The Structure of Scalar-Type Operators on L^p Spaces and Well-Bounded Operators on Hilbert Spaces

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To my parents
with honest love

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Abstract

It is known that every scalar-type spectral operator on a Hilbert space $\mathcal{H}$ is similar to a multiplication operator on some L^2 space. The purpose of the main theorem in Chapter 2 of this thesis is to show that every scalar-type spectral operator on an L^1 space whose spectral measure has finite multiplicity is similar to a multiplication operator on the same L^1 space provided that some conditions are satisfied. Also, we give conditions that make every scalar-type spectral operator on $L^2(\Omega, \Sigma_\Omega, \mu)$ similar to a multiplication operator on the same $L^2(\Omega, \Sigma_\Omega, \mu)$ space.

In 1954, Dunford proved that a bounded operator T on a Banach space X is spectral if and only if it has a canonical decomposition $T = S + Q$, where S is a scalar-type operator and Q is a quasinilpotent operator which commutes with S. In Chapter 3, we prove that if T is a well-bounded operator on a Hilbert space $\mathcal{H}$ then it has the form $T = A + Q$, where A is a self-adjoint operator and Q is a quasinilpotent operator such that $AQ - QA$ is quasinilpotent. Then we prove that if T is a trigonometrically well-bounded operator on $\mathcal{H}$ then it can be decomposed as $T = U(Q+I)$ where U is a unitary operator and Q is quasinilpotent such that $UQ - QU$ is also quasinilpotent. In Chapter 4 we prove that if T is an AC-operator with discrete spectrum on $\mathcal{H}$ then it can likewise be decomposed as a sum of a normal operator N and a quasinilpotent Q such that $NQ - QN$ is quasinilpotent. However, the converse of each of the last three theorems is not true in general.

In the final chapter we introduce a new class of operators on a Hilbert space $\mathcal{H}$ which is larger than the class of well-bounded operators on $\mathcal{H}$ and we call them operators with an AC_2-functional calculus. Then we give an example of an operator with an AC_2-functional calculus on $L^2([0,1])$ which can be decomposed as a sum of a self-adjoint operator and a quasinilpotent. We also discuss the possibility of decomposing every operator T with an AC_2-functional calculus on $\mathcal{H}$ into the sum of a self-adjoint operator A and a quasinilpotent operator Q such that $AQ - QA$ is quasinilpotent.

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Introduction

A bounded linear operator T on a complex Banach space X is said to be a spectral operator if there exists a spectral measure $E(\cdot)$ defined on the family of Borel sets $\Sigma_{\mathbb{C}}$ of the complex plane $\mathbb{C}$ such that
(i) $TE(\sigma) = E(\sigma)T$, $(\sigma \in \Sigma_{\mathbb{C}})$;
(ii) $\sigma(T|E(\sigma)X) \subset \overline{\sigma}$, $(\sigma \in \Sigma_{\mathbb{C}})$,
where the spectral measure is a mapping $E(\cdot) : \Sigma_{\mathbb{C}} \to \mathcal{B}(X)$ such that
(i) $E(\sigma_1 \cap \sigma_2) = E(\sigma_1)E(\sigma_2)$, $(\sigma_1, \sigma_2 \in \Sigma_{\mathbb{C}})$;
(ii) $E(\cup_{n=1}^{\infty}\sigma_n)x = \sum_{n=1}^{\infty} E(\sigma_n)x$, $(\sigma_n \in \Sigma_{\mathbb{C}}, \sigma_n \cap \sigma_m = \emptyset \text{ if } n \neq m, x \in X)$;
(iii) $E(\mathbb{C}) = I$.
Dunford [14] has shown that a spectral operator T can be decomposed as $T = \int \lambda E(d\lambda) + Q$, where Q is a quasinilpotent operator that commutes with the operator $\int \lambda E(d\lambda)$. If $Q = 0$, then $T = \int \lambda E(d\lambda)$ is called a scalar-type spectral operator.

A bounded linear operator T on a complex Banach space X is said to be well-bounded if, for some compact interval $J = [a, b]$,

$$\| p(T) \| \leq K\{|p(b)| + \mathrm{var}_J p\},$$

for some constant K and all polynomials p. Scalar-type spectral operators with real spectra satisfy similar inequalities with the $\mathrm{var}_J p$ term omitted and hence are well-bounded. Here $\mathrm{var}_J p = \sup_{\mathcal{P}(J)} \sum_{j=1}^{n} |p(t_j) - p(t_{j-1})| = \int_a^b |p'(t)|dt$.

In 1960, Smart [36] introduced well-bounded operators on a Banach space X as a natural analogue of self-adjoint operators on Hilbert space, and since then much work has been done to study these operators and to examine the relationship between well-bounded and scalar-type spectral operators on a Banach space X, and the effect of the geometry of the Banach space on this relation.

Smart [36] and Ringrose [33] have shown that T is well-bounded on a reflexive Banach space X if and only if there is a family $\{E(\lambda) : \lambda \in \mathbb{R}\}$ of projections on X such that, for some compact interval $J = [a, b]$,

(i) $\| E(\lambda) \| \leq K$, $\lambda \in \mathbb{R}$;
(ii) $E(\lambda)E(\mu) = E(\mu)E(\lambda) = E(\min\{\lambda, \mu\}), \lambda, \mu \in \mathbb{R}$;
(iii) $\lim_{\lambda \to \mu^+} E(\lambda)x = E(\mu)x$, $x \in X$;
(iv) $\lim_{\lambda \to \mu^-} E(\lambda)$ exists in the strong operator topology;
(v) $E(\lambda) = 0$, $\lambda < a$; $E(\lambda) = I$, $\lambda \geq b$;
(vi) $T = \int_J \lambda dE(\lambda)$,

where the integral exists as a strong limit of Riemann sums.
On a general Banach space, well-bounded operators which have such a spectral family representation are called well-bounded operators of type (B). Since any Hilbert space $\mathcal{H}$ is reflexive, every well-bounded operator on a Hilbert space $\mathcal{H}$ is of type (B). The well-bounded operators considered in this thesis act on Hilbert spaces and hence we shall only be concerned with well-bounded operators of type (B).

This thesis is devoted to proving representation theorems for scalar-type spectral operators on the classical Banach spaces L^p and also decomposition theorems for well-bounded operators and related operators on Hilbert spaces.

The spectral theorem for normal operators on $\mathcal{H}$ is stated in ([10], Theorem IX.4.6) as follows. If N is a normal operator on $\mathcal{H}$, then there is a measure space $(\Lambda, \Sigma_\Lambda, \nu)$ and a function φ in $L^\infty(\Lambda, \Sigma_\Lambda, \nu)$ such that N is unitarily equivalent to the multiplication operator M_φ on $L^2(\Lambda, \Sigma_\Lambda, \nu)$. In Chapter 2 of this thesis we prove that if μ is a Borel measure on a locally compact separable complete metric space Ω and Ω_d is the set of the supports of the atoms in Ω and $\Omega_c = \Omega \backslash \Omega_d$, then T is a scalar-type spectral operator on $L^1(\Omega, \Sigma_\Omega, \mu)$ with spectral measure of finite multiplicity N and card $\Omega_d = \sum_{j=1}^{N} j$ card ({eigenvalues of multiplicity j}) and either $E(\sigma_c(T))$ and $\mu(\Omega_c)$ are both zero or both nonzero if and only if there exists a bounded measurable essentially at most N-to-one function $\varphi : \Omega \to \mathbb{C}$ and an invertible $S : L^1(\mu) \to L^1(\mu)$ such that $T = S^{-1}M_\varphi S$. Here $\sigma_c(T)$ is the continuous spectrum of T and by ([16], Corollary XV.8.3.4), $\sigma_c(T) = \sigma(T)\backslash\sigma_p(T)$, where $\sigma_p(T)$ is the point spectrum of T. Also, we give conditions that make a scalar-type spectral operator on $L^2(\Omega, \Sigma_\Omega, \mu)$ similar to a multiplication operator on the *same* $L^2(\Omega, \Sigma_\Omega, \mu)$.

Suppose that T is a bounded operator on a Hilbert space $\mathcal{H}$. Then T is said to be trigonometrically well-bounded if there is a well-bounded operator A on $\mathcal{H}$ such that $T = e^{iA}$. Trigonometrically well-bounded operators are a generalisation of unitary operators in the context of well-boundedness and were introduced by

Berkson and Gillespie in [6]. The concept was introduced originally for Banach space operators but we shall only consider here trigonometrically well-bounded operators on Hilbert space. An operator on a Hilbert space (or indeed a reflexive Banach space) is trigonometrically well-bounded if and only if it has an $AC(\mathbb{T})$-functional calculus, where $AC(\mathbb{T})$ is the algebra of absolutely continuous functions on the unit circle $\mathbb{T}$. In [5], Berkson and Gillespie generalised the concept of normal operators on Hilbert space in the context of well-boundedness by introducing the concept of AC-operators as those operators which possess a functional calculus for the absolutely continuous functions on some rectangle in $\mathbb{C}$. They showed that these operators can be characterised by having the unique expression $T = A + iB$ where A and B are commuting well-bounded operators on $\mathcal{H}$. (In fact, they established this for operators on a reflexive Banach space.)

West [39] proved that a Riesz operator on a Hilbert space $\mathcal{H}$ can be expressed as the sum of a compact operator and a quasinilpotent operator. In Chapter 3, we prove that any well-bounded operator T on a Hilbert space $\mathcal{H}$ can be expressed as the sum of a self-adjoint operator A and a quasinilpotent operator Q such that $AQ - QA$ is also quasinilpotent. Then we prove that a trigonometrically well-bounded operator T on $\mathcal{H}$ can be decomposed as $T = U(Q + I)$ where U is a unitary operator and Q is quasinilpotent and $UQ - QU$ is also quasinilpotent. In Chapter 4 we prove that an AC-operator with discrete spectrum on $\mathcal{H}$ can be decomposed as a sum of a normal operator N and a quasinilpotent Q where $NQ - QN$ is also quasinilpotent. On the other hand, it is well-known that on a Banach space X the sum of a compact operator K and a quasinilpotent operator Q is a Riesz operator whether or not K and Q commute. We prove in Chapter 3 that on a Hilbert space $\mathcal{H}$ the sum of a self-adjoint operator A and a nonzero quasinilpotent operator Q is *not* well-bounded if A and Q commute. We also prove in Chapter 4 that on a Hilbert space $\mathcal{H}$ the sum of a normal operator N and a nonzero quasinilpotent operator Q is *not* an AC-operator if N and Q commute. In a similar fashion, we prove that if U is a unitary operator on a Hilbert space $\mathcal{H}$ and Q is a nonzero quasinilpotent such that $UQ = QU$ then $U(Q + I)$ is *not* a trigonometrically well-bounded operator on $\mathcal{H}$.

In the final chapter, we introduce operators with an AC_2-functional calculus on a Hilbert space $\mathcal{H}$ as a new class of operators on $\mathcal{H}$ larger than the class of well-bounded operators. We give an example of an operator T with an AC_2-functional calculus on $L^2([0, 1])$ which can be decomposed as a sum of a self-adjoint operator and a quasinilpotent. We also discuss the difficulties in proving that every operator with an AC_2-functional calculus can be decomposed as a sum of a self-adjoint operator and a quasinilpotent.

Much of our notation and terminology is standard and is therefore not introduced in the text if the meaning is clear from the context. We have however included a list of some of the notation at the end of the thesis. We note also that the statements of many definitions and results are incomplete in the sense that certain blanket assumptions may apply to some of the symbols. These are usually introduced at the beginning of each section of this thesis.

Chapter 1

Preliminaries

1.1 Well-Bounded Operators and AC-Operators on $\mathcal{H}$

In this section we collect the known items we shall need from the theory of well-bounded operators and AC-operators. As we are interested in well-bounded operators acting only on a Hilbert space $\mathcal{H}$, we shall restrict the discussion to a Hilbert space setting.

Let $J = [a, b]$ be a compact interval of the real line $\mathbb{R}$. We denote by $AC(J)$ the Banach algebra of absolutely continuous complex valued functions on J with the norm $\| f \|_{AC(J)} = |f(b)| + \text{var}_J f$, and denote by $AC(\mathbb{T})$ the Banach algebra of absolutely continuous complex valued functions on the unit circle $\mathbb{T}$ with the norm $\| f \|_{AC(\mathbb{T})} = |f(1)| + \text{var}_{\mathbb{T}} f$. Here $\text{var}_{\mathbb{T}} f = \sup_{\mathcal{P}([0,2\pi])} \sum_{j=1}^{n} |f(e^{it_j}) - f(e^{it_{j-1}})| = \int_0^{2\pi} |f'(e^{it})| dt$, where $\mathcal{P}([0, 2\pi])$ is the set of all partitions of $[0, 2\pi]$. Let $\mathcal{H}$ be a Hilbert space, and $\mathcal{B}(\mathcal{H})$ the Banach algebra of bounded operators on $\mathcal{H}$.

Definition 1.1.1 ([13], Definition 15.1). *Let $T \in \mathcal{B}(\mathcal{H})$. We say that T is a well-bounded operator on $\mathcal{H}$ implemented by (K, J) if there are a compact interval J and a real constant K such that*

$$\| p(T) \| \leq K \| p \|_{AC(J)} \text{ for all polynomials } p \text{ on } J. \tag{1.1.1}$$

As was mentioned in the introduction, when we say that T is a well-bounded operator on $\mathcal{H}$ it follows that T is well-bounded of type (B) and hence T has an integral representation with respect to a spectral family of projections.

Lemma 1.1.1 ([13], Lemma 15.2). *Let T be a well-bounded operator on $\mathcal{H}$ with natural algebra homomorphism $\varphi : p \to p(T)$ from $\mathcal{P}(J)$ into $\mathcal{B}(\mathcal{H})$. Let K and J be chosen so that (1.1.1) is satisfied. Then φ has a unique extension to an algebra homomorphism (also denoted by) $\varphi : f \to f(T)$ from $AC(J)$ into $\mathcal{B}(\mathcal{H})$ such that*

(i) $\| f(T) \| \leq K \| f \|_{AC(J)}$ *for all* $f \in AC(J)$,
(ii) if $S \in \mathcal{B}(\mathcal{H})$ *and* $ST = TS$ *then* $Sf(T) = f(T)S$ *for all* $f \in AC(J)$.
(Note that φ *is a norm continuous representation of* $AC(J)$ *on* $\mathcal{H}$ *such that* φ *sends the identity map id to* T *and the constant function 1 to the identity operator* I *on* $\mathcal{H}$*. In this event,* φ *is called an* $AC(J)$*-functional calculus for* T*.)*

Definition 1.1.2 ([6], Definition 2.18). *An operator* $T \in \mathcal{B}(\mathcal{H})$ *is said to be trigonometrically well-bounded if there exists a well-bounded operator* A *on* $\mathcal{H}$ *such that* $T = e^{iA}$.

Theorem 1.1.1 ([7], Theorem 2.20). *Let* $T \in \mathcal{B}(\mathcal{H})$*. Then* T *is trigonometrically well-bounded if and only if* T *has an* $AC(\mathbb{T})$*-functional calculus, i.e., if there exists a norm continuous homomorphism* $\varphi : AC(\mathbb{T}) \to \mathcal{B}(\mathcal{H})$ *such that* φ *sends the identity map id to* T *and the constant function 1 to* I.

In Chapter 4 of this thesis we prove a decomposition theorem for operators of the form $T = A + iB$ where A and B are commuting and well-bounded. In order to do this, we use the notions of bounded variation and absolute continuity of a function of two variables given in Cartesian form, defined in [5] and [4] as follows.

Definition 1.1.3. *Suppose that* $J = [a, b]$ *and* $K = [c, d]$ *are two fixed intervals in* $\mathbb{R}$*, and that* Λ *is a rectangular partition of* $J \times K$:

$$a = s_0 < s_1 < \cdots < s_n = b, \quad c = t_0 < t_1 < \cdots < t_m = d.$$

For a function $f : J \times K \to \mathbb{C}$*, define*

$$V_\Lambda(f) = \sum_{i=1}^{n} \sum_{j=1}^{m} |f(s_i, t_j) - f(s_i, t_{j-1}) - f(s_{i-1}, t_j) + f(s_{i-1}, t_{j-1})|.$$

The variation of f *is defined to be*

$$\mathrm{var}_{J\times K} f = \sup\{V_\Lambda(f) : \Lambda \textit{ is a rectangular partition of } J \times K\}.$$

The function f *is said to be of bounded variation on* $J \times K$ *if each of the numbers*

$$\mathrm{var}_{J\times K} f, \quad \mathrm{var}_J f(\cdot, d), \quad \mathrm{var}_K f(b, \cdot)$$

is finite.

Define $|||\cdot|||$ on the set $BV(J \times K)$ of all functions $f : J \times K \to \mathbb{C}$ of bounded variation as follows:

$$|||f||| = |f(b,d)| + \mathrm{var}_J f(\cdot, d) + \mathrm{var}_K f(b, \cdot) + \mathrm{var}_{J\times K} f.$$

Then $|||\cdot|||$ is a norm on $BV(J \times K)$ and $BV(J \times K)$ is a Banach algebra under this norm.

Remark 1.1.1. *If f is a C^2-function, then $f \in BV(J \times K)$ and*

$$|||f||| = |f(b,d)| + \int_a^b |\frac{\partial f}{\partial s}(s,d)|ds + \int_c^d |\frac{\partial f}{\partial t}(b,t)|dt + \int\int_{J\times K} |\frac{\partial^2 f}{\partial s \partial t}(s,t)|dsdt.$$

Definition 1.1.4. *A function $f : J \times K \to \mathbb{C}$ is said to be absolutely continuous if the following two conditions are satisfied:*

(1) *given $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\sum_{R\in\mathcal{R}} \mathrm{var}_R\, f < \epsilon$$

whenever $\mathcal{R}$ is a finite collection of non-overlapping subrectangles of $J \times K$ with $\sum_{R\in\mathcal{R}} m(R) < \delta$, where m denotes Lebesgue measure on $\mathbb{R}^2$;

(2) *the marginal functions $f(\cdot, d)$ and $f(b, \cdot)$ are absolutely continuous functions on J and K respectively.*

Theorem 1.1.2 ([5], Theorem 4). *The set $AC(J \times K)$ of all absolutely continuous functions $f : J \times K \to \mathbb{C}$ is a Banach subalgebra of $BV(J \times K)$, and is the closure in $BV(J \times K)$ of the polynomials in two real variables on $J \times K$.*

Theorem 1.1.3 ([5], Theorem 5). *Let $T \in \mathcal{B}(\mathcal{H})$. Then $T = A + iB$, where A and B are commuting well-bounded operators on $\mathcal{H}$, if and only if T has an $AC(J \times K)$-functional calculus, i.e., there exists a norm continuous algebra homomorphism $\varphi : AC(J \times K) \to \mathcal{B}(\mathcal{H})$ such that $\varphi(u + iv) = T$.*
Here $u, v \in AC(J \times K)$ are defined by $u(s,t) = s$ and $v(s,t) = t$.

Definition 1.1.5 ([5]). *An operator $T \in \mathcal{B}(\mathcal{H})$ is called an AC-operator if it can be written uniquely as $T = A + iB$, where A and B are commuting well-bounded operators on $\mathcal{H}$.*

Theorem 1.1.4 ([5], Lemma 4). *Let A and B be commuting well-bounded operators on $\mathcal{H}$ and let $S \in \mathcal{B}(\mathcal{H})$ commute with $A + iB$. Then S commutes with both A and B.*

Theorem 1.1.5 ([5], Corollary p.320). *The class of trigonometrically well-bounded operators is a subclass of the class of AC-operators.*

1.2 Local Spectral Theory

In this section we shall present the aspects of local spectral theory we shall need in the decomposition process in Chapters 3 and 4. Most of the definitions and theorems presented in this section are from [25] and [9] and they hold for any bounded linear operator T acting on any complex Banach space X. However, we prefer to state them in the Hilbert space setting as we only need them in this special case.

Definition 1.2.1 ([25], Definition 1.2.9). *An operator $T \in \mathcal{B}(\mathcal{H})$ is said to have the single-valued extension property, abbreviated SVEP, if, for every open set $U \subset \mathbb{C}$, the only analytic solution $f : U \to \mathcal{H}$ of the equation $(T-\lambda I)f(\lambda) = 0$ for all $\lambda \in U$ is the zero function on U.*

Definition 1.2.2 ([25]). *For an operator $T \in \mathcal{B}(\mathcal{H})$ having the SVEP, and for $x \in \mathcal{H}$, define the local resolvent set $\rho_T(x)$ of T at the point x to be the set of elements $\alpha \in \mathbb{C}$ such that there exists an analytic function $f_x : \lambda \to f_x(\lambda)$ defined in a neighbourhood U_α of α, with values in $\mathcal{H}$, which satisfies*

$$(T - \lambda I)f_x(\lambda) = x \text{ for all } \lambda \in U_\alpha.$$

In particular, $\rho(T) \subset \rho_T(x)$ for each $x \in \mathcal{H}$. The local spectrum $\sigma_T(x)$ of T at x is then defined as

$$\sigma_T(x) = \mathbb{C} \backslash \rho_T(x).$$

The local spectral subspaces of T are defined by

$$X_T(F) = \{x \in \mathcal{H} : \sigma_T(x) \subset F\} \text{ for all } F \subset \mathbb{C}.$$

The subspaces $X_T(F)$ defined above are T-hyperinvariant subspaces of $\mathcal{H}$ in the sense that they are invariant under any operator that commutes with T. Also, the analytic solutions occurring in the definition of the local resolvent set above are unique for all $x \in \mathcal{H}$ if T has the SVEP, and in this case, they define an analytic function f_x on all of $\rho_T(x)$, which is the maximal analytic extension of $(T - \lambda I)^{-1}x$ from $\rho(T)$ to $\rho_T(x)$.

Theorem 1.2.1 ([9], Proposition 1.1.2). *Let $T \in \mathcal{B}(\mathcal{H})$ be an operator having the SVEP. Then*
(i) $F_1 \subset F_2$ implies $X_T(F_1) \subset X_T(F_2)$,
(ii) $X_T(F)$ is a linear subspace (not necessarily closed) of $\mathcal{H}$,
(iii) $\sigma_T(x) = \emptyset$ if and only if $x = 0$,
(iv) $\sigma_T(Ax) \subset \sigma_T(x)$ for every $A \in \mathcal{B}(\mathcal{H})$ with $AT = TA$,
(v) $\sigma_T(f_x(\lambda)) = \sigma_T(x)$ for every $x \in \mathcal{H}$ and $\lambda \in \rho_T(x)$.

Theorem 1.2.2 ([25], Proposition 1.2.20). *Suppose that the operator $T \in \mathcal{B}(\mathcal{H})$ has the SVEP, and that $F \subset \mathbb{C}$ is a closed set for which the space $X_T(F)$ is closed. Then*

$$\sigma(T|X_T(F)) \subset F \cap \sigma(T).$$

Moreover, the operator $T/X_T(F)$ induced by T on the quotient space $\mathcal{H}/X_T(F)$ has the SVEP, and the identity $\sigma_T(x) = \sigma_{T|X_T(F)}(x)$ holds for all $x \in X_T(F)$.

Definition 1.2.3 ([25], Definition 1.1.1). *An operator $T \in \mathcal{B}(\mathcal{H})$ is called decomposable if for every open cover $\{U, V\}$ of $\sigma(T)$, there exist T-invariant closed linear subspaces Y and Z of $\mathcal{H}$ for which*

$$\sigma(T|Y) \subset U, \quad \sigma(T|Z) \subset V, \quad \textit{and } \mathcal{H} = Y + Z.$$

If $T \in \mathcal{B}(\mathcal{H})$ is a decomposable operator, then for any closed subset F of $\mathbb{C}$ the local spectral subspace $X_T(F)$ is closed ([25], Theorems 1.2.7 and 1.2.19).

Definition 1.2.4 ([9], Definition 3.1.1). *Let Ω be a set in the complex plane. An algebra $\mathcal{A}$ of $\mathbb{C}$-valued functions defined on Ω is called normal if for every open finite covering $\{G_i\}_{i=1}^n$ of $\overline{\Omega}$ there exist functions $f_i \in \mathcal{A}$ such that*
(i) $f_i(\Omega) \subset [0,1], (1 \le i \le n)$,
(ii) $\operatorname{supp}(f_i) \subset G_i, (1 \le i \le n)$, where $\operatorname{supp}(f_i) = \overline{\{\lambda \in \Omega : f_i(\lambda) \neq 0\}}$,
(iii) $\sum_{i=1}^n f_i = 1$ on Ω.

Definition 1.2.5 ([9], Definition 3.5.1). *An algebra $\mathcal{A}$ of $\mathbb{C}$-valued functions defined on $\Omega \subset \mathbb{C}$ is called topologically admissible if*
(i) id $\in \mathcal{A}$ and $1 \in \mathcal{A}$, where id is the identity function id$(\lambda) = \lambda$ and 1 is the constant function 1,
(ii) $\mathcal{A}$ is normal,
(iii) $\mathcal{A}$ is endowed with a locally convex topology τ such that if $\{f_n\}_{n=1}^\infty \subset \mathcal{A}$ is a Cauchy sequence in τ and $f_n(\lambda) \to 0$ for every $\lambda \in \Omega$, then $f_n \to 0$ in τ,
(iv) for every $f \in \mathcal{A}$ and every $\xi \notin \operatorname{supp}(f)$, the function

$$f_\xi(\lambda) = \begin{cases} \frac{f(\lambda)}{\xi - \lambda} & \textit{for } \lambda \in \Omega \backslash \{\xi\}, \\ 0 & \textit{for } \lambda \in \Omega \cap \{\xi\} \end{cases}$$

belongs to $\mathcal{A}$, and the mapping $\xi \to f_\xi$ of $\mathbb{C} \backslash \operatorname{supp}(f)$ into $\mathcal{A}$ is continuous.

In the light of ([9], Definitions 3.1.3, 3.1.18 and 3.5.3 and Theorem 3.5.4), we make the following definition.

Definition 1.2.6. *Let $\mathcal{A}$ be a topologically admissible algebra. An operator $T \in \mathcal{B}(\mathcal{H})$ is called $\mathcal{A}$-scalar if there exists a continuous algebra homomorphism $\varphi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ such that $\varphi(1) = I$ and $\varphi(id) = T$.*

In particular then, using ([9], Theorem 3.5.4), every $\mathcal{A}$-scalar operator in the sense of the previous definition is $\mathcal{A}$-scalar in the sense of ([9], Definition 3.1.18).

Theorem 1.2.3 ([9], Theorems 3.1.10 and 3.1.19). *Every $\mathcal{A}$-scalar operator on $\mathcal{H}$ is decomposable and has the SVEP.*

Remark 1.2.1. *By ([19], Corollary 4.5), the algebras $AC(J), AC(\mathbb{T}), AC(J\times K)$ are topologically admissible algebras. Hence well-bounded operators, trigonometrically well-bounded operators and AC-operators are $AC(J), AC(\mathbb{T}), AC(J\times K)$-scalar operators respectively, and therefore they are decomposable and have the SVEP, and hence their local spectral subspaces are closed.*

We shall also use the following theorems in the proof of our results in Chapters 3 and 4 in the particular cases when the $\mathcal{A}$-scalar operator is a well-bounded operator, a trigonometrically well-bounded operator or an AC-operator.

Theorem 1.2.4 ([9], Proposition 3.1.12). *Suppose that $T \in \mathcal{B}(\mathcal{H})$ is an $\mathcal{A}$-scalar operator and $\varphi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is its algebra homomorphism defined by $\varphi(f) = f(T)$ for all $f \in \mathcal{A}$ and let $x \in \mathcal{H}$. If $\sigma_T(x) \cap \operatorname{supp}(f) = \emptyset$ for some $f \in \mathcal{A}$, then $f(T)x = 0$.*

Theorem 1.2.5 ([9], Proposition 3.3.10 and Corollary 3.3.12). *Suppose that $T \in \mathcal{B}(\mathcal{H})$ is an $\mathcal{A}$-scalar operator, φ is its algebra homomorphism, and Y is a closed linear subspace of $\mathcal{H}$ which is invariant under T and invariant under the range of φ. Then the operator $T|Y$ and the induced operator $S = T/Y$ on the quotient space $\mathcal{H}/Y$ are also $\mathcal{A}$-scalar operators.*

Theorem 1.2.6 ([25], Proposition 1.2.22). *Let $T \in \mathcal{B}(\mathcal{H})$ be a decomposable operator on $\mathcal{H}$, and let $F \subset \mathbb{C}$ be closed. Then the induced operator $S = T/X_T(F)$ on the quotient space $Y = \mathcal{H}/X_T(F)$ satisfies*

$$\sigma(S) \subset \overline{\sigma(T)\backslash F}.$$

Moreover, the local spectral subspaces of S are given by $Y_S(E) = QX_T(E)$ for all closed sets $E \subset \mathbb{C}$ that contain F, where $Q : \mathcal{H} \to Y$ denotes the canonical quotient mapping.

Theorem 1.2.7 ([13], Proposition 1.34). *Let Y be a closed subspace of $\mathcal{H}$. Then there is a linear isometry $U : \mathcal{H}/Y \to Y^{\perp}$ which is given by*

$$U(x+Y) = Px \text{ for all } x \in \mathcal{H},$$

where P is the orthogonal projection onto $Y^{\perp}$.

1.3 Normed Köthe Spaces and L^p Spaces

In this section we shall summarise briefly some notions and results needed in Chapter 2. Throughout, let $\mathcal{B}$ be a σ-complete Boolean algebra of projections on a Banach space X with uniform bound M for the norms of the projections in $\mathcal{B}$. Regarding $\mathcal{B}$ as the range of a spectral measure $E(\cdot)$ defined on the family Σ_Λ of Borel sets of its Stone space Λ, Bade ([2], Theorem 3.1) has shown that for each f_0 in X, there exists a linear functional f_0^* in X^* such that the measure $(E(\cdot)f_0, f_0^*)$ is positive, and the countably additive vector valued measure $E(\cdot)f$ is absolutely continuous with respect to the scalar measure $\nu(\cdot) = (E(\cdot)f_0, f_0^*)$. In [16] it is shown that for every bounded Borel function φ, the integral $T_\varphi = \int_\Lambda \varphi(\lambda)E(d\lambda)$ exists in the uniform operator topology and satisfies

$$\| T_\varphi \| \leq 4M \sup\{|\varphi(\lambda)| : \lambda \in \Lambda\}.$$

If X contains an element e with the property that

$$X = \operatorname{clm}\{E(\cdot)e : E(\cdot) \in \mathcal{B}\},$$

then X is called a cyclic Banach space with a cyclic vector e. In this case, we also say that the spectral measure $E(\cdot)$ has a cyclic vector e. Here we denote by $\operatorname{clm} S$ the closed linear span of the set S, where $S \subset X$.

Definition 1.3.1 ([21]). *If $(\Lambda, \Sigma_\Lambda, \nu)$ is a measure space, $\mathcal{M}^+$ is the set of $[0,\infty]$-valued measurable functions defined on Λ, and $\mathcal{N}$ is the set of ν-null sets, then a function $\rho : \mathcal{M}^+ \to [0,\infty]$ is called a function norm on $(\Lambda, \Sigma_\Lambda, \nu)$ if it satisfies the following:*

$$\begin{array}{rl} (i) & \rho(f+g) \leq \rho(f) + \rho(g), \\ (ii) & \rho(\alpha f) = \alpha\rho(f), \\ (iii) & \rho(f) \leq \rho(g) \text{ if } f \leq g \text{ a.e.}, \\ (iv) & \rho(f) = 0 \text{ if and only if } f = 0 \text{ a.e.}, \end{array}$$

for all $f, g \in \mathcal{M}^+$ and all $\alpha \in [0,\infty]$, where we adopt the convention that $0.\infty = 0$.

Given such a function norm ρ, let $\mathcal{L}_\rho = \{f \in \mathcal{M} : \rho(|f|) < \infty\}$ and $L_\rho(\nu) = \mathcal{L}_\rho/\mathcal{N}$. Then ρ induces a norm on $L_\rho(\nu)$ defined by $\rho(f) = \rho(|f|)$ and $L_\rho(\nu)$ is called a normed Köthe space based on $(\Lambda, \Sigma_\Lambda, \nu)$.
We sometimes refer to L_ρ as a normed Köthe space and suppress ν in the notation.

We shall not give more details about the theory of normed Köthe spaces in general; we only need to depend on a representation theorem due to Gillespie ([21], Theorem 3.4) and an isomorphic characterisation theorem of L_ρ proved by Tzafriri ([38], Proposition 6) to prove our first result (Theorem 2.1.1).

The following result follows immediately from ([21], Theorem 3.4) and ([40], Theorem 2, p.485).

Theorem 1.3.1. *Suppose that X is a reflexive cyclic Banach space with a cyclic vector f_0, and let ν be the measure defined at the start of this section on Λ by $\nu(\cdot) = (E(\cdot)f_0, f_0^*)$ where f_0^* is a Bade functional for f_0, and define*

$$\rho : \mathcal{M}^+ \to [0, \infty] \text{ by } \rho(f) = \sup\{\| T_\varphi f_0 \| : \varphi \in \mathcal{L}^\infty, |\varphi| \le f\},$$

where $\mathcal{M}^+$ is the set of $[0,\infty]$-valued measurable functions defined on Λ and $\mathcal{L}^\infty$ is the set of bounded measurable functions defined on Λ. Then ρ is a function norm on $(\Lambda, \Sigma_\Lambda, \nu)$, and there exists a bicontinuous linear isomorphism U of X onto $L_\rho(\nu)$ such that

$$T_\varphi = U^{-1} M_\varphi U \text{ for all } \varphi \in \mathcal{L}^\infty, \text{ and } Uf_0 = 1.$$

In particular, $E(\sigma) = U^{-1} M_{\chi_\sigma} U$ for all $\sigma \in \Sigma_\Lambda$.

We shall make use of ([38], Proposition 6). Stated in the special case we need, it is as follows.

Theorem 1.3.2. *Let $1 \le p < \infty$. Then a normed Köthe space $L_\rho(\nu)$ is isomorphic to $L^p(\nu)$, and the isomorphism from $L_\rho(\nu)$ onto $L^p(\nu)$ leaves all characteristic functions invariant, provided that for each sequence of disjoint nonzero elements $g_n \in L_\rho(\nu)$, the basis $\{g_n/\rho(g_n)\}$ is equivalent to the natural basis of ℓ^p, i.e., there exist constants A and B, independent of $\{g_n\}$, such that*

$$A\rho\left(\sum_{n=1}^{N} \alpha_n g_n/\rho(g_n)\right) \le \left(\sum_{n=1}^{N} |\alpha_n|^p\right)^{1/p} \le B\rho\left(\sum_{n=1}^{N} \alpha_n g_n/\rho(g_n)\right) \qquad (1.3.1)$$

for all $\alpha_n \in \mathbb{C}$ and all $N \in \mathbb{N}$.

If $x(\{\alpha_n\}, \{g_n\})$ and $y(\{\alpha_n\}, \{g_n\})$ are non-negative real valued functions, where $\{\alpha_n\}$ is a sequence of scalars and $\{g_n\}$ is a sequence in some Banach space, then we shall write $x(\{\alpha_n\}, \{g_n\}) \sim y(\{\alpha_n\}, \{g_n\})$ if there exist positive constants A and B, independent of $\{g_n\}$, such that $Ax(\{\alpha_n\}, \{g_n\}) \le y(\{\alpha_n\}, \{g_n\}) \le Bx(\{\alpha_n\}, \{g_n\})$ for all $\{\alpha_n\}$, and we shall use this in the proof of our first result (Theorem 2.1.1). In particular, (1.3.1) will be written as follows:

$$\rho\left(\sum_{n=1}^{N} \alpha_n g_n/\rho(g_n)\right) \sim \left(\sum_{n=1}^{N} |\alpha_n|^p\right)^{1/p}.$$

Also, in the proof of our first result (Theorem 2.1.1) we shall need the following part of ([37], Lemma 1) which was stated for cyclic Banach spaces but which holds in any Banach space X.

Lemma 1.3.1. *For each $x \in X$, define $|x|$ by*

$$|x| = \sup\{\| T_\varphi x \|: \varphi \in \mathcal{L}^\infty, |\varphi| \leq 1\}.$$

Then $|\cdot|$ is a norm on X equivalent to the original norm $\|\cdot\|$, and $\| x \| \leq |x| \leq 4M \| x \|$.

Cyclic spaces were introduced by Bade [2] and [3] in connection with the multiplicity theory for spectral operators on Banach spaces. In what follows, we shall summarise some definitions and results concerning Boolean algebras of projections and their multiplicity theory which are mostly due to Bade [2] and [3]. Let $\mathcal{B}$ be a complete Boolean algebra of projections on a Banach space X throughout the definitions and results stated in this page.

Definition 1.3.2 ([3]). *The projection $\wedge\{E \in \mathcal{B} : Ex = x\}$ is called the carrier projection of x. The projection $E \in \mathcal{B}$ will be said to satisfy the countable chain condition, or to be countably decomposable, if every family of disjoint projections in $\mathcal{B}$ bounded by E is at most countable. Let $\mathcal{C}$ be the set of all $E \in \mathcal{B}$ satisfying this condition. The Boolean algebra $\mathcal{B}$ is called countably decomposable if $\mathcal{B} = \mathcal{C}$.*

We note that if G is the carrier projection of x and $0 \neq F \leq G$, then $Fx \neq 0$.

Lemma 1.3.2 ([3]). *If X is separable, then $\mathcal{B}$ is countably decomposable and every $E \in \mathcal{B}$ is the carrier projection of a vector in X.*

Lemma 1.3.3 ([12], Lemma 1 and [16], p.1958). *If T is a scalar-type spectral operator on X with spectral measure $E(\cdot)$, then*

$$\mathcal{M}(f_0) = \text{clm}\{E(\sigma)f_0 : \sigma \in \Sigma_{\mathbb{C}}\}$$

is separable, and if X is separable, then $\sigma_p(T)$ is countable.

Definition 1.3.3 ([3]). *A multiplicity function is a function m from $\mathcal{B}$ to the cardinal numbers such that $m(0) = 0$, and $m(\vee_\alpha E_\alpha) = \vee_\alpha m(E_\alpha)$ and if $E \in \mathcal{B}$ is countably decomposable then m(E) is the smallest number of cyclic subspaces whose closed span equals the range of E. We say that $E \in \mathcal{B}$ has uniform multiplicity n if m(F) $= n$ whenever $0 \neq F \leq E$.*

Theorem 1.3.3 ([3], Theorem 2.3). *Let m be a multiplicity function on $\mathcal{B}$. Then there is a unique family $\{E_n\}$ of disjoint elements in $\mathcal{B}$, for n running over the cardinals $\leq m(I)$, such that*

(i) $I = \vee E_n$,

(ii) if $E_n \neq 0$, then E_n has uniform multiplicity n.

Two Banach spaces X and Y are called isomorphic if there is an invertible operator from X onto Y. The distance coefficient $d(X,Y)$ of two isomorphic Banach spaces is defined by $\inf(\| T \| \| T^{-1} \|)$ where the inf is taken over all invertible operators T from X onto Y.
The following definition is due to Lindenstrauss and Pelczynski [26] and introduces a new class of Banach spaces $\mathcal{L}_p$ which is larger than the class of L^p spaces.

Definition 1.3.4 ([27]). *Let $1 \leq p \leq \infty$ and $1 \leq \lambda < \infty$. A Banach space X is said to be an $\mathcal{L}_{p,\lambda}$ space if for every finite dimensional subspace B of X there is a finite dimensional subspace C of X such that $C \supset B$ and $d(C, \ell_n^p) \leq \lambda$ where $n = \dim C$.*
A Banach space is said to be an $\mathcal{L}_p$ space, $1 \leq p \leq \infty$, if it is an $\mathcal{L}_{p,\lambda}$ space for some $\lambda < \infty$.

