The Transmission of Water Waves under Surface Obstacles

by

Duncan Neill Robertson

A thesis submitted for the degree of Doctor of Philosophy to the

University of Edinburgh

1984

ABSTRACT

A sinusoidal wave train travelling on the surface of an ocean of great depth is considered to be incident on a class of partly submerged circular cylinders whose generators are parallel to the wave crests and whose cross-sections pass through two fixed points on the mean surface at angles α (measured through the fluid) which may be acute, right or obtuse. The ocean is assumed incompressible and of constant density and, in addition, viscous effects, surface tension and variations of atmospheric pressure are neglected. The linear theory of water waves is then employed to carry out a comparative study of three different methods of determining the transmissio coefficient (T) for the class of obstacles mentioned, namely,

(a) the method of multipole expansions (Ursell (1949))

(b) the method of matched asymptotic expansions (Leppington (1973))

(c) the null field method for water waves (Martin (1981)).

In the case $\alpha = 90^\circ$, (a) is used to obtain numerical values of T for $0.01 \leqslant N \leqslant 20$ (where $N = ka$, $\frac{2\pi}{k}$ = wavelength, a = cylindrical semi-beam) and, by means of (b), three terms are added to Leppington's (1973a) asymptoti formula for T, these terms being of orders $\frac{1}{N^5}$, $\frac{(\log N)^2}{N^6}$ and $\frac{\log N}{N^6}$. Comparison of the values of T obtained using (a) for $8 \leqslant N \leqslant 20$ and the complete fifth order asymptotics establishes the existence of a region of overlap. In the case where α is obtuse, similar comparison, using (c) and the first two terms of Alker's (1977) asymptotic result, produces positive evidence of the existence of a similar region (N is taken up to 10 in these cases). Numerical values of T (and R, the reflection coefficient) are found for $45^\circ \leqslant \alpha \leqslant 165^\circ$ and $0.01 \leqslant N \leqslant 10$. The extension of the asymptotics in the case $\alpha = 90^\circ$ reveals striking examples of the cohesion of the method of matched asymptotic expansions as propounded by Leppington for water wave scattering and radiation problems.

This thesis has been composed by myself and it has not been submitted in any previous application for a degree. The work reported here was executed by myself, unless otherwise stated.

ACKNOWLEDGEMENTS

I wish to record my grateful thanks to the following organisations and individuals:

the University of Edinburgh for the award of a Chalmers Research Scholarship;

Dr. J. Martin of the Mathematics Department for many illuminating mathematical discussions and unfailing help and encouragement in general;

the staff of the advisory service of Edinburgh Regional Computing Centre for their generous help with computing queries (the computations were carried out on an ICL 2900 at the Centre);

the typists, Mrs. G. Gregory (in particular) and Mrs. M. Clark of Napier College, Edinburgh, and Mrs. E. Millar of the Mathematics Department of Edinburgh University, for facing a daunting task with equanimity;

the staff at the Print Room of Napier College for their help with the reprints;

J. Christie of the Canongate Venture for the binding.

Finally, special thanks to my wife, Susannah, without whose patience and understanding this work could not have been completed.

CONTENTS

CHAPTER 1

(Introductory)

CHAPTER 2

(Multipole expansions applied to the semi-submerged circular cylinder)

CHAPTER 3

(General description of the method of matched asymptotic expansions as applied to scatterers whose surface piercing tangents are vertical)

CHAPTER 4

(The classical wave maker family of problems)

CHAPTER 1

§1.1. Introduction

The main part of this thesis is devoted to the problem of the scattering of sea-waves in two dimensions. It is postulated that a long-crested sinusoidal wave train is incident upon a fixed partly immersed cylinder of general cross-section (the generators of the cylinder being parallel to the wave crests) and it is required to find the ratios of the (complex) amplitudes of the transmitted and reflected waves to that of the incident wave. These ratios (called the transmission and reflection coefficients respectively) will be seen, in the case of steady state oscillations, to be functions of the cylindrical geometry under consideration and also of the wave number of the incident wave. The reason for their importance lies in the fact that the squares of their moduli are measures of the proportions of energy transmitted through and reflected from the cylindrical obstacle in the form of wave trains, and the investigation of the nature of their dependence on obstacle and incident wave has been the subject of a considerable number of papers in recent years. Particular attention will be given in this work to the delicate problem of calculating the small proportion of energy which is transmitted in the case when the wavelengths are small compared to a typical dimension of the obstacle.

In sections §1.3-§1.5 the three main methods for tackling such problems are discussed and reference is made to some advances achieved later in the thesis by the use of two of these in particular (the multipole expansion method and the method of matched asymptotic expansions). Surveys of the literature are given in these sections and the mathematical notation to be employed later is also introduced with reference to Figs. 1 and 2. In section §1.6 the layout of the thesis is summarised. Before these matters are examined in detail, a summary is given of the assumptions underlying the mathematical model of the situation and the boundary value problem to be discussed hereafter is set out.

