SEMIGROUPS OF WELL-BOUNDED OPERATORS AND MULTIPLIERS

by

David J. Ralph

Thesis submitted for the degree of
Doctor of Philosophy
University of Edinburgh
1977

UNIV. EDINBURGH

CONTENTS

PREFACE

In 1960 Smart ([30]) introduced the concept of a well-bounded operator on a Banach space. Around the same time Krabbe ([21], [22]) obtained spectral theorems for certain bounded and unbounded operators on $L^p(\underline{R})$; the case where the bounded operators are of semigroup type is of especial interest. We have attempted to ~~inter~~ interpret these results in the light of the theory of well-bounded operators, developed in [30], [26], [27], [3] and [32], which we summarise in Chapter 1.

We have developed the theory of an unbounded analogue of the well-bounded operator, which we term "well-boundable". This material is presented in Chapter 2. A well-boundable operator has real spectrum, and possesses a bounded spectral family with respect to which it satisfies a Riemann-Stieltjes form of the spectral theorem. We prove the uniqueness of such a family (Theorem 2.2.8), identify the spectrum of the well-boundable operator as the support of its spectral family (Theorem 2.2.13) and obtain versions for well-boundable operators of various other standard spectral theory results. This enables us to prove a generalisation to well-bounded operators of type B of the Hille-Sz.-Nagy theorem (Theorem 2.3.2).

The main theorem of Chapter 3, Theorem 3.2.4, shows that, for a real-valued Fourier transform multiplier of $L^p(G)$, the existence and uniform boundedness of the appropriate multiplier projections is necessary and sufficient for the multiplier to define a well-bounded operator. Examples and counterexamples are given.

In Chapter 4 we study well-bounded Riesz operators. The Riesz operators form a larger class of operators which satisfy the Riesz theory of compact operators, but we are able to prove, in Theorem 4.2.3, that a well-bounded Riesz operator is necessarily compact. As an application, we use this theorem and our characterisation of well-bounded multiplier operators to show that the singular multiplier of Figà-Talamanca and Gaudry does not define a well-bounded operator.

The layout of this thesis is as follows. Each chapter is divided into three or four sections, numbered consecutively within each chapter only. All cross-references give the full chapter, section and result number. Display numbers are consecutive within each section only; on only one or two occasions is a display belonging to another section referred to.

ACKNOWLEDGEMENTS

I am sincerely grateful to my supervisor, Dr T. A. Gillespie, for initially drawing my attention to the topics treated here, and for his regular advice, encouragement and criticism, without which this thesis could not have reached its final form.

Throughout the three years of my study in Edinburgh I have been supported by a studentship from the Science Research Council.

CHAPTER 1

PRELIMINARIES

In this chapter we review the basic notions used in the thesis. The theory of well-bounded operators requires careful discussion of Stieltjes integration in a vector-valued setting, a definitive account of which was given by Spain [32], based on ideas of Krabbe [23]. We outline this material in §1.2. In §1.3 we summarise the theory of well-bounded operators.

§1.1. Basic notions

We denote the real numbers, the complex numbers, the integers, the rationals and the unit circle by $\underline{R}$, $\underline{C}$, $\underline{Z}$, $\underline{Q}$ and $\underline{T}$ respectively. Throughout the thesis X will be a complex Banach space. The norm of an element $x \in X$ will be denoted by $\|x\|$. By a linear subspace of X we shall mean simply a subset of X which is itself a linear space; we do not assume a linear subspace to be closed.

If X and Y are Banach spaces, a linear operator between X and Y will be a linear map $T : \mathcal{D}(T) \to Y$, where the domain $\mathcal{D}(T)$ is a linear subspace of X. When $X = Y$ we refer to T as a linear operator on X. The set of (everywhere defined) bounded linear operators between X and Y will be denoted by $B(X,Y)$, and $B(X,X)$ will be abbreviated to $B(X)$. We denote the dual of X by X^*, and the adjoint of $T \in B(X,Y)$ by T^*. If $\phi \in X^*$ and $x \in X$, then we shall sometimes use $\langle\phi,x\rangle$ to stand for the evaluation $\phi(x)$. We denote the spectrum of T by $\sigma(T)$ and the resolvent set of T by $\rho(T)$.

A linear subspace $X_1 \subset X$ is invariant for a linear operator T on X if, for all $x \in X_1 \cap \mathfrak{D}(T)$, $Tx \in X_1$. The restriction $T|X_1$ of T to an invariant subspace X_1 is the operator $T|X_1$: $X_1 \cap \mathfrak{D}(T) \to X_1$ given by $(T|X_1)x = Tx$ $(x \in X_1 \cap \mathfrak{D}(T))$. If for the linear operators S, T we have $\mathfrak{D}(S) \subset \mathfrak{D}(T)$ and $Sx = Tx$ $(x \in \mathfrak{D}(S))$ then we shall write $S \subset T$.

If $\Omega \subset \underline{C}$ and $\mathfrak{F}$ is an algebra of complex-valued functions on Ω which contains the unit function $1 : \omega \to 1$ $(\omega \in \Omega)$ and the identity function $j : \omega \to \omega$ $(\omega \in \Omega)$, then an $\mathfrak{F}$-functional calculus for an operator T on X is a mapping $f \to f(T)$ from $\mathfrak{F}$ to operators on X, which takes 1 to the identity operator I and j to T, and has the following properties:

(i) $(\lambda f)(T) = \lambda(f(T))$ $(\lambda \in \underline{C})$;

(ii) $(f+g)(T) \supset f(T) + g(T)$;

(iii) $(fg)(T) \supset f(T)g(T)$,

for all $f, g \in \mathfrak{F}$.

If $\mathfrak{A}$ is a subset of $B(X)$, then $\mathfrak{A}'$ will denote the commutant of $\mathfrak{A}$, that is, the subalgebra of $B(X)$ consisting of all $T \in B(X)$ such that $AT = TA$ for every A in $\mathfrak{A}$. The commutant of $\mathfrak{A}'$ is denoted by $\mathfrak{A}''$.

§1.2. Integration theory

DEFINITION 1.2.1. We shall be considering the following spaces of complex-valued functions:

(i) $BV[a,b]$ is the space of all functions of bounded variation on the compact real interval $[a,b]$. $BV[a,b]$ becomes a Banach algebra when given either of the equivalent norms

$$|||f|||_{[a,b]} = |f(b)| + \operatorname*{var}_{[a,b]} f ,$$

$$|||f|||'_{[a,b]} = \sup_{t\in[a,b]} |f(t)| + \operatorname*{var}_{[a,b]} f \quad (f \in BV[a,b]) ,$$

where $\operatorname*{var}_{[a,b]} f$ is the total variation of f over $[a,b]$. It is obvious that the norm $|||\cdot|||'_{[a,b]}$ is submultiplicative, but the fact that $|||\cdot|||_{[a,b]}$ is also submultiplicative requires proof. An indirect and exceedingly complicated proof, which gives much additional information, is given in [29], Theorem 5.2. We give an elementary proof in an appendix to this chapter.

(ii) NBV[a,b] is the subalgebra of BV[a,b] consisting of those functions which are continuous on the left at each point of $(a,b]$.

(iii) $AC[a,b] \subset BV[a,b]$ is the algebra of absolutely continuous functions on $[a,b]$. For $f \in AC[a,b]$, we can write

$$\operatorname*{var}_{[a,b]} f = \int_a^b |f'(t)|dt .$$

Hence the polynomials are norm dense in $AC[a,b]$.

(iv) $BV(\underline{R})$ is the Banach algebra of functions on $\underline{R}$ which have finite total variation. We use the norm

$$|||f|||_{\underline{R}} = \sup_{t\in\underline{R}} |f(t)| + \operatorname*{var}_{\underline{R}} f \quad (f \in BV(\underline{R})) .$$

(v) $LBV(\underline{R})$ is the space of functions $f : \underline{R} \to \underline{C}$ such that f is of bounded variation on each compact interval.

$NLBV(\underline{R})$ consists of those functions in $LBV(\underline{R})$ which are continuous on the left at each point of $\underline{R}$.

$LAC(\underline{R}) \subset LBV(\underline{R})$ consists of those functions which are absolutely continuous on each compact interval.

DEFINITION 1.2.2. A subdivision of the compact real interval $[a,b]$ is a finite sequence $\underline{t} = \{t_k\}_{k=0}^{m}$ of points of $[a,b]$ such that

$$a = t_0 < t_1 < \dots < t_m = b .$$

We denote the set of all subdivisions of $[a,b]$ by $\mathcal{P}[a,b]$ (abbreviated to $\mathcal{P}$ when $[a,b]$ is understood).

A marked partition of $[a,b]$ is a pair $(\underline{t},\underline{t}^*)$, where $\underline{t} \in \mathcal{P}[a,b]$ and $\underline{t}^*$ is a sequence $\{t_k^*\}_{k=1}^{m}$ such that $t_k^* \in [t_{k-1},t_k]$ $(k=1,\dots,m)$. We denote the set of all marked partitions of $[a,b]$ by $\mathcal{P}^*[a,b]$.

If $\underline{s}$ and $\underline{t}$ are subdivisions, we say that $\underline{s}$ is a refinement of $\underline{t}$ if $\underline{t}$ is a subsequence of $\underline{s}$. The relation $\leqslant$, where $\underline{s} \leqslant \underline{t}$ if and only if $\underline{t}$ is a refinement of $\underline{s}$, is a partial order on $\mathcal{P}[a,b]$. We order $\mathcal{P}^*[a,b]$ by setting $(\underline{s},\underline{s}^*) \leqslant (\underline{t},\underline{t}^*)$ if and only if $\underline{s} \leqslant \underline{t}$.

We consider the subsets $\mathcal{P}_r^*[a,b]$ and $\mathcal{P}_i^*[a,b]$ of $\mathcal{P}^*[a,b]$, where

$(\underline{t},\underline{t}^*) \in \mathcal{P}_r^*[a,b]$ if and only if $t_k^* = t_k$ for all k ;

$(\underline{t},\underline{t}^*) \in \mathcal{P}_i^*[a,b]$ if and only if $t_k \in (t_{k-1},t_k)$ for all k .

Under the order $\leqslant$, $\mathcal{P}, \mathcal{P}^*, \mathcal{P}_r^*, \mathcal{P}_i^*$ are all directed sets, and $\mathcal{P}_r^*$ and $\mathcal{P}_i^*$ are cofinal in $\mathcal{P}^*$.

DEFINITION 1.2.3. Let f and g be functions on $[a,b]$, one taking complex values, and the other taking values either in $\underline{C}$ or in $B(X)$ for some Banach space X . For each $(\underline{t},\underline{t}^*) \in \mathcal{P}^*[a,b]$, let

$$\sum(f,\Delta g,\underline{t}, \underline{t}^*) = \sum_{k=1}^{m} f(t_k^*)(g(t_k) - g(t_{k-1})) .$$

Then the following integrals are defined as net limits in the strong operator topology, whenever the limits exist:

$$\text{(i)} \int_a^b f dg = \lim_{(\underline{t},\underline{t}^*)\in\mathcal{P}^*} \sum (f, \Delta g, \underline{t}, \underline{t}^*) \; ;$$

$$\text{(ii)} \int_{[a,b]}^{r} f dg = \lim_{(\underline{t},\underline{t}^*)\in\mathcal{P}_r^*} \sum (f, \Delta g, \underline{t}, \underline{t}^*) \; ;$$

$$\text{(iii)} \int_{[a,b]}^{i} f dg = \lim_{(\underline{t},\underline{t}^*)\in\mathcal{P}_i^*} \sum (f, \Delta g, \underline{t}, \underline{t}^*) \; .$$

<u>DEFINITION 1.2.4</u>. Let E be a $B(X)$-valued function on an interval containing $[a,b]$. For $g \in BV[a,b]$, let

$$\mathcal{P}^*(g) = \begin{cases} \mathcal{P}^*[a,b] & \text{if } g \in NBV[a,b] \\ \mathcal{P}_i^*[a,b] & \text{if } g \in BV[a,b] \backslash NBV[a,b] \; . \end{cases}$$

Then, if the limit exists in the strong operator topology, we define the integral

$$\oint_{[a,b]} E dg = \lim_{(\underline{t},\underline{t}^*)\in\mathcal{P}^*(g)} \sum (E, \Delta g, \underline{t}, \underline{t}^*) \; .$$

<u>PROPOSITION 1.2.5</u> ([32], Theorem 1). If $E : \underline{R} \to B(X)$ is a function such that

(i) $\lim_{s \to t+0} E(s)x = E(t)x \quad (x \in X, \; t \in \underline{R})$;

(ii) $\lim_{s \to t-0} E(s)x$ exists in $X \quad (x \in X, \; t \in \underline{R})$;

(iii) $E(t) = 0 \; (t < a)$; $E(t) = E(b) \; (t \geq b)$;

then, for each $g \in BV[a,b]$, $\oint_{[a,b]} E dg$ exists and

$$\Big\| \oint_{[a,b]} E dg \Big\| \leq \sup_{t \in [a,b]} \|E(t)\| \operatorname*{var}_{[a,b]} g \; .$$

PROPOSITION 1.2.6 ([32], Theorem 2). Let E be as in 1.2.5, and let $\{g_\alpha\}$ be a bounded net in $BV[a,b]$ and $g \in BV[a,b]$ a function such that $g_\alpha(t) \to g(t)$ $(t \in [a,b])$. Then

$$\oint_{[a,b]} E dg = \lim_{\alpha} \oint_{[a,b]} E dg_\alpha$$

in the strong operator topology.

DEFINITION 1.2.7. For E as in 1.2.5 and $g \in BV[a,b]$, let

$$\int_{[a,b]}^{\oplus} g dE = g(b)E(b) - \oint_{[a,b]} E dg \; .$$

PROPOSITION 1.2.8 ([32], Theorem 3 and Lemma 6). Let E be as in 1.2.5, and $g \in BV[a,b]$. Then

$$\int_{[a,b]}^{\oplus} g dE = \begin{cases} g(a)E(a) + \int_a^b g dE & (g \in NBV[a,b]) \\ g(a)E(a) + \int_{[a,b]}^{r} g dE & (g \in BV[a,b] \backslash NBV[a,b]) \; , \end{cases}$$

and

$$\Big\| \int_{[a,b]}^{\oplus} g dE \Big\| \leq \sup_{t \in [a,b]} \|E(t)\| \; |||g|||_{[a,b]} \; .$$

N.B. When applying the operator $\int_{[a,b]}^{\oplus} g dE$ to a vector x we shall use the notation $\int_{[a,b]}^{\oplus} g(\lambda)dE(\lambda)x$, with similar variants for the other integrals defined.

§1.3. Summary of the theory of well-bounded operators

DEFINITION 1.3.1. An operator $T \in B(X)$ is well-bounded if there exist a compact interval $[a,b]$ and a constant $K > 0$ such that

$$\|p(T)\| \leqslant K\|\|p\|\|_{[a,b]} \qquad (1)$$

for every complex polynomial p . If (1) is satisfied, then we shall say that the well-bounded operator T is implemented by $(K,[a,b])$.

Obviously, if T is implemented by $(K,[a,b])$, then it is also implemented by $(K,[a',b'])$ whenever $a' \leqslant a$, $b' \geqslant b$.

It follows immediately from Definition 1.3.1, and the fact that the polynomials are dense in $AC[a,b]$, that there exists a continuous $AC[a,b]$-functional calculus into $B(X)$ for T ; furthermore, $f(T) \in \{T\}''$, for every $f \in AC[a,b]$.

It is also immediate that, if T is well-bounded and implemented by $(K,[a,b])$, then so is T^* , and the functional calculus for T^* is related to that for T by $f(T^*) = f(T)^*$ $(f \in AC[a,b])$.

DEFINITION 1.3.2. A spectral family for X is a projection-valued function $E : \underline{R} \to B(X)$ which satisfies the following conditions:

(i) $\|E(\lambda)\| \leqslant K$ $(\lambda \in \underline{R})$, for some constant $K < \infty$;

(ii) $E(\lambda)E(\mu) = E(\mu)E(\lambda) = E(\min(\lambda,\mu))$ $(\lambda,\mu \in \underline{R})$;

(iii) (a) $\lim_{\mu\to\lambda+0} E(\mu)x = E(\lambda)x$ $(\lambda \in \underline{R}, x \in X)$;

(b) $\lim_{\mu\to\lambda-0} E(\mu)x$ exists in X $(\lambda \in \underline{R}, x \in X)$;

(iv) $\lim_{\lambda\to-\infty} E(\lambda)x = 0$ $(x \in X)$; $\lim_{\lambda\to+\infty} E(\lambda)x = x$ $(x \in X)$.

If E satisfies the stronger condition

(iv′) there exist $a,b \in \underline{R}$ such that

$$E(\lambda) = 0 \ (\lambda < a) \ ; \ E(\lambda) = I \ (\lambda \geqslant b) ,$$

then we shall say that E has <u>compact support</u>.

<u>Remark</u>. There is some redundancy here. Conditions (ii)-(iv) together imply (i). If X is reflexive, (i) and (ii) imply the existence of $\lim_{\mu\to\lambda+0} E(\mu)x$ and $\lim_{\mu\to\lambda-0} E(\mu)x$, for all $\lambda \in \underline{R}$, $x \in X$, by a well-known theorem of Lorch ([25], Theorem 3.2).

The left-hand limit $E(\lambda-0)$ is itself a projection in $B(X)$ with norm at most K, for each $\lambda \in \underline{R}$.

We shall sometimes refer to a spectral family in this sense as a <u>strong</u> spectral family.

Well-bounded operators on reflexive spaces were characterised by Smart and Ringrose, [30] and [26], in the following way:

<u>PROPOSITION 1.3.3</u>. Let X be reflexive and $T \in B(X)$. Then T is well-bounded if and only if there exists a spectral family E, satisfying 1.3.2(iv′), such that

$$Tx = \int_{a-\theta}^{b+\theta} \lambda dE(\lambda)x \quad (x \in X) , \tag{2}$$

where $\theta > 0$ is arbitrary. We may write (2) in the alternative form

$$Tx = aE(a)x + \int_a^b \lambda dE(\lambda)x \quad (x \in X) . \tag{3}$$

The well-bounded operator T is then implemented by $(K,[a,b])$, and $f(T)$ is given by

$$f(T)x = f(a)E(a)x + \int_a^b f(\lambda)dE(\lambda)x \quad (x \in X,\ f \in AC[a,b]). \tag{4}$$

Furthermore, E is unique, and $E(\lambda) \in \{T\}'' \ (\lambda \in \underline{R})$.

When X is not reflexive, the characterisation is less simple, and in general we can only obtain projections in $B(X^*)$ with much weaker continuity properties.

<u>DEFINITION 1.3.4</u>. A <u>dual spectral family</u> for X is a projection-valued function $F : \underline{R} \to B(X^*)$ satisfying the following conditions:

(i) $\|F(\lambda)\| \leq K \ (\lambda \in \underline{R})$, for some constant $K < \infty$;

(ii) $F(\lambda)F(\mu) = F(\mu)F(\lambda) = F(\min(\lambda,\mu)) \ (\lambda,\mu \in \underline{R})$;

(iii) there exist $a,b \in \underline{R}$ such that

$$F(\lambda) = 0 \ (\lambda < a) \ ; \ F(\lambda) = I \ (\lambda \geq b) \ ;$$

(iv) for each $\phi \in X^*$, $x \in X$, the function $\lambda \to \langle F(\lambda)\phi,x\rangle$ is Lebesgue measurable;

(v) for any $\phi \in X^*$, $x \in X$ and $\mu \in [a,b)$, if

$\lim\limits_{h \to 0+} \frac{1}{h} \int_\mu^{\mu+h} \langle F(\lambda)\phi,x\rangle d\lambda$ exists, then the value of the limit is $\langle F(\mu)\phi,x\rangle$;

(vi) for each $x \in X$, the map $X^* \to L^\infty[a,b] = L^1[a,b]^*$ which sends ϕ to $(\lambda \to \langle F(\lambda)\phi,x\rangle)$ is continuous when both spaces are given the weak* topology.

Ringrose [27] introduced this concept and called it a "decomposition of the identity", but we prefer not to use the term in order to avoid confusion with the Colojoară-Foiaş theory of decomposable operators ([7], Chapter 2). Well-bounded operators are examples of "decomposable operators" in that sense.

PROPOSITION 1.3.5. $T \in B(X)$ is well-bounded if and only if there exists a dual spectral family F such that T is the unique operator in $B(X)$ satisfying

$$\langle\phi,Tx\rangle = b\langle\phi,x\rangle - \int_a^b \langle F(\lambda)\phi,x\rangle d\lambda \quad (\phi \in X^*\ ,\ x \in X)\ , \qquad (5)$$

where a,b are the numbers in 1.3.4(iii). T is then implemented by $(K,[a,b])$, and the functional calculus is given by

$$\langle\phi,f(T)x\rangle = f(b)\langle\phi,x\rangle - \int_a^b \langle F(\lambda)\phi,x\rangle f'(\lambda)d\lambda \qquad (6)$$
$$(\phi \in X^*\ ,\ x \in X\ ,\ f \in AC[a,b])\ .$$

Proof. The result is obtained by combining Theorems 1, 2 and 6 of [27].

Remark. If the well-bounded operator T is implemented by $(K,[a,b])$, then $\sigma(T) \subset [a,b]$ ([27], p.620, Corollary 1). The converse is also true (Lemma 4.2.1).

In general, the dual spectral family F associated with a well-bounded operator by (5) need not be unique, nor need the $F(\lambda)$'s be adjoints of projections in $B(X)$.

DEFINITION 1.3.6. A projection-valued function $E : \underline{R} \to B(X)$ is a weak spectral family if $E^* : \underline{R} \to B(X^*)$ (where $E^*(\lambda) = E(\lambda)^*$) is a dual spectral family.

PROPOSITION 1.3.7 ([27], Theorem 8). If E is a weak spectral family, then E^* is the unique dual spectral family associated with the well-bounded operator T given by

$$<\phi,Tx> = b<\phi,x> - \int_a^b <\phi,E(\lambda)x>d\lambda \quad (\phi \in X^* , x \in X) . \qquad (7)$$

Furthermore, $E(\lambda) \in \{T\}''$ $(\lambda \in \underline{R})$.

PROPOSITION 1.3.8 ([27], Theorem 9). If X is weakly complete, then a well-bounded operator $T \in B(X)$ has a unique dual spectral family if and only if is has a weak spectral family (i.e. if and only if there is a weak spectral family satisfying (7)).

A weak spectral family need not be a strong spectral family. (Examples are given in §6 of [3].)

DEFINITION 1.3.9. A well-bounded operator is said to be of type B if it has a strong spectral family which satisfies (7), i.e. if it has a (unique) dual spectral family which is the adjoint of a strong spectral family.

This concept was introduced in [3]. Well-bounded operators of type B are characterised as follows.

PROPOSITION 1.3.10 ([32], Theorem 5). Let $T \in B(X)$. Then the following statements are equivalent:

(i) T is a well-bounded operator of type B, implemented by $(K,[a,b])$.

(ii) there exists a (strong) spectral family $E : \underline{R} \to B(X)$, satisfying 1.3.2(iv′), such that

$$Tx = \int_{[a,b]}^{\oplus} \lambda dE(\lambda)x \quad (x \in X) ; \tag{8}$$

(iii) T is a well-bounded operator, implemented by $(K,[a,b])$, such that the functional calculus $\phi : AC[a,b] \to B(X)$ takes bounded sets to sets which are relatively compact in the weak operator topology;

(iv) T is a well-bounded operator, implemented by $(K,[a,b])$, such that, for each $x \in X$, the map $\phi_x : f \to f(T)x \ (f \in AC[a,b])$ is weakly compact;

(v) T is a well-bounded operator, implemented by $(K,[a,b])$, such that, for each $x \in X$, the map $\phi_x : f \to f(T)x \ (f \in AC[a,b])$ is compact.

If T satisfies (i)-(v), then $f(T)$ is given by

$$f(T)x = \int_{[a,b]}^{\oplus} f(\lambda)dE(\lambda)x \quad (x \in X) \tag{9}$$

for all $f \in AC[a,b]$.

Every well-bounded operator on a reflexive space is of type B. Berkson and Dowson [3] also define an intermediate concept, that of "type A", when there is a weak spectral family which need only satisfy 1.3.2(iii)(a), and not necessarily 1.3.2(iii)(b). However, the operator T given in Example 6.2 of [3] is of type A but not of type B, whereas $-T$ is not of type A; it follows from 1.3.10 (see 2.2.6) that, if T is of type B, then so is $-T$.

PROPOSITION 1.3.11 ([32], Theorem 6). Let $T \in B(X)$ be a well-bounded operator of type B, implemented by $(K,[a,b])$. Then the $AC[a,b]$-functional calculus for T extends to $BV[a,b]$, with the same norm K, and $f(T)$ is given by (9) for all $f \in BV[a,b]$. If $\{f_\alpha\}$ is a bounded net in $BV[a,b]$ converging pointwise to $f \in BV[a,b]$, then $f_\alpha(T)x \to f(T)x$ $(x \in X)$.

For an arbitrary well-bounded operator it is possible to obtain extensions of the homomorphism $f \to f(T^*) : AC[a,b] \to B(X^*)$ to the subalgebra of $NBV[a,b]$ consisting of those functions whose continuous singular parts vanish (see [27], Lemma 5). However, these extensions need not be unique.

§1.4. Appendix: BV[a,b] as a Banach algebra

We prove here the remark made in 1.2.1(i). We presume this proof to be known, but have been unable to locate it in the literature.

THEOREM 1.4.1. With the norm $|||\cdot|||_{[a,b]}$ defined in 1.2.1(i), $BV[a,b]$ is a Banach algebra.

Proof. It is slightly more convenient to prove the result for the norm

$$|||f|||^{*}_{[a,b]} = |f(a)| + \operatorname*{var}_{[a,b]} f \, .$$

From standard Banach algebra considerations, it is sufficient to work in the maximal ideal $BV_0[a,b]$ of functions vanishing at a, and to prove that $\operatorname*{var}_{[a,b]} fg \leq \operatorname*{var}_{[a,b]} f \operatorname*{var}_{[a,b]} g$ for all $f,g \in BV_0[a,b]$.

Let $f,g \in BV_0[a,b]$, and fix $\underline{t} \in \wp[a,b]$. We split f in the following way. Let

$$f_k(t) = \begin{cases} 0 & \overset{t_o}{\cancel{0}} \le t \le t_{k-1} \\ f(t) - f(t_{k-1}) & t_{k-1} \le t \le t_k \\ f(t_k) - f(t_{k-1}) & t \ge t_k \end{cases}$$

for $k=2,\ldots,m-1$, and

$$f_1(t) = \begin{cases} f(t) & 0 \le t \le t_1 \\ f(t_1) & t \ge t_1 \end{cases} ,$$

$$f_m(t) = \begin{cases} 0 & 0 \le t \le t_{m-1} \\ f(t) - f(t_{m-1}) & t \ge t_{m-1} \end{cases} .$$

Then

$$f = \sum_{k=1}^{m} f_k ,$$

$$\operatorname{var} f = \sum_{k=1}^{m} \operatorname*{var}_{[a,b]} f_k = \sum_{k=1}^{m} \operatorname*{var}_{[t_{k-1},t_k]} f_k ,$$

and

$$f_k(t_j) = \begin{cases} 0 & j < k \\ f(t_k) - f(t_{k-1}) & j \ge k \end{cases} .$$

Now,

$$\sum_{j=1}^{m} |f(t_j)g(t_j) - f(t_{j-1})g(t_{j-1})|$$

$$= \sum_{j=1}^{m} \left| \sum_{k=1}^{m} [f_k(t_j)g(t_j) - f_k(t_{j-1})g(t_{j-1})] \right|$$

$$\le \sum_{j=1}^{m} \sum_{k=1}^{m} |f_k(t_j)g(t_j) - f_k(t_{j-1})g(t_{j-1})|$$

$$= \sum_{k=1}^{m} \sum_{j=1}^{m} |f_k(t_j)g(t_j) - f_k(t_{j-1})g(t_{j-1})|$$

$$= \sum_{k=1}^{m} \Bigg(|[f(t_k) - f(t_{k-1})]g(t_k)|$$

$$+ \sum_{j=k+1}^{m} |[f(t_k) - f(t_{k-1})][g(t_j) - g(t_{j-1})]| \Bigg)$$

$$\leq \sum_{k=1}^{m} |f(t_k) - f(t_{k-1})| \Big(|g(t_k) - 0| + \sum_{j=k+1}^{m} |g(t_j) - g(t_{j-1})| \Big)$$

$$\leq \sum_{k=1}^{m} |f(t_k) - f(t_{k-1})| \underset{[a,b]}{\text{var}}\ g$$

$$\leq \underset{[a,b]}{\text{var}}\ f \ \underset{[a,b]}{\text{var}}\ g \ .$$

Since this is true for all $\underline{t} \in \mathcal{P}[a,b]$, it follows that $\underset{[a,b]}{\text{var}}\ fg \leq \underset{[a,b]}{\text{var}}\ f \ \underset{[a,b]}{\text{var}}\ g$ for all $f,g \in BV_0[a,b]$, and so the theorem is proved.