Theorem 1.3.4 ([27], Theorem 3.2). *Every complemented subspace of an $L^1(\mu)$ space is an $\mathcal{L}_1$ space.*

Throughout the rest of this section, let X be a complemented subspace of an $\mathcal{L}_1$ space, and $\mathcal{B}$ a complete Boolean algebra of projections on X. Regard $\mathcal{B}$ as the range of a spectral measure $E(\cdot)$ defined on the Borel sets Σ_Λ of a compact Hausdorff topological space Λ.
The following theorem is a result of Lindenstrauss and Pelczynski ([26], Corollary 8 of Theorem 6.1), but we need it in the following form.

Theorem 1.3.5 ([32], Theorem 2). *There exists a constant M_1 such that for every finite family of disjoint projections $E_k \in \mathcal{B}$ on X, $(k = 1, 2, ..., N)$*

$$\sum_{k=1}^{N} \| E_k x \| \leq M_1 \| (\sum_{k=1}^{N} E_k) x \| \quad \textit{for all } x \in X.$$

Theorem 1.3.6 ([32], Theorems 7 and 10). *Every cyclic subspace $\mathcal{M}(x)$ of X is complemented, and there exists a positive finite Borel measure ν on Λ such that $\mathcal{M}(x)$ is isomorphic to $L^1(\Lambda, \Sigma_\Lambda, \nu)$.*
Moreover, the image of the restriction of $\mathcal{B}$ to $\mathcal{M}(x)$ under this isomorphism is the Boolean algebra of projections consisting of multiplication by characteristic functions in $L^1(\Lambda, \Sigma_\Lambda, \nu)$.

Corollary 1.3.1 ([32], Corollary 11). *Assume that $\mathcal{B}$ is a countably decomposable Boolean algebra of projections on X having finite uniform multiplicity N.*

Then there exist N vectors $x_k \in X$, $k = 1, 2, \cdots, N$, *such that*

$$X = \mathcal{M}(x_1) \oplus \cdots \oplus \mathcal{M}(x_N).$$

Furthermore, there is a positive finite Borel measure space $(\Lambda', \Sigma_{\Lambda'}, \nu)$ *such that* X *is isomorphic to* $L^1(\Lambda', \Sigma_{\Lambda'}, \nu)$ *and under this isomorphism every* $E \in \mathcal{B}$ *corresponds to a multiplication by a characteristic function in* $L^1(\Lambda', \Sigma_{\Lambda'}, \nu)$. *Here* $\Lambda' = \Lambda_1 \cup \ldots\ldots \cup \Lambda_N$, $\Lambda_k = \Lambda$ *for* $k = 1, ..., N$ *and* $\nu = \nu_1 \oplus \ldots\ldots\ldots \oplus \nu_N$, $\nu_k(\cdot) = (E(\cdot)x_k, x_k^*)$ *with* x_k^* *a Bade functional for* x_k, $(k = 1, ..., N)$.

Finally, we shall state some definitions and isomorphism theorems from ([35], Chapter 15) required in the proofs of our results.

Definition 1.3.5. *Let* $(\Omega, \Sigma_\Omega, \mu)$ *be a finite measure space. If we consider the measure* μ *on the* σ*-algebra* $\mathcal{A} = \Sigma_\Omega/\mu-$*null sets, that is, if we fail to distinguish between sets of* Σ_Ω *which differ by a set of* μ*-measure zero, then* $(\mathcal{A}, \mu)$ *is called a measure algebra. The measure algebra* $(\mathcal{A}, \mu)$ *is called separable if* $\mathcal{A}$ *is separable in the metric on* $\mathcal{A}$ *defined by*

$$d(A, B) = \| \chi_A - \chi_B \|_{L^1(\mu)} .$$

Definition 1.3.6. *A mapping* Φ *of a measure algebra* $(\mathcal{A}, \mu)$ *onto a measure algebra* $(\mathcal{B}, \nu)$ *is called an isomorphism if* $\Phi(A^c) = [\Phi(A)]^c$, $\Phi(\cup_{i=1}^{\infty} A_i) = \cup_{i=1}^{\infty} \Phi(A_i)$ *and* $\mu(A) = \nu(\Phi(A))$ *for all* $A, A_i \in \mathcal{A}$. *Here* A^c *denotes the complement of the set* A.

The following theorem, which classifies separable nonatomic measure algebras, is due to Caratheodory.

Theorem 1.3.7 ([35], Chapter 15, Theorem 4). *Let* $(\mathcal{A}, \mu)$ *be a separable nonatomic measure algebra induced by the measure space* (Y, Σ_Y, μ) *with* $\mu(Y) = 1$. *Then there is an isomorphism* Φ *of* $(\mathcal{A}, \mu)$ *onto the measure algebra* $(\mathcal{B}_m/m$*-null sets,* $m)$ *induced by Lebesgue measure* m *on* $[0, 1]$ *by equating sets which differ by sets of Lebesgue measure zero.*

Definition 1.3.7. *A metric space* Y *is called topologically complete if it has an equivalent metric that makes it complete.*

Theorem 1.3.8 ([35], Chapter 15, p.414). *If Y is a complete metric space and $G \subset Y$ is a G_δ set, then G is topologically complete.*

Theorem 1.3.9 ([35], Chapter 15, Theorem 20). *Let (Y, Σ_Y) and (Z, Σ_Z) be topologically complete separable measure spaces, where Σ_Y and Σ_Z are the σ-algebras of Borel sets, and let $\mathcal{M}$ and $\mathcal{N}$ be the families of sets of measure zero in Σ_Y and Σ_Z. If Φ is an isomorphism of $\Sigma_Y/\mathcal{M}$ onto $\Sigma_Z/\mathcal{N}$, then there are sets $Y_0 \in \mathcal{M}$ and $Z_0 \in \mathcal{N}$ and a one-to-one mapping ψ of $Z \backslash Z_0$ onto $Y \backslash Y_0$ such that ψ and ψ^{-1} are measurable and $\Phi(A) = \psi^{-1}[A]$ modulo $\mathcal{N}$.*

We conclude this section by stating complete isometric and isomorphic classifications of $L^p(\mu)$ when μ is a finite measure and $L^p(\mu)$ is separable.

Theorem 1.3.10 ([24], Theorems 2.3 and 2.7). *Suppose that μ is a finite measure defined on (Ω, Σ_Ω) and $L^p(\mu)$ is separable. Then*
(I) $L^p(\mu)$ is linearly isometric and order isomorphic to exactly one of the following spaces:
(1) ℓ^p_n *if μ is purely atomic and the set of atoms is finite,*
(2) ℓ^p *if μ is purely atomic and the set of atoms is infinite,*
(3) $L^p([0,1])$ *if μ is purely nonatomic,*
(4) $(\ell^p_n \oplus L^p([0,1]))_p$ *if μ is neither purely atomic nor purely nonatomic and the set of atoms is finite,*
(5) $(\ell^p \oplus L^p([0,1]))_p$ *if μ is neither purely atomic nor purely nonatomic and the set of atoms is infinite.*
Here $(\ell^p \oplus L^p([0,1]))_p$ denotes the Banach space of all elements $(\{x_n\}, f)$ such that $\| (\{x_n\}, f) \| = [\| \{x_n\} \|^p_{\ell^p} + \| f \|^p_{L^p([0,1])}]^{\frac{1}{p}} < \infty$, with a similar meaning for $(\ell^p_n \oplus L^p([0,1]))_p$.
(II) $L^p(\mu)$ is isomorphic to exactly one of the following spaces:
(1) ℓ^p_n *if $L^p(\mu)$ is finite dimensional,*
(2) ℓ^p *if μ is purely atomic and $L^p(\mu)$ is infinite dimensional,*
(3) $L^p([0,1])$ *if μ is not purely atomic and $L^p(\mu)$ is infinite dimensional.*

Chapter 2

Scalar-Type Spectral Operators on L^p Spaces

The purpose of this chapter is to study the structure of scalar-type spectral operators on the classical Banach spaces $L^p(\Omega, \Sigma_\Omega, \mu)$ for $1 \leq p < \infty$ and to prove some representation theorems for them.

2.1 Scalar-Type Spectral Operators on Cyclic L^p Spaces

The first theorem of this section gives a necessary and sufficient condition for a reflexive cyclic Banach space X to be isomorphic to an L^p space, which we shall use together with Theorem 1.3.6 to prove that on an L^p space, $1 \leq p < \infty$, a scalar-type spectral operator whose spectral measure has a cyclic vector is similar to a multiplication operator on *some* L^p space if and only if a certain condition is satisfied. Then we give an example of a spectral measure with a cyclic vector which does not satisfy this condition on L^p for $1 < p < \infty$, $p \neq 2$, and we use Theorem 1.3.5 to deduce that on an L^1 space *every* scalar-type spectral operator whose spectral measure has a cyclic vector is similar to a multiplication operator on *some* L^1 space. We also give some conditions which make every scalar-type spectral operator on an L^p space, $1 \leq p < \infty$, similar to a multiplication operator on the *same* L^p space.

Condition $(*)_p$. Let $E(\cdot)$ be a spectral measure defined on a measure space $(\Lambda, \Sigma_\Lambda)$ and acting on a Banach space X. We shall say that the spectral measure $E(\cdot)$ satisfies Condition $(*)_p$ if there exists a constant $K_p > 0$ such that the inequalities

$$K_p^{-1} \parallel \sum_{n=1}^{N} E(\sigma_n) f \parallel_X^p \leq \sum_{n=1}^{N} \parallel E(\sigma_n) f \parallel_X^p \leq K_p \parallel \sum_{n=1}^{N} E(\sigma_n) f \parallel_X^p$$

hold for each $f \in X$ and for any disjoint sets $\sigma_1, \sigma_2, \cdots, \sigma_N$ in Σ_Λ with $N \in \mathbb{N}$. Using the notation given after Theorem 1.3.2, the previous inequalities will be written as follows:

$$\| \sum_{n=1}^{N} E(\sigma_n) f \|_X^p \sim \sum_{n=1}^{N} \| E(\sigma_n) f \|_X^p .$$

We begin by showing that a reflexive cyclic Banach space X is isomorphic to an L^p space if and only if the associated spectral measure $E(\cdot)$ on X satisfies Condition $(*)_p$ for the same p, and under this isomorphism the image of $E(\sigma) f_0$ in X is χ_σ in L^p, where $\sigma \in \Sigma_\Lambda$, and f_0 is the cyclic vector for X.

Theorem 2.1.1. *Suppose that X is a reflexive cyclic Banach space with a cyclic vector f_0 for the spectral measure $E(\cdot) : \Sigma_\Lambda \to \mathcal{B}(X)$, and let $1 < p < \infty$. Then there exist a positive finite Borel measure ν on $(\Lambda, \Sigma_\Lambda)$ and an isomorphism $V : X \to L^p(\nu)$ such that $E(\sigma) = V^{-1} M_{\chi_\sigma} V$ for all $\sigma \in \Sigma_\Lambda$ if and only if the spectral measure $E(\cdot)$ satisfies Condition $(*)_p$.*

Proof. Suppose that there exist a positive finite Borel measure ν on $(\Lambda, \Sigma_\Lambda)$ and an isomorphism $V : X \to L^p(\nu)$ such that $E(\sigma) = V^{-1} M_{\chi_\sigma} V$ for all $\sigma \in \Sigma_\Lambda$. Let $\sigma_1, \sigma_2, \cdots, \sigma_N$ be any disjoint sets in Σ_Λ with $N \in \mathbb{N}$ and let $f \in X$. Then

$$\begin{aligned}
\sum_{n=1}^{N} \| E(\sigma_n) f \|_X^p &= \sum_{n=1}^{N} \| V^{-1} M_{\chi_{\sigma_n}} V f \|_X^p \\
&\leq \sum_{n=1}^{N} \| V^{-1} \|^p \| M_{\chi_{\sigma_n}} V f \|_{L^p(\nu)}^p \\
&= \| V^{-1} \|^p \sum_{n=1}^{N} \int_{\sigma_n} |Vf|^p d\nu \\
&= \| V^{-1} \|^p \int_{\cup_{n=1}^{N} \sigma_n} |Vf|^p d\nu \\
&= \| V^{-1} \|^p \| M_{\chi_{(\cup_{n=1}^{N} \sigma_n)}} V f \|_{L^p(\nu)}^p \\
&\leq \| V^{-1} \|^p \| V \|^p \| V^{-1} M_{\chi_{(\cup_{n=1}^{N} \sigma_n)}} V f \|_X^p \\
&= \| V^{-1} \|^p \| V \|^p \| E(\cup_{n=1}^{N} \sigma_n) f \|_X^p \\
&= K_p \| \sum_{n=1}^{N} E(\sigma_n) f \|_X^p, \text{ where } K_p = \| V^{-1} \|^p \| V \|^p .
\end{aligned}$$

Similarly, we can show that

$$\| \sum_{n=1}^{N} E(\sigma_n) f \|_X^p \leq K_p \sum_{n=1}^{N} \| E(\sigma_n) f \|_X^p .$$

Hence $E(\cdot)$ satisfies Condition $(*)_p$.

Conversely, suppose that the spectral measure $E(\cdot)$ satisfies Condition $(*)_p$. Then if $\{\sigma_n\}_{n=1}^N$ is any fixed sequence of disjoint elements in Σ_Λ, we have

$$\| \sum_{n=1}^{N} E(\sigma_n) f \|_X^p \sim \sum_{n=1}^{N} \| E(\sigma_n) f \|_X^p \quad \text{for all } f \in X.$$

In particular, taking $f = \sum_{m=1}^N \gamma_m E(\sigma_m) f_0$, we get

$$\| \sum_{n=1}^{N} \gamma_n E(\sigma_n) f_0 \|_X^p \sim \sum_{n=1}^{N} \| \gamma_n E(\sigma_n) f_0 \|_X^p .$$

By Theorem 1.3.1, there exists a bicontinuous linear isomorphism U of X onto $L_\rho(\nu)$, where ν is the measure defined on Λ by $\nu(\cdot) = (E(\cdot)f_0, f_0^*)$ with f_0^* a Bade functional for f_0 defined at the start of section 1.3 and ρ is the function norm defined there as follows:

$$\rho(f) = \sup\{\| T_\psi f_0 \|_X : \psi \in \mathcal{L}^\infty, |\psi| \le f\}.$$

Moreover,

$$E(\sigma) = U^{-1} M_{\chi_\sigma} U \text{ for all } \sigma \in \Sigma_\Lambda \text{ and } U f_0 = 1.$$

Thus $X \cong L_\rho(\nu)$. We want to prove that for each sequence of disjoint nonzero elements $g_n \in L_\rho(\nu)$ we have

$$\left(\rho(\sum_{n=1}^{N} \alpha_n \frac{g_n}{\rho(g_n)})\right)^p \sim \sum_{n=1}^{N} |\alpha_n|^p$$

so that by Theorem 1.3.2, $L_\rho(\nu) \cong L^p(\nu)$ and the isomorphism from $L_\rho(\nu)$ onto $L^p(\nu)$ leaves all characteristic functions invariant. The proof will then be complete.

Let $\{g_n\} \in L_\rho(\nu)$ be any sequence of nonzero disjoint elements, i.e., g_n have disjoint supports $\sigma_n \in \Sigma_\Lambda$. Then the functions $f_n = \frac{g_n}{\rho(g_n)}$ have disjoint supports $\sigma_n \in \Sigma_\Lambda$ and $\rho(f_n) = 1$. We have to show that

$$\left(\rho(\sum_{n=1}^{N} \alpha_n f_n)\right)^p \sim \sum_{n=1}^{N} |\alpha_n|^p.$$

Suppose that $f_n = \sum_{m=1}^{N_n} \beta_{nm} \chi_{\sigma_{nm}}$, where $\sigma_{nm} \in \Sigma_\Lambda$ are pairwise disjoint. Then using the norms in Theorem 1.3.1 and Lemma 1.3.1, we have

$$\begin{aligned} \rho(f_n) &= \sup\{\| T_\psi f_0 \|_X : \psi \in \mathcal{L}^\infty, |\psi| \le f_n\} \\ &= \sup\{\| T_{\varphi f_n} f_0 \|_X : \varphi \in \mathcal{L}^\infty, |\varphi| \le 1\} \\ &= \sup\{\| T_\varphi T_{f_n} f_0 \|_X : \varphi \in \mathcal{L}^\infty, |\varphi| \le 1\} \\ &= |T_{f_n} f_0|. \end{aligned}$$

Thus we have

$$
\begin{aligned}
\left(\rho(\sum_n \alpha_n f_n)\right)^p &= |T_{\sum_n \alpha_n f_n} f_0|^p \\
&= |\sum_n \alpha_n T_{f_n} f_0|^p \\
&\sim \| \sum_n \alpha_n T_{f_n} f_0 \|_X^p \\
&= \| \sum_n \alpha_n \int f_n(\lambda) E(d\lambda) f_0 \|_X^p \\
&= \| \sum_n \alpha_n (\sum_m \beta_{nm} E(\sigma_{nm}) f_0) \|_X^p \\
&= \| \sum_{n,m} \alpha_n \beta_{nm} E(\sigma_{nm}) f_0 \|_X^p \\
&\sim \sum_{n,m} \| \alpha_n \beta_{nm} E(\sigma_{nm}) f_0 \|_X^p \\
&= \sum_n |\alpha_n|^p (\sum_m \| \beta_{nm} E(\sigma_{nm}) f_0 \|_X^p) \\
&\sim \sum_n |\alpha_n|^p \| \sum_m \beta_{nm} E(\sigma_{nm}) f_0 \|_X^p \\
&= \sum_n |\alpha_n|^p \| T_{f_n} f_0 \|_X^p \\
&\sim \sum_n |\alpha_n|^p |T_{f_n} f_0|^p \\
&= \sum_n |\alpha_n|^p [\rho(f_n)]^p \\
&= \sum_n |\alpha_n|^p.
\end{aligned}
$$

Since the step functions are dense in $L_\rho(\nu)$, the proof is now complete. □

Notice that in the proof of the previous theorem, the proof of the necessity of Condition $(*)_p$ holds in any Banach space X and is also valid when $p = 1$.
A direct consequence of the previous theorem and Theorem 1.3.6 is the following.

Corollary 2.1.1. *Suppose that $L^p(\mu)$, $1 \leq p < \infty$, is a cyclic Banach space with a cyclic vector f_0 for the spectral measure $E(\cdot) : \Sigma_\Lambda \to \mathcal{B}(L^p(\mu))$. Then there exist a positive finite Borel measure ν on $(\Lambda, \Sigma_\Lambda)$ and an isomorphism $V : L^p(\mu) \to L^p(\nu)$ such that $E(\sigma) = V^{-1} M_{\chi_\sigma} V$ for all $\sigma \in \Sigma_\Lambda$ if and only if the spectral measure $E(\cdot)$ satisfies Condition $(*)_p$.*

The next important result gives a necessary and sufficient condition for a scalar-type spectral operator on an L^p space, $1 \leq p < \infty$, whose spectral measure has a cyclic vector, to be similar to a multiplication operator on *some* L^p space.

The result for $p = 2$ is well-known to be true ([10], Theorem IX.3.4 and [13], Theorem 8.3).

Theorem 2.1.2. *Let T be a scalar-type spectral operator on $L^p(\Omega, \Sigma_\Omega, \mu)$, where $1 \leq p < \infty$, and suppose that the spectral measure $E(\cdot)$ of T has a cyclic vector f_0. Then the following are equivalent.*
(i) The operator T is similar to a multiplication operator M_ψ on some L^p space, where ψ is a bounded measurable essentially one-to-one function.
(ii) The spectral measure $E(\cdot)$ satisfies Condition $(*)_p$.

Proof. If (i) holds, then (ii) can be proved in the same way as in the proof of the previous theorem, just replace M_{χ_σ} by $M_{\chi_{\psi^{-1}(\sigma)}}$.

Suppose that (ii) holds, i.e., $E(\cdot)$ satisfies Condition $(*)_p$. Then, by Corollary 2.1.1, there exists an isomorphism $U : L^p(\mu) \to L^p(\nu)$ such that $Uf_0 = 1$ and $E(\sigma) = U^{-1}M_{\chi_\sigma}U$ for any Borel subset σ of $\sigma(T)$ where $\nu(\cdot) = (E(\cdot)f_0, f_0^*)$ is defined on $(\sigma(T), \Sigma_{\sigma(T)})$. Here f_0^* is a Bade functional for f_0. By Lemma 1.3.3, $L^p(\mu)$ is separable and $\sigma_p(T)$ is countable. Since $\sigma_r(T) = \emptyset$ by ([16], Corollary XV.8.3.4), we have $\Lambda_0 = \sigma_c(T) = \sigma(T)\backslash\sigma_p(T)$ is a G_δ-set. Thus Λ_0 is topologically complete by Theorem 1.3.8, and the measure $E(\cdot)E(\Lambda_0)f_0$ is nonatomic.
In the case $E(\Lambda_0)f_0 \neq 0$, we may assume that $\nu_c(\Lambda_0) = (E(\Lambda_0)f_0, f_0^*) = 1$, where $\nu = \nu_c \oplus \nu_d$ is the decomposition of ν into its continuous and discrete parts. Then by Theorem 1.3.7, there exists an isomorphism Φ_c of (Σ_{Λ_0}/ν_c-null sets, ν_c) onto the measure algebra ($\mathcal{B}_m/m$-null sets, m) induced by Lebesgue measure m on $[0, 1]$. Let Σ_{Λ_0} denote the restriction of Σ_Λ to Λ_0. Since $(\Lambda_0, \Sigma_{\Lambda_0}, \nu_c)$ and $([0, 1], \mathcal{B}_m, m)$ are topologically complete separable measure spaces, Σ_{Λ_0} and $\mathcal{B}_m$ are their Borel sets, and Φ_c is an isomorphism of Σ_{Λ_0}/ν_c-null sets onto $\mathcal{B}_m/m$-null sets, we know that by Theorem 1.3.9 there exist $X_0 \subset \Lambda_0$ and $Y_0 \subset [0, 1]$ such that $\nu_c(X_0) = 0 = m(Y_0)$ and there exists a one-to-one onto point mapping $\psi_c : [0, 1]\backslash Y_0 \to \Lambda_0\backslash X_0$ such that ψ_c and ψ_c^{-1} are measurable and $\Phi_c(A) = \psi_c^{-1}[A]$ and $m(\Phi_c(A)) = \nu_c(A)$ for all $A \in \Sigma_{\Lambda_0}/\nu_c$-null sets, i.e., $m(\sigma) = \nu_c(\psi_c(\sigma))$ for all $\sigma \in \mathcal{B}_m/m$-null sets. Define $\psi_d : \Gamma \to \sigma_p(T)$ where $\Gamma \subset \mathbb{N}$ such that $\psi_d(n) = \lambda_n$, $\lambda_n \in \sigma_p(T)$. Here $\sigma_p(T) = \{\lambda_n : n \in \Gamma\}$, the λ_n s are distinct and Γ is either $\emptyset$, finite or $\mathbb{N}$. Define the measure m_d on Γ as follows: $m_d(\{n\}) = \nu_d(\{\lambda_n\}) = (E(\{\lambda_n\})f_0, f_0^*)$. Let

$$\psi = \begin{cases} \psi_c & \text{on } [0, 1]\backslash Y_0, \\ \psi_d & \text{on } \Gamma. \end{cases}$$

Then ψ is a one-to-one point mapping of $([0, 1]\backslash Y_0) \cup \Gamma$ onto $\sigma(T)\backslash X_0$ such that ψ and ψ^{-1} are measurable.

Let $f = \sum_{j=1}^{N} \alpha_j \chi_{\Delta_j} \in L^p(\nu)$, where $\Delta_j = \sigma_j \cup \{\lambda_{i_1^{(j)}}, \dots, \lambda_{i_k^{(j)}}, \dots\dots\} \subset \sigma(T)$ are disjoint, $\sigma_j \subset \Lambda_0 = \sigma_c(T)$ and $\{\lambda_{i_1^{(j)}}, \dots, \lambda_{i_k^{(j)}}, \dots\dots\} \subset \sigma_p(T)$. Then

$$\begin{aligned}
\| f \|_{L^p(\nu)}^p &= \sum_{j=1}^{N} |\alpha_j|^p \nu(\Delta_j) \\
&= \sum_{j=1}^{N} |\alpha_j|^p (\nu_c(\sigma_j) + \nu_d(\{\lambda_{i_1^{(j)}}, \dots\})) \\
&= \sum_{j=1}^{N} |\alpha_j|^p (m(\psi_c^{-1}(\sigma_j)) + m_d(\{i_1^{(j)}, \dots\})) \\
&= \| \sum_{j=1}^{N} \alpha_j \chi_{\psi^{-1}(\Delta_j)} \|_{L^p[0,1] \oplus \ell_w^p(\Gamma)}^p \\
&= \| f \circ \psi \|_{L^p[0,1] \oplus \ell_w^p(\Gamma)}^p .
\end{aligned}$$

Here $\ell_w^p(\Gamma)$ is the weighted ℓ^p space corresponding to the weight $w = \{w_k\}$ on Γ given by $w_k = (E(\{\lambda_k\})f_0, f_0^*)$, i.e., $\ell_w^p(\Gamma) = L^p(\Gamma, \Sigma_\Gamma, \mu)$ with $\mu(\{\gamma_k\}) = w_k$ for each $\gamma_k \in \Gamma$. Thus, $\| f \|_{L^p(\nu)} = \| f \circ \psi \|_{L^p[0,1] \oplus \ell_w^p(\Gamma)}$ for all $f \in L^p(\sigma(T), \Sigma_{\sigma(T)}, \nu)$ such that $f = \sum_{j=1}^{N} \alpha_j \chi_{\Delta_j} \in L^p(\nu)$ with Δ_j disjoint Borel subsets of $\sigma(T)$. Since the set of all such functions in $L^p(\nu)$ is dense in $L^p(\nu)$, we know that $U_0 : f \to f \circ \psi$ can be extended by continuity to all of $L^p(\nu)$, and also this extension is an isometry from $L^p(\nu)$ onto $L^p[0,1] \oplus \ell_w^p(\Gamma)$, and $V = U_0 \circ U : L^p(\mu) \to L^p[0,1] \oplus \ell_w^p(\Gamma)$ is an isomorphism such that $VE(\sigma)f_0 = \chi_{\psi^{-1}(\sigma)}$ for all Borel $\sigma \subset \sigma(T)$.

We want to prove that $VTE(\sigma)f_0 = M_\psi VE(\sigma)f_0$ for all Borel subsets σ of $\sigma(T)$ so that $T = V^{-1} M_\psi V$ because $\text{clm}\{E(\sigma)f_0 : \sigma \in \Sigma_{\sigma(T)}\} = L^p(\mu)$.

Let $\epsilon > 0$ be given, and let $\{\sigma_1, \sigma_2, \dots, \sigma_N\}$ be any Borel partition of $\sigma(T)$ such that $\sup\{|\lambda - \lambda'| : \lambda, \lambda' \in \sigma_j\} < \epsilon$ for $1 \le j \le N$. Fix $\sigma \in \Sigma_{\sigma(T)}$.
Since $T = \int_{\sigma(T)} \lambda E(d\lambda)$, we have $TE(\sigma)f_0 = \int_{\sigma(T)} \lambda E(d\lambda)E(\sigma)f_0$. Hence choosing $\lambda_j \in \sigma_j \cap \sigma$ we have

$$\begin{aligned}
\| TE(\sigma)f_0 - \sum_{j=1}^{N} \lambda_j E(\sigma_j)E(\sigma)f_0 \|_{L^p(\mu)} &= \| \int (\lambda - \sum_{j=1}^{N} \lambda_j \chi_{\sigma_j}(\lambda))E(d\lambda)E(\sigma)f_0 \|_{L^p(\mu)} \\
&\le 4M^2 \max_{1 \le j \le N} \sup_{\lambda \in \sigma_j} |\lambda - \lambda_j| \, \| f_0 \| \\
&< 4M^2 \epsilon \, \| f_0 \|,
\end{aligned}$$

where $M = \sup\{\| E(\sigma) \| : \sigma \in \Sigma_{\sigma(T)}\}$.

Thus, denoting the identity map on $\mathbb{C}$ by id, we have

$$
\begin{aligned}
&\| VTE(\sigma)f_0 - M_\psi VE(\sigma)f_0 \|_{L^p[0,1]\oplus\ell^p_w(\Gamma)} \\
&\leq \| VTE(\sigma)f_0 - V\sum_{j=1}^{N}\lambda_j E(\sigma_j)E(\sigma)f_0 \| + \| V\sum_{j=1}^{N}\lambda_j E(\sigma_j)E(\sigma)f_0 - M_\psi VE(\sigma)f_0 \| \\
&\leq \| V \|\| TE(\sigma)f_0 - \sum_{j=1}^{N}\lambda_j E(\sigma_j)E(\sigma)f_0 \| + \| \sum_{j=1}^{N}\lambda_j \chi_{\psi^{-1}(\sigma_j\cap\sigma)} - \psi\chi_{\psi^{-1}(\sigma)} \| \\
&< 4M^2\epsilon \| V \|\| f_0 \| + \| \sum_{j=1}^{N}\lambda_j\chi_{\sigma_j\cap\sigma}\circ\psi - (id\circ\psi)\chi_{\psi^{-1}(\sigma)} \| \\
&= 4M^2\epsilon \| V \|\| f_0 \| + \| \sum_{j=1}^{N}\lambda_j\chi_{\sigma_j\cap\sigma}\circ\psi - (id\circ\psi)(\chi_\sigma\circ\psi) \| \\
&= 4M^2\epsilon \| V \|\| f_0 \| + \| \sum_{j=1}^{N}\lambda_j\chi_{\sigma_j\cap\sigma}\circ\psi - ((id)(\chi_\sigma))\circ\psi \| \\
&= 4M^2\epsilon \| V \|\| f_0 \| + \| \left(\sum_{j=1}^{N}\lambda_j\chi_{\sigma_j}\chi_\sigma - (id)(\chi_\sigma)\right)\circ\psi \|_{L^p[0,1]\oplus\ell^p_w(\Gamma)} \\
&= 4M^2\epsilon \| V \|\| f_0 \| + \| \sum_{j=1}^{N}(\lambda_j\chi_{\sigma_j} - id)\chi_\sigma \|_{L^p(\nu)} \\
&\leq 4M^2\epsilon \| V \|\| f_0 \| + \max_{1\leq j\leq N}\sup_{\lambda\in\sigma_j}|\lambda-\lambda_j|(\nu(\sigma))^{\frac{1}{p}} \\
&= 4M^2\epsilon \| V \|\| f_0 \| + \max_{1\leq j\leq N}\sup_{\lambda\in\sigma_j}|\lambda-\lambda_j|(E(\sigma)f_0, f_0^*)^{\frac{1}{p}} \\
&< \epsilon(4M^2 \| V \|\| f_0 \| + M^{\frac{1}{p}} \| f_0 \|^{\frac{1}{p}} \| f_0^* \|^{\frac{1}{p}}).
\end{aligned}
$$

Since $\epsilon > 0$ was arbitrary, $T = V^{-1}M_\psi V$.

In the case $E(\Lambda_0)f_0 = 0$, i.e., $\nu_c(\Lambda_0) = 0$, we get $T = V_d^{-1}M_{\psi_d}V_d$, where $V_d = U_0^d \circ U^d : L^p(\mu) \to \ell^p_w(\Gamma)$ is an isomorphism. □

The following result follows immediately from Theorems 1.3.5 and 2.1.2.

Corollary 2.1.2. *Suppose that T is a scalar-type spectral operator on $L^1(\Omega, \Sigma_\Omega, \mu)$ and that the spectral measure of T has a cyclic vector. Then T is similar to a multiplication operator M_ψ on some L^1 space, where ψ is a bounded measurable essentially one-to-one function.*

Next, we give an example to show that there exists a spectral measure with a cyclic vector which does not satisfy Condition $(*)_p$ on L^p for $1 < p < \infty$, $p \neq 2$. Hence, not every scalar-type spectral operator whose spectral measure has a cyclic vector on an L^p space, $1 < p < \infty$, $p \neq 2$, is similar to a multiplication operator.

Example 2.1.1. *The Haar system gives a spectral measure with a cyclic vector on $L^p[0,1]$, $1 < p < \infty$, which does not satisfy Condition $(*)_p$ for $p \neq 2$.*

Indeed, let $h_{n,j}$ be the Haar functions defined in ([30], p.150) as follows: $h_{0,0} = 1$ and for $n \geq 1$, $j = 1, \cdots, 2^{n-1}$,

$$h_{n,j} = \begin{cases} 1 & \text{on } [\frac{2j-2}{2^n}, \frac{2j-1}{2^n}), \\ -1 & \text{on } [\frac{2j-1}{2^n}, \frac{2j}{2^n}), \\ 0 & \text{otherwise.} \end{cases}$$

Let $\Lambda = \{(n,j) :$ either $n = j = 0$ or $n, j \in \mathbb{N}$ with $1 \leq j \leq 2^{n-1}\}$. Define $E(\cdot)$ on Λ as follows:

$$E(\sigma) = \sum_{(n,j)\in\sigma} E_{n,j} \text{ for all } \sigma \subset \Lambda,$$

where $E_{n,j}$ is defined by

$$E_{n,j}f = 2^{n-1}(\int_0^1 f h_{n,j}) h_{n,j}.$$

It is easy to check that $E_{n,j}^2 = E_{n,j}$, $E_{n,j}E_{m,k} = 0$ for all $(n,j), (m,k) \in \Lambda$ with $(n,j) \neq (m,k)$.

Fix $f_0 = \alpha_{00}h_{0,0} + \sum_{m=1}^{\infty}\sum_{k=1}^{2^{m-1}} \alpha_{m,k}h_{m,k}$ with $\alpha_{m,k} > 0$, $\sum_{m=1}^{\infty}\sum_{k=1}^{2^{m-1}} \alpha_{m,k} < \infty$. Then $E_{n,j}f_0 = \alpha_{n,j}h_{n,j}$. Since the linear span of the Haar system contains all the characteristic functions of the dyadic intervals, we have

$$\begin{aligned} \operatorname{clm}\{E(\sigma)f_0 : \sigma \subset \Lambda\} &= \operatorname{clm}\{h_{n,j} : (n,j) \in \Lambda\} \\ &= L^p[0,1], \quad 1 < p < \infty. \end{aligned}$$

In ([30], p.156) it is proved that the Haar system is an unconditional basis for $L^p[0,1]$, $1 < p < \infty$. Thus $E(\cdot)$ is a spectral measure with a cyclic vector. However, this spectral measure does not satisfy Condition $(*)_p$ for $1 < p < \infty$, $p \neq 2$. For let $f = \sum_{n=1}^{N}\sum_{j=1}^{2^{n-1}} a_{n,j}h_{n,j} \in L^p[0,1]$, $1 < p < \infty$, where $a_{n,j} = 1$ for $n = 1, \ldots, N$ and $j = 1, \ldots, 2^{n-1}$. Then

$$\sum_{n=1}^{N}\sum_{j=1}^{2^{n-1}} E_{n,j}f = \sum_{n=1}^{N}\sum_{j=1}^{2^{n-1}} h_{n,j} = \sum_{n=1}^{N} r_n,$$

where $r_n(t) = \operatorname{sign}\sin(2^n\pi t)$ is the sequence of Rademacher functions in $L^p[0,1]$. Now, Khintchine's inequality ([29], Theorem 2.b.3) states that for every $1 \leq p < \infty$ there exist positive constants A_p and B_p such that, for every choice of scalars $\{a_n\}_{n=1}^{m}$,

$$A_p \left(\sum_{n=1}^{m} |a_n|^2\right)^{\frac{1}{2}} \leq \left(\int_0^1 \left|\sum_{n=1}^{m} a_n r_n(t)\right|^p dt\right)^{\frac{1}{p}} \leq B_p \left(\sum_{n=1}^{m} |a_n|^2\right)^{\frac{1}{2}}.$$

Using Khintchine's inequality, we have

$$\begin{aligned}\| \sum_{n=1}^{N} \sum_{j=1}^{2^{n-1}} E_{n,j} f \|_{L^p[0,1]}^p &= \| \sum_{n=1}^{N} r_n \|_{L^p[0,1]}^p \\ &\sim (\sum_{n=1}^{N} |1|^2)^{\frac{p}{2}} \\ &= N^{\frac{p}{2}}.\end{aligned}$$

However,

$$\begin{aligned}\| E_{n,j} f \|_{L^p[0,1]}^p &= \| E_{n,j} (\sum_{n=1}^{N} \sum_{j=1}^{2^{n-1}} a_{n,j} h_{n,j}) \|_{L^p[0,1]}^p \\ &= \| a_{n,j} E_{n,j} h_{n,j} \|_{L^p[0,1]}^p \\ &= \| h_{n,j} \|_{L^p[0,1]}^p \\ &= 2^{1-n}.\end{aligned}$$

Thus,

$$\sum_{n=1}^{N} \sum_{j=1}^{2^{n-1}} \| E_{n,j} f \|_{L^p[0,1]}^p = \sum_{n=1}^{N} \sum_{j=1}^{2^{n-1}} 2^{1-n} = N.$$

Therefore, for $p \neq 2$,

$$\| \sum_{n=1}^{N} \sum_{j=1}^{2^{n-1}} E_{n,j} f \|_{L^p[0,1]}^p \sim N^{\frac{p}{2}} \nsim N = \sum_{n=1}^{N} \sum_{j=1}^{2^{n-1}} \| E_{n,j} f \|_{L^p[0,1]}^p .$$

Next, we prove that Condition $(*)_2$ is always satisfied by any spectral measure on any L^2 space. However, not every scalar-type spectral operator with a cyclic vector on L^2 is similar to a multiplication operator on the **same** L^2 space. For example, the bilateral shift U on ℓ^2 is not similar to any diagonal operator on ℓ^2 because $\sigma_p(U) = \emptyset$.

We mention here that it is proved in [31] that on an $\mathcal{L}_1, \mathcal{L}_2$ or $\mathcal{L}_\infty$ space a similar version of Condition $(*)_p$ ($p = 1, 2$ or ∞ respectively) is satisfied by any spectral measure acting on these spaces and that these spaces are the only spaces on which the similar version of Condition $(*)_p$ ($p = 1, 2$ or ∞ respectively) is satisfied by any spectral measure acting on them.