§1.2. The Mathematical Model

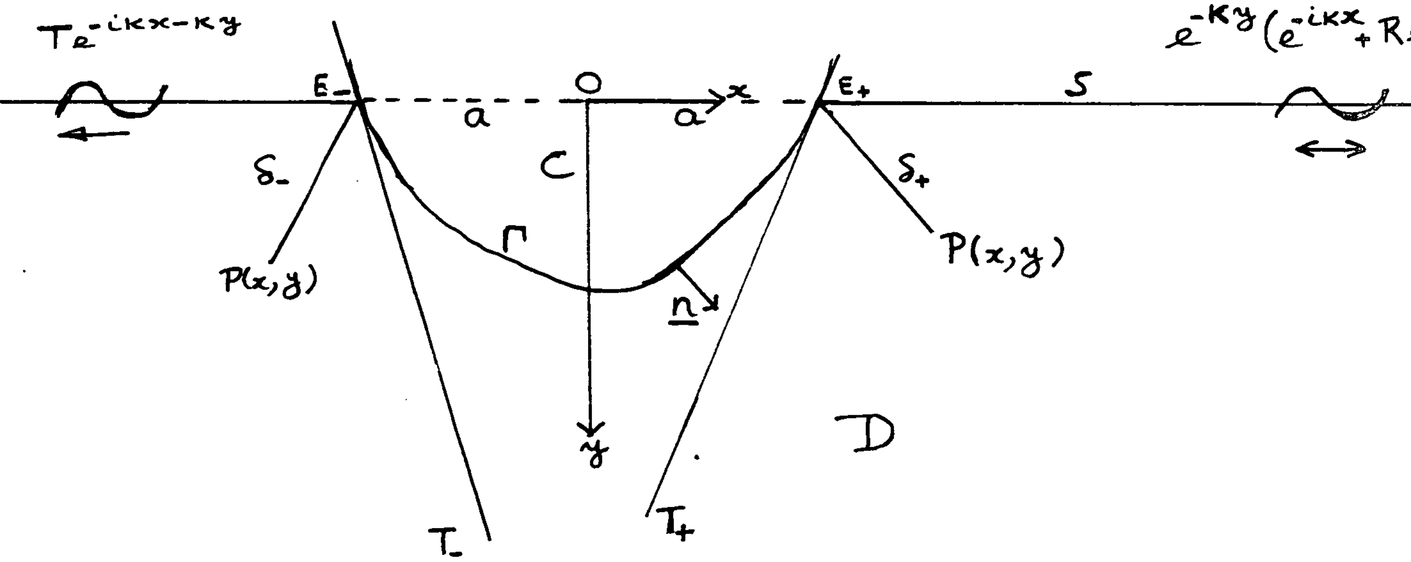
The usual assumptions of linearised water wave theory are employed and viscous effects, surface tension and variations of atmospheric pressure are neglected. In addition, the ocean is taken to be of infinite depth, incompressible and of constant density. If, therefore, the motion is assumed to have started from rest, then the previous assumptions imply that, throughout the subsequent motion, its original irrotational nature will be preserved. There will exist, in consequence, a scalar velocity potential $w(\underline{r},t)$, where t is the time measured from any suitable instant and $\underline{r}$ is the position vector of a point in the flow field relative to an origin fixed in space.

Attention will be confined to periodic states for which $w(\underline{r},t) = \mathrm{Re}[W(\underline{r})\, e^{-i\sigma t}]$, (where $\frac{2\pi}{\sigma}$ is the period of the wave motion and W is, in general, a complex function of position). To describe the problem in mathematical terms axes are set up as shown in Fig. 1 with Ox on the undisturbed water surface pointing towards the incoming wave, Oy vertically downwards and Oz chosen so that the system of axes is right handed. By appropriate choice of time origin and of scales of length and time, the potential of the incoming wave can then be taken as $\exp[-ky - i(kx + \sigma t)]$ (where $k = \frac{\sigma^2}{g}$ is the wave number and g is the acceleration due to gravity).

Thus the mathematical formulation of the problem is to find a

Fig. 1

Diagram and Axes for General Transmission Problem



C = profile of submerged part of cylinder

$E_\pm$ = points of intersection of cylinder cross-section with undisturbed water surface

$T_\pm$ = tangents at $E_\pm$

D = fluid domain

S = undisturbed water surface

Γ = boundary curve for C

a = cylindrical semi-beam

$\delta_\pm$ = distance of a general point in the fluid domain from $E_\pm$

$\underline{n}$ = unit vector in direction of outward normal at a general point of Γ

function $W(\underline{r}) = W(x,y)$ continuous and twice differentiable in the fluid domain satisfying the following spatial boundary value problem (see Fig. 1 for the notation):

$$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} = 0 \qquad \text{in } D,$$

$$kW + \frac{\partial W}{\partial y} = 0 \qquad \text{on } S,$$

$$\frac{\partial W}{\partial n} = 0 \qquad \text{on } \Gamma,$$

$$|\nabla W| \to 0 \quad \text{as } y \to +\infty,$$

As $x \to +\infty$, $W(x,y) \sim \exp(-ikx - ky) + R\exp(ikx - ky)$,

As $x \to -\infty$, $W(x,y) \sim T\exp(-ikx - ky)$

(where R and T are the (complex) reflection and transmission coefficients respectively).

It is usual, at this point, to introduce the scattered potential $\phi(x,y) = W(x,y) - \exp(-ikx - ky)$ so that the associated boundary value problem for ϕ is:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad \text{in } D, \tag{1.1}$$

$$k\phi + \frac{\partial \phi}{\partial y} = 0 \qquad \text{on } S, \tag{1.2}$$

$$\frac{\partial \phi}{\partial n} = -\frac{\partial}{\partial n}[\exp(-ikx - ky)] \quad \text{on } \Gamma, \tag{1.3}$$

$$|\nabla \phi| \to 0 \quad \text{as } y \to +\infty, \tag{1.4}$$

As $x \to +\infty$, $\phi(x,y) \sim R\exp(ikx - ky)$, (1.5)

As $x \to -\infty$, $\phi(x,y) \sim (T - 1)\exp(-ikx - ky)$. (1.6)

It is convenient also to express equation (1.3) in terms of the conjugate stream function $\psi(x,y)$ for then the equation can be integrated along Γ to give the value of ψ at any point of Γ in terms of its value, e.g. at E_+. Specifically, if arc length is measured positively from E_+ (as origin) towards E_-, then $\frac{\partial \phi}{\partial n} = \frac{\partial \psi}{\partial s}$

and the integrated form of (1.3) is

$$\psi(P) - \psi(E_+) = - \int_{E_+}^{P} \frac{\partial}{\partial n}[\exp(-ikx - ky)]\, ds \qquad (1.7)$$

(where P is any point on Γ and the line integral is taken along Γ between E_+ and P).

