\xi_\rho \|^2 &= \langle \Pi_\rho((a_j-\lambda_j)^*(a_j-\lambda_j))\xi_\rho , \xi_\rho \rangle \\ &= \rho((a_j-\lambda_j)^*(a_j-\lambda_j)) \\ &= 0 \end{aligned}$$

by (1) , i.e.,

$$(\lambda_1, \ldots, \lambda_n) \in J\sigma_p(\Pi(a_1), \ldots, \Pi(a_n)).$$

Hence, by theorem 6.5.,

$$(\bar\lambda_1, \ldots, \bar\lambda_n) \in J\sigma_{ap}(\Pi(a_1)^*, \ldots, \Pi(a_n)^*).$$

Therefore, by theorem 1.4., there exists a pure state ρ' of $\Pi(\mathcal{A})$ such that

$$\rho'(A\,\Pi(a_j)^*) = \bar\lambda_j\, \rho'(A) \qquad (\forall A \in \Pi(\mathcal{A});\ j=1,\ldots,n)$$

Define g on $\mathcal{A}$ by

$$g(x) = (\rho' \circ \Pi)(x) \qquad (\forall x \in \mathcal{A}).$$

A similar calculation as that given in proposition 3.1. of chapter I shows that g is left-multiplicative with respect to each a_j^* with

$$g(a_j^*) = \bar\lambda_j \qquad (j=1,\ldots,n).$$

Hence

$$(\bar\lambda_1, \ldots, \bar\lambda_n) \in J\sigma_{ap}(a_1^*, \ldots, a_n^*).$$

The converse may be proved similarly.

Finally, the last part follows from the first part and definition 4.1.(d).

This completes the proof.

6.6.1. Corollary- Let $\mathcal{A}$ be a C^*-algebra such that every irreducible representation of $\mathcal{A}$ is finite-dimensional. Then

$$\sigma(a) = \sigma_{ap}(a) = \overline{\sigma_{ap}(a^*)} \qquad (\forall a \in \mathcal{A}).$$

6.6.2 Remarks- (a). Examples of C^*-algebras all of whose representations are finite-dimensional include the class of n-homogeneous C^*-algebras and, in particular, C^*-algebras of the form

$$\mathcal{A} = \mathcal{C}_{\mathbb{C}}(X) \otimes M_n(\mathbb{C})$$

where X is a compact Hausdorff space.

(b). In the case where $\mathcal{A}$ is a V.N.algebra, the condition

$$\sigma(a) = \sigma_{ap}(a) \qquad (\forall a \in \mathcal{A})$$

is equivalent to the condition:

$\mathcal{A}$ has a uniformly dense invertible group.

(c.f.,[15; theorem 5],and theorem 6.3.).

On the other hand, this is no longer the case for arbitrary C^*-algebras; in fact, with

$$\mathcal{A} = \mathcal{C}_{\mathbb{C}}(X) \otimes M_n(\mathbb{C})$$

corollary 6.6.1. and remark (a) above imply that

$$\sigma(a) = \sigma_{ap}(a) \qquad (\forall a \in \mathcal{A}).$$

However, it may be shown that $\mathcal{A}$ has uniformly dense invertible group if and only if, the topological dimension of $\hat{\mathcal{A}}$ is at most 1 ($\hat{\mathcal{A}}$ denotes the set of unitary equivalence classes of irreducible representations of $\mathcal{A}$, with the Jacobson topology; it may be identified with X) ; c.f. [44; proposition 2].

Nevertheless, condition (1) above entails analagous results to those of proposition 5 of [44] , even in the absence of the uniform density of the invertible group; for example:

Let $\mathcal{A}$ be a C^*-algebra such that

$$\sigma(a) = \sigma_{ap}(a) \qquad (\forall a \in \mathcal{A}).$$

Let $x \in \mathcal{A}$.

The following conditions are equivalent:

(i)- x is invertible .

(ii)- $f(x^*x) > 0$ for each state f of $\mathcal{A}$.

(iii)- $\rho(x^*x) > 0$ for each pure state ρ of $\mathcal{A}$.

Proof- If , for a state (resp. pure state) f , we have

$$f(x^*x) = 0$$

then, by proposition 1.5., x is not invertible.

Conversely, if x is not invertible, then, since

$$\sigma(x) = \sigma_{ap}(x) ,$$

it follows that $0 \in \sigma_{ap}(x)$; hence, by proposition 1.5., there exists a state(resp. pure state) f such that $f(x^*x) = 0$.

This completes the proof.

We do not know whether the following equality (which holds true in any commutative C^*- algebra) is valid in all finite V.N. algebras:

$$J\sigma(a_1, \dots, a_n) = J\sigma_{ap}(a_1, \dots, a_n)$$

where $a_1, \ldots, a_n$ are pair-wise commuting elements of $\mathcal{A}$.

However, we have the following result:

6.7. Proposition- Let $\mathcal{A}$ be a finite V.N.algebra, let $a_1, \ldots, a_n$ be pair-wise commuting elements of $\mathcal{A}$, and let

$$(\lambda_1, \ldots, \lambda_n) \in J\sigma(a_1, \ldots, a_n) \tag{1}$$

Then, either

$$(\lambda_1, \ldots, \lambda_n) \in J\sigma_{ap}(a_1, \ldots, a_n) \tag{2}$$

or, there exists $(\mu_1, \ldots, \mu_n) \in \mathbb{C}^n$ such that

$$(\mu_1, \ldots, \mu_n) \in J\sigma_{ap}(a_1, \ldots, a_n)$$

and

$$\sum_{j=1}^{n} \mu_j = \sum_{j=1}^{n} \lambda_j .$$

Proof- Suppose that (1) holds and (2) does not hold. Then, by definition 4.1.(d),

$$(\bar{\lambda}_1, \ldots, \bar{\lambda}_n) \in J\sigma_{ap}(a_1^*, \ldots, a_n^*)$$

Hence, by proposition 3.1.(1),

$$\sum_{j=1}^{n} \bar{\lambda}_j \in \sigma_{ap}\Big(\sum_{j=1}^{n} a_j^*\Big),$$

so that, by proposition 6.2.(ii), we have

$$\sum_{j=1}^{n} \lambda_j \in \sigma_{ap}\Big(\sum_{j=1}^{n} a_j\Big)$$

The result now follows from proposition 3.1.(1).

We close this section with the following result concerning cohyponormal operators.

Recall that an operator a is said to be cohyponormal if it satisfies the following inequality:

$$aa^* \geq a^*a$$

6.8. Proposition- Let $a_1, \ldots, a_n$ be cohyponormal operators in a C^*-algebra $\mathcal{A}$. Then

$$J\sigma_{ap}(a_1, \ldots, a_n) = J\sigma(a_1, \ldots, a_n)$$

Proof- Let

$$(\mu_1, \ldots, \mu_n) \in J\sigma_{ap}(a_1^*, \ldots, a_n^*)$$

By proposition 5.1., there exists a state f of $\mathcal{A}$ such that

$$f\left(\sum_{j=1}^{n} (a_j - \bar{\mu}_j)(a_j^* - \mu_j)\right) = 0 .$$

Since

$$(a_j - \bar{\mu}_j)(a_j^* - \mu_j) \geqslant (a_j^* - \mu_j)(a_j - \bar{\mu}_j) \qquad (j = 1, \ldots, n),$$

it follows that

$$f\left(\sum_{j=1}^{n} (a_j^* - \mu_j)(a_j - \bar{\mu}_j)\right) = 0,$$

i.e.,

$$(\bar{\mu}_1, \ldots, \bar{\mu}_n) \in J\sigma_{ap}(a_1, \ldots, a_n)$$

by proposition 5.1. .

The result now follows from definition 4.1.(d) .

Chapter III

Spectral Mapping Theorem for Joint Approximate Point Spectrum

§1. The joint approximate point spectrum of infinite families of operators.

1.1. Introduction- The main purpose of this chapter is to prove a spectral mapping theorem for the joint approximate point spectrum of commuting elements of C^*- algebras. In the present section, the notion of the joint approximate point spectrum of finite families of operators is extended to that of the approximate point spectrum of any (not necessarily finite) family of operators. We shall also consider the questions of existence and compactness.

Throughout the rest of this chapter, the letter Γ denotes an indexing set which may or may not be infinite.

The following definition is given in [29] and [51] .

1.2. Definition- Let $\mathcal{A}$ be a C^*-algebra, let $(a_\tau)_{\tau\in\Gamma}$ be a Γ -tuple of elements of $\mathcal{A}$, and let $(\lambda_\tau)_{\tau\in\Gamma}$ be a Γ -tuple of complex numbers.

We say that $(\lambda_\tau)_{\tau\in\Gamma}$ is in the joint approximate point spectrum of $(a_\tau)_{\tau\in\Gamma}$ if and only if the left ideal of $\mathcal{A}$ generated by the set of elements

$$\{ a_\tau - \lambda_\tau \ : \ \tau\in\Gamma \}$$

is a proper left ideal of $\mathcal{A}$.

The joint approximate point spectrum of $(a_\tau)_{\tau\in\Gamma}$ will be denoted by $J\sigma_{ap}\,(a_\tau)_{\tau\in\Gamma}$

In what follows, the left ideal of $\mathcal{A}$ generated by a set of elements $\{a_\tau : \tau\in\Gamma\}$ will be denoted by $\sum_{\tau\in\Gamma} \mathcal{A} a_\tau$

1.3. Theorem- Let $\{a_\tau : \tau \in \Gamma\}$ be a family of elements of a C^*-algebra $\mathcal{A}$, and let $(\lambda_\tau)_{\tau\in\Gamma} \in \mathbb{C}^\Gamma$.

(a)- Let f be a state (respectively, pure state) of $\mathcal{A}$. The following conditions are equivalent:

(i)-

$$f(x a_\tau) = f(x) f(a_\tau) = \lambda_\tau f(x) \qquad (\forall x \in \mathcal{A}, \forall \tau \in \Gamma).$$

(ii)-

$$f(a_\tau) = \lambda_\tau \quad \& \quad f(a_\tau^* a_\tau) = |\lambda_\tau|^2 \qquad (\forall \tau \in \Gamma).$$

(iii)-

$$f((a_\tau - \lambda_\tau)^*(a_\tau - \lambda_\tau)) = 0 \qquad (\forall \tau \in \Gamma).$$

(b)- The following conditions are equivalent:

(i)-

$$(\lambda_\tau)_{\tau\in\Gamma} \in J\sigma_{ap}\,(a_\tau)_{\tau\in\Gamma}$$

(ii)- There exists a state f of $\mathcal{A}$ such that

$$f(x a_\tau) = f(x) f(a_\tau) = \lambda_\tau f(x) \qquad (\forall x \in \mathcal{A}, \forall \tau \in \Gamma)$$

(iii)- There exists a pure state ρ of $\mathcal{A}$ such that

$$\rho(x a_\tau) = \rho(x)\rho(a_\tau) = \lambda_\tau \rho(x) \qquad (\forall x \in \mathcal{A}, \forall \tau \in \Gamma)$$

(c)- A necessary and sufficient condition that

$$(\lambda_\tau)_{\tau\in\Gamma} \in J\sigma_{ap}\,(a_\tau)_{\tau\in\Gamma}$$

is that any one of the conditions a(i)- a(iii), b(i), b(ii), and b(iii) be satisfied.

Proof- (a)- An application of Cauchy-Schwartz inequality for positive linear functionals proves part (a); c.f. chapter II, Remark 1.4.1.(a).

(b)- Let b(i) be satisfied. Then, there exists a maximal left ideal $\mathcal{M}$ of $\mathcal{A}$ such that

$$\sum_{\tau\in\Gamma} \mathcal{A}(a_\tau-\lambda_\tau) \subseteq \mathcal{M} .$$

Let

$$\mathcal{M} = \{ x \in \mathcal{A} : \rho(x^* x) = 0 \} ,$$

where $\rho \in \mathcal{P}(\mathcal{A})$.

For each $\tau \in \Gamma$, we have

$$\mathcal{A}(a_\tau - \lambda_\tau) \subseteq \mathcal{M} ;$$

hence, as in the proof of Theorem 1.4.(a), (a(i) $\Longrightarrow$ a(ii)) of chapter II, we get

$$\rho((a_\tau-\lambda_\tau)^*(a_\tau-\lambda_\tau)) = 0 \qquad (\forall \tau\in\Gamma) .$$

Hence, by part (a),

$$\rho(x a_\tau) = \rho(x)\rho(a_\tau) = \lambda_\tau \rho(x) \qquad (\forall\tau\in\Gamma,\ \forall x\in\mathcal{A}).$$

Therefore, b(i) $\Longrightarrow$ b(iii)

Conversely, suppose b(iii) is satisfied, and let

$$\mathcal{M} = \{ x \in \mathcal{A} : \rho(x^* x) = 0 \} .$$

It is easily verified, as in Theorem 1.4. (a(v) $\Longrightarrow$ a(i)) of chapter II, that for each $\tau \in \Gamma$,

$$\mathcal{A}(a_\tau - \lambda_\tau) \subseteq \mathcal{M} ;$$

hence, since $\mathcal{M}$ is a left ideal, we get

$$\sum_{\tau\in\Gamma} \mathcal{A}(a_\tau - \lambda_\tau) \subseteq \mathcal{M} .$$

Therefore, since $\mathcal{M}$ is a proper left ideal, it follows that b(iii) $\Longrightarrow$ b(i).

The equivalence of b(ii) and b(iii) is proved in the same way as that of a(iv) and a(v) of Theorem 1.4. of chapter II.

This completes the proof of part (b).

Finally, part (c) follows from parts (a) and (b).

This completes the proof.

The following example shows that, in contrast to the case of a finite number of operators, condition b(i) of Theorem 1.4. of chapter II is not equivalent to any one of the conditions of the above theorem, if Γ is an infinite set.

1.4. Example- Let $\mathcal{H}$ be an infinite-dimensional Hilbert space, let A be a non-invertible positive operator on $\mathcal{H}$, and suppose that A has dense range. For each n (n=0,1,2,...) , let

$$A_n = (A)^{\frac{1}{2^n}}$$

Since $0 \in \sigma_{ap}(A)$, there exists a state f of $\mathcal{BL}(\mathcal{H})$ such that

$$f(XA) = f(X)f(A) = 0 \qquad (\forall X \in \mathcal{BL}(\mathcal{H})).$$

Hence (c.f. the remark immediately preceding proposition 1.6. of chapter II)

$$f(XA_n) = f(X)f(A_n) = 0 \qquad (\forall X \in \mathcal{BL}(\mathcal{H});\ n=0,1,2,\dots),$$

i.e.,

$$(0,0,\dots) \in J\sigma_{ap}(A_0, A_1, \dots).$$

On the other hand, suppose that the following were true:

$$\forall \epsilon > 0\ \exists x \in \mathcal{H} \text{ s.t. } x \neq 0\ \&\ \|A_n x\| \leq \epsilon \|x\| \qquad (n=0,1,2,\dots).\ (i)$$

Let Q be the projection onto the range of A . Then, for each $x \in \mathcal{H}$

$$A_n x \to Qx \qquad \text{as } n \to \infty$$

[53; §1, lemma 2]. However, since A has dense range, we have $Q = I$. Therefore, (i) is impossible.

Let a_1 and a_2 be operators in a C^*-algebra. It is a simple consequence of the definition of the joint approximate point spectrum that

$$(\lambda_1, \lambda_2) \in J\sigma_{ap}(a_1, a_2)$$

if and only if

$$(\lambda_2, \lambda_1) \in J\sigma_{ap}(a_2, a_1) .$$

The following proposition gives an analogous result for the joint approximate point spectrum of any family of operators.

1.5. Proposition- Let $\{a_\tau : \tau \in \Gamma\}$ be a family of elements of a C^*-algebra $\mathcal{A}$, let $(\lambda_\tau)_{\tau \in \Gamma} \in \mathbb{C}^\Gamma$, and let Θ be a bijective mapping of Γ onto itself. Then,

$$(\lambda_\tau)_{\tau \in \Gamma} \in J\sigma_{ap}(a_\tau)_{\tau \in \Gamma} \tag{1}$$

if and only if

$$(\lambda_{\Theta(\tau)})_{\tau \in \Gamma} \in J\sigma_{ap}(a_{\Theta(\tau)})_{\tau \in \Gamma} .$$

Proof- Suppose that (1) does not hold; then

$$\sum_{\tau \in \Gamma} \mathcal{A}(a_\tau - \lambda_\tau) = \mathcal{A} \ ;$$

hence, there exists a finite set of indeces $\tau_1, \tau_2, \ldots, \tau_n$ such that

$$\sum_{j=1}^{n} \mathcal{A}(a_{\tau_j} - \lambda_{\tau_j}) = \mathcal{A}$$

Since Θ is bijective, there exist elements $\tau_1', \tau_2', \ldots, \tau_n'$ in Γ such that

$$\tau_j = \Theta(\tau_j') \qquad (j = 1, 2, \ldots, n) .$$

Hence

$$\sum_{j=1}^{n} \mathcal{A}\,(a_{\Theta(\tau_j')} - \lambda_{\Theta(\tau_j')}) = \mathcal{A},$$

i.e.,

$$(\lambda_{\Theta(\tau)})_{\tau\in\Gamma} \notin J\sigma_{ap}\, (a_{\Theta(\tau)})_{\tau\in\Gamma} .$$

The converse is proved in a similar way.

This completes the proof.

The following proposition generalizes proposition 1.7. of chapter II to the case of infinite families of operators.

1.5. Proposition- Let $\{a_\tau : \tau\in\Gamma\}$ be a family of elements of a C^*-algebra $\mathcal{A}$, and let φ be a character on $C^*((a_\tau)_{\tau\in\Gamma})$. Then

$$(\varphi(a_\tau))_{\tau\in\Gamma} \in J\sigma_{ap}\ (a_\tau)_{\tau\in\Gamma}$$

Proof- Suppose not; then, there exists a finite set of indices $\tau_1, \tau_2, \dots, \tau_n$ in Γ such that

$$\sum_{j=1}^{n} \mathcal{A}\,(a_{\tau_j} - \varphi(a_{\tau_j})\,) = \mathcal{A}$$

Since the restriction of φ to $C^*(a_{\tau_1},\dots, a_{\tau_n})$ is a character on $C^*(a_{\tau_1},\dots, a_{\tau_n})$; this contradicts proposition 1.7. of chapter II.

This completes the proof.

§2. Existence and Compactness of $J\sigma_{ap}\ (a_\tau)_{\tau\in\Gamma}$.

2.1. Theorem- Let $\{a_\tau : \tau\in\Gamma\}$ be a family of elements of a C^*-algebra $\mathcal{A}$, and suppose that for each finite subfamily $\{a_\tau : \tau\in\Gamma_0\}$ we have

$$J\sigma_{ap}\ (a_\tau)_{\tau\in\Gamma_0} \neq \emptyset . \qquad (1)$$

Then

$$J\sigma_{ap}\ (a_\tau)_{\tau\in\Gamma} \neq \emptyset . \qquad (2)$$

In particular, the conclusion holds if the family $\{a_\tau : \tau \in \Gamma\}$ is mutually commuting.

Proof- The following proof is based on [10; proposition 5].