Theorem 2.1.3. *Condition $(*)_2$ is satisfied by any spectral measure on any Hilbert space $\mathcal{H}$.*

Proof. Suppose that Σ_Λ is the family of Borel subsets of Λ, $E(\cdot) : \Sigma_\Lambda \to \mathcal{B}(\mathcal{H})$ is a spectral measure, $\{\sigma_n\}$ is a disjoint sequence of elements of Σ_Λ and $\{r_n\}$ is the orthonormal sequence of Rademacher functions in $L^2([0,1])$. Fix $x \in \mathcal{H}$ and $t \in [0,1]$. Then

$$\begin{aligned}
\| \sum_{n=1}^{N} r_n(t) E(\sigma_n) x \| &= \| \sum_{n=1}^{N} \pm E(\sigma_n) x \| \\
&= \| \sum_{n=1}^{N} \pm E(\sigma_n) E(\cup_1^N \sigma_n) x \| \\
&\leq 2M \| E(\cup_1^N \sigma_n) x \|,
\end{aligned}$$

where $\pm$ are from the definition of the Rademacher functions $r_n(t) = \operatorname{sign} \sin(2^n \pi t)$, and $M = \sup\{\| E(\sigma) \| : \sigma \in \Sigma_{\sigma(T)}\}$.
Also we have

$$\begin{aligned}
\| E(\cup_1^N \sigma_n) x \| &= \| \sum_{n=1}^{N} E(\sigma_n) x \| \\
&= \| \left(\sum_{n=1}^{N} r_n(t) E(\sigma_n) \right) \left(\sum_{n=1}^{N} r_n(t) E(\sigma_n) \right) x \| \\
&\leq \| \sum_{n=1}^{N} \pm E(\sigma_n) \| \| \sum_{n=1}^{N} r_n(t) E(\sigma_n) x \| \\
&\leq 2M \| \sum_{n=1}^{N} r_n(t) E(\sigma_n) x \| .
\end{aligned}$$

Hence,

$$\| E(\cup_{n=1}^N \sigma_n) x \| \sim \| \sum_{n=1}^{N} r_n(t) E(\sigma_n) x \| \quad \text{for all } t \in [0,1].$$

Therefore we have

$$\begin{aligned}
\| \sum_{n=1}^{N} E(\sigma_n) x \|^2 &= \| E(\cup_{n=1}^N \sigma_n) x \|^2 \\
&\sim \int_0^1 \| \sum_{n=1}^{N} r_n(t) E(\sigma_n) x \|^2 \, dt \\
&= \int_0^1 \left(\sum_{n=1}^{N} r_n(t) E(\sigma_n) x, \sum_{n=1}^{N} r_n(t) E(\sigma_n) x \right) dt \\
&= \sum_{n=1}^{N} \| E(\sigma_n) x \|^2 .
\end{aligned}$$

□

Next, we present the main theorem of this section giving some conditions which make a scalar-type spectral operator on L^p, $1 \leq p < \infty$, similar to a multiplication operator on the *same* L^p space. A preliminary lemma is required.

Lemma 2.1.1. *Suppose that μ is a regular Borel measure on a locally compact Hausdorff space (Ω, Σ_Ω). If $\varphi : \Omega \to \mathbb{C}$ is a bounded measurable essentially one-to-one function, then the multiplication operator $M_\varphi : L^p(\Omega, \Sigma_\Omega, \mu) \to L^p(\Omega, \Sigma_\Omega, \mu)$, $1 \leq p < \infty$, has a cyclic vector for its spectral measure.*

Proof. Suppose that $\varphi : \Omega \to \mathbb{C}$ is a bounded measurable essentially one-to-one function. Since the measure μ is σ-finite, by ([28], p.98) we have

$$L^p(\Omega, \Sigma_\Omega, \mu) = \operatorname{clm}\{M_{\chi_\sigma} 1 : \sigma \in \Sigma_\Omega\}.$$

We want to prove that given $\sigma \in \Sigma_\Omega$, there exists $\tau \in \Sigma_\mathbb{C}$ such that $\varphi^{-1}(\tau)$ differs from σ by a set of measure zero.
Fix $\sigma \in \Sigma_\Omega$. By Lusin's Theorem ([22], p.242), for each $n \in \mathbb{N}$ there exists compact $K_n \subset \sigma$ such that φ is continuous on K_n and $\mu(\sigma \backslash K_n) < \frac{1}{n}$. Since φ is continuous on K_n and K_n is compact, $\varphi(K_n)$ is compact and hence Borel in $\mathbb{C}$. Thus, $\tau = \cup_{n=1}^\infty \varphi(K_n) \in \Sigma_\mathbb{C}$. But since $\varphi : \Omega \to \mathbb{C}$ is essentially one-to-one, $\varphi^{-1}(\tau) = \cup_{n=1}^\infty K_n \subset \sigma$ and $\mu(\sigma \backslash \varphi^{-1}(\tau)) = \mu(\sigma \backslash \cup_{n=1}^\infty K_n) = 0$. So $\varphi^{-1}(\tau) = \sigma$ up to a set of measure zero. Therefore, up to sets of measure zero, we have

$$\{\sigma : \sigma \in \Sigma_\Omega\} = \{\varphi^{-1}(\tau) : \tau \in \Sigma_\mathbb{C}\}.$$

So we have

$$L^p(\Omega, \Sigma_\Omega, \mu) = \operatorname{clm}\{M_{\chi_{\varphi^{-1}(\tau)}} 1 : \tau \in \Sigma_\mathbb{C}\}.$$

Therefore, the spectral measure of M_φ has a cyclic vector. □

The proof of the main theorem of this section now follows.

Theorem 2.1.4. *Suppose that Ω is a locally compact complete metric space, Σ_Ω is the σ-algebra of Borel subsets of Ω and μ is a Borel measure on (Ω, Σ_Ω). Suppose also that T is a scalar-type spectral operator with spectral measure $E(\cdot)$ on $L^p(\Omega, \Sigma_\Omega, \mu)$, $1 \leq p < \infty$. Then (a) and (b) are equivalent.*
(a) The operator T is similar to M_φ on the same $L^p(\Omega, \Sigma_\Omega, \mu)$, where $\varphi : \Omega \to \mathbb{C}$ is a bounded measurable essentially one-to-one function.
(b) The following two conditions are satisfied:
(i) The spectral measure $E(\cdot)$ has a cyclic vector f_0 and satisfies Condition $(*)_p$,
(ii) card Ω_d = card $\sigma_p(T)$ and either $\mu(\Omega_c)$ and $E(\sigma_c(T))$ are both zero or both nonzero, where Ω_d is the set of the supports of the atoms in Ω and $\Omega_c = \Omega \backslash \Omega_d$.

Proof. Throughout this proof we shall follow the same notation introduced in the proof of Theorem 2.1.2. Suppose that (*b*) holds and that $E(\sigma_c(T)) \neq 0$. Since condition (i) of (*b*) is satisfied, by Theorem 2.1.2 there exists a bounded one-to-one point mapping ψ of $([0, 1] \backslash Y_0) \cup \Gamma$ onto $\sigma(T) \backslash X_0$ such that ψ and ψ^{-1} are measurable and there exists an isomorphism $V : L^p(\Omega, \Sigma_\Omega, \mu) \to L^p([0,1]) \oplus \ell^p_w(\Gamma)$ such that $VE(\sigma)f_0 = \chi_{\psi^{-1}(\sigma)}$ for all Borel $\sigma \subset \sigma(T)$ and such that $T = V^{-1}M_\psi V$. Here $\Gamma \subset \mathbb{N}$, and card $\Gamma =$ card $\sigma_p(T)$.

Now, let $L^p(\Omega, \Sigma_\Omega, \mu) = L^p(\Omega_c, \Sigma_{\Omega_c}, \mu_c) \oplus L^p(\Omega_d, \Sigma_{\Omega_d}, \mu_d)$ be the decomposition of $L^p(\Omega, \Sigma_\Omega, \mu)$ into its continuous and discrete parts. Since $E(\sigma_c(T)) \neq 0$, by condition (ii) of (b) we have $\mu(\Omega_c) \neq 0$, and hence we may assume (by appropriate scaling) that $\mu_c(\Omega_c) = 1$. Thus by Theorem 1.3.7, there exists an isomorphism Φ^1_c of the separable nonatomic measure algebra (Σ_{Ω_c}/μ_c-null sets, μ_c) onto the measure algebra ($\mathcal{B}_m/m$-null sets, m) induced by Lebesgue measure m on $[0,1]$. Since Ω_d is countable, $\Omega_c = \Omega \backslash \Omega_d$ is a G_δ-set in the complete metric space Ω. Thus Ω_c is topologically complete by Theorem 1.3.8. Since $(\Omega_c, \Sigma_{\Omega_c}, \mu_c)$ and $([0,1], \mathcal{B}_m, m)$ are topologically complete separable measure spaces, Σ_{Ω_c} and $\mathcal{B}_m$ are their Borel sets, and Φ^1_c is an isomorphism of Σ_{Ω_c}/μ_c-null sets onto $\mathcal{B}_m/m$-null sets, by Theorem 1.3.9 there exist $X_1 \subset \Omega_c$ and $Y_1 \subset [0,1]$ such that $\mu_c(X_1) = 0 = m(Y_1)$ and there exists a one-to-one onto point mapping $\psi^1_c : [0,1] \backslash Y_1 \to \Omega_c \backslash X_1$ such that ψ^1_c and $(\psi^1_c)^{-1}$ are measurable and $\Phi^1_c(A) = (\psi^1_c)^{-1}[A]$ and $m(\Phi^1_c(A)) = \mu_c(A)$ for all $A \in \Sigma_{\Omega_c}/\mu_c$-null sets, i.e., $m(\sigma) = \mu_c(\psi^1_c(\sigma))$ for all $\sigma \in \mathcal{B}_m/m$-null sets. Using condition (ii) of (b), we know that card $\Omega_d =$ card $\sigma_p(T) =$ card Γ. Define $\psi^1_d : \Gamma \to \Omega_d$ such that $\psi^1_d(n) = \gamma_n$, $\gamma_n \in \Omega_d$. Here $\Omega_d = \{\gamma_n : n \in \Gamma\}$, the γ_n s are distinct and Γ is either $\emptyset$, finite or $\mathbb{N}$. Define the measure m^1_d on Γ as follows: $m^1_d(\{n\}) = \mu_d(\{\gamma_n\})$. Let

$$\psi_1 = \begin{cases} \psi^1_c & \text{on } [0,1] \backslash Y_1, \\ \psi^1_d & \text{on } \Gamma. \end{cases}$$

Then ψ_1 is a one-to-one point mapping of $([0,1] \backslash Y_1) \cup \Gamma$ onto $\Omega \backslash X_1$ such that ψ_1 and ψ_1^{-1} are measurable. If $\Delta \in \Sigma_\Omega$, then $\Delta = \Delta_c \cup \Delta_d$ where $\Delta_c \in \Sigma_{\Omega_c}$ and $\Delta_d \in \Sigma_{\Omega_d}$ and, denoting by m' the measure $m \oplus m^1_d$ on $[0,1] \cup \Gamma$, we have

$$\mu(\Delta) = \mu_c(\Delta_c) + \mu_d(\Delta_d) = m\left(\left(\psi^1_c\right)^{-1}(\Delta_c)\right) + m^1_d\left(\left(\psi^1_d\right)^{-1}(\Delta_d)\right) = m'\left(\psi_1^{-1}(\Delta)\right).$$

If $f = \sum_{j=1}^N \alpha_j \chi_{\Delta_j} \in L^p(\mu)$, where Δ_j are disjoint in Ω, then

$$\begin{aligned} \| f \|^p_{L^p(\mu)} &= \sum_{j=1}^N |\alpha_j|^p \mu(\Delta_j) = \sum_{j=1}^N |\alpha_j|^p m'(\psi_1^{-1}(\Delta_j)) \\ &= \| \sum_{j=1}^N \alpha_j \chi_{\psi_1^{-1}(\Delta_j)} \|^p_{L^p[0,1] \oplus \ell^p_\omega(\Gamma)} \\ &= \| f \circ \psi_1 \|^p_{L^p[0,1] \oplus \ell^p_\omega(\Gamma)} . \end{aligned}$$

Here $\ell^p_\omega(\Gamma)$ is the weighted ℓ^p space corresponding to the weight $\omega = \{\omega_k\}$ on Γ given by $\omega_k = \mu_d(\{\gamma_k\})$. Hence $W : f \to f \circ \psi_1$ can be extended by continuity to an isometry from $L^p(\mu)$ onto $L^p[0,1] \oplus \ell^p_\omega(\Gamma)$. Let $\varphi = \psi \circ \psi_1^{-1}$: $\Omega \backslash X_1 \to \sigma(T)\backslash X_0$. Define $J : L^p[0,1] \oplus \ell^p_w(\Gamma) \to L^p[0,1] \oplus \ell^p_\omega(\Gamma)$ as follows: $J(f, \{x_k\}) = (f, \{(\frac{w_k}{\omega_k})^{\frac{1}{p}} x_k\})$. Then J is an isometry from $L^p[0,1] \oplus \ell^p_w(\Gamma)$ onto $L^p[0,1] \oplus \ell^p_\omega(\Gamma)$ and $J^{-1}WM_\varphi f = M_\psi J^{-1}Wf$ for all $f \in L^p(\mu)$.
Therefore, $T = V^{-1}M_\psi V = V^{-1}J^{-1}WM_\varphi W^{-1}JV = S^{-1}M_\varphi S$, where $S = W^{-1}JV : L^p(\mu) \to L^p(\mu)$ is invertible and $\varphi : \Omega \to \mathbb{C}$ is essentially one-to-one point mapping which is bounded and measurable on Ω.

If $E(\sigma_c(T)) = 0$, then by condition (ii) of (b), $\mu(\Omega_c) = 0$. So as before we get $T = V_d^{-1}M_{\psi_d}V_d$, where $V_d = U_0^d \circ U^d : L^p(\mu) \to \ell^p_w(\Gamma)$ is an isomorphism. Put $\varphi_d = \psi_d \circ (\psi_d^1)^{-1} : \Omega_d \to \sigma_p(T)$. Then $J_d^{-1}W_dM_{\varphi_d}f = M_{\psi_d}J_d^{-1}W_df$ for all $f \in L^p(\mu)$. Let $S_d = W_d^{-1}J_dV_d : L^p(\mu) \to L^p(\mu)$. Then $T = S_d^{-1}M_{\varphi_d}S_d$.

Conversely, suppose that (a) holds, i.e., T is similar to M_φ on the same $L^p(\mu)$. Then there exists an invertible $S : L^p(\mu) \to L^p(\mu)$ such that $T = S^{-1}M_\varphi S$, where $\varphi : \Omega \to \mathbb{C}$ is essentially one-to-one point mapping which is bounded and measurable on Ω. By Lemma 2.1.1, the spectral measure of M_φ has a cyclic vector and hence the spectral measure of T has a cyclic vector. Therefore, condition (i) of (b) is satisfied as we can easily show that the spectral measure of T satisfies Condition $(*)_p$ in the same way as in the proof of Theorem 2.1.1.

To prove condition (ii) of (b), notice that $\sigma_p(T) = \sigma_p(M_\varphi)$ and $\sigma_c(T) = \sigma_c(M_\varphi)$, so it suffices to prove that card $\sigma_p(M_\varphi) =$ card Ω_d and that $F(\sigma_c(M_\varphi)) = 0$ if and only if $\mu(\Omega_c) = 0$, where $F(\cdot)$ is the spectral measure of M_φ.
In ([23], p.226), it is proved that $\sigma_p(M_\varphi) = \{\lambda \in \mathbb{C} : \mu(\varphi^{-1}(\lambda)) > 0\}$ holds on L^2, but it holds on all L^p, $1 \le p < \infty$. In fact, if $f \in L^p(\mu)$, and $\varphi(t)f(t) = \lambda f(t)$ a.e., then $\varphi(t) = \lambda$ a.e. whenever $f(t) \neq 0$. This implies that in order for λ to be an eigenvalue of M_φ, the function φ must take the value λ on a set of positive measure, i.e., $\mu(\varphi^{-1}(\lambda)) > 0$. Conversely, if $\mu(\varphi^{-1}(\lambda)) > 0$, then $\varphi(t) = \lambda$ on a set $M \subset \varphi^{-1}(\lambda)$ of finite positive measure. So $M_\varphi \chi_M = \lambda\chi_M$, where $\chi_M \in L^p(\mu), \chi_M \neq 0$. Thus, λ is an eigenvalue of M_φ. So, we have

$$\begin{aligned}\sigma_p(M_\varphi) &= \{\lambda \in \mathbb{C} : \mu(\varphi^{-1}(\lambda)) > 0\} \\ &= \{\lambda \in \mathbb{C} : \varphi^{-1}(\lambda) \text{ is an atom in } \Omega\} \\ &= \{\lambda \in \mathbb{C} : \varphi^{-1}(\lambda) \in \Omega_d\}.\end{aligned}$$

Since φ is essentially one-to-one, card $\sigma_p(M_\varphi)=$ card Ω_d. Also, since

$\sigma(M_\varphi) =$ essrange $\varphi = \{\lambda \in \mathbb{C} :$ for each neighbourhood N of $\lambda, \mu(\varphi^{-1}(N)) > 0\}$,

we have

$$\begin{aligned}\sigma_c(M_\varphi) &= \sigma(M_\varphi)\backslash\sigma_p(M_\varphi)\\ &= \{\lambda \in \text{essrange } \varphi : \mu(\varphi^{-1}(\lambda)) = 0\}\\ &= \{\lambda \in \text{essrange } \varphi : \varphi^{-1}(\lambda) \notin \Omega_d\}\\ &= \{\lambda \in \text{essrange } \varphi : \varphi^{-1}(\lambda) \in \Omega_c\}.\end{aligned}$$

Thus we have

$$F(\sigma_c(M_\varphi)) = F\left(\text{essrange } \varphi \cap \varphi(\Omega_c)\right) = F\left(\varphi(\Omega_c)\right) = M_{\chi_{\varphi^{-1}(\varphi(\Omega_c))}} = M_{\chi_{\Omega_c}}.$$

Therefore, $F(\sigma_c(M_\varphi)) = 0$ if and only if $\chi_{\Omega_c} = 0$ a.e. (μ) if and only if $\mu(\Omega_c) = 0$. □

In case $p = 2$, $E(\cdot)$ always satisfies Condition $(*)_2$ according to Theorem 2.1.3 and the assumption of the existence of a cyclic vector is not important. We can use Theorem IX.4.6 instead of Theorem IX.3.4 in [10] to prove that every scalar-type spectral operator on L^2 is similar to a multiplication operator on the *same* L^2 space provided that only a modified version of condition (ii) of (b) of the previous theorem is satisfied.
In case $p = 1$, $E(\cdot)$ always satisfies Condition $(*)_1$ according to Theorem 1.3.5, so we can omit this condition in the previous theorem. The omission of the condition of the existence of a cyclic vector in the previous theorem will be discussed in the next section.

2.2 Scalar-Type Spectral Operators on L^1 Spaces whose Spectral Measures have Finite Multiplicity

Throughout this section suppose that Ω is a locally compact separable complete metric space, Σ_Ω is the σ-algebra of Borel subsets of Ω and μ is a Borel measure on (Ω, Σ_Ω). By ([1], p.847) the topology on Ω has a countable basis. So by ([22], p.168, Theorem B), μ is separable. Hence $L^1(\mu)$ is separable by ([40], Theorem 2, p.137). In this section we generalise the main result of the previous section. Specifically, we prove that every scalar-type spectral operator on an L^1 space, whose spectral measure has finite multiplicity N and satisfies a similar condition to condition (ii) of (b) of Theorem 2.1.4, is similar to a multiplication operator on the same L^1 space. We start with two preliminary definitions.

Definition 2.2.1. *A function $\varphi : \Omega \to \mathbb{C}$ is said to be essentially N-to-one on Ω if there exists a partition of Ω (up to sets of measure zero) into N disjoint sets*

$\Omega_1, \Omega_2, \cdots, \Omega_N$ *such that* $\varphi|\Omega_j$ *is essentially one-to-one for each* j *and the ranges* $\varphi(\Omega_j)$ *are equal to within sets of measure zero.*

Definition 2.2.2. *A function* $\varphi : \Omega \to \mathbb{C}$ *is said to be essentially at most N-to-one on* Ω *if there exists a partition of* Ω *into* N *disjoint sets* $\Omega_1, \Omega_2, \cdots, \Omega_N$ *such that* φ *is essentially i-to-one on* Ω_i*, for* $i = 1, 2, \cdots, N$.

The following lemma holds on any L^p space, $1 \leq p < \infty$, but we shall prove it on L^1 as we are only concerned with operators on L^1 in this section.

Lemma 2.2.1. *If* $\varphi : \Omega \to \mathbb{C}$ *is a bounded measurable essentially 2-to-one function, then the multiplication operator* $M_\varphi : L^1(\Omega, \Sigma_\Omega, \mu) \to L^1(\Omega, \Sigma_\Omega, \mu)$ *has no cyclic vector for its spectral measure.*

Proof. Suppose that $\varphi : \Omega \to \mathbb{C}$ is a bounded measurable essentially 2-to-one function and that M_φ has a cyclic vector f_0 for its spectral measure $F(\cdot)$. Then by the Weirstrass approximation Theorem and since $f(M_\varphi) = \int_{\sigma(M_\varphi)} f(\lambda)F(d\lambda)$ for $f \in C(\sigma(M_\varphi))$, we have

$$\begin{aligned} L^1(\mu) &= \text{clm}\{F(\sigma)f_0 : \sigma \in \Sigma_{\mathbb{C}}\} \\ &= \text{clm}\{\int_{\sigma(M_\varphi)} f(\lambda)F(d\lambda)f_0 : f \in C(\sigma(M_\varphi))\} \\ &= \text{clm}\{f(M_\varphi) : f \in C(\sigma(M_\varphi))\} \\ &= \text{clm}\{p(M_\varphi, M_{\overline{\varphi}})f_0 : p \text{ a polynomial}\} \\ &= \text{clm}\{p(\varphi, \overline{\varphi})f_0 : p \text{ a polynomial}\}. \end{aligned} \tag{2.2.1}$$

Fix $\tau \in \Sigma_{\sigma(M_\varphi)}$. Since φ is essentially 2-to-one, there exist 2 disjoint sets of positive measure $\Omega_1, \Omega_2 \subset \Omega$ such that, up to sets of measure zero, $\varphi(\Omega_1) = \varphi(\Omega_2) = \tau$. Choose $g \in L^1(\mu)$ such that $g \equiv 0$ on Ω_1 and $|g| > \epsilon$ on Ω_2. Then by (2.2.1), there exists a sequence of polynomials p_n such that $p_n(\varphi, \overline{\varphi})f_0 \to g$ pointwise a.e. in Ω. So $p_n(\varphi(\omega), \overline{\varphi}(\omega)) \to 0$ pointwise a.e. in Ω_1 and $p_n(\varphi(\omega), \overline{\varphi}(\omega)) \nrightarrow 0$ pointwise a.e. in Ω_2. Thus, $p_n(z, \overline{z}) \to 0$ on τ and $p_n(z, \overline{z}) \nrightarrow 0$ on τ. This contradiction completes the proof. □

Now, we prove the main theorem of this section which shows that a similar condition to condition (ii) of (*b*) of Theorem 2.1.4 is necessary and sufficient for a scalar-type spectral operator on an L^1 space, whose spectral measure has finite multiplicity, to be similar to a multiplication operator on the *same* L^1 space.

Theorem 2.2.1. *Suppose that* $T \in \mathcal{B}(L^1(\Omega, \Sigma_\Omega, \mu))$. *Then the following are equivalent.*

(I) The operator T *is scalar-type spectral on* $L^1(\Omega, \Sigma_\Omega, \mu)$ *with spectral measure*

$E(\cdot)$ of finite multiplicity N and card $\Omega_d = \sum_{j=1}^{N} j$ card ({eigenvalues of multiplicity j}) and either $\mu(\Omega_c)$ and $E(\sigma_c(T))$ are both zero or both nonzero.

(II) There exist a bounded measurable essentially at most N-to-one function $\varphi : \Omega \to \sigma(T)$ and an invertible $S : L^1(\mu) \to L^1(\mu)$ such that $T = S^{-1}M_\varphi S$.

Proof. Throughout this proof we shall follow the same method and the same notation as in the proofs of Theorems 2.1.2 and 2.1.4. For simplicity we shall prove the theorem for the case N=2. Suppose that T is a scalar-type spectral operator on $L^1(\Omega, \Sigma_\Omega, \mu)$ whose spectral measure $E(\cdot)$ has multiplicity 2 and that card $\Omega_d = \sum_{j=1}^{2} j$ card ({eigenvalues of multiplicity j}) and either $\mu(\Omega_c)$ and $E(\sigma_c(T))$ are both zero or both nonzero. Then $I = E(\sigma(T))$ has multiplicity 2. So by Theorem 1.3.3, there is a unique partition of $\sigma(T)$ into disjoint Λ_1, Λ_2 such that

(i) $I = E(\Lambda_1) \vee E(\Lambda_2)$,

(ii) $E(\Lambda_1)$ has uniform multiplicity 1 and $E(\Lambda_2)$ has uniform multiplicity 2 if $E(\Lambda_i) \neq 0$ for $i = 1, 2$.

Let $T_1 = T|E(\Lambda_1)L^1(\mu)$. By Theorem 1.3.4, $E(\Lambda_1)L^1(\mu)$ is an $\mathcal{L}_1$ space. Since $I|E(\Lambda_1)L^1(\mu) = E(\Lambda_1)$ has multiplicity 1, $E(\Lambda_1)L^1(\mu) = \mathcal{M}(x_1)$ for some $x_1 \in E(\Lambda_1)L^1(\mu)$. By Theorem 1.3.6, there exists a positive finite Borel measure space $(\Lambda_1, \Sigma_{\Lambda_1}, \nu_1)$ such that $\mathcal{M}(x_1)$ is isomorphic to $L^1(\Lambda_1, \Sigma_{\Lambda_1}, \nu_1)$, and if $U_1 : \mathcal{M}(x_1) \to L^1(\nu_1)$ is the induced isomorphism, then $U_1 E(\sigma)E(\Lambda_1)x_1 = \chi_\sigma$. Since $E(\Lambda_1)L^1(\mu)$ is separable, $\sigma_p(T_1)$ is countable, and so $\sigma_c(T_1) = \sigma(T_1)\backslash\sigma_p(T_1)$ is a G_δ set, and $\nu_1^c = \nu_1|\sigma_c(T_1)$ has no atoms, and we may assume that $\nu_1^c(\sigma_c(T_1)) = \frac{1}{2}$. Then using Theorems 1.3.7 and 1.3.9 we get a one-to-one onto point mapping $\psi_1^c : [0, \frac{1}{2}]\backslash Y_1 \to \sigma_c(T_1)\backslash X_1$ such that $m(\sigma) = \nu_1^c(\psi_1^c(\sigma))$ for all Borel $\sigma \subset [0, \frac{1}{2}]$. Define $\psi_1^d : \Gamma_1 \subset \mathbb{N} \to \sigma_p(T_1)$ by $\psi_1^d(n) = \lambda_n^{(1)}$, $\lambda_n^{(1)} \in \sigma_p(T_1)$.

Let $T_2 = T|E(\Lambda_2)L^1(\mu)$. By Theorem 1.3.4, $E(\Lambda_2)L^1(\mu)$ is an $\mathcal{L}_1$ space. Since $E(\Lambda_2)L^1(\mu)$ is separable, by Lemma 1.3.2, the Boolean algebra $\mathcal{B}_2$ generated by the spectral measure $E(\cdot)E(\Lambda_2)$ is countably decomposable. Since $I|E(\Lambda_2)L^1(\mu) = E(\Lambda_2)$ has uniform multiplicity 2, by Corollary 1.3.1 there exist 2 vectors $x_{21}, x_{22} \in E(\Lambda_2)L^1(\mu)$ such that

$$\begin{aligned} E(\Lambda_2)L^1(\mu) &= \mathcal{M}(x_{21}) \oplus \mathcal{M}(x_{22}) \\ &\cong L^1(\sigma(T_2), \Sigma_{\sigma(T_2)}, \nu_{21}) \oplus L^1(\sigma(T_2), \Sigma_{\sigma(T_2)}, \nu_{22}) \\ &= L^1(\sigma(T_2) \cup \sigma(T_2), \Sigma_{\sigma(T_2)\cup\sigma(T_2)}, \nu_2) \end{aligned}$$

where $\nu_{2i}(\cdot) = (E(\cdot)E(\Lambda_2)x_{2i}, x_{2i}^*)$ and $\nu_2 = \nu_{21} \oplus \nu_{22}$, and there exists an isomorphism $U_2 : E(\Lambda_2)L^1(\mu) \to L^1(\nu_2)$ such that

$$U_2(E(\sigma_1)E(\Lambda_2)x_{21} \oplus E(\sigma_2)E(\Lambda_2)x_{22}) = \chi_{\sigma_1} \oplus \chi_{\sigma_2}.$$

Let

$$U = U_1 \oplus U_2 : L^1(\mu) \to L^1(\nu), \text{ where } \nu = \nu_1 \oplus \nu_2.$$

Since $E(\Lambda_2)L^1(\mu)$ is separable, $\sigma_p(T_2)$ is countable, and $\sigma_c(T_2) = \sigma(T_2)\backslash\sigma_p(T_2)$ is a G_δ set, and ν_{21}^c and ν_{22}^c have no atoms on $\sigma_c(T_2)$, and we may assume that $\nu_{2i}(\sigma_c(T_2)) = \frac{1}{4}$ for $i = 1, 2$. Since $(\Sigma_{\sigma_c(T_2)}, \nu_{21}^c)$ is a separable nonatomic measure algebra with $\nu_{21}^c(\sigma_c(T_2)) = \frac{1}{4}$, by Theorem 1.3.7 there exists an isomorphism Φ_{21}^c of $(\Sigma_{\sigma_c(T_2)}/\nu_{21}^c\text{-null sets}, \nu_{21}^c)$ onto the measure algebra $(\mathcal{B}_m/m\text{-null sets}, m)$ induced by Lebesgue measure m on $[\frac{1}{2}, \frac{3}{4}]$. Then by Theorem 1.3.9, there exists a one-to-one onto mapping $\psi_{21}^c : [\frac{1}{2}, \frac{3}{4}]\backslash Y_{21} \to \sigma_c(T_2)\backslash X_{21}$ where $\nu_{21}^c(X_{21}) = m(Y_{21}) = 0$ such that ψ_{21}^c and $(\psi_{21}^c)^{-1}$ are measurable, $\Phi_{21}^c(A) = (\psi_{21}^c)^{-1}[A]$ $a.e.[m]$ and $m(\sigma) = \nu_{21}^c(\psi_{21}^c(\sigma))$ for all Borel $\sigma \subset [\frac{1}{2}, \frac{3}{4}]$. Similarly, there exists a one-to-one onto mapping $\psi_{22}^c : [\frac{3}{4}, 1]\backslash Y_{22} \to \sigma_c(T_2)\backslash X_{22}$ where $\nu_{22}^c(X_{22}) = m(Y_{22}) = 0$ such that ψ_{22}^c and $(\psi_{22}^c)^{-1}$ are measurable, and $m(\sigma) = \nu_{22}^c(\psi_{22}^c(\sigma)$ for all Borel $\sigma \subset [\frac{3}{4}, 1]$. Define $\psi_{2i}^d : \Gamma_{2i} \subset \mathbb{N} \to \sigma_p(T_2)$ by $\psi_{2i}^d(n) = \lambda_n^{(2)}$, $\lambda_n^{(2)} \in \sigma_p(T_2)$, for $i = 1, 2$. Here $\Gamma_{21} = \Gamma_{22}$ and $\sigma_p(T_2) = \{\lambda_n^{(2)} : n \in \Gamma_{2i}\}$.
Define $\psi : ([0,1]\backslash(Y_1 \cup Y_{21} \cup Y_{22})) \cup \Gamma_1 \cup \Gamma_{21} \cup \Gamma_{22} \to \sigma(T)$ as follows:

$$\psi = \begin{cases} \psi_1^c & \text{on } [0, \frac{1}{2}], \\ \psi_{21}^c & \text{on } [\frac{1}{2}, \frac{3}{4}], \\ \psi_{22}^c & \text{on } [\frac{3}{4}, 1], \\ \psi_1^d & \text{on } \Gamma_1, \\ \psi_{21}^d & \text{on } \Gamma_{21}, \\ \psi_{22}^d & \text{on } \Gamma_{22}. \end{cases}$$

Then ψ is a bounded measurable essentially at most 2-to-one function on $[0,1] \cup \Gamma_1 \cup \Gamma_{21} \cup \Gamma_{22}$ and $U_0 : f \to f \circ \psi$ is an isometry from $L^1(\nu)$ onto $L^1([0,1]) \oplus \ell_w^1(\Gamma)$, where $\Gamma = \Gamma_1 \cup \Gamma_{21} \cup \Gamma_{22}$. So we have
card Γ = card Γ_1 + 2 card Γ_{2i} = card $\sigma_p(T_1)$ + 2 card $\sigma_p(T_2)$ = card Ω_d. Then $V = U_0 \circ U : L^1(\mu) \to L^1([0,1]) \oplus \ell_w^1(\Gamma)$ is an isomorphism such that $T = V^{-1}M_\psi V$. Then we can continue the proof exactly as the proof of Theorem 2.1.4.

Conversely, suppose that there exist a bounded measurable essentially at most 2-to-one function $\varphi : \Omega \to \sigma(T)$ and an invertible $S : L^1(\mu) \to L^1(\mu)$ such that $T = S^{-1}M_\varphi S$. Then there exists a partition of Ω into disjoint sets Ω_1, Ω_2 such that φ is essentially one-to-one on Ω_1 and φ is essentially 2-to-one on Ω_2. Let

$\varphi_1 = \varphi|\Omega_1$ and $\varphi_2 = \varphi|\Omega_2$. Then we have

$$L^1(\Omega, \Sigma_\Omega, \mu) = L^1(\Omega_1, \Sigma_{\Omega_1}, \mu_1) \oplus L^1(\Omega_2, \Sigma_{\Omega_2}, \mu_2)$$

and

$$T \cong M_\varphi = M_{\varphi_1} \oplus M_{\varphi_2}.$$

Since $M_\varphi L^1(\Omega_i) \subset L^1(\Omega_i)$ and $T = S^{-1}M_\varphi S$, we have $T(S^{-1}L^1(\Omega_i)) \subset S^{-1} L^1(\Omega_i)$ for $i = 1, 2$.

Let $T_i = T|S^{-1}L^1(\Omega_i, \Sigma_{\Omega_i}, \mu_i)$ for $i = 1, 2$. Then $T_i \cong M_{\varphi_i}$ for $i = 1, 2$. We know that $\sigma(T_i) = \sigma(M_{\varphi_i}) = \text{essrange}(\varphi_i)$, so we have

$$E(\sigma(T_i))L^1(\Omega) = S^{-1}M\chi_{\varphi_i^{-1}(\sigma(T_i))}SL^1(\Omega) = S^{-1}L^1(\Omega_i) \text{ for } i = 1, 2.$$

We claim that $I|S^{-1}L^1(\Omega_i) = E(\sigma(T_i))$ has uniform multiplicity i for $i = 1, 2$.

Since φ_1 is essentially one-to-one, by Lemma 2.1.1 we know that $M_{\varphi_1} : L^1(\Omega_1) \to L^1(\Omega_1)$ has a cyclic vector for its spectral measure and $\mathcal{M}(\varphi_1) = S^{-1}L^1(\Omega_1)$, that is,
$I|S^{-1}L^1(\Omega_1) = E(\sigma(T_1))$ has uniform multiplicity 1.
Let $\Omega_2 = \Omega_{21} \cup \Omega_{22}$ be a partition of Ω_2 such that $\varphi_{2i} = \varphi_2|\Omega_{2i}$ is essentially one-to-one on Ω_{2i}, for $i = 1, 2$. Then by Lemma 2.1.1, $M_{\varphi_{2i}} : L^1(\Omega_{2i}) \to L^1(\Omega_{2i})$ has a cyclic vector for its spectral measure and $\mathcal{M}(\varphi_{2i}) = L^1(\Omega_{2i})$. Hence, $\mathcal{M}(\varphi_{21}) \oplus \mathcal{M}(\varphi_{22}) = L^1(\Omega_2)$, i.e., $I|S^{-1}L^1(\Omega_2) = E(\sigma(T_2))$ has multiplicity 2 because $M_{\varphi_2} : L^1(\Omega_2) \to L^1(\Omega_2)$ has no cyclic vector for its spectral measure according to Lemma 2.2.1 as φ_2 is essentially 2-to-one. If $F(\cdot)$ is the spectral measure of M_φ and $0 < F(\sigma_0) \leq F(\sigma(T_2))$, i.e., $\emptyset \neq \sigma_0 \subset \sigma(T_2)$, then

$$\begin{aligned} F(\sigma_0)L^1(\Omega_2) &= F(\sigma_0)\mathcal{M}(\varphi_{21}) \oplus F(\sigma_0)\mathcal{M}(\varphi_{22}) \\ &= \mathcal{M}(\varphi_{21}|\varphi_{21}^{-1}(\sigma_0)) \oplus \mathcal{M}(\varphi_{22}|\varphi_{22}^{-1}(\sigma_0)), \end{aligned}$$

and $M_{\varphi_2|\varphi_2^{-1}(\sigma_0)}$ has no cyclic vector for its spectral measure according to Lemma 2.2.1. Therefore, $I|S^{-1}L^1(\Omega_2) = E(\sigma(T_2))$ has uniform multiplicity 2.

Now, to prove that card $\Omega_d = \sum_{j=1}^{2} j$ card ({eigenvalues of multiplicity j}) and either $\mu(\Omega_c)$ and $E(\sigma_c(T))$ are both zero or both nonzero, we use the proof of Theorem 2.1.4. Notice that $\varphi_1 = \varphi|\Omega_1$ is essentially one-to-one on Ω_1, so

$$\text{card } \sigma_p(M_{\varphi_1}) = \text{ card } (\Omega_1)_d \text{ and } F(\sigma_c(M_{\varphi_1})) = 0 \text{ if and only if } \mu((\Omega_1)_c) = 0.$$

Similarly, since $\varphi_{2i} = \varphi|\Omega_{2i}$, $i = 1, 2$, is essentially one-to-one on Ω_{2i},

$$\text{card } \sigma_p(M_{\varphi_{2i}}) = \text{ card } (\Omega_{2i})_d \text{ and } F(\sigma_c(M_{\varphi_{2i}})) = 0 \text{ if and only if } \mu((\Omega_{2i})_c) = 0.$$

Therefore, we have

$$\begin{aligned}\text{card } \Omega_d &= \text{ card } (\Omega_1)_d + \text{ card } (\Omega_{21})_d + \text{ card } (\Omega_{22})_d \\ &= \text{ card } \sigma_p(M_{\varphi_1}) + \text{ card } \sigma_p(M_{\varphi_{21}}) + \text{ card } \sigma_p(M_{\varphi_{22}}) \\ &= \text{ card } (\{\text{eigenvalues of multiplicity } 1\}) + 2 \text{ card } (\{\text{eigenvalues of multiplicity } 2\})\end{aligned}$$

and $F(\sigma_c(T)) = 0$ if and only if $F(\sigma_c(M_\varphi)) = 0$
if and only if $F(\sigma_c(M_{\varphi_1})) = 0$ and $F(\sigma_c(M_{\varphi_{2i}})) = 0$
if and only if $\mu((\Omega_1)_c) = 0$ and $\mu((\Omega_{2i})_c) = 0$ if and only if $\mu(\Omega_c) = 0$.

The proof is similar to the above if $2 < N < \infty$. □

2.3 Scalar-Type Spectral Operators on L^1 Spaces whose Spectral Measures have Infinite Multiplicity

In this section we give a non-trivial example of a scalar-type spectral operator on $L^1([0,1])$, whose spectral measure has infinite uniform multiplicity, which is a multiplication operator. (Of course, the identity operator is a trivial example of such an operator.) Then we give an example of a scalar-type spectral operator on ℓ^1, whose spectral measure has infinite nonuniform multiplicity, which is a diagonal operator on ℓ^1. Then we ask whether every scalar-type spectral operator on L^1, whose spectral measure has infinite (uniform) multiplicity, is similar to a multiplication operator on L^1. We also discuss the difficulties in producing an example of a scalar-type spectral operator on L^1 whose spectral measure has infinite (uniform) multiplicity which is not similar to a multiplication operator.

Example 2.3.1.