John (1950) has proved that such problems have unique solutions provided two other conditions are satisfied. These are

(a) the profile C should be completely contained within the two verticals at E_+ and E_- (it is probable that this condition can be relaxed but this will not be of concern here);

(b) certain edge conditions are satisfied at E_+ and E_-.

These conditions are usually stated in the form

$$\delta_+ \frac{\partial\phi}{\partial\delta_+} \to 0 \text{ as } \delta_+ \to 0 \qquad (1.8)$$

$$\delta_- \frac{\partial\phi}{\partial\delta_-} \to 0 \text{ as } \delta_- \to 0 \qquad (1.9)$$

(Stoker (1957), Chapter 5, interprets these conditions as stating, in mathematical terms, that waves do not break at the edges; certainly, if these conditions were not satisfied, there would be a net flux of liquid through arbitrarily small circular arcs centred at the edges indicating the existence of a source or sink there).

Equations (1.1)-(1.9) are of the same form as those associated with the forced harmonic motion of a cylinder of general cross-section (although, in this case, ϕ would represent the radiation potential and the coefficients of the wave terms in (1.5), (1.6) would be the complex amplitudes of the waves generated at $\pm\infty$). Indeed, for heaving, swaying and rolling modes the right hand side of (1.3) would be simply replaced by $v\underline{j}.\underline{n}$, $v\underline{i}.\underline{n}$ and $w(\underline{k}\times\underline{r}).\underline{n}$ respectively (where $\underline{i}$,

$\underline{j}$, $\underline{k}$ are unit vectors along Ox, Oy, Oz respectively and $Re(v\underline{j}e^{-i\sigma t})$, $Re(v\underline{i}e^{-i\sigma t})$ is the velocity in the heaving, swaying mode while $Re(w\underline{k}e^{-i\sigma t})$ is the angular velocity in the rolling mode. Newman (1975) has, in fact, derived equations relating the reflection and transmission coefficients of the scattering problem to the phases of the symmetric and anti-symmetric radiated waves and has shown that, in the cases of bodies symmetric about x = 0, R and T can be determined precisely from the radiation phase angles in heave and sway. This has similarities to the situation in the general two dimensional periodic problem of a freely floating body in the presence of an incident wave (which in linear theory can be treated as a superposition of a radiation and a scattering problem), where the Haskind relations (see e.g. Newman (1977)) enable the exciting forces in heave and sway and the roll exciting moment to be calculated provided the forced wave potential in the corresponding mode can be found for given forcing. Thus, in the general case, the solution of the diffraction problem can be avoided in the calculation of exciting forces and moments. In this work, however, the transmission problem will be tackled directly.

It will be convenient here to give also the complex form of the boundary value problem under consideration.

Let a complex variable

$$z = x + jy$$

and a complex potential

$$f(z) = \phi(x,y) + j\psi(x,y) \quad \text{be introduced}$$

(where j is a complex unit treated independently from i).

The problem then becomes that of finding a function f(z) analytic in D and such that

$$Im_j(f'(z) - jkf(z)) = 0 \text{ on } Im_j(z) = 0 \qquad (|z| \geqslant a),$$

$$\mathrm{Im}_j\,(f(z) - f(a)) = -\int_{E_+}^{P} \frac{\partial}{\partial n}[\exp(-ikx - ky)]\,ds \quad \text{on } \Gamma,$$

(where P is the point of affix z)

$|f(z)|$ is bounded for all z in D ,

As $x \to +\infty$, $\mathrm{Re}_j\,[f(z)] \sim R\exp(ikx - ky)$,

As $x \to -\infty$, $\mathrm{Re}_j\,[f(z)] \sim (T - 1)\exp(-ikx - ky)$.

In subsequent chapters there will be occasion to refer to the problem in both forms, but for the present the form (1.1)-(1.9) will be used in discussing the application of the methods of

(a) multipole expansions

(b) integral equations

(c) matched asymptotic expansions (for short waves)

to the solution of the problem.

§1.3. The method of multipole expansions

The method was originated by Ursell (1949) in solving the problem of the heaving motion of a semi-submerged circular cylinder and similar methods may be used for the transmission problem (see Chapter 2).

In the (more complicated) transmission problem the potential is represented as a superposition of

(a) a term representing a line source along the z-axis;

(b) a term representing a line dipole along the z-axis;

(c) an infinite series of terms representing wave-free line multipole potentials.

The individual terms of the solution are chosen specially to satisfy (1.1), (1.2) and (1.4) and the wave-free multipoles are, in addition, chosen so that they die off to zero at infinity, leaving the wave terms in (1.5) and (1.6) to be provided by the source and dipole terms. The coefficients in the infinite series i.e.

the strengths of the source, the dipole and the various multipoles are determined by satisfying (1.3). Once this is achieved it must be verified that the resulting infinite series is uniformly convergent and twice differentiable term by term in D. If this is so, then the unique solution of the problem has been established.

It should be noted that the infinite set of wave-free multipoles consists of two distinct subsets, one containing multipole potentials which are symmetric with respect to the plane $x = 0$ and the other containing anti-symmetric terms. In the heaving case, the radiation potential is represented by the superposition of a suitable source term together with the symmetric multipoles, while, in the swaying case, it consists of a dipole term plus a linear combination of anti-symmetric multipoles.

Martin (1971) has discussed the cases of the swaying circular cylinder and the rolling elliptic cylinder, pointing out that, in the latter case, a different combination of multipoles from that used in the circular case is required and that, in general, different combinations will be required for different cylindrical geometries. With regard to the convergence properties, Ursell (1949) has proved inverse cube convergence of the multipole expansion for the case of a heaving circular cylinder and Martin (using similar methods to Ursell) has shown that, in each of the three basic modes he considers, the rth term in the multipole expansion is smaller than $\frac{\lambda}{r^3}$ where λ is a function of k x a typical length in the profile and other non dimensional geometrical parameters, thus ensuring uniform convergence of the expansion in these cases. (A similar property is derived for the multipole expansions in the transmission problem in Appendix A). However, a proof that the series converges for all three modes of motion and an arbitrary shaped cylinder has not been given, though Vugts (1970) states that it is acceptable that this will be the case.