Partially order the set $\mathcal{F}$ of all finite subsets of Γ with respect to inclusion. By (1) and theorem 1.4. of chapter II, for each element Γ_0 of $\mathcal{F}$, there exists a state f_{Γ_0} of $\mathcal{A}$ such that

$$f_{\Gamma_0}(x a_\tau) = f_{\Gamma_0}(x) f_{\Gamma_0}(a_\tau) \qquad (\forall \tau \in \Gamma_0 ; \forall x \in \mathcal{A}) \quad (3)$$

By the compactness of $E(\mathcal{A})$, the net $\{f_{\Gamma_0} ; \Gamma_0 \in \mathcal{F}\}$ has a limit point f in $E(\mathcal{A})$; we claim that

$$f(x a_\tau) = f(x) f(a_\tau) \qquad (\forall \tau \in \Gamma ; \forall x \in \mathcal{A}) \quad (4)$$

For let τ_0 be an arbitrary element of Γ; for each element Γ_0 of $\mathcal{F}$ with $\Gamma_0 \supseteq \{\tau_0\}$ we have, by (3),

$$f_{\Gamma_0}(x a_{\tau_0}) = f_{\Gamma_0}(x) f_{\Gamma_0}(a_{\tau_0}) \qquad (\forall x \in \mathcal{A})$$

hence, for each arbitrary but fixed element x of $\mathcal{A}$, we have

$$f(x a_{\tau_0}) = f(x) f(a_{\tau_0})$$

Since τ_0 and x were arbitrary, this proves (4).

Hence, by theorem 1.3.,

$$(f(a_\tau))_{\tau \in \Gamma} \in J\sigma_{ap}(a_\tau)_{\tau \in \Gamma}$$

This completes the proof.

The following theorem is the main result needed in the next section where the spectral mapping theorem for the joint approximate point spectrum is proved.

The following convention will be adopted:

Let Γ (resp. Λ) be an indexing set, let $(a_\tau)_{\tau \in \Gamma}$ (resp. $(b_\nu)_{\nu \in \Lambda}$)

be a Γ-tuple (resp. Λ-tuple) of elements of $\mathcal{A}$, and let $(\lambda_\tau)_{\tau\in\Gamma}$ (resp. $(\mu_\nu)_{\nu\in\Lambda}$) be a Γ-tuple (resp. Λ-tuple) of complex numbers. The expression

$$\left((\lambda_\tau)_{\tau\in\Gamma}, (\mu_\nu)_{\nu\in\Lambda} \right) \in J\sigma_{ap}\left((a_\tau)_{\tau\in\Gamma}, (b_\nu)_{\nu\in\Lambda} \right) \qquad (1)$$

means that the left ideal of $\mathcal{A}$ generated by the set of elements

$$\left\{ (a_\tau - \lambda_\tau), (b_\nu - \mu_\nu) : \tau\in\Gamma, \nu\in\Lambda \right\}$$

is a proper left ideal of $\mathcal{A}$.

As in theorem 1.3., it is easily verified thst (1) is equivalent to the existence of a state f of $\mathcal{A}$ such that for each τ (resp. ν) f is left-multiplicative with respect to a_τ (resp. b_ν) with $f(a_\tau) = \lambda_\tau$ (resp. $f(b_\nu) = \mu_\nu$).

2.2. Theorem- Let $(a_\tau)_{\tau\in\Gamma}$ be a Γ-tuple of elements of $\mathcal{A}$, let $(b_\nu)_{\nu\in\Lambda}$ be a mutually commuting Λ-tuple of elements of $\mathcal{A}$, and suppose that

$$b_\nu a_\tau = a_\tau b_\nu \qquad (\forall \tau\in\Gamma, \forall \nu\in\Lambda)$$

Then, given

$$(\lambda_\tau)_{\tau\in\Gamma} \in J\sigma_{ap}\ (a_\tau)_{\tau\in\Gamma}$$

there exists a Λ-tuple $(\mu_\nu)_{\nu\in\Lambda}$ of complex numbers such that

$$\left((\lambda_\tau)_{\tau\in\Gamma}, (\mu_\nu)_{\nu\in\Lambda} \right) \in J\sigma_{ap}\left((a_\tau)_{\tau\in\Gamma}, (b_\nu)_{\nu\in\Lambda} \right)$$

Proof- The argument is essentially the same as that given in theorem 2.5. of chapter II. We shall therefore merely outline the proof.

Let Π be the universal representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$. For each $\tau\in\Gamma$, let

$$Q_\tau = supp\ (\Pi(a_\tau)^* - \bar{\lambda}_\tau),$$

and put

$$Q = I - \bigvee_{\tau \in \Gamma} Q_\tau ,$$

and

$$\mathcal{K} = Q(\mathcal{H}) .$$

Then, $\mathcal{K}$ is a non-zero closed subspace of $\mathcal{H}$. Since each b_ν $(\nu \in \Lambda)$ commutes with each a_τ $(\tau \in \Gamma)$, it follows that $\mathcal{K}$ is invariant under each $\pi(b_\nu)$. For each $\nu \in \Lambda$ let $\pi(b_\nu)'$ be the restriction of $\pi(b_\nu)$ to $\mathcal{K}$. Since the operators b_ν commute, theorem 2.1. implies the existence of a Λ-tuple of complex numbers such that

$$(\mu_\nu)_{\nu \in \Lambda} \in J\sigma_{ap}(\pi(b_\nu)')_{\nu \in \Lambda} .$$

The argument may now be completed as in theorem 2.5. of chapter II This completes the proof.

The rest of this section is devoted to a brief study of the compactness of the joint approximate point spectrum of infinite families of operators. While a similar argument as that presented in corollary 2.2.1. of chapter I shows that $J\sigma_{ap}(a_\tau)_{\tau \in \Gamma}$ is compact whenever it is non-empty (for example, the proof in [10; proposition 2] may be adapted to yield the result) we shall, instead, use theorem 5.3. of chapter II to give a simpler proof.

For the rest of this section, $(a_\tau)_{\tau \in \Gamma}$ denotes a fixed Γ-tuple of elements of $\mathcal{A}$, and it will be assumed that the joint approximate point spectrum of $(a_\tau)_{\tau \in \Gamma}$ is non-empty.

2.3. Theorem- The joint approximate point spectrum of $(a_\tau)_{\tau \in \Gamma}$ is a compact subset of $\mathbb{C}^\Gamma$.

Proof- Since

$$J\sigma_{ap}(a_\tau)_{\tau \in \Gamma} \subseteq \prod_{\tau \in \Gamma} (\sigma_{ap}(a_\tau))$$

and since each set on the right-hand side is compact, it is sufficient to prove that $J\sigma_{ap}(a_\tau)_{\tau\in\Gamma}$ is a closed subset of $\mathbb{C}^\Gamma$.

Let

$$(\mu_\tau)_{\tau\in\Gamma} \in \mathbb{C}^\Gamma \setminus J\sigma_{ap}(a_\tau)_{\tau\in\Gamma}$$

By definition 1.2., there exists a finite set of indices $\tau_1, \tau_2, \ldots, \tau_n$ in Γ such that

$$\sum_{j=1}^{n} \mathcal{A}(a_{\tau_j} - \mu_{\tau_j}) = \mathcal{A},$$

that is,

$$(\mu_{\tau_1}, \ldots, \mu_{\tau_n}) \notin J\sigma_{ap}(a_{\tau_1}, \ldots, a_{\tau_n})$$

By theorem 5.3. of chapter II, there exists open balls

$$B(\mu_{\tau_1}; \epsilon_1), \ldots, B(\mu_{\tau_n}; \epsilon_n),$$

where, for each $j = 1, 2, \ldots, n$,

$$B(\mu_{\tau_j}; \epsilon_j) = \{\lambda \in \mathbb{C} : |\mu_{\tau_j} - \lambda| < \epsilon_{\tau_j}\}$$

such that

$$\left(\prod_{j=1}^{n} B(\mu_{\tau_j}; \epsilon_j)\right) \cap J\sigma_{ap}(a_{\tau_1}, \ldots, a_{\tau_n}) = \emptyset \qquad (1)$$

Let $\mathcal{U}$ be the open set in $\mathbb{C}^\Gamma$ defined by

$$\mathcal{U} = \prod_{\tau\in\Gamma} \mathcal{U}_\tau,$$

where

$$\mathcal{U}_\tau = \begin{cases} \mathbb{C} & \text{if } \tau \neq \tau_j \quad (j=1,\ldots,n) \\ B(\mu_{\tau_j}; \epsilon_j) & \text{if } \tau = \tau_j \quad (j=1,\ldots,n). \end{cases}$$

Then, by (1) and the definition of $\mathcal{U}$, we have

$$(J\sigma_{ap}(a_\tau)_{\tau\in\Gamma}) \cap \mathcal{U} = \emptyset;$$

since $\mathcal{U}$ is an open subset of $\mathbb{C}^\Gamma$ containing $(\mu_\tau)_{\tau\in\Gamma}$, it follows that $\mathbb{C}^\Gamma \setminus J\sigma_{ap}(a_\tau)_{\tau\in\Gamma}$ is an open subset of $\mathbb{C}^\Gamma$.

This completes the proof.

For an alternative proof, see [51; proposition 1.9.].

§3. The Spectral Mapping Theorem-

3.1. Introduction- Let a be an element of a Banach algebra $\mathcal{A}$, and let p and q be complex polynomials such that q has no zeros on the spectrum of a. Then, with

$$f(a) = p(a)\, q(a)^{-1} = q(a)^{-1} p(a)$$

the usual spectral mapping theorem states that

$$\sigma(f(a)) = f(\sigma(a)) = \{ p(\lambda)\, q(\lambda)^{-1} : \lambda \in \sigma(a) \} ;$$

c.f., [6].

The purpose of this section is to prove a similar result for the joint approximate point spectrum of any commuting family of elements of a C^*-algebra. As a result, we shall also prove a spectral mapping theorem for the joint spectrum of operators.

Since the joint approximate point spectrum of operators may, in general, be empty, we shall assume that the operators under consideration mutually commute. We remark, however, that the commutativity assumption may be replaced by any other condition which entails the conclusion of theorem 2.2. .

Throughout the rest of this section, Γ will denote a fixed indexing set which may or may not be finite.

The main theorem of this section is as follows:

3.2. Theorem- Let $(a_\tau)_{\tau\in\Gamma}$ be a pair-wise commuting Γ-tuple of elements of a C^*-algebra $\mathcal{A}$, and let $(p_\tau)_{\tau\in\Gamma}$, and $(q_\tau)_{\tau\in\Gamma}$ be Γ-tuples of complex polynomials.

(a)- We have

$$\widetilde{J}\sigma_{ap}\left((a_\tau)_{\tau\in\Gamma}, (p_\tau(a_\tau))_{\tau\in\Gamma}\right) =$$

$$\left\{\left((\lambda_\tau)_{\tau\in\Gamma}, (p_\tau(\lambda_\tau))_{\tau\in\Gamma}\right) : (\lambda_\tau)_{\tau\in\Gamma} \in \widetilde{J}\sigma_{ap}(a_\tau)_{\tau\in\Gamma}\right\}.$$

In particular,

$$\widetilde{J}\sigma_{ap}\,(p_\tau(a_\tau))_{\tau\in\Gamma} =$$

$$\left\{(p_\tau(\lambda_\tau))_{\tau\in\Gamma} : (\lambda_\tau)_{\tau\in\Gamma} \in \widetilde{J}\sigma_{ap}(a_\tau)_{\tau\in\Gamma}\right\}$$

(b)- If for each $\tau \in \Gamma$, q_τ has no zeros on the approximate point spectrum of a_τ, then $q_\tau(a_\tau)$ is left-invertible; and if $q_\tau(a_\tau)^{-1}$ is any left inverse for $q_\tau(a_\tau)$ then

$$\widetilde{J}\sigma_{ap}\left((a_\tau)_{\tau\in\Gamma}, (q_\tau(a_\tau)^{-1}p_\tau(a_\tau))_{\tau\in\Gamma}\right)$$

is non-empty and equals

$$\left\{\left((\lambda_\tau)_{\tau\in\Gamma}, (q_\tau(\lambda_\tau)^{-1}p_\tau(\lambda_\tau))_{\tau\in\Gamma}\right) : (\lambda_\tau)_{\tau\in\Gamma} \in \widetilde{J}\sigma_{ap}(a_\tau)_{\tau\in\Gamma}\right\}$$

(c)- If for each $\tau \in \Gamma$, q_τ has no zeros on the spectrum of a_τ, then $q_\tau(a_\tau)$ is invertible; and if $q_\tau(a_\tau)^{-1}$ is the inverse of $q_\tau(a_\tau)$, then, with

$$s_\tau(\xi) = p_\tau(\xi)\,q_\tau(\xi)^{-1} \qquad (\tau\in\Gamma;\ \xi\in\sigma(a_\tau))$$

and

$$s_\tau(a_\tau) = p_\tau(a_\tau)\,q_\tau(a_\tau)^{-1} = q_\tau(a_\tau)^{-1}p_\tau(a_\tau) \qquad (\tau\in\Gamma)$$

we have

$$\widetilde{J}\sigma_{ap}\left((a_\tau)_{\tau\in\Gamma}, (s_\tau(a_\tau))_{\tau\in\Gamma}\right) =$$

$$\left\{\left((\lambda_\tau)_{\tau\in\Gamma}, (s_\tau(\lambda_\tau))_{\tau\in\Gamma}\right) : (\lambda_\tau)_{\tau\in\Gamma} \in \widetilde{J}\sigma_{ap}(a_\tau)_{\tau\in\Gamma}\right\}.$$

In particular,

$$J\sigma_{ap}(\mathfrak{s}_\tau(a_\tau))_{\tau\in\Gamma} = \{(\mathfrak{s}_\tau(\lambda_\tau))_{\tau\in\Gamma} : (\lambda_\tau)_{\tau\in\Gamma} \in J\sigma_{ap}(a_\tau)_{\tau\in\Gamma}\}.$$

The proof will be based on the following two preliminary results:

3.3. Lemma- Let a be an element of a C^*-algebra $\mathcal{A}$, let f be be a state of $\mathcal{A}$ which is left-multiplicative with respect to a with $f(a)=\lambda$, and let p be a complex polynomial. Then, f is left-multiplicative with respect to $p(a)$ and

$$f(p(a)) = p(f(a)) = p(\lambda).$$

Proof- Let

$$p(\xi) = \sum_{k=0}^{n} \alpha_k \xi^k \qquad (\xi \in \mathbb{C};\ \alpha_k \in \mathbb{C},\ k=1,\dots,n).$$

Then, for each $x \in \mathcal{A}$, we have

$$f(x\,p(a)) = f\left(\sum_{k=0}^{n} \alpha_k x a^k\right)$$

$$= \sum_{k=0}^{n} \alpha_k \lambda^k f(x)$$

since f is left-multiplicative with respect to a. Hence,

$$f(x\,p(a)) = f(x)\,p(\lambda) \qquad (\forall x \in \mathcal{A}),$$

and

$$f(p(a)) = p(\lambda) \quad .$$

This completes the proof.

3.4. Proposition- Let a be an element of a C^*-algebra $\mathcal{A}$, let f be a state of $\mathcal{A}$ which is left-multiplicative with respect to a with $f(a)=\lambda$, and let p and q be complex polynomials.

(a)- If q has no zeros on the approximate point spectrum of a then $q(a)$ is left-invertible; and if $q(a)^{-1}$ is any left inverse for $q(a)$ then, f is left-multiplicative with respect to $q(a)^{-1}p(a)$ and

$$f(q(a)^{-1}p(a)) = q(f(a))^{-1}p(f(a)) = q(\lambda)^{-1}p(\lambda).$$

(b)- If q has no zeros on the spectrum of a then $q(a)$ is invertible; and if $q(a)^{-1}$ is the inverse of $q(a)$ then, with

$$s(\lambda) = p(\lambda)\, q(\lambda)^{-1} \qquad (\lambda \in \sigma(a)),$$

and

$$s(a) = p(a)\, q(a)^{-1} = q(a)^{-1}p(a),$$

we have that f is left-multiplicative with respect to $s(a)$ and

$$f(s(a)) = s(f(a)) = q(\lambda)^{-1}p(\lambda)$$

Remark- By theorem 2.2. of chapter I, we have $\lambda \in \sigma_{ap}(a)$. Hence, since $0 \notin q(\sigma_{ap}(a))$, the expression $q(\lambda)^{-1}$ is meaningful.

Proof- (a). It is clear that $q(a)$ is left-invertible.

Let now l be a left inverse for $q(a)$. By lemma 3.3., f is left-multiplicative with respect to $q(a)$. Hence,

$$1 = f(1) = f(l\,q(a)) = f(l)\,f(q(a)) = q(\lambda)\,f(l) \quad ;$$

hence, we have,

$$f(l) = q(\lambda)^{-1} \tag{1}$$

Next, we have

$$1 = f(1) = f(\,q(a)^* l^* l\, q(a))$$
$$= |q(\lambda)|^2 f(l^* l)$$

by lemma 3.3., so that

$$f(l^* l) = |q(\lambda)|^{-2} \qquad (2)$$

Therefore, by (1), (2), and theorem 2.2. of chapter I, f is left-multiplicative with respect to l ; hence, if x is an arbitrary element of $\mathcal{A}$, we have (with $l = q(a)^{-1}$)

$$f(x\, q(a)^{-1} p(a)) = f(x\, q(a)^{-1})\, f(p(a))$$
$$= f(x) f(q(a)^{-1})\, f(p(a))$$
$$= q(\lambda)^{-1} p(\lambda) f(x).$$

This completes the proof of part (a).

(b)- This is clearly a special case of part (a).

This completes the proof.

We now turn to the proof of theorem 3.2. .

(a)- Let

$$(\lambda_\tau)_{\tau\in\Gamma} \in J\sigma_{ap}\,(a_\tau)_{\tau\in\Gamma}$$

By theorem 1.3., there exists $f \in E(\mathcal{A})$ such that

$$f(x a_\tau) = f(x) f(a_\tau) = \lambda_\tau f(x) \qquad (\forall x\in\mathcal{A}, \forall \tau\in\Gamma) \qquad (1)$$

Hence, by lemma 3.3.,

$$f(x\, p_\tau(a_\tau)) = f(x) f(p_\tau(a_\tau)) = p_\tau(\lambda_\tau) f(x) \quad (\forall x\in\mathcal{A}, \forall\tau\in\Gamma) \qquad (2)$$

Therefore, by (1), (2), and theorem 1.3.b(ii), we have

$$(\,(\lambda_\tau)_{\tau\in\Gamma}, (p_\tau(\lambda_\tau))_{\tau\in\Gamma}) \in J\sigma_{ap}((a_\tau)_{\tau\in\Gamma}, (p_\tau(a_\tau))_{\tau\in\Gamma})$$

Conversely, suppose that

$$((\lambda_\tau)_{\tau\in\Gamma},\ (\mu_\tau)_{\tau\in\Gamma}) \in J\sigma_{ap}((a_\tau)_{\tau\in\Gamma},\ (p_\tau(a_\tau))_{\tau\in\Gamma})\ ;$$

then, there exists $f \in E(\mathcal{A})$ such that for each $\tau \in \Gamma$, f is left-multiplicative with respect to a_τ and $p_\tau(a_\tau)$ with

$$f(a_\tau) = \lambda_\tau \qquad (\forall \tau \in \Gamma) \qquad (3)$$

and

$$f(p_\tau(a_\tau)) = \mu_\tau \qquad (\forall \tau \in \Gamma)$$

But (3) implies (as in the proof of lemma 3.3.) that

$$f(p_\tau(a_\tau)) = p_\tau(\lambda_\tau) \qquad (\forall \tau \in \Gamma)$$

Hence

$$\mu_\tau = p_\tau(\lambda_\tau) \qquad (\forall \tau \in \Gamma).$$

This completes the proof of the first part of (a).