Define M_φ on $L^1([0,1])$ as multiplication by φ, where φ is defined on $[0,1]$ as follows:

$$\varphi = 1 \text{ on } [\frac{1}{2}, 1], \varphi = \frac{1}{2} \text{ on } [\frac{1}{3}, \frac{1}{2}), \varphi = \frac{1}{3} \text{ on } [\frac{1}{4}, \frac{1}{3}), \text{ and so on.}$$

Then the spectral measure for M_φ has infinite uniform multiplicity.

Example 2.3.2.

Consider the diagonal operator T defined on ℓ^1 as follows:

$$T = \text{diag}\{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \cdots\cdots\}.$$

Then

$$\sigma(T) = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\cdots\} \cup \{0\},$$

and, if $\{e_j\}$ is the standard basis of ℓ^1, the spectral measure for T is given by

$$E(\{1\})f = f_1 e_1, \text{ so } E(\{1\})\ell^1 = \mathcal{M}(e_1),$$

$$E(\{\frac{1}{2}\})f = f_2 e_2 + f_3 e_3, \text{ so } E(\{\frac{1}{2}\})\ell^1 = \mathcal{M}(e_2) \oplus \mathcal{M}(e_3),$$

$$E(\{\frac{1}{3}\})f = f_4 e_4 + f_5 e_5 + f_6 e_6, \text{ so } E(\{\frac{1}{3}\})\ell^1 = \mathcal{M}(e_4) \oplus \mathcal{M}(e_5) \oplus \mathcal{M}(e_6),$$

and so on, where $f = (f_1, f_2, f_3, \cdots\cdots)$.

Hence, the spectral measure of T on ℓ^1 has infinite nonuniform multiplicity.

Perhaps more difficult is constructing an example of a scalar-type spectral operator on L^1 whose spectral measure has infinite multiplicity or infinite uniform multiplicity which is not similar to a multiplication operator as we know that proving that two operators are not similar is very hard.

On the other hand, it would be interesting to know whether every scalar-type spectral operator on L^1 whose spectral measure has infinite (uniform) multiplicity is similar to a multiplication operator. Suppose that T is a scalar-type spectral operator on L^1 whose spectral measure has infinite multiplicity and that card $\Omega_d = \sum_{j=1}^{\infty} j$ card ($\{$eigenvalues of multiplicity $j\}$) and either $\mu(\Omega_c)$ and $E(\sigma_c(T))$ are both zero or both nonzero. Then by Theorem 1.3.3, there exists a unique disjoint family $\{\sigma_n\}$ of subsets of $\sigma(T)$ such that

(i) $I = \vee_{n=1}^{\infty} E(\sigma_n)$,

(ii) if $E(\sigma_n) \neq 0$, then $E(\sigma_n)$ has uniform multiplicity n.

We considered the following two cases.

(1) If $E(\sigma_\infty) = 0$, as in Example 2.3.2, we know that for each $n \in \mathbb{N}$, $I|E(\sigma_n)L^1(\mu)$ has finite uniform multiplicity n and by Theorem 2.2.1, $T_n = T|E(\sigma_n)L^1(\mu)$ is similar to a multiplication operator, i.e., there exists a bounded measurable essentially n-to-one function $\varphi_n : \Omega_n \to \sigma(T_n)$ and an invertible $S_n : E(\sigma_n)L^1(\mu) \to E(\sigma_n)L^1(\mu)$ such that $T_n = S_n^{-1} M_{\varphi_n} S_n$. However, we could not control $\sup_n \parallel S_n \parallel$ and $\sup_n \parallel S_n^{-1} \parallel$ in order to say that $T = \oplus_1^\infty T_n = (\oplus_1^\infty S_n^{-1})(\oplus_1^\infty M_{\varphi_n})(\oplus_1^\infty S_n)$. Just in the special case when the constant M_1 in Theorem 1.3.5 equals 1 we get $\sup_n \parallel S_n \parallel \leq 1$ and $\sup_n \parallel S_n^{-1} \parallel \leq 1$, and in such a situation we can see that T is similar to a multiplication operator on $L^1(\mu)$.

(2) If $E(\sigma_\infty) = I$, as in Example 2.3.1, we have

$$\text{clm}\{E(\sigma)f_n : 1 \leq n \leq N, \sigma \subset \sigma(T)\} \neq L^1(\mu) \text{ for any finite } N,$$

so we could not use the L^1 theory to continue the proof.

If cases (1) and (2) were proved, then the extension to the general case when $0 < E(\sigma_\infty) < I$ might be possible.

Chapter 3

Decomposition of Well-Bounded and Trigonometrically Well-Bounded Operators on $\mathcal{H}$

In [17], Fong and Lam proved that a well-bounded operator on a Hilbert space $\mathcal{H}$ with a contractive AC(J)-functional calculus is a self-adjoint operator on $\mathcal{H}$. In this chapter, we show that every well-bounded operator on a Hilbert space $\mathcal{H}$ is a quasinilpotent perturbation of a self-adjoint operator and we obtain a similar result for trigonometrically well-bounded operators on a Hilbert space $\mathcal{H}$. We also consider the problem of determining which quasinilpotent perturbations of self-adjoint operators are well-bounded and highlight some of the difficulties involved in tackling this.

3.1 Well-Bounded Operators on $\mathcal{H}$

The aim of this section is to examine the structure of well-bounded operators acting on a Hilbert space $\mathcal{H}$. Specifically, we prove that a well-bounded operator T on $\mathcal{H}$ can be decomposed as a sum of a self-adjoint operator A and a quasinilpotent operator Q such that $AQ - QA$ is quasinilpotent. In the decomposition process we super-diagonalise along the direct sum of local spectral subspaces of T, and then show that the remainder is a quasinilpotent operator. Before doing this we need the following important lemma.

Lemma 3.1.1. *Let T have an AC-functional calculus on $[a, b]$ such that*

$$\| f(T) \| \leq K \| f \|_{AC([a,b])} \text{ for all } f \in AC([a, b])$$

and suppose that

$$\sigma(T) \subset [a_1, b_1] \subset [a, b].$$

(1) *If* $f \in AC([a,b])$ *and* $f \equiv 0$ *on* $[a_1, b_1]$, *then* $f(T) = 0$.

(2) *The operator* T *has an* AC*-functional calculus on* $[a_1, b_1]$ *such that*

$$\| f(T) \| \leq K \| f \|_{AC([a_1,b_1])} \text{ for all } f \in AC([a_1, b_1]).$$

Proof. (1) Let $f \in AC([a,b])$ be such that $f \equiv 0$ on $[a_1, b_1]$, and fix $x \in \mathcal{H}$ and $\epsilon > 0$. Let f_ϵ be a continuous function on $[a,b]$ defined as follows:

$$f_\epsilon = \begin{cases} 0 & \text{on } [a_1 - \frac{\epsilon}{2}, b_1 + \frac{\epsilon}{2}], \\ \text{linear} & \text{on } [a_1 - \epsilon, a_1 - \frac{\epsilon}{2}] \cup [b_1 + \frac{\epsilon}{2}, b_1 + \epsilon], \\ f & \text{on } [a, a_1 - \epsilon] \cup [b_1 + \epsilon, b]. \end{cases}$$

Since

$$\sigma_T(x) \subset \sigma(T) \subset [a_1, b_1] \text{ and } [a_1, b_1] \cap \operatorname{supp}(f_\epsilon) = \emptyset,$$

we have

$$\sigma_T(x) \cap \operatorname{supp}(f_\epsilon) = \emptyset.$$

Hence, by Theorem 1.2.4, $f_\epsilon(T)x = 0$. Since $x \in \mathcal{H}$ was arbitrary, $f_\epsilon(T) = 0$. Thus, in order to prove that $f(T) = 0$ we only need to show that $f_\epsilon(T) \to f(T)$ in the AC-norm as $\epsilon \to 0$. We have

$$\begin{aligned} \| f - f_\epsilon \|_{AC([a,b])} &= \operatorname{var}_{[a_1 - \frac{\epsilon}{2}, a_1]} f + \operatorname{var}_{[b_1, b_1 + \frac{\epsilon}{2}]} f \\ &+ \operatorname{var}_{[a_1 - \epsilon, a_1 - \frac{\epsilon}{2}]}(f - f_\epsilon) + \operatorname{var}_{[b_1 + \frac{\epsilon}{2}, b_1 + \epsilon]}(f - f_\epsilon) \\ &\leq \operatorname{var}_{[a_1 - \frac{\epsilon}{2}, a_1]} f + \operatorname{var}_{[b_1, b_1 + \frac{\epsilon}{2}]} f \\ &+ \operatorname{var}_{[a_1 - \epsilon, a_1 - \frac{\epsilon}{2}]} f + \operatorname{var}_{[a_1 - \epsilon, a_1 - \frac{\epsilon}{2}]} f_\epsilon + \operatorname{var}_{[b_1 + \frac{\epsilon}{2}, b_1 + \epsilon]} f + \operatorname{var}_{[b_1 + \frac{\epsilon}{2}, b_1 + \epsilon]} f_\epsilon \\ &= \int_{a_1 - \frac{\epsilon}{2}}^{a_1} |f'| + \int_{b_1}^{b_1 + \frac{\epsilon}{2}} |f'| + \int_{a_1 - \epsilon}^{a_1 - \frac{\epsilon}{2}} |f'| + |f_\epsilon(a_1 - \epsilon)| \\ &+ \int_{b_1 + \frac{\epsilon}{2}}^{b_1 + \epsilon} |f'| + |f_\epsilon(b_1 + \epsilon)| \\ &\leq 2\epsilon \sup_{[a_1 - \epsilon, a_1] \cup [b_1, b_1 + \epsilon]} |f'| + |f(a_1 - \epsilon)| + |f(b_1 + \epsilon)|. \end{aligned}$$

So $f - f_\epsilon \in AC([a,b])$ and hence, using the functional calculus of T, we have

$$\begin{aligned} \| (f - f_\epsilon)(T) \| &\leq K \| f - f_\epsilon \|_{AC([a,b])} \\ &\leq K(2\epsilon \sup_{[a_1 - \epsilon, a_1] \cup [b_1, b_1 + \epsilon]} |f'| + |f(a_1 - \epsilon)| + |f(b_1 + \epsilon)|). \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, $f_\epsilon(T) \to f(T)$ in the AC-norm as $\epsilon \to 0$. Thus, $f(T) = 0$.

(2) Let $f \in AC([a_1, b_1])$ be a polynomial. Extend f continuously to $\tilde{f}$ as follows:

$$\tilde{f} = \begin{cases} f(a_1) & \text{on } [a, a_1], \\ f & \text{on } [a_1, b_1], \\ f(b_1) & \text{on } [b_1, b]. \end{cases}$$

Then $\tilde{f} - f \equiv 0$ on $[a_1, b_1]$. So by the first part of this lemma, $\tilde{f}(T) = f(T)$. Since $\tilde{f} \in AC([a,b])$, $\| \tilde{f}(T) \| \leq K \| \tilde{f} \|_{AC([a,b])}$. But

$$\| \tilde{f} \|_{AC([a,b])} = \| f \|_{AC([a_1,b_1])} .$$

So we have

$$\| f(T) \| = \| \tilde{f}(T) \| \leq K \| \tilde{f} \|_{AC([a,b])} = K \| f \|_{AC([a_1,b_1])} .$$

Therefore, T has the required AC-functional calculus on $[a_1, b_1]$. □

Now, we show that every well-bounded operator T on $\mathcal{H}$ can be expressed as a sum of a self-adjoint operator A and a quasinilpotent operator Q such that $AQ - QA$ is quasinilpotent.

Theorem 3.1.1. *Let T be a well-bounded operator on a Hilbert space $\mathcal{H}$ implemented by (K, J), where $J = [a, b]$, i.e.,*

$$\| f(T) \| \leq K \| f \|_{AC([a,b])} \ \textit{for all}\ f \in AC([a,b]).$$

(i) For each $\lambda \in \mathbb{R}$, let $X_\lambda = \{x \in \mathcal{H} : \sigma_T(x) \subset [a, \lambda]\}$, and let $E(\lambda)$ be the orthogonal projection of $\mathcal{H}$ onto X_λ. Then $E(\cdot)$ is a spectral family concentrated on $[a, b]$ and gives a spectral measure.
(ii) Define $A = \int_{[a,b]}^{\oplus} \lambda dE(\lambda)$ and let $Q = T - A$. Then A is self-adjoint, Q is quasinilpotent and $AQ - QA$ is also quasinilpotent.

Proof. First, we want to show that $\sigma(T) \subset [a, b]$. Suppose that $\lambda \notin [a, b]$, and let $f_\lambda(t) = \frac{1}{\lambda - t}$, $t \in [a, b]$. Then

$$\begin{aligned} \| f_\lambda \|_{AC(J)} &= |f_\lambda(b)| + \int_a^b |f'_\lambda(t)| dt \\ &= \frac{1}{|\lambda - b|} + \int_a^b \frac{1}{(\lambda - t)^2} dt \\ &= \frac{1}{|\lambda - b|} + \frac{1}{\lambda - b} - \frac{1}{\lambda - a} < \infty. \end{aligned}$$

Thus, $f_\lambda \in AC([a,b])$, and hence $\| f_\lambda(T) \| \leq K \| f_\lambda \|_{AC(J)} < \infty$. Therefore, $f_\lambda(T) \in \mathcal{B}(\mathcal{H})$. Since

$$(\lambda - t).f_\lambda(t) = 1 = f_\lambda(t).(\lambda - t),$$

we have

$$(\lambda - T).f_\lambda(T) = I = f_\lambda(T).(\lambda - T).$$

Thus, $\lambda \in \rho(T)$, i.e., $\lambda \notin \sigma(T)$. Therefore, $\sigma(T) \subset [a, b]$.

Next, we want to show that $E(\cdot)$ is a spectral family concentrated on $[a, b]$. We have $E(\lambda)$ is an orthogonal projection for each $\lambda \in \mathbb{R}$ and satisfies the following:

(1) $\sup_{\lambda \in \mathbb{R}} \parallel E(\lambda) \parallel \leq 1$.

(2) $X_\lambda = \{x \in \mathcal{H} : \sigma_T(x) \subset [a, \lambda]\} = X_T([a, \lambda])$ is a local spectral subspace of T which is a T-hyperinvariant closed linear subspace of $\mathcal{H}$ according to Remark 1.2.1.
If $\lambda_1 \leq \lambda_2$, then $[a, \lambda_1] \subset [a, \lambda_2]$, so by Theorem 1.2.1(i), $X_T([a, \lambda_1]) \subset X_T([a, \lambda_2])$, i.e., $X_{\lambda_1} \subset X_{\lambda_2}$. Hence, $E(\lambda_1)E(\lambda_2) = E(\lambda_2)E(\lambda_1) = E(\lambda_1)$.

(3) If $\lambda < a$, $X_\lambda = \{x \in \mathcal{H} : \sigma_T(x) \subset [a, \lambda] = \emptyset\} = \{0\}$ by Theorem 1.2.1(iii), so $E(\lambda) = 0$. If $\lambda \geq b$, then $X_\lambda = \mathcal{H}$, so $E(\lambda) = I$.

(4) If $\{\lambda_n\}$ is a decreasing sequence converging to $\lambda \in \mathbb{R}$, then $\{E(\lambda_n)\}$ is a decreasing sequence of orthogonal projections on $\mathcal{H}$, so $\{E(\lambda_n)\}$ is strongly operator convergent to the orthogonal projection onto

$$\begin{aligned} \cap_{n=1}^{\infty} E(\lambda_n)(\mathcal{H}) &= \cap_{n=1}^{\infty} X_{\lambda_n} \\ &= \cap_{n=1}^{\infty} X_T([a, \lambda_n]) \\ &= X_T(\cap_{n=1}^{\infty} [a, \lambda_n]) \\ &= X_T([a, \lambda]) \\ &= X_\lambda. \end{aligned}$$

Hence, $\lim_{\lambda_n \to \lambda^+} E(\lambda_n)x = E(\lambda)x$ for all $x \in \mathcal{H}$.

(5) If $\{\lambda_n\}$ is an increasing sequence converging to λ, then $\{E(\lambda_n)\}$ is an increasing sequence of orthogonal projections on $\mathcal{H}$, so $\{E(\lambda_n)\}$ is strongly operator convergent to the orthogonal projection of $\mathcal{H}$ onto $\overline{\cup_{n=1}^{\infty} E(\lambda_n)\mathcal{H}} = \overline{\cup_{n=1}^{\infty} X_{\lambda_n}}$. Hence, $\lim_{\lambda_n \to \lambda^-} E(\lambda_n)x$ exists for all $x \in \mathcal{H}$.

Therefore, $E(\cdot)$ is a spectral family on $\mathcal{H}$ concentrated on $[a, b]$.

Now, assume that $X_a = \{0\}$ and bisect $[a,b]$ into 2 subintervals $[a,\lambda_1^1]$ and $[\lambda_1^1, b]$ and let $X_1^1 = (E(\lambda_1^1) - E(a))\mathcal{H}$ and $X_2^1 = (E(b) - E(\lambda_1^1))\mathcal{H}$. Then

$$\mathcal{H} = X_{\lambda_1^1} \oplus X_{\lambda_1^1}^{\perp} = X_1^1 \oplus X_2^1.$$

Since $X_1^1 = X_{\lambda_1^1} = X_T([a,\lambda_1^1])$ is T-invariant, T can be represented by the matrix

$$T = \begin{bmatrix} T_{11}^1 & T_{12}^1 \\ 0 & T_{22}^1 \end{bmatrix}$$

relative to the above decomposition of $\mathcal{H}$.

Since T is an $AC(J)$-scalar operator, by Remark 1.2.1 we know that T has the SVEP and $F = [a,\lambda_1^1] \subset \mathbb{C}$ is a closed set for which the space $X_1^1 = X_{\lambda_1^1} = X_T([a,\lambda_1^1])$ is closed. Thus by Theorem 1.2.2, $\sigma(T|X_1^1) \subset F \cap \sigma(T) \subset [a,\lambda_1^1]$. But $T|X_1^1 = T_{11}^1$.
Hence

$$\sigma(T_{11}^1) \subset [a,\lambda_1^1].$$

By Theorem 1.2.7, there exists an invertible isometry $U : \mathcal{H}/X_{\lambda_1^1} \to X_{\lambda_1^1}^{\perp}$ defined by $U^{-1}x = x + X_{\lambda_1^1}$ for all $x \in X_{\lambda_1^1}^{\perp}$.
Let $T/X_1^1 = S : \mathcal{H}/X_1^1 \to \mathcal{H}/X_1^1$ be defined by $S(x + X_1^1) = Tx + X_1^1$. We want to prove that $U^{-1}T_{22}^1U = S = T/X_1^1$.
Let $x \in X_2^1$, then $U^{-1}x = x + X_1^1$, so we have

$$\begin{aligned} SU^{-1}x &= S(x + X_1^1) \\ &= Tx + X_1^1 \\ &= \begin{bmatrix} T_{11}^1 & T_{12}^1 \\ 0 & T_{22}^1 \end{bmatrix} \begin{bmatrix} 0 \\ x \end{bmatrix} + X_1^1 \\ &= \begin{bmatrix} T_{12}^1 x \\ T_{22}^1 x \end{bmatrix} + X_1^1 \\ &= T_{22}^1 x + X_1^1 \\ &= U^{-1}T_{22}^1 x. \end{aligned}$$

Hence, $U^{-1}T_{22}^1U = T/X_1^1$, i.e., $T_{22}^1 \cong T/X_1^1$.

Since T is an $AC(J)$-scalar operator, by Remark 1.2.1 the operator T is decomposable and $[a,\lambda_1^1] \subset \mathbb{C}$ is closed, so by Theorem 1.2.6 the induced operator T/X_1^1 on the quotient space $\mathcal{H}/X_1^1$ satisfies

$$\sigma(T/X_1^1) \subset \overline{\sigma(T)\backslash[a,\lambda_1^1]}, \text{ i.e., } \sigma(T/X_1^1) \subset [\lambda_1^1, b].$$

But

$$T/X_1^1 \cong T_{22}^1, \text{ so } \sigma(T_{22}^1) \subset [\lambda_1^1, b].$$

Therefore, we have

$$\sigma(T_{11}^1) \subset [\lambda_0^1, \lambda_1^1] \text{ and } \sigma(T_{22}^1) \subset [\lambda_1^1, \lambda_2^1]$$

$$\text{i.e., } \sigma(T_{jj}^1) \subset [\lambda_{j-1}^1, \lambda_j^1] \text{ for } j = 1, 2,$$

where $\lambda_0^1 = a$ and $\lambda_2^1 = b$.

Then, continuing this bisection method, we get at the n^{th} step
$\lambda_j^n = a + j\frac{b-a}{2^n}$ for $j = 0, 1, 2, \cdots, 2^n$; $X_j^n = (E(\lambda_j^n) - E(\lambda_{j-1}^n))\mathcal{H}$;
$\mathcal{H}$ has the decomposition $\mathcal{H} = X_1^n \oplus X_2^n \oplus \cdots\cdots \oplus X_{2^n}^n$ and
$E(\lambda_j^n)\mathcal{H} = X_1^n \oplus X_2^n \oplus \cdots\cdots \oplus X_j^n$ is T-hyperinvariant for $j = 1, 2, \cdots, 2^n$.
So T can be represented by the matrix

$$T = \begin{bmatrix} T_{11}^n & T_{12}^n & \cdots & \cdots & T_{12^n}^n \\ 0 & T_{22}^n & \cdots & \cdots & T_{22^n}^n \\ 0 & 0 & T_{33}^n & \cdots & T_{32^n}^n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & T_{2^n 2^n}^n \end{bmatrix}$$

relative to this decomposition of $\mathcal{H}$. Fix $j \in \{1, 2, \cdots\cdots, 2^n\}$ and let

$$C_j^n = \begin{bmatrix} T_{11}^n & T_{12}^n & \cdots & \cdots & T_{1j}^n \\ 0 & T_{22}^n & \cdots & \cdots & T_{2j}^n \\ 0 & 0 & T_{33}^n & \cdots & T_{3j}^n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & T_{jj}^n \end{bmatrix}.$$

Since $X_{\lambda_j^n} = E(\lambda_j^n)\mathcal{H} = X_1^n \oplus X_2^n \oplus \cdots \oplus X_j^n = X_T([a, \lambda_j^n])$ is T-hyperinvariant, $\mathcal{H}$ can be decomposed as

$$\mathcal{H} = X_{\lambda_j^n} \oplus X_{\lambda_j^n}^{\perp}$$

and T can be represented by the matrix

$$T = \begin{bmatrix} C_j^n & * \\ 0 & * \end{bmatrix}$$

relative to this decomposition of $\mathcal{H}$, where * is an operator entry. Thus

$$C_j^n = T|X_{\lambda_j^n} = T|(X_1^n \oplus X_2^n \oplus \cdots \oplus X_j^n) = T|X_T([a, \lambda_j^n]).$$

By Theorem 1.2.5, C_j^n is an AC(J)-scalar operator. So by Remark 1.2.1, C_j^n has the SVEP and $X_T([a, \lambda_j^n])$ is closed. Thus by Theorem 1.2.2,

$$\sigma(C_j^n) = \sigma(T|X_T([a, \lambda_j^n])) \subset [a, \lambda_j^n] \cap \sigma(T) \subset [a, \lambda_j^n].$$

We have

$$X_{\lambda_j^n} = X_1^n \oplus X_2^n \oplus \cdots \oplus X_{j-1}^n \oplus X_j^n \text{ and } X_{\lambda_{j-1}^n} = X_1^n \oplus X_2^n \oplus \cdots \oplus X_{j-1}^n.$$

So $X_{\lambda_j^n}$ can be decomposed as

$$X_{\lambda_j^n} = X_{\lambda_{j-1}^n} \oplus X_j^n$$

and relative to this decomposition of $X_{\lambda_j^n}$, the operator C_j^n can be represented by the matrix

$$C_j^n = \begin{bmatrix} C_{j-1}^n & * \\ 0 & T_{jj}^n \end{bmatrix}.$$

Thus we have $T_{jj}^n \cong C_j^n / X_{\lambda_{j-1}^n}$ on the quotient space $X_{\lambda_j^n}/X_{\lambda_{j-1}^n}$. By Theorem 1.2.5, $C_j^n/X_{\lambda_{j-1}^n}$ is an AC(J)-scalar operator, so by Remark 1.2.1, it is decomposable and hence by Theorem 1.2.6,

$$\sigma(T_{jj}^n) = \sigma(C_j^n/X_{\lambda_{j-1}^n}) \subset \overline{\sigma(C_j^n)\backslash[a, \lambda_{j-1}^n]} \subset [\lambda_{j-1}^n, \lambda_j^n].$$

Now, since $E(\lambda)$ is self-adjoint for each $\lambda \in [a, b]$,

$$A = \int_{[a,b]}^{\oplus} \lambda dE(\lambda) = \begin{bmatrix} A_{11}^n & 0 & \cdots & \cdots & 0 \\ 0 & A_{22}^n & 0 & \cdots & 0 \\ 0 & 0 & A_{33}^n & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & A_{2^n 2^n}^n \end{bmatrix}$$

is self-adjoint because it is the strong limit of self-adjoint operators.

So we have

$$Q = T - A = \begin{bmatrix} T_{11}^n - A_{11}^n & T_{12}^n & \cdots & \cdots & T_{12^n}^n \\ 0 & T_{22}^n - A_{22}^n & \cdots & \cdots & T_{22^n}^n \\ 0 & 0 & T_{33}^n - A_{33}^n & \cdots & T_{32^n}^n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & T_{2^n 2^n}^n - A_{2^n 2^n}^n \end{bmatrix},$$

$$\sigma(Q) \subset \sigma(T_{11}^n - A_{11}^n) \cup \cdots\cdots \cup \sigma(T_{2^n 2^n}^n - A_{2^n 2^n}^n),$$

$$\sigma(T_{jj}^n) \subset [\lambda_{j-1}^n, \lambda_j^n] \text{ and } \sigma(A_{jj}^n) \subset [\lambda_{j-1}^n, \lambda_j^n] \text{ for } j = 1, \cdots, 2^n.$$

Now, let $\epsilon > 0$ be given. Choose $\beta > 1$ such that $\frac{K}{\beta-1}(\frac{\beta}{\beta-1}) < 1$. Here K is the constant defined for the functional calculus of T as in the statement of the theorem. Choose a partition $\mathcal{P}$ of $[a, b]$ such that

$$\mathcal{P} = \{a, a + \frac{b-a}{2^N}, a + 2\frac{b-a}{2^N}, \cdots\cdots, b\}$$

and such that

$$\frac{b-a}{2^N} < \min\{\frac{\epsilon}{\beta}, \frac{\epsilon}{2 \parallel T \parallel}\}.$$

Since

$$\sigma(Q) \subset \sigma(T_{11}^N - A_{11}^N) \cup \cdots\cdots \cup \sigma(T_{2^N 2^N}^N - A_{2^N 2^N}^N),$$

it is enough to show that

$$\sigma(T_{jj}^N - A_{jj}^N) \subset \{\lambda : |\lambda| \leq \epsilon\} \text{ for } j = 1, 2, \cdots, 2^N$$

to prove that Q is quasinilpotent.

Fix $j \in \{1, 2, \cdots, 2^N\}$ and suppose that $|\lambda| > \epsilon > \beta\frac{b-a}{2^N}$. We want to show that $\lambda - (T_{jj}^N - A_{jj}^N)$ is invertible. We have

$$\begin{aligned}
\lambda - (T_{jj}^N - A_{jj}^N) &= \lambda - ((T_{jj}^N - \lambda_j^N I_{jj}^N) + (\lambda_j^N I_{jj}^N - A_{jj}^N)) \\
&= (\lambda - (T_{jj}^N - \lambda_j^N I_{jj}^N)) - (\lambda_j^N I_{jj}^N - A_{jj}^N) \\
&= (\lambda - (T_{jj}^N - \lambda_j^N I_{jj}^N))(I_{jj}^N - (\lambda - (T_{jj}^N - \lambda_j^N I_{jj}^N))^{-1}(\lambda_j^N I_{jj}^N - A_{jj}^N)).
\end{aligned}$$

We know that $\lambda - (T_{jj}^N - \lambda_j^N I_{jj}^N)$ is invertible because

$$\sigma(T_{jj}^N - \lambda_j^N I_{jj}^N) \subset [\lambda_{j-1}^N - \lambda_j^N, \lambda_j^N - \lambda_j^N] = [-\frac{b-a}{2^N}, 0].$$

Thus we only need to prove that

$$\| (\lambda - (T_{jj}^N - \lambda_j^N I_{jj}^N))^{-1}(\lambda_j^N I_{jj}^N - A_{jj}^N) \| < 1,$$

to complete the proof that Q is quasinilpotent.

Since $T_{jj}^N - \lambda_j^N I_{jj}^N$ has an AC-functional calculus on $[a, b] - \lambda_j^N$ and

$$\sigma(T_{jj}^N - \lambda_j^N I_{jj}^N) \subset [-\frac{b-a}{2^N}, 0],$$

by Lemma 3.1.1 we know that $T_{jj}^N - \lambda_j^N I_{jj}^N$ has an AC-functional calculus on $[-\frac{b-a}{2^N}, 0]$. Thus we have

$$\| f(T_{jj}^N - \lambda_j^N I_{jj}^N) \| \leq K \| f \|_{AC([-\frac{b-a}{2^N}, 0])} \text{ for all } f \in AC([-\frac{b-a}{2^N}, 0]).$$

Since $f_0(s) = (\lambda - s)^{-1}$ is in $AC([-\frac{b-a}{2^N}, 0])$, we have

$$\| f_0(T_{jj}^N - \lambda_j^N I_{jj}^N) \| \leq K \| f_0 \|_{AC([-\frac{b-a}{2^N}, 0])} .$$

Hence,

$$\| (\lambda - (T_{jj}^N - \lambda_j^N I_{jj}^N))^{-1} \| \leq K \| (\lambda - s)^{-1} \|_{AC([-\frac{b-a}{2^N}, 0])} .$$

But

$$\| (\lambda - s)^{-1} \|_{AC([-\frac{b-a}{2^N},0])} \leq \int_{-\frac{b-a}{2^N}}^{0} \frac{1}{|\lambda - s|^2} ds + \sup_{s\in[-\frac{b-a}{2^N},0]} \frac{1}{|\lambda - s|}$$
$$\leq \frac{(2^N)^2}{(\beta-1)^2(b-a)^2}\frac{b-a}{2^N} + \frac{2^N}{(\beta-1)(b-a)}$$
$$= \frac{2^N}{(\beta-1)(b-a)}(\frac{\beta}{\beta-1}).$$

Thus

$$\| (\lambda - (T_{jj}^N - \lambda_j^N I_{jj}^N))^{-1} \| \leq K\frac{2^N}{(\beta-1)(b-a)}(\frac{\beta}{\beta-1}).$$

Since $A_{jj}^N - \lambda_j^N I_{jj}^N$ is self-adjoint and

$$\sigma(A_{jj}^N - \lambda_j^N I_{jj}^N) \subset [\lambda_{j-1}^N - \lambda_j^N, \lambda_j^N - \lambda_j^N] = [-\frac{b-a}{2^N}, 0],$$

we have $\| A_{jj}^N - \lambda_j^N I_{jj}^N \| = r(A_{jj}^N - \lambda_j^N I_{jj}^N) \leq \frac{b-a}{2^N}$. Therefore, we have

$$\| (\lambda - (T_{jj}^N - \lambda_j^N I_{jj}^N))^{-1}(\lambda_j^N I_{jj}^N - A_{jj}^N) \| \leq \| (\lambda - (T_{jj}^N - \lambda_j^N I_{jj}^N))^{-1} \| \| \lambda_j^N I_{jj}^N - A_{jj}^N \|$$
$$\leq K\frac{2^N}{(\beta-1)(b-a)}(\frac{1}{\beta-1}+1)\frac{b-a}{2^N}$$
$$= \frac{K}{\beta-1}(\frac{\beta}{\beta-1}) < 1.$$

Hence, $\lambda - (T_{jj}^N - A_{jj}^N)$ is invertible for $|\lambda| > \epsilon$, i.e.,

$$\sigma(T_{jj}^N - A_{jj}^N) \subset \{\lambda : |\lambda| \leq \epsilon\} \text{ for } j = 1, 2, \cdots, 2^N.$$

Therefore, $\sigma(Q) = \{0\}$, i.e., Q is quasinilpotent.

Finally, we want to prove that $AQ - QA$ is quasinilpotent. We have

$$AQ - QA = AT - TA$$
$$= \begin{bmatrix} A_{11}^N T_{11}^N - T_{11}^N A_{11}^N & A_{11}^N T_{12}^N - T_{12}^N A_{22}^N & \cdots & \cdots & A_{11}^N T_{12^N}^N - T_{12^N}^N A_{2^N 2^N}^N \\ 0 & A_{22}^N T_{22}^N - T_{22}^N A_{22}^N & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & A_{2^N 2^N}^N T_{2^N 2^N}^N - T_{2^N 2^N}^N A_{2^N 2^N}^N \end{bmatrix}.$$

Thus,

$$\sigma(AQ - QA) \subset \cup_{j=1}^{2^N} \sigma(A_{jj}^N T_{jj}^N - T_{jj}^N A_{jj}^N).$$

Fix $j \in \{1, 2, \cdots, 2^N\}$. We have

$$\| A_{jj}^N T_{jj}^N - T_{jj}^N A_{jj}^N \| = \| (A_{jj}^N - \lambda_j^N) T_{jj}^N + T_{jj}^N (\lambda_j^N - A_{jj}^N) \|$$
$$\leq \| A_{jj}^N - \lambda_j^N I_{jj}^N \| \| T_{jj}^N \| + \| T_{jj}^N \| \| \lambda_j^N I_{jj}^N - A_{jj}^N \|$$
$$= 2 \| T_{jj}^N \| \| A_{jj}^N - \lambda_j^N I_{jj}^N \| .$$

Since T_{jj}^N is either a restriction of T to a closed invariant subspace of $\mathcal{H}$ or a compression of T, we have

$$\| T_{jj}^N \| \leq \| T \| .$$

Hence,

$$\begin{aligned} r(A_{jj}^N T_{jj}^N - T_{jj}^N A_{jj}^N) &\leq \| A_{jj}^N T_{jj}^N - T_{jj}^N A_{jj}^N \| \\ &\leq 2 \| T_{jj}^N \| \| A_{jj}^N - \lambda_j^N I_{jj}^N \| \\ &\leq 2 \| T \| \frac{b-a}{2^N} \\ &< 2 \| T \| \frac{\epsilon}{2 \| T \|} \\ &= \epsilon. \end{aligned}$$

Therefore, $AQ - QA$ is quasinilpotent.

If $X_a = E(a)\mathcal{H} \neq \{0\}$, $E(a)\mathcal{H} = \ker(T - aI)$ and

$$\mathcal{H} = E(a)\mathcal{H} \oplus (E(\lambda_1^n) - E(a))\mathcal{H} \oplus (E(\lambda_2^n) - E(\lambda_1^n))\mathcal{H} \oplus \cdots \oplus (E(\lambda_{2^n}^n) - E(\lambda_{2^n-1}^n))\mathcal{H}$$

and relative to this decomposition of $\mathcal{H}$, the operators T and A are represented by the matrices

$$T = \begin{bmatrix} a & * & \cdots & \cdots & \cdots & * \\ 0 & T_{11}^n & T_{12}^n & \cdots & \cdots & T_{12^n}^n \\ 0 & 0 & T_{22}^n & \cdots & \cdots & T_{22^n}^n \\ 0 & 0 & 0 & T_{33}^n & \cdots & T_{32^n}^n \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & T_{2^n 2^n}^n \end{bmatrix}, A = \begin{bmatrix} a & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & A_{11}^n & 0 & \cdots & \cdots & 0 \\ 0 & 0 & A_{22}^n & 0 & \cdots & 0 \\ 0 & 0 & 0 & A_{33}^n & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & A_{2^n 2^n}^n \end{bmatrix}.$$

Thus,

$$Q = T - A = \begin{bmatrix} 0 & * & \cdots & \cdots & \cdots & * \\ 0 & T_{11}^n - A_{11}^n & T_{12}^n & \cdots & \cdots & T_{12^n}^n \\ 0 & 0 & T_{22}^n - A_{22}^n & \cdots & \cdots & T_{22^n}^n \\ 0 & 0 & 0 & T_{33}^n - A_{33}^n & \cdots & T_{32^n}^n \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & T_{2^n 2^n}^n - A_{2^n 2^n}^n \end{bmatrix}$$

is quasinilpotent according to the first part of this proof. Similarly, we can show that $AQ - QA$ is quasinilpotent. This completes the proof. □

The following question arises. Suppose that A is self-adjoint, Q is quasinilpotent and $AQ - QA$ is quasinilpotent. What extra conditions imply that $A + Q$ is well-bounded? We shall return to discuss the difficulties involved in section 3 of this chapter.

3.2 Trigonometrically Well-Bounded Operators on $\mathcal{H}$

In this section we prove that any non-unitary trigonometrically well-bounded operator T on a Hilbert space $\mathcal{H}$ can be decomposed as $T = U(Q_1+I)$ and as $T = (Q_2+I)U$ where U is a unitary operator, Q_1 and Q_2 are non-equal quasinilpotent operators and for each $j = 1, 2$, we have $UQ_j - Q_jU$ is also quasinilpotent. In order to prove this, we utilise the following lemma.

Lemma 3.2.1. *Let T have an AC-functional calculus on $\mathbb{T}$ such that*

$$\| f(T) \| \leq K \| f \|_{AC(\mathbb{T})} \text{ for all } f \in AC(\mathbb{T})$$

and suppose that

$$\sigma(T) \subset \Gamma \subset \mathbb{T}, \text{ where } \Gamma = \{e^{it} : 0 \leq a \leq t \leq b \leq 2\pi\}.$$

(1) *If $f \in AC(\mathbb{T})$ and $f \equiv 0$ on Γ, then $f(T) = 0$.*
(2) *The operator T has an AC-functional calculus on Γ such that*

$$\| f(T) \| \leq 2K \| f \|_{AC(\Gamma)} \text{ for all } f \in AC(\Gamma),$$

where $AC(\Gamma)$ is the Banach algebra of absolutely continuous complex valued functions on Γ with the norm $\| f \|_{AC(\Gamma)} = |f(e^{ib})| + \text{var}_\Gamma f$.

Proof. The proof of the first part is almost the same as that of Lemma 3.1.1. So we only need to prove that T has an AC-functional calculus on Γ such that

$$\| f(T) \| \leq 2K \| f \|_{AC(\Gamma)} \text{ for all } f \in AC(\Gamma).$$

Let $f \in AC(\Gamma)$ be a trigonometric polynomial. Extend f continuously to $\tilde{f}$ on $\mathbb{T}$ as follows:

$$\tilde{f}(e^{it}) = \begin{cases} f(e^{it}) & \text{if } a \leq t \leq b, \\ \text{linear} & \text{if } 0 \leq t \leq a, \\ f(e^{ib}) & \text{if } b \leq t \leq 2\pi. \end{cases}$$

Then we have $\tilde{f} \in AC(\mathbb{T})$ and $\text{var}_\mathbb{T}\tilde{f} = \text{var}_\Gamma f + |f(e^{ia}) - f(e^{ib})|$. Also we have $\tilde{f} - f \equiv 0$ on Γ. So by the first part of this lemma, $\tilde{f}(T) = f(T)$. Thus we have

$$\begin{aligned} \| f(T) \| = \| \tilde{f}(T) \| &\leq K \| \tilde{f} \|_{AC(\mathbb{T})} \\ &= K\{|\tilde{f}(e^{i(2\pi)})| + \text{var}_\mathbb{T}\tilde{f}\} \\ &= K\{|f(e^{ib})| + \text{var}_\Gamma f + |f(e^{ia}) - f(e^{ib})|\} \\ &\leq K\{|f(e^{ib})| + \text{var}_\Gamma f + \text{var}_\Gamma f\} \\ &\leq 2K \| f \|_{AC(\Gamma)} . \end{aligned}$$

Hence, T has the required AC-functional calculus on Γ. ☐

Now, we prove that any non-unitary trigonometrically well-bounded operator T on $\mathcal{H}$ can be decomposed as $T = U(Q_1 + I)$ and as $T = (Q_2 + I)U$ where U is a unitary operator, Q_1 and Q_2 are non-equal quasinilpotent operators and for each $j = 1, 2$, we have $UQ_j - Q_jU$ is also quasinilpotent.