In conclusion, it should be stressed that, while convergence of the multipole series occurring in the transmission problem has been proved, considerable numerical difficulties are experienced in the calculation of the transmission coefficient as N increases. Nevertheless, it is seen in Chapter 2 that use of multipole expansions of up to 80 terms, coupled with numerical routines which maximise computer accuracy, produces three reliable significant figures in T for values of N well beyond the range of those previously examined. Meanwhile, consideration is now given to the second method of tackling the basic problem.

§1.4. The integral equation (I.E.) method

The numerical difficulties inherent in satisfying equation (1.3) for large N using multipole expansions and a desire to elucidate the physical processes involved in the scattering of short surface waves led to the development of the I.E. method by Ursell (1961). The ground work had been laid down by John (1950) and Ursell himself (1953). A brief description of the method is now given and the notation for Chapter 6 (on the null field equations) is introduced.

Suppose a Green's function $G(x,y;\ \xi,\eta)$ (also denoted by $G(P,Q)$) can be found for the domain D such that :

$$G(P,Q) = G(Q,P),$$

$$\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} = 0 \quad \text{in } D \qquad (x,y) \neq (\xi,\eta),$$

$$kG + \frac{\partial G}{\partial y} = 0 \quad \text{on } S,$$

$$\frac{\partial G}{\partial r} - ikG \to 0 \quad \text{as } r = \sqrt{x^2 + y^2} \to \infty,$$

$$G(x,y;\ \xi,\eta) = \tfrac{1}{2} \log\left[(x-\xi)^2 + (y-\eta)^2\right] + G_1(x,y;\ \xi,\eta),$$

where $G_1(x,y;\ \xi,\eta)$ is regular throughout D (see e.g. John (1950) p.100).

By applying Green's theorem to the functions ϕ and G in the region bounded by S,Γ, the semi-circle at infinity and a small circle centred on the point P(x,y) (assumed to lie in D but not on Γ), it can be proved that

$$2\pi\ \phi(P) = \int_{\Gamma} [G(P,q) \frac{\partial}{\partial n_q} \phi(q) - \phi(q) \frac{\partial}{\partial n_q} G(P,q)]\ ds_q ,$$

where q is a general point of Γ, $\frac{\partial}{\partial n_q}$ denotes differentiation along the normal to Γ from q into D and ds_q is the length of a line element at q.

By use of equation (1.3) the above equation can be rearranged as

$$2\pi\phi(P) + \int_{\Gamma} \phi(q) \frac{\partial}{\partial n_q}[G(P,q)]\, ds_q = -\int_{\Gamma} G(P,q) \frac{\partial}{\partial n_q}[\exp(-ikx_q - y_q)]\, ds_q \qquad (1.1$$

where (x_q, y_q) are the coordinates of q. It follows that the value of the potential at any point in the fluid domain which is not on Γ, will be known if the value of the potential on Γ can be found, or if a Green's function can be constructed satisfying also $\frac{\partial G}{\partial n} = 0$ on Γ. The latter approach leads to a problem which, in practice, is as difficult as the original one posed so it is necessary to examine the possiblity of determining the value of the potential on Γ.

In theory this can be achieved by applying Green's theorem as before with the field point P now occupying a position p on the curve Γ, the point p this time being surrounded by a small semi-circular arc centred on it. The result is a Fredholm integral equation of the second kind for the values of ϕ on Γ viz.

$$\pi\phi(p) + \int_{\Gamma} \phi(q) \frac{\partial}{\partial n_q}[G(P,q)]\, ds_q = -\int_{\Gamma} G(P,q) \frac{\partial}{\partial n_q}[\exp(-ikx_q - ky_q)]\, ds_q ,$$

and an attempt may be made to solve this by one of the standard methods for such equations (e.g. iteration) or by a numerical approach. However, although this is feasible for long wavelengths, problems again arise when N is large because the kernel of the I.E. $\left(\frac{\partial G}{\partial n}\right)$ contains the rapidly oscillatory term exp(-ikx - ky) when John's fundamental G.F. is employed. As a consequence iteration proves impractical while attempts at a numerical solution run into the same kind of difficulties as are experienced using multipole expansions. In addition, it is known that the above integral equation of the second kind is singular at a certain infinite discrete set of frequencies corresponding to the eigenvalues of a related interior problem (John 1950). This is purely a consequence of the method of solution since it is also known that the original boundary value problem has a unique solution for all frequencies provided the union of Γ and its image in the free surface is a convex, twice differentiable curve.

Use of the null field equations (to be discussed in Chapter 6) helps to remove the difficulties but these equations have only recently been developed for water wave problems (Martin, 1981) and, in any case, are not amenable to analytic solution.

Ursell's resolution of the dilemma lay in modifying John's Green's Function by subtracting from it a linear combination of source and dipole terms specially constructed to achieve cancellation of the rapidly oscillatory term. The resulting kernel is then small when N is large and iterative methods can be applied to give an asymptotic form for ϕ on the cylinder when N is large. Substitution of this form in equation (1.10) then enables the development of ϕ in the fluid domain to be found for large N and, in particular, by letting $x \to -\infty$ the leading term of the asymptotic form of the transmitted wave can be derived.
By this means, Ursell proves rigorously the result

$$T \sim \frac{2i}{\pi N^4} \exp(-2iN) \text{ as } N \to \infty$$

for a semi-submerged circular cylinder.

(N.B. Ursell (1961), p.655, suggests the result

$$T \sim -\frac{2i}{\pi N^4} \exp(-2iN)$$

but Leppington (1973a), p.141, points out that a sign error exists in formula (6.1), p.650, of Ursell's work).