CHAPTER 2

SEMIGROUPS OF WELL-BOUNDED OPERATORS

In this chapter we discuss an unbounded analogue of the well-bounded operator. Such an operator, to be termed well-boundable, satisfies a form of the spectral theorem, with respect to a spectral family whose support (Definition 2.1.1) may be an unbounded subset of $\underline{R}$. The construction of the well-boundable operator may be thought of as the development of integrals similar to those of §1.2, but with an unbounded range of integration. It is also a parallel for well-bounded operators to the passage from bounded to unbounded scalar type spectral operators considered in [10], Chapter XVIII. In §2.1 we carry out the construction, and develop a functional calculus for the well-boundable operator.

In §2.2 we employ the functional calculus and various standard techniques to develop the spectral theory of the well-boundable operator. The main results are that the spectral family is unique (Theorem 2.2.8), and that its support is equal to the spectrum of the operator (Theorem 2.2.13).

There are some interesting examples, to be discussed in Chapter 3, of well-bounded operators which constitute strongly continuous semigroups. The final theorem of this chapter (Theorem 2.3.2) is a generalisation to semigroups of well-bounded operators of type B of the Hille-Sz.-Nagy theorem on strongly continuous semigroups of self-adjoint operators. It turns out that the infinitesimal generator of such a semigroup is a well-boundable operator.

§2.1. Construction of unbounded operators

In this section E will be a fixed spectral family for X, with the bound K as in 1.3.2(i).

DEFINITION 2.1.1. If E is a spectral family, then the support of E, denoted by supp E, is the set

$$\operatorname{supp} E = \underline{R} \setminus \{\lambda \in \underline{R} : E \text{ is constant in a neighbourhood of } \lambda\} .$$

Obviously supp E is always closed; in §1.3 we assumed supp E to be compact, but we now drop that assumption.

The following lemma is fundamental in ensuring that our unbounded operators have the correct domains. Before stating it, let us remark that if E is a spectral family, and

$$T : \bigcup_{a,b\in\underline{R}} (E(b) - E(a)) X \to X$$

is a linear operator, then $E(\lambda)\mathcal{D}(T) \subset \mathcal{D}(T)$ $(\lambda \in \underline{R})$. (Obviously $\mathcal{D}(T)$ is a linear space.) If, in addition, $E(\lambda)Tx = TE(\lambda)x$ $(\lambda \in \underline{R},\ x \in \mathcal{D}(T))$, then $T(E(b) - E(a))X \subset (E(b) - E(a))X$ $(a,b \in \underline{R})$.

LEMMA 2.1.2. Let E be a spectral family for X, and

$$T_0 : \bigcup_{a,b\in\underline{R}} (E(b) - E(a)) X \to X$$

a linear operator. Suppose that $E(\lambda)T_0x = T_0E(\lambda)x$ $(\lambda \in \underline{R},\ x \in \mathcal{D}(T_0))$, and that the restrictions $T_0|(E(b) - E(a))X$ are all bounded. Let $\{[a_n,b_n]\}$ be an increasing sequence of bounded real intervals, such that $a_n \to -\infty$ and $b_n \to +\infty$. Define the operator T by

$$\mathcal{D}(T) = \{x : \lim T_0(E(b_n) - E(a_n))x \text{ exists in } X\},$$

$$Tx = \lim T_0(E(b_n) - E(a_n))x \quad (x \in \mathcal{D}(T)) .$$

Then T is a closed, densely defined linear operator, independent of the choice of sequence $\{[a_n, b_n]\}$ and satisfying

(i) $E(\lambda)\mathfrak{D}(T) \subset \mathfrak{D}(T) \quad (\lambda \in \underline{R})$;

(ii) $TE(\lambda)x = E(\lambda)Tx \quad (\lambda \in \underline{R}\ ,\ x \in \mathfrak{D}(T))$;

(iii) $T(E(b) - E(a))x = T_0(E(b) - E(a))x \quad (a,b \in \underline{R}\ ,\ x \in X)$.

Proof. Clearly T is linear. Choose $a,b \in \underline{R}$, $x \in X$, and let $y = (E(b) - E(a))x$. There exists an N such that $[a,b] \subset [a_n,b_n]$ for all $n \geq N$. If $n \geq N$, then we have

$$\begin{aligned} T_0(E(b_n) - E(a_n))y &= T_0(E(b_n) - E(a_n))(E(b) - E(a))x \\ &= T_0(E(b) - E(a))x \\ &= T_0 y \ . \end{aligned}$$

Therefore $\lim T_0(E(b_n) - E(a_n))y$ exists and equals $T_0 y$. Hence $\mathfrak{D}(T_0) \subset \mathfrak{D}(T)$ and (iii) is proved.

For all $x \in X$, $E(b)x \to x$ as $b \to +\infty$ and $E(a)x \to 0$ as $a \to -\infty$, so $\mathfrak{D}(T_0)$, and hence also $\mathfrak{D}(T)$, is dense in X . Since

$$\begin{aligned} T_0(E(b_n) - E(a_n))E(\lambda)x &= E(\lambda)T_0(E(b_n) - E(a_n))x \\ &\to E(\lambda)Tx \quad , \end{aligned}$$

for all $x \in \mathfrak{D}(T)$, $\lambda \in \underline{R}$, we have (i) and (ii).

To check that T is closed, let $\{x_m\}$ be a sequence in $\mathfrak{D}(T)$ such that $x_m \to x$ and $Tx_m \to y$. For fixed n, $T_0(E(b_n) - E(a_n))x_m = (E(b_n) - E(a_n))Tx_m$, by (ii), so $T_0(E(b_n) - E(a_n))x_m \to (E(b_n) - E(a_n))y$ as $m \to \infty$. But since $T_0|(E(b_n) - E(a_n))X$ is bounded, $T_0(E(b_n) - E(a_n))x_m$ also tends to $T_0(E(b_n) - E(a_n))x$, as $m \to \infty$. Therefore, for all n ,

$$\begin{aligned} T_0(E(b_n) - E(a_n))x &= (E(b_n) - E(a_n))y \\ &\to y \end{aligned}$$

as $n \to \infty$, and so $x \in \mathfrak{D}(T)$ and $Tx = y$. Therefore T is closed.

For the uniqueness, suppose that $\{[c_n', d_n]\}$ is another increasing sequence of real intervals, with $c_n \to -\infty$ and $d_n \to +\infty$, and let the corresponding operator be $\hat{T}$. Let $x \in \mathfrak{D}(\hat{T})$. Then

$$\begin{aligned}\hat{T}x &= \lim T_0(E(d_n) - E(c_n))x \\ &= \lim T(E(d_n) - E(c_n))x \\ &= Tx ,\end{aligned}$$

since T is closed. So $\hat{T} \subset T$. Similarly $T \subset \hat{T}$, giving $T = \hat{T}$.

We now aim to give a meaning to the expression $\int_{\underline{R}}^{\oplus} \alpha dE$ when E is an arbitrary spectral family, and $\alpha \in LBV(\underline{R})$. It will turn out to be equal to a closed operator T_α constructed by the method of Lemma 2.1.2. There are two possible ways to construct this integral. Firstly, we can consider the strong limit as $a \to \infty$ of integrals over $[-a,a]$. Alternatively, we can look at limits of nets of Stieltjes sums over partitions of the extended real line $\overline{\underline{R}} = [-\infty,\infty]$. This second method may not work if $\alpha \notin BV(\underline{R})$. However, we shall show in 2.1.14 that when $\alpha \in BV(\underline{R})$ the two versions coincide, and the common value is a bounded operator.

Recall from 1.2.8 the value of the integral

$$\int_{[a,b]}^{\oplus} \alpha(\lambda) dE(\lambda)x = \begin{cases} \alpha(a)E(a)x + \int_a^b \alpha(\lambda) dE(\lambda)x & (\alpha \in NBV[a,b]) \\ \alpha(a)E(a)x + \int_{[a,b]}^{r} \alpha(\lambda) dE(\lambda)x & (\alpha \in BV[a,b] \setminus NBV[a,b]) \end{cases} .$$

In subsequent arguments we shall use $\int_{[a,b]}^{*} \alpha dE$ to denote the integral which is defined by $\int_a^b \alpha dE$ when $\alpha \in NBV[a,b]$ and by

$\int_{[a,b]}^{r} \alpha dE$ when $\alpha \in BV[a,b]\backslash NBV[a,b]$.

<u>DEFINITION 2.1.3.</u> For $\alpha \in LBV(\underline{R})$, $x \in X$, let

$$\int_{\underline{R}}^{\oplus} \alpha(\lambda)dE(\lambda)x = \lim_{u\to\infty} \int_{[-u,u]}^{*} \alpha(\lambda)dE(\lambda)x$$

whenever the limit exists. When $\alpha \in NLBV(\underline{R})$, we shall denote this integral by $\int_{-\infty}^{\infty} \alpha(\lambda)dE(\lambda)x$.

<u>DEFINITION 2.1.4.</u> For $\alpha \in LBV(\underline{R})$, define

$$T_{\alpha}^{(0)} : \bigcup_{a,b\in\underline{R}} (E(b) - E(a)) X = \bigcup_{u>0} (E(u) - E(-u)) X \to X$$

by

$$T_{\alpha}^{(0)}x = \int_{[-u,u]}^{\oplus} \alpha(\lambda)dE(\lambda)x \quad (x \in (E(u) - E(-u))X) \quad .$$

If $x \in (E(u) - E(-u))X$ and $x \in (E(v) - E(-v))X$, where $0 < u < v$, then the integrals over $[-u,u]$ and $[-v,v]$ have the same value, since

$$\int_{[-v,v]}^{\oplus} \alpha(\lambda)dE(\lambda)x = \alpha(-v)E(-v)x + \int_{[-v,v]}^{*} \alpha(\lambda)dE(\lambda)x$$

$$= 0 + \left(\int_{[-v,-u]}^{*} + \int_{[-u,u]}^{*} + \int_{[u,v]}^{*} \right) \alpha(\lambda)dE(\lambda)x$$

$$= 0 + \int_{[-u,u]}^{\oplus} \alpha(\lambda)dE(\lambda)x - \alpha(-u)E(-u)x + 0$$

$$= \int_{[-u,u]}^{\oplus} \alpha(\lambda) dE(\lambda)x \quad ,$$

and so $T_\alpha^{(0)}$ is well-defined.

Clearly $E(\lambda)T_\alpha^{(0)}x = T_\alpha^{(0)}E(\lambda)x \quad (\lambda \in \underline{R} \text{ , } x \in \mathfrak{D}(T_\alpha^{(0)}))$, and, if $[a,b] \subset [-u,u]$, then

$$\|T_\alpha^{(0)}x\| = \| \int_{[-u,u]}^{\oplus} \alpha(\lambda) dE(\lambda)x\| \leq K\|\|\alpha\|\|_{[-u,u]}\|x\| \qquad (1)$$

for all $x \in (E(b) - E(a))X$, by 1.2.8, and so $\overset{T_\alpha^{(0)}}{\cancel{T}}|(E(b) - E(a))X$ is bounded for all $a,b \in \underline{R}$.

The operator $T_\alpha^{(0)}$ therefore satisfies the hypotheses of Lemma 2.1.2; hence there exists a closed, densely defined linear operator T_α , extending $T_\alpha^{(0)}$, corresponding to the operator T of 2.1.2.

We now consider the operators $T_\alpha^{(u)}$ $(u > 0)$, where

$$T_\alpha^{(u)}x = \int_{[-u,u]}^{\oplus} \chi_{(-u,u]}(\lambda)\alpha(\lambda) dE(\lambda)x \quad (x \in X) \quad .$$

(Here χ_A denotes the indicator function of the set A ; $\chi_A(\lambda)=1$ if $\lambda \in A$, and $\chi_A(\lambda)=0$ if $\lambda \notin A$.) It follows from the next proposition and (1) that $T_\alpha^{(u)} \in B(X)$ for all $\alpha \in LBV(\underline{R})$, $u > 0$.

<u>PROPOSITION 2.1.5</u>. For all $\alpha \in LBV(\underline{R})$, $u > 0$ and $x \in X$,

$$\begin{aligned} T_\alpha^{(u)}x &= \int_{[-u,u]}^{\oplus} \alpha(\lambda) dE(\lambda)(E(u) - E(-u))x \\ &= T_\alpha^{(0)}(E(u) - E(-u))x \\ &= \int_{[-u,u]}^{\oplus} \alpha(\lambda) dE(\lambda)x - \alpha(-u)E(-u)x \quad . \end{aligned}$$

Proof. Letting $(\underline{w},\underline{w}^*)$ range over $\wp^*[-u,u]$ (or over $\wp^*_r[-u,u]$ if $\chi_{(-u,u]}\alpha \notin \mathrm{NBV}[-u,u]$), we have

$$
\begin{aligned}
T^{(u)}_\alpha x &= 0 + \int^*_{[-u,u]} \chi_{(-u,u]}(\lambda)\alpha(\lambda)dE(\lambda)x \\
&= \lim \sum(\chi_{(-u,u]}\alpha, \Delta E, \underline{w}, \underline{w}^*)x \\
&= \lim \chi_{(-u,u]}(w_1^*)\alpha(w_1^*)(E(w_1)-E(-u))x \\
&\qquad + \lim \sum_{k=2}^{m} \alpha(w_k^*)(E(w_k)-E(w_{k-1}))x \quad .
\end{aligned}
$$

$$
\begin{aligned}
\int^{\oplus}_{[-u,u]} \alpha(\lambda)dE(\lambda)(E(u)-E(-u))x &= \alpha(-u)E(-u)(E(u)-E(-u))x \\
&\qquad + \int^*_{[-u,u]} \alpha(\lambda)dE(\lambda)(E(u)-E(-u))x \\
&= 0 + \lim \sum(\alpha, \Delta E, \underline{w}, \underline{w}^*)x \quad .
\end{aligned}
$$

The difference between corresponding sums in the two nets is $\chi_{(-u,u]}(w_1^*)\alpha(w_1^*)(E(w_1)-E(-u))x$, which tends to zero, proving equality between the first two members of the assertion.

The fact that

$$
\int^{\oplus}_{[-u,u]} \alpha(\lambda)dE(\lambda)(E(u)-E(-u))x = T^{(0)}_\alpha(E(u)-E(-u))x
$$

is immediate by definition.

Finally,

$$
\begin{aligned}
\int^{\oplus}_{[-u,u]} \alpha(\lambda)dE(\lambda)x - T^{(u)}_\alpha x &= \int^{\oplus}_{[-u,u]} \chi_{\{-u\}}(\lambda)\alpha(\lambda)dE(\lambda)x \\
&= \alpha(-u)E(-u)x + \int_{-u}^{u} \chi_{\{-u\}}(\lambda)\alpha(\lambda)dE(\lambda)x \\
&= \alpha(-u)E(-u)x \quad .
\end{aligned}
$$

COROLLARY 2.1.6. For $\alpha \in \mathrm{LBV}(\underline{R})$, $x \in \mathcal{D}(T_\alpha)$ if and only if $\lim T_\alpha^{(u)}x$ exists in X as $u \to \infty$. In that case $T_\alpha x = \lim T_\alpha^{(u)}x$.

Proof. The result follows from the first two equalities in 2.1.5.

COROLLARY 2.1.7. If $\alpha \in \mathrm{LBV}(\underline{R})$, then $x \in \mathcal{D}(T_\alpha)$ if and only if $\int_{\underline{R}}^{\oplus} \alpha(\lambda)dE(\lambda)x$ exists, in which case $T_\alpha x = \int_{\underline{R}}^{\oplus} \alpha(\lambda)dE(\lambda)x$.

Proof. The result follows from 2.1.6, the definition of $\int_{\underline{R}}^{\oplus}$, and the last equality in 2.1.5.

We now consider integrals defined as net limits of Stieltjes sums over subdivisions of $\overline{\underline{R}}$.

DEFINITION 2.1.8. A subdivision of $\overline{\underline{R}}$ is a finite sequence $\underline{t} = \{t_k\}_{k=0}^{m}$ of points such that

$$-\infty = t_0 < t_1 < \dots < t_m = +\infty \ .$$

The set of all subdivisions of $\overline{\underline{R}}$ will be denoted by $\mathcal{P}(\underline{R})$.

A marked partition of $\overline{\underline{R}}$ is a pair $(\underline{t},\underline{t}^*)$, where $\underline{t} \in \mathcal{P}(\underline{R})$ and $\underline{t}^*$ is a sequence $\{t_k^*\}_{k=1}^{m}$ such that $t_k^* \in [t_{k-1},t_k]$ $(k=1,\dots m)$; we allow $t_1^*=-\infty$, $t_m^*=+\infty$. $\mathcal{P}^*(\underline{R})$ denotes the set of all marked partitions of $\overline{\underline{R}}$. The relation $\leq$ and the subsets $\mathcal{P}_r^*(\underline{R})$, $\mathcal{P}_i^*(\underline{R})$ have similar meanings to those in 1.2.2, and sums $\sum(f,\Delta g,\underline{t},\underline{t}^*)$ are defined analogously to those in 1.2.3.

DEFINITION 2.1.9. For $\alpha \in \mathrm{BV}(\underline{R})$, let

$$\mathcal{P}^*(\alpha,\underline{R}) = \begin{cases} \mathcal{P}^*(\underline{R}) & \text{if } \alpha \in \mathrm{NBV}(\underline{R}) \\ \mathcal{P}_r^*(\underline{R}) & \text{if } \alpha \in \mathrm{BV}(\underline{R})\setminus\mathrm{NBV}(\underline{R}) \end{cases} \ .$$

We define $S(\alpha, dE, \underline{R})$ to be the integral

$$S(\alpha, dE, \underline{R}) = \lim_{(\underline{t},\underline{t}^*) \in \mathcal{P}^*(\alpha,\underline{R})} \sum(\alpha, \Delta E, \underline{t}, \underline{t}^*)$$

in the strong operator topology. (We take $\alpha(\underline{+}\infty) = \lim_{t \to \underline{+}\infty} \alpha(t)$, $E(-\infty) = 0$, $E(+\infty) = I$.)

PROPOSITION 2.1.10. Let (a,b) be any bounded open interval and $\phi : (a,b) \to \underline{R}$ an increasing homeomorphism with $\lim_{t \to a+0} \phi(t) = -\infty$, $\lim_{t \to b-0} \phi(t) = +\infty$. Then the mapping $(\underline{t},\underline{t}^*) \to (\underline{v},\underline{v}^*)$, where $v_k = \phi(t_k)$, $v_k^* = \phi(t_k^*)$, determines a one-to-one correspondence from $\mathcal{P}^*[a,b]$ onto $\mathcal{P}^*(\underline{R})$, which preserves the partial order, and also takes $\mathcal{P}_r^*[a,b]$, $\mathcal{P}_i^*[a,b]$ onto $\mathcal{P}_r^*(\underline{R})$, $\mathcal{P}_i^*(\underline{R})$ respectively.

PROPOSITION 2.1.11. Let ϕ be as in 2.1.10. For each $\alpha \in BV(\underline{R})$, define $\tilde{\alpha} \in BV[a,b]$ by

$$\tilde{\alpha}(\lambda) = \alpha(\phi(\lambda)) \quad (\lambda \in (a,b)) \quad ; \quad \tilde{\alpha}(a) = \alpha(-\infty) \quad ; \quad \tilde{\alpha}(b) = (+\infty) .$$

Similarly, set

$$\tilde{E}(\lambda) = E(\phi(\lambda)) \quad (\lambda \in (a,b)) \quad ; \quad \tilde{E}(a) = 0 \quad ; \quad \tilde{E}(b) = I .$$

Then $S(\alpha, dE, \underline{R})$ exists and equals $\int_{[a,b]}^{\oplus} \tilde{\alpha}\, d\tilde{E}$.

Proof. The result follows immediately from 2.1.10, since $\alpha \to \tilde{\alpha}$ preserves the continuity properties of α , and so the corresponding limits are taken over the correct nets of Stieltjes sums.

LEMMA 2.1.12. For $\alpha \in BV(\underline{R})$ and $u > 0$, let $\alpha_u = \chi_{(-u,u]}\alpha$. Then

$$S(\alpha, dE, \underline{R}) = \lim_{u \to \infty} S(\alpha_u, dE, \underline{R}) ,$$

in the strong operator topology.

__Proof.__ The net $\{\alpha_u\}_{u>0}$ is bounded in the topology of $BV(\underline{R})$, with $\alpha_u(t) \to \alpha(t)$ as $u \to \infty$, for all $t \in \underline{R}$. Consequently $\{\widetilde{\alpha}_u\}_{u>0}$ is a bounded net in $BV[a,b]$ with

$$\widetilde{\alpha}_u(\lambda) \to \widetilde{\alpha}(\lambda) - \hat{\alpha}(\lambda) \quad (\lambda \in [a,b]) ,$$

where

$$\hat{\alpha}(\lambda) = \begin{cases} \widetilde{\alpha}(a) = \alpha(-\infty) & (\lambda = a) \\ \widetilde{\alpha}(b) = \alpha(+\infty) & (\lambda = b) \\ 0 & (a < \lambda < b) . \end{cases}$$

By 1.2.6, therefore,

$$\oint_{[a,b]} \widetilde{E}d\widetilde{\alpha} = \lim_{u\to\infty} \oint_{[a,b]} \widetilde{E}d\widetilde{\alpha}_u + \oint_{[a,b]} \widetilde{E}d\hat{\alpha}$$

$$= \lim_{u\to\infty} \widetilde{\alpha}_u(b)\widetilde{E}(b) - \lim_{u\to\infty} \int_{[a,b]}^{\oplus} \widetilde{\alpha}_u d\widetilde{E} + \oint_{[a,b]} \widetilde{E}d\hat{\alpha} .$$

We have $\widetilde{\alpha}_u(b) = 0 \; (u > 0)$, and

$$\oint_{[a,b]} \widetilde{E}d\hat{\alpha} = \lim_{(\underline{t},\underline{t}^*)\in\mathcal{P}_r^*[a,b]} \sum (E, \Delta\hat{\alpha}, \underline{t}, \underline{t}^*)$$

$$= \lim \, (\widetilde{\alpha}(b)\widetilde{E}(t_m^*) - \widetilde{\alpha}(a)\widetilde{E}(t_1^*))$$

$$= \alpha(\infty)I .$$

Thus

$$\int_{[a,b]}^{\oplus} \widetilde{\alpha}d\widetilde{E} = \widetilde{\alpha}(b)\widetilde{E}(b) + \lim_{u} \int_{[a,b]}^{\oplus} \widetilde{\alpha}_u d\widetilde{E} - \oint_{[a,b]} \widetilde{E}d\hat{\alpha}$$

$$= \alpha(\infty)I + \lim_{u} \int_{[a,b]}^{\oplus} \widetilde{\alpha}_u d\widetilde{E} - \alpha(\infty)I$$

$$= \lim_{u} S(\alpha_u, dE, \underline{R}) ,$$

in the strong operator topology. The result therefore follows from 2.1.11.

LEMMA 2.1.13. If $\alpha \in BV[-u,u]$ is considered as a member of $BV(\underline{R})$ by letting $\alpha(t) = 0 \ (t \notin [-u,u])$, then

$$\text{(i)} \ S(\alpha, dE, \underline{R}) = \int_{[-u,u]}^{\oplus} \alpha dE - \alpha(-u)E(-u-0) ;$$

$$\text{(ii)} \ S(\chi_{(-u,u]}\alpha, dE, \underline{R}) = S(\alpha, dE, \underline{R}) - \alpha(-u)(E(-u)-E(-u-0)) .$$

In particular, if $\beta \in BV(\underline{R})$, then

$$\text{(iii)} \ S(\beta_u, dE, \underline{R}) = \int_{[-u,u]}^{\oplus} \beta_u dE .$$

Proof. Consider $\underline{w} = \{w_k\}_{k=0}^{m} \in \mathcal{P}(\underline{R})$, where $w_1 = -u$ and $w_{m-1} = u$. A typical refinement of $\underline{w}$ is of the form $\underline{v}$, where

$$-\infty = v_0 < v_1 < \ldots < v_{n_1} = -u < v_{n_1+1} < \ldots$$

$$\ldots < v_{n_{m-1}} = u < \ldots < v_{n_m} = +\infty ,$$

and $v_{n_j} = w_j \ (j=1,2,\ldots,m)$. A corresponding sum is

$$\sum(\alpha, \Delta E, \underline{v}, \underline{v}^*) = \alpha(v^*_{n_1})(E(-u)-E(v_{n_1-1})) + \sum_{k=n_1+1}^{n_{m-1}} \alpha(v^*_k)(E(v_k)-E(v_{k-1}))$$

$$+ \alpha(v^*_{n_{m-1}+1})(E(v_{n_{m-1}+1})-E(u)) ,$$

all other terms vanishing. The last term tends to 0 , since E is right-continuous, and the sum in the middle tends to $\int_{[-u,u]}^{*} \alpha dE$.

If α is left-continuous on $\underline{R}$, then $\alpha(t) = 0 \ (t \leq -u)$, and if not then each $v^*_{n_1} = -u$, so in either case the first term tends to $\alpha(-u)(E(-u)-E(-u-0))$. Therefore

$$S(\alpha,dE,\underline{R}) = \int_{[-u,u]}^{*} \alpha dE + \alpha(-u)(E(-u)-E(-u-0))$$

$$= \int_{[-u,u]}^{\oplus} \alpha dE - \alpha(-u)E(u-0) \quad ,$$

and so (i) is proved. To obtain (ii), we see that

$$S(\chi_{(-u,u]}\alpha,dE,\underline{R}) = \int_{[-u,u]}^{\oplus} \chi_{(-u,u]}\alpha dE$$

$$= \int_{[-u,u]}^{\oplus} \alpha dE - \alpha(-u)E(-u) \quad ,$$

by 2.1.5. Part (iii) follows immediately from (i) .

We summarise the result for $\alpha \in BV(\underline{R})$ in the following theorem:

<u>THEOREM 2.1.14</u>. For each $\alpha \in BV(\underline{R})$, $\mathfrak{D}(T_\alpha) = X$ and

$$T_\alpha x = \lim_{u\to\infty} T^{(0)}_\alpha (E(u)-E(-u))x$$

$$= \int_{\underline{R}}^{\oplus} \alpha(\lambda) dE(\lambda) x$$

$$= S(\alpha,dE,\underline{R})x \quad (x \in X) \ .$$

Thus $T_\alpha \in B(X)$, and moreover $\|T_\alpha\| \leq K|||\alpha|||_{\underline{R}}$.