Theorem 3.2.1. *Let T be a non-unitary trigonometrically well-bounded operator on a Hilbert space $\mathcal{H}$ with an AC-functional calculus on $\mathbb{T}$ satisfying*

$$\| f(T) \| \leq K \| f \|_{AC(\mathbb{T})} \text{ for all } f \in AC(\mathbb{T}).$$

(i) For each $\lambda \in [0, 2\pi]$, let $\Gamma_\lambda = \{e^{it} : 0 \leq t \leq \lambda\}$, $X_\lambda = \{x \in \mathcal{H} : \sigma_T(x) \subseteq \Gamma_\lambda\}$ and let $F(\lambda)$ be the orthogonal projection of $\mathcal{H}$ onto X_λ. Then $F(\cdot)$ is a spectral family concentrated on $[0, 2\pi]$ and gives rise to a self-adjoint spectral measure on the Borel subsets of the unit circle $\mathbb{T}$.
(ii) Define $U = \int_{[0,2\pi]}^{\oplus} e^{i\lambda} dF(\lambda)$ and let $Q_1 = U^{-1}T - I$ and $Q_2 = TU^{-1} - I$. Then U is a unitary operator and Q_1, Q_2 are quasinilpotent operators.
Moreover, $UQ_i - Q_iU$ is quasinilpotent for $i = 1, 2$, and $Q_1 \neq Q_2$.

Proof. As in the proof of Theorem 3.1.1, there is no loss of generality in assuming that $X_0 = \{0\}$, so that $F(0) = 0$. Bisect $[0, 2\pi]$ into 2 subintervals $[0, \lambda_1^1]$ and $[\lambda_1^1, 2\pi]$ and let $X_1^1 = (F(\lambda_1^1) - F(0))\mathcal{H}$ and $X_2^1 = (F(2\pi) - F(\lambda_1^1))\mathcal{H}$. Then

$$\mathcal{H} = X_{\lambda_1^1} \oplus X_{\lambda_1^1}^{\perp} = X_1^1 \oplus X_2^1.$$

So T can be represented by the matrix

$$T = \begin{bmatrix} T_{11}^1 & T_{12}^1 \\ 0 & T_{22}^1 \end{bmatrix}.$$

relative to this decomposition of $\mathcal{H}$.

Since T is an AC($\mathbb{T}$)-scalar operator, by Remark 1.2.1 the operator T has the SVEP and $\Gamma_{\lambda_1^1} = \{e^{it} : 0 \leq t \leq \lambda_1^1\} \subset \mathbb{C}$ is a closed set for which the space $X_1^1 = X_{\lambda_1^1} = X_T(\Gamma_{\lambda_1^1})$ is closed. Thus by Theorem 1.2.2, $\sigma(T|X_1^1) \subset \Gamma_{\lambda_1^1} \cap \sigma(T) \subset \Gamma_{\lambda_1^1} \cap \mathbb{T} = \Gamma_{\lambda_1^1}$. But $T|X_1^1 = T_{11}^1$. Hence, $\sigma(T_{11}^1) \subset \Gamma_{\lambda_1^1}$.
Since T is an $AC(\mathbb{T})$-scalar operator, by Remark 1.2.1 the operator T is decomposable and $\Gamma_{\lambda_1^1} \subset \mathbb{C}$ is closed, and hence by Theorem 1.2.6, T/X_1^1 on the quotient space $\mathcal{H}/X_1^1$ satisfies

$$\sigma(T/X_1^1) \subset \overline{\sigma(T) \backslash \Gamma_{\lambda_1^1}} \subset \{e^{it} : \lambda_1^1 \leq t \leq 2\pi\}.$$

But $T/X_1^1 \cong T_{22}^1$, so

$$\sigma(T_{22}^1) \subset \{e^{it} : \lambda_1^1 \leq t \leq \lambda_2^1\}.$$

Therefore, we have

$$\sigma(T_{jj}^1) \subset \{e^{it} : \lambda_{j-1}^1 \leq t \leq \lambda_j^1\} \text{ for } j = 1, 2 \text{ where } \lambda_0^1 = 0 \text{ and } \lambda_2^1 = 2\pi.$$

Then, continuing this bisection method, we get at the n^{th} step
$\lambda_j^n = j\frac{2\pi}{2^n}$ for $j = 0, 1, 2, \cdots\cdots, 2^n$; $X_j^n = (F(\lambda_j^n) - F(\lambda_{j-1}^n))\mathcal{H}$;
$\mathcal{H}$ has the decomposition $\mathcal{H} = X_1^n \oplus X_2^n \oplus \cdots\cdots \oplus X_{2^n}^n$ and
$F(\lambda_j^n)\mathcal{H} = X_1^n \oplus X_2^n \oplus \cdots\cdots \oplus X_j^n$ is T-hyperinvariant for $j = 1, 2, \cdots, 2^n$.
So T can be represented by the matrix

$$T = \begin{bmatrix} T_{11}^n & T_{12}^n & \cdots & \cdots & T_{12^n}^n \\ 0 & T_{22}^n & \cdots & \cdots & T_{22^n}^n \\ 0 & 0 & T_{33}^n & \cdots & T_{32^n}^n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & T_{2^n2^n}^n \end{bmatrix}$$

relative to this decomposition of $\mathcal{H}$ and

$$\sigma(T_{jj}^n) \subset \{e^{it} : \lambda_{j-1}^n \leq t \leq \lambda_j^n\} \text{ for } j = 1, 2, \cdots, 2^n.$$

Define

$$V = U^{-1} = \int_0^{2\pi} e^{-i\lambda} dF(\lambda) = \begin{bmatrix} V_{11}^n & 0 & \cdots & \cdots & 0 \\ 0 & V_{22}^n & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & V_{2^n2^n}^n \end{bmatrix}.$$

Then we have

$$Q_1 = U^{-1}T - I = \begin{bmatrix} V_{11}^n T_{11}^n - I_{11}^n & V_{11}^n T_{12}^n & \cdots & \cdots & V_{11}^n T_{12^n}^n \\ 0 & V_{22}^n T_{22}^n - I_{22}^n & \cdots & \cdots & V_{22}^n T_{22^n}^n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & V_{2^n2^n}^n T_{2^n2^n}^n - I_{2^n2^n}^n \end{bmatrix}.$$

Hence

$$\sigma(Q_1) \subset \sigma(V_{11}^n T_{11}^n - I_{11}^n) \cup \cdots\cdots \cup \sigma(V_{2^n2^n}^n T_{2^n2^n}^n - I_{2^n2^n}^n).$$

Let $\epsilon > 0$ be given. Choose a partition $\mathcal{P}$ of $[0, 2\pi]$ such that for each $j \in \{1, 2, \cdots\cdots, 2^N\}$, $\lambda_j^N - \lambda_{j-1}^N = \frac{2\pi}{2^N} < \min\{\frac{\epsilon}{4}, \frac{\epsilon}{2\|T\|}\}$ and $\text{length}(\Gamma_j^N) < \frac{\epsilon}{4K}$, where $\Gamma_j^N = \{e^{it} : \lambda_{j-1}^N \leq t \leq \lambda_j^N\}$.
Since

$$\sigma(Q_1) \subset \sigma(V_{11}^N T_{11}^N - I_{11}^N) \cup \cdots\cdots \cup \sigma(V_{2^N2^N}^N T_{2^N2^N}^N - I_{2^N2^N}^N),$$

it is enough to prove that

$$\sigma(V_{jj}^N T_{jj}^N - I_{jj}^N) \subset \{\lambda : |\lambda| \leq \epsilon\} \text{ for all } j \in \{1, 2, \cdots, 2^N\}$$

to complete the proof that Q_1 is quasinilpotent.

Fix $j \in \{1, 2, \cdots, 2^N\}$. We have

$$U_{jj}^N = \int_{\lambda_{j-1}^N}^{\lambda_j^N} e^{i\lambda} dF(\lambda), \quad \sigma(U_{jj}^N) \subset \{e^{it} : \lambda_{j-1}^N \leq t \leq \lambda_j^N\}$$

and

$$\sigma(T_{jj}^N) \subset \{e^{it} : \lambda_{j-1}^N \leq t \leq \lambda_j^N\}.$$

So we have

$$\begin{aligned} r(V_{jj}^N T_{jj}^N - I_{jj}^N) &\leq \| V_{jj}^N T_{jj}^N - I_{jj}^N \| \\ &= \| V_{jj}^N (T_{jj}^N - U_{jj}^N) \| \\ &\leq \| V_{jj}^N \| \| T_{jj}^N - U_{jj}^N \| \\ &\leq \| T_{jj}^N - \gamma_j^N I_{jj}^N \| + \| \gamma_j^N I_{jj}^N - U_{jj}^N \| \\ &= \| T_{jj}^N - \gamma_j^N I_{jj}^N \| + \| (\gamma_j^N)^{-1} U_{jj}^N - I_{jj}^N \|, \text{ where } \gamma_j^N = e^{\lambda_j^N}. \end{aligned}$$

Since

$$\sigma(U_{jj}^N) \subset \{e^{it} : \lambda_{j-1}^N \leq t \leq \lambda_j^N\},$$

we have

$$\sigma((\gamma_j^N)^{-1} U_{jj}^N) \subset \{e^{it} : -(\lambda_j^N - \lambda_{j-1}^N) \leq t \leq 0\}.$$

Since $(\gamma_j^N)^{-1} U_{jj}^N$ is unitary, there is a self-adjoint operator A_{jj}^N such that $(\gamma_j^N)^{-1} U_{jj}^N = e^{iA_{jj}^N}$ and

$$\sigma(A_{jj}^N) \subset [-(\lambda_j^N - \lambda_{j-1}^N), 0] \subset [-\frac{\epsilon}{4}, 0].$$

Since A_{jj}^N is self-adjoint, $\| A_{jj}^N \| = r(A_{jj}^N) \leq \frac{\epsilon}{4}$ and we have

$$\begin{aligned} (\gamma_j^N)^{-1} U_{jj}^N - I_{jj}^N &= e^{iA_{jj}^N} - I_{jj}^N \\ &= iA_{jj}^N + \frac{(iA_{jj}^N)^2}{2!} + \frac{(iA_{jj}^N)^3}{3!} + \cdots\cdots\cdots . \end{aligned}$$

Hence

$$\begin{aligned} \| (\gamma_j^N)^{-1} U_{jj}^N - I_{jj}^N \| &\leq \sum_{n=1}^{\infty} \frac{\| A_{jj}^N \|^n}{n!} \\ &\leq \sum_{n=1}^{\infty} \frac{(\frac{\epsilon}{4})^n}{n!} \\ &< \frac{\epsilon}{4} \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = \frac{\epsilon}{2}. \end{aligned}$$

Since T has an AC-functional calculus on $\mathbb{T}$, and T_{jj}^N is either a restriction of T to a closed hyperinvariant subspace of $\mathcal{H}$ or a quotient of a restriction of T, it follows that T_{jj}^N has an AC-functional calculus on $\mathbb{T}$.

Since T_{jj}^N has an AC-functional calculus on $\mathbb{T}$ and $\sigma(T_{jj}^N) \subset \Gamma_j^N$, by Lemma 3.2.1 we know that T_{jj}^N has an AC-functional calculus on Γ_j^N such that

$$\| f(T_{jj}^N) \| \leq 2K \| f \|_{AC(\Gamma_j^N)} \text{ for all } f \in AC(\Gamma_j^N).$$

Since $f_0(z) = \gamma_j^N - z$ is in $AC(\Gamma_j^N)$, we have

$$\begin{aligned} \| \gamma_j^N I_{jj}^N - T_{jj}^N \| &\leq 2K(|f_0(\gamma_j^N)| + \mathrm{var}_{\Gamma_j^N} f_0) \\ &= 2K(0 + \mathrm{length}(\Gamma_j^N)) \\ &< 2K\frac{\epsilon}{4K} = \frac{\epsilon}{2}. \end{aligned}$$

Hence

$$\begin{aligned} r(V_{jj}^N T_{jj}^N - I_{jj}^N) &\leq \| T_{jj}^N - \gamma_j^N I_{jj}^N \| + \| \gamma_j^N I_{jj}^N - U_{jj}^N \| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Therefore, Q_1 is quasinilpotent. Similarly, we can prove that Q_2 is quasinilpotent.

Now we want to prove that $UQ_1 - Q_1U$ is quasinilpotent. We have

$$\begin{aligned} &UQ_1 - Q_1U = U^{-1}(UT - TU) \\ &= \begin{bmatrix} V_{11}^N(U_{11}^N T_{11}^N - T_{11}^N U_{11}^N) & * & \cdots & * \\ 0 & \cdots & \cdots & * \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & V_{2^N 2^N}^N(U_{2^N 2^N}^N T_{2^N 2^N}^N - T_{2^N 2^N}^N U_{2^N 2^N}^N) \end{bmatrix}, \end{aligned}$$

and

$$\begin{aligned} &r(V_{jj}^N(U_{jj}^N T_{jj}^N - T_{jj}^N U_{jj}^N)) \\ &\leq \| V_{jj}^N(U_{jj}^N T_{jj}^N - T_{jj}^N U_{jj}^N) \| \\ &\leq \| V_{jj}^N \| \| U_{jj}^N T_{jj}^N - T_{jj}^N U_{jj}^N \| \\ &\leq \| (U_{jj}^N - \gamma_j^N) T_{jj}^N + T_{jj}^N(\gamma_j^N - U_{jj}^N) \| \\ &\leq 2 \| U_{jj}^N - \gamma_j^N \| \| T_{jj}^N \| \\ &< 2\frac{\epsilon}{2 \| T \|} \| T \| = \epsilon. \end{aligned}$$

Hence, $UQ_1 - Q_1U$ is quasinilpotent. Similarly, we can prove that $UQ_2 - Q_2U$ is quasinilpotent.

Finally, we want to show that $Q_1 \neq Q_2$. In fact, if $Q_1 = Q_2$, then $T = U(Q_1 + I) = (Q_1 + I)U$. Thus, $UQ_1 = Q_1U$ and hence by Theorem 4.2.3, T is not trigonometrically well-bounded. This contradiction completes the proof that $Q_1 \neq Q_2$. □

3.3 Is The Sum of a Self-Adjoint and a Quasinilpotent Well-Bounded?

We finish the chapter by discussing the validity of the converse of Theorem 3.1.1. First, we show by an example that the converse of Theorem 3.1.1 is not true in general. Then we prove that the sum of a self-adjoint operator A and a nonzero quasinilpotent Q is not well-bounded if A and Q commute. Then we conclude the section by stating several open questions as an attempt to formulate some kind of converse to Theorem 3.1.1 by imposing some further conditions on the self-adjoint operator A and the quasinilpotent operator Q.

Example 3.3.1. *Suppose that $A = \lambda I$ for some $\lambda \in \mathbb{R}$ and Q is any nonzero quasinilpotent operator on $\mathcal{H}$. We claim that $T = A + Q$ is not well-bounded. In fact, if T were well-bounded, then since A is a self-adjoint operator on $\mathcal{H}$ with finite spectrum, which commutes with T, by ([20], Lemma 3) it follows that $Q = T - A$ is well-bounded. However, by ([19], Corollary 2.12), the only quasinilpotent well-bounded operator is the zero operator. Hence T cannot be well-bounded.*

Theorem 3.3.1. *If A is a self-adjoint operator on $\mathcal{H}$ and Q is a nonzero quasinilpotent operator on $\mathcal{H}$ such that $AQ = QA$, then $T = A + Q$ is not well-bounded.*

Proof. Suppose that A is a self-adjoint operator on $\mathcal{H}$ and Q is a nonzero quasinilpotent operator on $\mathcal{H}$ such that $AQ = QA$ and $T = A + Q$ is well-bounded. Then A is a scalar-type spectral operator on $\mathcal{H}$ with real spectrum such that $AT = TA$. Thus by ([20], Theorem p.171), $Q = T - A$ is well-bounded on $\mathcal{H}$, which is a contradiction to ([19], Corollary 2.12). Therefore, T is not well-bounded.

□

From the above theorem we conclude that the problem of formulating a partial converse to Theorem 3.1.1 is very complicated as we must suppose that $AQ \neq QA$ which would imply that we should get an expression for $p(A + Q)$ whose norm is difficult to control. Therefore, we shall leave the following questions open.

Question 3.3.1. *Assume that $A = \int \lambda E(d\lambda)$ is a self-adjoint operator on $\mathcal{H}$ and Q is a quasinilpotent operator such that $QE(\lambda) = E(\lambda)QE(\lambda)$ and $AQ - QA$ is a quasinilpotent operator on $\mathcal{H}$. Is $A + Q$ well-bounded?*

Question 3.3.2. *The same as the last question after adding the extra condition that $Q^2 = 0$.*

Question 3.3.3. *When is the sum of a self-adjoint and a quasinilpotent scalar-type spectral operator?*

Question 3.3.4. *Suppose that T is a scalar-type spectral operator on $\mathcal{H}$ with real spectrum. Then T is well-bounded. So by Theorem 3.1.1, there exists a self-adjoint operator A_1 on $\mathcal{H}$ such that $T = A_1 + Q$. Also, there exists another self-adjoint operator A_2 on $\mathcal{H}$ such that $T = B^{-1}A_2B$. What is the relationship between A_1 and A_2?*

Chapter 4

Decomposition of AC-Operators on $\mathcal{H}$

AC-operators were defined earlier in Chapter 1 as a natural generalisation, in the context of well-boundedness, of normal operators on a Hilbert space $\mathcal{H}$. In this chapter, we prove that certain AC-operators on $\mathcal{H}$ with discrete spectrum are in fact quasinilpotent perturbations of normal operators on $\mathcal{H}$. Then we prove that the sum of a normal operator N on $\mathcal{H}$ and a nonzero quasinilpotent operator Q is not an AC-operator if N and Q commute and we use this result to prove that if U is a unitary operator on $\mathcal{H}$ and Q is a nonzero quasinilpotent operator which commutes with U then $U(Q+I)$ is not trigonometrically well-bounded on $\mathcal{H}$. We also discuss the main obstacle to proving that all AC-operators acting on a Hilbert space $\mathcal{H}$ are quasinilpotent perturbations of normal operators.

4.1 AC-Operators with Discrete Spectrum

In this section, we prove that any AC-operator T on a Hilbert space $\mathcal{H}$ with discrete spectrum of a specific type can be decomposed as a sum of a normal operator N and a quasinilpotent operator Q such that $NQ-QN$ is also a quasinilpotent operator on $\mathcal{H}$. Throughout this section, we shall need the strong operator topology (SOT) which is defined in ([11], p.37) as a topology on $\mathcal{B}(\mathcal{H})$ induced by the family of seminorms $\{p_x : x \in \mathcal{H}\}$, where $p_x(T) = \| Tx \|$ for $T \in \mathcal{B}(\mathcal{H})$. A sequence $\{T_n\}$ of operators in $\mathcal{B}(\mathcal{H})$ is said to be strongly operator convergent to T, or $T_n \to T$ strongly, if $\| T_n x - Tx \| \to 0$ for all $x \in \mathcal{H}$. An infinite series of operators in $\mathcal{B}(\mathcal{H})$ is said to be strong operator topology convergent (SOT convergent), or converges strongly in $\mathcal{B}(\mathcal{H})$, if its sequence of partial sums converges strongly in $\mathcal{B}(\mathcal{H})$. The SOT convergence of an infinite series of operators on $\mathcal{H}$ will be denoted by st-$\sum$. We start with the following lemma which is required in the proof of the main result.

Lemma 4.1.1. *Let T have an AC-functional calculus on $J \times K$ such that*

$$\| f(T) \| \leq M|||f|||_{AC(J\times K)} \text{ for all } f \in AC(J \times K)$$

and suppose that

$$\sigma(T) \subset J_1 \times K_1 \subset J \times K.$$

Then T has an AC-functional calculus on $J_1 \times K_1$ such that

$$\| f(T) \| \leq M|||f|||_{AC(J_1\times K_1)} \text{ for all } f \in AC(J_1 \times K_1).$$

Proof. Let $f \in AC(J_1 \times K_1)$ be a polynomial, where $J_1 = [a_1, b_1] \subset [a, b] = J$ and $K_1 = [c_1, d_1] \subset [c, d] = K$. Extend f continuously to $\tilde{f}$ on $J \times K$ as follows:

$$\tilde{f}(x,y) = \begin{cases} f(x,y) & \text{on } [a_1,b_1] \times [c_1,d_1], \\ f(a_1,c_1) & \text{on } [a,a_1] \times [c,c_1], \\ f(a_1,d_1) & \text{on } [a,a_1] \times [d_1,d], \\ f(b_1,c_1) & \text{on } [b_1,b] \times [c,c_1], \\ f(b_1,d_1) & \text{on } [b_1,b] \times [d_1,d], \\ f(a_1,y) & \text{on } [a,a_1] \times [c_1,d_1], \\ f(b_1,y) & \text{on } [b_1,b] \times [c_1,d_1], \\ f(x,c_1) & \text{on } [a_1,b_1] \times [c,c_1], \\ f(x,d_1) & \text{on } [a_1,b_1] \times [d_1,d]. \end{cases}$$

This extension $\tilde{f}$ of f is defined by taking the value of f at the boundary of $[a_1, b_1] \times [c_1, d_1]$ nearest to the point under consideration. Then $\tilde{f} - f \equiv 0$ on $J_1 \times K_1$ so, as in the proof of part (1) of Lemma 3.1.1, we can see that

$$\tilde{f}(T) = f(T).$$

Since $\tilde{f} \in AC(J \times K)$, we have

$$\| \tilde{f}(T) \| \leq M|||\tilde{f}|||_{AC(J\times K)}.$$

We want to prove that

$$|||\tilde{f}|||_{AC(J\times K)} = |||f|||_{AC(J_1\times K_1)}$$

so that

$$\| f(T) \| = \| \tilde{f}(T) \| \leq M|||\tilde{f}|||_{AC(J\times K)} = M|||f|||_{AC(J_1\times K_1)},$$

i.e., T has an AC-functional calculus on $J_1 \times K_1$ with the required constant. The proof of the lemma will then be complete.

We have

$$|||\tilde{f}|||_{AC(J\times K)} = |\tilde{f}(b,d)| + \mathrm{var}_J\tilde{f}(\cdot,d) + \mathrm{var}_K\tilde{f}(b,\cdot) + \mathrm{var}_{J\times K}\tilde{f},$$

where

$$|\tilde{f}(b,d)| = |f(b_1,d_1)|,$$

$$\begin{aligned}\mathrm{var}_J\tilde{f}(\cdot,d) &= \sup_{\mathcal{P}(J)}\sum_{i=1}^{n}|\tilde{f}(x_i,d)-\tilde{f}(x_{i-1},d)|\\ &= \sup_{\mathcal{P}(J_1)}\sum_{i=1}^{n}|f(x_i,d_1)-f(x_{i-1},d_1)|\\ &= \mathrm{var}_{J_1}f(\cdot,d_1),\end{aligned}$$

$$\begin{aligned}\mathrm{var}_K\tilde{f}(b,\cdot) &= \sup_{\mathcal{P}(K)}\sum_{j=1}^{m}|\tilde{f}(b,y_j)-\tilde{f}(b,y_{j-1})|\\ &= \sup_{\mathcal{P}(K_1)}\sum_{j=1}^{m}|f(b_1,y_j)-f(b_1,y_{j-1})|\\ &= \mathrm{var}_{K_1}f(b_1,\cdot)\end{aligned}$$

and

$$\mathrm{var}_{J\times K}\tilde{f} = \sup_{\mathcal{P}(J\times K)}\sum_{i=1}^{n}\sum_{j=1}^{m}|\Delta_{ij}(\tilde{f})|,$$

where

$$\Delta_{ij}(\tilde{f}) = \tilde{f}(x_i,y_j)-\tilde{f}(x_i,y_{j-1})-\tilde{f}(x_{i-1},y_j)+\tilde{f}(x_{i-1},y_{j-1}).$$

It is clear that $\Delta_{ij}(\tilde{f})=0$ on $[a,a_1]\times[c,c_1]$, $[a,a_1]\times[d_1,d]$, $[b_1,b]\times[c,c_1]$ and $[b_1,b]\times[d_1,d]$.

On $[a,a_1]\times[c_1,d_1]$ and $[b_1,b]\times[c_1,d_1]$ we have

$$\Delta_{ij}(\tilde{f}) = \{\tilde{f}(x_i,y_j)-\tilde{f}(x_{i-1},y_j)\}+\{\tilde{f}(x_{i-1},y_{j-1})-\tilde{f}(x_i,y_{j-1})\}=0.$$

On $[a_1,b_1]\times[c,c_1]$ and $[a_1,b_1]\times[d_1,d]$ we have

$$\Delta_{ij}(\tilde{f}) = \{\tilde{f}(x_i,y_j)-\tilde{f}(x_i,y_{j-1})\}+\{\tilde{f}(x_{i-1},y_{j-1})-\tilde{f}(x_{i-1},y_j)\}=0.$$

Thus, $\Delta_{ij}(\tilde{f})=0$ outside $J_1\times K_1$. Hence, using the fact that refining a particular partition will not decrease the corresponding sum $\sum_{i=1}^{n}\sum_{j=1}^{m}|\Delta_{ij}(\tilde{f})|$, we have

$$\begin{aligned}\mathrm{var}_{J\times K}\tilde{f} &= \sup_{\mathcal{P}(J\times K)}\sum_{i=1}^{n}\sum_{j=1}^{m}|\Delta_{ij}(\tilde{f})|\\ &= \sup_{\mathcal{P}(J_1\times K_1)}\sum_{i=1}^{n}\sum_{j=1}^{m}|\Delta_{ij}(f)|\\ &= \mathrm{var}_{J_1\times K_1}f.\end{aligned}$$

Therefore, we have

$$\begin{aligned}|||\tilde{f}|||_{AC(J\times K)} &= |\tilde{f}(b,d)| + \mathrm{var}_J\tilde{f}(\cdot,d) + \mathrm{var}_K\tilde{f}(b,\cdot) + \mathrm{var}_{J\times K}\tilde{f} \\ &= |f(b_1,d_1)| + \mathrm{var}_{J_1}f(\cdot,d_1) + \mathrm{var}_{K_1}f(b_1,\cdot) + \mathrm{var}_{J_1\times K_1}f \\ &= |||f|||_{AC(J_1\times K_1)}.\end{aligned}$$

□

Suppose that for each $n \in \mathbb{N}$, P_n and Q_n are projections on a Hilbert space $\mathcal{H}$ such that $P_nQ_m = Q_mP_n$ for all $n, m \in \mathbb{N}$, $P_nP_m = 0$ for $n \neq m$, st-$\sum_1^\infty P_n = I$, $Q_nQ_m = 0$ for $n \neq m$ and st-$\sum_1^\infty Q_n = I$. Suppose also that $a = \lambda_1 < \lambda_2 < \cdots\cdots \uparrow b$, $J = [a,b]$, and $c = \mu_1 < \mu_2 < \cdots\cdots \uparrow d$, $K = [c,d]$. Then, according to the proof of ([20], Lemma 1), $A =$ st-$\sum_1^\infty \lambda_n P_n$, and $B =$ st-$\sum_1^\infty \mu_n Q_n$ are two commuting well-bounded operators on $\mathcal{H}$. So $T = A + iB$ is an AC-operator on $\mathcal{H}$. To find the spectrum of such an operator we need the following definition and lemma.

Definition 4.1.1. *Suppose that for each $n, m \in \mathbb{N}$, $x_{nm} \in \mathcal{H}$ and that $x \in \mathcal{H}$. We shall say that $\sum_{m=1}^\infty \sum_{n=1}^\infty x_{nm} = x$ if given $\epsilon > 0$, there exist $M_0, N_0 \in \mathbb{N}$ such that*

$$\| \sum_{m=1}^{M}\sum_{n=1}^{N} x_{nm} - x \| < \epsilon \textit{ for all } N > N_0, M > M_0.$$

Notice that if $\sum_{m=1}^\infty \sum_{n=1}^\infty x_{nm} = x$, then by definition $\sum_{n=1}^\infty \sum_{m=1}^\infty x_{nm} = x$. Throughout the rest of this section, we shall denote by $BV(\mathbb{N})$ the set of all sequences of scalars $\{\beta_n\}_{n=1}^\infty$ satisfying $\mathrm{var}_{\mathbb{N}}\{\beta_n\}_{n=1}^\infty < \infty$, where

$$\mathrm{var}_{\mathbb{N}}\{\beta_n\}_{n=1}^\infty = \sum_{n=1}^{\infty} |\beta_n - \beta_{n+1}|,$$

and denote by $BV(\mathbb{N}\times\mathbb{N})$ the set of all sequences of scalars $\{\beta_{nm}\}$ satisfying $\mathrm{var}_{\mathbb{N}\times\mathbb{N}}\{\beta_{nm}\} < \infty$, where

$$\mathrm{var}_{\mathbb{N}\times\mathbb{N}}\{\beta_{nm}\} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} |\beta_{nm} - \beta_{n+1,m} - \beta_{n,m+1} + \beta_{n+1,m+1}|.$$

The set $BV(\mathbb{N})$ was introduced in ([15], p.239). Note that if $\mathrm{var}_{\mathbb{N}\times\mathbb{N}}\{\beta_{nm}\} < \infty$ and $\epsilon > 0$ is given, then there exists $N \in \mathbb{N}$ such that

$$(i)\ \mathrm{var}_{[N+1,\infty)\times\mathbb{N}}\{\beta_{nm}\} = \sum_{n=N+1}^{\infty}\sum_{m=1}^{\infty} |\beta_{nm} - \beta_{n+1,m} - \beta_{n,m+1} + \beta_{n+1,m+1}| < \epsilon;$$

$$(ii)\ \mathrm{var}_{\mathbb{N}\times[N+1,\infty)}\{\beta_{nm}\} = \sum_{n=1}^{\infty}\sum_{m=N+1}^{\infty} |\beta_{nm} - \beta_{n+1,m} - \beta_{n,m+1} + \beta_{n+1,m+1}| < \epsilon;$$

$$(iii)\ \mathrm{var}_{[N+1,\infty)\times[N+1,\infty)}\{\beta_{nm}\} = \sum_{n=N+1}^{\infty}\sum_{m=N+1}^{\infty} |\beta_{nm} - \beta_{n+1,m} - \beta_{n,m+1} + \beta_{n+1,m+1}| < \epsilon.$$

Lemma 4.1.2. *Suppose that for each $n \in \mathbb{N}$, P_n and Q_n are projections on a Hilbert space $\mathcal{H}$ such that $P_nQ_m = Q_mP_n$ for all $n, m \in \mathbb{N}$, st-$\sum_{n=1}^{\infty} P_n = I$ and st-$\sum_{n=1}^{\infty} Q_n = I$. Suppose also that the sequence $\{\beta_{nm}\}$ satisfies the following conditions:*

(i) $\{\beta_{1m}\}_{m=1}^{\infty} \in BV(\mathbb{N})$,

(ii) $\{\beta_{n1}\}_{n=1}^{\infty} \in BV(\mathbb{N})$,

(iii) $\{\beta_{nm}\} \in BV(\mathbb{N} \times \mathbb{N})$.

Then

$$(a) \quad \sum_{m=1}^{\infty}(\sum_{n=1}^{\infty} \beta_{nm}P_nQ_mx) \text{ and } \sum_{n=1}^{\infty}(\sum_{m=1}^{\infty} \beta_{nm}P_nQ_mx) \text{ exist for all } x \in \mathcal{H};$$

$$(b) \quad \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \beta_{nm}P_nQ_mx \text{ exists and equals the two repeated sums in (a) for all } x \in \mathcal{H}.$$

Notice that if $\{\beta_{nm}\} = \{\beta_n\} \in BV(\mathbb{N})$, then $\{\beta_{nm}\}$ satisfies the conditions of the above lemma, and hence st $- \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \beta_nP_nQ_m$ exists in $\mathcal{B}(\mathcal{H})$. We shall use this in the proof of the main theorem of this section where $\{\beta_{nm}\} = \{\lambda_n\} \in BV(\mathbb{N})$ or $\{\beta_{nm}\} = \{\mu_m\} \in BV(\mathbb{N})$.

Proof. For each $n, m \in \mathbb{N}$, let $A_n = \sum_{k=1}^{n} P_k$, $A_0 = 0$, $B_m = \sum_{\ell=1}^{m} Q_\ell$, $B_0 = 0$. Since $\{\beta_{n1}\}_{n=1}^{\infty} \in BV(\mathbb{N})$ and $\text{var}_{\mathbb{N}\times\mathbb{N}}\{\beta_{nm}\} < \infty$, $\{\beta_{nm}\}_{n=1}^{\infty} \in BV(\mathbb{N})$ for all $m \in \mathbb{N}$ because $\text{var}_{\mathbb{N}}\{\beta_{nm}\}_{n=1}^{\infty} \le \text{var}_{\mathbb{N}}\{\beta_{n1}\}_{n=1}^{\infty} + \text{var}_{\mathbb{N}\times\mathbb{N}}\{\beta_{nk}\}$. Similarly, since $\{\beta_{1m}\}_{m=1}^{\infty} \in BV(\mathbb{N})$ and $\text{var}_{\mathbb{N}\times\mathbb{N}}\{\beta_{\ell m}\} < \infty$, $\{\beta_{nm}\}_{m=1}^{\infty} \in BV(\mathbb{N})$ for all $n \in \mathbb{N}$. Fix $m \in \mathbb{N}$. Abel's summation lemma gives

$$\sum_{n=1}^{N} \beta_{nm}P_n = \sum_{n=1}^{N}(\beta_{nm} - \beta_{n+1,m})A_n + \beta_{N+1,m}A_N \text{ for } N = 1, 2, \cdots\cdots. \tag{4.1.1}$$

Now st-$\sum_{n=1}^{\infty} P_n = I$, so by the principle of uniform boundedness, the partial sums of the series $\sum_{n=1}^{\infty} P_n$ are bounded in norm, i.e., the sequence $\{A_n\}$ is bounded in norm. Since $\{\beta_{nm}\}_{n=1}^{\infty} \in BV(\mathbb{N})$ and $\{A_n\}$ is bounded in norm, $\sum_{n=1}^{\infty}(\beta_{nm} - \beta_{n+1,m})A_n$ is norm convergent. Since $\{\beta_{Nm}\}$ is convergent to β_m, say, as $N \to \infty$ ($\{\beta_{Nm}\}$ is a Cauchy sequence because $\{\beta_{nm}\}_{n=1}^{\infty} \in BV(\mathbb{N})$) and $A_N \to I$ strongly as $N \to \infty$, the sequence $\{\beta_{N+1,m}A_N\}$ converges strongly as $N \to \infty$ to β_mI in $\mathcal{B}(\mathcal{H})$. Thus $\sum_{n=1}^{\infty} \beta_{nm}P_n$ converges strongly and therefore, as $N \to \infty$, (4.1.1) becomes

$$\text{st-}\sum_{n=1}^{\infty} \beta_{nm}P_n = \sum_{n=1}^{\infty}(\beta_{nm} - \beta_{n+1,m})A_n + \beta_mI. \tag{4.1.2}$$

Since (4.1.2) holds for all $m \in \mathbb{N}$ and $P_nQ_m = Q_mP_n$ for all $n, m \in \mathbb{N}$, we get

$$\sum_{m=1}^{M}(\text{st-}\sum_{n=1}^{\infty} \beta_{nm}P_nQ_m) = \sum_{m=1}^{M}(\sum_{n=1}^{\infty}(\beta_{nm} - \beta_{n+1,m})A_nQ_m) + \sum_{m=1}^{M} \beta_mQ_m. \tag{4.1.3}$$

By applying Abel's lemma again, (4.1.3) gives

$$\sum_{m=1}^{M}(\text{st-}\sum_{n=1}^{\infty}\beta_{nm}P_nQ_m) = \sum_{m=1}^{M}(\sum_{n=1}^{\infty}\{(\beta_{nm}-\beta_{n+1,m})-(\beta_{n,m+1}-\beta_{n+1,m+1})\}A_nB_m)$$
$$+\sum_{n=1}^{\infty}(\beta_{n,M+1}-\beta_{n+1,M+1})A_nB_M+\sum_{m=1}^{M}(\beta_m-\beta_{m+1})B_m$$
$$+\beta_{M+1}B_M \text{ for } M=1,2,\cdots\cdots.$$

By the principle of uniform boundedness, the sequence $\{B_m\}$ is bounded in norm. Since $\text{var}_{\mathbb{N}\times\mathbb{N}}\{\beta_{nm}\}<\infty$ and $\{A_n\}$ and $\{B_m\}$ are bounded in norm, $\sum_{m=1}^{\infty}(\sum_{n=1}^{\infty}\{(\beta_{nm}-\beta_{n+1,m})-(\beta_{n,m+1}-\beta_{n+1,m+1})\}A_nB_m)$ is norm convergent. Since $\sum_{n=1}^{\infty}(\beta_{n,M+1}-\beta_{n+1,M+1})A_n$ is norm convergent and $B_M\to I$ strongly as $M\to\infty$, the sequence $\sum_{n=1}^{\infty}(\beta_{n,M+1}-\beta_{n+1,M+1})A_nB_M$ converges strongly as $M\to\infty$. Since $\{\beta_{nm}\}_{m=1}^{\infty}\in BV(\mathbb{N})$ for all $n\in\mathbb{N}$ with a uniform (in n) bound on its $BV(\mathbb{N})$ norm because $\text{var}_{\mathbb{N}\times\mathbb{N}}\{\beta_{nm}\}<\infty$ and $\{\beta_{nm}\}_{m=1}^{\infty}\to\beta_m$ as $n\to\infty$, it follows that $\{\beta_m\}_{m=1}^{\infty}\in BV(\mathbb{N})$. Since $\{\beta_m\}_{m=1}^{\infty}\in BV(\mathbb{N})$ and $\{B_m\}$ is bounded in norm, $\sum_{m=1}^{\infty}(\beta_m-\beta_{m+1})B_m$ is norm convergent. Since $\{\beta_M\}$ is convergent and $B_M\to I$ strongly as $M\to\infty$, the sequence $\{\beta_{M+1}B_M\}$ converges strongly in $\mathcal{B}(\mathcal{H})$. Therefore, $\sum_{m=1}^{\infty}(\sum_{n=1}^{\infty}\beta_{nm}P_nQ_m)$ converges strongly in $\mathcal{B}(\mathcal{H})$.

Similarly, we can see that $\sum_{n=1}^{\infty}(\sum_{m=1}^{\infty}\beta_{nm}P_nQ_m)$ converges strongly in $\mathcal{B}(\mathcal{H})$.