To prove the second part, let

$$(\mu_\tau)_{\tau\in\Gamma} \in J\sigma_{ap}(p_\tau(a_\tau))_{\tau\in\Gamma}\ .$$

Since the set

$$\{a_\tau,\ p_\gamma(a_\gamma)\ :\ \tau, \gamma \in \Gamma\}$$

is a mutually commuting set, there exists (theorem 2.2.) a Γ-tuple $(\lambda_\tau)_{\tau\in\Gamma}$ of complex numbers such that

$$((\lambda_\tau)_{\tau\in\Gamma},\ (\mu_\tau)_{\tau\in\Gamma}) \in J\sigma_{ap}((a_\tau)_{\tau\in\Gamma},\ (p_\tau(a_\tau))_{\tau\in\Gamma})\ .$$

Hence, by the first part of (a),

$$(\mu_\tau)_{\tau\in\Gamma} = (p_\tau(\lambda_\tau))_{\tau\in\Gamma}\ .$$

Conversely, let

$$(\lambda_\tau)_{\tau\in\Gamma} \in J\sigma_{ap}(a_\tau)_{\tau\in\Gamma}$$

Then,

$$\left((\lambda_\tau)_{\tau\in\Gamma}, (p_\tau(\lambda_\tau))_{\tau\in\Gamma} \right) \in J\sigma_{ap}\left((a_\tau)_{\tau\in\Gamma}, (p_\tau(a_\tau))_{\tau\in\Gamma} \right);$$

hence, in particular,

$$(p_\tau(\lambda_\tau))_{\tau\in\Gamma} \in J\sigma_{ap}(p_\tau(a_\tau))_{\tau\in\Gamma}.$$

This completes the proof of part (a).

(b)- The proof is much the same as the proof of the first part of (a), the only difference being the use of proposition 3.4. instead of lemma 3.3. . We omit the details.

(c)- The first part is clearly a special case of part (b).

To prove the second part, let

$$(\nu_\tau)_{\tau\in\Gamma} \in J\sigma_{ap}(s_\tau(a_\tau))_{\tau\in\Gamma}$$

Since for each $\tau \in \Gamma$, $q_\tau(a_\tau)^{-1}$ is the two-sided inverse for $q_\tau(a_\tau)$, the set

$$\{ a_\tau, s_\gamma(a_\gamma) : \tau, \gamma \in \Gamma \}$$

is a commuting family of elements of $\mathcal{A}$. Hence, by theorem 2.2., there exists a Γ-tuple $(\lambda_\tau)_{\tau\in\Gamma}$ of complex numbers such that

$$\left((\lambda_\tau)_{\tau\in\Gamma}, (\nu_\tau)_{\tau\in\Gamma} \right) \in J\sigma_{ap}\left((a_\tau)_{\tau\in\Gamma}, (s_\tau(a_\tau))_{\tau\in\Gamma} \right).$$

Hence, by the first part of (c),

$$(\nu_\tau)_{\tau\in\Gamma} = (s_\tau(\lambda_\tau))_{\tau\in\Gamma}.$$

The converse may be proved as in the second part of (a).

This completes the proof of the theorem.

The above theorem throws some light on the structure of the joint approximate point spectrum of operators. More specifically, part (b) shows that, whereas in order that the joint approximate point spectrum of, say two operators be non-empty it is sufficient that the operators commute, the commutativity condition is far from necessary. For instance, let a be a left-invertible operator with left inverse a_l^{-1} such that a and a_l^{-1} do not commute (e.g., let a be a non unitary isometry); then, taking $p(\xi)=1$ and $q(\xi)=\xi$ in theorem 3.2.(b), we have that the joint approximate point spectrum of a and a_l^{-1} is non-empty, and in fact

$$J\sigma_{ap}(a, a_l^{-1}) = \{(\lambda, \lambda^{-1}) : \lambda \in \sigma_{ap}(a)\} .$$

In fact, we have the following corollary:

3.5. Corollary- Let $(a_\tau)_{\tau\in\Gamma}$ be a commuting Γ-tuple of isometries in a C^*-algebra $\mathcal{A}$. There exists a 1-1 correspondence between the set of characters on $C^*((a_\tau)_{\tau\in\Gamma})$ and the set of points in $J\sigma_{ap}(a_\tau)_{\tau\in\Gamma}$.

Proof- For each $\tau\in\Gamma$, define the complex polynomials $p_\tau(\xi)$ and $q_\tau(\xi)$ by

$$p_\tau(\xi) = 1 \quad , \quad q_\tau(\xi) = \xi$$

Since each a_τ is left-invertible, q_τ has no zeros on the approximate point spectrum of a_τ ; hence, by theorem 3.2.(b),

$$J\sigma_{ap}\left((a_\tau)_{\tau\in\Gamma}, (a_\tau^*)_{\tau\in\Gamma} \right) = \left\{ ((\lambda_\tau)_{\tau\in\Gamma}, (\lambda_\tau^{-1})_{\tau\in\Gamma}) : (\lambda_\tau)_{\tau\in\Gamma} \in J\sigma_{ap}(a_\tau)_{\tau\in\Gamma} \right\} \tag{1}$$

Let now $(\lambda_\tau)_{\tau\in\Gamma}$ be an arbitrary point of $J\sigma_{ap}(a_\tau)_{\tau\in\Gamma}$. By (1) and the remark immediately preceding theorem 2.2., there exists a state f of $\mathcal{A}$ such that for each τ, f is left-multiplicative with respect to a_τ and a_τ^* with

$$f(a_\tau) = \lambda_\tau \qquad (\forall \tau \in \Gamma).$$

Hence, if x is an arbitrary element of $\mathcal{A}$, we have

$$\begin{aligned} f(a_\tau x) &= \overline{f(x^* a_\tau^*)} \\ &= \overline{f(x^*)}\ \overline{f(a_\tau^*)} \\ &= f(x) f(a_\tau) \qquad (\forall \tau \in \Gamma) \end{aligned}$$

so that f is also right-multiplicative with respect to each a_τ. Therefore, the restriction g of f to $C^*((a_\tau)_{\tau\in\Gamma})$ is a character on $C^*((a_\tau)_{\tau\in\Gamma})$ with

$$g(a_\tau) = f(a_\tau) = \lambda_\tau$$

The converse is proposition 1.5. .

This completes the proof.

Let $(a_\tau)_{\tau\in\Gamma}$ be a mutually commuting Γ-tuple of elements of a C^*-algebra $\mathcal{A}$.

The joint spectrum of $(a_\tau)_{\tau\in\Gamma}$, denoted by $J\sigma\,(a_\tau)_{\tau\in\Gamma}$ is defined to be the set

$$J\sigma_{ap}\,(a_\tau)_{\tau\in\Gamma} \cup J\sigma_r\,(a_\tau)_{\tau\in\Gamma}$$

where, $J\sigma_r\,(a_\tau)_{\tau\in\Gamma}$ is the joint right spectrum of $(a_\tau)_{\tau\in\Gamma}$ defined by

$$J\sigma_r\,(a_\tau)_{\tau\in\Gamma} = \left\{ (\lambda_\tau)_{\tau\in\Gamma} \in \mathbb{C}^\Gamma : \sum_{\tau\in\Gamma} (a_\tau - \lambda_\tau)\mathcal{A} \neq \mathcal{A} \right\}.$$

When Γ is a finite set, the above definition coïncides with definition 4.1.(d) of chapter II. Note also that the joint spectrum of $(a_\tau)_{\tau\in\Gamma}$ is non-empty (theorem 2.1.), and compact (theorem 2.3.).

Clearly, the analogue of theorem 3.2. holds for the joint right spectrum of $(a_\tau)_{\tau\in\Gamma}$; the statement of the relevant theorem may be obtained from the statement of theorem 3.2. by replacing "(joint) approximate point spectrum" with "(joint) right spectrum" throughout.

For each τ, let p_τ and q_τ, and s_τ be defined as in theorem 3.2.(c), and let

$$(\lambda_\tau)_{\tau\in\Gamma} \in J\sigma\,(a_\tau)_{\tau\in\Gamma}\,.$$

Then, either

$$(\lambda_\tau)_{\tau\in\Gamma} \in J\sigma_{ap}\,(a_\tau)_{\tau\in\Gamma}\,,$$

or

$$(\lambda_\tau)_{\tau\in\Gamma} \in J\sigma_r\,(a_\tau)_{\tau\in\Gamma}\,.$$

Hence, by theorem 3.2.(c) and its analogue for the joint right spectrum, we have

$$(s_\tau(\lambda_\tau))_{\tau\in\Gamma} \in J\sigma_{ap}\,(s_\tau(a_\tau))_{\tau\in\Gamma}$$

or

$$(s_\tau(\lambda_\tau))_{\tau\in\Gamma} \in J\sigma_r\,(s_\tau(a_\tau))_{\tau\in\Gamma}$$

respectively. Thus, in either case, we have

$$(s_\tau(\lambda_\tau))_{\tau\in\Gamma} \in J\sigma\,(s_\tau(a_\tau))_{\tau\in\Gamma}$$

by the above definition.

A similar reasoning shows that if

$$(\mu_\tau)_{\tau\in\Gamma} \in J\sigma\,(s_\tau(a_\tau))_{\tau\in\Gamma}$$

then, there exists

$$(\lambda_\tau)_{\tau\in\Gamma} \in J\sigma\,(a_\tau)_{\tau\in\Gamma}$$

such that

$$(\mu_\tau)_{\tau\in\Gamma} = (s_\tau(\lambda_\tau))_{\tau\in\Gamma}\,.$$

We have thus proved the following spectral mapping theorem for

for the joint spectrum:

3.6. Theorem- Let $(a_\tau)_{\tau\in\Gamma}$ be a pair-wise commuting Γ-tuple of elements of a C^*-algebra $\mathcal{A}$, let $(p_\tau)_{\tau\in\Gamma}$, and $(q_\tau)_{\tau\in\Gamma}$ be Γ-tuples of complex polynomials such that for each τ, q_τ has no zeros on the spectrum of a_τ, and let

$$s_\tau(\xi) = p_\tau(\xi)\, q_\tau(\xi)^{-1} \qquad (\xi \in \sigma(a_\tau))$$

and

$$s_\tau(a_\tau) = p_\tau(a_\tau)\, q_\tau(a_\tau)^{-1} = q_\tau(a_\tau)^{-1}\, p_\tau(a_\tau).$$

Then

$$J\sigma(s_\tau(a_\tau))_{\tau\in\Gamma} = \left\{ (s_\tau(\lambda_\tau))_{\tau\in\Gamma} : (\lambda_\tau)_{\tau\in\Gamma} \in J\sigma(a_\tau)_{\tau\in\Gamma} \right\}.$$

We close this chapter with the following remarks concerning theorem 3.6. . In the case of a finite number of commuting elements of a complex unital Banach algebra, the polynomial spectral mapping theorem (i.e., theorem 3.6. with no q_τ present) was proved in [19; proposition 3]. The same result was extended to the case of any commuting family of elements of a complex unital Banach algebra in [29]. See also [4; § 1.1.], where theorem 3.6. is proved in the case of a finite family of mutually commuting elements of a complex unital Banach algebra.

Characters and Finite Operators

§1. Finite operators.

1.1. Introduction- This chapter consists mainly of the applications of the results developed in the previous chapters to the theory of characters and finite operators. The present section is devoted mainly to the definition and certain general properties of finite operators; examples of specific classes of finite operators will be given in §4.

1.2. Definition- Let $\mathcal{A}$ be a C^*-algebra, let $\mathcal{B}$ be a C^*-subalgebra of $\mathcal{A}$, and let $f \in E(\mathcal{A})$. The state f is said to have the trace-like property relative to $\mathcal{B}$ provided that

$$f(xb) = f(bx) \qquad (\forall x \in \mathcal{A},\ \forall b \in \mathcal{B}) \quad (i)$$

[43].

An element a of $\mathcal{A}$ is said to be finite provided that for each $x \in \mathcal{A}$, there exists a state f_x of $\mathcal{A}$ such that

$$f_x(xa) = f_x(ax) \qquad (ii)$$

[55].

Let $a_1, \ldots, a_n$ be a finite number of elements of a C^*-algebra $\mathcal{A}$. Recall that the joint numerical range of $a_1, \ldots, a_n$ denoted by $JV(a_1, \ldots, a_n)$ is the set of n-tuples of complex numbers $(\lambda_1, \ldots, \lambda_n)$ such that

$$\lambda_j = f(a_j) \qquad (j = 1, \ldots, n)$$

for some $f \in E(\mathcal{A})$ [7; Definition 11, §2]. In particular, the numerical range of a single operator a is the set

$$V(a) = \{ \lambda \in \mathbb{C} : \lambda = f(a) \text{ for some } f \in E(\mathcal{A}) \}.$$

When $\mathcal{A} = \mathcal{BL}(\mathcal{H})$, and $a = A \in \mathcal{BL}(\mathcal{H})$, the numerical range of A as defined above is precisely the closure of the ordinary numerical range of A defined by

$$\{ \langle Ax, x \rangle : x \in \mathcal{H} \ \& \ \|x\| = 1 \} .$$

Thus, to say that a is finite is equivalent to saying that, for each $x \in \mathcal{A}$, $0 \in V(ax - xa)$. Much more is true, however:

1.3. Theorem- Let a be a finite element of a C^*-algebra $\mathcal{A}$. Then there exists $f \in E(\mathcal{A})$ such that

$$f(ax) = f(xa) \qquad (\forall x \in \mathcal{A}) .$$

Proof- [55; Theorem 4].

Using theorem 1.3., we may now relate definitions 1.2.(i), and 1.2.(ii) as follows:

1.4. Proposition- Let a be an element ef a C^*-algebra $\mathcal{A}$. Then a is finite if and only if there exists a state f of $\mathcal{A}$ such that f has the trace-like property with respect to $C^*(a)$:

$$f(xb) = f(bx) \qquad (\forall x \in \mathcal{A}, \ \forall b \in C^*(a)) \quad (1).$$

Proof- The sufficiency is obvious.

To prove the necessity, let (by theorem 1.3.) f be a state of which satisfies

$$f(ax) = f(xa) \qquad (\forall x \in \mathcal{A}) \quad (2).$$

Since f is self-adjoint, we have

$$f(a^*x) = f(xa^*) \qquad (\forall x \in \mathcal{A}) \quad (3).$$

Using (2) and (3) in succession, it follows that for each non-negative integer n, we have

$$f(a^n x) = f(x a^n) \qquad (\forall x \in \mathcal{A}) \quad (4),$$

and

$$f(a^{*n}x) = f(x a^{*n}) \qquad (\forall x \in \mathcal{A}) \qquad (5).$$

Let now $q_1^{i_1} \ldots q_k^{i_k}$ be an arbitrary product of non-negative integral powers of a and a^*; here, each q_j ($1 \le j \le k$) is either a or a^*, and each i_j ($1 \le j \le k$) is a non-negative integer. Then, for each $x \in \mathcal{A}$, we have

$$\begin{aligned} f(q_1^{i_1} \cdots q_k^{i_k} x) &= f(q_2^{i_2} \cdots q_k^{i_k} x q_1^{i_1}) \\ &= \ldots \\ &= f(x q_1^{i_1} \cdots q_k^{i_k}) \end{aligned}$$

where equality in, e.g., the first line follows form (4) if $q_1 = a$, and from (5) if $q_1 = a^*$. Hence, by linearity and continuity, f satisfies (1) .

This completes the proof.

The following proposition gives a necessary and sufficient condition for an operator to be finite.

1.5. Proposition- Let a be an element of a C^*-algebra $\mathcal{A}$, and let $a = a_1 + i a_2$ be the decomposition of a into its self-adjoint parts.

A necessary and sufficient condition that a be finite is that for each $x \in \mathcal{A}$,

$$(0,0) \in JV((a_1x - xa_1), (a_2x - xa_2)) \qquad (1)$$

Proof- Let (1) be satisfied for each $x \in \mathcal{A}$. then,for each $x \in \mathcal{A}$, there exists $f \in E(\mathcal{A})$ such that

$$f(a_j x - x a_j) = 0 \qquad (j = 1, 2).$$

Hence

$$\begin{aligned} f(ax - xa) &= f((a_1x - xa_1) + i(a_2x - xa_2)) \\ &= 0, \end{aligned}$$

so that a is finite.

Conversely, let a be finite. By theorem 1.3., there exists $f \in E(\mathcal{A})$ such that

$$f(ax - xa) = 0 \qquad (\forall x \in \mathcal{A}),$$

i.e.,

$$f(a_1 x - x a_1) = f(i(a_2 x - x a_2)) \qquad (\forall x \in \mathcal{A}) \ (2).$$

Let x be an arbitrary self-adjoint element of $\mathcal{A}$; since x and a_2 are self-adjoint, the operator $i(a_2 x - x a_2)$ is self-adjoint; hence, the right- hand side of (2) is a real number. Similarly, the left-hand side of (2) is purely immaginary. Hence,

$$f(a_1 x - x a_1) = f(i(a_2 x - x a_2)) = 0 \qquad (3).$$

If now x is an arbitrary element of $\mathcal{A}$, with $x = x_1 + i x_2$ the decomposition of x into its self-adjoint parts, we have, by (3),

$$f(a_1 x_1 - x_1 a_1) = f(a_2 x_1 - x_1 a_2) = 0,$$

and

$$f(a_1 x_2 - x_2 a_1) = f(a_2 x_2 - x_2 a_2) = 0,$$

so that

$$f(a_1 x - x a_1) = f(a_2 x - x a_2) = 0,$$

i.e., (1) holds. This completes the proof.

Let $x \in \mathcal{A}$ be a singular,non-finite operator, and let $x = x_1 + i x_2$ be the decomposition of x into its self- adjoint parts. Then, there exists $f \in E(\mathcal{A})$ such that $f(x) = 0$. Hence, $f(x_1) = f(x_2) = 0$. However, since x is not finite,there exists no state of $\mathcal{A}$ which has the trace- like property relative to $C^*(x_1, x_2)$. This shows that the answer to the following question is in the negative:

Question: Let h_1 and h_2 be self- adjoint elements of $\mathcal{A}$, and let

$$(0,0) \in JV(h_1, h_2).$$

Does there exist a state f of $\mathcal{A}$ such that for each $x \in \mathcal{A}$,

$$f(h_j x) = f(x h_j) \qquad (j=1,2)\ ?$$

In connection with the above question, we remark that, if k is a self-adjoint element of a C^*-algebra $\mathcal{A}$, and if $0 \in V(k)$ then there exists a state f of $\mathcal{A}$ such that $f(k) = 0$ and

$$f(kx) = f(xk) \qquad (\forall x \in \mathcal{A}).$$

For, the numerical range of a self-adjoint operator is the convex hull of its spectrum [7; corollary 11, §5]; the condition $0 \in V(k)$ then implies that there exists $\lambda_j \in \sigma(k)$ $(j=1,\dots,n)$ such that

$$\sum_{j=1}^{n} \alpha_j \lambda_j = 0 \qquad (\alpha_j \in \mathbb{R}_+ ;\ \sum_{j=1}^{n} \alpha_j = 1).$$

Since k is self-adjoint, corresponding to each λ_j, there exists $\varphi_j \in \Phi_{C^*(k)}$ such that

$$\lambda_j = \varphi_j(k) \qquad (j=1,\dots,n).$$

Extend each φ_j to a state f_j of $\mathcal{A}$; by the Cauchy-Schwartz inequality, we have

$$f_j(kx) = f_j(xk) = \lambda_j f(x) \qquad (j=1,\dots,n;\ \forall x \in \mathcal{A}).$$

If now we define a state f of $\mathcal{A}$ by

$$f = \sum_{j=1}^{n} \alpha_j f_j ,$$

we have

$$f(k) = \sum_{j=1}^{n} \alpha_j f_j(k) = \sum_{j=1}^{n} \alpha_j \lambda_j = 0 ,$$

and

$$f(xk) = \sum_{j=1}^{n} \alpha_j f_j(xk) = \sum_{j=1}^{n} \alpha_j f_j(x) f_j(k)$$
$$= f(kx) .$$

This proves our assertion.