<u>Proof</u>. For each $x \in X$ and $u > 0$ we have, by 2.1.5 and 2.1.13(iii),

$$\begin{aligned} T_\alpha^{(u)}x &= T_\alpha^{(0)}(E(u)-E(-u))x \\ &= \int_{[-u,u]}^{\oplus} \alpha_u(\lambda)dE(\lambda)x \\ &= S(\alpha_u,dE,\underline{R})x \ . \end{aligned}$$

Letting $u \to \infty$, therefore, the equalities in the statement follow from 2.1.6, 2.1.7 and 2.1.12.

To obtain the bound for $\|T_\alpha\|$, note that from (1) of 2.1.4 we have

$$\|T_\alpha^{(0)}x\| \leq K|\|\alpha\||_{[-u,u]}\|x\| \leq K|\|\alpha\||_{\underline{R}}\|x\| \quad (x \in (E(u)-E(-u))X\ ) ,$$

and therefore $\|T_\alpha^{(0)}x\| \leq K|\|\alpha\||_{\underline{R}}\|x\| \quad (x \in \mathcal{D}(T_\alpha^{(0)}))$; hence in this case T_α is just the extension by continuity of $T_\alpha^{(0)}$ to X, and so $\|T_\alpha\| \leq K|\|\alpha\||_{\underline{R}}$.

We next obtain the multiplicative property of the representation $\alpha \to T_\alpha$.

<u>PROPOSITION 2.1.15</u>. If $\alpha,\beta \in LBV(\underline{R})$, $x \in X$ and $u > 0$, then $T_{\alpha\beta}^{(u)}x = T_\alpha^{(u)}T_\beta^{(u)}x$.

<u>Proof</u>. Let $F^{(u)}(\lambda) = E(\lambda)|(E(u)-E(-u))X \quad (\lambda \in \underline{R})$. Then

$$F^{(u)}(\lambda) = \begin{cases} 0|(E(u)-E(-u))X & (\lambda \leq -u) \\ (E(\lambda)-E(-u))|(E(u)-E(-u))X & (-u < \lambda < u) \\ I|(E(u)-E(-u))X & (\lambda \geq u) \ . \end{cases}$$

Since $F^{(u)}$ clearly has the required continuity property, it is a spectral family for $(E(u)-E(-u))X$, with $\operatorname{supp} F^{(u)} \subset [-u,u]$. Hence, by 1.3.10 and 1.3.11, the operator $S^{(u)} \in B((E(u)-E(-u))X)$, where

$$S^{(u)}x = \int_{[-u,u]}^{\oplus} \lambda dF^{(u)}(\lambda)x \quad (x \in (E(u)-E(-u))X\ ) ,$$

is a well-bounded operator of type B, with functional calculus given by

$$f(S^{(u)})x = \int_{[-u,u]}^{\oplus} f(\lambda)dF^{(u)}(\lambda)x \quad (f \in BV[-u,u],\ x \in (E(u)-E(-u))X\ ) .$$

Since, when $y = (E(u)-E(-u))x$,

$$\int_{[-u,u]}^{\oplus} \alpha(\lambda)dF^{(u)}(\lambda)y = \int_{[-u,u]}^{\oplus} \alpha(\lambda)dE(\lambda)(E(u)-E(-u))x = T_{\alpha}^{(u)}x$$

by 2.1.5, it follows that

$$T_{\alpha\beta}^{(u)}x = \int_{[-u,u]}^{\oplus} \alpha(\lambda)\beta(\lambda)dF^{(u)}(\lambda)y$$

$$= \int_{[-u,u]}^{\oplus} \alpha(\lambda)dF^{(u)}(\lambda)(E(u)-E(-u)) \int_{[-u,u]}^{\oplus} \beta(\lambda)dF^{(u)}(\lambda)y$$

$$= T_{\alpha}^{(u)}T_{\beta}^{(u)}x .$$

<u>PROPOSITION 2.1.16</u>. For all $\alpha,\beta \in LBV(\underline{R})$,

(i) $T_{\alpha+\beta} \supset T_{\alpha} + T_{\beta}$;

(ii) $T_{\alpha\beta} \supset T_{\alpha}T_{\beta}$ and $\mathfrak{D}(T_{\alpha}T_{\beta}) = \mathfrak{D}(T_{\alpha\beta}) \cap \mathfrak{D}(T_{\beta})$.

<u>Proof</u>. (i) If $x \in \mathfrak{D}(T_{\alpha} + T_{\beta}) = \mathfrak{D}(T_{\alpha}) \cap \mathfrak{D}(T_{\beta})$, then

$$T_{\alpha+\beta}^{(n)}x = T_{\alpha}^{(n)}x + T_{\beta}^{(n)}x \rightarrow T_{\alpha}x + T_{\beta}x .$$

Therefore $x \in \mathfrak{D}(T_{\alpha+\beta})$ and $T_{\alpha+\beta}x = T_{\alpha}x + T_{\beta}x$.

(ii) If $x \in \mathfrak{D}(T_{\alpha}T_{\beta})$ then for $n=1,2,\ldots$,

$$\begin{aligned} T_{\alpha\beta}^{(n)}x &= T_{\alpha}^{(n)}T_{\beta}^{(n)}x \\ &= T_{\alpha}^{(0)}(E(n)-E(-n))T_{\beta}^{(0)}(E(n)-E(-n))x \\ &= T_{\alpha}(E(n)-E(-n))T_{\beta}(E(n)-E(-n))x \\ &= T_{\alpha}(E(n)-E(-n))T_{\beta}x \end{aligned}$$

$$= (E(n)-E(-n))T_\alpha T_\beta x$$
$$\to T_\alpha T_\beta x \; ,$$

using 2.1.15, 2.1.5, 2.1.2(iii), and finally 2.1.2(ii) twice, noting that $x \in \mathfrak{D}(T_\beta)$ and $T_\beta x \in \mathfrak{D}(T_\alpha)$. Therefore $\lim_n T^{(n)}_{\alpha\beta}\, x$ exists and equals $T_\alpha T_\beta x$, so $x \in \mathfrak{D}(T_{\alpha\beta})$ and $T_{\alpha\beta}x = T_\alpha T_\beta x$.

If $x \in \mathfrak{D}(T_{\alpha\beta}) \cap \mathfrak{D}(T_\beta)$, then

$$\begin{aligned} T^{(n)}_\alpha T_\beta x &= T^{(n)}_\alpha (E(n)-E(-n))T_\beta x \\ &= T^{(n)}_\alpha T_\beta (E(n)-E(-n))x \\ &= T^{(n)}_\alpha T^{(n)}_\beta (E(n)-E(-n))x \\ &= T^{(n)}_{\alpha\beta}\, (E(n)-E(-n))x \\ &= T^{(n)}_{\alpha\beta}\, x \\ &\to T_{\alpha\beta}x \; , \end{aligned}$$

using 2.1.5, 2.1.2(ii), 2.1.5 and 2.1.2(ii), 2.1.15 and 2.1.5 in the respective steps. Therefore $\lim_n T^{(n)}_\alpha T_\beta x$ exists and equals $T_{\alpha\beta}x$, so $T_\beta x \in \mathfrak{D}(T_\alpha)$ and $T_\alpha T_\beta x = T_{\alpha\beta}x$.

<u>PROPOSITION 2.1.17</u>. $T_1 = I$.

<u>Proof</u>. Let a,b and $\widetilde{E}$ be as in 2.1.11. Then

$$\begin{aligned} T_1 &= S(1,dE,\underline{R}) \\ &= \int^{\oplus}_{[a,b]} 1d\widetilde{E} \\ &= \widetilde{E}(a) + \int_a^b 1d\widetilde{E} \\ &= 0 + \widetilde{E}(b) - \widetilde{E}(a) \\ &= I \; . \end{aligned}$$

In the applications we shall need to use certain integration by parts, change of variable and change of order of integration processes. We have not investigated the best possible results along these lines, but Lemmas 2.1.18, 2.1.20 and 2.1.25 will suffice for the applications we shall make.

<u>LEMMA 2.1.18</u>. If $\alpha \in LBV(\underline{R})$ with $\alpha(t)$ bounded as $t \to -\infty$, and $\alpha(t) \to 0$ as $t \to \infty$, then

$$\int_{\underline{R}}^{\oplus} \alpha(\lambda) dE(\lambda)x = -\lim_{u\to\infty} \oint_{[-u,u]} E(\lambda)x d\alpha(\lambda) \quad (x \in \mathfrak{D}(T_\alpha)) .$$

<u>Proof</u>. If $x \in \mathfrak{D}(T_\alpha)$, then

$$\begin{aligned} \int_{\underline{R}}^{\oplus} \alpha(\lambda) dE(\lambda)x &= \lim_{u\to\infty} \int_{[-u,u]}^{*} \alpha(\lambda) dE(\lambda)x \\ &= \lim_{u\to\infty} \left[\int_{[-u,u]}^{\oplus} \alpha(\lambda) dE(\lambda)x - \alpha(-u)E(-u)x \right] \quad (2) \\ &= \lim_{u\to\infty} \left[\alpha(u)E(u)x - \oint_{[-u,u]} E(\lambda)x d\alpha(\lambda) \right] \quad (3) \\ &= -\lim_{u\to\infty} \oint_{[-u,u]} E(\lambda)x d\alpha(\lambda) , \end{aligned}$$

the equality between (2) and (3) following from 1.2.7.

<u>DEFINITION 2.1.19</u>. For $\alpha \in NLBV(\underline{R})$ and $x \in \mathfrak{D}(T_\alpha)$, we define $\int_{-\infty}^{\infty} E(\lambda)x d\alpha(\lambda)$ to be $\lim_{u\to\infty} \int_{-u}^{u} E(\lambda)x d\alpha(\lambda)$.

Thus, if α satisfies the hypotheses of 2.1.18, we have

$$\int_{\underline{R}}^{\oplus} \alpha(\lambda)dE(\lambda)x \;=\; -\int_{-\infty}^{\infty} \overset{E(\lambda)x\,d\alpha(\lambda)}{\cancel{\alpha(\lambda)dE(\lambda)x}} \;.$$

It is easily seen that the method used in 2.1.11 generalises in the following way:

<u>LEMMA 2.1.20</u>. If $\phi : [c,d] \to [a,b]$ (where a,b,c,d may take the values $\pm\infty$) is an increasing homeomorphism, and E and F are spectral families, with supports contained respectively in $[a,b]$ and $[c,d]$, such that $F(\lambda) = E(\phi(\lambda))$ $(\lambda \in [c,d])$, then for each $\alpha \in BV[a,b]$ we have

$$\int_{[a,b]}^{*} \alpha(\lambda)dE(\lambda) \;=\; \int_{[c,d]}^{*} \alpha(\phi(\lambda))dF(\lambda) \;,$$

where, if either interval is infinite, $\int^{*}$ is a net integral with a similar meaning to the finite case (see remarks before 2.1.3).

<u>Proof</u>. $(\underline{t},\underline{t}^*) \to (\underline{w},\underline{w}^*)$, where $w_k = \phi(t_k)$ and $w_k^* = \phi(t_k^*)$ sets up an order-preserving correspondence between $\wp^*[c,d]$ and $\wp^*[a,b]$, and since ϕ is a homeomorphism, the integrals for α and $\alpha\circ\phi$ are taken over correctly corresponding nets of Stieltjes sums.

<u>LEMMA 2.1.21</u>. Let $[a,b]$ be a compact interval and $g : [a,b] \to \underline{C}$ a continuously differentiable function. Then, for each $x \in X$, $\int_a^b g'(\lambda)E(\lambda)x d\lambda$ exists and equals $\int_a^b E(\lambda)x dg(\lambda)$.

<u>Proof</u>. Let $\epsilon > 0$, and let $(\underline{s},\underline{s}^*) \in \wp^*[a,b]$ be such that

$$\|\sum (E,dg,\underline{t},\underline{t}^*)x - \int_a^b E(\lambda)xdg(\lambda)\| < \epsilon$$

whenever $(\underline{t},\underline{t}^*) \geq (\underline{s},\underline{s}^*)$. Let $\delta > 0$ be such that $|g'(s)-g'(t)| < \epsilon$ if $|s-t| < \delta$ and $s,t \in [a,b]$. Let $\underline{v}$ be a refinement of $\underline{s}$ such that $\max(v_i - v_{i-1}) < \delta$. Let $(\underline{u},\underline{u}^*) \in \mathcal{P}^*[a,b]$ with $\underline{u} \geq \underline{v}$. Then, if $j(\lambda) \equiv \lambda$,

$$\begin{aligned} &\|\sum (E,dg,\underline{u},\underline{u}^*)x - \sum (g'E,dj,\underline{u},\underline{u}^*)x\| \\ &= \|\sum E(u_i^*)(g'(u_i^{**})-g'(u_i^*))(u_i-u_{i-1})x\| \\ &\leq K\|x\|\epsilon(b-a) \quad , \end{aligned}$$

each u_i^{**} being a point in (u_{i-1},u_i) whose existence is given by the mean value theorem. Thus

$$\|\sum (g'E,dj,\underline{u},\underline{u}^*)x - \int_a^b E(\lambda)xdg(\lambda)\| \leq (K\|x\|(b-a) + 1)\epsilon$$

whenever $\underline{u} \geq \underline{v}$, giving the required result.

COROLLARY 2.1.22. If $f : [a,b] \to \underline{C}$ is continuous, then $\int_a^b f(\lambda)E(\lambda)xd\lambda$ exists.

Proof. Take $g(\lambda) = \int_a^\lambda f(t)dt$ in 2.1.21.

LEMMA 2.1.23. Let $f : \underline{R} \to \underline{C}$ be continuous, with $\int_{-\infty}^{\infty} |f| < \infty$. Then $\lim_{u\to\infty} \int_{-u}^{u} f(\lambda)E(\lambda)xd\lambda$ exists for each $x \in X$, defining a bounded linear operator, to be denoted by $\int_{-\infty}^{\infty} f(\lambda)E(\lambda)d\lambda$. A similar result holds with $\underline{R}$ replaced by $(-\infty,b]$.

Proof. Let $g(\lambda) = \int_{-\infty}^{\lambda} f(t)dt$. Then g is continuously differentiable and belongs to $BV(\underline{R})$. Using 2.1.21, we have

$$\begin{aligned}\int_{-u}^{u} f(\lambda)E(\lambda)xd\lambda &= \int_{-u}^{u} E(\lambda)xdg(\lambda) \\ &= g(u)E(u)x - g(-u)E(-u)x - \int_{-u}^{u} g(\lambda)dE(\lambda)x \\ &\to \left(\int_{-\infty}^{\infty} f(\lambda)d\lambda\right) x - \int_{\underline{R}}^{\oplus} g(\lambda)dE(\lambda)x \quad .\end{aligned}$$

This gives the required result.

COROLLARY 2.1.24. If $g : \underline{R} \to \underline{C}$ is a function such that g' is continuous and integrable, then

$$\int_{-\infty}^{\infty} E(\lambda)xdg(\lambda) = \int_{-\infty}^{\infty} g'(\lambda)E(\lambda)xd\lambda \quad .$$

A similar result holds with $\underline{R}$ replaced by $(-\infty,b]$.

LEMMA 2.1.25. Let $g : \underline{R} \times [c,d] \to \underline{C}$ ($[c,d]$ compact) be continuous, and let $x \in X$. Suppose that

(i) $\int_{-\infty}^{\infty}\int_{c}^{d} |g(t,u)|\,dudt < \infty$;

(ii) for each $u \in [c,d]$ the function $g(\cdot,u)$ is integrable over $(-\infty,\infty)$, and the X-valued function

$$u \to \int_{-\infty}^{\infty} g(t,u)E(t)xdt$$

is continuous on $[c,d]$. Then the integrals

$$\int_{c}^{d} \left[\int_{-\infty}^{\infty} g(t,u)E(t)xdt\right] du$$

and

$$\int_{-\infty}^{\infty}\left[\int_{c}^{d} g(t,u)du\right]E(t)x dt$$

are defined and equal. A similar result holds with $\underline{R}$ replaced by $(-\infty,b]$.

Proof. The first integral exists since the integrand $\int_{-\infty}^{\infty} g(t,u)E(t)x dt$ is, by hypothesis, a continuous X-valued function on $[c,d]$. The second is defined, by 2.1.23, since the function

$$t \to \int_{c}^{d} g(t,u)du$$

is continuous and integrable on $\underline{R}$. For each $x^* \in X^*$,

$$\langle x^*, \int_{c}^{d}\left[\int_{-\infty}^{\infty} g(t,u)E(t)x dt\right] du\rangle = \int_{c}^{d}\left[\int_{-\infty}^{\infty}\langle x^*, g(t,u)E(t)x\rangle dt\right] du$$

and

$$\langle x^*, \int_{-\infty}^{\infty}\left[\int_{c}^{d} g(t,u)du\right] E(t)x dt\rangle = \int_{-\infty}^{\infty}\left[\int_{c}^{d}\langle x^*, g(t,u)E(t)x\rangle du\right] dt .$$

Since $\langle x^*, g(t,u)E(t)x\rangle \leq K\|x^*\| \, \|x\| \, |g(t,u)|$, hypothesis (i) and Fubini's theorem, together with the Hahn-Banach theorem, give the required equality of integrals.

§2.2. Spectral theory of well-boundable operators

DEFINITION 2.2.1. A closed linear operator $T : \mathcal{D}(T) \subset X \to X$ (in general unbounded) is well-boundable if there exists a spectral family $E : \underline{R} \to B(X)$ for which $T = T_j$, where j is the function $j(\lambda) \equiv \lambda$ $(\lambda \in \underline{R})$.

The aim of this section is to demonstrate the uniqueness of the spectral family of a well-boundable operator, and to identify $\sigma(T)$

as the support of the spectral family of T. Until the uniqueness has been proved, we shall describe a spectral family E satisfying 2.2.1 as a "spectral family for T". As in §2.1, we shall denote $\sup_{\lambda\in\underline{R}} \|E(\lambda)\|$ by K.

If $T : \mathfrak{D}(T) \subset X \to X$ is a closed linear operator, and $\mu \in \rho(T)$, then we shall denote the resolvent, $(\mu I-T)^{-1}$, by $R(\mu;T)$.

<u>PROPOSITION 2.2.2</u>. If $T : \mathfrak{D}(T) \subset X \to X$ is a well-boundable operator and E is a spectral family for T, then $\sigma(T) \subset \underline{R}$ and, for $\mu \in \underline{C}\backslash\underline{R}$, $R(\mu;T)$ is given by

$$R(\mu;T)x = \int_{\underline{R}}^{\oplus} (\mu-\lambda)^{-1} dE(\lambda)x \qquad (x \in X) .$$

Furthermore,

$$\|R(\mu;T)\| = O(|\mathrm{Im}\ \mu|^{-1}) .$$

<u>Proof</u>. Let $\alpha(\lambda) = \mu-\lambda$, $\beta(\lambda) = (\mu-\lambda)^{-1}$ $(\lambda \in \underline{R})$. Note that

$$\underset{\underline{R}}{\mathrm{var}}\ \beta = \int_{-\infty}^{\infty} \frac{d\lambda}{|\mu-\lambda|^2} = \frac{\pi}{|\mathrm{Im}\ \mu|} ,$$

so $\beta \in BV(\underline{R})$. By 2.1.16, $T_{\alpha\beta} \supset T_\alpha T_\beta$ and $\mathfrak{D}(T_\alpha T_\beta) = \mathfrak{D}(T_{\alpha\beta}) \cap \mathfrak{D}(T_\beta)$. Using 2.1.17, this gives $I = T_1 \supset T_{\mu-j}T_{(\mu-j)^{-1}}$ and $\mathfrak{D}(T_{\mu-j}T_{(\mu-j)^{-1}}) = \mathfrak{D}(T_1) \cap \mathfrak{D}(T_{(\mu-j)^{-1}}) = X$. Thus

$$I = (\mu I-T)T_{(\mu-j)^{-1}} . \qquad (1)$$

Similarly, $I = T_1 \supset T_{(\mu-j)^{-1}}T_{\mu-j}$ and $\mathfrak{D}(T_{(\mu-j)^{-1}}T_{\mu-j}) = \mathfrak{D}(T_1) \cap \mathfrak{D}(T_{\mu-j}) = \mathfrak{D}(T)$. Thus

$$x = T_{(\mu-j)^{-1}}(\mu I-T)x \quad (x \in \mathfrak{D}(T)) . \qquad (2)$$

Combining (1) and (2), we have

$$T_{(\mu-j)^{-1}} = R(\mu;T) = \int_{\underline{R}}^{\oplus} (\mu-\lambda)^{-1} dE(\lambda) .$$

By 2.1.14,

$$\begin{aligned} \|R(\mu;T)\| &\leq K\|\|(\mu-j)^{-1}\|\|_{\underline{R}} \\ &= K(|\operatorname{Im}\mu|^{-1} + \int_{-\infty}^{\infty} \frac{d\lambda}{|\mu-\lambda|^2}) \\ &= K(1+\pi)|\operatorname{Im}\mu|^{-1} \quad . \end{aligned}$$

<u>DEFINITION 2.2.3</u>. A closed linear operator T is said to satisfy <u>condition (G_1)</u> if $\sigma(T) \subset \underline{R}$ and there exists a constant K_1 such that $|\operatorname{Im}\mu|\ \|R(\mu;T)\| \leq K_1 \quad (\mu \in \underline{C}\backslash\underline{R})$.

This definition is due to Bartle [1]. Proposition 2.2.2 therefore says that a well-boundable operator satisfies (G_1) .

<u>DEFINITION 2.2.4</u>. For $x \in X$, an analytic extension of $R(\mu;T)x$ is an analytic function $F : \mathfrak{D}(F) \supset \rho(T) \to X$ (where $\mathfrak{D}(F)$ is open), such that

$$(\mu I-T)F(\mu) = x \quad (\mu \in \mathfrak{D}(F)) \quad . \tag{3}$$

A point $\mu_0 \in \underline{C}$ is said to be in the <u>analytic point spectrum</u> of T if there exist a neighbourhood V of μ_0 and an analytic function $G : V \to X$, such that $G(\mu_0) \neq 0$, and

$$(\mu I-T)G(\mu) = 0 \quad (\mu \in V) \quad . \tag{4}$$

Equivalently, μ_0 is in the analytic point spectrum of T if there exists a vector $x \in X$ for which $(\mu \to R(\mu;T)x)$ has analytic extensions F_1, F_2 satisfying (3) but not agreeing at μ_0 , nor, therefore, on some neighbourhood V of μ_0 , $V \supset \mathfrak{D}(F_1) \cap \mathfrak{D}(F_2)$.

T has the <u>single-valued extension property</u> (s.v.e.p.) if its analytic point spectrum is empty. If T has s.v.e.p., then, for each $x \in X$, $R(\mu;T)x$ has a unique maximal analytic extension $\tilde{x}(\mu;T)$, whose domain $\rho(x;T)$ is called the <u>local resolvent set</u> of

x. The complement, $\sigma(x;T)$, of $\rho(x;T)$ is called the <u>local spectrum</u> of x.

<u>REMARKS 2.2.5</u>. (i) If $\sigma(T)$ is nowhere dense in $\underline{C}$, then T has s.v.e.p. In particular, this holds when $\sigma(T) \subset \underline{R}$.

(ii) If T has s.v.e.p., then it is implicit in the definition that $\tilde{x}(\mu;T) \in \mathcal{D}(T)$ $(x \in X, \mu \in \rho(x;T))$. $\tilde{x}(\mu;T)$ is defined for all $x \in X$, not merely for $x \in \mathcal{D}(T)$.

(iii) If T has s.v.e.p. and $A \in B(X)$ commutes with T, then $(Ax)^{\sim}(\mu;T) = A\tilde{x}(\mu;T)$ $(\mu \in \rho(x;T))$, $A\tilde{x}(\mu;T)$ being analytic, and so $\sigma(Ax;T) \subset \sigma(x;T)$.

<u>PROPOSITION 2.2.6</u>. Let $T \in B(X)$ be a well-bounded operator of type B, with spectral family E.

(i) If $E(\lambda) = 0$ $(\lambda < a)$ and $E(\lambda) = I$ $(\lambda \geqslant b)$, then the functional calculus $\phi : BV[a,b] \to B(X)$ for T gives

$$\phi(\chi_{[a,\mu]}) = E(\mu) \quad (a \leqslant \mu \leqslant b) .$$

(ii) $-T$ is also a well-bounded operator of type B, with spectral family E' given by

$$E'(\lambda) = I - E(-\lambda-0) \quad (\lambda \in \underline{R}) .$$

(iii) $(E(\nu)-E(\mu-0))X = \{x \in X : \sigma(x;T) \subset [\mu,\nu]\}$ $(\mu \leqslant \nu)$.

<u>Proof</u>. (i) We have, noting that $\chi_{[a,\mu]} \in NBV[a,b]$,

$$\chi_{[a,\mu]}(T)x = \int_{[a,b]}^{\oplus} \chi_{[a,\mu]}(\lambda)dE(\lambda)x$$

$$= E(a)x + \int_a^b \chi_{[a,\mu]}(\lambda)dE(\lambda)x$$

$$= E(a)x + \int_a^{\mu} dE(\lambda)x$$

$$= E(\mu)x \quad (a \leqslant \mu \leqslant b,\ x \in X) \ .$$

(ii) It follows from 1.3.11 that

$$\chi_{[a,\mu)}(T) = E(\mu-0) \quad (a < \mu \leqslant b) \ .$$

Consider the map $f \to \tilde{f}$, where $\tilde{f}(\lambda) = f(-\lambda)$. We see that the maps

$$AC[-b,-a] \xrightarrow{\sim} AC[a,b] \xrightarrow{\psi} B(X)$$

satisfy

$$\begin{tikzcd} j \arrow[r, "\sim"] & -j \arrow[r, "\psi"] & -T \\ 1 \arrow[r, "\sim"] & 1 \arrow[r, "\psi"] & I \ , \end{tikzcd}$$

where $j(\lambda) = \lambda$ $(\lambda \in [-b,-a])$. Therefore $\tilde{\psi} = \psi\circ\sim$ is a functional calculus for the well-bounded operator $-T$, and it clearly satisfies any of the compactness properties (iii)-(v) of 1.3.10. Hence $-T$ is of type B, and $\tilde{\psi}$ gives

$$\begin{aligned} E'(-\lambda) = \tilde{\psi}(\chi_{[-b,-\lambda]}) &= \chi_{[-b,-\lambda]}(-T) \\ &= I - \chi_{(-\lambda,-a]}(-T) \\ &= I - \chi_{[a,\lambda)}(T) \\ &= I - E(\lambda-0) \quad (a < \lambda \leqslant b) \ , \end{aligned}$$

with $E'(-\lambda) = I$ $(\lambda \leqslant a)$ and $E'(-\lambda) = 0$ $(\lambda > b)$. Thus (ii) is proved.

(iii) We have $E(\lambda)X = \{x \in X : \sigma(x;T) \subset (-\infty,\lambda]\}$ $(\lambda \in \underline{R})$, from [3], Theorem 5.7. It follows that $E'(\lambda)X = \{x \in X : \sigma(x;-T) \subset (-\infty,\lambda]\}$ $(\lambda \in \underline{R})$. Therefore, if $x \in (E(\nu)-E(\mu-0))X$, then $x \in E(\nu)X$ and $x \in E'(-\mu)X$, hence $\sigma(x;T) \subset (-\infty,\nu]$ and $\sigma(x;-T) \subset (-\infty,-\mu]$, which implies $\sigma(x;T) \subset [\mu,\nu]$.

Conversely, if $\sigma(x;T) \subset [\mu,\nu]$, then $x \in E(\nu)X$; also,

$\sigma(x;-T) \subset [-\nu,-\mu]$, so $x \in E'(-\mu)X$. Hence $x \in E(\nu)(I-E(\mu-0)X = (E(\nu)-E(\mu-0))X$.

PROPOSITION 2.2.7. Let $T : \mathfrak{D}(T) \subset X \to X$ be a well-boundable operator and E a spectral family for T. Then

(i) $E'(\lambda) = I - E(-\lambda-0)$ is a spectral family for the well-boundable operator $-T$.