Ursell also predicted that for any shape Γ intersecting the free water surface at right angles the value of T will be of the order of $\frac{1}{k^4}$ as $k \to \infty$ and this is borne out by Leppington (1973a), p.140, using the formal method of matched asymptotic expansions (see §1.5).

Holford (1964a, b) applies the I.E. method to the finite dock problem (on an infinite ocean) and derives rigorously a sequence of results giving the leading asymptotic forms of the virtual mass and damping coefficient for this geometry in heave and roll when $N \to \infty$.

Holford also mentions that the finite dock problem can be tackled by formulating the problem in terms of a new potential function θ where

$$\frac{\partial\theta}{\partial y} = k\phi + \frac{\partial\phi}{\partial y} \quad .$$

This method reduces the problem to solving an I.E. with a particularly simple kernel (see Holford, 1964a, pp.963-965, for a description) and has been employed by Sparenberg (1957) and MacCamy (1961) following Rubin (1954) who was its originator. However, Holford points out that the same problems arise for short wavelengths using this method as existed in the case of Ursell's original I.E. description.

Finally, in this section it may be remarked that an I.E. method was used by Ursell (1947) to solve the scattering problem for a fixed vertical barrier extending from a point above the mean free surface to a distance d below it and that John (1948) (using complex variables and a differential equation approach) has solved the equivalent problem for a barrier inclined at an angle $\frac{\pi}{2n}$ to the mean free surface when n is an integer. In the same work he also considers the case of a submerged infinite vertical barrier, a problem first discussed by Dean (1945).

§1.5. The Method of Matched Asymptotic Expansions

(a) The integral equation method described in the previous section has the advantage of providing a rigorous derivation of the leading term in the asymptotics of the amplitude of the radiated and scattered waves but is not by its nature suited to the derivation of higher order terms. The reason is that the "wave-makers" in the I.E. method are curved surfaces i.e. the wave coefficients are integrals along arcs of Γ (in Ursell's case the arc for the transmitted wave was $r = a, \frac{\pi}{2} \leqslant \theta \leqslant \pi$). Indeed, higher order terms involve (at least) double integrals along these arcs whose asymptotic evaluation for large N proves intractable. The advantage of the method of matched asymptotic expansions lies in a simplification of the geometry of the "wave makers" whereby they become of the "classical" type i.e. the integral coefficients of the wave terms are along a straight line from 0 to ∞. Higher order terms can now be dealt with since the asymptotics of the double integrals involved in the higher order wave terms can be found using the thorem in Appendix B. The other main advantage of the method of matched asymptotic expansions is that its general underlying philosophy is capable of application to a wide variety of different forms of Γ (including cases where T_+ and T_- are not normal to the free water surface). Before a general description of the method is given, a summary of results obtained by its use in problems involving the radiation and scattering of water waves is presented.

The main contribution has come from Leppington in a series of three papers (1972) and (1973a, b). In (1972) he tackles the finite dock problem for infinite and finite depth, extending, in the former case, the results derived rigorously by Holford (1964) for the amplitude of a radiated wave and verifying the efficiency of the method in deriving the first order reflection and transmission coefficient for the scattering problem; in addition, first order estimates of R and T are obtained in the case of scattering by a T-shaped dock. For finite depth, explicit results are also given for radiation and scattering by a finite dock.

(1973b) concerns itself with curved geometries where Γ is locally smooth and convex at the two intersection points with the fluid and the intersection is normal; again infinite and finite depth are considered. For infinite depth a first order form for T is obtained in the general case, and this is checked against Ursell's (1961) result for the semi-submerged circular cylinder. Other special cases considered are the semi-ellipse and circle with vertical keel. An extension of Ursell's result for the semi-circle is also suggested, viz.

$$T = \exp(-2ia/\varepsilon)(2i/\pi)[(\varepsilon/a)^4 - 4/\pi(\varepsilon/a)^5 \log(\varepsilon/a) + O(\varepsilon/a)^5] \quad (\varepsilon = 1/k)$$

(one of the main results of the thesis (in Chap. 5) is the derivation of the next term in the expansion for the purposes of comparing the asymptotic form of T with the values obtained using multipole expansions in Chap. 2).

Again, for finite and infinite depth (but for the radiation problem this time) a general first order result is obtained for the amplitudes of the waves radiated to $\pm\infty$ for the heaving case. Comparison is made with the results derived rigorously by Ursell (1953) (for infinite depth) and Rhodes-Robinson (1970a, b), (1972) (for finite depth). Agreement is obtained in each case.

In (1973b) attention is turned to three dimensional problems and explicit first order asymptotics worked out for the amplitudes of radiated waves in the cases of a heaving and rolling circular dock and a heaving hemisphere. In addition some conjectures are made concerning the relation of the reflection and transmission coefficients in three dimensional problems to the corresponding values in the case of two dimensional acoustic scattering by a cylinder with the same cross-section as is formed by the intersection of the three dimensional obstacle with the free surface.

Following Leppington, Alker (1975) has extended Ursell's rigorous result for the amplitude of the wave radiated to infinity by a heaving semi-circular cylinder to terms of order $\varepsilon^3 \log \varepsilon$ and ε^3 and Ayad and Leppington (1977) have discussed the case of plane vertical barriers (of finite depth and width). In addition, Alker (1977) has derived estimates for R and T in the case of scattering by a circular cylinder whose centre is not on the mean water surface.

(b) <u>Description of the method of matched asymptotic expansions as applied to scattering problems in two dimensions</u>

(It is assumed in this section for exactness that T_+ and T_- are normal to the undisturbed water surface.)

In the short wave limit consideration is given to the asymptotic form of the solution of the boundary value problem when $\varepsilon(=\frac{1}{k}) \to 0$. The problem may be kept within the bounds of linear theory by ensuring that the waves under consideration are always such that the ratios of their heights to their lengths are vanishingly small as the wavelength approaches zero, i.e. the wave slopes are adjusted appropriately as the wavelength is shortened.