Now, fix $x\in\mathcal{H}$ and let $z=\sum_{m=1}^{\infty}(\sum_{n=1}^{\infty}\beta_{nm}P_nQ_mx)$. We want to show that given $\epsilon>0$, there exist $M_0,N_0\in\mathbb{N}$ such that

$$\parallel\sum_{m=1}^{M}\sum_{n=1}^{N}\beta_{nm}P_nQ_mx-z\parallel<\epsilon \text{ for all } N>N_0, M>M_0.$$

We have

$$\parallel\sum_{m=1}^{M}\sum_{n=1}^{N}\beta_{nm}P_nQ_mx-z\parallel$$
$$=\parallel\sum_{m=1}^{M}(\sum_{n=N+1}^{\infty}\beta_{nm}P_nQ_mx)+\sum_{m=M+1}^{\infty}(\sum_{n=1}^{\infty}\beta_{nm}P_nQ_mx)\parallel$$
$$\leq\parallel\sum_{m=1}^{M}(\sum_{n=N+1}^{\infty}\beta_{nm}P_nQ_mx)\parallel+\parallel\sum_{m=M+1}^{\infty}(\sum_{n=1}^{\infty}\beta_{nm}P_nQ_mx)\parallel.$$

Abel's summation lemma gives

$$\sum_{n=N+1}^{N'}\beta_{nm}P_n=\sum_{n=N+1}^{N'}(\beta_{nm}-\beta_{n+1,m})A_n+\beta_{N'+1,m}A_{N'}-\beta_{N+1,m}A_N \text{ for } N'\geq N+1.$$

As $N' \to \infty$, $\{\beta_{N'+1,m}A_{N'}\} \to \beta_m I$. Thus, since $\sum_1^\infty \beta_{nm}P_n$ converges strongly, we have

$$\text{st-}\sum_{n=N+1}^{\infty} \beta_{nm}P_n = \sum_{n=N+1}^{\infty} (\beta_{nm} - \beta_{n+1,m})A_n + \beta_m I - \beta_{N+1,m}A_N.$$

Applying Abel's lemma again, we get, for $M = 1, 2, \cdots\cdots$,

$$\begin{aligned}
&\sum_{m=1}^{M}(\sum_{n=N+1}^{\infty} \beta_{nm}P_nQ_mx) \\
&= \sum_{m=1}^{M}(\sum_{n=N+1}^{\infty} (\beta_{nm} - \beta_{n+1,m})A_nQ_mx) + \sum_{m=1}^{M}\beta_m Q_m x - \sum_{m=1}^{M}\beta_{N+1,m}A_NQ_mx \\
&= \sum_{m=1}^{M}(\sum_{n=N+1}^{\infty} \{(\beta_{nm} - \beta_{n+1,m}) - (\beta_{n,m+1} - \beta_{n+1,m+1})\}A_nB_mx) \\
&+ \sum_{n=N+1}^{\infty} (\beta_{n,M+1} - \beta_{n+1,M+1})A_nB_Mx + \sum_{m=1}^{M}(\beta_m - \beta_{m+1})B_mx \\
&+ \beta_{M+1}B_Mx - \beta_{N+1,M+1}A_NB_Mx - \sum_{m=1}^{M}(\beta_{N+1,m} - \beta_{N+1,m+1})A_NB_mx \\
&= \sum_{m=1}^{M}(\sum_{n=N+1}^{\infty} \{(\beta_{nm} - \beta_{n+1,m}) - (\beta_{n,m+1} - \beta_{n+1,m+1})\}A_nB_mx) \\
&+ \sum_{n=N+1}^{\infty} (\beta_{n,M+1} - \beta_{n+1,M+1})A_nB_Mx + B_M(\beta_{M+1}I - \beta_{N+1,M+1}A_N)x \\
&+ \sum_{m=1}^{M}(\beta_m - \beta_{m+1})B_m(I - A_N)x + A_N\sum_{m=1}^{M}\{(\beta_m - \beta_{m+1}) - (\beta_{N+1,m} - \beta_{N+1,m+1})\}B_mx
\end{aligned}$$

Let $\epsilon > 0$ be given. Since $\{A_n\}_{n=1}^\infty$ and $\{B_m\}_{m=1}^\infty$ are bounded in norm, there exists $K > 0$ such that

$$\| A_n \| \leq K \text{ for all } n \in \mathbb{N} \text{ and } \| B_m \| \leq K \text{ for all } m \in \mathbb{N}.$$

Since $\sum_{m=1}^\infty(\sum_{n=1}^\infty \beta_{nm}P_nQ_m)$ converges strongly in $\mathcal{B}(\mathcal{H})$, there exists $M_1 \in \mathbb{N}$ such that

$$\| \sum_{m=M+1}^{\infty} (\sum_{n=1}^{\infty} \beta_{nm}P_nQ_mx) \| < \frac{\epsilon}{2} \text{ for all } M > M_1.$$

Since $\mathrm{var}_{\mathbb{N}\times\mathbb{N}}\{\beta_{nm}\} < \infty$, there exists $N_1 \in \mathbb{N}$ such that

$$\mathrm{var}_{[N+1,\infty)\times\mathbb{N}}\{\beta_{nm}\} < \frac{\epsilon}{20K^2 \| x \|} \text{ for all } N > N_1.$$

Since $\mathrm{var}_{\mathbb{N}}\{\beta_{n,1}\}_{n=1}^\infty < \infty$, there exists $N_2 \in \mathbb{N}$ such that

$$\mathrm{var}_{[N+1,\infty)}\{\beta_{n,1}\} < \frac{\epsilon}{20K^2 \| x \|} \text{ for all } N > N_2.$$

Since for each $M \in \mathbb{N}$, $\{\beta_{N+1,M+1}\} \to \beta_{M+1}$ as $N \to \infty$ and $\beta_{M+1} \to \beta$, say, as $M \to \infty$ (because $\text{var}_{\mathbb{N}}\{\beta_{M+1}\} < \infty$), it follows that $\{\beta_{N+1,M+1}\} \to \beta$ as $N \to \infty$ and $M \to \infty$. Since $\{A_N\} \to I$ strongly as $N \to \infty$, the sequence $\{\beta_{N+1,M+1}A_N\} \to \beta I$ strongly as $N \to \infty$ and $M \to \infty$. Thus, there exist $N_3 \in \mathbb{N}$ and $M_2 \in \mathbb{N}$ such that

$$\| \beta x - \beta_{N+1,M+1}A_N x \| < \frac{\epsilon}{20K} \text{ for all } N > N_3 \text{ and for all } M > M_2.$$

Since $\beta_{M+1} \to \beta$ as $M \to \infty$, there exists $M_3 \in \mathbb{N}$ such that

$$|\beta_{M+1} - \beta| < \frac{\epsilon}{20K \| x \|} \text{ for all } M > M_3.$$

Since $\{A_N\} \to I$ strongly as $N \to \infty$, there exists $N_4 \in \mathbb{N}$ such that

$$\| x - A_N x \| < \frac{\epsilon}{10K \, \text{var}_{\mathbb{N}}\{\beta_m\}_{m=1}^{\infty}} \text{ for all } N > N_4.$$

Let $N_0 = \max\{N_1, N_2, N_3, N_4\}$ and $M_0 = \max\{M_1, M_2, M_3\}$. Then, if $N > N_0$ and $M > M_0$, we have

$$
\begin{aligned}
&\| \sum_{m=1}^{M} (\sum_{n=N+1}^{\infty} \beta_{nm} P_n Q_m x) \| \\
&\le \sum_{m=1}^{M} (\sum_{n=N+1}^{\infty} |(\beta_{nm} - \beta_{n+1,m}) - (\beta_{n,m+1} - \beta_{n+1,m+1})| \| A_n \| \| B_m \| \| x \|) \\
&+ \sum_{n=N+1}^{\infty} |\beta_{n,M+1} - \beta_{n+1,M+1}| \| A_n \| \| B_M \| \| x \| + \| B_M \| \| \beta_{M+1} x - \beta_{N+1,M+1} A_N x \| \\
&+ \sum_{m=1}^{M} |\beta_m - \beta_{m+1}| \| B_m \| \| x - A_N x \| \\
&+ \| A_N \| \sum_{m=1}^{M} |(\beta_m - \beta_{m+1}) - (\beta_{N+1,m} - \beta_{N+1,m+1})| \| B_m \| \| x \| \\
&< K^2 \| x \| \text{var}_{[N+1,\infty)\times\mathbb{N}}\{\beta_{nm}\} \\
&+ K^2 \| x \| \text{var}_{[N+1,\infty)}\{\beta_{n,M+1}\} + K \| \beta_{M+1} x - \beta_{N+1,M+1} A_N x \| \\
&+ K \, \text{var}_{\mathbb{N}}\{\beta_m\}_{m=1}^{\infty} \| x - A_N x \| + K^2 \| x \| \text{var}_{[N+1,\infty)\times\mathbb{N}}\{\beta_{nm}\} \\
&\le K^2 \| x \| \text{var}_{[N+1,\infty)\times\mathbb{N}}\{\beta_{nm}\} \\
&+ K^2 \| x \| (\text{var}_{[N+1,\infty)}\{\beta_{n,1}\} + \text{var}_{[N+1,\infty)\times\mathbb{N}}\{\beta_{nm}\}) \\
&+ K(\| \beta x - \beta_{N+1,M+1} A_N x \| + \| \beta_{M+1} I - \beta I \| \| x \|) \\
&+ K \, \text{var}_{\mathbb{N}}\{\beta_m\}_{m=1}^{\infty} \| x - A_N x \| + K^2 \| x \| \text{var}_{[N+1,\infty)\times\mathbb{N}}\{\beta_{nm}\} \\
&< \frac{\epsilon}{2}.
\end{aligned}
$$

Therefore, if $N > N_0$ and $M > M_0$, then

$$\begin{aligned}
&\| \sum_{m=1}^{M}\sum_{n=1}^{N}\beta_{nm}P_nQ_mx - z \| \\
&\leq \| \sum_{m=1}^{M}(\sum_{n=N+1}^{\infty} \beta_{nm}P_nQ_mx) \| + \| \sum_{m=M+1}^{\infty} (\sum_{n=1}^{\infty}\beta_{nm}P_nQ_mx) \| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

The proof of the lemma is now complete. □

Now we prove the main theorem of this chapter which decomposes any AC-operator on $\mathcal{H}$ with discrete spectrum of a certain type as a sum of a normal operator N and a quasinilpotent operator Q such that $NQ-QN$ is also quasinilpotent.

Theorem 4.1.1. *Suppose that for each $n \in \mathbb{N}$, P_n and Q_n are projections on a Hilbert space $\mathcal{H}$ such that $P_nQ_m = Q_mP_n$ for all $n, m \in \mathbb{N}$, $P_nP_m = 0$ for $n \neq m$, st-$\sum_1^\infty P_n = I$, $Q_nQ_m = 0$ for $n \neq m$ and st-$\sum_1^\infty Q_n = I$. Suppose also that $a = \lambda_1 < \lambda_2 < \cdots\cdots \uparrow b$, $J = [a,b]$, and $c = \mu_1 < \mu_2 < \cdots\cdots \uparrow d$, $K = [c,d]$, so that $A =$ st-$\sum_1^\infty \lambda_nP_n$ and $B =$ st-$\sum_1^\infty \mu_nQ_n$ are two commuting well-bounded operators on $\mathcal{H}$ and so $T = A + iB$ is an AC-operator with functional calculus satisfying*

$$\| f(T) \| \leq M \, |||f|||_{AC(J\times K)} \quad \text{for all } f \in AC(J \times K).$$

Then T has a canonical decomposition $T = N + Q$, where N is a normal operator and Q is a quasinilpotent operator. Moreover, $NQ - QN$ is quasinilpotent.

Proof. Notice that $\{\lambda_n\}_{n=1}^\infty \in BV(\mathbb{N})$ and $\{\mu_m\}_{m=1}^\infty \in BV(\mathbb{N})$. So by Lemma 4.1.2, we have st-$\sum_{n=1}^\infty\sum_{m=1}^\infty \lambda_nP_nQ_m$ and st-$\sum_{n=1}^\infty\sum_{m=1}^\infty \mu_mP_nQ_m$ exist in $\mathcal{B}(\mathcal{H})$. Thus, we have

$$\begin{aligned}
T = A + iB &= (\text{st-}\sum_{n=1}^{\infty}\lambda_nP_n) + i(\text{st-}\sum_{m=1}^{\infty}\mu_mQ_m) \\
&= (\text{st-}\sum_{n=1}^{\infty}\lambda_nP_n)(\text{st-}\sum_{m=1}^{\infty}Q_m) + i(\text{st-}\sum_{n=1}^{\infty}P_n)(\text{st-}\sum_{m=1}^{\infty}\mu_mQ_m) \\
&= (\text{st-}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\lambda_nP_nQ_m) + i(\text{st-}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\mu_mP_nQ_m) \text{ (by Lemma 4.1.2)} \\
&= \text{st-}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(\lambda_n + i\mu_m)P_nQ_m \\
&= \text{st-}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\gamma_{nm}P_nQ_m, \text{ where } \gamma_{nm} = \lambda_n + i\mu_m.
\end{aligned}$$

First, we want to prove that

$$\sigma(T) = \overline{\{\gamma_{nm} : P_n Q_m \neq 0\}}.$$

Let $n_0, m_0 \in \mathbb{N}$ be such that $P_{n_0} Q_{m_0} \neq 0$. Then there exists $x_0 \in \mathcal{H}$ such that $P_{n_0} Q_{m_0} x_0 \neq 0$. Since $T = \text{st-}\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \gamma_{nm} P_n Q_m$, we have $T(P_{n_0} Q_{m_0} x_0) = \gamma_{n_0 m_0}(P_{n_0} Q_{m_0} x_0)$. So $\gamma_{n_0 m_0} \in \sigma_p(T) \subset \sigma(T)$. Since $n_0, m_0 \in \mathbb{N}$ are arbitrary such that $P_{n_0} Q_{m_0} \neq 0$, and $\sigma(T)$ is closed, we have

$$\overline{\{\gamma_{nm} : P_n Q_m \neq 0\}} \subset \sigma(T).$$

Now, we want to prove that

$$\sigma(T) \subset \overline{\{\gamma_{nm} : P_n Q_m \neq 0\}}.$$

Suppose that $\gamma = \lambda + i\mu \notin \overline{\{\gamma_{nm} : P_n Q_m \neq 0\}}$. Then either $\gamma = \gamma_{n_0 m_0}$ with $P_{n_0} Q_{m_0} = 0$ or γ is a boundary point for $\{\gamma_{nm} : P_n Q_m = 0\}$ or there exists an $\epsilon > 0$ such that

$$|\gamma - \gamma_{nm}| > \epsilon \text{ for all } \gamma_{nm}.$$

We want to show that in all cases $\gamma \notin \sigma(T)$, i.e., $(\gamma - T)^{-1}$ exists in $\mathcal{B}(\mathcal{H})$.
(1) Let γ be such that there exists an $\epsilon > 0$ satisfying

$$|\gamma - \gamma_{nm}| > \epsilon \text{ for all } \gamma_{nm}.$$

Setting $\beta_{nm} = (\gamma - \gamma_{nm})^{-1}$ and doing some calculations, we get

$$\begin{aligned} \text{var}_{\mathbb{N}\times\mathbb{N}}\{\beta_{nm}\} &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |\beta_{nm} - \beta_{n+1,m} - \beta_{n,m+1} + \beta_{n+1,m+1}| \\ &\leq C_\epsilon (b-a)(d-c) < \infty, \text{ where } C_\epsilon = \frac{2|\gamma| + 2|b + id|}{\epsilon^4}. \end{aligned}$$

Similarly, we can show that $\{\beta_{1m}\}, \{\beta_{n1}\} \in BV(\mathbb{N})$. Hence, by Lemma 4.1.2,

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \beta_{nm} P_n Q_m x \text{ exists for all } x \in \mathcal{H}.$$

Then we can easily see that

$$(\gamma - T)^{-1} = \text{st-}\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \beta_{nm} P_n Q_m \in \mathcal{B}(\mathcal{H}), \text{ i.e., } \gamma \notin \sigma(T).$$

(2) Let $\gamma = \gamma_{n_0 m_0}$ with $P_{n_0} Q_{m_0} = 0$. Decompose $\mathcal{H}$ as follows:

$$\mathcal{H} = P_1 \mathcal{H} + P_2 \mathcal{H} + \cdots + P_{n_0 - 1} \mathcal{H} + P_{n_0} \mathcal{H} + (I - P_1 - P_2 - \cdots - P_{n_0}) \mathcal{H}.$$

Then, as before, $\sum_{n=1}^{n_0-1}\sum_{m=1}^{\infty}(\gamma_{n_0m_0}-\gamma_{nm})^{-1}P_nQ_mx$ exists for all $x\in\mathcal{H}$ and $\sum_{n=n_0+1}^{\infty}\sum_{m=1}^{\infty}(\gamma_{n_0m_0}-\gamma_{nm})^{-1}P_nQ_mx$ exists for all $x\in\mathcal{H}$. Also, since $T=$ st-$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\gamma_{nm}P_nQ_m$ and $P_{n_0}Q_{m_0}=0$, we have

$$T|P_{n_0}\mathcal{H}=\text{st-}\sum_{m=1}^{\infty}\gamma_{n_0m}P_{n_0}Q_m=\text{st-}\sum_{m=1}^{m_0-1}\gamma_{n_0m}P_{n_0}Q_m+\text{st-}\sum_{m=m_0+1}^{\infty}\gamma_{n_0m}P_{n_0}Q_m.$$

Then, also as before,

$$\sum_{m=1}^{m_0-1}(\gamma_{n_0m_0}-\gamma_{n_0m})^{-1}P_{n_0}Q_mx+\sum_{m=m_0+1}^{\infty}(\gamma_{n_0m_0}-\gamma_{n_0m})^{-1}P_{n_0}Q_mx \text{ exists for all } x\in\mathcal{H}.$$

Thus, $(\gamma_{n_0m_0}-T)^{-1}\in\mathcal{B}(\mathcal{H})$ and hence $\gamma_{n_0m_0}\notin\sigma(T)$.
(3) If γ is a boundary point of $\{\gamma_{nm}:P_nQ_m=0\}$, then we can show that $\gamma\notin\sigma(T)$ in the same way as in case (2).
Therefore,

$$\sigma(T)=\overline{\{\gamma_{nm}:P_nQ_m\neq 0\}}.$$

Let $\gamma_n=\lambda_n+id$, $\delta_m=b+i\mu_m$ and $\eta=b+id$ and let

$$S=\overline{\{\gamma_{nm}:n,m\in\mathbb{N}\}}.$$

Note that

$$S=\{\gamma_{nm}:n,m\in\mathbb{N}\}\cup\{\gamma_n:n\in\mathbb{N}\}\cup\{\delta_m:m\in\mathbb{N}\}\cup\{\eta\}.$$

Also, notice that $\sigma(T)\subset S$ is discrete of a special type so that we can order the elements of S to get a corresponding increasing sequence of $R_\gamma\subset S$ which will give an increasing sequence of local spectral subspaces $X_T(R_\gamma)$. Then the orthogonal projections E_γ onto $X_T(R_\gamma)$ will give us the spectral measure we need to define the normal operator N. Define the total order $\prec$ on S as follows:

$$\begin{aligned}
\gamma_{11}&\prec\gamma_{12}\prec\gamma_{13}\prec\gamma_{14}\prec\cdots\prec\gamma_1\prec\gamma_{21}\prec\gamma_{31}\prec\cdots\cdots\cdots\prec\delta_1\\
&\prec\gamma_{22}\prec\gamma_{23}\prec\gamma_{24}\prec\cdots\prec\gamma_2\prec\gamma_{32}\prec\gamma_{42}\prec\cdots\cdots\cdots\prec\delta_2\\
&\prec\gamma_{33}\prec\gamma_{34}\prec\gamma_{35}\prec\cdots\prec\gamma_3\prec\gamma_{43}\prec\gamma_{53}\prec\cdots\cdots\cdots\prec\delta_3\\
&\prec\cdots\cdots\cdots\\
&\prec\gamma_{kk}\prec\gamma_{k,k+1}\prec\cdots\prec\gamma_k\prec\gamma_{k+1,k}\prec\gamma_{k+2,k}\prec\gamma_{k+3,k}\prec\cdots\prec\delta_k\\
&\prec\cdots\cdots\cdots\\
&\prec\eta=b+id.
\end{aligned}$$

Given $\gamma\in S$, let $R_\gamma=\{\gamma'\in S:\gamma'\preceq\gamma\}$. Because of the discrete nature of S, each R_γ is a closed subset of $\mathbb{C}$. In fact, we have the following cases (where we identify $\mathbb{C}$ with $\mathbb{R}^2$ for simplicity).

(1) If $\gamma \in S$ is such that $\gamma = \gamma_{1j}$, then $R_{\gamma_{1j}} = \{\gamma_{11}, \gamma_{12}, \cdots, \gamma_{1j}\}$.

(2) If $\gamma \in S$ is such that $\gamma = \gamma_{ij}$, $j \geq i$, then

$$R_{\gamma_{ij}} = [\{\lambda_1, \lambda_2, \cdots, \lambda_{i-1}\} \times (\{\mu_1, \mu_2, \cdots\cdots\} \cup \{d\})] \cup [(\{\lambda_1, \lambda_2, \cdots\cdots\} \cup \{b\}) \times \{\mu_1, \mu_2, \cdots, \mu_{i-1}\}] \cup [\{\lambda_i\} \times \{\mu_i, \mu_{i+1}, \cdots, \mu_j\}].$$

(3) If $\gamma \in S$ is such that $\gamma = \gamma_i$, then

$$\begin{aligned} R_{\gamma_i} = & [\{\lambda_1, \lambda_2, \cdots, \lambda_{i-1}, \lambda_i\} \times (\{\mu_1, \mu_2, \cdots\cdots\} \cup \{d\})] \\ & \cup [(\{\lambda_1, \lambda_2, \cdots\cdots\} \cup \{b\}) \times \{\mu_1, \mu_2, \cdots, \mu_{i-1}\}]. \end{aligned}$$

(4) If $\gamma \in S$ is such that $\gamma = \gamma_{ij}$, $j < i$, then

$$\begin{aligned} R_{\gamma_{ij}} &= [\{\lambda_1, \lambda_2, \cdots, \lambda_j\} \times (\{\mu_1, \mu_2, \cdots\cdots\} \cup \{d\})] \cup [(\{\lambda_1, \lambda_2, \cdots\cdots\} \cup \{b\}) \\ &\quad \times \{\mu_1, \mu_2, \cdots, \mu_{j-1}\}] \cup [\{\lambda_{j+1}, \lambda_{j+2}, \cdots, \lambda_i\} \times \{\mu_j\}] \\ &= R_{\gamma_j} \cup [\{\lambda_{j+1}, \lambda_{j+2}, \cdots, \lambda_i\} \times \{\mu_j\}]. \end{aligned}$$

(5) If $\gamma \in S$ is such that $\gamma = \delta_j = b + i\mu_j$, then

$$\begin{aligned} R_{\delta_j} = & [\{\lambda_1, \lambda_2, \cdots, \lambda_j\} \times (\{\mu_1, \mu_2, \cdots\cdots\} \cup \{d\})] \\ & \cup [(\{\lambda_1, \lambda_2, \cdots\cdots\} \cup \{b\}) \times \{\mu_1, \mu_2, \cdots, \mu_{j-1}, \mu_j\}]. \end{aligned}$$

(6) If $\gamma = \eta = b + id$, then $R_\eta = S$.

Define

$$X_\gamma = \text{clm}\{P_n Q_m \mathcal{H} : \gamma_{nm} \preceq \gamma\} \text{ for } \gamma \in S.$$

We want to prove that in each of the above 6 cases we have

$$X_\gamma = X_T(R_\gamma) = \{x \in \mathcal{H} : \sigma_T(x) \subset R_\gamma\}.$$

Case 2. Suppose that $\gamma \in S$ is such that $\gamma = \gamma_{i_0 j_0}$ with $j_0 \geq i_0$. We want to prove that

$$\{x \in \mathcal{H} : \sigma_T(x) \subset R_{\gamma_{i_0 j_0}}\} = X_{\gamma_{i_0 j_0}} = \text{clm}\{P_n Q_m \mathcal{H} : \gamma_{nm} \preceq \gamma_{i_0 j_0}\}.$$

Let $x \in \mathcal{H}$ be such that

$$x \in X_{\gamma_{i_0 j_0}} = \text{clm}\{P_n Q_m \mathcal{H} : \gamma_{nm} \preceq \gamma_{i_0 j_0}\}.$$

We want to show that $x \in X_T(R_{\gamma_{i_0 j_0}})$, i.e., $\sigma_T(x) \subset R_{\gamma_{i_0 j_0}}$.

Fix $z \in R^c_{\gamma_{i_0 j_0}}$. Then there exists a $\delta > 0$ such that

$$|z - \gamma_{ij}| > \delta \text{ for } 1 \leq i \leq i_0 - 1 \text{ and for all } j \in \mathbb{N},$$

$$|z - \gamma_{ij}| > \delta \text{ for } 1 \leq j \leq i_0 - 1 \text{ and for all } i \in \mathbb{N},$$

and

$$|z - \gamma_{i_0 j}| > \delta \text{ for } 1 \leq j \leq j_0.$$

Define $f_z : V_\delta \to \mathcal{H}$, where $V_\delta = \{w \in \mathbb{C} : |w - z| < \delta\}$ is a neighbourhood of z in $R^c_{\gamma_{i_0 j_0}}$, as follows:

$$f_z(w) = \sum_{j=1}^{\infty} \frac{P_1 Q_j x}{w - \gamma_{1j}} + \sum_{i=2}^{\infty} \frac{P_i Q_1 x}{w - \gamma_{i1}} + \sum_{j=2}^{\infty} \frac{P_2 Q_j x}{w - \gamma_{2j}} + \sum_{i=3}^{\infty} \frac{P_i Q_2 x}{w - \gamma_{i2}} + \cdots\cdots\cdots$$
$$+ \sum_{j=i_0-1}^{\infty} \frac{P_{i_0-1} Q_j x}{w - \gamma_{i_0-1,j}} + \sum_{i=i_0}^{\infty} \frac{P_i Q_{i_0-1} x}{w - \gamma_{i,i_0-1}} + \sum_{j=i_0}^{j_0} \frac{P_{i_0} Q_j x}{w - \gamma_{i_0,j}} \text{ for all } w \in V_\delta.$$

Then $f_z : V_\delta \to \mathcal{H}$ is analytic and

$$\begin{aligned}
&(w - T) f_z(w) \\
&= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (w - \gamma_{nm}) P_n Q_m f_z(w) \\
&= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (w - \gamma_{nm}) P_n Q_m \Big(\sum_{j=1}^{\infty} \frac{P_1 Q_j x}{w - \gamma_{1j}}\Big) + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (w - \gamma_{nm}) P_n Q_m \Big(\sum_{i=2}^{\infty} \frac{P_i Q_1 x}{w - \gamma_{i1}}\Big) \\
&+ \cdots\cdots\cdots \\
&+ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (w - \gamma_{nm}) P_n Q_m \Big(\sum_{i=i_0}^{\infty} \frac{P_i Q_{i_0-1} x}{w - \gamma_{i,i_0-1}}\Big) + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (w - \gamma_{nm}) P_n Q_m \Big(\sum_{j=i_0}^{j_0} \frac{P_{i_0} Q_j x}{w - \gamma_{i_0,j}}\Big) \\
&= \sum_{m=1}^{\infty} (w - \gamma_{1m}) Q_m P_1 \Big(\sum_{j=1}^{\infty} \frac{Q_j x}{w - \gamma_{1j}}\Big) + \sum_{n=1}^{\infty} (w - \gamma_{n1}) P_n Q_1 \Big(\sum_{i=2}^{\infty} \frac{P_i x}{w - \gamma_{i1}}\Big) \\
&+ \cdots\cdots\cdots \\
&+ \sum_{n=1}^{\infty} (w - \gamma_{n,i_0-1}) P_n Q_{i_0-1} \Big(\sum_{i=i_0}^{\infty} \frac{P_i x}{w - \gamma_{i,i_0-1}}\Big) + \sum_{m=1}^{\infty} (w - \gamma_{i_0,m}) P_{i_0} Q_m \Big(\sum_{j=i_0}^{j_0} \frac{Q_j x}{w - \gamma_{i_0,j}}\Big) \\
&= x.
\end{aligned}$$

Hence, there exists an analytic function $f_z : V_\delta \to \mathcal{H}$ such that

$$(w - T) f_z(w) = x \text{ for all } w \in V_\delta \subset R^c_{\gamma_{i_0 j_0}}.$$

Therefore, $z \notin \sigma_T(x)$, and $x \in X_T(R_{\gamma_{i_0 j_0}})$. Hence, $X_{\gamma_{i_0 j_0}} \subset X_T(R_{\gamma_{i_0 j_0}})$.

Now, let $x \in X_T(R_{\gamma_{i_0 j_0}})$, i.e., $\sigma_T(x) \subset R_{\gamma_{i_0 j_0}}$. So, if $i > i_0$ and $j \geq i_0$, or $i = i_0$ and $j > j_0$, then $\gamma_{ij} \in \rho_T(x)$.

Thus, there exist analytic functions $f_{ij} : V_{ij} \to \mathcal{H}$, where V_{ij} is a neighbourhood of γ_{ij} such that

$$(t - T)f_{ij}(t) = x \text{ for all } t \in V_{ij}.$$

We want to show that

$$x \in X_{\gamma_{i_0 j_0}} = \text{clm}\{P_n Q_m \mathcal{H} : \gamma_{nm} \preceq \gamma_{i_0 j_0}\}.$$

If not, then there exist P_{k_0}, Q_{m_0} ($k_0 > i_0$ and $m_0 \geq i_0$, or $k_0 = i_0$ and $m_0 > j_0$) such that $P_{k_0} Q_{m_0} x \neq 0$.
Since

$$(\gamma_{k_0 m_0} - T) f_{k_0 m_0}(\gamma_{k_0 m_0}) = x,$$

we have

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\gamma_{k_0 m_0} - \gamma_{nm}) P_n Q_m f_{k_0 m_0}(\gamma_{k_0 m_0}) = x.$$

Thus

$$P_{k_0} Q_{m_0} \Big(\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\gamma_{k_0 m_0} - \gamma_{nm}) P_n Q_m (f_{k_0 m_0}(\gamma_{k_0 m_0}))\Big) = P_{k_0} Q_{m_0} x,$$

which implies that

$$0 = (\gamma_{k_0 m_0} - \gamma_{k_0 m_0}) P_{k_0} Q_{m_0} (f_{k_0 m_0}(\gamma_{k_0 m_0})) = P_{k_0} Q_{m_0} x \neq 0,$$

a contradiction. Hence,

$$x \in X_{\gamma_{i_0 j_0}} = \text{clm}\{P_n Q_m \mathcal{H} : \gamma_{nm} \preceq \gamma_{i_0 j_0}\}.$$

Therefore,

$$X_{\gamma_{i_0 j_0}} = X_T(R_{\gamma_{i_0 j_0}}) = \{x \in \mathcal{H} : \sigma_T(x) \subset R_{\gamma_{i_0 j_0}}\}.$$

The proof of the other cases is similar to the proof of case 2 and will be omitted.

Now, let E_γ be the orthogonal projection of $\mathcal{H}$ onto $X_\gamma = X_T(R_\gamma)$. Then since we have

$$R_\gamma \subset R_{\gamma'} \text{ for } \gamma \preceq \gamma',$$

by Theorem 1.2.1(i) it follows that

$$X_T(R_\gamma) \subset X_T(R_{\gamma'}) \text{ for } \gamma \preceq \gamma'.$$

Thus

$$E_\gamma \leq E_{\gamma'} \text{ for } \gamma \preceq \gamma'.$$

Therefore, we have

$$\begin{aligned} E_{\gamma_{11}} &\le E_{\gamma_{12}} \le E_{\gamma_{13}} \le E_{\gamma_{14}} \le \cdots \le E_{\gamma_1} \le E_{\gamma_{21}} \le E_{\gamma_{31}} \le \cdots\cdots\cdots \le E_{\delta_1} \\ &\le E_{\gamma_{22}} \le E_{\gamma_{23}} \le E_{\gamma_{24}} \le \cdots \le E_{\gamma_2} \le E_{\gamma_{32}} \le E_{\gamma_{42}} \le \cdots\cdots\cdots \le E_{\delta_2} \\ &\le E_{\gamma_{33}} \le E_{\gamma_{34}} \le E_{\gamma_{35}} \le \cdots \le E_{\gamma_3} \le E_{\gamma_{43}} \le E_{\gamma_{53}} \le \cdots\cdots\cdots \le E_{\delta_3} \\ &\le \cdots\cdots\cdots\cdots \\ &\le E_{\gamma_{kk}} \le E_{\gamma_{k,k+1}} \le \cdots \le E_{\gamma_k} \le E_{\gamma_{k+1,k}} \le E_{\gamma_{k+2,k}} \le \cdots\cdots\cdots \le E_{\delta_k} \\ &\le \cdots\cdots\cdots\cdots \\ &\le E_\eta = E_{b+id} = I. \end{aligned}$$

So we can define F_{ij} as follows:

$$F_{ij} = \begin{cases} E_{\gamma_{ij}} & \text{if } i = j = 1, \\ E_{\gamma_{ij}} - E_{\delta_{j-1}} & \text{if } i = j > 1, \\ E_{\gamma_{ij}} - E_{\gamma_{i-1}} & \text{if } i = j+1, \\ E_{\gamma_{ij}} - E_{\gamma_{i-1,j}} & \text{if } i > j+1, \\ E_{\gamma_{ij}} - E_{\gamma_{i,j-1}} & \text{if } i < j. \end{cases}$$

Then F_{ij} are commuting mutually disjoint orthogonal projections and

$$\text{clm}_{ij}\{F_{ij}\mathcal{H}\} = \mathcal{H}.$$

Define $\Xi : \Sigma_{\mathbb{C}} \to \mathcal{B}(\mathcal{H})$, where $\Sigma_{\mathbb{C}}$ is the σ-algebra of Borel subsets of $\mathbb{C}$, as follows:

$$\Xi(\Delta) = \sum_{\gamma_{ij} \in \Delta} F_{ij}.$$

We claim that Ξ is a spectral measure defined on the Borel subsets of $\mathbb{C}$. In fact, $\Xi(\Delta)$ is an orthogonal projection for each $\Delta \in \Sigma_{\mathbb{C}}$, $\Xi(\emptyset) = 0$, $\Xi(\mathbb{C}) = I$, and using the disjointness property of the projections F_{ij} we have, for any $\Delta_1, \Delta_2 \in \Sigma_{\mathbb{C}}$,

$$\Xi(\Delta_1).\Xi(\Delta_2) = \sum_{\gamma_{nm} \in \Delta_1} F_{nm} \cdot \sum_{\gamma_{st} \in \Delta_2} F_{st} = \sum_{\gamma_{\ell k} \in \Delta_1 \cap \Delta_2} F_{\ell k} = \Xi(\Delta_1 \cap \Delta_2).$$

If $\{\Delta_k\}_{k=1}^{\infty}$ is a sequence of pairwise disjoint sets in $\Sigma_{\mathbb{C}}$, then

$$\Xi(\cup_{k=1}^{\infty} \Delta_k) = \sum_{\gamma_{nm} \in \cup_{k=1}^{\infty} \Delta_k} F_{nm} = \sum_{k=1}^{\infty} \Big(\sum_{\gamma_{nm} \in \Delta_k} F_{nm} \Big) = \sum_{k=1}^{\infty} \Xi(\Delta_k).$$

Hence, Ξ is a spectral measure on the Borel subsets of $\mathbb{C}$, and therefore

$$N = \int \gamma \Xi(d\gamma) = \sum_{ij} \gamma_{ij} F_{ij}$$

is a normal operator.

Now, we want to prove that $Q = T - N$ is quasinilpotent. We have

$$\gamma_1 = \lambda_1 + id, X_{\gamma_1} = P_1\mathcal{H} \text{ and } E_{\gamma_1} = \text{ the orthogonal projection onto } X_{\gamma_1}.$$

Let

$$T_{11}^1 = T|X_{\gamma_1}$$

and let

$$\mathcal{H} = (E_{\gamma_1}\mathcal{H}) \oplus (E_{\gamma_1}\mathcal{H})^\perp = X_{\gamma_1} \oplus X_{\gamma_1}^\perp = X_1^1 \oplus X_2^1.$$

Since $X_1^1 = X_{\gamma_1} = X_T(R_{\gamma_1})$ is T-invariant, T can be represented by the matrix

$$T = \begin{bmatrix} T_{11}^1 & T_{12}^1 \\ 0 & T_2 \end{bmatrix}$$

relative to this decomposition of $\mathcal{H}$, where we put $T_2 = T_{22}^1$ for simplicity of notation.

Since T is an $AC(J \times K)$-scalar operator, by Remark 1.2.1 the operator T has the SVEP and $R_{\gamma_1} = \{a\} \times (\{\mu_1, \mu_2, \cdots\cdots\} \cup \{d\}) \subset \mathbb{C}$ is a closed set for which the space $X_{\gamma_1} = X_T(R_{\gamma_1})$ is closed. So by Theorem 1.2.2,

$$\sigma(T|X_{\gamma_1}) \subset R_{\gamma_1} \cap \sigma(T) \subset \{\gamma_{11}, \gamma_{12}, \gamma_{13}, \gamma_{14}, \cdots\cdots\} \cup \{\gamma_1\}.$$

But

$$T|X_{\gamma_1} = T_{11}^1,$$

so we have

$$\sigma(T_{11}^1) \subset \{\gamma_{11}, \gamma_{12}, \gamma_{13}, \gamma_{14}, \cdots\cdots\} \cup \{\gamma_1\}.$$

Since T is an $AC(J \times K)$-scalar operator, by Remark 1.2.1 the operator T is decomposable and $R_{\gamma_1} = \{a\} \times (\{\mu_1, \mu_2, \cdots\cdots\} \cup \{d\}) \subset \mathbb{C}$ is a closed set and hence by Theorem 1.2.6,

$$\sigma(T/X_{\gamma_1}) \subset \overline{\sigma(T) \backslash R_{\gamma_1}},$$

where T/X_{γ_1} is the operator induced by T on the quotient space $\mathcal{H}/X_{\gamma_1}$. But $T/X_{\gamma_1} \cong T_2$, so we have

$$\begin{aligned}
\sigma(T_2) \subset\ & \{\gamma_{21}, \gamma_{31}, \gamma_{41}, \gamma_{51}, \cdots\cdots\} \cup \{\delta_1\} \\
& \cup \{\gamma_{22}, \gamma_{23}, \gamma_{24}, \gamma_{25}, \cdots\cdots\} \cup \{\gamma_2\} \cup \{\gamma_{32}, \gamma_{42}, \gamma_{52}, \cdots\cdots\} \cup \{\delta_2\} \\
& \cup \cdots\cdots\cdots \\
& \cup \{\gamma_{kk}, \gamma_{k,k+1}, \gamma_{k,k+2}, \gamma_{k,k+3}, \cdots\cdots\} \cup \{\gamma_k\} \cup \{\gamma_{k+1,k}, \gamma_{k+2,k}, \gamma_{k+3,k}, \cdots\cdots\} \cup \{\delta_k\} \\
& \cup \cdots\cdots\cdots \\
& \cup \{b + id\}.
\end{aligned}$$

Also, we have

$\delta_1 = b+i\mu_1, X_{\delta_1} = (P_1+Q_1-P_1Q_1)\mathcal{H}$ and $E_{\delta_1} =$ the orthogonal projection onto X_{δ_1}.