(ii) $(E(\nu)-E(\mu-0))X = \{x \in X : \sigma(x;T) \subset [\mu,\nu]\}$ $(\mu \leqslant \nu)$.

Proof. (i) Firstly, note that E' actually is a spectral family, the order and continuity properties being obvious. Let S be the well-boundable operator associated with the spectral family E'. Suppose that $x \in \mathfrak{D}(T)$; then $Tx = \lim_n T_j^{(u_n)}x$, where

$$T_j^{(u_n)}x = \int_{[-u_n,u_n]}^{\oplus} \lambda dE(\lambda)(E(u_n)-E(-u_n))x$$

and $\{u_n\}$ is any increasing sequence of positive numbers such that $u_n \to \infty$. We recall from 2.1.2 that Tx is independent of the actual choice of the sequence $\{u_n\}$. In view of the Lemma on p.330 of [30], the function $\lambda \to E(\lambda)x$ has only countably many discontinuities. It is possible, therefore, to choose $\{u_n\}$ in such a way that $E(\pm u_n)x - E(\pm u_n-0)x = 0$ for all n. (This choice of $\{u_n\}$ depends on x, of course.)

Each $T_j^{(u_n)}|(E(u_n)-E(-u_n))X$ is well-bounded of type B and has spectral family $E(\lambda)|(E(u_n)-E(-u_n))X$; consequently each $-T_j^{(u_n)}|(E(u_n)-E(-u_n))X$ is also of type B and has spectral family $E'(\lambda)|(E(u_n)-E(-u_n))X = E'(\lambda)|(E'(u_n-0)-E'(-u_n-0))X$. Hence, with the obvious notation,

$$
\begin{aligned}
-T_j^{(u_n)}x &= \int_{[-u_n,u_n]}^{\oplus} \lambda dE'(\lambda)(E(u_n)-E(-u_n))x \\
&= \int_{[-u_n,u_n]}^{\oplus} \lambda dE'(\lambda)(E'(u_n)-E'(-u_n))x \\
&= S_j^{(u_n)}x \ .
\end{aligned}
$$

Since $-T_j^{(u_n)}x \to -Tx$, it follows that $x \in \mathfrak{D}(S)$ and $Sx = -Tx$. By interchanging the roles of S and T , and noting that $(E')' = E$, we see that $S = -T$.

(ii) Suppose $\sigma(x;T) \subset [\mu,\nu]$. Then

$$(\lambda I-T)\tilde{x}(\lambda;T) = x \quad (\lambda \notin [\mu,\nu]) \ .$$

Using 2.1.2(ii) and 2.2.5(iii), we see that for any $\kappa_1,\kappa_2 \in \underline{R}$,

$$
\begin{aligned}
(\lambda I-T)(E(\kappa_2)-E(\kappa_1))\tilde{x}(\lambda;T) &= (E(\kappa_2)-E(\kappa_1))(\lambda I-T)\tilde{x}(\lambda;T) \\
&= (E(\kappa_2)-E(\kappa_1))x \ .
\end{aligned}
$$

Hence $\sigma((E(\kappa_2)-E(\kappa_1))x \ ; \ T|(E(\kappa_2)-E(\kappa_1))X) \subset [\mu,\nu]$. Since $T|(E(\kappa_2)-E(\kappa_1))X$ has unique spectral family $E|(E(\kappa_2)-E(\kappa_1))X$, it follows from 2.2.6(iii) that

$$(E(\kappa_2)-E(\kappa_1))x \ \in \ (E(\nu)-E(\mu-0))(E(\kappa_2)-E(\kappa_1))X \ ,$$

hence

$$(E(\kappa_2)-E(\kappa_1))x = (E(\nu)-E(\mu-0))(E(\kappa_2)-E(\kappa_1))x \ .$$

Letting $\kappa_1 \to -\infty$, $\kappa_2 \to +\infty$ gives $x \in (E(\nu)-E(\mu-0))X$.

Conversely, if $x \in (E(\nu)-E(\mu-0))X$, then, since the spectrum of the type B operator $T|(E(\nu)-E(\mu-0))X$ is contained in $[\mu,\nu]$, we have

$$(\lambda I-T)R(\lambda;T|(E(\nu)-E(\mu-0))X)x = x \quad (\lambda \notin [\mu,\nu]) \ ,$$

and therefore $\sigma(x;T) \subset [\mu,\nu]$.

THEOREM 2.2.8. The spectral family of a well-boundable operator is unique.

Proof. Let $T : \mathfrak{D}(T) \subset X \to X$ be well-boundable, and E, F be spectral families for T. Choose real numbers μ, ν with $\mu < \nu$, and let

$$M_{\mu,\nu} = \{x \in X : \sigma(x;T) \subset [\mu,\nu]\} .$$

Then, by 2.2.7,

$$M_{\mu,\nu} = (E(\nu)-E(\mu-0))X = (F(\nu)-F(\mu-0))X .$$

Moreover, in view of 2.2.5(iii), $M_{\mu,\nu}$ is an invariant subspace for both $E(\lambda)$ and $F(\lambda)$, for all $\lambda \in \underline{R}$.

Consider the spectral family G, where

$$G(\lambda) = E(\lambda)|M_{\mu,\nu} \quad (\lambda \in \underline{R}) ,$$

so that

$$G(\lambda) = \begin{cases} I|M_{\mu,\nu} & \lambda \geq \nu \\ (E(\lambda)-E(\mu-0))|M_{\mu,\nu} & \mu \leq \lambda < \nu \\ 0 & \lambda < \mu . \end{cases}$$

If $x \in M_{\mu,\nu}$, then

$$\int_{[\mu,\nu]}^{\oplus} \lambda dG(\lambda)x = \mu(E(\mu)-E(\mu-0))x + \int_{\mu}^{\nu} \lambda dE(\lambda)x$$

$$= \mu(E(\mu)-E(\mu-0))x + \int_{\mu}^{\nu} \lambda dE(\lambda)(E(\nu)-E(\mu-0))x . \quad (5)$$

Now $M_{\mu,\nu} \subset \mathfrak{D}(T)$; if $x \in M_{\mu,\nu}$, then by 2.1.6, $Tx = \lim_{n\to\infty} T^{(n)}x$, where

$$T^{(n)}x = \int_{[-n,n]}^{\oplus} \lambda dE(\lambda)(E(n)-E(-n))x .$$

Suppose that $n > \nu$ and $-n < \mu$. Then

$$T^{(n)}x = -nE(-n)(E(n)-E(-n))x + \int_{-n}^{n}\lambda dE(\lambda)(E(n)-E(-n))x$$

$$= \int_{-n}^{n}\lambda dE(\lambda)(E(n)-E(-n))x \quad . \qquad (6)$$

We have

$$\begin{aligned}(E(n)-E(-n))x &= (E(n)-E(-n))(E(\nu)-E(\mu-0))x \\ &= (E(\nu)-E(\mu-0))x \\ &= x \quad . \qquad (7)\end{aligned}$$

Therefore

$$\int_{-n}^{\mu}\lambda dE(\lambda)(E(n)-E(-n))x$$

$$= \lim_{(\underline{\lambda},\underline{\lambda}^*)\in\mathcal{P}^*[-n,\mu]} \lambda_m^*(E(\mu)-E(\lambda_{m-1}))(E(\nu)-E(\mu-0))x$$

$$= \mu(E(\mu)-E(\mu-0))x \quad , \qquad (8)$$

and, since $E(\lambda)(E(\nu)-E(\mu-0))x = 0 \ (\lambda \geqslant \nu)$,

$$\int_{\nu}^{n}\lambda dE(\lambda)(E(n)-E(-n))x = 0 \quad . \qquad (9)$$

On comparing (5) with (6)-(9) , we obtain

$$T^{(n)}x = \int_{[\mu,\nu]}^{\oplus}\lambda dG(\lambda)x \quad (x \in M_{\mu,\nu} \ , \ n > \max(-\mu,\nu)) \quad .$$

Therefore, by 1.3.10, $T|M_{\mu,\nu}$ is a well-bounded operator of type B with spectral family $G = E|M_{\mu,\nu}$.

The above calculation can be repeated with E replaced by F throughout, yielding that $F|M_{\mu,\nu}$ is also a spectral family for $T|M_{\mu,\nu}$. Since the spectral family for $T|M_{\mu,\nu}$ is unique, we have

$$E(\lambda)y = F(\lambda)y \quad (y \in M_{\mu,\nu} , \lambda \in \underline{R}) .$$

Therefore

$$E(\lambda)(E(\nu)-E(\mu-0))x = F(\lambda)(E(\nu)-E(\mu-0))x \quad (x \in X , \lambda \in \underline{R}) .$$

Letting $\nu \to +\infty$, $\mu \to -\infty$ gives $E(\lambda)x = F(\lambda)x$ $(x \in X , \lambda \in \underline{R})$, and so $E = F$.

<u>THEOREM 2.2.9</u>. Let $T : \mathfrak{D}(T) \subset X \to X$ be a well-boundable operator with spectral family E . Then, for each $\lambda \in \underline{R}$,

$$\begin{aligned}(E(\lambda)-E(\lambda-0))X &= \{x \in X : \sigma(x;T) \subset \{\lambda\} \} \\ &= \{x \in X : x \in \mathfrak{D}(T) \text{ and } Tx = \lambda x\} .\end{aligned}$$

<u>Proof</u>. Equality between the first two members was proved in 2.2.7.

If $(\lambda I-T)x = 0$, then

$$\tilde{x}(\mu;T) = \frac{x}{\mu-\lambda}$$

satisfies

$$(\mu I-T)\tilde{x}(\mu;T) = x \quad (\mu \neq \lambda) ,$$

and so $\sigma(x;T) \subset \{\lambda\}$.

Conversely, suppose $\sigma(x;T) \subset \{\lambda\}$. Since $x \in (E(\lambda)-E(\lambda-0))X$, $x \in \mathfrak{D}(T)$. We have

$$(\mu I-T)\tilde{x}(\mu;T) = x \quad (\mu \neq \lambda) .$$

Therefore, since E commutes with T ,

$$(\mu I-T)(E(n)-E(-n))\tilde{x}(\mu;T) = (E(n)-E(-n))x \quad (\mu \neq \lambda) .$$

As in 2.2.5(iii), then,

$$\sigma((E(n)-E(-n))x ; T|(E(n)-E(-n))X) \subset \{\lambda\} .$$

Since $T|(E(n)-E(-n))X$ is a well-bounded operator of type B, it follows from Theorem 4.3(iii) of [3] that

$$T(E(n)-E(-n))x = \lambda(E(n)-E(-n))x \quad .$$

Now let $n \to \infty$; since T is closed, we have $Tx = \lambda x$.

The next theorem is a special case of a result proved by Kocan ([18], Corollary 1.15(i)), where it is proved under the assumption that T is an operator satisfying a condition (G_n) which includes (G_1). We give here a proof depending on the properties of well-boundable operators, and not involving use of the integrals I_r considered in [18].

<u>THEOREM 2.2.10</u>. Let $T : \mathfrak{D}(T) \subset X \to X$ be a well-boundable operator, and let $x \in X$ such that $\sigma(x;T) \subset [a,b]$, where $-\infty < a < b < \infty$. Let Γ be a closed, rectifiable, positively oriented Jordan contour with $\sigma(x;T)$ contained in the interior of Γ . Then

$$x = \frac{1}{2\pi i}\int_\Gamma \tilde{x}(\mu;T)d\mu \quad .$$

<u>Proof</u>. Let E be the spectral family of T . For $n=1,2,\ldots$, denote $(E(n)-E(-n))x$ by x_n and $T|(E(n)-E(-n))X$ by T_n . By the same argument as in 2.2.9, we have

$$\sigma(x_n;T) \subset [a,b] \quad ,$$

$$(\mu I-T)(E(n)-E(-n))\tilde{x}(\mu;T) = (\mu I-T_n)(E(n)-E(-n))\tilde{x}(\mu;T)$$

$$= x_n \quad ,$$

and

$$\tilde{x}_n(\mu;T) = \tilde{x}_n(\mu;T_n) = (E(n)-E(-n))\tilde{x}(\mu;T) \quad (\mu \notin [a, b]) \quad .$$

Therefore

$$\frac{1}{2\pi i}\int_\Gamma \tilde{x}_n(\mu;T)d\mu = \frac{1}{2\pi i}\int_\Gamma \tilde{x}_n(\mu;T_n)d\mu$$

$$= \frac{1}{2\pi i}\int_{\Gamma_n} \tilde{x}_n(\mu;T_n)d\mu \quad ,$$

where Γ_n is a suitable contour in $\{z \in \underline{C} : |z| > n\}$ with Γ in the interior of Γ_n. But, since T_n is a bounded operator with $\sigma(T_n) \subset [-n,n]$, $\tilde{x}_n(\mu;T_n) = R(\mu;T)x_n$ on Γ_n, and so

$$\frac{1}{2\pi i}\int_\Gamma \tilde{x}_n(\mu;T)d\mu = \frac{1}{2\pi i}\int_{\Gamma_n} R(\mu;T)x_n d\mu = x_n \quad .$$

As $n \to \infty$, $x_n \to x$, and

$$\int_\Gamma \tilde{x}_n(\mu;T)d\mu = (E(n)-E(-n))\int_\Gamma \tilde{x}(\mu;T)d\mu$$

$$\to \int_\Gamma \tilde{x}(\mu;T)d\mu \quad ,$$

and so the result follows.

<u>LEMMA 2.2.11</u>. Let $f : \underline{T} \to X$ be bounded and have left and right hand limits at each point. Let

$$\tilde{f}(re^{it}) = \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta-t)f(e^{i\theta})d\theta \quad (0 \leq r < 1 , 0 \leq t < 2\pi) \quad ,$$

where P_r is the Poisson kernel $P_r(t) = \dfrac{1-r^2}{1-2r\cos t + r^2}$. If $z = re^{it}$ tends to $z_0 = e^{it_0}$ along a continuously differentiable path in $\{z : |z| < 1\}$ whose tangent at z_0 is radial, then

$$\tilde{f}(z) \to \tfrac{1}{2}[f(e^{i(t_0+0)}) + f(e^{i(t_0-0)})] \quad .$$

Proof. This is a standard result for scalar-valued functions ([34], Theorem III.6.15). We sketch the argument here, giving details where the validity for vector-valued functions needs to be stressed.

Let

$$\mu_r(\delta) = \max_{\delta \leqslant t \leqslant \pi} P_r(t) \quad (0 < \delta \leqslant \pi) .$$

In view of the estimate

$$P_r(t) \leqslant \frac{A(1-r)}{t^2} \quad (0 < t \leqslant \pi , 0 \leqslant r < 1)$$

([34], III.6.9), it follows that

$$\mu_r(\delta) \leqslant \frac{A(1-r)}{\delta^2} \quad (0 < \delta \leqslant \pi , 0 \leqslant r < 1) ,$$

hence

$$\mu_r(\delta) \to 0 \text{ as } r \to 1 \tag{10}$$

for each <u>fixed</u> δ .

For each $z \in \underline{T}$, consider the funotion ϕ_z , where

$$\phi_z(u) = \frac{f(e^{iu}z) + f(e^{-iu}z) - 2f(z)}{2} \quad (u \neq 0)$$

$$\phi_z(0) = 0 .$$

ϕ_{z_0} is continuous at $u = 0$, provided we normalise $f(z_0) = \frac{1}{2}[f(e^{i(t_o+0)}) + f(e^{i(t_o-0)})]$ $(z_0 = e^{it_o})$.

Suppose f is continuous at z_0 . Let $\epsilon > 0$. Then there is an arc $C = \{e^{it} : t_1 \leqslant t \leqslant t_2\}$, with $t_1 < t_0 < t_2$, and a $\delta = \delta(\epsilon)$, independent of $z \in C$, such that

$$\|\phi_z(u)\| < \frac{\epsilon}{2} \quad (0 \leqslant |u| < \delta , z \in C) .$$

Then by standard manipulation, taking into account that $P_r(\cdot)$ is an even function and $\frac{1}{2\pi}\int_0^{2\pi} P_r(t)dt = 1$, we reach

$$\tilde{f}(re^{it}) - f(e^{it}) = \frac{1}{2\pi}\int_0^{2\pi} P_r(u)\phi_z(u)du \quad ,$$

where $z = e^{it}$. Therefore

$$\begin{aligned}\|\tilde{f}(re^{it}) - f(e^{it})\| &\leq \frac{1}{2\pi}\frac{\epsilon}{2}\int_0^{\delta} P_r(u)du + \frac{\mu_r(\delta)}{2\pi}\int_{\delta}^{\pi}\|\phi_z\|_\infty dt \\ &\leq \frac{\epsilon}{2} + \frac{\mu_r(\delta)}{2\pi}.\pi.2\|f\|_\infty . \qquad (11) \\ &< \epsilon ,\end{aligned}$$

provided r is near 1 , since the right hand side of (11) is independent of $z \in C$, and by (10) its second term tends to zero as $r \to 1$, independently of z . This shows that the function equal to $\tilde{f}(re^{it})$ for $0 \leq r < 1$ and $f(e^{it})$ when $r = 1$ is continuous at z_0 .

Now suppose f is not continuous at z_0 . Without loss of generality we may assume $z_0 = 1$. We again assume $f(1)$ to be normalised as the average of the upper and lower limits along $\underline{T}$. Let

$$d = f(e^{i(0+0)}) - f(e^{i(0-0)})$$

be the discontinuity at 1 , and let

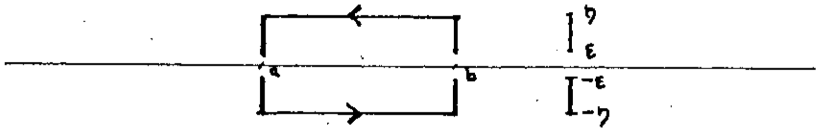
$$\phi(t) \sim \sum^{\infty} \frac{\sin nt}{n} \quad .$$

Then, exactly as in [34], p.98, we apply the result of the case where f is continuous at 1 to the function $g = f - \frac{\phi}{\pi} d$, which satisfies

$$\tilde{f}(re^{it}) = \tilde{g}(re^{it}) + \arctan\left(\frac{r \sin t}{1 - r\cos t}\right)\frac{d}{\pi} \quad . (12)$$

Along a continuously differentiable path, $\arctan\left(\frac{r \sin t}{1 - r\cos t}\right)$ tends to the angle between the tangents to the path and ~~to~~ the radius of the circle at 1 . The result now follows from (12) .

<u>THEOREM 2.2.12.</u> Let $T : \mathfrak{D}(T) \subset X \to X$ be a well-boundable operator with spectral family E. Let $-\infty < a < b < \infty$ and $0 < \epsilon < \eta$, and let the contour Δ_ϵ be the union of the directed polygonal contours $[b+i\epsilon, b+i\eta, a+i\eta, a+i\epsilon]$ and $[a-i\epsilon, a-i\eta, b-i\eta, b-i\epsilon]$.



Then

$$\frac{1}{2}\,[E(b) + E(b-0) \overset{-E(a)}{\cancel{+E(a)}} \overset{-E(a-0)}{\cancel{+E(a-0)}}]x = \lim_{\epsilon\to 0} \frac{1}{2\pi i} \int_{\Delta_\epsilon} R(\mu;T)x\,d\mu \quad (x \in X).$$

<u>Remark:</u> Kocan ([19], Appendix) proved this formula under the assumption that E is strongly continuous.

<u>Proof.</u> For $\mu \notin \underline{R}$ we have, by 2.2.2,

$$R(\mu;T)x = \int_{\underline{R}}^{\oplus} (\mu-\lambda)^{-1} dE(\lambda)x = \int_{-\infty}^{\infty} (\mu-\lambda)^{-1} dE(\lambda)x \quad (x \in X) \quad .$$

Since $\lambda \to (\mu-\lambda)^{-1}$ satisfies the hypotheses of 2.1.18 and 2.1.24, and is in $NBV(\underline{R})$, we have

$$R(\mu;T)x = -\int_{-\infty}^{\infty} \frac{E(\lambda)x}{(\mu-\lambda)^2}\,d\lambda \quad (x \in X\ ,\ \mu \notin \underline{R}) \quad . \qquad (13)$$

Therefore

$$\frac{1}{2\pi i} \int_{\Delta_\epsilon} R(\mu;T)x\,d\mu = -\frac{1}{2\pi i} \int_{\Delta_\epsilon} \left[\int_{-\infty}^{\infty} \frac{E(\lambda)x}{(\mu-\lambda)^2}\,d\lambda\right] d\mu \quad . \qquad (14)$$

To justify the change of order of integration in (14), we put $g(\lambda,\mu) = (\mu-\lambda)^{-2}$ in 2.1.25. Condition 2.1.25(i) becomes $\int_{-\infty}^{\infty}\int_{\Delta_\epsilon} \frac{1}{|\mu-\lambda|^2}\,dm(\mu)d\lambda < \infty$, where m is linear measure on Δ_ϵ; it is easily verified that this integral is finite. For each $\mu \in \Delta_\epsilon$,

$\int_{-\infty}^{\infty} \frac{d\lambda}{|\mu-\lambda|^2} = \frac{\pi}{|\text{Im } \mu|} < \infty$, and the continuity requirement is given by (13), so 2.1.25(ii) is also satisfied. Therefore

$$\frac{1}{2\pi i}\int_{\Delta_\epsilon} R(\mu;T)x\,d\mu = -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\left[\int_{\Delta_\epsilon}\frac{d\mu}{(\mu-\lambda)^2}\right]E(\lambda)x\,d\lambda \quad .$$

We have

$$-\frac{1}{2\pi i}\int_{\Delta_\epsilon}\frac{d\mu}{(\mu-\lambda)^2} = \frac{1}{2\pi i}\left[\int_{a+i\epsilon}^{b+i\epsilon} - \int_{a-i\epsilon}^{b-i\epsilon} \frac{d\mu}{(\mu-\lambda)^2}\right], \quad (15)$$

since the integrals round the rectangles $[b+i\epsilon, b+i\eta, a+i\eta, a+i\epsilon]$ and $[b-i\epsilon, a-i\epsilon, a-i\eta, b-i\eta]$ are ~~zreo~~ zero. Therefore, evaluating the right hand side of (15),

$$-\frac{1}{2\pi i}\int_{\Delta_\epsilon}\frac{d\mu}{(\mu-\lambda)^2} = \frac{\epsilon}{\pi}\left[\frac{1}{(b-\lambda)^2+\epsilon^2} - \frac{1}{(a-\lambda)^2+\epsilon^2}\right] ,$$

and so

$$\frac{1}{2\pi i}\int_{\Delta_\epsilon} R(\mu;T)x\,d\mu = \frac{\epsilon}{\pi}\int_{-\infty}^{\infty}\frac{E(\lambda)x\,d\lambda}{(b-\lambda)^2+\epsilon^2} - \frac{\epsilon}{\pi}\int_{-\infty}^{\infty}\frac{E(\lambda)x\,d\lambda}{(a-\lambda)^2+\epsilon^2} \quad .$$

In order to obtain the limit as $\epsilon \to 0$, we make the conformal transformation

$$\phi \; : \; z \to \frac{1+iz}{1-iz}$$

of $\underline{C}$, which takes $\overline{\underline{R}}$ to $\underline{T}$, and the upper half plane to the unit disc. If $\lambda \in \underline{R}$, then

$$\phi(\lambda) = \frac{1+i\lambda}{1-i\lambda} = e^{i\theta(\lambda)} , \quad \text{where} \quad -\pi < \theta(\lambda) < \pi ,$$

and if $z = a+i\epsilon$, $\epsilon > 0$, then $\phi(z) = re^{it}$, where $r < 1$, $0 \leqslant t < 2\pi$. Let

$$F(\theta) = E(\phi^{-1}(e^{i\theta})) \quad (-\pi < \theta < \pi) \quad .$$

Then

$$\frac{\epsilon}{\pi}\int_{-\infty}^{\infty}\frac{E(\lambda)xd\lambda}{(a-\lambda)^2+\epsilon^2} \quad \text{becomes} \quad \frac{1}{2\pi}\int_{-\pi}^{\pi}P_r(\theta-t)F(\theta)xd\theta \quad ,$$

since

$$\frac{1}{2\pi}\frac{d\theta}{d\lambda} = \frac{1}{\pi}\frac{1}{1+\lambda^2}$$

and

$$\frac{\epsilon(1+\lambda^2)}{(a-\lambda)^2+\epsilon^2} = \mathrm{Re}\left[\frac{e^{i\theta}+re^{it}}{e^{i\theta}-re^{it}}\right] = P_r(\theta-t)$$

As $\epsilon \to 0$, $a+i\epsilon \to a \in \underline{R}$ along a vertical line; therefore $re^{it} \to e^{i\theta(a)}$ along a (continuously differentiable) path in $\{z : |z| < 1\}$ whose tangent at $e^{i\theta(a)}$ is radial, since the transformation ϕ is conformal. It now follows from Lemma 2.2.11 that the limit as $\epsilon \to 0$ is $\frac{1}{2}[F(\theta(a)) + F(\theta(a)-0)]x = \frac{1}{2}[E(a)+E(a-0)]x$, thus completing the proof of the theorem.

Remark. The formula proved in 2.2.12 can be used to provide an alternative proof of the uniqueness of the spectral family for a well-boundable operator.

THEOREM 2.2.13. Let $T : \mathfrak{D}(T) \subset X \to X$ be a well-boundable operator with spectral family E . Then $\sigma(T) = \operatorname{supp} E$.

Proof. Let $\mu \in \underline{R}$ and $\delta > 0$ such that E is constant on $[\mu-\delta, \mu+\delta]$. Then for all $\alpha \in LBV(\underline{R})$,

$$\int_{\underline{R}}^{\oplus} \chi_{(\mu-\delta,\mu+\delta]}(\lambda)\alpha(\lambda)dE(\lambda) = 0 \quad .$$

In particular,

$$\int_{\underline{R}}^{\oplus} \chi_{(\mu-\delta,\mu+\delta]}(\lambda)dE(\lambda)x = 0 \quad (x \in X) \quad .$$

Let

$$\alpha(\lambda) = \begin{cases}(\mu-\lambda)^{-1} & \lambda \notin (\mu-\delta,\mu+\delta] \\ 0 & \lambda \in (\mu-\delta,\mu+\delta]\end{cases}$$

$$\beta(\lambda) = \mu-\lambda \quad (\lambda \in \underline{R})$$

$$\gamma(\lambda) = 1 - \chi_{(\mu-\delta,\mu+\delta]}(\lambda) \quad (\lambda \in \underline{R}) \quad .$$

Then $\alpha\beta = \gamma$, and $T_\gamma = I$. Applying the argument of 2.2.2, and noting that $\alpha \in BV(\underline{R})$, we see that $T_\alpha = R(\mu;T)$. Therefore $\mu \in \rho(T)$.

Conversely, suppose $\mu \in \rho(T)$. Then, since no point in a neighbourhood of μ is an eigenvalue of T , 2.2.9 shows that E is continuous in a neighbourhood $(\mu-\delta,\mu+\delta)$ of μ . Choose a,b such that $\mu-\delta < a < \mu < b < \mu+\delta$, and let $\eta > 0$. If Δ is the rectangular contour joining $\{a\underline{+}i\eta, b\underline{+}i\eta\}$, then a neighbourhood of Δ and its interior is contained in $\rho(T)$. Let Δ_ϵ be the contour defined in 2.2.12. Then, by 2.2.12,

$$\begin{aligned}(E(b)-E(a))x &= \lim_{\epsilon\to 0} \frac{1}{2\pi i}\int_{\Delta_\epsilon} R(\lambda;T)x d\lambda \\ &= \frac{1}{2\pi i}\int_{\Delta} R(\lambda;T)x d\lambda \\ &= 0 \quad (x \in X) \quad ,\end{aligned}$$

and so E is constant in $(\mu-\delta,\mu+\delta)$.

<u>THEOREM 2.2.14</u>. If $\alpha \in LAC(\underline{R})$, then $\sigma(T_\alpha) \supset \alpha(\sigma(T))$.