The results obtained show that the transmission coefficient is a function of $\frac{\varepsilon}{a}$, where a is the semi-width of the scatterer so that their validity may be implied in the case of waves whose lengths are small in comparison with the dimensions of the scatterer in the direction of the wave motion. This physical interpretation of the mathematical limiting process $\varepsilon \to 0$ is adopted by Holford (1964).

It can be seen from (1.2) that if ε is set formally equal to zero, then the highest derivative term is lost, and the free surface condition becomes simply $\phi = 0$ so that the far field form of the potential cannot be achieved since such a problem does not permit the existence of surface waves. It is clear, therefore, that our problem in the short wave limit is of the singular perturbation type and that the asymptotic form of the solution as $\varepsilon \to 0$ cannot be represented uniformly throughout the whole fluid domain by a single asymptotic series of Poincaré form. In other words, it is not possible to find functions $a_r(\varepsilon)$, $\phi_r(x,y)$ and numbers $A(M)$ independent of x, y and ε such that

(i) $a_{r+1}(\varepsilon) = o(a_r(\varepsilon)) \quad$ as $\varepsilon \to 0$

(ii) for each integer $M \geqslant 0$

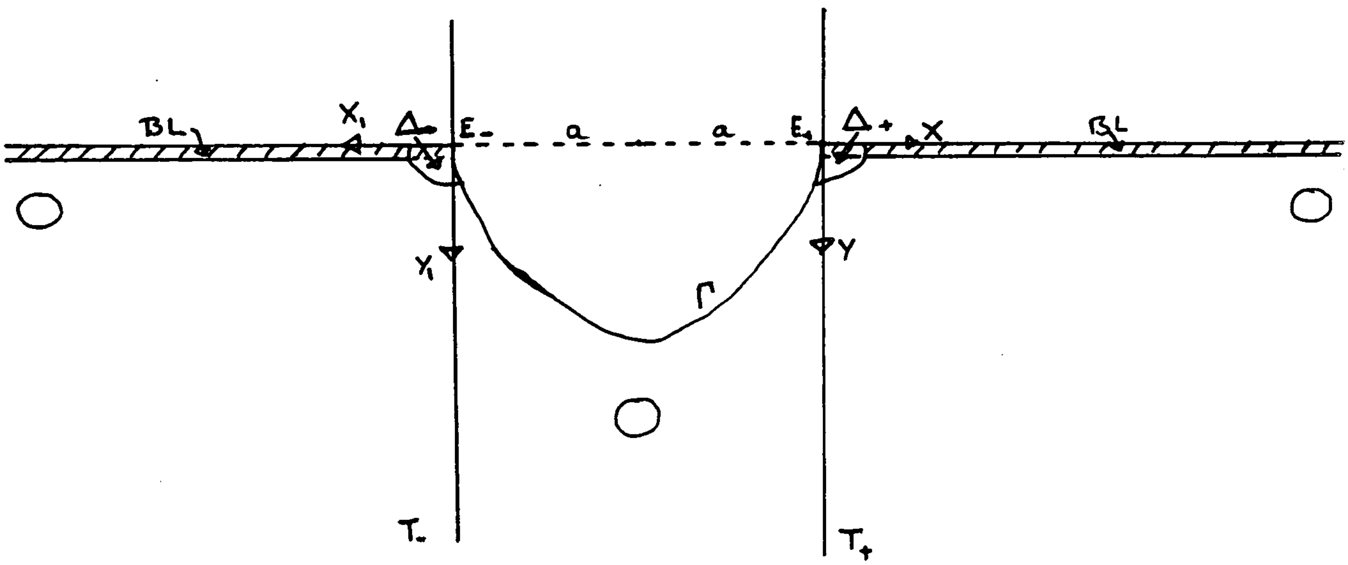
$$\left| \phi(x,y;\varepsilon) - \sum_{r=0}^{M} a_r(\varepsilon)\phi_r(x,y) \right| \leqslant A(M)\left|a_{M+1}(\varepsilon)\right|$$

for all (x,y) in the fluid domain under consideration.

In such cases the method is to find two (or more) asymptotic series which approximate to $\phi(x,y;\varepsilon)$ in different parts of D but complement each other in a sense which is contained within a

Fig. 2

Sub-Division of the Fluid Domain for Application of the Method of Matched Asynptotic Expansions



BL = boundary layer (width of order ε)

Δ_+ = right inner region $(\delta_+ \ll a)$

Δ_- = left inner region $(\delta_- \ll a)$

O = outer region $(\delta_\pm \gg \varepsilon)$

matching principle to be described later.

Intuitively, two regions can immediately be identified in which different forms of solution may be expected:

(i) a boundary layer with thickness of order ε in which wave effects are detectable, and

(ii) an outer region, many wavelengths from the free surface in which wave effects will be negligible.

Within these two main divisions, further subdivisions are necessary as shown in Fig. 2. These additional domains, $\Delta_\pm$, the right and left inner regions, consist of points which are very near $E_\pm$ on the a-scale, i.e. $\delta_\pm \ll a$. Hence, as measured on a length scale in these regions, the curvature of the cylinder $(\frac{1}{a})$ will be negligible and the effect of the cylinder will be indistinguishable from that of a vertical barrier along $T_\pm$. This simplification in the geometry of the problem enables the boundary condition on the curved surface in $\Delta_\pm$ to be replaced by an equivalent condition stated on $T_\pm$. Indeed, the potentials, in the perturbation series for the potential in $\Delta_\pm$, turn out to be solutions of the classical wave maker problem, each potential in the series being determined by the velocity distribution induced on $T_\pm$ by a potential appearing earlier in the series (or by the incoming wave in the case of the leading term of the scattered potential in Δ_+). This result has the important consequence that the reflection and transmission coefficients can be determined to a given order in ε from perturbation expansions in $\Delta_\pm$ which are of lower order in ε and is another of the factors enabling progress to be made beyond the limits of the integral equation method.