In later sections, we shall also be interested in families of finite operators. The following two results contain the relevant information in this direction.

1.6. Proposition- Let $\{a_\tau : \tau \in \Gamma\}$ be a family of elements of a C^*-algebra $\mathcal{A}$. A necessary and sufficient condition that there exist $f \in E(\mathcal{A})$ with trace-like property relative to $C^*(a_\tau)_{\tau\in\Gamma}$ is that there exist $f \in E(\mathcal{A})$ which satisfies

$$f(xa_\tau) = f(a_\tau x) \qquad (\forall \tau \in \Gamma,\ \forall x \in \mathcal{A}).$$

1.7. Proposition- Let $\{a_\tau : \tau \in \Gamma\}$ be a family of elements of a C^*-algebra $\mathcal{A}$. Suppose that for each finite subfamily $\{a_\tau : \tau \in \Gamma_0\}$ there exists $f \in E(\mathcal{A})$ such that

$$f(xa_\tau) = f(a_\tau x) \qquad (\forall \tau \in \Gamma_0,\ \forall x \in \mathcal{A}).$$

Then, there exists $g \in E(\mathcal{A})$ such that

$$g(xb) = g(bx) \qquad (\forall b \in C^*(a_\tau)_{\tau\in\Gamma},\ \forall x \in \mathcal{A}).$$

Proposition 1.6. may be proved in the same way as proposition 1.4., while proposition 1.7. may be proved in essentially the same way as theorem 2.1. of chapter III. We omit the details.

Further consideration of finite operators suggests the consideration of the following three questions: Let a be an element of a C^*-algebra $\mathcal{A}$;

(i)- Does there exist a nontrivial bounded linear functional f on $\mathcal{A}$ such that

$$f(ax) = f(xa) \qquad (\forall x \in \mathcal{A}) \quad (1)\ ?$$

(ii)- Does there exist a state f of $\mathcal{A}$ which satisfies (1) above?

(iii)- Does there exist a pure state f of $\mathcal{A}$ which satisfies (1) , It is clear that (iii) $\Rightarrow$(ii) $\Rightarrow$(i). The equivalence of (i) and (ii) was first proved by J.W. Bunce in [11; proposition 5]. On the other hand, we shall give an example to show that (ii) does not , in general, imply (iii), and then show that under certain conditions a restricted form of (iii) is equivalent to (ii).

Let$(\mathcal{B}, \|\cdot\|)$ be a finite-dimensional C^*-algebra with norm $\|\cdot\|$. Since $\mathcal{B}$ is the algebraic linear span of its unitaries, there exist unitary elements $u_1, \ldots, u_n$ in $\mathcal{B}$ such that the u_j's are linearly independent, and such that every element x of $\mathcal{B}$ can be written as a unique complex linear combination of the u_j's :

$$x = \sum_{j=1}^{n} \alpha_j u_j \qquad (\alpha_j \in \mathbb{C} , \ j=1, \ldots, n) .$$

We call the set $\{\alpha_1, \ldots, \alpha_n\}$ the associated sequence of x .

Define a new norm $\|\cdot\|_1$ on $\mathcal{B}$ by

$$\|x\|_1 = \sum_{j=1}^{n} |\alpha_j| \qquad (\forall x \in \mathcal{B})$$

where $\{\alpha_1, \ldots, \alpha_n\}$ is the associated sequence of x . Since $\mathcal{B}$ is finite-dimensional, the new norm $\|\cdot\|_1$ is equivalent to the original norm $\|\cdot\|$. Hence, there exists a positive number M such that

$$\|x\|_1 \leq M \|x\| \qquad (\forall x \in \mathcal{B}).$$

In particular, if u is a unitary element of $\mathcal{B}$ then the associated sequence satisfies

$$\sum_{j=1}^{n} |\alpha_j| = \|u\|_1 \leq M \|u\| = M$$

since u is unitary.

We now have the following result:

1.8. Theorem- Let $\mathcal{B}$ be a finite-dimensional C^*-subalgebra of a C^*-algebra $\mathcal{A}$. There exists a state f of $\mathcal{A}$ such that

$$f(xb) = f(bx) \qquad (\forall x \in \mathcal{A},\ \forall b \in \mathcal{B}).$$

Proof- We shall give a proof using the fixed point theorem of Kakutani [24; theorem V.10.8].

Let $\mathcal{U}$ be the unitary group of $\mathcal{B}$, and let $\{u_1, \ldots, u_n\} = S$ be a finite set of linearly independent elements of $\mathcal{U}$ such that $\mathcal{B}$ is the algebraic linear span of S. By the preceding remarks, there exists a positive number M such that if u is any element of $\mathcal{U}$ with associated sequence $\{\alpha_1, \ldots, \alpha_n\}$ then

$$|\alpha_j| \le M \qquad (j = 1, \ldots, n) \qquad (1).$$

Let u be an arbitrary but fixed element of $\mathcal{U}$. For each $f \in \mathcal{A}'$ define a mapping $T_u f$ on $\mathcal{A}$ by

$$(T_u f)(x) = f(u^* x u) \qquad (\forall x \in \mathcal{A}) \quad (2).$$

Clearly, $T_u f \in \mathcal{A}'$; further, it is easily verified that the map

$$T_u : \mathcal{A}' \to \mathcal{A}'$$

defined by

$$f \to T_u f$$

is linear. Let

$$\mathcal{T} = \{ T_u : u \in \mathcal{U} \};$$

we claim that

(i) $\mathcal{T}$ is a group of linear transformations on $\mathcal{A}'$;

(ii) $\mathcal{T}(E(\mathcal{A})) \subseteq E(\mathcal{A})$; and

(iii) $\mathcal{T}$ is equicontinuous on $E(\mathcal{A})$.

To prove (i), let u and w be elements of $\mathcal{U}$, let $f \in \mathcal{A}'$, and let $x \in \mathcal{A}$. Then, by (2),

$$\begin{aligned}(T_u T_w) f(x) &= T_u f(w^* x w) \\ &= f((wu)^* x (wu)) \\ &= T_v f(x),\end{aligned}$$

where

$$v = wu \in \mathcal{U}.$$

To prove (ii), let $f \in E(\mathcal{A})$, $u \in \mathcal{U}$, and let $x^* x$ be an arbitrary positive element of $\mathcal{A}$; then

$$T_u f(x^* x) = f((xu)^* (xu)) \geq 0,$$

and

$$T_u f(1) = f(u^* u) = 1,$$

so that $T_u f \in E(\mathcal{A})$.

Finally, we prove (iii). Let

$$V = \{ f \in \mathcal{A}' : |f(x_k)| \leq \epsilon \quad (k=1,\dots,m) \}$$

be an arbitrary w^* neighbourhood of zero in $\mathcal{A}'$ where

$$\epsilon > 0 \quad \& \quad x_j \in \mathcal{A} \qquad (j=1,\dots,m)$$

Let

$$y_{j,k,s} = u_j^* x_k u_s \qquad (s,j=1,\dots,n;\ k=1,\dots,m).$$

and define a w^*-neighbourhood of zero W in $\mathcal{A}'$ by

$$W = \{ f \in \mathcal{A}' : |f(y_{j,k,s})| \leq \frac{\epsilon}{M^2 n^2} \ (s,j=1,\dots,n;\ k=1,\dots,m) \}.$$

We shall show that if $f, g \in E(\mathcal{A})$ with $(f-g) \in W$ then

$$T_u(f-g) \in V. \qquad (\forall u \in \mathcal{U})$$

Let u be an arbitrary element of $\mathcal{U}$ with associated sequence $\{\alpha_1, \dots, \alpha_n\}$. Then, for each k, $1 \leq k \leq m$, we have

$$\begin{aligned}
|T_u(f-g)(x_k)| &= |(f-g)(u^* x_k u)| \\
&= \Big|(f-g)\Big(\sum_{j=1}^{n} \bar{\alpha}_j u_j^*\Big) x_k \Big(\sum_{j=1}^{n} \alpha_j u_j\Big)\Big| \\
&= \Big|\sum_{j=1}^{n} \sum_{s=1}^{n} \bar{\alpha}_j \alpha_s (f-g)(u_j^* x_k u_s)\Big| \\
&\leq \sum_{j=1}^{n} \sum_{s=1}^{n} |\alpha_j| |\alpha_s| \, |(f-g)(u_j^* x_k u_s)| \\
&\leq \sum_{j=1}^{n} \sum_{s=1}^{n} M^2 \cdot \frac{\epsilon}{M^2 n^2} = \epsilon ;
\end{aligned}$$

hence, $T_u(f-g) \in V$. This proves (iii).

It now follows from the Kakutani fixed point theorem that the group $\mathcal{J}$ has a fixed point in $E(\mathcal{A})$, i.e., there exists $f \in E(\mathcal{A})$ such that for $\forall u \in \mathcal{U}$,

$$T_u f(x) = f(x) \qquad (\forall x \in \mathcal{A}),$$

i.e.,

$$f(u^* x u) = f(x) \qquad (\forall u \in \mathcal{U}, \ \forall x \in \mathcal{A}).$$

In particular, replacing x by ux, we get

$$f(ux) = f(xu) \qquad (\forall u \in \mathcal{U}, \ \forall x \in \mathcal{A}).$$

Since $\mathcal{B}$ is the algebraic linear span of its unitaries, this shows that

$$f(bx) = f(xb) \qquad (\forall b \in \mathcal{B}, \ \forall x \in \mathcal{A}).$$

This completes the proof.

1.8.1. Corollary- Let $\mathcal{A}$ be an approximately finite-dimensional C^*-algebra. There exists $f \in E(\mathcal{A})$ such that

$$f(xy) = f(yx) \qquad (\forall x \ \forall y \in \mathcal{A}). \quad (1)$$

Proof- Let

$$\mathcal{A} = \overline{\bigcup_{n=1}^{\infty} \mathcal{A}_n}$$

where each $\mathcal{A}_n$ is a finite-dimensional C^*-subalgebra of $\mathcal{A}$ with

$$\mathcal{A}_n \subseteq \mathcal{A}_{n+1} \qquad (n=1, 2, \ldots) \qquad (2) .$$

Partially order the set $\mathcal{F}$ of all finite subsets of the set of positive integers with respect to inclusion; for each $F \in \mathcal{F}$ let

$$m_F = \max \{ n : n \in F \} .$$

By theorem 1.8., for each F, there exists a state f_F of $\mathcal{A}$ such that

$$f_F(ab) = f_F(ba) \qquad (\forall a, \forall b \in \mathcal{A}_{m_F}) \quad (3) .$$

Let f be a w^*-limit point of the net $\{ f_F ; F \in \mathcal{F} \}$. It is then easily verified , using (2), (3), and the definition of f, that

$$f(xy) = f(yx) \qquad (\forall x \forall y \in \bigcup_{n=1}^{\infty} \mathcal{A}_n),$$

hence, by continuity, (1) holds.

This completes the proof.

Let now $\mathcal{A} = M_n(\mathbb{C})$ $(n \geq 2)$. Then $\mathcal{A}$ is singly generated by an invertible operator a [53 ; page 84]. By theorem 1.8., there exists $f \in E(\mathcal{A})$ such that

$$f(xy) = f(yx) \qquad (\forall x \forall y \in \mathcal{A}) \quad (1).$$

On the other hand, if there were a pure state f satisfying (1) above then f would have to be a character on $C^*(a) = M_n(\mathbb{C})$. [43 ; proposition 5] . Since a is invertible, the kernel o f would be a non-trivial closed two-sided ideal of $M_n(\mathbb{C})$. But this is impossible, since $M_n(\mathbb{C})$ is simple.

The above example shows that the implication (ii) $\Longrightarrow$ (iii) mentioned immediately before theorem 1.8. does not hold in general.

For the next theorem, we shall need the following result, which is due to Anderson [2].

Let $\mathcal{H}$ be a separable Hilbert space, and let $\mathcal{B}$ be a separable C^*- subalgebra of the Calkin algebra $\mathcal{C}alk(\mathcal{H}) = \mathcal{BL}(\mathcal{H}) / \mathcal{KL}(\mathcal{H})$. Then, any state of $\mathcal{B}$ may be extended to a pure state of $\mathcal{C}alk(\mathcal{H})$.

In the next theorem, we shall assume that $\mathcal{A} = \mathcal{BL}(\mathcal{H})$ for some separable Hilbert space $\mathcal{H}$. If a is an element of $\mathcal{A}$, then $C^*(a + \mathcal{KL}(\mathcal{H}))$ is singly generated, hence it is a separable C^*-subalgebra of $\mathcal{C}alk(\mathcal{H})$.

1.9. Theorem- Let $a \in \mathcal{A}$, and suppose that there exists $f \in E(\mathcal{A})$ such that

$$f(bc) = f(cb) \qquad (\forall b\ \forall c \in C^*(a))$$

and

$$f(\mathcal{KL}(\mathcal{H})) = \{0\} \qquad (1).$$

Then, there exists $\rho \in \mathcal{P}(\mathcal{A})$ such that

$$\rho(bc) = \rho(cb) \qquad (\forall b\ \forall c \in C^*(a)) \quad (2).$$

Proof- Let $f_1 = f|_{C^*(a)}$; define $\tilde{f}_1$ on $C^*(a + \mathcal{KL}(\mathcal{H}))$ by

$$\tilde{f}_1(b + \mathcal{KL}(\mathcal{H})) = f_1(b) \qquad (\forall b \in C^*(a))$$

By (1), $\tilde{f}_1$ is well-defined; further, since $f_1 \in E(C^*(a))$, it is easily verified that $\tilde{f}_1 \in E(C^*(a + \mathcal{BL}(\mathcal{H})))$. Hence, by Anderson's theorem, there exists $\tilde{\rho} \in \mathcal{P}(\mathcal{C}alk(\mathcal{H}))$ such that

$$\tilde{\rho}(b + \mathcal{KL}(\mathcal{H})) = \tilde{f}_1(b + \mathcal{KL}(\mathcal{H})) \qquad (\forall b \in C^*(a)).$$

Define a linear functional ρ on $\mathcal{A}$ by

$$\rho(x) = \tilde{\rho}(x + \mathcal{KL}(\mathcal{H})) \qquad (\forall x \in \mathcal{A}).$$

By [22; 2.11.8.(ii)], ρ is a pure state of $\mathcal{A}$, and it is then easily verified that ρ satisfies (2).

This completes the proof.

We present an example to illustrate theorem 1.9. .

1.9.1. Example- Let $\mathcal{A} = \mathcal{BL}(\mathcal{H})$ where $\mathcal{H}$ is an infinite-dimensional separable Hilbert space, and let a be the unilateral forward shift. It may be proved that $C^*(a)/\mathcal{KL}(\mathcal{H})$ is isometric and $*$-isomorphic to the C^*-algebra of complex valued continuous functions on the unit circle [18; theorem 2], and that $\mathcal{KL}(\mathcal{H})$ is the smallest closed two-sided ideal of $C^*(a)$.

In order to find a state f of $\mathcal{A}$ which satisfies the hypotheses of theorem 1.9., we shall use the Schauder-Tychonoff fixed point theorem [24; theorem V.10.5].

Let

$$S = \{ f \in E(\mathcal{A}) : f(\mathcal{KL}(\mathcal{H})) = \{0\} \} .$$

Then, S is a w^*-compact convex subset of $E(\mathcal{A})$. For each $f \in S$ define a mapping $T_a f$ on $\mathcal{A}$ by

$$T_a f(x) = f(a^* x a) \qquad (\forall x \in \mathcal{A}) .$$

It is easily verified that $T_a f \in E(\mathcal{A})$; further, if x is any compact operator then $a^* x a$ is again compact (since $\mathcal{KL}(\mathcal{H})$ is a two-sided ideal of $\mathcal{BL}(\mathcal{H})$), so that

$$T_a f(\mathcal{KL}(\mathcal{H})) = \{0\} \qquad (\forall f \in S) .$$

It follows that the map T_a defined on S by

$$f \rightarrow T_a f \qquad (f \in S)$$

maps S into itself. Finally, we show that T_a is w^*-continuous. Let $\{f_\alpha\}$ be a net of elements in S and soppose that

$$f_\alpha \xrightarrow{w^*} f_0 \qquad (f_0 \in S) .$$

Then, for each $x \in \mathcal{A}$ we have

$$f_\alpha(a^* x a) \rightarrow f_0(a^* x a) ,$$

i.e.,

$$T_a f_\alpha(x) \to T_a f_0(x),$$

so that

$$T_a f_\alpha \xrightarrow{w^*} T_a f_0 .$$

Thus, T_a is a w^*-continuous mapping of the w^*-compact convex set S into itself; by the Schauder-Tychonoff fixed point theorem, there exists $f \in S$ such that $T_a f = f$, i.e.,

$$f(a^* x a) = f(x) \qquad (\forall x \in \mathcal{A}).$$

In particular, replacing x by ax we get

$$f(ax) = f(xa) \qquad (\forall x \in \mathcal{A}),$$

since $a^* a = 1$. Hence, by proposition 1.4., we have

$$f(bx) = f(xb) \qquad (\forall b \in C^*(a), \forall x \in \mathcal{A}),$$

and, also $f(\mathcal{K}\mathcal{L}(\mathcal{A})) = \{0\}$, since $f \in S$.

Thus, the hypotheses of theorem 1.9. are satisfied.

We close this section with the following theorem concerning commutative C^*-algebras.

1.10. Theorem- Let $\mathcal{B}$ be an abelian C^*-subalgebra of a C^*-algebra $\mathcal{A}$. There exists a state f of $\mathcal{A}$ such that

$$f(bx) = f(xb) \qquad (\forall x \in \mathcal{A}, \forall b \in \mathcal{B}).$$

Proof- Let $\mathcal{U}$ be the unitary group of $\mathcal{B}$. For each $u \in \mathcal{U}$ and each $f \in E(\mathcal{A})$ define a map $T_u f$ on $\mathcal{A}$ by

$$T_u f(x) = f(u^* x u) \qquad (\forall x \in \mathcal{A}).$$

For each $u \in \mathcal{U}$ define T_u on $E(\mathcal{A})$ by

$$T_u : f \to T_u f ,$$

and let

$$\mathcal{J} = \{ T_u : u \in \mathcal{U} \}.$$

Since $\mathcal{U}$ is commutative, it is easily verified that $\mathcal{J}$ is a commuting family of affine mappings, mapping the w^*-compact, convex set $E(\mathcal{A})$ into itself; further, a similar reasoning as that of example 1.9.1. shows that each T_u is w^*-continuous; hence, by the Markov-Kakutani fixed point theorem [24; theoremV 10.6.] there exists a fixed point, i.e., there exists $f \in E(\mathcal{A})$ such that

$$f(u^* x u) = f(x) \qquad (\forall x \in \mathcal{A}, \forall u \in \mathcal{U}).$$

hence,

$$f(bx) = f(xb) \qquad (\forall x \in \mathcal{A}, \forall b \in \mathcal{B}).$$

This completes the proof.

§ 2. Characters.