<u>Proof</u>. Let $\lambda_0 \in \sigma(T)$. Then the function $\beta : \lambda \to \alpha(\lambda_0)-\alpha(\lambda)$ is also in $LAC(\underline{R})$. Let $\epsilon > 0$. By the definition of absolute continuity, there exists $\delta_1 > 0$ such that, if $\{(s_k,t_k)\}_{k=1}^{m}$ is a disjoint collection of intervals contained in some fixed compact

interval $[a,b]$ with $\sum(t_k-s_k) < \delta_1$, then $\sum|\alpha(t_k)-\alpha(s_k)| < \frac{\epsilon}{2}$. Consequently, if $\underline{t} \in [\lambda_0-\frac{\delta_1}{2},\lambda_0+\frac{\delta_1}{2}]$, then $\sum|\alpha(t_k)-\alpha(t_{k-1})| = \sum|\beta(t_k)-\beta(t_{k-1})| < \frac{\epsilon}{2}$; therefore the variation of β over $[\lambda_0-\frac{\delta_1}{2},\lambda_0+\frac{\delta_1}{2}]$ is bounded by $\frac{\epsilon}{2}$. It follows also that, if $|\lambda-\lambda_0| < \delta_1$, then $|\alpha(\lambda)-\alpha(\lambda_0)| < \frac{\epsilon}{2}$. Therefore there exists $\delta > 0$ such that $|||\beta|||_J < \epsilon$, where $J = [\lambda_0-\delta,\lambda_0+\delta]$.

By 2.2.13, there exist λ_1,λ_2 with $\lambda_0-\delta \leq \lambda_1 < \lambda_0 < \lambda_2 \leq \lambda_0+\delta$ and $E(\lambda_1) \neq E(\lambda_2)$. Hence there is a $y \in X$ such that $E(\lambda_1)y = 0$ and $E(\lambda_2)y = y \neq 0$. Then

$$\begin{aligned} \|(\alpha(\lambda_0)I-T_\alpha)y\| &= \|\int_{\underline{R}}^{\oplus}(\alpha(\lambda_0)-\alpha(\lambda))dE(\lambda)y\| \\ &= \|\int_{\lambda_1}^{\lambda_2}(\alpha(\lambda_0)-\alpha(\lambda))dE(\lambda)y\| \\ &\leq K|||\beta|||_J\|y\| \\ &< K\epsilon\|y\| , \end{aligned}$$

and so $\alpha(\lambda_0) \in \sigma(T_\alpha)$.

§2.3. An extension of the Hille-Sz.-Nagy theorem to well-bounded operators

We recall some basic notions from the theory of semigroups of operators.

By a strongly continuous semigroup in $B(X)$ we mean a set $\{T(t)\}_{t\geq 0} \subset B(X)$ which satisfies

$$\left.\begin{array}{l}\text{(i) } T(s+t) = T(s)T(t) \quad (s,t \geq 0)\ ; \\ \text{(ii) } T(0) = I\ ; \\ \text{(iii) } T(\cdot)x \text{ is continuous for each } x \in X\ .\end{array}\right\} \qquad (1)$$

(The apparently weaker hypothesis of right continuity for each $T(\cdot)x$ is in fact equivalent: see [5], Proposition 1.1.2.) It follows from the uniform boundedness theorem that $\|T(t)\|$ is bounded on every compact interval. We also have

$$\omega_0 = \lim_{t\to\infty} \frac{\log\|T(t)\|}{t} < \infty\ . \qquad (2)$$

A strongly continuous semigroup $\{T(t)\}$ is characterised by its <u>infinitesimal generator</u> A, where

$$Ax = \lim_{t\to 0+} \frac{T(t)x - x}{t}$$

whenever the limit exists. The operator is closed, and $\mathfrak{D}(A)$ is dense in X ; $A \in B(X)$ if and only if $t \to T(t)$ is norm continuous. The spectrum of A is contained in the left half plane $\{\lambda : \text{Re}\,\lambda \leq \omega_0\}$, where ω_0 is as in (2), and the resolvent is given by

$$R(\lambda;A)x = \int_0^\infty e^{-\lambda t}T(t)x\,dt \quad (\text{Re}\lambda > \omega_0,\ x \in X)\ , \qquad (3)$$

([5], Theorem 1.3.5).

$T(t)$ can be recovered from A by

$$T(t)x = \lim_{\lambda\to\infty} e^{-\lambda t}\sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!}[\lambda R(\lambda;A)]^k x \quad (x \in X)\ ,$$

([5], Proposition 1.3.11).

The well-known theorem of Hille and Sz.-Nagy ([16], Theorem 22.3.1) states that if X is a Hilbert space, and $\{T(t)\}$ is a strongly continuous semigroup of self-adjoint operators, then A

is self-adjoint, and

$$T(t) = \int_{\underline{R}} e^{\lambda t} K(d\lambda) ,$$

where K is the spectral measure of A. This representation for $T(t)$ is unique. Sz.-Nagy generalised the result to normal operators ([16], Theorem 22.4.2). A similar result for scalar type spectral operators on a weakly complete space was given by Sourour (see [2] and [31]). We give here a version of the theorem for well-bounded operators of type B.

<u>PROPOSITION 2.3.1</u>. Let A and B be two linear operators on X. Suppose there exists $\mu \in \rho(A) \cap \rho(B)$ such that $R(\mu;A) = R(\mu;B)$. Then $A = B$.

<u>Proof</u>. Denote the common value $R(\mu;A) = R(\mu;B)$ by Q. Then

$$Q(\mu I-A)x = x \quad (x \in \mathfrak{D}(A)),$$
$$Q(\mu I-B)x = x \quad (x \in \mathfrak{D}(B)),$$
$$(\mu I-A)Qx = x \quad (x \in X),$$
$$(\mu I-B)Qx = x \quad (x \in X).$$

In particular, $Qx \in \mathfrak{D}(A) \cap \mathfrak{D}(B)$ and $AQx = BQx$, for all $x \in X$. Consequently $x \in \mathfrak{D}(A)$ implies $x = Q(\mu I-A)x \in \mathfrak{D}(B)$, and $Ax = AQ(\mu I-A)x = BQ(\mu I-A)x = Bx$. Therefore $A \subset B$. Similarly $B \subset A$, so $A = B$.

<u>THEOREM 2.3.2</u>. Let $\{T(t)\}_{t \geq 0}$ be a strongly continuous semigroup in $B(X)$, with each $T(t)$ a well-bounded operator of type B whose spectral family is E_t. Let A be the infinitesimal generator of $\{T(t)\}$. Then A is well-boundable, and its spectral family is

G , where $G(\lambda) = E_1(e^{\lambda})$ $(\lambda \in \underline{R})$. Furthermore,

$$T(t)x = \int_{-\infty}^{\infty} e^{\lambda t} dG(\lambda)x \quad (x \in X \, , \; t \geq 0) \quad , \qquad (4)$$

and G is the unique spectral family which satisfies (4) .

<u>Proof</u>. Since $\sigma(T(t)) = \sigma(T(t/2)^2) = [\sigma(T(t/2))]^2$, by the spectral mapping theorem, we have $\sigma(T(t)) \subset \underline{R}^+$ $(t > 0)$. Choose $a > 0$ such that $\sigma(T(1)) \subset [0,a]$. Let $a_n = a^{1/n}$; then $\sigma(T(1/n)) \subset [0,a_n]$, $n=1,2,\ldots$. We have

$$T(1)x = \int_{[0,a]}^{\oplus} \lambda dE_1(\lambda)x \quad (x \in X) \quad ,$$

$$T(1/n)x = \int_{[0,a_n]}^{\oplus} \lambda dE_{1/n}(\lambda)x \quad (x \in X) \quad .$$

By the functional calculus in 1.3.10,

$$T(1)x = [T(1/n)^n]x = \int_{[0,a_n]}^{\oplus} \lambda^n dE_{1/n}(\lambda)x \quad (x \in X) \quad .$$

We claim that

$$E_{1/n}(\lambda) = E_1(\lambda^n) \quad (\lambda \geq 0 \; ; \; n=1,2,\ldots) \quad . \qquad (5)$$

To see this, let

$$F(\lambda) = E_{1/n}(\lambda^{1/n}) \quad (\lambda \geq 0) \quad ,$$

$$F(\lambda) = E_{1/n}(\lambda) = 0 \quad (\lambda < 0) \quad .$$

Then F is a spectral family, and for each $x \in X$

$$\int_{[0,a]}^{\oplus} \lambda dF(\lambda)x = \lim_{(\underline{\lambda},\underline{\lambda}^*) \in \mathcal{P}^*[0,a]} \sum \lambda_k^*(F(\lambda_k) - F(\lambda_{k-1}))x$$

$$= \lim_{(\underline{\lambda},\underline{\lambda}^*) \in \mathcal{P}^*[0,a]} \sum \lambda_k^*(E_{1/n}(\lambda_k^{1/n}) - E_{1/n}(\lambda_{k-1}^{1/n}))x$$

$$= \lim_{(\underline{\mu},\underline{\mu}^*)\in\wp^*[0,a_n]} \sum \mu_k^{*n}(E_{1/n}(\mu_k)-E_{1/n}(\mu_{k-1}))x$$

$$= \int_{[0,a_n]}^{\oplus} \lambda^n dE_{1/n}(\lambda)x \quad ,$$

using the correspondence between $\wp^*[0,a]$ and $\wp^*[0,a_n]$ (2.1.20) .

Therefore

$$\int_{[0,a]}^{\oplus} \lambda dF(\lambda)x = \int_{[0,a_n]}^{\oplus} \lambda^n dE_{1/n}(\lambda)x$$

$$= T(1)x$$

$$= \int_{[0,a]}^{\oplus} \lambda dE_1(\lambda)x \quad (x \in X) \quad .$$

Since the spectral family of $T(1)$ is unique, $F = E_1$, and so (5) holds. It follows that

$$T(1/n)x = \int_{[0,a_n]}^{\oplus} \lambda dE_{1/n}(\lambda)x$$

$$= \int_{[0,a_n]}^{\oplus} (\lambda^n)^{1/n} dE_{1/n}(\lambda)x$$

$$= \int_{[0,a]}^{\oplus} \mu^{1/n} dE_1(\mu)x \quad (x \in X \; ; \; n=1,2,\dots) \quad ,$$

and therefore, by the functional calculus,

$$T(m/n)x = \int_{[0,a]}^{\oplus} \mu^{m/n} dE_1(\mu)x \quad (x \in X)$$

for each rational $m/n > 0$.

If $t > 0$ and $q_n \to t$, where $\{q_n\}$ is a sequence in Q^+, then $\mu^{q_n} \to \mu^t$ $(\mu \in [0,a])$. Since $\{|||\mu^{q_n}|||_{[0,a]}\}$ are bounded, it follows from 1.3.11 that

$$T(q_n)x \to \int_{[0,a]}^{\oplus} \mu^t dE_1(\mu)x \quad (x \in X)$$

as $n \to \infty$. Hence, by the strong continuity of $T(t)$,

$$T(t)x = \int_{[0,a]}^{\oplus} \mu^t dE_1(\mu)x \quad (t > 0 \,;\, x \in X) \quad .$$

We next consider G, defined by $G(\lambda) = E_1(e^\lambda)$. It is clear that $G(\lambda) = I$ $(\lambda \geq \log a)$ and that G inherits the order, continuity and boundedness properties from E_1, so in order to show that G is a spectral family we have only to establish that $\lim_{\lambda \to -\infty} G(\lambda)x = 0$ $(x \in X)$. For this we need to show $E_1(0) = 0$. Let $x \in E_1(0)X$. Then $x = E_1(0)x$, and

$$T(t)x = \int_{[0,a]}^{\oplus} \mu^t dE_1(\mu)E_1(0)x$$

$$= 0.E_1(0)x + \int_0^a \mu^t dE_1(\mu)E_1(0)x$$

$$= 0 \quad (t > 0) \quad .$$

By the strong continuity of $T(t)$, $x = T(0)x = \lim_{t \to 0} T(t)x = 0$. Therefore $E_1(0) = 0$.

It follows from 2.1.20 that

$$T(t)x = \int_{[0,a]}^{\oplus} \mu^t dE_1(\mu)x$$

$$= \int_0^a \mu^t dE_1(\mu)x$$

$$= \int_{-\infty}^{\log a} e^{\lambda t} dG(\lambda)x$$

$$= \int_{-\infty}^{\infty} e^{\lambda t} dG(\lambda)x \quad (t > 0 ; x \in X) \quad ,$$

since $G(\lambda) = I \ (\lambda \geq \log a)$.

Let H be any spectral family such that

$$T(t)x = \int_{-\infty}^{\infty} e^{\lambda t} dH(\lambda)x \quad (t > 0 ; x \in X) \quad . \tag{6}$$

Form the corresponding well-boundable operator B , where

$$Bx = \int_{-\infty}^{\infty} \lambda dH(\lambda)x \quad (x \in \mathfrak{D}(B)) \tag{7}$$

and $\mathfrak{D}(B)$ is precisely the set of $x \in X$ for which (7) converges. By Theorem 2.2.14, $\sigma(T(1)) \supset \exp(\sigma(B))$. Since $\sigma(B) = \text{supp } H$, by 2.2.13, and $T(1)$ is bounded, there exists $d \in \underline{R}$ such that $H(\lambda) = I \ (\lambda \geq d)$. Hence, when $\mu > d$, similar considerations to 2.2.13 allow us to write

$$R(\mu;B)x = \int_{-\infty}^{d} (\mu-\lambda)^{-1} dH(\lambda)x \quad (x \in X) \quad .$$

Using (3) , and substituting for $T(t)$ from (6) , we obtain

$$R(\mu;A)x = \int_0^{\infty} \left[\int_{-\infty}^{\infty} e^{(\lambda-\mu)t} \chi_{(-\infty,d]}(\lambda) dH(\lambda)x \right] dt$$

for all real $\mu > \omega_0$. Let $\mu > \max(d,\omega_0)$. For each fixed $t \geq 0$, the function $h_t(\lambda) = e^{(\lambda-\mu)t}\chi_{(-\infty,d]}(\lambda)$ is bounded and left-continuous and vanishes at $+\infty$. Therefore we can apply 2.1.18, obtaining

$$\begin{aligned}
R(\mu;A)x &= -\int_0^\infty \left[\int_{-\infty}^\infty H(\lambda)x\,dh_t(\lambda)\right]dt \\
&= -\int_0^\infty \left[\int_{-\infty}^d H(\lambda)x\,dh_t(\lambda) - e^{(d-\mu)t}x\right]dt \\
&= -\int_0^\infty \left[\int_{-\infty}^d te^{(\lambda-\mu)t}H(\lambda)x\,d\lambda\right]dt + \frac{x}{\mu-d} \\
&= -\lim_{N\to\infty}\int_0^N \left[\int_{-\infty}^d te^{(\lambda-\mu)t}H(\lambda)x\,d\lambda\right]dt + \frac{x}{\mu-d} ,
\end{aligned}$$

using 2.1.24 in the penultimate step. Let

$$g(\lambda,t) = te^{(\lambda-\mu)t} \quad (-\infty < \lambda \leqslant d\ ,\ 0 \leqslant t \leqslant N) \ .$$

It is easily verified that g satisfies 2.1.25(i), and that $g(\cdot,t)$ is integrable over $(-\infty,d]$ for every $t \in [0,N]$. Furthermore

$$\begin{aligned}
\int_{-\infty}^d te^{(\lambda-\mu)t}H(\lambda)x\,d\lambda &= \int_{-\infty}^d H(\lambda)x\,dh_t(\lambda) = \int_{-\infty}^\infty H(\lambda)x\,dh_t(\lambda) + h_t(d)x \\
&= \cancel{-\int_{-\infty}^d e^{(\lambda-\mu)t}dH(\lambda)x} = -\int_{-\infty}^\infty h_t(\lambda)dH(\lambda)x + e^{(d-\mu)t}x \\
&= -e^{-\mu t}T(t)x + e^{(d-\mu)t}x
\end{aligned}$$

which is a continuous function of t. Thus the conditions of 2.1.25 are satisfied, and so

$$\begin{aligned}
R(\mu;A)x &= -\lim_{N\to\infty}\int_{-\infty}^d \left[\int_0^N te^{(\lambda-\mu)t}dt\right]H(\lambda)x\,d\lambda + \frac{x}{\mu-d} \\
&= +\lim_{N\to\infty}\int_{-\infty}^d \left[\frac{Ne^{-(\mu-\lambda)N}}{\mu-\lambda} + \frac{e^{-(\mu-\lambda)N}}{(\mu-\lambda)^2} - \frac{1}{(\mu-\lambda)^2}\right]H(\lambda)x\,d\lambda \ . \quad (8) \\
&\quad + \frac{x}{\mu-d}
\end{aligned}$$

Since

$$\int_{-\infty}^d \frac{Ne^{-(\mu-\lambda)N}}{\mu-\lambda}\,d\lambda \leqslant \frac{N}{\mu-d}\int_{-\infty}^d e^{-(\mu-\lambda)N}d\lambda = \frac{e^{-(\mu-d)N}}{\mu-d} \to 0$$

and

$$\int_{-\infty}^{d} \frac{e^{-(\mu-\lambda)N}}{(\mu-\lambda)^2}\, d\lambda \leq \frac{1}{(\mu-d)^2}\int_{-\infty}^{d} e^{-(\mu-\lambda)N} d\lambda = \frac{e^{-(\mu-d)N}}{N(\mu-d)^2} \to 0 \quad ,$$

the first two members of the right-hand side of (8) tend to zero. Therefore, using 2.1.18,

$$\begin{aligned} R(\mu;A)x &= -\int_{-\infty}^{d} \frac{H(\lambda)x}{(\mu-\lambda)^2}\, d\lambda + \frac{x}{\mu-d} \\ &= \int_{-\infty}^{\infty} (\mu-\lambda)^{-1} \chi_{(-\infty,d]}(\lambda) dH(\lambda)x \\ &= \int_{-\infty}^{d} (\mu-\lambda)^{-1} dH(\lambda)x \\ &= R(\mu;B)x \quad (\mu > \max(d,\omega_0) \ ; \ x \in X) \ . \end{aligned}$$

Therefore, by 2.3.1, $A = B$.

Thus A is a well-boundable operator with spectral family H , for any H satisfying (6). Since G is such a spectral family, and the spectral family of a well-boundable operator is unique, G must be the spectral family of A and the unique spectral family which satisfies (4). This completes the proof of the theorem.

CHAPTER 3

WELL-BOUNDED OPERATORS AND MULTIPLIERS

In this chapter we discuss a number of interesting examples of well-bounded operators which are multiplier operators, i.e. bounded operators on some $L^p(G)$ which commute with all translations. The relevant notions are reviewed in §3.1. Well-bounded multiplier operators are characterised in §3.2; the criterion is simply the existence and uniform boundedness of certain multiplier projections. In §3.3 we assemble some known facts about the existence of such projections to obtain examples and counterexamples on well-bounded multiplier operators. Some of these operators satisfy the semigroup property, and so Theorem 2.3.2 can be applied to them.

Certain of the operators considered here have been studied by previous authors, notably G. L. Krabbe [20-22], who obtained spectral theorems both for bounded and for a class of unbounded operators. In §3.4 we use our theorem on semigroups of well-bounded operators to clarify these earlier results.

§3.1. Multipliers

Let G be a locally compact abelian group, with dual group Γ. We denote the pairing between elements of g and Γ by (γ,x), $\gamma \in \Gamma$, $x \in G$. Choose Haar measures m on G and η on Γ, normalised so that the Plancherel identity holds. For $1 \leq p < \infty$, let $L^p(G)$ denote the usual spaces of equivalence classes of functions modulo null sets. For $p = \infty$, the appropriate space to use is the space of equivalence classes of bounded measurable functions modulo locally null sets. (This technicality takes account of the possibility that m may fail to be σ-finite: for an example and discussion on this point, see [15], note 11.33, Theorem 12.2 and Definition 12.11.)

The Fourier transform

$$\hat{f}(\gamma) = \int_{\underline{G}} \overline{(\gamma,x)} f(x) dm(x) \quad (\gamma \in \Gamma)$$

is defined for all $f \in L^1(G)$. The Hausdorff-Young theorem states that if $1 \leq p \leq 2$ and $f \in L^1(G) \cap L^p(G)$, then $\|\hat{f}\|_{p'} \leq \|f\|_p$ (where $\frac{1}{p} + \frac{1}{p'} = 1$), and therefore the mapping $f \to \hat{f}$ extends by continuity to an operator in $B(L^p(G), L^{p'}(G))$, which will also be denoted by $f \to \hat{f}$. When $p > 2$, $\hat{f}$ exists for all f in the dense subspace $L^2(G) \cap L^p(G)$ of $L^p(G)$, although it need not then be an element of $L^{p'}(G)$ and the transform will not, in general, extend to the whole of $L^p(G)$.

By a multiplier operator in $B(L^p(G))$ $(1 \leq p < \infty)$ we shall mean a bounded linear operator which commutes with each translation operator. Multiplier operators are characterised by the following:

PROPOSITION 3.1.1 ([4], Theorem 4.4). An operator $T \in B(L^p(G))$ $(1 \leq p < \infty)$ is a multiplier operator if and only if there exists a bounded measurable function $\phi : \Gamma \to \underline{C}$ such that

$$(Tf)^\wedge = \phi \hat{f} \qquad (1)$$

for all $f \in L^2(G) \cap L^p(G)$. The equivalence class of ϕ in $L^\infty(\Gamma)$ is unique.

The function ϕ of 3.1.1 will be referred to as a p-multiplier. If $1 \leq p \leq 2$, relation (1) holds for all $f \in L^p(G)$. For each p we denote the set of p-multipliers by $M_p(\Gamma)$; if $\phi \in M_p(\Gamma)$, then the multiplier operator in $B(L^p(G))$ satisfying (1) will be denoted by T_ϕ (or by $T_\phi^{(p)}$ when the particular value of p is to be stressed).

In the proof of 3.1.1, the following duality relation is established. We shall require this result in §3.2.

PROPOSITION 3.1.2. If $1 < p < \infty$, then $M_p(\Gamma) = M_{p'}(\Gamma)$. If the operators corresponding to the multiplier ϕ are $T_\phi^{(p)}$, $T_\phi^{(p')}$ with norms $\|T_\phi^{(p)}\|_p$, $\|T_\phi^{(p')}\|_{p'}$ respectively, $(Sf)(x) = f(-x)$ for any measurable function $f : G \to \underline{C}$, and $\langle f,g\rangle = \int_G fg$, then

(i) $\langle T_\phi^{(p)} Sf, Sg\rangle = \langle f, T_\phi^{(p')} g\rangle \quad (f \in L^p(G),\ g \in L^{p'}(G))$;

(ii) $\|T_\phi^{(p)}\|_p = \|T_\phi^{(p')}\|_{p'}$.

Proof. See [4], §4.6, or [24], Theorem 4.1.2.

It is well-known that $M_2(\Gamma) = L^\infty(\Gamma)$, and that $M_1(\Gamma)$ is the set of all Fourier-Stieltjes transforms of bounded regular complex Borel measures on G. If G is infinite, then $M_{p_1}(\Gamma)$ is properly contained in $M_{p_2}(\Gamma)$ when $\left|\frac{1}{2} - \frac{1}{p_1}\right| > \left|\frac{1}{2} - \frac{1}{p_2}\right|$ ([24], Theorem 4.5.5).

Remark. We have reserved the term "multiplier" for the function ϕ rather than the operator T_ϕ; many authors define a multiplier as an operator in $B(L^p(G))$ which commutes with translations.

§3.2. Characterisation of well-bounded multiplier operators

LEMMA 3.2.1. Let (Ω,Σ,μ) be a measure space and $\{f_n\}$ a sequence in $L^p(\Omega,\Sigma,\mu)$ $(1 \leq p < \infty)$.

(i) If $\|f_n-f\|_p \to 0$, then there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k}(\omega) \to f(\omega)$ almost everywhere (a.e.).

(ii) If $\|f_n-f\|_p \to 0$ and $f_n(\omega) \to g(\omega)$ a.e., then $f(\omega) = g(\omega)$ a.e.

Proof. (i) Combine III.3.6(i) and III.6.13(a) of [9]. Part (ii) follows immediately.

PROPOSITION 3.2.2. Let $\phi \in M_p(\Gamma)$ $(1 \leq p < \infty)$ be real-valued, and suppose that $T_\phi \in B(L^p(G))$ is well-bounded. Let T_ϕ be implemented by $(K,[a,b])$, where $[a,b]$ is large enough so that $[-\|\phi\|_\infty, \|\phi\|_\infty] \subset [a,b]$. Then the $AC[a,b]$-functional calculus for T_ϕ is given by

$$[\alpha(T_\phi)f]^\wedge = (\alpha\circ\phi)\hat{f} \qquad (\alpha \in AC[a,b],\ f \in L^p(G)\cap L^2(G)) \ .$$

Proof. In view of 3.1.2, we may assume $1 \leq p \leq 2$. For each $\alpha \in AC[a,b]$, $\alpha(T_\phi) \in \{T_\phi\}''$ and so commutes with translations. Hence there exists $\phi_\alpha \in M_p(\Gamma)$ such that $[\alpha(T_\phi)f]^\wedge = \phi_\alpha\hat{f}$ for all $f \in L^p(G)$. If q is a polynomial then obviously $[q(T_\phi)f]^\wedge = (q\circ\phi)\hat{f}$. Let $\{q_n\}$ be a sequence of polynomials such that $|||q_n-\alpha|||_{[a,b]} \to 0$. Then it follows that $q_n(\lambda) \to \alpha(\lambda)$ for all $\lambda \in [a,b]$, and so

$$q_n(\phi(\gamma))\hat{f}(\gamma) \to \alpha(\phi(\gamma))\hat{f}(\gamma) \qquad (\gamma \in \Gamma\overset{\text{a.e.}}{,}\ f \in L^p(G)) \ .$$

We also have, for each $f \in L^p(G)$,

$$\|(q_n\circ\phi)\hat{f} - \phi_\alpha\hat{f}\|_{p'} = \|[q_n(T_\phi)f]^\wedge - [\alpha(T_\phi)f]^\wedge\|_{p'}$$

$$\begin{aligned} &\leq \|q_n(T_\phi)f - \alpha(T_\phi)f\| \\ &\leq K\||q_n - \alpha\||_{[a,b]}\|f\| \\ &\to 0 \ . \end{aligned}$$

By 3.2.1(ii), therefore, $\psi_\alpha(\gamma)\hat{f}(\gamma) = \alpha(\phi(\gamma))\hat{f}(\gamma)$ a.e., and so $\psi_\alpha\hat{f} = (\alpha\cdot\phi)\hat{f}$ as elements of $L^{p'}(\Gamma)$, for all $f \in L^p(G)$.

<u>LEMMA 3.2.3</u>. In the notation of 1.2.3, if

$$\lim_{(\underline{t},\underline{t}^*)\in\mathcal{P}^*} \sum(f,\Delta g,\underline{t},\underline{t}^*) = h \quad ,$$

then there is a sequence $(\underline{t}^{(n)},\underline{t}^{*(n)})$ in $\mathcal{P}^*$, with $\max_k (t_k^{(n)}-t_{k-1}^{(n)}) \to 0$ as $n \to \infty$, such that $\sum(f,\Delta g,\underline{t}^{(n)},\underline{t}^{*(n)}) \to h$.