In the outer region which consists of points many wavelengths from the free surface ($y \gg \varepsilon$) the perturbation series for the potential is developed by first formally putting $\varepsilon = 0$ in the boundary value problem. This leads to a condition $\phi = 0$ at the surface so that the problem is a homogeneous one and it is necessary for uniqueness of solution to apply the matching principle (see §3.5) between O and Δ_+ (this turns out to be equivalent physically to specifying that the outer potential has a multipole singularity at E_+). Subsequent development of the outer perturbation series is obtained by matching with the solution in Δ_+ and substituting the series formally in the surface condition $\phi + \varepsilon\, \phi y = 0$ on $y = 0$ and equating terms of various orders in ε to zero. This leads to classical boundary value problems for the potential coefficients in the outer series. Once these have been solved, the development of the perturbation series on Δ_- can be determined leading to the far field wave form at $-\infty$ and hence to the asymptotic form of the transmission coefficient.

These general remarks will be expanded in Chapter 3 when the qualitative ideas given here will be expressed in more quantitative mathematical terms.

Finally, it must be emphasised that, although a wide variety of water wave problems have proved susceptible to successful solution using matched asymptotic expansions, the method is, nevertheless, a formal one. However, in many cases the formal expansions derived do turn out to be actual approximations in certain regions of the fluid domain.

§1.6. Layout of the thesis

Chapter 2 describes the calculation of numerical values of T for a semi-submerged circular cylinder (using multipole expansions) for values of N well beyond the range previously considered. Comparison with Ursell's and Leppington's asymptotic forms are inconclusive. Large relative differences can occur even when the absolute difference is within the order of the error term (see Tables 7, 8 and graphs 7, 8). This leads to use of the method of matched asymptotic expansions to derive the next term in the asymptotic expansion of T and Chapters 3 and 4 lay the groundwork for the achievement of this in Chapter 5. Chapter 3 contains a general mathematical description of the method as applied to scatterers which are perpendicular to the free surface and the necessity for a detailed examination of "classical wave maker" type problems is explained. This forms the subject matter of Chapter 4. In Chapter 6 the null field equations are used to provide numerical values of the transmission coefficient for a class of obstacles which intersect the free surface at an angle to the vertical. Comparison is made with Alker's (1977) asymptotic result for short waves. Chapter 7 contains the derivation of the first two sixth order terms in the asymptotics of the transmission coefficient for a semi-submerged circular cylinder while Chapter 8, finally, contains a summary of the work and conclusions of the thesis.

§2.1. Introduction

The problem of determining the transmission coefficient T for a half-immersed circular cylinder in regular beam seas can be solved using the multipole expansion method of Ursell (described in §1.3).

In outline the method adopted is to separate symmetric and anti-symmetric problems and solve each by expressing the stream function as a series containing a source/dipole term together with wave-free combinations of multipoles having appropriate symmetry. The resulting series is truncated and the boundary condition (1.7) on the cylinder is imposed at a finite number of appropriately chosen points.

By systematically increasing the number of points, a sequence of approximations to the complex transmission coefficient $\{T(M;ka)\}$ is obtained (where M is the number of points used). For each ka, the sequence is extended till the pattern of variations between successive terms is such as to allow the inference of the value of the limit of the sequence to a useful number of significant figures. Thereafter, comparison of this value with any value $T(M;ka)$ obtained at an earlier point in the sequence enables an indication of the accuracy at this point to be given (see §2.6).

It is found that multipole expansions of less than 10 terms give good accuracy for values of ka up to order unity but that, thereafter, such abbreviated series often fail to produce even one correct significant figure. Indeed, for values of ka in the range 6 to 20, it was found necessary (in general) to use multipole expansions of up to 80 terms (coupled with extreme computer accuracy) to be sure of just 3 significant figures for the limit. Comparison of the values obtained in this range with the asymptotic formulae of Ursell and Leppington indicates the need for completion of the fifth-order asymptotics.

§2.2. The multipole form of the solution to the transmission problem for a semi-submerged circular cylinder

The problem is considered in the form (1.1)-(1.9).

The coordinates are first re-scaled by setting

$$X = kx \text{ and } Y = ky$$

so that also $R = kr$ (where R and r are the radial distances from O in scaled and unscaled form). Similarly $R_+ = k\delta_+$ and $R_- = k\delta_-$ (in an obvious notation (see Fig. 1), while a new potential function $\phi_s(X,Y)$ may be defined by

$$\phi_s(X,Y) = \phi\left(\frac{X}{k}, \frac{Y}{k}\right) .$$

In terms of ϕ_s, the problem now takes the form

$$\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}\right)\phi_s(X,Y) = 0 \qquad (R \geqslant ka,\ Y \geqslant 0) \tag{2.1}$$

$$\frac{\partial}{\partial Y}\phi_s(X,0) + \phi_s(X,0) = 0 \qquad (|X| \geqslant ka) \tag{2.2}$$

$$\frac{\partial\phi_s}{\partial R} = -\frac{\partial}{\partial R}[\exp(-iX - Y)] \quad \text{on } R = ka \tag{2.3}$$

$$|\nabla\phi_s| \to 0 \text{ as } Y \to +\infty \qquad \text{(for all X)} \tag{2.4}$$

$$\text{As } X \to +\infty,\ \phi_s(X,Y) \sim R\exp(iX - Y) \tag{2.5}$$

$$\text{As } X \to -\infty,\ \phi_s(X,Y) \sim (T-1)\exp(-iX - Y) \tag{2.6}$$

$$R_+\frac{\partial\phi_s}{\partial R_+} \to 0 \text{ as } R_+ \to 0 \tag{2.7}$$

$$R_-\frac{\partial\phi_s}{\partial R_-} \to 0 \text{ as } R_- \to 0 . \tag{2.8}$$

It can be seen, therefore, that the problem is equivalent to determining the transmission coefficient for a wave of fixed length incident upon a semi-submerged circular cylinder of radius ka. The value of T will, therefore, depend only on the value of the non-dimensional parameter $N = ka$.