2.1. Introduction- In this section we study the theory of characters on C^*-algebras generated by families of operators. Using the methods developed in chapters I and II, we shall show that the concept of the joint approximate point spectrum of operators is directly related to the theory of characters, even in the case of a C^*-algebra generated by a single operator a : the existence of a character on $C^*(a)$ is equivalent to the non-emptiness of the joint approximate point spectrum of a and a^*.

Let $\mathcal{A}$ be a C^*-algebra, and let $\mathcal{B}$ be a C^*-subalgebra of $\mathcal{A}$ (we assume, as usual, that $\mathcal{B}$ contains the identity of $\mathcal{A}$).

A character on $\mathcal{B}$ is a self-adjoint multiplicative linear functional φ on $\mathcal{B}$ such that $\varphi(1) = 1$. (Actually, the condition that φ be self- adjoint is redundant; c.f. [6; §16, proposition 3]).

The set of all characters on $\mathcal{B}$ will be denoted by $\Phi_{\mathcal{B}}$.

Let $\varphi \in \Phi_{\mathcal{B}}$ and let x^*x be an arbitrary positive element of $\mathcal{B}$. Since φ is self-adjoint and multiplicative, we have

$$\varphi(x^*x) = |\varphi(x)|^2 = 1,$$

so that φ is, in fact, positive. Hence, since $\varphi(1) = 1$, it follows that $\varphi \in E(\mathcal{B})$.

Next, suppose that

$$\varphi = (1-\alpha) f_1 + \alpha f_2 \qquad (0 < \alpha < 1)$$

where $f_1, f_2 \in E(\mathcal{B})$. Then, for each $x \in \mathcal{B}$ we have

$$\begin{aligned} 0 &= \varphi((x-\varphi(x))^*(x-\varphi(x))) \\ &= (1-\alpha) f_1((x-\varphi(x))^*(x-\varphi(x))) + \\ &\quad \alpha\, f_2((x-\varphi(x))^*(x-\varphi(x))), \end{aligned}$$

so that

$$\begin{aligned} 0 &= f_1((x-\varphi(x))^*(x-\varphi(x))) \\ &= f_2((x-\varphi(x))^*(x-\varphi(x))). \end{aligned}$$

Hence

$$f_1(x-\varphi(x)) = f_2(x-\varphi(x)) = 0,$$

i.e.,

$$f_1(x) = f_1(\varphi(x)) = \varphi(x),$$

and

$$f_2(x) = f_2(\varphi(x)) = \varphi(x),$$

It follows that $\varphi \in \mathcal{P}(\mathcal{B})$. Thus, every element of $\Phi_{\mathcal{B}}$ is a pure state of $\mathcal{B}$.

The following result relates the theory of characters to that of finite operators.

2.2. Proposition- Let a be an element of a C^*-algebra $\mathcal{A}$.

(i)- If there exists a character on $C^*(a)$ then a is a finite element of $\mathcal{A}$.

(ii)- If there exists a pure state ρ of $\mathcal{A}$ such that

$$\rho(ax) = \rho(xa) \qquad (\forall x \in \mathcal{A})$$

then

$$\rho\big|_{C^*(a)} \in \bar{\Phi}_{C^*(a)} \; .$$

Proof- (i). Let $\varphi \in \bar{\Phi}_{C^*(a)}$; since $\varphi \in \mathcal{P}(C^*(a))$, there exists a pure state f of $\mathcal{A}$ such that

$$f\big|_{C^*(a)} = \varphi \; .$$

If now x is an arbitrary element of $\mathcal{A}$, we have, by the Cauchy-Schwartz inequality,

$$\begin{aligned} |f(xa) - f(x)f(a)|^2 &= |f(x(a-\varphi(a)))|^2 \\ &\leq f(xx^*)\, f((a-\varphi(a))^*(a-\varphi(a))) \\ &= 0, \end{aligned}$$

since $f\big|_{C^*(a)} = \varphi \in \bar{\Phi}_{C^*(a)}$.

Similarly,

$$f(ax) = f(a)f(x) \qquad (\forall x \in \mathcal{A}).$$

This proves (i).

(ii)- Let ρ have the stated property in (ii). By proposition 1.4., we have

$$\rho(xb) = \rho(bx) \qquad (\forall x \in \mathcal{A},\ \forall b \in C^*(a)).$$

The result now follows from [43; proposition 5].

This completes the proof.

The following lemma will be used in our future work without specific mention. The proof is simple and is omitted.

2.3. Lemma- Let $\{a_\tau : \tau \in \Gamma\}$ be a family of elements of a C^*-algebra $\mathcal{A}$. Suppose that there exists $\varphi \in E(\mathcal{A})$ (or $E(C^*(a))$) such that

$$\varphi(x a_\tau) = \varphi(a_\tau x) = \varphi(x)\varphi(a_\tau) \qquad (\forall \tau \in \Gamma, \forall x \in C^*(a_\tau)_{\tau\in\Gamma})$$

Then

$$\varphi|_{C^*(a_\tau)_{\tau\in\Gamma}} \in \Phi_{C^*(a_\tau)_{\tau\in\Gamma}}.$$

We now turn to the question of necessary and sufficient conditions for the existence of characters.

2.4. Theorem- Let $\{a_\tau : \tau \in \Gamma\}$ be a family of elements of a C^*-algebra $\mathcal{A}$ and let $(\lambda_\tau)_{\tau\in\Gamma}$ and $(\mu_\tau)_{\tau\in\Gamma}$ be Γ-tuples of complex numbers.

(a)- Suppose that

$$((\lambda_\tau)_{\tau\in\Gamma}, (\mu_\tau)_{\tau\in\Gamma}) \in J\sigma_{ap}((a_\tau)_{\tau\in\Gamma}, (a_\tau^*)_{\tau\in\Gamma}) \qquad (1).$$

Then

$$\mu_\tau = \bar{\lambda}_\tau \qquad (\forall \tau \in \Gamma)$$

and there exists $\varphi \in \Phi_{C^*(a_\tau)_{\tau\in\Gamma}}$ such that

$$\varphi(a_\tau) = \lambda_\tau \qquad (\forall \tau \in \Gamma).$$

(b)- A necessary and sufficient condition that there be a character on $C^*(a_\tau)_{\tau\in\Gamma}$ is that

$$J\sigma_{ap}((a_\tau)_{\tau\in\Gamma}, (a_\tau^*)_{\tau\in\Gamma}) \neq \emptyset$$

Proof- (a). Let a(1) be satisfied; then, there exists a state f of $\mathcal{A}$ such that

$$f(x a_\tau) = \lambda_\tau f(x) \qquad (\forall x \in \mathcal{A}, \forall \tau \in \Gamma),$$

and

$$f(xa_\tau^*) = \mu_\tau f(x) \qquad (\forall x \in \mathcal{A}, \forall \tau \in \Gamma).$$

In particular, for each $\tau \in \Gamma$ and each $b \in C^*(a_\tau)_{\tau\in\Gamma}$ we have

$$\lambda_\tau = f(a_\tau) = \overline{f(a_\tau^*)} = \bar{\mu}_\tau ,$$

and

$$f(a_\tau b) = \overline{f(b^* a_\tau^*)} = \overline{f(b^*)}\ \overline{f(a_\tau^*)} = \lambda_\tau f(b) .$$

Hence, for each $\tau \in \Gamma$, f is also right-multiplicative with respect to a_τ with $f(a_\tau) = \lambda_\tau$. It follows that the restriction of f to $C^*(a_\tau)_{\tau\in\Gamma}$ is a character.

This proves (a).

(b). Suppose that φ is a character on $C^*(a_\tau)_{\tau\in\Gamma}$. Extend φ to a pure state f of $\mathcal{A}$. It follows, as in the proof of proposition 2.2.(i) that

$$f(x(a_\tau - \varphi(a_\tau))) = 0 \qquad (\forall x \in \mathcal{A}, \forall \tau \in \Gamma),$$

and

$$f(x(a_\tau^* - \overline{\varphi(a_\tau)})) = 0 \qquad (\forall x \in \mathcal{A}, \forall \tau \in \Gamma).$$

Hence, by the remark immediately preceding theorem 2.2. of chapter III, we have

$$((\varphi(a_\tau))_{\tau\in\Gamma}, (\overline{\varphi(a_\tau)})_{\tau\in\Gamma}) \in$$

$$J\sigma_{ap}((a_\tau)_{\tau\in\Gamma}, (a_\tau^*)_{\tau\in\Gamma}).$$

The converse follows from part (a).

This completes the proof.

2.4.1. Remark- Let $\mathcal{B}$ be a C^*-subalgebra of a C^*-algebra $\mathcal{A}$, and suppose that there exists a character φ on $\mathcal{B}$. Then, for each indexed family $\{ b_\nu : \nu \in \Lambda \}$ of elements of $\mathcal{B}$ we have

$$\mathcal{J}\sigma_{ap}((b_\nu)_{\nu\in\Lambda}, (b_\nu^*)_{\nu\in\Lambda}) \neq \emptyset,$$

and, in particular,

$$\mathcal{J}\sigma_{ap}\ (b_\nu)_{\nu\in\Lambda} \neq \emptyset.$$

This is because the restriction of φ to $C^*(b_\nu)_{\nu\in\Lambda}$ is a character and the assertion then follows from theorem 2.4. (b).

In particular, with the notations and hypotheses of theorem 2.4.(a) we have that

$$\mathcal{J}\sigma_{ap}((a_\tau)_{\tau\in\Gamma}, (a_\tau^* a_\tau)_{\tau\in\Gamma}) \neq \emptyset \qquad (1).$$

The question now arises as to whether every point belonging to the left- hand side of (1) gives rise to a character on $C^*(a_\tau)_{\tau\in\Gamma}$. The answer is no in general as the following example shows.

Let b be a non-unitary isometric operator on an infinite-dimensional Hilbert space $\mathcal{H}$, and let $a = b^*$. Since b does not have dense range (otherwise, it would be invertible since it is bounded below), there exists a unit vector ξ in $\mathcal{H}$ such that $b^*\xi = 0$; let $f = \omega_\xi$ be the vector state defined by ξ; then

$$f(a^*a) = f(bb^*) = \langle b^*\xi, b^*\xi \rangle = 0;$$

hence

$$(0,0) \in \mathcal{J}\sigma_{ap}\ (a, a^*a)$$

(proposition 1.6. of chapter II). However, there is no character φ on $C^*(a)$ with $\varphi(a)=0$; otherwise, we would have

$$1 = \varphi(1) = \varphi(a^*a) = |\varphi(a)|^2 = 0$$

which is absurd.

On the other hand, we have the following result:

2.5. Theorem- Let $\{ a_\tau : \tau \in \Gamma \}$ be a family of elements of a C^*-algebra $\mathcal{A}$, and let $(\lambda_\tau)_{\tau\in\Gamma}$ and $(\mu_\tau)_{\tau\in\Gamma}$ be Γ-tuples of complex numbers. Suppose that

$$\left((\lambda_\tau)_{\tau\in\Gamma}, (\mu_\tau)_{\tau\in\Gamma} \right) \in$$

$$\widetilde{\sigma}_{ap}\left((a_\tau)_{\tau\in\Gamma}, (a_\tau^* a_\tau)_{\tau\in\Gamma} \right) \tag{1}$$

If

$$\lambda_\tau \neq 0 \qquad (\forall \tau \in \Gamma) \tag{2}$$

then there exists $\varphi \in \Phi_{C^*(a_\tau)_{\tau\in\Gamma}}$ such that

$$\varphi(a_\tau) = \lambda_\tau \qquad (\forall \tau \in \Gamma) .$$

Proof- By (1), there exists a state f of $\mathcal{A}$ such that

$$f(x a_\tau) = \lambda_\tau f(x) \qquad (\forall x \in \mathcal{A},\ \forall \tau \in \Gamma) \tag{3}$$

and

$$f(x\, a_\tau^* a_\tau) = \mu_\tau f(x) \qquad (\forall x \in \mathcal{A},\ \forall \tau \in \Gamma) \tag{4}$$

Then, for each $\tau \in \Gamma$ we have

$$\mu_\tau = f(a_\tau^* a_\tau) = |f(a_\tau)|^2 = |\lambda_\tau|^2 \tag{5}$$

hence, if x is an arbitrary element of $\mathcal{A}$ and τ is an arbitrary element of Γ, we have

$$\begin{aligned} f(a_\tau x) &= \overline{f(x^* a_\tau^*)} \\ &= \frac{1}{\lambda_\tau} \overline{f(x^* a_\tau^* a_\tau)} && \text{(by (2) and (3))} \\ &= \frac{1}{\bar{\lambda}_\tau} f(x) f(a_\tau^* a_\tau) && \text{(by (4))} \\ &= \lambda_\tau f(x) \end{aligned}$$

by (5), so that f is also right-multiplicative with respect to each a_τ. We may now take $\varphi = f \mid C^*(a_\tau)_{\tau\in\Gamma}$.

This completes the proof.

We shall prove two more general theorems concerning the existence of characters.

2.6. Theorem- Let $\{a_\tau : \tau\in\Gamma\}$ be a family of elements of a C^*-algebra $\mathcal{A}$, and let

$$(\lambda_\tau)_{\tau\in\Gamma} \in \mathcal{J}\sigma_{ap}(a_\tau)_{\tau\in\Gamma} .$$

Suppose that there exists a Γ-tuple $(\alpha_\tau)_{\tau\in\Gamma}$ of positive numbers such that

$$((0_\tau)_{\tau\in\Gamma}, (0_\tau)_{\tau\in\Gamma}) \in \mathcal{J}\sigma_{ap}((h_\tau)_{\tau\in\Gamma}, (k_\tau)_{\tau\in\Gamma}) \qquad (1)$$

where, for each τ, $0_\tau = 0$, and

$$h_\tau = ((a_\tau - \lambda_\tau)^*(a_\tau - \lambda_\tau))^{\alpha_\tau}$$

and

$$k_\tau = ((a_\tau - \lambda_\tau)(a_\tau - \lambda_\tau)^*)^{\alpha_\tau} .$$

Then, there exists a character φ on $C^*(a_\tau)_{\tau\in\Gamma}$ such that

$$\varphi(a_\tau) = \lambda_\tau \qquad (\forall \tau\in\Gamma) .$$

Proof- By (1), there exists $f \in E(\mathcal{A})$ such that

$$f(h_\tau) = f(k_\tau) = 0 \qquad (\forall\tau\in\Gamma) \quad (2) .$$

Let τ be an arbitrary but fixed element of Γ. In the following argument, we shall assume that the corresponding positive number α_τ is less than or equal to 1 ; the case where $\alpha_\tau > 1$ is settled similarly.

By the functional calculus, there exist non-negative integers

m and n such that

$$\left((a_\tau-\lambda_\tau)^*(a_\tau-\lambda_\tau)\right)^{\frac{1}{2^m}} \leq h_\tau$$

and

$$\left((a_\tau-\lambda_\tau)(a_\tau-\lambda_\tau)^*\right)^{\frac{1}{2^n}} \leq k_\tau \quad ;$$

hence, by (2),

$$f\left(\left((a_\tau-\lambda_\tau)^*(a_\tau-\lambda_\tau)\right)^{\frac{1}{2^m}}\right)=0$$

and

$$f\left(\left((a_\tau-\lambda_\tau)(a_\tau-\lambda_\tau)^*\right)^{\frac{1}{2^n}}\right)=0 \quad .$$

It follows (c.f. the remark immediately preceding proposition 1.6. of chapter II) that

$$f\left((a_\tau-\lambda_\tau)^*(a_\tau-\lambda_\tau)\right)=0$$

and

$$f\left((a_\tau-\lambda_\tau)(a_\tau-\lambda_\tau)^*\right)=0 \quad .$$

Therefore, by remark 1.4.1.(a) of chapter II, f is left-multiplicative with respect to a_τ and a_τ^* with $f(a_\tau)=\lambda_\tau$. Since τ was arbitrary, this completes the proof.

2.6.1. Corollary- Let a be an element of a C^*-algebra $\mathcal{A}$ and suppose that the following condition holds:

$$\forall \lambda \in \sigma_{ap}(a), \ \exists\, \alpha, M \in \mathbb{R}_+ \quad \text{such that}$$

$$\left((a-\lambda)(a^*-\bar\lambda)\right)^{\alpha} \leq M\left((a^*-\bar\lambda)(a-\lambda)\right)^{\alpha} \qquad (1)$$

Then, every state of $\mathcal{A}$ which is left-multiplicative with respect to a is also right-multiplicative with respect to a .

Proof- Let $f \in E(\mathcal{A})$ be left-multiplicative with respect to a

with $f(a)=\lambda$; then $\lambda\in\sigma_{ap}(a)$ (chapter I, theorem 2.2.a(iv)); hence, there exists positive numbers α and M such that (1) above holds.

Let now $\bar{\mathcal{A}}$ be the enveloping V.N. algebra of $\mathcal{A}$, and extend f to a state $\bar{f}$ of $\bar{\mathcal{A}}$. By lemma 4.2.b(ii) of chapter I, there exists an element b of $\bar{\mathcal{A}}$ such that

$$((a-\lambda)(a^*-\bar{\lambda}))^{\alpha/2}=b((a^*-\bar{\lambda})(a-\lambda))^{\alpha/2} \qquad (2).$$

Now

$$\bar{f}((a^*-\bar{\lambda})(a-\lambda))=f((a^*-\bar{\lambda})(a-\lambda))=0$$

since f is left-multiplicative with respect to a and $f(a)=\lambda$. Hence by the functional calculus, we have

$$\bar{f}(((a^*-\bar{\lambda})(a-\lambda))^{\alpha})=0 \qquad (3).$$

Therefore, by (2), (3), and the Cauchy-Schwartz inequality, we have

$$\bar{f}(((a-\lambda)(a^*-\bar{\lambda}))^{\alpha})=0 .$$

It now follows, as in the proof of theorem 2.6., that $\bar{f}$ is left-multiplicative with respect to a and a^* with $\bar{f}(a)=\lambda$. Since $\bar{f}|_{C^*(a)}=f|_{C^*(a)}$,this completes the proof.

2.7. Theorem- Let $\{a_\tau : \tau\in\Gamma\}$ be a family of elements of a C^*-algebra $\mathcal{A}$. Suppose that there exists a state f of $\mathcal{A}$ such that

$$|f(a_\tau)|=\|a_\tau\| \qquad (\forall\tau\in\Gamma).$$

Then

$$f|_{C^*(a_\tau)_{\tau\in\Gamma}}\in\Phi_{C^*(a_\tau)_{\tau\in\Gamma}} .$$

Proof- For each $x \in \mathcal{A}$ and each $\tau \in \Gamma$ we have

$$|f(xa_\tau) - f(x)f(a_\tau)|^2 =$$
$$|f(x(a_\tau - f(a_\tau)))|^2 \leq$$
$$f(xx^*)\, f((a_\tau^* - \overline{f(a_\tau)})(a_\tau - f(a_\tau))) \,;$$

since

$$|f(a_\tau)|^2 \leq f(a_\tau^* a_\tau) \leq \|a_\tau\|^2 = |f(a_\tau)|^2,$$

it follows that

$$f(xa_\tau) - f(x)f(a_\tau) = 0$$

so that f is left-multiplicative with respect to each a_τ .
Similarly, f is right-multiplicative with respect to each a_τ .
This completes the proof.

§3. Fully Charactered Operators.

3.1. Introduction- Let a be a normal operator in a C^*-algebra $\mathcal{A}$. It is well-known that there exists a 1-1 correspondence between the set of characters on $C^*(a)$ and the spectrum of a . Now, for a normal operator, we have $\sigma(a) = \sigma_{ap}(a)$; further, any pure state that is left-multiplicative with respect to a is also right-multiplicative with respect to a . This, together with proposition 1.2.3. of chapter I, motivates the following definition :

3.2. Definition- Let a be an element of a C^*-algebra $\mathcal{A}$. We say that a is fully charactered provided that every pure state of $\mathcal{A}$ which is left-multiplicative with respect to a is also right-multiplicative with respect to a .