<u>Proof</u>. We construct the sequence inductively as follows. Let $\underline{t}^{(1)}$ be any partition such that $\max (t_k^{(1)}-t_{k-1}^{(1)}) < \frac{1}{2}$. For each $n = 1,2,\ldots$, suppose we have found a partition $\underline{t}^{(n)}$ such that $\max_k (t_k^{(n)}-t_{k-1}^{(n)}) < 2^{-n}$. The Stieltjes sums converge, by assumption, so there exists $\underline{s}^{(n)} \in \mathcal{P}$ such that $\|\sum(f,\Delta g,\underline{s},\underline{s}^*) - h\| < 2^{-n}$ for all $(\underline{s},\underline{s}^*) \in \mathcal{P}^*$ such that $\underline{s} \geq \underline{s}^{(n)}$. Since $\mathcal{P}$ is a directed set, there is a $\underline{u}^{(n)} \in \mathcal{P}$ with $\underline{u}^{(n)} \geq \underline{t}^{(n)}$, $\underline{u}^{(n)} \geq \underline{s}^{(n)}$. We can find a refinement $\underline{t}^{(n+1)}$ of $\underline{u}^{(n)}$ with $\max_k (t_k^{(n+1)} - t_{k-1}^{(n+1)}) < 2^{-n-1}$. Then $\underline{t}^{(n+1)} \geq \underline{u}^{(n)} \geq \underline{s}^{(n)}$, and so any $(\underline{t}^{(n+1)},\underline{t}^{*(n+1)}) \in \mathcal{P}^*$ satisfies

$$\|\sum(f,\Delta g,\underline{t}^{(n+1)},\underline{t}^{*(n+1)}) - h\| < 2^{-n} \quad .$$

Thus we have partitions $\{\underline{t}^{(n)}\}_{n=1}^{\infty}$ such that $\max_k (t_k^{(n)}-t_{k-1}^{(n)}) < 2^{-n}$ $(n \geq 1)$ and $\|\sum(f,\Delta g,\underline{t}^{(n)},\underline{t}^{*(n)}) - h\| < 2^{-n+1}$ $(n \geq 2)$, and so the result is proved.

<u>THEOREM 3.2.4</u>. Let $\phi \in M_p(\Gamma)$ $(1 < p < \infty)$ be real-valued. Then T_ϕ is well-bounded if and only if each $\chi_{(-\infty,\lambda]}\circ\phi \in M_p(\Gamma)$ and there exists $K < \infty$ such that $\|T_{\chi_{(-\infty,\lambda]}\circ\phi}\| \leq K$ $(\lambda \in \underline{R})$. The spectral family E of T_ϕ is then given by $E(\lambda) = T_{\chi_{(-\infty,\lambda]}\circ\phi}$ $(\lambda \in \underline{R})$. (Of course, T_ϕ is self-adjoint when $p = 2$.)

<u>Proof</u>. As in 3.2.2, we may assume $1 < p \leq 2$. Suppose T_ϕ is well-bounded, and implemented by $(K,[a,b])$, where $[a,b]$ is large enough so that $[-\|\phi\|_\infty, \|\phi\|_\infty] \subset [a,b]$. Since $E(\lambda) \in \{T_\phi\}''$ for all $\lambda \in \underline{R}$, each $E(\lambda)$ commutes with all translations. Therefore, for each $\lambda \in \underline{R}$, there exists $\psi_\lambda \in M_p(\Gamma)$ such that $[E(\lambda)f]^\wedge = \psi_\lambda \hat{f}$ $(f \in L^p(G))$. We need to show that $\psi_\lambda = \chi_{(-\infty,\lambda]}\circ\phi$ locally a.e.

Since $L^p(G)$ is reflexive, T_ϕ is of type B, and therefore there is a $BV[a,b]$-functional calculus for T_ϕ. By 2.2.6(i)

$$E(\lambda) = \chi_{[a,\lambda]}(T_\phi) \quad (a \leq \lambda < b) .$$

Fix $\lambda \in [a,b)$. Let $\beta_n \in AC[a,b]$ be the function which is 1 on $[a,\lambda]$, 0 on $[\lambda+\frac{b-\lambda}{n}, b]$ and linear on $[\lambda, \lambda+\frac{b-\lambda}{n}]$. Then $\beta_n(\mu) \to \chi_{[a,\lambda]}(\mu)$ for all $\mu \in [a,b]$, and $|||\beta_n|||_{[a,b]} = 1$ for all n. By 1.3.11, therefore,

$$\|\beta_n(T_\phi)f - \chi_{[a,\lambda]}(T_\phi)f\|_p \to 0 \quad (f \in L^p(G)) .$$

For each n, by 3.2.2,

$$[\beta_n(T_\phi)f]^\wedge = (\beta_n\circ\phi)\hat{f} \quad (f \in L^p(G)) ,$$

and so

$$\begin{aligned} \|(\beta_n\circ\phi)\hat{f} - \psi_\lambda\hat{f}\|_{p'} &= \|[\beta_n(T_\phi)f]^\wedge - [E(\lambda)f]^\wedge\|_{p'} \\ &\leq \|\beta_n(T_\phi)f - E(\lambda)f\|_p \end{aligned}$$

$$= \|\beta_n(T_\phi)f - \chi_{[a,\lambda]}(T_\phi)f\|_p$$
$$\to 0 \ .$$

Since $\beta_n(\mu) \to \chi_{[a,\lambda]}(\mu)$ for all $\mu \in [a, b]$, we also have that

$$\beta_n(\phi(\gamma))\hat{f}(\gamma) \to \chi_{[a,\lambda]}(\phi(\gamma))\hat{f}(\gamma) \quad (\gamma \in \Gamma \ , \ f \in L^p(G)) \ .$$

From 3.2.1(ii) it follows that $\psi_\lambda \hat{f} = (\chi_{[a,\lambda]} \circ \phi)\hat{f}$ in $L^{p'}(\Gamma)$, for all $f \in L^p(G)$. Therefore, taking into account the assumption on $[a, b]$ and $\|\phi\|_\infty$, we have $E(\lambda) = T_{\chi_{(-\infty,\lambda]}\circ\phi}$ $(\lambda \in \underline{R})$.

Conversely, suppose the stated condition is satisfied. For each $\lambda \in \underline{R}$, let $L_\lambda = \{\gamma \in \Gamma : \phi(\gamma) \leq \lambda\}$ and set

$$E(\lambda) = T_{\chi_{L_\lambda}} = T_{\chi_{(-\infty,\lambda]}\cdot\phi} \ . \tag{1}$$

By hypothesis,

$$\|E(\lambda)\| \leq K \quad (\lambda \in \underline{R}) \ . \tag{2}$$

Obviously (1) implies

$$E(\lambda)E(\mu) = E(\min(\lambda,\mu)) \quad (\lambda,\mu \in \underline{R}) \ . \tag{3}$$

It is also clear that, if $a = \operatorname*{ess\,inf}_{\gamma\in\Gamma} \phi(\gamma)$ and $b = \operatorname*{ess\,sup}_{\gamma\in\Gamma} \phi(\gamma)$, then

$$E(\lambda) = 0 \ (\lambda < a) \ ; \ E(\lambda) = I \ (\lambda \geq b) \ . \tag{4}$$

Properties (2) , (3) and the reflexivity of $L^p(G)$ imply, by Lorch's theorem ([25], Theorem 3.2), that thelimits $E(\lambda+0)$ and $E(\lambda-0)$ exist in the strong operator topology, for all $\lambda \in \underline{R}$. We claim that

$$E(\lambda)f = E(\lambda+0)f \quad (f \in L^p(G) \ , \ \lambda \in \underline{R}) \ . \tag{5}$$

To obtain (5) , let $g = \lim_{\mu\to\lambda+0} E(\mu)f \in L^p(G)$. Let $\{\mu_n\}$ be a sequence of real numbers decreasing to λ . Then

$$\|E(\mu_n)f - g\|_p \to 0 \quad .$$

Consequently

$$\|(\chi_{(-\infty,\mu_n]} \circ \phi)\hat{f} - \hat{g}\|_{p'} \to 0 \quad ,$$

and so by 3.2.1 a subsequence $\{\mu_{n_k}\}$ exists for which

$$\chi_{(-\infty,\mu_{n_k}]}(\phi(y))\hat{f}(y) \to \hat{g}(y) \quad \text{a.e.}$$

But $\chi_{(-\infty,\mu_{n_k}]}(x) \to \chi_{(-\infty,\lambda]}(x)$ $(x \in \underline{R})$, hence

$$\begin{aligned}[E(\lambda)f]^\wedge(y) &= \chi_{(-\infty,\lambda]}(\phi(y))\hat{f}(y) \\ &= \lim \chi_{(-\infty,\mu_{n_k}]}(\phi(y))\hat{f}(y) \\ &= \hat{g}(y) \quad \text{a.e.}\end{aligned}$$

Thus $[E(\lambda)f]^\wedge = \hat{g}$ in $L^{p'}(\Gamma)$, hence by the injectivity of the Fourier transform, $E(\lambda)f = g$.

Properties (2)-(5) mean that E is a spectral family, so by 1.3.3,

$$T = aE(a) + \int_a^b \lambda dE(\lambda)$$

exists, the integral being strongly convergent, and is a well-bounded operator. We must show that $T = T_\phi$.

For any $f \in L^p(G)$,

$$Tf = aE(a)f + \underset{(\underline{\lambda},\underline{\lambda}^*) \in \mathcal{P}^*[a,b]}{\|\cdot\|_p\text{-lim}} \sum_{k=1}^{l(\underline{\lambda})} \lambda_k^*(E(\lambda_k)-E(\lambda_{k-1}))f \quad ,$$

$l(\underline{\lambda})$ being the number of intervals of the subdivision $\underline{\lambda} = \{\lambda_k\}_{k=0}^{l(\underline{\lambda})}$.

Therefore,

$$(Tf)^\wedge = a[E(a)f]^\wedge + \underset{(\underline{\lambda},\underline{\lambda}^*) \in \mathcal{P}^*[a,b]}{\|\cdot\|_p\text{-lim}} \sum_{k=1}^{l(\underline{\lambda})} \lambda_k^*[(E(\lambda_k)-E(\lambda_{k-1}))f]^\wedge \quad .$$

By 3.2.3 there exists a sequence $\{\underline{\lambda}^{(n)}, \underline{\lambda}^{*(n)}\}$ in $\mathcal{P}^*[a,b]$ with $\max_k (\lambda_k^{(n)}-\lambda_{k-1}^{(n)}) \to 0$, such that

$$(Tf)^\wedge = a[E(a)f]^\wedge + \|\cdot\|_{p}\text{,-}\lim_{n\to\infty} \sum_{k=1}^{l} \lambda^{*(n)}_k[(E(\lambda_k^{(n)})-E(\lambda_{k-1}^{(n)}))f]^\wedge \quad .$$

By 3.2.1(i) there is a subsequence, also to be denoted by $\{\underline{\lambda}^{(n)}, \underline{\lambda}^{*(n)}\}$, such that

$$a[E(a)f]^\wedge(y) + \sum_{k=1}^{l} \lambda^{*(n)}_k[(E(\lambda_k^{(n)})-E(\lambda_{k-1}^{(n)}))f]^\wedge(y)$$

$$\to (Tf)^\wedge(y) \text{ a.e.} \qquad (6)$$

Each of the expressions on the left hand side of (6) is of the form

$$a\chi_{(-\infty,a]}(\phi(y))\hat{f}(y) + \sum_{k=1}^{l} \lambda^*_k[\chi_{L_{\lambda_k}}(y) - \chi_{L_{\lambda_{k-1}}}(y)]\hat{f}(y) \quad . \qquad (7)$$

If $\phi(y) > a$, then, for each $\underline{\lambda}$, $\lambda_{k_0-1} < \phi(y) \leq \lambda_{k_0}$ for some k_0 . Thus $y \in L_{\lambda_k}$ for $k \geq k_0$, and $y \notin L_{\lambda_k}$ for $k < k_0$, and so (7) has the value $\lambda^*_{k_0}\hat{f}(y)$. For each n , $\lambda^{(n)}_{k_0-1} < \phi(y) \leq \lambda^{(n)}_{k_0}$ and $\lambda^{(n)}_{k_{0-1}} \leq \lambda^{*(n)}_{k_0} \leq \lambda^{(n)}_{k_0}$, so since $\max_k (\lambda_k^{(n)}-\lambda_{k-1}^{(n)}) \to 0$, it follows that $\lambda^{*(n)}_{k_0} \to \phi(y)$. If $\phi(y) = a$, then for each subdivision (7) has the value $a[E(a)f]^\wedge(y) = a\hat{f}(y) = \phi(y)\hat{f}(y)$, all other terms vanishing.

Therefore the left hand side of (7) tends to $\phi(y)\hat{f}(y)$ a.e., and so $(Tf)^\wedge(y) = (T_\phi f)^\wedge(y)$ a.e. Since this is true for all $f \in L^p(G)$, we have $T = T_\phi$.

Finally, we relate the result for $p > 2$ to that for the conjugate index by using 3.1.2. Since $T^{(p)}_\phi$ is similar, via the invertible isometry S, to $T_\phi^{(p')*}$, it is well-bounded. Furthermore, the spectral family of $T_\phi^{(p)}$ is $\lambda \to E(\lambda)^{(p)}$, $E(\lambda)^{(p)}$ being similar in the same way to $E(\lambda)^{(p')*}$, for all $\lambda \in \underline{R}$.

COROLLARY 3.2.5. Let $\phi : \underline{R} \to \underline{R}$ be bounded and piecewise monotone; that is, there exists a finite set of real points $\{x_j\}_{j=1}^{m-1}$, together with $x_0 = -\infty$, $x_m = +\infty$, such that ϕ is monotone and bounded on each interval $K_j = (x_{j-1}, x_j)$ $(j = 1,2,\ldots,m)$. (The values $\phi(x_j)$ $(j = 1,2,\ldots,m-1)$ can be assigned arbitrarily.) Then $\phi \in M_p(\underline{R})$ $(1 < p < \infty)$, and $T_\phi^{(p)}$ is a well-bounded operator.

Proof. Obviously $\phi \in BV(\underline{R})$, and so, by a theorem of Stečkin ([12], Theorem 6.2.5), $\phi \in M_p(\underline{R})$ $(1 < p < \infty)$. For each $\lambda \in \underline{R}$, the set L_λ introduced in the proof of Theorem 3.2.4 differs by only a finite set from a union $\bigcup_{j=1}^{m} L_{j,\lambda}$, where each $L_{j,\lambda}$ is a subinterval of K_j, possibly empty. Thus

$$\chi_{(-\infty,\lambda]}(\phi(y)) = \sum_{j=1}^{m} \chi_{L_{j,\lambda}}(y) \quad \text{a.e.}$$

Each $T_{\chi_{L_{j,\lambda}}}$ is the difference of two operators of the form P_s, where

$$(P_s f)^\wedge = \chi_{(-\infty,s)}\hat{f} \quad (s \in \underline{R},\ f \in L^1(\underline{R})\cap L^p(\underline{R})) .$$

Let

$$(U_s f)(x) = e^{-isx} f(x)$$

for almost all $x \in \underline{R}$, and all $s \in \underline{R}$, $f \in L^p(\underline{R})$. Then, for all $f \in L^1(\underline{R})\cap L^p(\underline{R})$,

$$(U_s f)^\wedge(y) = \hat{f}(y+s)$$

$$(P_s f)^\wedge(y) = \chi_{(-\infty,0)}(y-s)\hat{f}(y)$$

$$(U_s P_s f)^\wedge(y) = (P_s f)^\wedge(y+s)$$

$$\begin{aligned} &= \chi_{(-\infty,0)}(y)\hat{f}(y+s) \\ &= \chi_{(-\infty,0)}(y)(U_s f)^\wedge(y) \\ &= \tfrac{1}{2}((I-iH)U_s f)^\wedge(y) \quad \text{a.e.,} \end{aligned}$$

where H is the Hilbert transform operator, which is the multiplier operator corresponding to $\phi(y) = -i\,\mathrm{sgn}\,y$ $(y \in \underline{R})$. The fact that $\phi \in M_p(\underline{R})$ $(1 < p < \infty)$ is a theorem of M. Riesz ([12], Theorem 6.2.3). It follows that

$$P_s = \tfrac{1}{2}U_{-s}(I-iH)U_s \quad (s \in \underline{R})$$

and

$$\|P_s\|_p \leq \tfrac{1}{2}(1+\|H\|_p) \quad (s \in \underline{R}, \ 1 < p < \infty) .$$

Consequently, if we set

$$E^{(p)}(\lambda) = T^{(p)}_{\chi_{(-\infty,\lambda]}\cdot\phi} = \sum_{j=1}^{m} T_{\chi_{L_{j,\lambda}}} , \qquad (8)$$

then

$$\|E^{(p)}(\lambda)\|_p \leq m(1+\|H\|_p) < \infty \quad (\lambda \in \underline{R}) . \ (9)$$

(8) and (9) imply that the condition of Theorem 3.2.4 is satisfied, and so $T^{(p)}_\phi$ is well-bounded and has spectral family $E^{(p)}$.

Let bv be the space of all sequences $\{\xi_n\}_{n=-\infty}^{\infty}$ of complex numbers such that $\sum_{-\infty}^{\infty} |\xi_n - \xi_{n-1}| < \infty$. By another theorem of Stečkin ([12], Theorem 6.3.5), $bv \subset M_p(\underline{Z})$ $(1 < p < \infty)$.

<u>COROLLARY 3.2.6</u>. If $\xi = \{\xi_n\}_{n=-\infty}^{\infty}$ is a piecewise monotone bounded sequence of real numbers, then $\xi \in M_p(\underline{Z})$ $(1 < p < \infty)$ and $T^{(p)}_\xi$ is a well-bounded operator.

<u>Proof</u>. The proof is exactly analogous to Corollary 3.2.5, with the conjugate operator for functions on $\underline{T}$ replacing the Hilbert transform on $\underline{R}$.

COROLLARY 3.2.7. If $\psi : \underline{T} \to \underline{R}$ is piecewise monotone and bounded, then $\psi \in M_p(\underline{T})$ $(1 < p < \infty)$ and $T_\psi^{(p)}$ is a well-bounded operator.

Proof. In this case there is another Stečkin theorem which states that $BV(\underline{T}) \subset M_p(\underline{T})$, with $\|T_\theta^{(p)}\| \leq K_p \|\|\theta\|\|$ $(\theta \in BV(\underline{T}))$ ([12], Theorem 6.4.4). (It is immaterial which norm we give $BV(\underline{T})$.) In particular, when $\theta = \chi_{[0,\lambda]}$ $(0 \leq \lambda \leq 2\pi)$, $\|T_\theta\| \leq 2K_p$ and so it follows as in Corollary 3.2.5 that the appropriate projections $E(\lambda)$ $(\lambda \in \underline{R})$ will be uniformly bounded.

§3.3. Examples of well-bounded multiplier operators

In this section we give examples and counterexamples to illustrate 3.2.4 - 3.2.7.

THEOREM 3.3.1. (i) The Poisson operators $P(t)$, where

$$[P(t)f](x) = \frac{t}{\pi}\int_{-\infty}^{\infty} \frac{f(x-u)}{t^2+u^2}\,du \quad (0 < t < \infty,\ f \in L^p(\underline{R}))$$

$$P(0) = I$$

are well-bounded operators in $B(L^p(\underline{R}))$ $(1 < p < \infty)$.

(ii) If D is the operator on $L^p(\underline{R})$ $(1 < p < \infty)$ where

$$\mathfrak{D}(D) = \{f \in L^p(\underline{R}) \cap LAC(\underline{R}) : f' \in L^p(\underline{R})\},$$

$$Df = f' \quad (f \in \mathfrak{D}(D)),$$

and H is the Hilbert transform, then DH is a well-boundable operator on $L^p(\underline{R})$.

Proof. (i) The kernel $p(t;x) = \frac{1}{\pi}\frac{t}{t^2+x^2}$ has Fourier transform $\hat{p}(t;y) = e^{-t|y|}$, therefore

$$[P(t)f]^\wedge(y) = e^{-t|y|}\hat{f}(y) \quad (y \in \underline{R},\ f \in L^1(\underline{R}) \cap L^p(\underline{R}),\ 0 \leqslant t < \infty)$$

Since $e^{-t|y|}$ satisfies the condition of 3.2.5, it is immediate that $P(t)$ is well-bounded.

(ii) The operators $\{P(t)\}_{t \geqslant 0}$ form a strongly continuous semigroup in $B(L^p(\underline{R}))$, whose infinitesimal generator A is given by

$$\mathfrak{D}(A) = \{f \in L^p(\underline{R}) : Hf \in LAC(\underline{R}) \text{ and } (Hf)' \in L^p(\underline{R})\}$$

$$Af = -DHf \quad (f \in \mathfrak{D}(A)),$$

which also satisfies

$$\mathfrak{D}(A) = \{f \in L^p(\underline{R}) : (y \to |y|\hat{f}(y)) \in [L^p(R)]^\wedge\}$$

$$(Af)^\wedge(y) = -|y|f(y),$$

when $1 < p \leqslant 2$. These facts are proved in [5], Theorem 4.2.10 and Lemma 4.2.11. It follows from Theorem 2.3.2 that $-DH$ is well-boundable, and from 2.2.7(i) that DH is also well-boundable.

THEOREM 3.3.2. (i) The Gauss-Weierstrass operators $W(t)$, where

$$[W(t)f](x) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} f(x-u)e^{-u^2/4t}du \quad (0 < t < \infty,\ f \in L^p(\underline{R}))$$

$$W(0) = I$$

is a well-bounded operator in $B(L^p(\underline{R}))$ $(1 < p < \infty)$.

(ii) The operator D^2 on $L^p(\underline{R})$ $(1 < p < \infty)$, where D is as in 3.3.1, is well-boundable.

Proof. (i) The kernel $w(t;x) = \frac{1}{\sqrt{4\pi t}}e^{-x^2/4t}$ has Fourier transform $\hat{w}(t;y) = e^{-ty^2}$, therefore

$$[W(t)f]^\wedge(y) = e^{-ty^2}\hat{f}(y) \quad (y \in \underline{R},\ f \in L^1(\underline{R}) \cap L^p(\underline{R}),\ 0 \leqslant t < \infty).$$

Since e^{-ty^2} satisfies 3.2.5, the result follows.

(ii) The operators $\{W(t)\}_{t \geqslant 0}$ form a strongly continuous semigroup whose infinitesimal generator is D^2 ([5], Theorem 4.3.11), so the result again follows from 2.3.2.

THEOREM 3.3.3. The following periodic analogues of 3.3.1 and 3.3.2 hold:

(i) the periodic Poisson semigroup $\{T(t)\}$, where

$$[T(t)](\zeta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(1-e^{-2t})f(\zeta-\phi)}{1-2e^{-t}\cos\phi+e^{-2t}}\,d\phi$$

$$(f \in L^p(\underline{T}),\ \zeta \in \underline{T},\ 0 < t < \infty)$$

$$T(0) = I,$$

is a strongly continuous semigroup of well-bounded operators in $B(L^p(\underline{T}))$ $(1 < p < \infty)$ corresponding to the multipliers $n \to e^{-t|n|}$ in $M_p(\underline{Z})$;

(ii) the periodic Gauss-Weierstrass semigroup $\{V(t)\}$, where

$$[V(t)](\zeta) = \frac{1}{2\pi}\int_0^{2\pi} f(\zeta-\phi)\theta_3(\phi;t)d\phi \quad (f \in L^p(\underline{T}),\ \zeta \in \underline{T},\ 0 < t < \infty),$$

$$V(0) = I,$$

($\theta_3(\cdot;t)$ is the Jacobi theta function $\phi \to \sum_{k=-\infty}^{\infty} e^{-k^2t}e^{ik\phi}$), is a strongly continuous semigroup of well-bounded operators in $B(L^p(\underline{T}))$ $(1 < p < \infty)$ corresponding to the multipliers $n \to e^{-n^2t}$ in $M_p(\underline{Z})$;

(iii) the infinitesimal generators A of $\{T(t)\}$ and B of $\{V(t)\}$ are given by

$$\mathfrak{D}(A) = \{f \in L^p(\underline{T}) : \tilde{f} \in AC(\underline{T}) \text{ and } (\tilde{f})' \in L^p(\underline{T})\}$$

$$= \{f \in L^p(\underline{T}) : (n \to |n|\hat{f}(n)) \in [L^p(\underline{T})]^\wedge\},$$

$$Af = -(\tilde{f})' \quad (f \in \mathfrak{D}(A)),$$

$$(Af)^\wedge(n) = -|n|\hat{f}(n) \quad (f \in \mathfrak{D}(A),\ n \in \underline{Z}),$$

where $\tilde{f}$ is the conjugate function of f , and

$$\begin{aligned} \mathfrak{D}(B) &= \{f \in L^p(\underline{T}) : f,f' \in AC(\underline{T}) \text{ and } f'' \in L^p(\underline{T})\} \\ &= \{f \in L^p(\underline{T}) : (n \to n^2\hat{f}(n)) \in [L^p(\underline{T})]^\wedge\} \ , \\ Bf &= f'' \quad (f \in \mathfrak{D}(B)) \ , \\ (Bf)^\wedge(n) &= -n^2\hat{f}(n) \quad (f \in \mathfrak{D}(B) \ , \ n \in \underline{Z}) \ , \end{aligned}$$

and are well-boundable operators.

Proof. The relevant facts about $\{T(t)\}$ and $\{V(t)\}$ are given in [5], Proposition 1.5.1 and Theorems 1.5.3, 1.5.10. It is clear that the multiplier sequences $\{e^{-|n|t}\}$ and $\{e^{-n^2t}\}$ satisfy 3.2.6, so $T(t)$ and $V(t)$ are well-bounded. The fact that A and B are well-boundable follows from 2.3.2.

THEOREM 3.3.4. The Poisson and Gauss-Weierstrass operators on $L^p(R^n)$ $(1 < p < \infty \ , \ p \neq 2)$ given by

$$[P(t)f](x) = c_n t\int_{\underline{R}^n} \frac{f(x-u)}{(t^2+|u|^2)^{(n+1)/2}}\,du \quad (0 < t < \infty \ , \ f \in L^p(\underline{R}^n)) \ ,$$

where $c_n = \dfrac{\Gamma[(n+1)/2]}{\pi^{(n+1)/2}}$, and

$$[W(t)f](x) = \frac{1}{(4\pi t)^{n/2}} \int_{\underline{R}^n} f(x-u)e^{-|u|^2/4t}du \quad (0 < t < \infty \ , \ f \in L^p(\underline{R}^n)) \ ,$$

are not well-bounded if $n > 1$.

Proof. Let $1 < p < 2$. When expressed in the multiplier operator form $(T_\phi f)^\wedge = \phi\hat{f}$, $P(t)$ and $W(t)$ become

$$[P(t)f]^\wedge(y) = e^{-t|y|}\hat{f}(y)$$

$$[W(t)f]^\wedge(y) = e^{-ty^2}\hat{f}(y)$$

for all $f \in L^p(\underline{R}^n)$. Each set $\{y \in \underline{R}^n : \phi(y) \leqslant \lambda\}$, for both $P(t)$ and $W(t)$, is therefore the exterior of a sphere in $\underline{R}^n$. It was proved by Fefferman [13], that, if $B_n = \{x \in \underline{R}^n : |x| \leqslant 1\}$, then $\chi_{B_n} \notin M_p(\underline{R}^n)$ for any $p \neq 2$, if $n > 1$. It therefore follows from 3.2.4 that neither $P(t)$ nor $W(t)$ can be well-bounded when $n > 1$.

Not all real-valued functions in $BV(\underline{R})$ yield well-bounded operators when considered as multipliers. To obtain a counterexample, we need the following concept:

<u>DEFINITION 3.3.5</u>. A set of uniqueness for $L^p(\Gamma)$ is a measurable set $F \subset G$ such that, if $f \in L^1(G)$, $f = 0$ a.e. on $G \backslash F$, and $\hat{f} \in L^p(\Gamma)$, then $f = 0$ a.e.

<u>PROPOSITION 3.3.6</u>. If F is a set of uniqueness for $L^p(G)$ contained in a compact set, with $0 < \eta(F) < \infty$, then $\chi_F \notin M_p(\Gamma)$ $(1 < p < 2)$.

<u>Proof</u>. Let $H \supset F$ be a compact set with $\eta(H) < \infty$. It is a standard fact (see [28], §2.6) that there exists $h \in L^1(G) \cap L^p(G) \cap L^2(G)$ such that $\hat{h}$ is equal a.e. to a continuous function of compact support which takes the value 1 on H.