Let

$$X_1^2 = E_{\gamma_1}\mathcal{H}, X_2^2 = (E_{\delta_1} - E_{\gamma_1})\mathcal{H} \text{ and } X_3^2 = (E_\eta - E_{\delta_1})\mathcal{H}.$$

Then

$$\mathcal{H} = X_1^2 \oplus X_2^2 \oplus X_3^2.$$

Since $X_1^2 = X_1^1 = E_{\gamma_1}\mathcal{H} = X_T(R_{\gamma_1})$ and $X_1^2 \oplus X_2^2 = X_{\delta_1} = X_T(R_{\delta_1})$ are T-invariant, T can be represented by the matrix

$$T = \begin{bmatrix} T_{11}^1 & T_{12}^2 & T_{13}^2 \\ 0 & T_{22}^2 & T_{23}^2 \\ 0 & 0 & T_3 \end{bmatrix}$$

relative to this decomposition of $\mathcal{H}$, where we put $T_3 = T_{33}^2$ to simplify the notation.

Let

$$C_2 = \begin{bmatrix} T_{11}^1 & T_{12}^2 \\ 0 & T_{22}^2 \end{bmatrix}.$$

Since $X_1^2 \oplus X_2^2 = X_{\delta_1} = X_T(R_{\delta_1})$ is T-invariant, $\mathcal{H}$ can be decomposed as

$$\mathcal{H} = X_{\delta_1} \oplus X_{\delta_1}^{\perp},$$

and relative to this decomposition of $\mathcal{H}$, the operator T can be represented by the matrix

$$T = \begin{bmatrix} C_2 & * \\ 0 & T_3 \end{bmatrix}.$$

Thus

$$C_2 = T|X_{\delta_1} = T|X_T(R_{\delta_1}).$$

Since T is an $AC(J \times K)$-scalar operator, by Remark 1.2.1 the operator T has the SVEP and $X_T(R_{\delta_1})$ is closed. Thus by Theorem 1.2.2,

$$\sigma(C_2) = \sigma(T|X_T(R_{\delta_1})) \subset R_{\delta_1} \cap \sigma(T) \subset \{\gamma_{11}, \gamma_{12}, \gamma_{13}, \gamma_{14}, \cdots\cdots\} \cup \{\gamma_1\}$$
$$\cup \{\gamma_{21}, \gamma_{31}, \gamma_{41}, \gamma_{51}, \cdots\cdots\} \cup \{\delta_1\}.$$

Now, using the last statement of Theorem 1.2.2, we have

$$\sigma_T(x) = \sigma_{T|X_T(R_{\delta_1})}(x) = \sigma_{C_2}(x) \text{ for all } x \in X_T(R_{\delta_1}).$$

Thus, $X_{C_2}(R_{\gamma_1}) = X_T(R_{\gamma_1}) = X_{\gamma_1}$.

Since

$$X_{\delta_1} = X_{\gamma_1} \oplus X_2^2 \text{ and } C_2 = \begin{bmatrix} T_{11}^1 & T_{12}^2 \\ 0 & T_{22}^2 \end{bmatrix},$$

we have $T_{22}^2 \cong C_2/X_{\gamma_1}$ on the quotient space $X_{\delta_1}/X_{\gamma_1}$. By Theorem 1.2.5, C_2 is an $AC(J \times K)$-scalar operator, so it is decomposable and R_{γ_1} is a closed subset of $\mathbb{C}$. Hence by Theorem 1.2.6,

$$\begin{aligned}\sigma(T_{22}^2) = \sigma(C_2/X_{\gamma_1}) = \sigma(C_2/X_{C_2}(R_{\gamma_1})) &\subset \overline{\sigma(C_2)\backslash R_{\gamma_1}} \\ &\subset \{\gamma_{21}, \gamma_{31}, \gamma_{41}, \gamma_{51}, \cdots\cdots\} \cup \{\delta_1\}.\end{aligned}$$

Since $T_3 \cong T/X_{\delta_1}$ on the quotient space $\mathcal{H}/X_{\delta_1}$, by Theorem 1.2.6 we have

$$\begin{aligned}\sigma(T_3) &= \sigma(T/X_{\delta_1}) = \sigma(T/X_T(R_{\delta_1})) \subset \overline{\sigma(T)\backslash R_{\delta_1}} \\ &\subset \{\gamma_{22}, \gamma_{23}, \gamma_{24}, \gamma_{25}, \cdots\cdots\} \cup \{\gamma_2\} \cup \{\gamma_{32}, \gamma_{42}, \gamma_{52}, \cdots\cdots\} \cup \{\delta_2\} \\ &\cup \cdots\cdots\cdots \\ &\cup \{\gamma_{kk}, \gamma_{k,k+1}, \gamma_{k,k+2}, \gamma_{k,k+3}, \cdots\cdots\} \cup \{\gamma_k\} \cup \{\gamma_{k+1,k}, \gamma_{k+2,k}, \gamma_{k+3,k}, \cdots\cdots\} \cup \{\delta_k\} \\ &\cup \cdots\cdots\cdots \\ &\cup \{b + id\}.\end{aligned}$$

Then, continuing in this way we get at the $2k^{\text{th}}$ step

$$\mathcal{H} = X_1^{2k} \oplus X_2^{2k} \oplus \cdots\cdots \oplus X_{2k}^{2k} \oplus X_{2k+1}^{2k}$$

and $X_1^{2k} \oplus X_2^{2k} \oplus \cdots\cdots \oplus X_j^{2k}$ is T-invariant for $j = 1, 2, \cdots\cdots, 2k$ and hence, relative to this decomposition of $\mathcal{H}$, the operator T can be represented by the matrix

$$T = \begin{bmatrix} T_{11}^1 & * & \cdots & \cdots & * \\ 0 & T_{22}^2 & * & \cdots & \cdots \\ 0 & 0 & T_{33}^3 & * & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & T_{2k,2k}^{2k} & * \\ 0 & \cdots & \cdots & 0 & T_{2k+1} \end{bmatrix},$$

where

$$\sigma(T_{2i-1,2i-1}^{2i-1}) \subset \{\gamma_{ii}, \gamma_{i,i+1}, \gamma_{i,i+2}, \gamma_{i,i+3}, \cdots\cdots\} \cup \{\gamma_i\} \text{ for } i = 1, 2, \cdots, k,$$

$$\sigma(T_{2i,2i}^{2i}) \subset \{\gamma_{i+1,i}, \gamma_{i+2,i}, \gamma_{i+3,i}, \gamma_{i+4,i}, \cdots\cdots\} \cup \{\delta_i\} \text{ for } i = 1, 2, \cdots, k,$$

$$\begin{aligned}\sigma(T_{2k+1}) \subset &\{\gamma_{k+1,k+1}, \gamma_{k+1,k+2}, \gamma_{k+1,k+3}, \gamma_{k+1,k+4}, \cdots\cdots\} \cup \{\gamma_{k+1}\} \\ &\cup \{\gamma_{k+2,k+1}, \gamma_{k+3,k+1}, \gamma_{k+4,k+1}, \cdots\cdots\} \cup \{\delta_{k+1}\} \\ &\cup \cdots\cdots\cdots \cup \{b + id\}.\end{aligned}$$

Now, decompose $P_1\mathcal{H}$ as follows:

$$P_1\mathcal{H} = E_{\gamma_{11}}\mathcal{H} \oplus (E_{\gamma_1} - E_{\gamma_{11}})\mathcal{H} = X_{\gamma_{11}} \oplus (X_{\gamma_{11}}^{\perp} \cap X_{\gamma_1}) = (F_{11}\mathcal{H}) \oplus (F_{11}\mathcal{H})^{\perp}.$$

Let

$$T^{11} = T|X_{\gamma_{11}}.$$

Since $X_{\gamma_{11}} = X_T(R_{\gamma_{11}})$ is T-invariant, we have

$$T_{11}^1 = \begin{bmatrix} T^{11} & T_{12}^{11} \\ 0 & T_2^{11} \end{bmatrix},$$

where we let $T_2^{11} = T_{22}^{11}$ to simplify the notation later. Then using Theorems 1.2.2 and 1.2.6, we get

$$\sigma(T^{11}) \subset \{\gamma_{11}\},$$

and

$$\sigma(T_2^{11}) \subset \{\gamma_{12}, \gamma_{13}, \gamma_{14}, \cdots\cdots\} \cup \{\gamma_1\}.$$

Continuing in this way, we get at the ℓ^{th} step

$$T_{11}^1 = \begin{bmatrix} T^{11} & * & \cdots & \cdots & \cdots & * \\ 0 & T^{12} & * & \cdots & \cdots & * \\ 0 & 0 & T^{13} & * & \cdots & * \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & T^{1\ell} & * \\ 0 & \cdots & \cdots & \cdots & 0 & T_{\ell+1}^{1\ell} \end{bmatrix},$$

where

$$\sigma(T^{1j}) \subset \{\gamma_{1j}\} \text{ for } j = 1, 2, \cdots, \ell$$

and

$$\sigma(T_{\ell+1}^{1\ell}) \subset \{\gamma_{1,\ell+1}, \gamma_{1,\ell+2}, \gamma_{1,\ell+3}, \cdots\cdots\} \cup \{\gamma_1\}.$$

Similarly, we get

$$T_{22}^2 = \begin{bmatrix} T^{21} & * & \cdots & \cdots & \cdots & * \\ 0 & T^{31} & * & \cdots & \cdots & * \\ 0 & 0 & T^{41} & * & \cdots & * \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & T^{\ell 1} & * \\ 0 & \cdots & \cdots & \cdots & 0 & T_{\ell+1}^{\ell 1} \end{bmatrix},$$

where

$$\sigma(T^{i1}) \subset \{\gamma_{i1}\} \text{ for } i = 2, \cdots, \ell$$

and

$$\sigma(T_{\ell+1}^{\ell 1}) \subset \{\gamma_{\ell+1,1}, \gamma_{\ell+2,1}, \gamma_{\ell+3,1}, \cdots\cdots\} \cup \{\delta_1\}.$$

Therefore, in general we have, for $i = 1, 2, \cdots\cdots, k$,

$$T^{2i-1}_{2i-1,2i-1} = \begin{bmatrix} T^{ii} & * & \cdots & \cdots & \cdots & * \\ 0 & T^{i,i+1} & * & \cdots & \cdots & * \\ 0 & 0 & T^{i,i+2} & * & \cdots & * \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & T^{i\ell} & * \\ 0 & \cdots & \cdots & \cdots & 0 & T^{i\ell}_{\ell+1} \end{bmatrix},$$

where

$$\sigma(T^{ij}) \subset \{\gamma_{ij}\} \text{ for } j = i, i+1, i+2, \cdots, \ell$$

and

$$\sigma(T^{i\ell}_{\ell+1}) \subset \{\gamma_{i,\ell+1}, \gamma_{i,\ell+2}, \gamma_{i,\ell+3}, \cdots\cdots\} \cup \{\gamma_i\},$$

and

$$T^{2i}_{2i,2i} = \begin{bmatrix} T^{i+1,i} & * & \cdots & \cdots & \cdots & * \\ 0 & T^{i+2,i} & * & \cdots & \cdots & * \\ 0 & 0 & T^{i+3,i} & * & \cdots & * \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & T^{\ell i} & * \\ 0 & \cdots & \cdots & \cdots & 0 & T^{\ell i}_{\ell+1} \end{bmatrix},$$

where

$$\sigma(T^{ji}) \subset \{\gamma_{ji}\} \text{ for } j = i+1, \cdots\cdots, \ell$$

and

$$\sigma(T^{\ell i}_{\ell+1}) \subset \{\gamma_{\ell+1,i}, \gamma_{\ell+2,i}, \gamma_{\ell+3,i}, \cdots\cdots\} \cup \{\delta_i\}.$$

Let $\epsilon > 0$ be given. Choose $\beta > \sqrt{2}$ such that

$$M[\frac{\sqrt{2}}{\beta} + \frac{2\sqrt{2}}{(\beta-1)^2} + \frac{2\sqrt{2}}{(\beta-\sqrt{2})^3}] < 1,$$

where M is the constant defined in the statement of the Theorem.

Since $\lambda_n \uparrow b$ and $\mu_m \uparrow d$, we can choose an even integer $\mathcal{N} \in \mathbb{N}$ and a rectangular partition of $J \times K$ into subrectangles such that each subrectangle contains at most one point of $\{\gamma_{ij} : 1 \leq i, j \leq \mathcal{N}\}$ and such that

$$|b - \lambda_n| < \min\{\frac{\epsilon}{\beta}, \frac{\epsilon}{4 \parallel T \parallel}\} \text{ and } |d - \mu_m| < \min\{\frac{\epsilon}{\beta}, \frac{\epsilon}{4 \parallel T \parallel}\} \text{ for all } n, m \geq \mathcal{N},$$

so that

$$|(b + id) - \gamma_{nm}| < \min\{\sqrt{2}\frac{\epsilon}{\beta}, \frac{\sqrt{2}\epsilon}{4 \parallel T \parallel}\} \text{ for all } n, m \geq \mathcal{N}.$$

Since $Q = T - N$ and $N = \sum_{ij} \gamma_{ij} F_{ij}$, we have

$$\begin{aligned}\sigma(Q) \subset\ & \sigma(T^{11} - N^{11}) \cup \cdots\cdots\cdots \cup \sigma(T^{1\mathcal{N}} - N^{1\mathcal{N}}) \\ & \cup \sigma(T^{1\mathcal{N}}_{\mathcal{N}+1} - N^{1\mathcal{N}}_{\mathcal{N}+1}) \\ & \cup \sigma(T^{21} - N^{21}) \cup \cdots\cdots\cdots \cup \sigma(T^{\mathcal{N}1} - N^{\mathcal{N}1}) \\ & \cup \sigma(T^{\mathcal{N}1}_{\mathcal{N}+1} - N^{\mathcal{N}1}_{\mathcal{N}+1}) \\ & \cup \cdots\cdots\cdots\cdots\cdots\cdots \\ & \cup \sigma(T^{\mathcal{N}-1,\mathcal{N}-1} - N^{\mathcal{N}-1,\mathcal{N}-1}) \cup \sigma(T^{\mathcal{N}-1,\mathcal{N}} - N^{\mathcal{N}-1,\mathcal{N}}) \\ & \cup \sigma(T^{\mathcal{N}-1,\mathcal{N}}_{\mathcal{N}+1} - N^{\mathcal{N}-1,\mathcal{N}}_{\mathcal{N}+1}) \\ & \cup \sigma(T^{\mathcal{N},\mathcal{N}-1} - N^{\mathcal{N},\mathcal{N}-1}) \cup \sigma(T^{\mathcal{N},\mathcal{N}-1}_{\mathcal{N}+1} - N^{\mathcal{N},\mathcal{N}-1}_{\mathcal{N}+1}) \\ & \cup \sigma(T_{2\mathcal{N}-1} - N_{2\mathcal{N}-1}).\end{aligned}$$

We know that

$$\sigma(T^{ij} - N^{ij}) = \{0\} \text{ for all } 1 \le i, j \le \mathcal{N}.$$

So in order to see that Q is quasinilpotent, we need to prove the following:
(1) $\sigma(T^{i\mathcal{N}}_{\mathcal{N}+1} - N^{i\mathcal{N}}_{\mathcal{N}+1}) \subset \{\gamma : |\gamma| \le \epsilon\}$ for $i = 1, \cdots, \mathcal{N} - 1$,
(2) $\sigma(T^{\mathcal{N}i}_{\mathcal{N}+1} - N^{\mathcal{N}i}_{\mathcal{N}+1}) \subset \{\gamma : |\gamma| \le \epsilon\}$ for $i = 1, \cdots, \mathcal{N} - 1$,
(3) $\sigma(T_{2\mathcal{N}-1} - N_{2\mathcal{N}-1}) \subset \{\gamma : |\gamma| \le \epsilon\}$.

(1) Fix $i \in \{1, \cdots\cdots, \mathcal{N} - 1\}$. Then

$$\begin{aligned}\sigma(T^{i\mathcal{N}}_{\mathcal{N}+1} - \gamma_i I^{i\mathcal{N}}_{\mathcal{N}+1}) &\subset \{\gamma_{i,\mathcal{N}+1} - \gamma_i, \gamma_{i,\mathcal{N}+2} - \gamma_i, \gamma_{i,\mathcal{N}+3} - \gamma_i, \cdots\cdots\} \cup \{\gamma_i - \gamma_i\} \\ &\subset \{0\} \times [\mu_{\mathcal{N}} - d, 0].\end{aligned}$$

Since $T^{i\mathcal{N}}_{\mathcal{N}+1} - \gamma_i I^{i\mathcal{N}}_{\mathcal{N}+1}$ has an AC-functional calculus on $(J \times K) - \gamma_i$, and

$$\sigma(T^{i\mathcal{N}}_{\mathcal{N}+1} - \gamma_i I^{i\mathcal{N}}_{\mathcal{N}+1}) \subset \{0\} \times [\mu_{\mathcal{N}} - d, 0],$$

by Lemma 4.1.1 it follows that $T^{i\mathcal{N}}_{\mathcal{N}+1} - \gamma_i I^{i\mathcal{N}}_{\mathcal{N}+1}$ has an AC-functional calculus on $\{0\} \times [\mu_{\mathcal{N}} - d, 0]$ such that

$$\| f(T^{i\mathcal{N}}_{\mathcal{N}+1} - \gamma_i I^{i\mathcal{N}}_{\mathcal{N}+1}) \| \le M |||f|||_{AC(\{0\}\times[\mu_{\mathcal{N}}-d,0])} \text{ for all } f \in AC(\{0\} \times [\mu_{\mathcal{N}} - d, 0]).$$

Fix $\gamma \in \mathbb{C}$ such that $|\gamma| > \epsilon$. Since

$$f_0(z) = (\gamma - z)^{-1} \text{ is in } AC(\{0\} \times [\mu_{\mathcal{N}} - d, 0]),$$

we have

$$\| [\gamma - (T^{i\mathcal{N}}_{\mathcal{N}+1} - \gamma_i I^{i\mathcal{N}}_{\mathcal{N}+1})]^{-1} \| \le M |||f_0|||_{AC(\{0\}\times[\mu_{\mathcal{N}}-d,0])}.$$

But

$$
\begin{aligned}
|||f_0|||_{AC(\{0\}\times[\mu_{\mathcal{N}}-d,0])} &= |f_0(0,0)| + \int_{\mu_{\mathcal{N}}-d}^{0} \frac{1}{|\gamma - t|^2}dt \\
&\le \frac{1}{|\gamma|} + \int_{\mu_{\mathcal{N}}-d}^{0} \frac{1}{(|\gamma| - |t|)^2}dt \\
&< \frac{1}{\epsilon} + (\frac{\beta}{(\beta-1)\epsilon})^2 \frac{\epsilon}{\beta} \\
&= \frac{1}{\epsilon} + \frac{\beta}{(\beta-1)^2\epsilon} \\
&= \frac{\beta}{\epsilon}[\frac{1}{\beta} + \frac{1}{(\beta-1)^2}] \\
&< \frac{\beta}{\epsilon}[\frac{\sqrt{2}}{\beta} + \frac{2\sqrt{2}}{(\beta-1)^2} + \frac{2\sqrt{2}}{(\beta-\sqrt{2})^3}] \\
&< \frac{\beta}{\epsilon}\frac{1}{M}.
\end{aligned}
$$

Thus,

$$\| \left(\gamma - (T^{i\mathcal{N}}_{\mathcal{N}+1} - \gamma_i I^{i\mathcal{N}}_{\mathcal{N}+1})\right)^{-1} \| < \frac{\beta}{\epsilon}.$$

Since $N^{i\mathcal{N}}_{\mathcal{N}+1}$ is a normal operator, we have

$$\| N^{i\mathcal{N}}_{\mathcal{N}+1} - \gamma_i I^{i\mathcal{N}}_{\mathcal{N}+1} \| = r(N^{i\mathcal{N}}_{\mathcal{N}+1} - \gamma_i I^{i\mathcal{N}}_{\mathcal{N}+1}) \le |\mu_{\mathcal{N}} - d| < \frac{\epsilon}{\beta}.$$

Hence,

$$\| \left(\gamma - (T^{i\mathcal{N}}_{\mathcal{N}+1} - \gamma_i I^{i\mathcal{N}}_{\mathcal{N}+1})\right)^{-1} \left(N^{i\mathcal{N}}_{\mathcal{N}+1} - \gamma_i I^{i\mathcal{N}}_{\mathcal{N}+1}\right) \| < 1.$$

Thus, $I^{i\mathcal{N}}_{\mathcal{N}+1} - \left(\gamma - (T^{i\mathcal{N}}_{\mathcal{N}+1} - \gamma_i I^{i\mathcal{N}}_{\mathcal{N}+1})\right)^{-1} \left(N^{i\mathcal{N}}_{\mathcal{N}+1} - \gamma_i I^{i\mathcal{N}}_{\mathcal{N}+1}\right)$ is invertible. Hence

$$
\begin{aligned}
&\gamma - (T^{i\mathcal{N}}_{\mathcal{N}+1} - N^{i\mathcal{N}}_{\mathcal{N}+1}) = \gamma - \left((T^{i\mathcal{N}}_{\mathcal{N}+1} - \gamma_i I^{i\mathcal{N}}_{\mathcal{N}+1}) + (\gamma_i I^{i\mathcal{N}}_{\mathcal{N}+1} - N^{i\mathcal{N}}_{\mathcal{N}+1})\right) \\
&= \left(\gamma - (T^{i\mathcal{N}}_{\mathcal{N}+1} - \gamma_i I^{i\mathcal{N}}_{\mathcal{N}+1})\right) \left(I^{i\mathcal{N}}_{\mathcal{N}+1} - (\gamma - (T^{i\mathcal{N}}_{\mathcal{N}+1} - \gamma_i I^{i\mathcal{N}}_{\mathcal{N}+1}))^{-1}(\gamma_i I^{i\mathcal{N}}_{\mathcal{N}+1} - N^{i\mathcal{N}}_{\mathcal{N}+1})\right)
\end{aligned}
$$

is invertible. Therefore, $\gamma \notin \sigma(T^{i\mathcal{N}}_{\mathcal{N}+1} - N^{i\mathcal{N}}_{\mathcal{N}+1})$, and (1) is proved.

(2) is very similar to (1).

(3) We have

$$T_{2\mathcal{N}-1} - N_{2\mathcal{N}-1} = (T_{2\mathcal{N}-1} - \eta I_{2\mathcal{N}-1}) + (\eta I_{2\mathcal{N}-1} - N_{2\mathcal{N}-1}),$$

$$
\begin{aligned}
\sigma(T_{2\mathcal{N}-1}) &\subset \{\gamma_{ij} : i \ge \mathcal{N}, j \ge \mathcal{N}\} \cup \{\gamma_i : i \ge \mathcal{N}\} \cup \{\delta_j : j \ge \mathcal{N}\} \cup \{b + id\} \\
&\subset [\lambda_{\mathcal{N}}, b] \times [\mu_{\mathcal{N}}, d]
\end{aligned}
$$

and

$$
\begin{aligned}
\sigma(N_{2\mathcal{N}-1}) &\subset \{\gamma_{ij} : i \ge \mathcal{N}, j \ge \mathcal{N}\} \cup \{\gamma_i : i \ge \mathcal{N}\} \cup \{\delta_j : j \ge \mathcal{N}\} \cup \{b + id\} \\
&\subset [\lambda_{\mathcal{N}}, b] \times [\mu_{\mathcal{N}}, d].
\end{aligned}
$$

So

$$\| \eta I_{2\mathcal{N}-1} - N_{2\mathcal{N}-1} \| = r(\eta I_{2\mathcal{N}-1} - N_{2\mathcal{N}-1}) < \sqrt{2}\frac{\epsilon}{\beta},$$

and

$$\sigma(T_{2\mathcal{N}-1} - \eta I_{2\mathcal{N}-1}) \subset [\lambda_{\mathcal{N}} - b, 0] \times [\mu_{\mathcal{N}} - d, 0],$$

and by Lemma 4.1.1 it follows that $T_{2\mathcal{N}-1} - \eta I_{2\mathcal{N}-1}$ has an AC-functional calculus on $J_1 \times K_1 = [\lambda_{\mathcal{N}} - b, 0] \times [\mu_{\mathcal{N}} - d, 0]$.

Suppose that $|\gamma| > \epsilon$. Then

$$\begin{aligned}&\gamma - (T_{2\mathcal{N}-1} - N_{2\mathcal{N}-1})\\ &\quad = (\gamma - (T_{2\mathcal{N}-1} - \eta I_{2\mathcal{N}-1})\big(\big(I_{2\mathcal{N}-1} - (\gamma - (T_{2\mathcal{N}-1} - \eta I_{2\mathcal{N}-1}))^{-1}(\eta I_{2\mathcal{N}-1} - N_{2\mathcal{N}-1})\big)\end{aligned}$$

is invertible if

$$\| (\gamma - (T_{2\mathcal{N}-1} - \eta I_{2\mathcal{N}-1}))^{-1}(\eta I_{2\mathcal{N}-1} - N_{2\mathcal{N}-1}) \| < 1.$$

Since $\| \eta I_{2\mathcal{N}-1} - N_{2\mathcal{N}-1} \| < \sqrt{2}\frac{\epsilon}{\beta}$, we only need to prove that

$$\| (\gamma - (T_{2\mathcal{N}-1} - \eta I_{2\mathcal{N}-1}))^{-1} \| < \frac{\beta}{\sqrt{2}\epsilon}$$

to complete the proof that Q is quasinilpotent.

Since $f_1(z) = (\gamma - z)^{-1}$ is in $AC(J_1 \times K_1)$, we have

$$\| f_1(T_{2\mathcal{N}-1} - \eta I_{2\mathcal{N}-1}) \| \leq M|||f_1|||_{AC(J_1\times K_1)},$$

so

$$\| (\gamma - (T_{2\mathcal{N}-1} - \eta I_{2\mathcal{N}-1}))^{-1} \| \leq M|||f_1|||_{AC(J_1\times K_1)}.$$

But,

$$\begin{aligned}|||f_1|||_{AC(J_1\times K_1)} &= |f_1(0,0)| + \int_{\lambda_{\mathcal{N}}-b}^{0} |\frac{\partial f_1}{\partial s}(s,0)|ds + \int_{\mu_{\mathcal{N}}-d}^{0} |\frac{\partial f_1}{\partial t}(0,t)|dt \\ &\quad + \int\int_{J_1\times K_1} |\frac{\partial^2 f_1}{\partial s \partial t}(s,t)|dsdt \\ &= \frac{1}{|\gamma|} + \int_{\lambda_{\mathcal{N}}-b}^{0} \frac{1}{|\gamma - s|^2}ds + \int_{\mu_{\mathcal{N}}-d}^{0} \frac{1}{|\gamma - t|^2}dt + \int\int_{J_1\times K_1} |\frac{2}{|\gamma - z|^3}dsdt \\ &< \frac{1}{\epsilon} + 2\frac{\beta^2}{(\beta-1)^2\epsilon^2}\frac{\epsilon}{\beta} + 2\frac{\beta^3}{(\beta-\sqrt{2})^3\epsilon^3}\frac{\epsilon^2}{\beta^2} \\ &= \frac{\beta}{\sqrt{2}\epsilon}[\frac{\sqrt{2}}{\beta} + \frac{2\sqrt{2}}{(\beta-1)^2} + \frac{2\sqrt{2}}{(\beta-\sqrt{2})^3}] \\ &< \frac{\beta}{\sqrt{2}\epsilon}\frac{1}{M}.\end{aligned}$$

Hence,

$$\| [\gamma - (T_{2\mathcal{N}-1} - \eta I_{2\mathcal{N}-1})]^{-1} \| \leq \frac{\beta}{\sqrt{2}\epsilon}$$

and the proof that Q is quasinilpotent is complete.

Finally, we want to prove that $NQ - QN$ is quasinilpotent. We have

$$\begin{aligned} NQ - QN = NT - TN = \\ & \operatorname{diag}\{N^{11}T^{11} - T^{11}N^{11}, \cdots\cdots, N^{1\mathcal{N}}T^{1\mathcal{N}} - T^{1\mathcal{N}}N^{1\mathcal{N}}, \\ & N^{1\mathcal{N}}_{\mathcal{N}+1}T^{1\mathcal{N}}_{\mathcal{N}+1} - T^{1\mathcal{N}}_{\mathcal{N}+1}N^{1\mathcal{N}}_{\mathcal{N}+1}, \\ & N^{21}T^{21} - T^{21}N^{21}, \cdots\cdots, N^{\mathcal{N}1}T^{\mathcal{N}1} - T^{\mathcal{N}1}N^{\mathcal{N}1}, \\ & N^{\mathcal{N}1}_{\mathcal{N}+1}T^{\mathcal{N}1}_{\mathcal{N}+1} - T^{\mathcal{N}1}_{\mathcal{N}+1}N^{\mathcal{N}1}_{\mathcal{N}+1}, \\ & \cdots\cdots\cdots\cdots\cdots\cdots \\ & N^{\mathcal{N}-1,\mathcal{N}-1}T^{\mathcal{N}-1,\mathcal{N}-1} - T^{\mathcal{N}-1,\mathcal{N}-1}N^{\mathcal{N}-1,\mathcal{N}-1}, \\ & N^{\mathcal{N}-1,\mathcal{N}}T^{\mathcal{N}-1,\mathcal{N}} - T^{\mathcal{N}-1,\mathcal{N}}N^{\mathcal{N}-1,\mathcal{N}}, \\ & N^{\mathcal{N}-1,\mathcal{N}}_{\mathcal{N}+1}T^{\mathcal{N}-1,\mathcal{N}}_{\mathcal{N}+1} - T^{\mathcal{N}-1,\mathcal{N}}_{\mathcal{N}+1}N^{\mathcal{N}-1,\mathcal{N}}_{\mathcal{N}+1}, \\ & N^{\mathcal{N},\mathcal{N}-1}T^{\mathcal{N},\mathcal{N}-1} - T^{\mathcal{N},\mathcal{N}-1}N^{\mathcal{N},\mathcal{N}-1}, \\ & N^{\mathcal{N},\mathcal{N}-1}_{\mathcal{N}+1}T^{\mathcal{N},\mathcal{N}-1}_{\mathcal{N}+1} - T^{\mathcal{N},\mathcal{N}-1}_{\mathcal{N}+1}N^{\mathcal{N},\mathcal{N}-1}_{\mathcal{N}+1}, \\ & N_{2\mathcal{N}-1}T_{2\mathcal{N}-1} - T_{2\mathcal{N}-1}N_{2\mathcal{N}-1}\}. \end{aligned}$$

Now $T^{ij} = N^{ij} = \gamma_{ij}I^{ij}$, and $T^{i\mathcal{N}}_{\mathcal{N}+1}$, $T^{\mathcal{N}i}_{\mathcal{N}+1}$ and $T_{2\mathcal{N}-1}$ are compressions of T, so the norm of each one of them is $\leq \| T \|$. Hence, we have

$$r(N^{ij}T^{ij} - T^{ij}N^{ij}) = 0 < \epsilon,$$

$$\begin{aligned} r(N^{i\mathcal{N}}_{\mathcal{N}+1}T^{i\mathcal{N}}_{\mathcal{N}+1} - T^{i\mathcal{N}}_{\mathcal{N}+1}N^{i\mathcal{N}}_{\mathcal{N}+1}) &\leq \| N^{i\mathcal{N}}_{\mathcal{N}+1}T^{i\mathcal{N}}_{\mathcal{N}+1} - T^{i\mathcal{N}}_{\mathcal{N}+1}N^{i\mathcal{N}}_{\mathcal{N}+1} \| \\ &= \| (N^{i\mathcal{N}}_{\mathcal{N}+1} - \gamma_i)T^{i\mathcal{N}}_{\mathcal{N}+1} + T^{i\mathcal{N}}_{\mathcal{N}+1}(\gamma_i - N^{i\mathcal{N}}_{\mathcal{N}+1}) \| \\ &\leq 2 \| N^{i\mathcal{N}}_{\mathcal{N}+1} - \gamma_i \| \| T^{i\mathcal{N}}_{\mathcal{N}+1} \| \\ &= 2r(N^{i\mathcal{N}}_{\mathcal{N}+1} - \gamma_i) \| T^{i\mathcal{N}}_{\mathcal{N}+1} \| \\ &\leq 2|\mu_{\mathcal{N}} - d| \| T \| \\ &< 2\frac{\epsilon}{4 \| T \|} \| T \| < \epsilon, \end{aligned}$$

$$\begin{aligned} r(N^{\mathcal{N}i}_{\mathcal{N}+1}T^{\mathcal{N}i}_{\mathcal{N}+1} - T^{\mathcal{N}i}_{\mathcal{N}+1}N^{\mathcal{N}i}_{\mathcal{N}+1}) &\leq \| N^{\mathcal{N}i}_{\mathcal{N}+1}T^{\mathcal{N}i}_{\mathcal{N}+1} - T^{\mathcal{N}i}_{\mathcal{N}+1}N^{\mathcal{N}i}_{\mathcal{N}+1} \| \\ &= \| (N^{\mathcal{N}i}_{\mathcal{N}+1} - \delta_i)T^{\mathcal{N}i}_{\mathcal{N}+1} + T^{\mathcal{N}i}_{\mathcal{N}+1}(\delta_i - N^{\mathcal{N}i}_{\mathcal{N}+1}) \| \\ &\leq 2 \| N^{\mathcal{N}i}_{\mathcal{N}+1} - \delta_i \| \| T^{\mathcal{N}i}_{\mathcal{N}+1} \| \\ &= 2r(N^{\mathcal{N}i}_{\mathcal{N}+1} - \delta_i) \| T^{i\mathcal{N}}_{\mathcal{N}+1} \| \\ &\leq 2|\lambda_{\mathcal{N}} - b| \| T \| \\ &< 2\frac{\epsilon}{4 \| T \|} \| T \| < \epsilon \end{aligned}$$

and

$$\begin{aligned} r(N_{2\mathcal{N}-1}T_{2\mathcal{N}-1} - T_{2\mathcal{N}-1}N_{2\mathcal{N}-1}) &\leq \| N_{2\mathcal{N}-1}T_{2\mathcal{N}-1} - T_{2\mathcal{N}-1}N_{2\mathcal{N}-1} \| \\ &\leq 2 \| N_{2\mathcal{N}-1} - (b+id) \| \| T_{2\mathcal{N}-1} \| \\ &= 2r(N_{2\mathcal{N}-1} - (b+id)) \| T_{2\mathcal{N}-1} \| \\ &< 2\frac{\sqrt{2}\epsilon}{4 \| T \|} \| T \| < \epsilon. \end{aligned}$$

Hence, $NQ - QN$ is quasinilpotent.

□

4.2 General AC-Operators

In this section we discuss different approaches for extending the main result of the previous section to any AC-operator on $\mathcal{H}$. In particular, we attempt to prove the following apparently difficult conjecture.

Conjecture 4.2.1. *Any AC-operator T on a Hilbert space $\mathcal{H}$ can be decomposed in a canonical way as $T = N + Q$, where N is a normal operator and Q is a quasinilpotent operator such that $NQ - QN$ is also quasinilpotent.*

The main problem here is the nonexistence of a total order on $\sigma(T)$ if $\sigma(T)$ is not discrete. Thus it is not clear how to find a family of subspaces of $\mathcal{H}$ so that the orthogonal projections on them might give a spectral measure as in the preceding section to define the normal operator N on $\mathcal{H}$. On the other hand, there are some total orders on $\sigma(T)$ if $\sigma(T)$ is not discrete which will give a spectral measure which we need to define the normal operator N but will not allow us to squeeze $\sigma(T_{ij}^{\mathcal{N}} - N_{ij}^{\mathcal{N}})$ in a rectangle which is small enough to prove that $Q = T - N$ is quasinilpotent. We also considered the situation when one of the two commuting well-bounded operators which define the AC-operator has a discrete spectrum and the other one has a non-discrete spectrum but even in this case there are difficulties.

Also, as in the previous chapter we ask when the sum of a normal operator and a quasinilpotent operator on $\mathcal{H}$ is an AC-operator. Of course, in general, the sum of a normal operator and a quasinilpotent operator on $\mathcal{H}$ is not an AC-operator. In fact, Example 3.3.1 is an example of a normal operator and a quasinilpotent operator on $\mathcal{H}$ whose sum is not an AC-operator. However, Theorem 3.3.1 can be generalised to AC-operators as follows:

Theorem 4.2.1. *If N is a normal operator on $\mathcal{H}$ and Q is a nonzero quasinilpotent operator on $\mathcal{H}$ such that $NQ = QN$, then $T = N + Q$ is not an AC-operator on $\mathcal{H}$.*

Proof. Since N is normal, N can be expressed uniquely as $N = H + iK$ where H and K are self-adjoint and $HK = KH$. Since $NQ = QN$, we have $(H + iK)Q = Q(H + iK)$. So by [18], $HQ = QH$ and $KQ = QK$. Since $T = N + Q = H + iK + Q$, we have $HT = TH$, $KT = TK$ and $QT = TQ$.

If T were an AC-operator, then there would exist two commuting well-bounded operators A and B such that $T = A + iB$, and by Theorem 1.1.4, any operator $S \in \mathcal{B}(\mathcal{H})$ commuting with $T = A + iB$ must commute with A and B. Hence $HA = AH$, $KB = BK$ and $QA = AQ$.

Now, since H is a scalar-type spectral operator on $\mathcal{H}$ with real spectrum such that $AH = HA$, by ([20], Theorem p.171) we know that $A - H$ is well-bounded. Similarly, since K is a scalar-type spectral operator on $\mathcal{H}$ with real spectrum such that $BK = KB$, by ([20], Theorem p.171) we know that $K - B$ is well-bounded.

We have $A + iB = H + iK + Q$, so $A - H - Q = i(K - B)$. Since Q is a quasinilpotent that commutes with $A - H$ and $A - H$ is well-bounded, we have $\sigma(A - H - Q) = \sigma(A - H) \subset \mathbb{R}$. Hence $\sigma(i(K - B)) \subset \mathbb{R}$ and so $\sigma(K - B) \subset i\mathbb{R}$. However, $\sigma(K - B) \subset \mathbb{R}$ since $K - B$ is well-bounded. Therefore, $\sigma(K - B) = \{0\}$, showing that $K - B$ is a quasinilpotent well-bounded operator. Hence $K - B = 0$. Thus $A - H - Q = 0$, which implies that $Q = A - H$ is a well-bounded quasinilpotent operator and so $Q = 0$ which is a contradiction to our assumption that $Q \neq 0$. Therefore, T cannot be an AC-operator. □

Next, we prove a stronger version of the previous theorem using the fact that the property of being an AC-operator is invariant under a similarity transformation.

Theorem 4.2.2. *If S is a scalar-type spectral operator on $\mathcal{H}$ and Q is a nonzero quasinilpotent operator on $\mathcal{H}$ such that $SQ = QS$, then $T = S + Q$ is not an AC-operator on $\mathcal{H}$.*

Proof. Since S is a scalar-type spectral operator on $\mathcal{H}$, it is similar to a normal operator N on $\mathcal{H}$, i.e., $S = VNV^{-1}$ for some invertible operator V on $\mathcal{H}$. Since $T = S + Q$, we have $V^{-1}TV = N + V^{-1}QV$. Now $V^{-1}QV$ is a nonzero quasinilpotent operator that commutes with N, so by the previous theorem $V^{-1}TV$ is not an AC-operator. Therefore, T is not an AC-operator on $\mathcal{H}$. □

The following result shows that the converse to Theorem 3.2.1 is not true in general.