3.3. Proposition- Let a be a fully charactered element of $\mathcal{A}$, and let $f \in E(\mathcal{A})$. If f is left-multiplicative with respect to a then f is also right-multiplicative with respect to a.

Proof- By Remark 5.3.1. of Ch. I, f may be expressed as a w^*-limit of convex combinations of pure states f_{α_j} of $\mathcal{A}$ each of which is left-multiplicative with respect to a with $f_{\alpha_j}(a) = f(a)$ $(\forall \alpha_j)$. By definition 3.2., each f_{α_j}. is also right- multiplicative with respect to a; hence f is also right-multiplicative with respect to a.

This completes the proof.

3.3.1. Remarks- (a). Suppose that a is a fully charactered element of a C^*-algebra $\mathcal{A}$, and let $\mathcal{B}$ be a C^*-algebra containing $\mathcal{A}$. If f is a state of $\mathcal{B}$ which is left-multiplicative with respect to a, then f is also right-multiplicative with respect to a. For, by proposition 3.3.,

$$f|_{C^*(a)} \in \Phi_{C^*(a)} .$$

The Cauchy-Schwartz inequality now proves the required result.

Thus, the property that an operator is fully charactered is independent of the C^*-algebra containing the operator.

(b)- If a is fully charactered then a^* need not be fully charactered. For example, if a is a non-unitary isometry then, by corollary 2.6.1., a is fully charactered. However, a^* is not fully charactered (remark 2.4.1.).

3.4. Proposition- Let a be a fully charactered element of a C^*-algebra $\mathcal{A}$.

(i)- There exists a 1-1 correspondence between $\Phi_{C^*(a)}$ and $\sigma_{ap}(a)$.

(ii)- For each $\lambda \in \sigma_{ap}(a)$, we have

$$(\lambda, \bar{\lambda}) \in J\sigma_{ap}(a, a^*) .$$

In particular,

$$\lambda \in \sigma_{ap}(a) \Rightarrow \bar{\lambda} \in \sigma_{ap}(a^*) .$$

(iii)- Suppose that $\mathcal{A}$ acts in its universal representation on a Hilbert space $\mathcal{H}$, and let $a = u\,(a^*a)^{\frac{1}{2}}$ be the polar decomposition of a. Then

$$\mathcal{J}\sigma_{ap}\ (a, u) \neq \emptyset$$

and there is a 1-1 correspondence between $\mathcal{J}\sigma_{ap}\ (a, u^*)$ and $\Phi C^*(a, u)$.

Proof- (i). Let $\lambda \in \sigma_{ap}(a)$; by theorem 2.2. of chapter I, there exists a pure state f of $\mathcal{A}$ such that f is left-multiplicative with respect to a with $f(a) = \lambda$; by definition 3.2., $f|_{C^*(a)}$ is a character on $C^*(a)$.

Conversely, if φ is a character on $C^*(a)$ then, by proposition 1.2.3. of chapter I, we have $\varphi(a) \in \sigma_{ap}(a)$.

This proves (i).

(ii). Let $\lambda \in \sigma_{ap}(a)$; by part (i), there exists $\varphi \in \Phi_{C^*(a)}$ such that $\varphi(a) = \lambda$. Hence

$$(\lambda, \bar{\lambda}) \in \mathcal{J}\sigma_{ap}\ (a, a^*)$$

by the proof of theorem 2.4.. In particular, we have $\bar{\lambda} \in \sigma_{ap}(a^*)$.

This proves (ii).

(iii). Suppose first that there exists $\lambda \in \sigma_{ap}(a)$ with $\lambda \neq 0$. By part (i), there exists $\varphi \in \Phi_{C^*(a)}$ such that $\varphi(a) = \lambda$. Extend φ to a state f of $\overline{\mathcal{A}}^{\tau_\omega}$ (note that $u \in \overline{\mathcal{A}}^{\tau_\omega}$). Then, by the Cauchy-Schwartz inequality, we have

$$f(xa) = f(ax) = f(a)f(x) \qquad (\forall x \in \overline{\mathcal{A}}^{\tau_\omega})$$

so that

$$f(a) = f(uu^*a) = f(uu^*)f(a) \tag{1}$$

and

$$f(a^*) = f(u^*ua^*) = f(u^*u)\,f(a^*) \tag{2}$$

and

$$f(u^*)\,f(a) = f(u^*a) = f((a^*a)^{\frac{1}{2}}) = |f(a)| \tag{3}$$

Hence, since $\lambda = f(a) \neq 0$, we have

$$f(u^*u) = f(uu^*) = 1 \quad \& \quad |f(u)| = 1 .$$

It follows (as in the proof of theorem 2.7.) that

$$f|_{C^*(u)} \in \bar{\Phi}_{C^*(u)} .$$

Since

$$f|_{C^*(a)} \in \bar{\Phi}_{C^*(a)} ,$$

it follows that

$$f|_{C^*(a,u)} \in \bar{\Phi}_{C^*(a,u)} .$$

In particular,

$$(f(a), f(u)) \in J\sigma_{ap}(a,u) .$$

by proposition 1.7. of chapter II.

Next, suppose that $\sigma_{ap}(a) = \{0\}$. By part (ii), we have

$$(0,0) \in J\sigma_{ap}(a, a^*)$$

Since $\mathcal{A}$ is in its universal representation, it follows that

$$(0,0) \in J\sigma_p(a,a^*)$$

so, there exists a vector $\xi \in \mathcal{H}$ such that $\|\xi\| = 1$ and

$$a\xi = a^*\xi = 0 \tag{4}$$

Now u^*u is the support projection of $(a^*a)^{\frac{1}{2}}$, so [53; lemma 2] it is the strong limit of the sequence $\{(a^*a)^{\frac{1}{2^n}}\}_{n=1}^{\infty}$;

by (4), we have

$$(a^*a)^{\frac{1}{2^n}}\xi = 0 \qquad (n = 0, 1, 2, \dots) \; ;$$

hence $u^*u\xi = 0$. Similarly, $uu^*\xi = 0$. Therefore, the vector state ω_ξ defined by ξ is a character on $C^*(a, u)$ with

$$\omega_\xi(a) = \omega_\xi(u) = 0$$

In particular,

$$(0, 0) \in J\sigma_{ap}(a, u) .$$

This proves the first part of (iii).

To prove the second part, let

$$(\lambda, \mu) \in J\sigma_{ap}(a, u^*) .$$

Then, there exists $f \in E(\bar{\mathcal{A}}^{\tau_\omega})$ such that f is left-multiplicative with respect to a and u^*, with

$$f(a) = \lambda \text{ and } f(u^*) = \mu .$$

Since a is fully charactered, f is also right-multiplicative with respect to a (remark 3.3.1.(a)). Therefore, the equalities (1), (2), and (3) above hold. The proof may now be completed as in the proof of the first part by distinguishing between the cases $\lambda = 0$ and $\lambda \neq 0$.

Finally, if there exists a character φ on $C^*(a, u^*)$ then, by proposition 1.7. of chapter II, we have

$$(\varphi(a), \varphi(u^*)) \in J\sigma_{ap}(a, u^*).$$

This proves (iii) and completes the proof of the theorem.

Our next result concerns reduced V.N.algebras of given V.N. algebras which are singly generated by a Fully charactered operator.

3.5. Theorem- Let $\mathcal{A}$ be a V.N.algebra acting on a Hilbert space $\mathcal{H}$ and suppose that $\mathcal{A}$ is singly generated by a fully charactered operator A. If E is a central projection in $\mathcal{A}$ then, with

$$\mathcal{K} = E(\mathcal{H}) \quad \text{and} \quad B = EA|_{\mathcal{K}}$$

we have

(i) the operator B is a fully charactered element in the reduced V.N. algebra $E\mathcal{A}E$; and

(ii) there exists a 1-1 correspondence between $\overline{\Phi}_{C^*(B)}$ and $\sigma_{ap}(B)$.

Remark- The reduced V.N.algebra $E\mathcal{A}E$ is the set of operators T on $\mathcal{K}$ such that $T=(ES)|_{\mathcal{K}}$ for some $S \in \mathcal{A}$; c.f., [21; Ch. I, §2.].

Proof- Let g be a state of $E\mathcal{A}E$ such that g is left-multiplicative with respect to B, and put $g(B)=\lambda$. Define f on $\mathcal{A}$ by

$$f(X) = g((EX)|_{\mathcal{K}}) \qquad (\forall X \in \mathcal{A}) \qquad (1).$$

It is easily verified that $f \in E(\mathcal{A})$.

Let now X and Y be arbitrary elements of $\mathcal{A}$ and let ξ be an element of $\mathcal{K}$; then

$$\begin{aligned}((EXY)|_{\mathcal{K}})\xi &= (EXY)\xi \\ &= (EXE)(EYE)\xi \\ &= ((EX)|_{\mathcal{K}})((EY)|_{\mathcal{K}})\xi\end{aligned}$$

so that

$$(EXY)|_{\mathcal{K}} = ((EX)|_{\mathcal{K}})((EY)|_{\mathcal{K}}) \qquad (2).$$

Hence, by (1) and (2), we have

$$f(XY) = g\left(((EX)|_{\mathfrak{K}})((EY)|_{\mathfrak{K}})\right) . \qquad (\forall X \forall Y \in \mathcal{A})$$

If now X is an arbitrary element of $\mathcal{A}$,we have

$$\begin{aligned} f(XA) &= g(((EX)|_{\mathfrak{K}}) \cdot B) \\ &= g((EX)|_{\mathfrak{K}}) \cdot g(B) \\ &= \lambda f(X) \end{aligned}$$

so that f is left-multiplicative with respect to A with $f(A) = \lambda$. Since A is fully charactered, proposition 3.3. shows that f is a character on $C^*(A)$. It is then easily seen that g is a character on $C^*(B)$. In particular, g is also right-multiplicative with respect to B. This proves (i).

To prove (ii), we need merely note that the C^*-algebra $C^*(B)$ is singly generated by the fully charactered operator B ,and apply proposition 3.4.(i).

This completes the proof.

We conclude this section with an analogue of the Gelfand-Naimark theorem for normal operators.

The following lemma will be needed below.

3.6. Lemma- Let $\{ a_\tau : \tau \in \Gamma \}$ be a family of elements of a C^*-algebra $\mathcal{A}$,and suppose that each a_τ is fully charactered. Then, given

$$(\lambda_\tau)_{\tau \in \Gamma} \in J\sigma_{ap}\ (a_\tau)_{\tau\in\Gamma} \tag{1}$$

there exists $\varphi \in \Phi_{C^*(a_\tau)_{\tau\in\Gamma}}$ such that

$$\varphi(a_\tau) = \lambda_\tau \qquad (\forall \tau \in \Gamma).$$

Proof- Let (1) be satisfied; there exists a pure state f of $\mathcal{A}$ such that for each τ , f is left-multiplicative with respect to a_τ

with $f(a_\tau) = \lambda_\tau$ (theorem 1.3.(c) of chapter III). Since each a_τ is fully charactered, f is also right-multiplicative with respect to each a_τ. Hence, we may take $\varphi = f \mid C^*(a_\tau)_{\tau\in\Gamma}$.

This completes the proof.

3.7. Theorem- Let $\{a_\tau : \tau\in\Gamma\}$ be a family of elements of a C^*-algebra $\mathcal{A}$. Suppose that for each

$$(\lambda_\tau)_{\tau\in\Gamma} \in J\sigma_{ap}(a_\tau)_{\tau\in\Gamma}$$

there exists $\varphi\in \bar{\Phi}_{C^*(a_\tau)_{\tau\in\Gamma}}$ such that

$$\varphi(a_\tau) = \lambda_\tau \qquad (\forall\tau\in\Gamma);$$

let

$$\mathcal{J} = \left\{ x\in C^*(a_\tau)_{\tau\in\Gamma} : \varphi(x)=0 \quad (\forall\varphi\in\bar{\Phi}_{C^*(a_\tau)_{\tau\in\Gamma}}) \right\}.$$

Then, $\mathcal{J}$ is a closed two-sided ideal of $C^*(a_\tau)_{\tau\in\Gamma}$ and the C^*-algebra

$$C^*(a_\tau)_{\tau\in\Gamma} / \mathcal{J}$$

is isometrically isomorphic to the C^*-algebra of complex-valued continuous functions on $J\sigma_{ap}(a_\tau)_{\tau\in\Gamma}$.

Proof- This is essentially proved in [10; proposition 5]. For while proposition5 of [10] is proved for commuting families of hyponormal operators on a Hilbert space, an examination of the proof shows that the only property of a hyponormal operator used in the proof is that any family of hyponormal operators satisfies the condition expressed in the hypothesis of the present theorem. We therefore omit the full proof.

We remark that the conclusion of the above theorem holds trivially if $J\sigma_{ap}(a_\tau)_{\tau\in\Gamma} = \emptyset$; and that the conclusion holds for any family of fully charactered operators (lemma 3.6.).

Question: Is $\mathcal{J}$ a minimal closed two-sided ideal of $C^*(a_\tau)_{\tau\in\Gamma}$?

3.7.1. Corollary- Let a be a fully charactered element of a C^*-algebra $\mathcal{A}$ and let b be an element of $C^*(a)$ which commutes with b . Then with $\mathcal{J} = \{x \in C^*(b) : \varphi(x) = 0 \ (\forall \varphi \in \bar{\Phi}_{C^*(b)})\}$, we have

$$\forall \lambda \in \sigma_{ap}(b), \ \exists \varphi \in \bar{\Phi}_{C^*(b)} \ \text{s.t.} \ \varphi(b) = \lambda ,$$

and $C^*(b)/_{\mathcal{J}}$ is isometrically $*$-isomorphic to $\mathcal{C}_{\mathbb{C}}(\sigma_{ap}(b))$.

Proof- Let $\lambda \in \sigma_{ap}(b)$. Since a and b commute, there exists $\mu \in \sigma_{ap}(a)$ such that

$$(\lambda, \mu) \in J\sigma_{ap}(b, a) ;$$

hence, there exists $f \in E(\mathcal{A})$ such that f is left-multiplicative with respect to a and b with

$$f(a) = \mu \ \ \& \ \ f(b) = \lambda .$$

Since a is fully charactered, $f/_{C^*(a)} \in \bar{\Phi}_{C^*(a)}$; hence

$$f/_{C^*(b)} \in \bar{\Phi}_{C^*(b)} .$$

The second part follows from theorem 3.7. .

This completes the proof.

§ 4. Some Examples .

4.1. Introduction- In this section, we give some examples of fully charactered and finite operators. We shall also be interested in the existence of states f of a C^*-algebra $\mathcal{A}$ such that f satisfies the trace-like property expressed in definition 1.2.(i) with respect to the C^*-algebra generated by a fully charactered element a of $\mathcal{A}$,and such that f is not a character on $C^*(a)$.

4.2. Isometries- Let $\mathcal{A}$ be a C^*-algebra, and let $a \in \mathcal{A}$. The operator a is said to be an isometry, provided that $a^*a = 1$. By corollary 2.6.1. (with $\alpha = M = 1$), every isometry is fully charactered. In particular, a is a finite operator by proposition 2.2.(i).

Let $\mathcal{H}$ be an infinite-dimensional separable Hilbert space with basis $\{e_n\}_{n=0}^{\infty}$ and let u be the unilateral shift defined by

$$u e_n = e_{n+1} \qquad (n = 0, 1, 2, \dots).$$

Then u is an isometry, and it is well-known that $\sigma_{ap}(u) = \mathbb{T}$ where

$$\mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}.$$

[27; problem 67]. By theorem 3.7. $C^*(u)/\mathcal{J}$ is isometrically $*$-isomorphic to $\mathcal{C}_{\mathbb{C}}(\mathbb{T})$ where

$$\mathcal{J} = \{x \in C^*(u) : \varphi(x) = 0 \ (\forall \varphi \in \Phi_{C^*(u)})\}.$$

We now show that

$$\mathcal{KL}(\mathcal{H}) \subseteq \mathcal{J}$$

assuming, by [18], that $\mathcal{KL}(\mathcal{H}) \subseteq C^*(u)$.

Since $1 - uu^* \in \mathcal{KL}(\mathcal{H})$ and

$$\varphi(1 - uu^*) = 1 - |\varphi(u)|^2 = 0 \qquad (\forall \varphi \in \Phi_{C^*(u)})$$

we have

$$\mathcal{J} \cap \mathcal{KL}(\mathcal{H}) \neq \emptyset.$$

If there exists $k \in \mathcal{KL}(\mathcal{H})$ such that $\varphi(k) \neq 0$ for some $\varphi \in \Phi_{C^*(u)}$ then, the kernel of φ would be a non-trivial closed two-sided ideal of $\mathcal{KL}(\mathcal{H})$; this is impossible since $\mathcal{KL}(\mathcal{H})$ is simple; hence

$$\mathcal{KL}(\mathcal{H}) \subseteq \mathcal{J} \subseteq C^*(u).$$

The above result (namely, that $\mathcal{KL}(\mathcal{H}) \subseteq \mathcal{J}$) is in accordance with Coburn's result which proves that $\mathcal{KL}(\mathcal{H})$ is the smallest closed two-sided ideal of $C^*(u)$; c.f., [18].

Let a be a non-unitary isometry in a C^*-algebra $\mathcal{A}$. The following argument proves the existence of a state f such that

$$f(bx) = f(xb) \qquad (\forall x \in \mathcal{A},\ \forall b \in C^*(a))$$

and

$$f|_{C^*(a)} \notin \Phi_{C^*(a)} \quad .$$

Let

$$S = \{ f \in E(\mathcal{A}) : f(a) = 0 \}$$

(note that $S \neq \emptyset$ since a is non-unitary). Then S is a w^*-compact convex subset of $E(\mathcal{A})$ and the mapping T_a defined on S by

$$T_a f(x) = f(a^* x a) \qquad (\forall x \in \mathcal{A},\ \forall f \in S)$$

is a w^*-continuous mapping of S into itself. By the Schauder-Tychonoff fixed point theorem, there exists $f \in S$ such that

$$f(a^* x a) = f(x) \qquad (\forall x \in \mathcal{A}) .$$

It is then easily verified that f has the required properties.

4.3. Quasinormal Operators- The class of quasinormal operators was first introduced by A.Brown in [8].

An operator a is said to be quasinormal provided that a and $a^* a$ commute.

Let $a = u\,(a^*a)^{1/2}$ be the polar decomposition of a . Then, a is quasinormal if and only if u and $(a^*a)^{1/2}$ commute [27; problem 108]. hence

$$\begin{aligned} aa^* &= u\,(a^*a)^{\frac{1}{2}}\, u^*\, (a^*a)^{\frac{1}{2}} \\ &= (a^*a)^{1/2}\, uu^*\, (a^*a)^{\frac{1}{2}} \\ &\leq \| uu^* \| \;\; a^*a \\ &= a^*a \, . \end{aligned}$$

Hence, by corollary 2.6.1. (with $\alpha = M = 1$), every quasinormal operator is fully charactered.

We present an alternative proof of the existence of characters on $C^*(a)$, using theorem 2.5. .