If $\chi_F \in M_p(\Gamma)$, then

$$\chi_F \hat{f} \in [L^p(G)]^\wedge \quad (f \in L^p(G)) .$$

Hence $\chi_F = \chi_F \hat{h} \in [L^p(G)]^\wedge$, and so there exists $g \in L^p(G)$ with $\hat{g} = \chi_F$. But since $\chi_F \in L^1(\Gamma) \cap L^2(\Gamma)$, its inverse transform $\check{\chi}_F$ exists, and we have

$$\check{\chi}_F = (\hat{g})^\vee = g \in L^p(G) .$$

Therefore F satisfies the conditions $\chi_F \in L^1(\Gamma)$, $\chi_F = 0$ a.e. on

$\Gamma\backslash F$, and $\check{\chi}_F \in L^p(G)$ of 3.3.5. Since F is a set of uniqueness, we must have $\chi_F = 0$ a.e. But $\eta(F) > 0$, which is a contradiction. Therefore $\chi_F \notin M_p(\Gamma)$.

The existence of non-trivial sets of uniqueness was shown by a construction of Figà-Talamanca and Gaudry, reproduced in [24], Theorem 4.4.1:

PROPOSITION 3.3.7. Let G be a nondiscrete locally compact abelian group, and $H \subset G$ a measurable set such that $0 < m(H) < \infty$. If $\epsilon > 0$, then there exists a measurable set $F \subset H$ such that

(i) $m(F) > m(H) - \epsilon$;

(ii) F is a set of uniqueness for $L^p(\Gamma)$ $(1 \leq p < 2)$.

THEOREM 3.3.8. There is a real-valued function $\phi \in BV(\underline{R}) \subset M_p(\underline{R})$ such that $T_\phi \in B(L^p(\underline{R}))$ is not well-bounded $(1 < p < \infty)$.

Proof. Let F be a set of uniqueness as in Proposition 3.3.7. By 3.3.6, $\chi_F \notin M_p(\underline{R})$ $(1 < p < \infty)$. We refer to [24], pp. 103-108, to examine the construction of F : it is an intersection $F = \bigcap_{n=1}^{\infty} F_n$, where each F_n is a finite union of dyadic subintervals of $[0,1]$. The dyadic numbers being a countable set, we may assume that each F_n is a closed set. Then F is also closed, and so $F_0 = [0,1]\backslash(F\cup\{0,1\})$ is an open subset of $\underline{R}$, such that $\chi_{F_0} \notin M_p(\underline{R})$ $(1 < p < 2)$. Being open, F_0 can be expressed as a union $F_0 = \bigcup_{n=1}^{\infty} D_n$ of disjoint open intervals $D_n = (a_n, b_n)$.

Now let $\phi(x) = 0$ $(x \notin F_0)$, $\phi(\frac{1}{2}(a_n+b_n)) = n^{-2}$, with ϕ linear on each interval $(a_n, \frac{1}{2}(a_n+b_n))$ and $(\frac{1}{2}(a_n+b_n), b_n)$. Then

$\{x \in \underline{R} : \phi(x) \leq 0\} = F\cup(-\infty,0]\cup[1,\infty)$. If T_ϕ were well-bounded, then the spectral projection $E(0)$ would be $T_{\chi_{(-\infty,0]}\circ\phi}$; but this does not exist in $B(L^p(\underline{R}))$, since $\chi_{(-\infty,0]}$, $\chi_{[1,\infty)} \in M_p(\underline{R})$ $(1 < p < \infty)$ and $\chi_F \notin M_p(\underline{R})$.

It remains to check that $\phi \in BV(\underline{R})$. Let $\underline{t} \in \mathcal{P}[0,1]$. If t_k, $t_{k-1} \notin F_0$ then $\phi(t_k)-\phi(t_{k-1}) = 0$. If $t_k \in F_0$ and $t_{k-1} \notin F_0$, then $t_k \in (a_{n(k)},b_{n(k)})$, say, and so $|\phi(t_k)-\phi(t_{k-1})| = \phi(t_k) \leq n(k)^{-2}$. If $t_k \notin F_0$ and $t_{k-1} \in F_0$, then $|\phi(t_k)-\phi(t_{k-1})| \leq n(k-1)^{-2}$. Finally, if $t_k,t_{k-1} \in F_0$, then either $n(k) = n(k-1)$, in which case $|\phi(t_k)-\phi(t_{k-1})| \leq n(k)^{-2}$, or else $n(k) \neq n(k-1)$, in which case there exists an $s \in (t_{k-1},t_k)$ such that $\phi(s) = 0$, so that $|\phi(t_k)-\phi(t_{k-1})| \leq |\phi(t_k)-\phi(s)| + |\phi(s)-\phi(t_{k-1})| \leq n(k)^{-2} + n(k-1)^{-2}$. Therefore we see that $\underset{\underline{R}}{\text{var}}\, \phi = 2\sum_{n=1}^{\infty} n^{-2} < \infty$.

§3.4. Connection with results of Krabbe

The fact that operators of the type T_ϕ arising in 3.2.5 and 3.2.7 satisfy a Riemann-Stieltjes form of the spectral theorem is not new. Such theorems were established by Krabbe in [20] and [21], using direct calculations. Krabbe's results are more general than ours in that he is able to handle two-dimensional integrals by using spectral projections associated with rectangles, and so obtains results for complex-valued multipliers. However, for the real case, his treatment does not reveal that the mere existence and uniform boundedness of the spectral projections is itself necessary and sufficient for well-boundedness, thus, in view of 1.3.3, reducing the problem of finding a uniformly bounded spectral family for T_ϕ to the (difficult) harmonic analysis question of determining those sets whose characteristic functions are multipliers.

In [20] it is shown that for certain functions ϕ in $BV(\underline{T})$, namely those for which $\chi_J \cdot \phi \in BV(\underline{T})$ whenever J is a half-open rectangle or a singleton, then, for $1 < p < \infty$,

$$<g,T_\phi f> = \int \lambda d<g,E(\lambda)f> \quad (f \in l^p(\underline{Z}) \ , \ g \in l^{p'}(\underline{Z})) \qquad (1)$$

where $E(J) = T_{\chi_J \cdot \phi}$ for all such J, and the integral in (1) is a two-dimensional Stieltjes integral. However, the only functions actually identified in [20] as belonging to this class are those whose real and imaginary parts are piecewise monotone ([20], remarks on p.458 and Theorem 8.10). The special case $\phi(j) \equiv j \ (j \in [0,2\pi])$ is studied in [8], where it is shown that T_ϕ is well-bounded for this choice of ϕ; our general version is based on the ideas in [8].

The analogous results for the group $\underline{R}$ are established in [21]. If $\psi : \underline{R} \to \underline{C}$ is a bounded function whose real and imaginary parts are piecewise monotone, then the multiplier operator $T_\psi \in B(L^p(\underline{R}))$ $(1 < p < \infty)$ satisfies

$$T_\psi f = \int \lambda dE(\lambda) f \quad (f \in L^p(\underline{R})) \qquad (2)$$

where this time the integral in (2) is a strongly convergent two-dimensional Stieltjes integral ([21], Theorem 6.14). (We have not examined in detail the reason for the discrepancy between the weak convergence in (1) and strong convergence in (2).)

In [22] (Theorem 9.4, Corollaries 9.5, 9.9) it is shown that when Q is one of the operators iD, D^2 and DH (where D is as in 3.3.1) then

$$Qf = \int \lambda dE(\lambda)f \quad (f \in \mathfrak{D}(Q)) \tag{3}$$

for a oertain strongly convergent integral over $\underline{R}$ which is similar to our $\int_{\underline{R}}^{\oplus}$. A connection between the bounded and unbounded results is pointed out in [22], §9.10, via a form of funotional calclulus, but the intimacy of the relation between the results for the semigroups $\{P(t)\}$, $\{W(t)\}$ and their generators $-DH$, D^2 is obscured by the direot but independent constructions of the integrals in (2) and (3). In our case, however, Theorem 2.3.2, and in particular equation (4) of §2.3, give a direot interpretation of the form $T(t) = \exp(tA)$ when $\{T(t)\}$ is one of the semigroups $\{P(t)\}$, $\{W(t)\}$.

Finally, we remark that there is at present no satisfactory generalisation of the theory of well-bounded operators to operators with complex spectrum. In view of the above observations, such a theory would need to cover the complex-valued oases of Krabbe's results, and to permit an extension of Theorem 2.3.2 in analogy to the theorems for normal operators ([16], Theorem 22.4.2) and for scalar type operators (even for complex spectrum) ([31], Theorem 5.3). An interpretation of Krabbe's spectral theorem for iD (since D is the infinitesimal generator of the group of translations) might then be possible.

CHAPTER 4

WELL-BOUNDED RIESZ OPERATORS

The class of Riesz operators in $B(X)$ is defined in such a way that it consists of all those operators which satisfy the Riesz theory of compact operators. We summarise this theory below in §4.1; the additional hypothesis of well-boundedness permits a little simplification. The main result of this chapter is that a well-bounded Riesz operator is necessarily compact; this is proved in §4.2 (Theorem 4.2.3). As an application, in §4.3 we use this result to show that the singular multiplier in $M_p(\underline{Z})$ $(1 < p < 2)$ constructed by Figà-Talamanca and Gaudry [14], does not define a well-bounded operator.

§4.1. Preliminaries on Riesz operators

DEFINITION 4.1.1. An operator $T \in B(X)$ is a Fredholm operator if $\dim \ker T$ and $\dim X/TX$ are both finite.

The second of these conditions implies that TX is closed: see [6], Corollary 3.2.5.

If $\lambda I-T$ is Fredholm for all $\lambda \in \underline{C}\backslash\{0\}$ (which implies that T is not Fredholm unless $\dim X < \infty$), then T is called a Riesz operator.

We denote the set of Riesz operators in $B(X)$ by $R(X)$, and the compact operators by $K(X)$; of course, $K(X) \subset R(X)$.

PROPOSITION 4.1.2. (i) If $T \in B(X)$, then T is Fredholm if and only if $\widetilde{T}$, the image of T under the canonical map $B(X) \to B(X)/K(X)$, is invertible.

(ii) If $T \in B(X)$, then $T \in R(X)$ if and only if $\widetilde{T}$ is quasinilpotent.

(iii) For $T \in B(X)$, T is Fredholm if and only if T^* is Fredholm.

(iv) $T \in R(X)$ if and only if $T^* \in R(X^*)$.

Proof. (i) [6], Theorem 3.2.8.

(ii) follows from (i) and the observation that $\sigma(T) \neq \emptyset$.

(iii) follows from the definition and [6], Proposition 1.2.7.

(iv) follows from (iii) .

Remark. The question as to whether $T \in R(X)$ can always be expressed in the form $T = C + Q$, with C compact and Q quasinilpotent, is unsolved.

PROPOSITION 4.1.3. If $T \in R(X)$ and $\lambda \in \sigma(T)\setminus\{0\}$, then λ is a pole of $(\mu \to R(\mu;T))$ with order $p \neq 0$, and we have

$$X = \mathfrak{R}_\lambda \oplus \mathfrak{N}_\lambda \quad ,$$

where $\mathfrak{R}_\lambda = (\lambda I-T)^p X$ and $\mathfrak{N}_\lambda = \ker(\lambda I-T)^p$. This decomposition reduces T ; $(\lambda I-T)|\mathfrak{R}_\lambda$ is invertible, $(\lambda I-T)|\mathfrak{N}_\lambda$ is nilpotent and $\mathfrak{N}_\lambda$ is finite-dimensional. Furthermore, λ is an isolated point of $\sigma(T)$, and the spectral projection

$$P_\lambda = \frac{1}{2\pi i}\int_\Gamma R(\mu;T)d\mu \quad ,$$

where Γ is a contour in $\rho(T)$ separating λ from $\sigma(T)\setminus\{\lambda\}$, has range $\mathcal{N}_\lambda$ and nullspace $\mathcal{R}_\lambda$.

Proof. These results are all in [6], Chapter 3 . We use the the characterisation ([6], Theorem 3.2.2) as our definition, which is equivalent to ([6], Definition 3.1.1); the other results are in ([6], Lemma 3.4.2).

It follows that if $T \in R(X)$, then $\sigma(T)$ is a sequence $\{\lambda_n\}$ whose only limit point is zero. Unless $\dim X < \infty$, we have $0 \in \sigma(T)$, even if $\sigma(T)$ is finite, because T is not Fredholm, and therefore cannot be invertible.

If $Y \subset X$ and $Z \subset X^*$, let

$$Y^{\perp} = \{\phi \in X^* : \phi(y) = 0 \text{ for all } y \in Y\} ,$$

$$Z_{\perp} = \{x \in X : \phi(x) = 0 \text{ for all } \phi \in Z\}.$$

PROPOSITION 4.1.4. If $T \in R(X)$ is well-bounded, and $\lambda \in \sigma(T) \backslash \{0\}$, then

(i) $X = (\lambda I-T)X \oplus \ker(\lambda I-T)$;

(ii) $X^* = (\lambda I^*-T^*)X^* \oplus \ker(\lambda I^*-T^*)$;

(iii) $(\lambda I-T)X = [\ker(\lambda I^*-T^*)]_{\perp}$;

(iv) $(\lambda I^*-T^*)X^* = [\ker(\lambda I-T)]^{\perp}$;

(v) $\dim \ker (\lambda I-T) = \dim \ker (\lambda I^*-T^*)$;

(vi) P^*_λ has range $[(\lambda I-T)X]^{\perp} = \ker (\lambda I^*-T^*)$ and nullspace $[\ker (\lambda I-T)]^{\perp} = (\lambda I^*-T^*)X^*$.

Proof. (i) follows from repeated applications of [30], Lemma 2.7; alternatively, we can use [1], Theorem 7, since any well-bounded operator satisfies condition (G_1).

(ii) is immediate from (i) , since $T^* \in R(X^*)$ is also well-bounded.

(iii) is true for any operator with closed range ([33], Theorem 4.6-D).

(iv) We need only show that $[\ker(\lambda I-T)]^{\perp} \subset (\lambda I^*-T^*)X^*$, since the reverse inclusion is true for any bounded operator. Let $y^* \in [\ker(\lambda I-T)]^{\perp}$. For each $y \in (\lambda I-T)X$, define $\phi_0(y) = y^*(x)$, where $x \in X$ is any vector for which $(\lambda I-T)x = y$. Since $y^* \in [\ker(\lambda I-T)]^{\perp}$, $\phi_0(y)$ is well-defined, so ϕ_0 is a linear functional on $(\lambda I-T)X$. Since $(\lambda I-T)|(\lambda I-T)X$ is invertible, there is an $M > 0$ such that, for each $y \in (\lambda I-T)X$, there is an x with $(\lambda I-T)x = y$ and $\|x\| \leq M\|y\|$. Consequently ϕ_0 is continuous. By the Hahn-Banach theorem, we can extend ϕ_0 to $\phi \in X^*$. Then $\langle\phi,(\lambda I-T)x\rangle = \langle y^*,x\rangle$ $(x \in X)$, and so $y^* = (\lambda I^*-T^*)\phi$.

(v) The proof of this part is the same as that of the corresponding result for compact operators ([33], Theorem 5.5-H). We shall not present it in full, but make the following observation about its validity for Riesz operators. The proof proceeds by constructing a suitable finite rank perturbation A of T (resp. B of T^*), showing that $(\lambda I-A)^{-1}$ (resp. $(\lambda I-B)^{-1}$) exists, and obtaining a contradiction of the hypothesis $m < n$ (resp. $m > n$), where $m = \dim\ker(\lambda I-T)$, $n = \dim\ker(\lambda I^*-T^*)$. The fact that A and B are Riesz operators in our case follows from 4.1.2(i). Once we have shown that $\ker(\lambda I-A) = \{0\}$, then 4.1.3 implies that $(\lambda I-A)X = X$, and so $(\lambda I-A)^{-1}$ exists. A similar argument works for B. Consequently the procedure for obtaining the contradiction is still valid.

Finally, it should be noted that we have also proved $\dim\ker(\lambda I-T)^n$

$= \dim \ker (\lambda I^*-T^*)^n$, for all n , for any $T \in R(X)$, since by 4.1.2(ii) and the spectral radius formula any polynomial in T without constant term is a Riesz operator, and so $(\lambda I-T)^n = \lambda^n I-S$, where $S \in R(X)$. (N.B. we have not used well-boundedness in this part.)

(vi) For any projection P in a Banach space, $\ker P^* = (PX)^\perp$ and $P^*X^* = [(I-P)X]^\perp$. Therefore $\ker P^*_\lambda = [\ker (\lambda I-T)]^\perp = (\lambda I^*-T^*)X^*$ by (iv) , and $P^*_\lambda X^* = [(\lambda I-T)X]^\perp = [\ker (\lambda I^*-T^*)]_\perp^\perp \supset \ker (\lambda I^*-T^*)$, so in view of (ii) we must have $P^*_\lambda X^* = \ker (\lambda I^*-T^*)$.

<u>PROPOSITION 4.1.5</u>. Under the hypotheses of 4.1.4, $[(\lambda I-T)X]^*$ is isomorphic (but not isometric) to $(\lambda I^*-T^*)X^*$, and $[\ker (\lambda I-T)]^*$ is isomorphic (but not isometric) to $\ker (\lambda I^*-T^*)$.

<u>Proof</u>. $[(\lambda I-T)X]^*$ is isometrically isomorphic to $X^*/[(\lambda I-T)X]^\perp = X^*/\ker (\lambda I^*-T^*)$, which is clearly isomorphic to $(\lambda I^*-T^*)X^*$ because of 4.1.4(vi). It is easily seen that if $\phi \in (\lambda I^*-T^*)X^*$ and ϕ^0 is the image of ϕ in $[(\lambda I-T)X]^*$, then $\langle \phi,x\rangle = \langle \phi^0,x\rangle$ $(x \in (\lambda I-T)X)$. The other part is similar.

§4.2. Compactness of well-bounded Riesz operators

<u>NOTATION</u>. If $T \in R(X)$ is well-bounded, we shall denote the positive and negative parts of the spectrum respectively by $\{\lambda_i\}_{i\geq 1}$ and $\{\mu_j\}_{j\geq 1}$, arranged so that $\{\lambda_i\}_{i\geq 1}$ $[\{\mu_j\}_{j\geq 1}]$ is a decreasing [increasing] sequence, possibly finite, of positive [negative] numbers, which tends to zero if infinite. (Either sequence may be absent.) The spectral projections P_{λ_i} $[P_{\mu_j}]$ of §4.1 will henceforth be denoted by Q_i $[P_j]$.

LEMMA 4.2.1. Let $T \in B(X)$ be a well-bounded operator implemented by $(K,[a,b])$, and F a dual spectral family for T. Let $c = \inf \sigma(T)$ and $d = \sup \sigma(T)$. Then $F(\lambda) = 0 \ (\lambda < c)$ and $F(\lambda) = I^* \ (\lambda \geqslant d)$, so that T is implemented by $(K,[c,d])$.

Proof. The hypotheses of the lemma imply $a \leqslant c \leqslant d \leqslant b$. Suppose that $a \leqslant \lambda < c$. Then, by [27], Theorem 3, $\sigma(T^* | F(\lambda)X^*) \subset [a,\lambda]$. Since $\sigma(T^*)$ does not separate $\underline{C}$, $\sigma(T^*|F(\lambda)X^*) \subset \sigma(T^*) \subset [c,d]$ also. Therefore $\sigma(T^*|F(\lambda)X^*) = \emptyset$ and so $F(\lambda)X^* = \{0\}$ $(\lambda < c)$. Similarly, $\bigcap_{\mu<\lambda}(I-F(\mu))X^* = \{0\}$ if $d < \lambda \leqslant b$. Since $(I^*-F(\lambda))X^* \subset (I^*-F(\mu))X^*$ $(\mu < \lambda)$ it follows that $(I^*-F(\lambda))X^* = \{0\}$ $(d < \lambda \leqslant b)$. Therefore by 1.3.4(v) $F(\lambda)X^* = X^*$ $(\lambda \geqslant d)$. The fact that T is implemented by $(K,[c,d])$ follows from 1.3.5.

THEOREM 4.2.2. If $T \in R(X)$ is well-bounded and F is a dual spectral family for T, then

$$F(\lambda) = \begin{cases} 0 & \lambda < \mu_1 \\ P_1^* + \dots + P_n^* & \mu_n \leqslant \lambda < \mu_{n+1} \\ I^* - (Q_1^* + \dots + Q_n^*) & \lambda_{n+1} \leqslant \lambda < \lambda_n \\ I^* & \lambda \geqslant \lambda_1 \end{cases},$$

leaving $F(0)$ as the only value that is possibly not uniquely determined.

Proof. The required values of $F(\lambda)$ for $\lambda \notin [\mu_1,\lambda_1)$ are given by 4.2.1. Starting from the decomposition 4.1.4(i) with $\lambda = \mu_1$ we consider $T|(\mu_1 I-T)X$. This operator is well-bounded, by definition, and Riesz, by [6], Lemma 3.5.1. We can therefore repeat the decomposition, and since it is easily shown that

$$\ker(\mu_2 I-T) = \ker[(\mu_2 I-T)|(\mu_1 I-T)X] \quad ,$$

we have

$$X = (\mu_1 I-T)(\mu_2 I-T)X \oplus \ker(\mu_1 I-T) \oplus \ker(\mu_2 I-T) \quad .$$

After n applications of 4.1.4(i) we get

$$X = (\mu_1 I-T)(\mu_2 I-T)\cdots(\mu_n I-T)X \oplus \ker(\mu_1 I-T) \oplus \cdots \oplus \ker(\mu_n I-T) \quad . \quad (1)$$

Similarly,

$$X^* = (\mu_1 I^*-T^*)(\mu_2 I^*-T^*)\cdots(\mu_n I^*-T^*)X \oplus \ker(\mu_1 I^*-T^*) \oplus \cdots$$
$$\cdots \oplus \ker(\mu_n I^*-T^*) \quad . \quad (2)$$

Let

$$\begin{aligned} X_n &= (\mu_1 I-T)\cdots(\mu_n I-T)X \quad , \\ N_j &= \ker(\mu_j I-T) \quad (j = 1,\ldots,n) \quad , \\ (X^*)_n &= (\mu_1 I^*-T^*)\cdots(\mu_n I^*-T^*)X^* \quad , \\ M_j &= \ker(\mu_j I^*-T^*) \quad (j = 1,\ldots,n) \quad , \\ T_n &= T|X_n \quad , \\ U_j &= T|N_j \quad (j = 1,\ldots,n) \quad . \end{aligned}$$

Then

$$\begin{aligned} \sigma(T_n) &= \{\lambda_i\}_{i\geq 1} \cup \{\mu_j\}_{j\geq n+1} \cup \{0\} \quad , \\ \sigma(U_j) &= \{\mu_j\} \quad (j = 1,\ldots,n) \quad . \end{aligned}$$

The decomposition (1) reduces T, and T_n, U_j $(j = 1,\ldots,n)$ are all well-bounded Riesz operators. If F_n, G_j are any dual spectral families for T_n, U_j respectively, then we have, by 4.2.1,

$$\begin{aligned} F_n(\lambda) &= 0 \qquad \lambda < \mu_{n+1} \\ G_j(\lambda) &= \begin{cases} 0 & \lambda < \mu_j \\ I^*|M_j & \lambda \geq \mu_j \end{cases} \end{aligned} \qquad (3)$$

on identification of N_j^* with M_j .

The proof is complicated by the fact that, for an arbitrary spectral family F for T, we cannot assume a priori that $F(\lambda)P_j^* = P_j^*F(\lambda)$ $(j = 1,\dots,n,\ \lambda \in [\mu_1,\lambda_1])$, in which case the result would follow immediately. We shall show, however, that there is a dual spectral family F' for T, which agrees with F on $(-\infty,\mu_{n+1})$ and commutes with $P_1^*,\dots,P_n^*$. We do this by analysing the function $\omega_{x,\phi}$ introduced in [27], Lemma 3, which states that given any $x \in X$, $\phi \in X^*$, there exists a function $\omega_{x,\phi} \in L^\infty[\mu_1,\lambda_1]$, uniquely determined up to equivalence a.e., such that

$$\langle f(T^*)\phi,x\rangle = f(\lambda_1)\langle\phi,x\rangle - \int_{\mu_1}^{\lambda_1}\omega_{x,\phi}(\lambda)df(\lambda) \quad (f \in AC[\mu_1,\lambda_1]) \quad . \tag{4}$$

The equivalence class of $\omega_{x,\phi}$ depends linearly on both x and ϕ. For $x \in X$, we obtain from (1) a unique decomposition

$$x = x_0 + x_1 + \dots + x_n$$

where $x_0 \in X_n$, $x_j \in N_j$ $(j = 1,\dots,n)$. Analogously for $\phi \in X^*$ we obtain from (2) a unique decomposition

$$\phi = \phi_0 + \phi_1 + \dots + \phi_n$$

with $\phi_0 \in (X^*)_n$, $\phi_j \in M_j$ $(j = 1,\dots,n)$.

By 4.1.5 we can identify $(X_n)^*$ with $(X^*)_n$ and N_j^* with M_j, up to isomorphism. We shall write X_n^* for both $(X_n)^*$ and $(X^*)_n$, and $\langle\phi,x\rangle$ $(\phi \in X_n^*,\ x \in X_n)$ for the common evaluation. It is clear that

$$\langle T^*\phi,x\rangle = \langle T_n^*\phi,x\rangle \quad (\phi \in X_n^*,\ x \in X_n) \quad .$$

Therefore

$$\langle p(T^*)\phi,x\rangle = \langle p(T_n^*)\phi,x\rangle \quad (\phi \in X_n^*,\ x \in X_n) \quad ,$$

and so

$$\langle p(T_n{}^*)\phi,x\rangle \;=\; p(\lambda_1)\langle\phi,x\rangle - \int_{\mu_1}^{\lambda_1}\omega_{x,\phi}(\lambda)dp(\lambda)$$

for each polynomial p, from (4). Since

$$f \to \langle\phi,f(T_n)x\rangle - f(\lambda_1)\langle\phi,x\rangle + \int_{\mu_1}^{\lambda_1}\omega_{x,\phi}(\lambda)df(\lambda)$$

is a continuous linear functional on $AC[\mu_1,\lambda_1]$, which vanishes on polynomials, and the polynomials are dense in $AC[\mu_1,\lambda_1]$, it follows that

$$\langle f(T_n{}^*)\phi,x\rangle = f(\lambda_1)\langle\phi,x\rangle - \int_{\mu_1}^{\lambda_1}\omega_{x,\phi}(\lambda)df(\lambda) \quad (f \in AC[\mu_1,\lambda_1]) \;, \qquad (5)$$

for all $\phi \in X_n^*$, $x \in X_n$.