Theorem 4.2.3. *If U is a unitary operator on $\mathcal{H}$ and Q is a nonzero quasinilpotent operator on $\mathcal{H}$ such that $UQ = QU$, then $T = U(Q + I)$ is not a trigonometrically well-bounded operator on $\mathcal{H}$.*

Proof. Since $UQ = QU$, by ([8], Corollary 3 p.19) we have

$$0 \leq r(UQ) \leq r(U)r(Q) = 0, \text{ so } r(UQ) = 0, \text{ i.e., } UQ \text{ is quasinilpotent.}$$

Since U is unitary, U is normal and so the sum $U + UQ$ cannot be an AC-operator by Theorem 4.2.1 and therefore by Theorem 1.1.5, $T = U + UQ$ cannot be a trigonometrically well-bounded operator. □

Chapter 5

Operators with an AC_2-Functional Calculus on $\mathcal{H}$

Let $T = M_t + V$, where $M_t f(t) = tf(t)$ and $Vf(t) = \int_0^t f(s)ds$ with f defined on $[0, 1]$. It has been shown by Ringrose ([34], p.631) that T is a well-bounded operator as an operator acting on $L^1([0, 1])$. In the present chapter we consider the operator $T = M_t + V$ as an operator acting on $L^2([0, 1])$. We shall replace the L^1 norm by the L^2 norm of the derivative in the definition of the AC norm and call the new norm an AC_2 norm and introduce the notion of operators with an AC_2-functional calculus as a generalisation of operators with an AC-functional calculus, i.e., the well-bounded operators. Then we show that the operator $T = M_t + V$ is an operator with an AC_2-functional calculus on $L^2([0, 1])$ and that T can be decomposed in a canonical way as in Chapters 3 and 4 as a sum of a self-adjoint operator and a quasinilpotent operator. Of course, $T = M_t + V$ is also a decomposition of T as a sum of a self-adjoint operator and a quasinilpotent operator, but this decomposition is not canonical. Then we conclude the chapter by leaving as an open question whether or not every operator T with an AC_2-functional calculus on $\mathcal{H}$ is a sum of a self-adjoint operator and a quasinilpotent operator after highlighting the main obstacle to proving this.

5.1 Operators with an AC_2-Functional Calculus on $\mathcal{H}$

Let $J = [a, b]$ and let $AC_2([a, b]) = \{f \in AC([a, b]) : ||f||_{AC_2(J)} < \infty\}$, where

$$||f||_{AC_2(J)} = (\int_a^b |f'|^2)^{1/2} + ||f||_\infty.$$

We shall say that an operator $T \in \mathcal{B}(\mathcal{H})$ is an operator with an $AC_2(J)$-functional calculus if there exist a constant K and a compact interval $J = [a, b]$ such that

$$\| p(T) \| \leq K \| p \|_{AC_2(J)} \text{ for all polynomials } p.$$

In this section we give an example of an operator T on $L^2([0,1])$ with an $AC_2([0,1])$-functional calculus, which was inspired by an example due to Ringrose ([34], p.631), and which has the decomposition $T = A + Q$, where A is self-adjoint and Q is quasinilpotent and the construction of A is canonical as in Chapters 3 and 4. We begin with the following lemma.

Lemma 5.1.1. *The algebra $AC_2([a,b])$ with the norm $\| \cdot \|_{AC_2(J)}$ is a Banach algebra in which the complex polynomials are dense.*

Proof. Let $p, q \in AC_2([a,b])$. Then

$$\begin{aligned}
\| pq \|_{AC_2(J)} &= (\int_a^b |(pq)'|^2)^{\frac{1}{2}} + \| pq \|_\infty \\
&\leq (\int_a^b |pq' + p'q|^2)^{\frac{1}{2}} + \| p \|_\infty \| q \|_\infty \\
&= \| pq' + p'q \|_2 + \| p \|_\infty \| q \|_\infty \\
&\leq \| pq' \|_2 + \| p'q \|_2 + \| p \|_\infty \| q \|_\infty \\
&\leq \| p \|_\infty \| q' \|_2 + \| p' \|_2 \| q \|_\infty + \| p \|_\infty \| q \|_\infty \\
&\leq (\| p' \|_2 + \| p \|_\infty)(\| q' \|_2 + \| q \|_\infty) \\
&= \| p \|_{AC_2(J)} \| q \|_{AC_2(J)} \,.
\end{aligned}$$

Thus, $\| \cdot \|_{AC_2(J)}$ is an algebra norm.
Let $\{p_n\}$ be a Cauchy sequence in $AC_2(J)$ with respect to the norm $\| \cdot \|_{AC_2(J)}$, i.e., $\| p_n - p_m \|_{AC_2(J)} \to 0$ as $n, m \to \infty$, so

$$\| p_n' - p_m' \|_2 + \| p_n - p_m \|_\infty \to 0 \text{ as } n, m \to \infty.$$

It follows that $\| p_n' - p_m' \|_2 \to 0$ and $\| p_n - p_m \|_\infty \to 0$ as $n, m \to \infty$.
Since $\| p_n - p_m \|_\infty \to 0$ as $n, m \to \infty$, there is a continuous function f such that $p_n \to f$ uniformly. Hence, $\| p_n - f \|_\infty \to 0$ as $n \to \infty$.
Since $\| p_n' - p_m' \|_2 \to 0$ as $n, m \to 0$, and $L^2([a,b])$ is complete, there is a function $g \in L^2([a,b])$ such that $p_n' \to g$ in $L^2([a,b])$, i.e., $\| p_n' - g \|_2 \to 0$ as $n \to \infty$.
Also, since the convergence in $L^2([a,b])$ implies the convergence in $L^1([a,b])$, $p_n' \to g$ in $L^1([a,b])$, i.e., $\| p_n' - g \|_1 \to 0$ as $n \to \infty$, and hence $p_n \to f$ in $AC([a,b])$ because $\| p_n - f \|_{AC([a,b])} \leq \| p_n - f \|_\infty + \| p_n' - g \|_1 \to 0$ as $n \to \infty$. Since $AC([a,b])$ is complete and $\{p_n\}$ is a Cauchy sequence in $AC([a,b])$ converging to f, it follows that $f \in AC([a,b])$ and $f' = g \in L^2([a,b])$.
Therefore, $\| f \|_{AC_2(J)} = \| f' \|_2 + \| f \|_\infty < \infty$, i.e., $f \in AC_2([a,b])$, and

$$\begin{aligned}
\| p_n - f \|_{AC_2(J)} &= \| p_n' - f' \|_2 + \| p_n - f \|_\infty \\
&= \| p_n' - g \|_2 + \| p_n - f \|_\infty \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Hence, $p_n \to f$ in $AC_2(J)$. Therefore, $AC_2([a,b])$ with the norm $\|\cdot\|_{AC_2(J)}$ is a Banach algebra.

Now, if $f \in AC_2([a,b])$ is given, then $f' \in L^2([a,b])$. So there exists a sequence $\{q_n\}$ of polynomials such that

$$\| q_n - f' \|_2 = (\int_J |q_n - f'|^2)^{\frac{1}{2}} \to 0 \text{ as } n \to \infty.$$

Define

$$p_n(t) = -\int_t^b q_n(u)du + f(b).$$

Then p_n is a polynomial and

$$\begin{aligned}\| p_n - f \|_{AC_2(J)} &= \| q_n - f' \|_2 + \| p_n - f \|_\infty \\ &\leq \| q_n - f' \|_2 + \| p_n' - f' \|_1 + |(p_n - f)(b)| \\ &= \| q_n - f' \|_2 + \| p_n' - f' \|_1 \\ &\leq (1 + (b-a)^{\frac{1}{2}}) \| q_n - f' \|_2 \to 0 \text{ as } n \to \infty.\end{aligned}$$

Thus, $AC_2([a,b])$ is the completion of the polynomials with respect to $\|\cdot\|_{AC_2(J)}$.

□

Let $T \in \mathcal{B}(\mathcal{H})$ be an operator with an $AC_2(J)$-functional calculus. Since the polynomials are dense in $AC_2(J)$, the algebra homomorphism $p \to p(T)$ extends by continuity to give an $AC_2(J)$-functional calculus for T such that

$$\| f(T) \| \leq K \| f \|_{AC_2(J)} \text{ for all } f \in AC_2(J).$$

Now, we give an example of an operator defined on $L^2([0,1])$ with a contractive $AC_2([0,1])$-functional calculus which can be decomposed in a canonical way as a sum of a self-adjoint operator and a quasinilpotent operator.

Example 5.1.1. *Suppose that $T = M_t + V$ is defined on $\mathcal{H} = L^2([0,1])$, where*

$$M_t f(t) = tf(t), \text{ and } Vf(t) = \int_0^t f(s)ds \quad \text{for all } f \in L^2([0,1]).$$

Then the following hold.
(i) The operator T has an $AC_2([0,1])$-functional calculus such that

$$\| f(T) \| \leq \| f \|_{AC_2([0,1])} \text{ for all } f \in AC_2([0,1]).$$

(ii) For each $\lambda \in \mathbb{R}$, let $Y_\lambda = \{g \in L^2([0,1]) : \sigma_T(g) \subseteq [0,\lambda]\}$. Then Y_λ is a closed linear subspace of $\mathcal{H}$. Let $E(\lambda)$ be the orthogonal projection of $\mathcal{H}$ onto Y_λ. Then $E(\cdot)$ is a spectral family concentrated on $[0,1]$.
(iii) Define $A = \int \lambda dE(\lambda)$ and let $Q = T - A$. Then A is self-adjoint and Q is a Hilbert-Schmidt quasinilpotent operator.

Proof. Let $f \in L^2([0,1])$. Then

$$\begin{aligned}(VM_t)(f(t)) &= V(tf(t)) \\ &= \int_0^t sf(s)ds \\ &= [sVf(s)]_0^t - \int_0^t (Vf)(s)ds \\ &= t(Vf)(t) - V^2 f(t) \\ &= (M_t V - V^2)f(t).\end{aligned}$$

Thus, we have

$$M_t V - VM_t = V^2. \tag{5.1.1}$$

Using (5.1.1) we get

$$T^n = M_t^n + nM_t^{n-1}V \text{ for all } n \in \mathbb{N}. \tag{5.1.2}$$

Hence, for any polynomial p,

$$p(T) = p(M_t) + p'(M_t)V. \tag{5.1.3}$$

Let $f \in L^2([0,1])$. By (5.1.3) we have

$$p(T)f = p(t)f + p'(t)Vf.$$

So

$$\| p(T)f \|_2 \leq \| pf \|_2 + \| p'Vf \|_2 \leq \| p \|_\infty \| f \|_2 + \| p'Vf \|_2 .$$

But

$$\begin{aligned}\| p'Vf \|_2^2 &= \int_0^1 |p'(t) \int_0^t f(s)ds|^2 dt \\ &= \int_0^1 |p'(t)|^2 | \int_0^t f(s)ds|^2 dt \\ &\leq \int_0^1 |p'(t)|^2 (\int_0^t |f(s)|ds)^2 dt \\ &\leq \int_0^1 |p'(t)|^2 (\int_0^1 1.|f(s)|ds)^2 dt \\ &\leq \int_0^1 |p'(t)|^2 ((\int_0^1 1^2 ds)^{\frac{1}{2}} . (\int_0^1 |f(s)|^2 ds)^{\frac{1}{2}})^2 dt \\ &= \int_0^1 |p'(t)|^2 . (\int_0^1 |f(s)|^2 ds) dt \\ &= \| f \|_2^2 \int_0^1 |p'(t)|^2 dt \\ &= \| f \|_2^2 \| p' \|_2^2 .\end{aligned}$$

Thus,

$$\| p(T)f \|_2 \leq \| p \|_\infty \| f \|_2 + \| f \|_2 \| p' \|_2 .$$

So we have

$$\| p(T) \| \leq \| p \|_\infty + \| p' \|_2 = \| p \|_{AC_2([0,1])} .$$

Since the polynomials are dense in $AC_2([0,1])$, we get an AC_2-functional calculus for T such that

$$\| f(T) \| \leq \| f \|_{AC_2([0,1])} \text{ for all } f \in AC_2([0,1]).$$

Let $AC_2^0 = \{f \in AC_2([0,1]) : f(0) = 0\}$ and let $U : AC_2^0 \to L^2([0,1])$ be defined by $Uf = f'$. We have, for $f \in AC_2^0$,

$$\| Uf \|_2 = \| f' \|_2 \leq \| f \|_{AC_2([0,1])} \leq \| f' \|_2 + \| f' \|_1 + |f(0)| \leq 2 \| f' \|_2 = 2 \| Uf \|_2 .$$

If $g \in L^2([0,1])$, then $g \in L^1([0,1])$, so there exists $f \in AC([0,1])$ such that $f(t) = \int_0^s g(t)dt$ and $\| f' \|_2 = \| g \|_2 < \infty$. Thus $\| f \|_{AC_2([0,1])} < \infty$, i.e., $f \in AC_2([0,1])$. Clearly, $f(0) = 0$. Hence, $f \in AC_2^0$ and $Uf = g$. Thus, U is onto. Therefore, U is an isomorphism.

Let $\tilde{M_t} = M_t$ on AC_2^0. Then, for $f \in AC_2^0$,

$$\begin{aligned} U\tilde{M_t}f(t) = U(tf(t)) &= tf'(t) + f(t) \\ &= M_t f'(t) + Vf'(t) \\ &= Tf'(t) = TUf(t). \end{aligned}$$

So $U\tilde{M_t}U^{-1} = T$, i.e., $T \cong \tilde{M_t}$.

For each $\lambda \in [0,1]$, let

$$\begin{aligned} X_\lambda = X_{\tilde{M_t}}([0,\lambda]) &= \{f \in AC_2^0 : \sigma_{\tilde{M_t}}(f) \subset [0,\lambda]\} \\ &= \{f \in AC_2^0 : f = 0 \text{ on } [\lambda, 1]\}. \end{aligned}$$

Then we have

$$\begin{aligned} Y_\lambda = X_T([0,\lambda]) = UX_{\tilde{M_t}}([0,\lambda]) &= UX_\lambda \\ &= U\{f \in AC_2^0 : f = 0 \text{ on } [\lambda,1]\} \\ &= \{f' : f \in AC_2^0 \text{ and } f = 0 \text{ on } [\lambda,1]\} \\ &= \{g \in L^2([0,1]) : g = 0 \text{ on } [\lambda,1] \text{ and } \int_0^\lambda g = 0\}. \end{aligned}$$

Let

$$Z_\lambda = \{g \in L^2([0,1]) : g = 0 \text{ on } [\lambda,1]\}.$$

Fix any $f_0 \in Z_\lambda \backslash Y_\lambda$. Then $f_0 = 0$ on $[\lambda,1]$ and $\alpha = \int_0^\lambda f_0 \neq 0$. Let $g \in Z_\lambda$ and $\beta = \int_0^\lambda g$, then

$$g - \frac{\beta}{\alpha} f_0 = 0 \text{ on } [\lambda,1] \text{ and } \int_0^\lambda (g - \frac{\beta}{\alpha} f_0) = 0.$$

Thus $g - \frac{\beta}{\alpha} f_0 \in Y_\lambda$, i.e., $g \in Y_\lambda + \frac{\beta}{\alpha} f_0$. So $g \in \text{span}\{f_0, Y_\lambda\}$.
Thus, $Z_\lambda = \text{span}\{f_0, Y_\lambda\}$, i.e., Y_λ is a subspace of Z_λ of codimension 1.
Therefore, $Y_\lambda \oplus$ one dimensional subspace $= Z_\lambda$. The orthogonal projection of $L^2([0,1])$ onto Z_λ is $M_{\chi_{[0,\lambda]}}$. Let $E(\lambda)f = \chi_{[0,\lambda]}(f - \frac{1}{\lambda}\int_0^\lambda f)$ for $f \in L^2([0,1])$ and $\lambda > 0$. We claim that $E(\lambda)$ is the orthogonal projection of $L^2([0,1])$ onto Y_λ.
If $f \in Y_\lambda$, then $f = 0$ on $[\lambda, 1]$ and $\int_0^\lambda f = 0$, so

$$E(\lambda)f = \chi_{[0,\lambda]}(f - \frac{1}{\lambda}\int_0^\lambda f) = f.$$

Thus, $E(\lambda) = I$ on Y_λ.
Let $f \in L^2([0,1])$. We want to prove that $E(\lambda)f \in Y_\lambda$, i.e., $E(\lambda)f = 0$ on $[\lambda, 1]$ and $\int_0^\lambda E(\lambda)f = 0$.
If $t \in [\lambda, 1]$, then $\chi_{[0,\lambda]}(t) = 0$, so $E(\lambda)f = 0$ on $[\lambda, 1]$.

$$\begin{aligned}
\int_0^\lambda (E(\lambda)f)(s)ds &= \int_0^1 (E(\lambda)f)(s)ds \\
&= \int_0^1 \chi_{[0,\lambda]}(f - \frac{1}{\lambda}\int_0^\lambda f)(s)ds \\
&= \int_0^\lambda f(s)ds - \frac{1}{\lambda}\int_0^\lambda (\int_0^\lambda f(t)dt)ds \\
&= \int_0^\lambda f(s)ds - \int_0^\lambda f(t)dt = 0.
\end{aligned}$$

Thus, $E(\lambda)f \in Y_\lambda$ for all $f \in Y_\lambda$ and $E(\lambda)$ is idempotent.

$$\begin{aligned}
\| E(\lambda)f \|_2 &= (\int_0^1 |E(\lambda)f|^2)^{\frac{1}{2}} \\
&= (\int_0^1 |\chi_{[0,\lambda]}(f - \frac{1}{\lambda}\int_0^\lambda f)|^2)^{\frac{1}{2}} \\
&= (\int_0^\lambda |f - \frac{1}{\lambda}\int_0^\lambda f|^2)^{\frac{1}{2}} \\
&\leq (\int_0^\lambda |f|^2)^{\frac{1}{2}} + (\int_0^\lambda (\frac{1}{\lambda}|\int_0^\lambda f|)^2)^{\frac{1}{2}} \\
&\leq \| f \|_2 + (\lambda \frac{1}{\lambda^2}|\int_0^\lambda f|^2)^{\frac{1}{2}} \\
&\leq \| f \|_2 + \frac{1}{\lambda^{\frac{1}{2}}}\int_0^\lambda |f| \\
&\leq \| f \|_2 + \frac{1}{\lambda^{\frac{1}{2}}}(\int_0^\lambda |f|^2)^{\frac{1}{2}}(\int_0^\lambda 1^2)^{\frac{1}{2}} \\
&\leq \| f \|_2 + \| f \|_2 = 2 \| f \|_2 .
\end{aligned}$$

Thus, $E(\lambda)$ is a bounded projection onto Y_λ.

We have

$$\begin{aligned}((E(\lambda))^*f,g) &= (f,E(\lambda)g)\\ &= (f,\chi_{[0,\lambda]}(g-\frac{1}{\lambda}\int_0^\lambda g))\\ &= \int_0^1 f(t)\overline{\chi_{[0,\lambda]}(t)}\overline{(g(t)-\frac{1}{\lambda}\int_0^\lambda g)}dt\\ &= \int_0^\lambda f(t)\overline{g(t)}dt-(\frac{1}{\lambda}\int_0^\lambda f(t)dt)(\overline{\int_0^\lambda g(s)ds})\end{aligned}$$

and

$$\begin{aligned}(E(\lambda)f,g) &= (\chi_{[0,\lambda]}(f-\frac{1}{\lambda}\int_0^\lambda f),g)\\ &= \int_0^1 \chi_{[0,\lambda]}(t)(f(t)-\frac{1}{\lambda}\int_0^\lambda f)\overline{g(t)}dt\\ &= \int_0^\lambda f(t)\overline{g(t)}dt-(\frac{1}{\lambda}\int_0^\lambda f(s)ds)(\int_0^\lambda \overline{g(t)}dt).\end{aligned}$$

Thus, $(E(\lambda))^* = E(\lambda)$. Therefore, $E(\lambda)$ is the orthogonal projection of $L^2([0,1])$ onto Y_λ.

Next, we want to show that $E(\cdot)$ is a spectral family concentrated on $[0,1]$. We have $E(\lambda)$ is an orthogonal projection for each $\lambda \in \mathbb{R}$ and satisfies the following:

(1) $\sup_{\lambda\in\mathbb{R}} \| E(\lambda) \| \leq 1$.

(2) $Y_\lambda = \{g \in L^2([0,1]) : \sigma_T(g) \subset [0,\lambda]\} = X_T([0,\lambda])$ is a local spectral subspace of T which is a T-hyperinvariant closed linear subspace of $L^2([0,1])$.
If $\lambda_1 \leq \lambda_2$, then $[0,\lambda_1] \subset [0,\lambda_2]$, so $X_T([0,\lambda_1]) \subset X_T([0,\lambda_2])$, so $Y_{\lambda_1} \subset Y_{\lambda_2}$. Hence, $E(\lambda_1)E(\lambda_2) = E(\lambda_2)E(\lambda_1) = E(\lambda_1)$.

(3) If $\lambda \leq 0$, $X_\lambda = \{f \in AC_2^0 : f = 0 \text{ on } [0,1]\} = \{0\}$. So $Y_\lambda = UX_\lambda = \{0\}$. So $E(\lambda) = 0$ for $\lambda \leq 0$.
If $\lambda \geq 1$, then $X_\lambda = \{f \in AC_2^0 : \sigma_{\tilde{M}_t}(f) \subset [0,1]\} = AC_2^0$, so $Y_\lambda = L^2([0,1])$ and $E(\lambda) = I$.

(4) If $\{\lambda_n\}$ is a decreasing sequence converging to $\lambda \in \mathbb{R}$, then $\{E(\lambda_n)\}$ is a decreasing sequence of orthogonal projections on $\mathcal{H} = L^2([0,1])$. So $\{E(\lambda_n)\}$ is

strongly operator convergent to the orthogonal projection onto

$$\begin{aligned}\cap_{n=1}^{\infty} E(\lambda_n)(\mathcal{H}) &= \cap_{n=1}^{\infty} Y_{\lambda_n} \\ &= \cap_{n=1}^{\infty} X_T([0,\lambda_n]) \\ &= X_T(\cap_{n=1}^{\infty}[0,\lambda_n]) \\ &= X_T([0,\lambda]) \\ &= Y_\lambda.\end{aligned}$$

Hence, $\lim_{\lambda_n \to \lambda^+} E(\lambda_n)x = E(\lambda)x$ for all $x \in \mathcal{H}$.

(5) If $\{\lambda_n\}$ is an increasing sequence converging to λ, then $\{E(\lambda_n)\}$ is an increasing sequence of orthogonal projections on $\mathcal{H}$. So $\{E(\lambda_n)\}$ is strongly operator convergent to the orthogonal projection of $\mathcal{H}$ onto $\overline{\cup_{n=1}^{\infty} E(\lambda_n)\mathcal{H}} = \overline{\cup_{n=1}^{\infty} Y_{\lambda_n}}$. Hence, $\lim_{\lambda_n \to \lambda^-} E(\lambda_n)x$ exists for all $x \in \mathcal{H}$.

Therefore, $E(\cdot)$ is a spectral family on $\mathcal{H} = L^2([0,1])$ concentrated on $[0,1]$.

Since $A = \int_0^1 \lambda dE(\lambda)$, it is self-adjoint and

$$\begin{aligned}A = \int_0^1 \lambda dE(\lambda) &= [\lambda E(\lambda)]_0^1 - \int_0^1 E(\lambda)d\lambda \\ &= E(1) - \int_0^1 E(\lambda)d\lambda \\ &= I - \int_0^1 E(\lambda)d\lambda.\end{aligned}$$

Fix $f \in L^2([0,1])$. Then for almost all t,

$$\begin{aligned}Af(t) &= f(t) - \int_0^1 E(\lambda)f(t)d\lambda \\ &= f(t) - \int_0^1 \chi_{[0,\lambda]}(t)f(t)d\lambda + \int_0^1 \chi_{[0,\lambda]}(t)(\frac{1}{\lambda}\int_0^\lambda f)d\lambda \\ &= f(t) - \int_t^1 f(t)d\lambda + \int_t^1 (\frac{1}{\lambda}\int_0^\lambda f(u)du)d\lambda \\ &= f(t) - (1-t)f(t) + \int_t^1 (\frac{1}{\lambda}\int_0^\lambda f(u)du)d\lambda \\ &= tf(t) + \int_t^1 (\frac{1}{\lambda}\int_0^\lambda f(u)du)d\lambda \\ &= (M_t f)(t) + (Kf)(t).\end{aligned}$$

So $A = M_t + K$, where

$$\begin{aligned} Kf(t) &= \int_t^1 (\frac{1}{\lambda}\int_0^\lambda f(u)du)d\lambda \\ &= \int_0^t (\int_t^1 \frac{1}{\lambda} f(u)d\lambda)du + \int_t^1 (\int_u^1 \frac{1}{\lambda} f(u)d\lambda)du \\ &= \int_0^t [\ln\lambda]_t^1 f(u)du + \int_t^1 [\ln\lambda]_u^1 f(u)du \\ &= \int_0^t (-\ln t)f(u)du + \int_t^1 (-\ln u)f(u)du \\ &= \int_0^1 k(t,u)f(u)du, \end{aligned}$$

and

$$k(t,u) = \begin{cases} -\ln t & \text{if } 0 \le u \le t \le 1 \\ -\ln u & \text{if } 0 \le t \le u \le 1. \end{cases}$$

Thus, setting $Q = T - A$, we have

$$T = M_t + V = A + Q = M_t + K + Q.$$

Hence, $V = K + Q$, i.e, $Q = V - K$. So we have

$$\begin{aligned} (Qf)(t) &= (V - K)f(t) \\ &= \int_0^t f(u)du - \int_0^1 k(t,u)f(u)du \\ &= \int_0^t f(u)du - (\int_0^t (-\ln t)f(u)du + \int_t^1 (-\ln u)f(u)du) \\ &= \int_0^t (1+\ln t)f(u)du + \int_t^1 (\ln u)f(u)du \\ &= \int_0^1 q(t,u)f(u)du, \end{aligned}$$

where

$$q(t,u) = \begin{cases} 1+\ln t & \text{if } 0 \le u \le t \le 1 \\ \ln u & \text{if } 0 \le t \le u \le 1. \end{cases}$$

Then, after some calculations, we get

$$\| q \|_2^2 = \int_0^1 \int_0^1 |q(t,u)|^2 du dt = \frac{1}{2} < \infty.$$

So Q is a Hilbert-Schmidt operator. Thus, Q is compact. So, if $\alpha \in \sigma(Q)$ and $\alpha \neq 0$, then α is an eigenvalue of Q.

Suppose that $Qf = \alpha f$, $\alpha \neq 0$. Then for almost all t

$$(1+\ln t)\int_0^t f(u)du + \int_t^1 (\ln u)f(u)du = \alpha f(t). \tag{5.1.4}$$

Thus we can take f to be continuous and hence differentiable by (5.1.4). Differentiating (5.1.4) with respect to t, we get

$$\int_0^t f(u)du + tf(t) = \alpha t f'(t) \tag{5.1.5}$$

Differentiating (5.1.5) with respect to t, we get

$$\alpha t f''(t) + (\alpha - t)f'(t) - 2f(t) = 0. \tag{5.1.6}$$

Solving (5.1.6), we get

$$f(t) = a_0 \sum_{n=0}^{\infty} \frac{n+1}{\alpha^n n!} t^n = a_0(\frac{t}{\alpha} + 1)e^{\frac{t}{\alpha}}.$$

Then, after some calculations, we get $a_0 = 0$, so $f = 0$. Therefore, $\sigma(Q) = \{0\}$, i.e., Q is quasinilpotent. □

5.2 Future Work

In this final section we shall consider whether the idea of Example 5.1.1 can be extended to any operator with an $AC_2(J)$-functional calculus. So we ask the following question.

Question 5.2.1. *If T is any operator on $\mathcal{H}$ with an $AC_2(J)$-functional calculus, then is it true that $T = A + Q$ where A is self-adjoint and Q is quasinilpotent such that $AQ - QA$ is quasinilpotent?*

If we adapt (as we did in Example 5.1.1) the method we used in the proof of Theorem 3.1.1 which shows that every well-bounded operator can be decomposed as a sum of a self-adjoint operator and a quasinilpotent operator, we find that in general all the proof will work except the last step of making $\| (\lambda - s)^{-1} \|_{AC_2([-\frac{b-a}{2N}, 0])}$ small enough in order to be able to prove that Q is quasinilpotent. So we ask why the decomposition process works in the $M_t + V$ case? However, the proof that $AQ - QA$ is quasinilpotent will work exactly as in the proof of Theorem 3.1.1 as we did not use any AC norm in that part of the proof.

It is also of interest to note that our main motivating example $T = M_t + V$ is not a well-bounded operator on $L^2([0, 1])$.

We finish the thesis with the following:

Question 5.2.2. *Consider the Cesaro operator C on ℓ^2 defined by*

$$C(\{x_n\}) = \{x_1, \frac{x_1 + x_2}{2}, \frac{x_1 + x_2 + x_3}{3}, \cdots\cdots\}.$$

Does there exist an operator M on ℓ^2 given by

$$M(\{x_n\}) = \{\alpha_1 x_1, \alpha_2 x_2, \alpha_3 x_3, \cdots\cdots\}$$

such that $M + C$ is an operator with an AC_2-functional calculus?

Question 5.2.3. *Under what conditions is the sum of a self-adjoint operator and a quasinilpotent operator on a Hilbert space an operator with an AC_2-functional calculus?*

Question 5.2.4. *We shall say that an operator $T \in \mathcal{B}(\mathcal{H})$ is an AC_2-operator if $T = A + iB$ where A and B are two commuting operators on $\mathcal{H}$ with real spectrum and AC_2-functional calculi.*
The new class of AC_2-operators is larger than the class of AC-operators.
(i) To what extent is the decomposition $T = A + iB$ unique?
(ii) Can we decompose any AC_2-operator into a sum of a normal operator and a quasinilpotent?
(iii) Does an AC_2-operator have an appropriate AC_2-functional calculus on the smallest rectangle in $\mathbb{C}$ containing its spectrum?
(iv) What other properties we can extend from AC-operators to AC_2-operators?

Question 5.2.5. *Consider the class of operators $T \in \mathcal{B}(\mathcal{H})$ of the form $T = e^{iA}$ where A is an operator on $\mathcal{H}$ with an $AC_2([0, 2\pi]))$-functional calculus. This new class of operators is larger than the class of trigonometrically well-bounded operators on $\mathcal{H}$. What are the properties of the operators in this class? Can we generalise some of the properties of the trigonometrically well-bounded operators to the operators in this class?*

Notation

The following notation is used throughout the thesis.

$\mathbb{R}$	the real numbers
$\mathbb{C}$	the complex numbers
$\mathbb{N}$	the natural numbers
X	a complex Banach space
$\mathcal{H}$	a Hilbert space
T	a bounded linear operator
$T\|Y$	the restriction of T to Y
T/Y	the operator induced by T on the quotient space $\mathcal{H}/Y$
$\mathbb{T}$	the unit circle
X^*	the Banach space of continuous linear functionals on the Banach space X
(x, x^*)	the linear functional $x^* \in X^*$ evaluated at $x \in X$
$\mathcal{B}(X)$	the space of bounded linear operators on X
$\sigma(T)$	the spectrum of T
$\rho(T)$	the resolvent set of T
$\rho_T(x)$	the local resolvent of T at x
$\sigma_T(x)$	the local spectrum of T at x
$X_T(F)$	a local spectral subspace of $\mathcal{H}$
$\sigma_p(T)$	the point spectrum of T
$\sigma_c(T)$	the continuous spectrum of T
$\sigma_r(T)$	the residual spectrum of T
$r(T)$	the spectral radius of T
$\overline{\sigma}$	the closure of a set σ
Σ_σ	the family of Borel sets of σ
χ_σ	the characteristic function of a set σ

$BV(J \times K)$	the space of functions of bounded variation on $J \times K$
$AC(J \times K)$	the space of absolutely continuous functions on $J \times K$
$AC(\mathbb{T})$	the space of absolutely continuous functions on $\mathbb{T}$
$L^p(\Omega, \Sigma_\Omega, \mu)$	the space of equivalence classes of p-integrable Σ_Ω-measurable functions
$\ell^p(\Gamma)$	$L^p(\Gamma, \Sigma_\Gamma, \mu)$, where Γ is a discrete set and $\mu(\{\gamma\}) = 1$ for every $\gamma \in \Gamma$
ℓ^p	the space $L^p(\Gamma)$ where $\Gamma = \mathbb{N}$
ℓ^p_n	the space $L^p(\Gamma)$ where $\Gamma = \{1, 2, \cdots, n\}$
$\ell^p_w(\Gamma)$	$L^p(\Gamma, \Sigma_\Gamma, \mu)$, where Γ is a discrete set and $\mu(\{\gamma_k\}) = w_k$ for $\gamma_k \in \Gamma$ where $w = \{w_k\}_{k \in \Gamma}$
$\mathcal{L}_p$	see p.15
$L^\infty(\Omega, \Sigma_\Omega, \mu)$	the space of equivalence classes of essentially bounded Σ_Ω-measurable functions
$\mathcal{L}^\infty$	the set of bounded measurable functions on a measure space $(\Lambda, \Sigma_\Lambda)$
$\mathcal{B}$	a σ-complete Boolean algebra of projections on X
$\mathcal{M}(f_0)$	the cyclic subspace determined by f_0
clm S	the closed linear span of S, where S is a subset of a Banach space X
$E \vee F$	$E + F - EF$ with range clm$\{EX, FX\}$
$E \wedge F$	EF with range $EX \cap FX$
SVEP	single-valued extension property
T_φ	$\int_\Lambda \varphi(\lambda) E(d\lambda)$
M_φ	the multiplication operator by φ
$d(X, Y)$	$\inf\{\| T \| \| T^{-1} \| : T : X \to Y \text{ is invertible}\}$ where X and Y are Banach spaces
Condition$(*)_p$	see p.18
$L^p(\mu) \cong L^p(\nu)$	$L^p(\mu)$ is isomorphic to $L^p(\nu)$
$T \cong M_\varphi$	T is similar to M_φ

Bibliography

[1] M. B. Abrahamse and T. H. Kriete. The Spectral Multiplicity of a Multiplication Operator. *Indiana U. Math. J.*, 22:845–857, 1973.

[2] W. G. Bade. On Boolean Algebras of Projections and Algebras of Operators. *Trans. Amer. Math. Soc.*, 80:345–359, 1955.

[3] W. G. Bade. A Multiplicity Theory for Boolean Algebras of Projections in Banach Spaces. *Trans. Amer. Math. Soc.*, 92:508–530, 1959.

[4] E. Berkson, I. Doust, and T. A. Gillespie. Properties of AC-Operators. *Acta Sci. Math. (Szeged)*, 63:249–271, 1997.

[5] E. Berkson and T. A. Gillespie. Absolutely Continuous Functions of two Variables and Well-Bounded Operators. *J. London Math. Soc. (2)*, 30:305–321, 1984.

[6] E. Berkson and T. A. Gillespie. AC Functions on the Circle and Spectral Families. *J. Operator Theory*, 13:33–47, 1985.

[7] E. Berkson and T. A. Gillespie. Steckin's Theorem, Transference, and Spectral Decompositions. *Journal of Functional Analysis*, 70:140–170, 1987.

[8] F. F. Bonsall and J. Duncan. *Complete Normed Algebras.* Springer-Verlag, New York, 1973.

[9] Ion Colojoara and Ciprian Foias. *Theory of Generalized Spectral Operators.* Gordon and Breach, New York, 1968.

[10] J. B. Conway. *A Course in Functional Analysis.* Springer Verlag, New York, 1985.

[11] J. B. Conway. *A Course in Operator Theory.* American Mathematical Society, Providence, Rhode Island, 2000.

[12] H. R. Dowson. Restrictions of Scalar-Type Spectral Operators. *Bull. London Math. Soc.*, 10:305–309, 1978.

[13] H. R. Dowson. *Spectral Theory of Linear Operators.* Academic Press, London, 1978.

[14] N. Dunford. Spectral Operators. *Pacific J. Math*, 4:321–354, 1954.

[15] N. Dunford and J. T. Schwartz. *Linear Operators I.* Wiley, New York, 1964.

[16] N. Dunford and J. T. Schwartz. *Linear Operators III.* Wiley, New York, 1971.

[17] C. K. Fong and L. Lam. On Spectral Theory and Convexity. *Trans. Amer. Math. Soc.*, 264:59–75, 1981.

[18] B. Fuglede. A commutativity Theorem for Normal Operators. *Proc. Nat. Acad. Sci. U.S.A.*, 36:35–40, 1950.

[19] M. B. Ghaemi. On Some Classes of AC-Operators. *Glasgow Math. J.*, 44:329–338, 2002.

[20] T. A. Gillespie. Commuting Well-Bounded Operators on Hilbert Spaces. *Proc. Edinburgh Math. Soc.(2)*, 20:167–172, 1976.

[21] T. A. Gillespie. Cyclic Banach Spaces and Reflexive Operator Algebras. *Proc. Roy. Soc. Edinburgh*, 78A:225–235, 1978.

[22] P. R. Halmos. *Measure Theory.* Van Nostrand, Princeton, N. J., 1950.

[23] P. R. Halmos. *A Hilbert Space Problem Book.* Springer-Verlag, New York, 1982.

[24] H. E. Lacey and S. J. Bernau. Characterizations and Classifications of some Classical Banach Spaces. *Advances in Mathematics*, 12:367–401, 1974.

[25] Kjeld B. Laursen and Michael M. Neumann. *An Introduction to Local Spectral Theory.* Clarendon Press, Oxford, 2000.

[26] J. Lindenstrauss and A. Pelczynski. Absolutely Summing Operators in $\mathcal{L}_p$ Spaces and their Applications. *Studia Math.*, 29:275–326, 1968.

[27] J. Lindenstrauss and H. P. Rosenthal. The $\mathcal{L}_p$ Spaces. *Israel J. Math.*, 7:325–349, 1969.

[28] J. Lindenstrauss and L. Tzafriri. *Classical Banach Spaces. Lecture Notes in Mathematics (338).* Springer-Verlag, New York, 1973.

[29] J. Lindenstrauss and L. Tzafriri. *Classical Banach Spaces I.* Springer-Verlag, New York, 1977.

[30] J. Lindenstrauss and L. Tzafriri. *Classical Banach Spaces II.* Springer-Verlag, New York, 1979.

[31] J. Lindenstrauss and M. Zippin. Banach Spaces with sufficiently many Boolean Algebras of Projections. *J. Math. Anal. Appl.*, 25:309–320, 1969.

[32] C. A. McCarthy and L. Tzafriri. Projections in $\mathcal{L}_1$ and $\mathcal{L}_\infty$ Spaces. *Pacific J. Math.*, 26:529–546, 1968.

[33] J. R. Ringrose. On Well-Bounded Operators. *J. Austral. Math. Soc. Ser. A*, 1:334–343, 1960.

[34] J. R. Ringrose. On Well-Bounded Operators II. *Proc. London Math. Soc. (3)*, 13:613–638, 1963.

[35] H. L. Royden. *Real Analysis.* Macmillan, New York, 1988.

[36] D. R. Smart. Conditionally Convergent Spectral Expansions. *J. Austral. Math. Soc. Ser.A*, 1:319–333, 1960.

[37] L. Tzafriri. Reflexivity of Cyclic Banach Spaces. *Proc. Amer. Math. Soc.*, 22:61–68, 1969.

[38] L. Tzafriri. An Isomorphic Characterization of L_p and c_0 Spaces II. *Michigan Math. J.*, 18:21–31, 1971.

[39] T. T. West. The Decomposition of Riesz Operators. *Proc. London Math. Soc.(3)*, 16:737–752, 1966.

[40] A. C. Zaanen. *Integration.* North-Holland Publishing Company, Amsterdam, 1967.