Let $\lambda \in \sigma_{ap}(a)$; since a and a^*a commute, there exists $\mu \in \sigma_{ap}(a^*a)$ such that

$$(\lambda, \mu) \in J\sigma_{ap}(a, a^*a) .$$

If $\lambda \neq 0$ then, by theorem 2.5., there exists $\varphi \in \Phi_{C^*(a)}$ such that $\varphi(a) = \lambda$; if $\lambda = 0$ then there exists $f \in E(C^*(a))$ such that

$$f(a) = f(a^*a) = 0 ,$$

hence

$$0 \leq f(aa^*) \leq f(a^*a) = 0$$

so that $f \in \Phi_{C^*(a)}$ with $f(a) = 0$.

We remark that every isometry is quasinormal (since in this case $a^*a = 1$).

4.4. Hyponormal Operators- An operator a is said to be hyponormal provided that $a^*a \geqslant aa^*$ [27; no. 160].

By corollary 2.6.1., (with $\alpha = M = 1$), every hyponormal operator is fully charactered. Here, we present an alternative proof of the existence of characters on $C^*(a)$ referred to at the end of §1 of chapter I. Thus, it is sufficient to prove that

$$\forall \lambda \in \sigma_p(a), \ \exists \varphi \in \Phi_{C^*(a)} \text{ s.t. } \varphi(a) = \lambda .$$

Let $\lambda \in \sigma_p(a)$ with eigenvector ξ ($\|\xi\| = 1$). Then

$$\begin{aligned} \| a^*\xi - \bar{\lambda}\xi \|^2 &= \langle aa^*\xi, \xi \rangle - |\lambda|^2 \\ &\leq \langle a^*a\xi, \xi \rangle - |\lambda|^2 \\ &= 0 \end{aligned}$$

so that

$$(\lambda, \bar{\lambda}) \in J\sigma_{ap}(a, a^*).$$

The result now follows from theorem 2.4. .

The existence of characters on the C^*- algebra generated by a hyponormal operator acting on a Hilbert space was first proved by Bunce in [9] . The result was later extended to commuting families of hyponormal operators in [10] and [37].

We remark that every subnormal operator is hyponormal [27; no. 160]; that the partial isometry in the polar decomposition of a hyponormal operator is easily seen to be hyponormal; and that the example given in example 2.7.(a) of chapter I shows that there exist hyponormal operators which are not subnormal [27; problem 160].

4.5. Quasi- hyponormal Operators- An operator a is said to be quasi- hyponormal provided that

$$a^{*2} a^2 - (a^* a)^2 \geq 0$$

[49].

Let k be a positive integer. An operator a is said to be hyponormal of order k provided that a^k is quasi- hyponormal [39].

Let a be quasi- hyponormal, and let $f \in E(C^*(a))$ be left-multiplicative with respect to a with $f(a) \neq 0$. Then

$$\begin{aligned} |f(a)|^4 &= f(a^{*2} a^2) \\ &\geq f((a^* a)^2) \\ &= |f(a)|^2 f(aa^*) \\ &\geq |f(a)|^4 \end{aligned}$$

so that $f(aa^*) = |f(a)|^2$; hence, f is also right-multiplicative with respect to a and $f \in \bar{\Phi}_{C^*(a)}$.

In particular, if a quasi-hyponormal operator is left-invertible (i.e., if $0 \notin \sigma_{ap}(a)$) then a is fully charactered.

Essentially the same argument shows that, if a is a hyponormal operator of order 2^n for some integer $n \geq 0$, and if a state f is left-multiplicative with respect to a then f is also right-multiplicative with respect to a, provided that $f(a) \neq 0$.

We remark that every hyponormal operator is quasi-hyponormal. The converse is false as is shown by the operator

$$T = \begin{pmatrix} A & (A^*A)^{\frac{1}{2}} \\ 0 & 0 \end{pmatrix}$$

where A is a non-normal hyponormal operator with $\mathrm{Re}\, \sigma(A) \geq 0$ (e.g., take $A = U + 1$, where U is the unilateral shift); c.f. [39; theorems 2 and 3].

4.6. M-hyponormal and Dominant Operators- An operator a is said to be M-hyponormal provided that there exists a *real* number M such that for all $\lambda \in \mathbb{C}$,

$$(a-\lambda)(a^*-\bar{\lambda}) \leq M^2 (a^*-\bar{\lambda})(a-\lambda)$$

[54].

An operator a is said to be dominant provided that for each $\lambda \in \sigma(a)$,there exists a positive number M_λ such that

$$(a-\lambda)(a^*-\bar{\lambda}) \leq M_\lambda (a^*-\bar{\lambda})(a-\lambda)$$

[52].

By corollary 2.6.1., every M-hyponormal and every dominant operator is fully charactered.

We remark that every hyponormal operator is M-hyponormal, and every M-hyponormal operator is dominant. Example 2.7.(b) of chapter II shows that there exist M-hyponormal operators which are not hyponormal, and the example given in [52; example 2] shows that there exist dominant operators which are not M-hyponormal for any real number M .

4.7. Paranormal Operators- The class of paranormal operators was first introduced in [30] under the name "operators of class N". The definition and properties of paranormal operators was later extended to C^*-algebras in [36] and studied in [40].

An operator a is said to be paranormal, provided that

$$a^{*2}a^{2} - 2\lambda a^*a + \lambda^2 \geq 0$$

for all non-negative numbers λ .

Let k be a positive integer. A bounded linear operator A on a Hilbert space $\mathcal{H}$ is said to be a $C(N, k)$ operator, provided that

$$\|Ax\|^k \leq \|A^k x\| \qquad (\forall x \in \mathcal{H}, \|x\|=1).$$

[32], and [33]. Thus, a paranormal operator is a $C(N, 2)$ operator, and conversely.

Let $\mathcal{A}$ be a C^*-algebra, and let $a \in \mathcal{A}$ be paranormal. We use the Schauder-Tychonoff fixed point theorem to show that a is finite.

By the definition of paranormal operators we have, for each $b \in \mathcal{A}$, and for each $f \in E(\mathcal{A})$,

$$f(bb^*)\lambda^2 - 2f(ba^*ab^*)\lambda + f(ba^{*2}a^2b^*) \geq 0 \quad (\forall \lambda \geq 0)$$

so that

$$(f(ba^*ab^*))^2 \leq f(bb^*)f(ba^{*2}a^2b^*) \qquad (\forall b \in \mathcal{A}).$$

Let f be a state of $\mathcal{A}$ such that

$$f(aa^*) = \|aa^*\| .$$

Then,

$$f|_{C^*(aa^*)} \in \bar{\Phi}_{C^*(aa^*)} ;$$

hence,

$$\begin{aligned}(f(aa^*))^4 &= (f(aa^*aa^*))^2 \\ &\leq f(a\,a^{*2}a^2a^*)\,f(aa^*) \\ &= (f(aa^*))^3 f(a^*a)\end{aligned}$$

so that $f(a^*a) \geq f(aa^*)$; also

$$f(a^*a) \leq \|a^*a\| = \|aa^*\| = f(aa^*) ;$$

hence

$$f(a^*a) = f(aa^*) = \|a^*a\| .$$

For the rest of the argument we shall assume, without real loss of generality, that $\|a\| = 1$.

Let

$$S = \{ f \in E(\mathcal{A}) : f(a^*a) = f(aa^*) = 1 \} .$$

By the above remarks, $S \neq \emptyset$; further, it is easily verified that S is a w^*-compact convex subset of $E(\mathcal{A})$. For each $f \in S$ define $T_a f$ on $\mathcal{A}$ by

$$T_a f(x) = f(a^*xa) \qquad (\forall x \in \mathcal{A});$$

then $T_a f \in E(\mathcal{A})$; further,

$$T_a f(aa^*) = f(a^*aa^*a) \geq (f(a^*a))^2 = 1$$

and

$$f(a^*aa^*a) \leq \|a^*aa^*a\| = 1$$

so that $T_a f(a a^*) = 1$. Similarly,

$$T_a f(a^* a) = f(a^{*2} a^2) \geq (f(a^* a))^2 = 1,$$

and

$$f(a^{*2} a^2) \leq \| a^{*2} a^2 \| \leq 1,$$

so that $T_a f(a^* a) = 1$. It follows that the map $T_a : S \to S$ defined by $f \to T_a f$ maps S into itself. Also, it is easily seen that T_a is w*-continuous. Hence, there exists a fixed point i.e., there exists $f \in S$ such that

$$f(a^* x a) = f(x) \qquad (\forall x \in \mathcal{A}).$$

If now x is an arbitrary element of $\mathcal{A}$,we have

$$\begin{aligned} |f(xa) - f(ax)|^2 &= |f(xa) - f(a^* a x a)|^2 \\ &= |f((1 - a^* a) x a)|^2 \\ &\leq f((1 - a^* a)^2) f((xa)^*(xa)) \\ &= 0 \end{aligned}$$

since $f(a^* a) = 1$.

This completes the proof.

We remark that every $C(N, k)$ operator is normaloid (c.f. 4.8. for normaloid operators) [32] , so that every $C(N, k)$ operator is finite. Also, every quasi-hyponormal operator is paranormal, but the converse is false in general [49]. Finally, there exists a normaloid operator which is not paranormal [26].

4.8. Unimodular Contractions and Normaloid Operators- An operator a is said to be normaloid provided that

$$r(a) = \| a \|$$

where $r(a)$ denotes the spectral radius of a [27; no.174].

Let $\mathcal{A}$ be a C^*-algebra, and let $a \in \mathcal{A}$ be normaloid. Since

$$\|a\| = r(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\}$$

it follows that corresponding to each positive integer n, there exists $\lambda_n \in \sigma(a)$ such that

$$\|a\| - \frac{1}{n} \leq |\lambda_n| \leq \|a\| .$$

Since $\sigma(a)$ is compact, it follows that there exists $\lambda \in \sigma(a)$ such that

$$|\lambda| = \|a\| .$$

If $\lambda \in \sigma_{ap}(a)$ then, there exists a state f of $\mathcal{A}$ such that $f(a) = \lambda$ so, by theorem 2.7., we have

$$f|_{C^*(a)} \in \Phi_{C^*(a)} .$$

If, on the other hand, $\lambda \in \sigma_{ap}(a^*)$ then, since

$$r(a^*) = r(a) = \|a\| = \|a^*\|$$

we may apply the same reasoning to a^* and conclude that

$$f|_{C^*(a)} \in \Phi_{C^*(a)} .$$

Thus, we have proved the following result:

4.8.1. Theorem- Given a normaloid operator a, there exists a character φ on $C^*(a)$ such that $|\varphi(a)| = \|a\|$.

Alternative proofs of the above result may be found in [43] and [3; corollary 3.1.3.].

4.8.1.1. Corollary- Let $\mathcal{A}$ be a V.N.algebra, let $a \in \mathcal{A}$ be normaloid, and let $a = u(a^*a)^{\frac{1}{2}}$ be the polar decomposition of a. Then u is normaloid.

Proof- By theorem 4.8.1., there exists $\varphi \in \Phi_{C^*(a)}$ such that

$$|\varphi(a)| = \|a\|$$

Extend φ to a state f of $\mathcal{A}$; by the Cauchy-Schwartz inequality, we have

$$f(bx) = f(xb) \qquad (\forall x \in \mathcal{A},\ \forall b \in C^*(a)).$$

In particular, since $u \in \mathcal{A}$, we have

$$f((a^*a)^{\frac{1}{2}}) = f(u^*a) = f(u^*)f(a)$$

so that

$$|f(u^*)| = \frac{f((a^*a)^{\frac{1}{2}})}{|f(a)|} = \frac{\varphi((a^*a)^{\frac{1}{2}})}{|\varphi(a)|} = 1 .$$

Since u is a partial isometry, u^*u is a projection, so

$$\|u\|^2 = \|u^*u\| = 1 ;$$

hence

$$|f(u)| = |f(u^*)| = 1 = \|u\| .$$

Therefore, by theorem 2.7., we have

$$f|_{C^*(u)} \in \Phi_{C^*(u)}$$

and, in particular, $f(u) \in \sigma(u)$; hence

$$1 = |f(u)| \leq r(u) \leq \|u\| = 1 ,$$

i.e., u is normaloid.

This completes the proof.

We remark that there exist non-normaloid partial isometries; for example, let

$$u = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} .$$

We shall use the above result to give a decomposition theorem for arbitrary elements of V.N. algebras. The following lemma will be needed:

4.8.2. Lemma- Let x be an element of a C^*-algebra $\mathcal{A}$. There exist a unitary operator $u \in \mathcal{A}$, and a normaloid operator $a \in \mathcal{A}$ such that $x = ua$.

Proof- Let $\mathcal{A}_1^*$ denote the unit ball of the dual of $\mathcal{A}$:

$$\mathcal{A}_1^* = \{ f \in \mathcal{A}^* : \|f\| = 1 \} .$$

Assuming, without real loss of generality, that $\|x\|=1$, the Hahn-Banach theorem implies that the set S defined by

$$S = \{ f \in \mathcal{A}_1^* : f(x) = \|x\| = 1 \}$$

is non-empty. An elementary argument shows that S is a w^*-compact convex subset of $\mathcal{A}_1^*$; let f be an extreme point of S; then f is an extreme point of $\mathcal{A}_1^*$. For suppose

$$f = (1-\alpha) f_1 + \alpha f_2 \qquad (f_1, f_2 \in \mathcal{A}_1^*;\ 0 < \alpha < 1).$$

Then

$$1 = f(x) = (1-\alpha) f_1(x) + \alpha f_2(x).$$

so that

$$|f_j(x)| \leq 1 \qquad (j = 1, 2).$$

Since 1 is an extreme point of the unit ball in the complex plane, it follows that

$$f_1(x) = f_2(x) = 1$$

so that $f_1, f_2 \in S$. Since f was an extreme point of S, it follows that $f = f_1 = f_2$, so that f is an extreme point of $\mathcal{A}_1^*$.

Now by [45 ; lemma 4], there exists a unitary operator $u \in \mathcal{A}$ and a pure state g of $\mathcal{A}$ such that

$$f(y) = g(uy) \qquad (\forall y \in \mathcal{A}).$$

In particular,

$$1 = f(x) = g(ux) \leq \|ux\| \leq 1$$

so that

$$g(ux) = \|ux\| = 1 .$$

Hence, ux is normaloid, and $x = u^*(ux)$.

This completes the proof.

4.8.2.1. Corollary- Let x be an arbitrary element of a V.N. algebra $\mathcal{A}$. Then,

$$x = uvh$$

where u is unitary, v is a normaloid partial isometry, and h is a positive operator in $\mathcal{A}$.

Proof- Corollary 4.8.1.1., and lemma 4.8.2. .

The question now arises as to whether every normaloid operator is fully charactered. The following example shows that the answer is no, in general.

Let $\mathcal{A} = M_3(\mathbb{C})$, and define an operator A by the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \alpha \end{pmatrix}$$

where $\alpha \geq 3$. Then

$$A^* = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & \alpha \end{pmatrix}$$

and an elementary calculation shows that

$$A^*A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 5 & 0 \\ 0 & 0 & \alpha^2 \end{pmatrix}$$

The eigenvalues of the matrix A^*A are $3 \pm \sqrt{5}$,and α^2; hence, since $\alpha^2 \geq 3+\sqrt{5}$,we have

$$\| A \| = \| A^*A \|^{\frac{1}{2}} = \alpha .$$

Also, the eigenvalues of A are 1, 2, and α , so that

$$r(A) = \| A \| = \alpha ,$$

i.e., A is normaloid.

Suppose now that there exists a character φ on $C^*(A)$ with $\varphi(A)=1$. Then

$$(1,1) \in J\sigma_p (A, A^*)$$

which means that there exists a non-zero vector $x = (x_1, x_2, x_3)^t$ (t for transpose) such that $Ax = x$ and $A^*x = x$. Now

$$Ax = \begin{pmatrix} x_1+x_2 \\ 2x_2 \\ \alpha x_3 \end{pmatrix} , \quad A^*x = \begin{pmatrix} x_1 \\ x_1+x_2 \\ \alpha x_3 \end{pmatrix}$$

so that the only solution to the simultaneous equations $Ax = x$ and $A^*x = x$ is $(0,0,0)^t$.

Thus, corresponding to the point $1 \in \sigma(A) = \sigma_{ap}(A)$,there is no character on $C^*(A)$, so that A is not fully charactered.

We remark that the above example uses a general construction given in [25; theorem 1]:

If A is an operator acting on a Hilbert space then,there exists an operator B such that $A \oplus B$ is normaloid.

Finally, we shall consider a sub-class of the class of normaloid operators,first introduced by B. Russo in [48], whose elements are fully charactered.

Let a be an element of a C^*-algebra $\mathcal{A}$. We call a a unimodular contraction provided that $\|a\| \leq 1$ and

$$\sigma(a) \subseteq \{\lambda \in \mathbb{C} : |\lambda| = 1\}.$$

Clearly, every unimodular contraction is normaloid. See also [31].

Let now $f \in E(\mathcal{A})$ be left-multiplicative with respect to a with $f(a) = \lambda$. Then $\lambda \in \sigma_{ap}(a) \subseteq \sigma(a)$, so $|\lambda| = 1$. Since

$$1 = |\lambda| \leq \|a\| \leq 1$$

we have

$$|f(a)| = |\lambda| = \|a\| = 1.$$

Hence, by theorem 2.7., f is also right-multiplicative with respect to a. Thus, a is fully charactered.

We close this section with the following remark: There exists a normaloid operator a, such that a is not a $C(N,k)$ operator for any positive integer k; c.f. [33; theorem 1].

4.9. Weighted Shifts and Bi-normal Operators- Let $\mathcal{H}$ be an infinite-dimensional Hilbert space with basis $\{e_n\}_{n=0}^{\infty}$.

A (unilateral) weighted shift is an operator W which satisfies the relation

$$We_n = \alpha_n e_{n+1} \qquad (n = 0, 1, 2, \ldots)$$

for some bounded sequence of complex numbers $\{\alpha_n\}_{n=0}^{\infty}$ [27; no. 75].

Using the theory of Banach limits, it may be shown that there exists $f \in E(\mathcal{BL}(\mathcal{H}))$ such that

$$g(XB) = g(BX) \qquad (\forall X \in \mathcal{BL}(\mathcal{H}), \forall B \in \mathcal{B})$$

where $\mathcal{B}$ denotes the C^*-algebra generated by the set of all weighted shift operators; c.f. [12; §3].

An operator A is said to be bi-normal provided that A^*A

commutes with AA^*; c.f. [13] and [14].

It is easily verified that every weighted shift operator is bi-normal. In view of the above result for weighted shifts, the following question is of interest:

Question: Let $A \in \mathcal{BL}(\mathcal{H})$ be bi-normal. Is A finite?

4.10. Concluding Remarks- Througout the present section, we we have considered only the case of C^*-algebras generated by a single operator.The extension to the case of a C^*-algebra generated by a commuting family of operators follows immediately from lemma 3.6., provided that each operator is fully charactered. For example, if $\{a_\tau : \tau \in \Gamma\}$ is a commuting family of dominant operators (c.f.,4.6.), then the joint approximate point spectrum of $\{a_\tau : \tau \in \Gamma\}$ is non-empty (chapter II, corollary 2.5.1.), and for each

$$(\lambda_\tau)_{\tau\in\Gamma} \in J\sigma_{ap}\,(a_\tau)_{\tau\in\Gamma}$$

there exists a character φ on $C^*\,(a_\tau)_{\tau\in\Gamma}$ with

$$\varphi(a_\tau) = \lambda_\tau \qquad\qquad (\forall \tau \in \Gamma).$$

In particular, there exists $f \in E(\mathcal{A})$ such that

$$f(xb) = f(bx) \qquad (\forall x \in \mathcal{A},\ \forall b \in C^*(a_\tau)_{\tau\in\Gamma})$$

(proposition 2.2.(i)).