We require an expression like (5) to hold with $AC[\mu_1,\lambda_1]$ and $\int_{\mu_1}^{\lambda_1}$ replaced by $AC[\mu_{n+1},\lambda_1]$ and $\int_{\mu_{n+1}}^{\lambda_1}$, with the same $\omega_{x,\phi}$. To show this is possible, let $g \in AC[\mu_{n+1},\lambda_1]$ and let $f \in AC[\mu_1,\lambda_1]$ be equal to g on $[\mu_{n+1},\lambda_1]$ and identically equal to the constant $g(\mu_{n+1})$ on $[\mu_1,\mu_{n+1}]$. Then

$$\begin{aligned}\langle f(T_n{}^*)\phi,x\rangle &= f(\lambda_1)\langle\phi,x\rangle - \int_{\mu_{n+1}}^{\lambda_1}\omega_{x,\phi}(\lambda)df(\lambda)\\ &= g(\lambda_1)\langle\phi,x\rangle - \int_{\mu_{n+1}}^{\lambda_1}\omega_{x,\phi}(\lambda)dg(\lambda) \;. \qquad (6)\end{aligned}$$

In the notation of [27], Theorem 3, $f-g \in R_{\mu_{n+1}}$. Now, by 4.2.1, any dual spectral family for T_n is zero on $(-\infty,\mu_{n+1})$, so it follows from [27], Theorem 3(ii) that

$$X_n^* \;=\; \{\phi \in X_n^* : h(T_n{}^*)\phi = 0 \;\text{ for all }\; h \in R_{\mu_{n+1}}\} \;.$$

Therefore $(f-g)(T_n{}^*)\phi = 0$ for all $\phi \in X_n^*$, and so we can replace f by g on the left of (6) to obtain

$$<g(T_n{}^*)\phi,x> \ = \ g(\lambda_1)<\phi,x> - \int_{\mu_{n+1}}^{\lambda_1} \omega_{x,\phi}(\lambda)dg(\lambda)$$

$$(g \in AC[\mu_{n+1},\lambda_1] \ , \ \phi \in X_n^* \ , \ x \in X_n) \ .$$

Thus if $\omega_{x,\phi}^{(n)}$ is the function associated with $\phi \in X_n^*$, $x \in X_n$ by the well-bounded operator T_n, then we have $\omega_{x,\phi}(\lambda) = \omega_{x,\phi}^{(n)}(\lambda)$ a.e. in $[\mu_{n+1},\lambda_1]$. Since (5) gives

$$\begin{aligned} 0 \ &= \ <h(T_n{}^*)\phi,x> \ = \ <h(T^*)\phi,x> \\ &= \ h(\lambda_1)<\phi,x> - \int_{\mu_1}^{\lambda_1} \omega_{x,\phi}(\lambda)dh(\lambda) \\ &= \ - \int_{\mu_1}^{\mu_{n+1}} \omega_{x,\phi}(\lambda)dh(\lambda) \end{aligned}$$

for all $h \in R_{\mu_{n+1}}$, it follows that for $\phi \in X_n^*$, $x \in X_n$ we have $\omega_{x,\phi}(\lambda) = 0$ a.e. on $[\mu_1,\mu_{n+1}]$.

For $j = 1,\ldots,n$, if $x \in N_j$ and $\phi \in M_j$, then

$$<\phi,Tx> \ = \ <\phi,U_j x> \ = \ <T^*\phi,x> \ = \ <U_j{}^*\phi,x> \ = \ \mu_j<\phi,x>$$

and so clearly

$$\begin{aligned} <p(T^*)\phi,x> \ &= \ <p(U_j{}^*)\phi,x> \\ &= \ p(\mu_j)<\phi,x> \\ &= \ p(\lambda_1)<\phi,x> - \int_{\mu_j}^{\lambda_1} <\phi,x>dp(\lambda) \end{aligned}$$

for each polynomial p. It therefore follows that

$$\omega_{x,\phi}(\lambda) \ = \begin{cases} 0 & \text{a.e. on } [\mu_1,\mu_j] \\ <\phi,x> & \text{a.e. on } [\mu_j,\lambda_1] \end{cases}$$

In each of the cases $x \in N_j$, $\phi \in M_k$ $(j \neq k)$, $x \in X_n$, $\phi \in M_k$ $(k=1,\ldots,n)$ or $x \in N_j$ $(j=1,\ldots n)$, $\phi \in X_n^*$, it is easily verified that $\langle\phi,x\rangle = \langle\phi,Tx\rangle = 0$. Substitution in (4) gives

$$0 = \langle f(T^*)\phi,x\rangle = 0 - \int_{\mu_1}^{\lambda_1} \omega_{x,\phi}(\lambda)df(\lambda) \quad (f \in AC[\mu_1,\lambda_1]) ,$$

and so $\omega_{x,\phi}(\lambda) = 0$ a.e.

It follows from the bilinearity of $\omega_{x,\phi}$ and the foregoing calculations that

$$\omega_{x,\phi} = \sum_{j=0}^{n} \omega_{x_j,\phi_j}$$

$$= \begin{cases} \sum_{j=1}^{m} \langle\phi_j,x_j\rangle \quad \text{a.e. on } [\mu_m,\mu_{m+1}] \\ \sum_{j=1}^{n} \langle\phi_j,x_j\rangle + \omega_{x_0,\phi_0}^{(n)} \quad \text{on } [\mu_{n+1},\lambda_1] \end{cases}$$

for all $\phi \in X^*$, $x \in X$.

In particular $\lim_{h\to 0+} \int_{\lambda}^{\lambda+h} \omega_{x,\phi}(\mu)d\mu$ exists for all $\lambda \in [\mu_1,\mu_{n+1})$. Now, by the argument used in the proof of [27], Theorem 7, it follows that if F is any dual spectral family for T, then $\langle F(\lambda)\phi,x\rangle = \omega_{x,\phi}(\lambda)$ a.e., and so, by 1.3.4(v), $F(\lambda)$ is uniquely determined for $\lambda \in [\mu_1,\mu_{n+1})$. Now, since $P_jT = TP_j$ $(j=1,\ldots,n)$, it follows from [27], Theorem 6, that there is a dual spectral family F', necessarily coinciding with F on $[\mu_1,\mu_{n+1})$, such that $P_j^*F'(\lambda) = F'(\lambda)P_j^*$ $(j=1,2,\ldots,$ $\lambda \in [\mu_1,\lambda_1])$, and of course $T^*F'(\lambda) = F'(\lambda)T^*$ $(\lambda \in [\mu_1,\lambda_1])$.

Having established that F', T^*, and P_j^* all commute, we can now write

$$<P_j{}^*\phi, f(U_j)P_j x> = f(\lambda_1)<P_j{}^*\phi, P_j x> - \int_{\mu_1}^{\lambda_1} <P_j{}^* F'(\lambda) P_j{}^*\phi, P_j x> f'(\lambda) d\lambda$$
$$(j = 1, \ldots, n) \quad ,$$

with a similar expression holding with $I-P_1-\ldots-P_n$ and T_n in place of P_j and U_j . Then $F'P_j{}^* = P_j{}^*F'$ is the dual spectral family for U_j , and $F'(I^*-P_1{}^*-\ldots-P_n{}^*) = (I^*-P_1{}^*-\ldots-P_n{}^*)F'$ a dual spectral family for T_n . Since we know what these are on $[\mu_1, \mu_{n+1})$ (equations (3)), the values of $F = F'$ on $[\mu_1, \mu_{n+1})$ asserted by the theorem follow.

It is clear that a similar calculation can be done on $[\lambda_{n+1}, \lambda_1]$ which yields the values $I^*-Q_1{}^*-\ldots-Q_j{}^*$ on $[\lambda_{j+1}, \lambda_j)$.

Remark. Of course, the evaluation of $\omega_{x,\phi}$ on $[\mu_{n+1}, \lambda_1]$ is not actually needed to prove the theorem, but it provides a little more insight into what is going on.

THEOREM 4.2.3. If $T \in R(X)$ is well-bounded, then $T \in K(X)$. Furthermore,

$$Tx = \sum_{i\geq 1} \lambda_i Q_i x + \sum_{j\geq 1} \mu_j P_j x \quad (x \in X) \quad , \qquad (7)$$

each infinite sum in (7) converging in norm. (Either the positive or negative parts of (7) could be absent.)

Proof. Suppose, without loss of generality, that $\|T\| = 1$. For each positive $s \leq 1$, choose t such that $0 < t < s$ and that no eigenvalue of T lies in $(-s,-t]$ or in $[t,s)$. Let the function $\eta_{s,t} \in AC[-1,1]$ be defined by

$$\eta_{s,t}(\lambda) = \begin{cases} \lambda & -1 \leqslant \lambda \leqslant -s \\ \frac{s(\lambda+t)}{s-t} & -s \leqslant \lambda \leqslant -t \\ 0 & -t \leqslant \lambda \leqslant t \\ \frac{s(\lambda-t)}{s-t} & t \leqslant \lambda \leqslant s \\ \lambda & s \leqslant \lambda \leqslant 1 \end{cases} .$$

If $j(\lambda) \equiv \lambda$, then

$$j(\lambda) - \eta_{s,t}(\lambda) = \begin{cases} 0 & -1 \leqslant \lambda \leqslant -s \\ \frac{-t(\lambda+s)}{s-t} & -s \leqslant \lambda \leqslant -t \\ \lambda & -t \leqslant \lambda \leqslant t \\ \frac{t(s-\lambda)}{s-t} & t \leqslant \lambda \leqslant s \\ 0 & s \leqslant \lambda \leqslant 1 \end{cases} ,$$

and so $|||j-\eta_{s,t}|||_{[-1,1]} = 4t$. Therefore $\|j(T)-\eta_{s,t}(T)\| = \|T-\eta_{s,t}(T)\| \leqslant 4Kt < 4Ks \to 0$ as $s \to 0$. Once we have demonstrated that

$$\eta_{s,t}(T) = \sum_{\substack{i \\ \lambda_i \in [s,1]}} \lambda_i Q_i + \sum_{\substack{j \\ \mu_j \in [-1,-s]}} \mu_j P_j \tag{8}$$

it will follow that T is a norm limit of finite rank operators, and so is compact.

We know from 1.3.5 that for all $\phi \in X^*$, $x \in X$,

$$<\phi,\eta_{s,t}(T)x> = \eta_{s,t}(1)<\phi,x> - \int_{-1}^{1} <F(\lambda)\phi,x> d\eta_{s,t}(\lambda)$$

$$= <\phi,x> - \int_{-1}^{1} <F(\lambda)\phi,x> d\eta_{s,t}(\lambda) \quad . \tag{9}$$

$$\int_{-1}^{1}\langle F(\lambda)\phi,x\rangle\eta'_{s,t}(\lambda)d\lambda$$

$$= \left[\int_{-1}^{-s}+\int_{s}^{1}\right]\langle F(\lambda)\phi,x\rangle d\lambda + \left[\int_{-s}^{-t}+\int_{t}^{s}\right]\langle F(\lambda)\phi,x\rangle\frac{s}{s-t}d\lambda \ .$$

Let μ_1 , μ_2 , ... , μ_{m_s} be the eigenvalues of T in $[-1,-s]$, and λ_1 , λ_2 , ... , λ_{l_s} the eigenvalues in $[s,1]$. Then substitution from 4.2.2 gives the following equations

$$\begin{aligned}\int_{-1}^{-s}\langle F(\lambda)\phi,x\rangle d\lambda &= (\mu_2-\mu_1)\langle\phi,P_1x\rangle + (\mu_3-\mu_2)\langle\phi,(P_1+P_2)x\rangle + \dots\\ &\quad \dots + (\mu_{m_s}-\mu_{m_s-1})\langle\phi,(P_1+\dots+P_{m_s-1})x\rangle\\ &\quad + (-s-\mu_{m_s})\langle\phi,(P_1+\dots+P_{m_s})x\rangle\\ &= -(s+\mu_1)\langle\phi,P_1x\rangle - (s+\mu_2)\langle\phi,P_2x\rangle - \dots - (s+\mu_{m_s})\langle\phi,P_{m_s}x\rangle \ . \qquad (10)\end{aligned}$$

$$\begin{aligned}\int_{s}^{1}\langle F(\lambda)\phi,x\rangle d\lambda &= (1-\lambda_1)\langle\phi,x\rangle + (\lambda_1-\lambda_2)\langle\phi,(I-Q_1)x\rangle +\\ &\quad + (\lambda_2-\lambda_3)\langle\phi,(I-Q_1-Q_2)x\rangle + \dots\\ &\quad \dots + (\lambda_{l_s-1}-\lambda_{l_s})\langle\phi,(I-Q_1-\dots-Q_{l_s-1})x\rangle\\ &\quad + (\lambda_{l_s}-s)\langle\phi,(I-Q_1-\dots-Q_{l_s})x\rangle\\ &= (1-s)\langle\phi,x\rangle - (\lambda_1-s)\langle\phi,Q_1x\rangle - (\lambda_2-s)\langle\phi,Q_2x\rangle - \dots - (\lambda_{l_s}-s)\langle\phi,Q_{l_s}x\rangle \qquad (11)\end{aligned}$$

$$\begin{aligned}\int_{-s}^{-t}\langle F(\lambda)\phi,x\rangle\frac{s}{s-t}d\lambda &= \int_{-s}^{-t}\langle\phi,(P_1+\dots+P_{m_s})x\rangle\frac{s}{s-t}d\lambda\\ &= s\langle\phi,(P_1+\dots+P_{m_s})x\rangle \ . \qquad (12)\end{aligned}$$

$$\int_{t}^{s}\langle F(\lambda)\phi,x\rangle\frac{s}{s-t}d\lambda = \int_{t}^{s}\langle\phi,(I-Q_1-\dots-Q_{l_s})x\rangle\frac{s}{s-t}d\lambda$$

$$= s\langle\phi,x\rangle - s\langle\phi,(Q_1+\dots+Q_{l_s})x\rangle \quad . \tag{13}$$

Substituting (10)-(13) in (9) gives

$$\begin{aligned}\langle\phi,\eta_{s,t}(T)x\rangle &= \langle\phi,x\rangle + \sum_{j=1}^{m_s}(s+\mu_j)\langle\phi,P_jx\rangle - (1-s)\langle\phi,x\rangle \\ &\quad + \sum_{i=1}^{l_s}(\lambda_i-s)\langle\phi,Q_ix\rangle - s\sum_{j=1}^{m_s}\langle\phi,P_jx\rangle - s\langle\phi,x\rangle \\ &\quad + s\sum_{i=1}^{l_s}\langle\phi,Q_ix\rangle \\ &= \sum_{i=1}^{l_s}\lambda_i\langle\phi,Q_ix\rangle + \sum_{j=1}^{m_s}\mu_j\langle\phi,P_jx\rangle \quad ,\end{aligned}$$

for all $\phi \in X^*$, $x \in X$, and therefore (8) follows. This completes the proof of the theorem.

§4.3. Application: the singular multiplier

We first require a characterisation of compact multiplier operators on $L^p(\underline{T})$ $(1 \leq p < \infty)$. The following theorem is apparently well-known, and is hinted at in several places in the literature, but we have been unable to find a published proof.

THEOREM 4.3.1. Let $T \in B(L^p(\underline{T}))$ $(1 \leq p < \infty)$ be a multiplier operator. Then T is compact if and only if it is a norm limit of multiplier operators corresponding to sequences of finite support.

Proof. $(p > 1)$. Suppose T is compact. For $n = 1,2,\dots$, let U_n be the operator given by

$$(U_nf)^\wedge(m) = \begin{cases}\hat{f}(m) & |m| \leq n \\ 0 & |m| > n\end{cases} \quad .$$

Then $U_n T = TU_n$, and $\|U_n\| \leq K_p < \infty$ for all n. Given $\epsilon > 0$, for any $f \in L^p(\underline{T})$ there is a trigonometric polynomial q such that $\|f-q\| < \epsilon$. For sufficiently large n, $U_n q = q$, and so

$$\begin{aligned} \|U_n f - f\| &\leq \|U_n f - U_n q\| + \|U_n q - q\| + \|q-f\| \\ &< \epsilon\|U_n\| + 0 + \epsilon \\ &\leq (1+K_p)\epsilon \end{aligned}$$

for sufficiently large n. Therefore $U_n \to I$ strongly, and so $TU_n \to T$ strongly.

Let $\mathcal{B}$ be the unit ball of $L^p(\underline{T})$. Let $\epsilon > 0$. Then for each $g \in \mathcal{B}$ there exists $n(g)$ such that

$$\|U_n Tg - Tg\| < \epsilon \quad (n > n(g)) \quad .$$

Now $\{\{f \in L^p(\underline{T}) : \|Tg-f\| < \epsilon\} : g \in \mathcal{B}\}$ is an open cover of $\overline{T\mathcal{B}}$, so since T is compact there exist $g_1, \ldots, g_k$ such that

$$\overline{T\mathcal{B}} \subset \bigcup_{j=1}^{k} \{f \in L^p(\underline{T}) : \|Tg_j - f\| < \epsilon\} \quad .$$

Let $n_0 = \max_{j=1,\ldots,k} n(g_j)$. Let $f \in L^p(\underline{T})$ be any function with $\|f\| = 1$, so that $\|Tg_j - Tf\| < \epsilon$ for some j. Then $\|U_n Tg_j - Tg_j\| < \epsilon$ for all $n > n_0 \geq n(g_j)$, and so

$$\begin{aligned} \|U_n Tf - Tf\| &\leq \|U_n Tf - U_n Tg_j\| + \|U_n Tg_j - Tg_j\| + \|Tg_j - Tf\| \\ &< K_p\epsilon + \epsilon + \epsilon = (2+K_p)\epsilon \quad (n > n_0) \quad . \end{aligned}$$

Since this is true for all f such that $\|f\| = 1$, it follows that $U_n T \to T$ in norm.

The converse is obvious.

$(p = 1)$. $M_1(\underline{Z})$ is $[M(\underline{T})]^\wedge$, the algebra of Fourier-Stieltjes transforms of regular complex Borel measures on $\underline{T}$, and $\|T_{\hat{\mu}}\| = \|\mu\|$ $(\mu \in M(\underline{T}))$. The closure of the trigonometric polynomials in $M(\underline{T})$ is

$L^1(\underline{T})$. The result now follows from the theorem of [17].

If $\xi \in [L^1(\underline{T})]^\wedge$, say $\xi = \hat{f}$, then $T_\xi g = f * g \ (g \in L^p(\underline{T}))$, and $\|T_\xi\| \leqslant \|f\|_1$ ([15], Corollary 20.14). Since the trigonometric polynomials are dense in $L^1(\underline{T})$, it follows from this that we can replace multipliers of finite support (i.e. Fourier series of trigonometric polynomials) by $[L^1(\underline{T})]^\wedge$ in 4.3.1:

THEOREM 4.3.1'. Let $T \in B(L^p(\underline{T}))(1 \leqslant p < \infty)$ be a multiplier operator. Then T is compact if and only if it is a norm limit of multiplier operators of the form T_ξ , where $\xi \in [L^1(\underline{T})]^\wedge$.

The space of multipliers in $M_p(\underline{Z})$ giving compact multiplier operators will be denoted by $m_p(\underline{Z})$. (N.B. in the harmonic analysis literature, e.g. in [11], p.276, the characterisation 4.3.1' is usually taken as the definition of $m_p(\underline{Z})$.)

Since $M_p(\underline{Z}) \subset l^\infty(\underline{Z})$, $\|\phi\|_\infty \leqslant \|T\|_p \ (\phi \in M_p(\underline{Z}))$, and $[L^1(\underline{T})]^\wedge \subset c_0(\underline{Z})$, it is obvious that

$$m_p(\underline{Z}) \subset c_0(\underline{Z}) \cap M_p(\underline{Z}) \quad . \qquad (1)$$

The question arises as to whether the inclusion (1) is proper; in fact, this is so for $1 \leqslant p < 2$. When $p = 1$, there exist singular measures μ on $\underline{T}$ such that $\hat{\mu} \in c_0(\underline{Z})$, although $\hat{\mu} \notin m_1(\underline{Z}) = [L^1(\underline{T})]^\wedge$: we can take μ to be a Riesz product (see [34], §V.7). For $1 < p < 2$,the counterexample, constructed by Figà-Talamanca and Gaudry, is known as a singular multiplier. The construction of the singular multiplier $\phi_p \in M_p(\underline{Z})$ is as follows ([14], or [12], §9.3):

$$\phi_p(n) = \begin{cases} 0 & n \leqslant 0 \\ \hat{\rho}_{j-1}(n-2^{j-1})2^{-(j-1)/r} & n = 2^{j-1}, 2^{j-1}+1, \ldots, 2^j-1,\ j \geqslant 1 \end{cases} \qquad (2)$$

where in (2) $r = 2p/(2-p)$ and $\{\rho_j\}_{j=0}^{\infty}$ are the Rudin-Shapiro polynomials. These are defined inductively by

$$\rho_0 = \sigma_0 = 1$$

$$\rho_n(e^{it}) = \rho_{n-1}(e^{it}) + \exp(i2^{n-1}t)\sigma_{n-1}(e^{it})$$

$$\sigma_n(e^{it}) = \rho_{n-1}(e^{it}) - \exp(i2^{n-1}t)\sigma_{n-1}(e^{it}) \quad .$$

$\hat{\rho}_n$ is therefore supported on $\{0,1,2,\ldots,2^n-1\}$, where it takes only the values ± 1.

We use Theorem 4.2.3 to prove the (unsurprising) fact that $T_{\phi_p}^{(q)}$ is not well-bounded (unless $q = 2$).

THEOREM 4.3.2. For $1 < p < 2$, let $\phi_p \in M_p(\underline{Z})$ be as defined in (2). Then $T_{\phi_p}^{(q)}$ is not well-bounded for any q, $p \leqslant q < 2$.

Proof. The point of the example is, of course, that $\phi_p \notin m_p(\underline{Z})$, i.e. $T_{\phi_p}^{(p)}$ is not compact ([12], Theorem 9.3.5). However $\phi_p^2 \in m_q(\underline{Z})$ for all p,q, $1 < p,q < 2$. To see this, note that, since

$$\phi_p(n)^2 = \begin{cases} 0 & n \leqslant 0 \\ 2^{-2(j-1)/r} & n = 2^{j-1}, \ldots, 2^j-1,\ j \geqslant 1 \end{cases} ;$$

the sequence $\{\phi_p(n)^2\}_{n=1}^{\infty}$ is decreasing. Consequently, by the remarks before 3.2.6, $\phi_p^2 \in M_q(\underline{Z})$ $(1 < p,q < 2)$. If U_n is as in 4.3.1 and $(U_n T_{\phi_p^2} f)^\wedge(m) = \psi_p^{(n)}\hat{f}(m)$ $(f \in L^q(\underline{T})$, $m \in \underline{Z}$, $n = 1,2,\ldots)$ then $\psi_p^{(n)} \to \phi_p^2$ in the norm of bv, and therefore by [12], Theorem 6.3.5,

$\|U_n T_{\phi_p}^{(q)2} - T_{\phi_p}^{(q)2}\|_q \to 0$, so that, by 4.3.1, $T_{\phi_p}^{(q)2}$ is compact $(1 < p,q < 2)$.

Since $\phi_p^2 \in m_p(\underline{Z})$, $T_{\phi_p}^{(p)}$ is a Riesz operator, by 4.1.2(ii); but $T_{\phi_p}^{(p)}$ is not compact, so by 4.2.3 $T_{\phi_p}^{(p)}$ cannot be well-bounded.

It remains to show that $T_{\phi_p}^{(q)}$ is not well-bounded for $p < q < 2$. For suppose $T_{\phi_p}^{(q)}$ is well-bounded; then 3.2.4 enables us to identify its spectral family $E_p^{(q)}$: we have $E_p^{(q)}(\lambda) = T_{\chi_{(-\infty,\lambda]}\cdot\phi_p}^{(q)}$. Now the set of functions $\{\chi_{(-\infty,\lambda]}\circ\phi_p : \lambda \in \underline{R}\}$ consists of exactly the same functions as $\{\chi_{(-\infty,\lambda]}\circ\phi_q : \lambda \in \underline{R}\}$, since changing from ϕ_p to ϕ_q does not affect the fact that the weights $2^{-(j-1)/r}$ decrease, and the distribution of the signs remains the same. But, again by 3.2.4, if $\{E_p^{(q)}(\lambda) : \lambda \in \underline{R}\} = \{T_{\chi_{(-\infty,\lambda]}\circ\phi_p}^{(q)} : \lambda \in \underline{R}\} = \{T_{\chi_{(-\infty,\lambda]}\circ\phi_q}^{(q)} : \lambda \in \underline{R}\}$ is a bounded set in $B(L^q(\underline{T}))$, then $T_{\phi_q}^{(q)}$ is well-bounded, which is a contradiction. Therefore $T_{\phi_p}^{(q)}$ is not well-bounded $(p < q < 2)$.

REFERENCES

1. R. G. Bartle - "Spectral localization of operators in Banaoh spaces", Math. Ann. 153 (1964) 261-269.
2. E. Berkson - "Semigroups of scalar type operators and a theorem of Stone", Illinois J. Math. 10 (1966) 345-352.
3. ———— and H. R. Dowson - "On uniquely decomposable well-bounded operators", Proc. London Math. Soc. (3) 22 (1971) 339-358.
4. B. Brainerd and R. E. Edwards - "Linear operators which commute with translations, I", J. Australian Math. Soo. 6 (1966) 289-327.
5. P. L. Butzer and H. Berens - Semi-groups of operators and approximation, Springer, Berlin, 1967.
6. S. R. Caradus, W. E. Pfaffenberger and B. Yood - Calkin algebras and algebras of operators on Banach spaces, Dekker, New York, 1974.
7. I. Colojoara and C. Foias - Theory of generalized spectral operators, Gordon and Breach, New York, 1968.
8. H. R. Dowson and P. G. Spain - "An example in the theory of well-bounded operators", Proc. Amer. Math. Soc. 32 (1972) 205-208.
9. N. Dunford and J. T. Schwartz - Linear operators, Vol. I, Interscience, New York, 1958.
10. ———————— - ————————, Vol. III, ———, ————, 1971.
11. R. E. Edwards - Fourier series, Vol. II, Holt, Rinehart and Winston, New York, 1967.

12. ——————— and G. I. Gaudry - *Littlewood-Paley and multiplier theory*, Springer, Berlin, 1977.

13. C. Fefferman - "The multiplier problem for the ball", Ann. of Math. **94** (1971) 330-336.

14. A. Figà-Talamanca and G. I. Gaudry - "Multipliers of L^p which vanish at infinity", J. Functional analysis **7** (1971) 475-486.

15. E. Hewitt and K. A. Ross - *Abstract harmonic analysis*, Vol. I, Springer, Berlin, 1963.

16. E. Hille and R. S. Phillips - *Functional analysis and semi-groups*, Amer. Math. Soc. Colloq. Publ. Vol. 31, Providence, R.I., 1957.

17. J. W. Kitchen - "The almost periodic measures on a compact abelian group", Monatshefte für Math. **72** (1968) 217-219.

18. D. Kocan - "Spectral manifolds for a class of operators", Illinois J. Math. **10** (1966) 605-622.

19. ————— - "Strongly continuous spectral families", Annali di Mat. (4) **86** (1970) 31-42.

20. G. L. Krabbe - "Convolution operators that satisfy the spectral theorem", Math. Zeit. **70** (1959) 446-462.

21. ——————— - "Normal operators on the Banach space $L^p(-\infty,\infty)$, I - Bounded operators", Canadian J. Math. **13** (1961) 505-518.

22. ——————— - "———————————————————,

II - Unbounded operators", J. Math. Mech. **10** (1961) 111-133.

23. ——————— - "A Helly convergence theorem for Stieltjes integrals", Indag. Math. **27** (1965) 52-69.

24. R. Larsen - An introduction to the theory of multipliers, Springer, Berlin, 1971.

25. E. R. Lorch - "On a calculus of operators in reflexive vector spaces", Trans. Amer. Math. Soc. 45 (1939) 217-234.

26. J. R. Ringrose - "On well-bounded operators", J. Australian Math. Soc. 1 (1960) 344-356.

27. ——————— - "———————————, II", Proc. London Math. Soc. (3) 13 (1963) 613-638.

28. W. Rudin - Fourier analysis on groups, Interscience, New York, 1962.

29. W. H. Sills - "On absolutely continuous functions and the well-bounded operator", Pacific J. Math. 17 (1966) 349-366.

30. D. R. Smart - "Conditionally convergent spectral expansions", J. Australian Math. Soc. 1 (1960) 319-343.

31. A. R. Sourour - "Semigroups of scalar type operators on Banach spaces", Trans. Amer. Math. Soc. 200 (1974) 207-232.

32. P. G. Spain - "On well-bounded operators of type (B)", Proc. Edinburgh Math. Soc. (2) 18 (1972) 35-48.

33. A. E. Taylor - Introduction to functional analysis, Wiley, New York, 1958.

34. A. Zygmund - Trigononmetric series, Cambridge, 1959.