For non commuting families of operators, nothing can be said in general, as is shown by examples 2.1.(a) and 2.1.(b) of chapter II. Finally, we wish to pose three problems.

Problem 1. Let $\mathcal{A}$ be a C^*-algebra, and let $a, b \in \mathcal{A}$ be commuting finite operators. Is there a state f of $\mathcal{A}$ such that

$$f(xc) = f(cx) \qquad (\forall x \in \mathcal{A},\ \forall c \in C^*(a,b))?$$

Let us call an operator a fully charactered in the weak sense provided that the following condition holds:

$$\forall \lambda \in \sigma_{ap}(a), \ \exists \varphi \in \overline{\Phi}_{C^*(a)} \text{ s.t. } \varphi(a) = \lambda .$$

By proposition 3.4.(i), every fully charactered operator is fully charactered in the weak sense.

Problem 2. Suppose a is fully charactered in the weak sense. Is a fully charactered ?

Problem 3. Let a and b be two commuting operators, each of which is fully charactered in the weak sense. Is there a character on $C^*(a, b)$?

Appendix I

The Joint Approximate Point Spectrum of Elements of Finite-dimensional C^*-algebras.

The purpose of this appendix is to prove theorem 6.5. of chapter II, that is to prove the following theorem:

Theorem A- Let $\mathcal{A}$ be a C^*-algebra acting on a finite-dimensional Hilbert space $\mathcal{H}$, let $a_1, \ldots, a_n$ be mutually commuting elements of $\mathcal{A}$, and let $\lambda_1, \ldots, \lambda_n$ be complex numbers. Then

$$\sum_{j=1}^{n} \mathcal{A}(a_j - \lambda_j) \neq \mathcal{A}$$

if and only if

$$\sum_{j=1}^{n} (a_j - \lambda_j)\mathcal{A} \neq \mathcal{A}.$$

The following observations will simplify the statement of theorem A.

First, since the joint approximate point spectrum of operators is independent of the C^*-algebra containing the operators (Chapter II, proposition 1.2.1.), we may assume that $\mathcal{A} = \mathcal{BL}(\mathcal{H})$.

Next, since the conditions

$$(\lambda_1, \ldots, \lambda_n) \in J\sigma_{ap}(a_1, \ldots, a_n)$$

and

$$(0, \ldots, 0) \in J\sigma_{ap}(a_1 - \lambda_1, \ldots, a_n - \lambda_n)$$

are equivalent, we may assume that

$$\lambda_j = 0 \qquad (j = 1, \ldots, n).$$

Finally, since $\mathcal{H}$ is finite-dimensional, we may assume that

$$\mathcal{A} = \mathcal{BL}(\mathcal{H}) = M_m(\mathbb{C}) \qquad (m \geq 1).$$

Thus, it is sufficient to prove the following theorem:

Theorem B. Let $\mathcal{A} = M_m(\mathbb{C})$, and let $a_1, \dots, a_n$ be mutually commuting elements of $\mathcal{A}$. Then

$$\sum_{j=1}^{n} \mathcal{A} a_j = \mathcal{A}$$

if and only if

$$\sum_{j=1}^{n} a_j \mathcal{A} = \mathcal{A} .$$

Before presenting the proof of theorem B, we recall some definitions and results from the theory of rings and modules.

Let $\mathcal{A}$ be a ring, and let M be an $\mathcal{A}$-module.

A finite chain of submodules of M is a sequence M_i $(0 \leq i \leq n)$ of submodules of M such that

$$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_{n-1} \supsetneq M_n = 0 .$$

The length of the chain is n.

A composition series of M is a maximal chain, that is a chain in which no extra submodules can be inserted.

Suppose that a module M has a composition series of length n for some non-negative integer n. Then, every composition series of M has length n, and the lenght of M, denoted by $\ell(M)$, is defined to be n.

Let $\mathcal{A}$ be a ring, let $a \in \mathcal{A}$, and let $r(a)$ denote the right annihilator of :

$$r(a) = \{ x \in \mathcal{A} : ax = 0 \} .$$

Then, the map $\theta : \mathcal{A} \longrightarrow a\mathcal{A}$ defined by $x \longrightarrow ax$ induces a module isomorphism of $\mathcal{A}/r(a)$ and $a\mathcal{A}$. If, further, $\mathcal{A}$ has finite length then, so do $a\mathcal{A}$ and $\mathcal{A}/r(a)$, and the following holds:

$$\ell(a\mathcal{A}) = \ell(\mathcal{A}) - \ell(r(a)) .$$

Finally, we remark that, if $\mathcal{A}$ is the ring of $m \times m$- matrices then $\ell(\mathcal{A}) = m$, and, in particular, (1) holds for every element a of $\mathcal{A}$.

Reference: [1; §13].

We now turn to the proof of theorem B. We begin with

Lemma 1.- Let $a_1, \ldots, a_n$ be pair-wise commuting elements of $\mathcal{A}$, and suppose that

$$a_i \mathcal{A} \oplus r(a_i) = \mathcal{A} \qquad (i = 1, \ldots, n) \qquad (1).$$

Then

$$\left(\sum_{i=1}^{n} a_i \mathcal{A} \right) \cap \left(\bigcap_{i=1}^{n} r(a_i) \right) = \{0\} \qquad (2).$$

Proof- The proof is by induction on n. For $n = 1$ the proof follows from (1).

Suppose now that the result is true for any $n-1$ commuting elements of $\mathcal{A}$, and let $a_1, \ldots, a_n$ satisfy the hypotheses of the lemma. Then, by the induction hypothesis, we have

$$\left(\sum_{i=2}^{n} a_i \mathcal{A} \right) \cap \left(\bigcap_{i=2}^{n} r(a_i) \right) = \{0\} \qquad (3).$$

Since

$$a_1 \mathcal{A} \oplus r(a_1) = \mathcal{A}$$

we have, for each i ($2 \leq i \leq n$),

$$\begin{aligned} a_i \mathcal{A} &= a_i a_1 \mathcal{A} + a_i r(a_1) \\ &\subseteq a_1 \mathcal{A} + a_i r(a_1), \end{aligned}$$

since a_i and a_1 commute; hence,

$$\begin{aligned} \sum_{i=1}^{n} a_i \mathcal{A} &= a_1 \mathcal{A} + \sum_{i=2}^{n} a_i \mathcal{A} \\ &\subseteq a_1 \mathcal{A} + \sum_{i=2}^{n} a_i r(a_1) \end{aligned}$$

since the right-hand side of the above inclusion is obviously contained in the left-hand side, we get

$$\sum_{i=1}^{n} a_i \mathcal{A} = a_1\mathcal{A} + \sum_{i=2}^{n} a_i r(a_1) \tag{4}$$

We claim that the sum in (4) is, in fact, a direct sum. For, let

$$x \in \sum_{i=2}^{n} a_i r(a_1) ;$$

then,

$$x = \sum_{i=2}^{n} a_i y_i$$

where

$$y_i \in r(a_1) \qquad (i=2,\dots,n);$$

hence,

$$a_1 x = \sum_{i=2}^{n} a_1 a_i y_i = \sum_{i=2}^{n} a_i a_1 y_i = 0,$$

so $x \in r(a_1)$.

Thus, by (1),

$$\sum_{i=1}^{n} a_i \mathcal{A} = a_1\mathcal{A} \oplus \sum_{i=2}^{n} a_i r(a_i) \tag{5}$$

Let now

$$y \in \Big(\sum_{i=1}^{n} a_i \mathcal{A}\Big) \cap \Big(\bigcap_{i=1}^{n} r(a_i)\Big) \tag{6}$$

By (5), there exist $y_1, \dots, y_n$ in $\mathcal{A}$ such that

$$y_i \in r(a_1) \qquad (i=2,\dots,n),$$

and

$$y = a_1 y_1 + \sum_{i=2}^{n} a_i y_i .$$

By (6), we have $y \in r(a_1)$. Also

$$a_1\Big(\sum_{i=2}^{n} a_i y_i\Big) = \sum_{i=2}^{n} a_i (a_1 y_i) = 0 ;$$

hence

$$a_1 y_1 = y - \sum_{i=2}^{n} a_i y_i \in r(a_1) ;$$

therefore,

$$a_1 y_1 \in a_1\mathcal{A} \cap r(a_1) = \{0\} .$$

Hence

$$y = \sum_{i=2}^{n} a_i y_i \quad .$$

By (6), we also have

$$y \in \bigcap_{i=2}^{n} r(a_i)$$

hence,

$$y \in \left(\sum_{i=2}^{n} a_i \mathcal{A} \right) \cap \left(\bigcap_{i=2}^{n} r(a_i) \right) = \{0\}$$

by (3).

This completes the proof.

Corollary 2. - With the hypotheses of lemma 1 , suppose further that

$$\sum_{i=1}^{n} a_i \mathcal{A} = \mathcal{A} \quad .$$

Then

$$\sum_{i=1}^{n} \mathcal{A} a_i = \mathcal{A} \quad .$$

Proof- By lemma 1, we get

$$\bigcap_{i=1}^{n} (r(a_i)) = \{0\}$$

Hence

$$r \left(\sum_{i=1}^{n} \mathcal{A} a_i \right) = \{0\} ,$$

i.e.,

$$\sum_{i=1}^{n} \mathcal{A} a_i = \mathcal{A} .$$

This completes the proof.

We shall need one more lemma.

Lemma 3. Let $a_1 , \ldots , a_n$ be commuting elements of $\mathcal{A}$;

then, there exists a positive integer k such that

$$a_i^k \mathcal{A} \oplus r(a_i^k) = \mathcal{A} \qquad (i=1,\dots,n) \qquad (1).$$

Proof- For each i ($1 \leq i \leq n$), we have

$$r(a_i) \subseteq r(a_i^2) \subseteq \dots \subseteq r(a_i^s) \subseteq \dots \qquad (s=1,2,\dots);$$

hence, there exists a positive integer k_i such that

$$r(a_i^{k_i}) = r(a_i^{k_i+s}) \qquad (s=1,2,\dots).$$

Let $k = \max\limits_{1 \leq i \leq n} \{k_i\}$; then

$$r(a_i^k) = r(a_i^{k+s}) \qquad (i=1,\dots,n;\ s=1,2,\dots) \quad (2)$$

We shall now prove that (1) above holds for k.

For the rest of this argument, let i be an arbitrary but fixed element of $\{1,\dots,n\}$.

We first show that

$$(a_i^k \mathcal{A}) \cap (r(a_i^k)) = \{0\} \qquad (3)$$

Let

$$x \in (a_i^k \mathcal{A}) \cap (r(a_i^k)).$$

Then $x = a_i^k y$ for some $y \in \mathcal{A}$; so

$$a_i^{2k} y = a_i^k (a_i^k y) = a_i^k x = 0$$

since $x \in r(a_i^k)$. Hence

$$y \in r(a_i^{2k}) = r(a_i^k)$$

by (2). Therefore $x = a_i^k y = 0$.

This proves (3).

Next, we show that

$$a_i^k + r(a_i^k) = \mathcal{A} \qquad (4).$$

We have(see the remarks immediately preceding lemma 1)

$$\ell(\mathcal{A}) = \ell(a_i^k \mathcal{A}) + \ell(r(a_i^k)).$$

Also

$$\ell(a_i^k\mathcal{A} \oplus r(a_i^k)) = \ell(a_i^k\mathcal{A}) + \ell(r(a_i^k)).$$

Hence

$$\ell(\mathcal{A}) = \ell(a_i^k\mathcal{A} \oplus r(a_i^k)),$$

so that

$$\mathcal{A} = a_i^k\mathcal{A} \oplus r(a_i^k).$$

Since i was arbitrary, this completes the proof.

We are now ready for the proof of theorem B.

Proof of theorem B.(T.Lenagan)- Suppose that

$$\sum_{i=1}^{n} a_i\mathcal{A} = \mathcal{A} \qquad (1).$$

By lemma (3), there exists a positive integer k such that

$$\mathcal{A} = a_i^k\mathcal{A} \oplus r(a_i^k) \qquad (i=1,\dots,n) \qquad (2).$$

Now, (1) implies that

$$\sum_{i=1}^{n} a_i^k\mathcal{A} = \mathcal{A} \qquad (3).$$

For, multiplying both sides of (1) by a_1, we get

$$\begin{aligned} a_1\mathcal{A} &= a_1^2\mathcal{A} + \sum_{i=2}^{n} a_1 a_i\mathcal{A} \\ &\subseteq a_1^2\mathcal{A} + \sum_{i=2}^{n} a_i\mathcal{A} \end{aligned}$$

since a_1 and a_i commute; hence

$$\mathcal{A} = \sum_{i=1}^{n} a_i\mathcal{A} \subseteq a_1^2\mathcal{A} + \sum_{i=2}^{n} a_i\mathcal{A} \subseteq \mathcal{A},$$

so that

$$a_1^2\mathcal{A} + \sum_{i=2}^{n} a_i\mathcal{A} = \mathcal{A}.$$

An easy induction shows that

$$a_1^k\mathcal{A} + \sum_{i=2}^{n} a_i\mathcal{A} = \mathcal{A}.$$

Applying the same reasoning to the n commuting elements $b_1, \dots, b_n$

of $\mathcal{A}$, defined by

$$b_1 = a_2\,, \quad b_2 = a_1^k\,, \quad b_m = a_m \quad (3 \leq m \leq n)\,,$$

we obtain

$$a_1^k \mathcal{A} + a_2^k \mathcal{A} + \sum_{i=3}^{n} a_i \mathcal{A} = \mathcal{A}\,.$$

By induction, it follows that (3) holds.

Now, by (2) and (3), the n commuting elements $a_1^k, \ldots, a_n^k$ satisfy the hypotheses of corollary 2. Hence,

$$\sum_{i=1}^{n} \mathcal{A} a_i^k = \mathcal{A}\,.$$

Since for each i ($1 \leq i \leq n$), we have

$$\mathcal{A} a_i^k \subseteq \mathcal{A} a_i\,,$$

it follows that

$$\mathcal{A} = \sum_{i=1}^{n} \mathcal{A} a_i^k \subseteq \sum_{i=1}^{n} \mathcal{A} a_i \subseteq \mathcal{A}\,;$$

hence $\sum_{i=1}^{n} \mathcal{A} a_i = \mathcal{A}$.

The converse may be established by a similar reasoning, by replacing right annihilators with left annihilators, and repeating the above arguments.

This completes the proof.

We remark that the assumption $\mathcal{A} = M_m(\mathbb{C})$ may be considerably weakened; for example, the result remains true if $\mathcal{A}$ is any artinian ring (c.f. [28;theorem 1]).

Finally, we present an alternative proof of theorem B. based on the following result:

Let A , B , C , and D be complex n×n matrices and suppose that C and D commute. Then, the 2n×2n matrix K defined by

$$K = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is invertible if and only if

$$\det(AD - BC) \neq 0 .$$

For the proof we refer to [27; solution 56].

Alternative proof of theorem B.- We restrict the proof to the case of two commuting n×n matrices A and B.

Define the 2n×2n matrices M, N, and S by

$$M = \begin{pmatrix} A & -B \\ B^* & A^* \end{pmatrix}, \quad N = \begin{pmatrix} A^* & B^* \\ -B & A \end{pmatrix}, \quad S = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$$

where I_n is the n×n identity matrix. An elementary calculation shows that

$$S = S^{-1} \text{ and } SMS^{-1} = N .$$

Hence

$$\sigma(N) = \sigma(SMS^{-1}) = \sigma(M)$$

since similar operators have the same spectrum[27; problem 60]. Hence, by the above result, we have

$$\det(AA^* + BB^*) = 0 \text{ if and only if } \det(A^*A + B^*B) = 0$$

i.e.,

$$AA^* + BB^* \text{ is invertible} \qquad \text{(i)}$$

if and only if

$$A^*A + B^*B \text{ is invertible} \qquad \text{(ii)} .$$

By proposition 1.5. of chapter II, (i) and (ii) are equivalent to

$$(0,0) \notin J\sigma_{ap}(A^*, B^*)$$

and

$$(0,0) \notin J\sigma_{ap}(A, B)$$

respectively.

This completes the proof.

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Index of Terminology

Index of Notations

$JV(a_1,\dots,a_n)$	The joint numerical range of $a_1,\dots,a_n$:	112
$\ker \pi$	The kernel of π.	13
$\mathcal{K}\mathcal{L}(\mathcal{H})$	The ideal of compact operators on $\mathcal{H}$.	13
$\ell(M)$	The length of a module M	162
$[\mathcal{L}]$	The closed linear subspace spanned by $\mathcal{L}$.	7
$M_n(\mathbb{C})$	The ring of complex n×n matrices .	46
$\mathbb{N}^+$	The positive integers .	38
$\mathcal{P}(\mathcal{A})$	The set of pure states of $\mathcal{A}$.	1
$P(S)$	The property $P(S)$.	73
$(P_{A,\lambda})$	The property $P_{A,\lambda}$.	27
$\bigvee_{j=1}^{n} Q_j$	The supremum of projections $Q_1,\dots,Q_n$.	48
$\bigvee_{\tau\in\Gamma} Q_\tau$	The supremum of projections $\{Q_\tau\ ,\ \tau\in\Gamma\}$	99
$r(A)$	The spectral radius of A.	61
$r(a)$	The right annihilator of a.	162
range A	The range of A.	17
$\bar{S}$	$=\{\bar{\lambda} : \lambda\in S\}$.	84
$S_1 - S_2$	$=\{\lambda_1-\lambda_2 : \lambda_1\in S_1\ \&\ \lambda_2\in S_2\}$.	68
$S_1\setminus S_2$	The set theoretic difference of S_1 and S_2.	100
$\mathrm{Span}(\mathcal{L})$	The linear span of $\mathcal{L}$.	7
Supp A	The right support of A.	17
$\bigoplus_{j=1}^{n} T_j$	The direct sum of operators $T_1,\dots,T_n$.	58
$V(a)$	The numerical rangr of a.	112
$\tilde{\mathfrak{X}}$	The quotient space of $\mathfrak{X}$.	76
$\prod_{j=1}^{n}\sigma_{ap}(a_j)$	$=\{\lambda_1\cdots\lambda_n : \lambda_j\in\sigma_{ap}(a_j)\ \ (j=1,\dots,n)\}$.	57
$\prod_{\tau\in\Gamma}(\sigma_{ap}(a_\tau))$	The Cartesian product of the sets $\sigma_{ap}(a_\tau)$.	99
$\sigma(A)$	The spectrum of A.	2
$\sigma_p(A)$	The point spectrum of A.	16
$\sigma_{ap}(A)$	The approximate point spectrum of A.	2
$\sum_{\tau\in\Gamma}\mathcal{A}a_\tau$	The left ideal of $\mathcal{A}$ generated by $\{a_\tau : \tau\in\Gamma\}$.	91
$\sum_{j=1}^{n}\sigma_{ap}(a_j)$	$=\{\lambda_1+\dots+\lambda_n : \lambda_j\in\sigma_{ap}(a_j)\ \ (j=1,\dots,n)$.	57

$\tau_w, \tau_{\sigma w}, \tau_s, \tau_{\sigma s}$	The weak, ultraweak, strong,and ultrastrong operator topologies, respectively.	17
τ_n	The norm topology.	17
$\bar{\Phi}_{\mathcal{A}}$	The set of characters on $\mathcal{A}$.	127
ω_η	The vector state defined by η.	25
$\langle \cdot , \cdot \rangle$	Inner product on a Hilbert space .	4