If a complex variable $z = X + jY$ is introduced, it can be easily verified (if $z \neq 0$) that in the first quadrant $X \geqslant 0$, $Y \geqslant 0$ the real parts of the complex-valued functions

$$\text{(a)}\quad e_m(z) = \frac{j}{(2m-1)z^{2m-1}} - \frac{1}{z^{2m}} \qquad (m \in Z^+)$$

$$\text{(b)}\quad s(z) = e^{jz}\int_z^\infty \frac{e^{-jt}}{t}\,dt + j\pi e^{jz}$$

$$\text{(c)}\quad w(z) = e^{jz}$$

are harmonic, bounded at infinity and satisfy the free surface condition (which in complex form is $\mathrm{Im}_j(D - j)f(z) = 0$ if $\mathrm{Im}_j(z) = 0$). In addition they have the property that

$$\mathrm{Re}_j\,[f'(z)] = 0 \text{ if } \mathrm{Re}_j(z) = 0\ .$$

They are indeed the fundamental solutions of the vertical barrier problem which are bounded at infinity (see §4.5). It follows that they can be extended into the second quadrant as functions whose real parts are even in X by means of the equation

$$f_{ext}(z) = \overline{f(-\bar{z})}\ .$$

This leads to a set of functions which in the whole half plane $Y \geqslant 0$ (except at $z = 0$) are harmonic, bounded at infinity and satisfy the free surface condition, viz.

$$e_m(z) = \frac{j}{(2m-1)z^{2m-1}} - \frac{1}{z^{2m}} \qquad (m \in Z^+) \tag{2.9}$$

$$s(z) = \begin{cases} e^{jz}\displaystyle\int_z^\infty \frac{e^{-jt}}{t}\,dt + j\pi e^{jz} & \mathrm{Re}(z) > 0 \\ e^{jz}\displaystyle\int_{-z}^\infty \frac{e^{jt}}{t}\,dt - j\pi e^{jz} & \mathrm{Re}(z) < 0 \end{cases} \tag{2.10}$$

$$w(z) = e^{jz}\ . \tag{2.11}$$

Similarly the real parts of the derivatives of these functions are odd in X and otherwise satisfy the same conditions as the functions themselves. Thus it is postulated that the solution of the problem can be expressed as the real part of a complex potential F(z) where

$$F(z) = \alpha s(z) + Aw(z) + \sum_{m=1}^{\infty} a_m e_m(z) + \beta s'(z) + Bw'(z) + \sum_{m=1}^{\infty} b_m e_m'(z) \tag{2.12}$$

and the coefficients in the above expansion are independent of j (though they may depend on i). Clearly this form of solution exhibits the potential as the sum of two parts, one even in X which will be denoted by $\mathrm{Re}_j(F_E(z))$ and the other odd in X denoted by $\mathrm{Re}_j(F_0(z))$. Thus

$$F_E(z) = \alpha s(z) + Aw(z) + \sum_{m=1}^{\infty} a_m e_m(z) \tag{2.13}$$

and $$F_0(z) = \beta s'(z) + Bw'(z) + \sum_{m=1}^{\infty} b_m e_m'(z) \tag{2.14}$$

(The form (2.12) may be derived rigorously by setting $\zeta = f'(z) - jf(z)$ and using Laurent's theorem after suitable continuations to derive the form of solution for ζ in complex form then integrating to find the form of f. The methods are similar to those used in Ursell (1950) for the case of a submerged circular cylinder, see also §6.4 after (6.11).)

With reference now to (2.12) and (2.9)-(2.11), it is easily seen that as $z \to +\infty$ in $X > 0$

$$F(z) \sim \alpha j\pi e^{jz} + Ae^{jz} + \beta(-\pi e^{jz}) + B(je^{jz})$$

so that

$$\mathrm{Re}_j\,[F(z)] \sim -\alpha\pi e^{-Y}\sin X + Ae^{-Y}\cos X - \beta\pi e^{-Y}\cos X - Be^{-Y}\sin X$$

ie $$\mathrm{Re}_j\,[F(z)] \sim e^{-Y}\,[\,(A - \beta\pi)\cos X - (B + \alpha\pi)\sin X\,] \quad . \tag{2.15}$$

The condition (2.5) requires that this must be of the form (constant).exp(iX - Y) so it is necessary that

$$B + \alpha\pi = -i(A - \beta\pi)$$

or $$B + iA = -\pi(\alpha - i\beta) \ .$$

Similarly it can be shown that as $z \to \infty$ in $X < 0$

$$\mathrm{Re}_j(F(z)) \sim e^{-Y}\,[\,(A + \beta\pi)\cos X + (\alpha\pi - B)\sin X\,] \tag{2.16}$$

whence application of condition (2.6) requires that

$$\alpha\pi - B = -i(A + \beta\pi)$$

or $$B - iA = \pi(\alpha + i\beta) \quad .$$

These two conditions give $B = i\pi\beta$ and $A = i\pi\alpha$, and substitution in (2.12) leads to a modified form of the complex potential satisfying the appropriate wave conditions at $\pm\infty$,

$$F(z) = \alpha[s(z) + i\pi w(z)] + \sum_{m=1}^{\infty} a_m e_m(z) + \beta[s'(z) + i\pi w'(z)]$$
$$+ \sum_{m=1}^{\infty} b_m e_m'(z) \quad . \tag{2.17}$$

In addition, reference to (2.15) and (2.16) shows that (in terms of α and β) the wave forms at $\pm\infty$ are respectively $\pi(i\alpha - \beta)\exp(iX - Y)$ and $\pi(i\alpha + \beta)\exp(iX - Y)$ whence the reflection and transmission coefficients are

$$R = \pi(i\alpha - \beta) \tag{2.18}$$

and $$T = 1 + \pi(i\alpha + \beta) \quad . \tag{2.19}$$

The numerical calculation of α and β depends on satisfying the boundary condition on the submerged half of the cylindrical surface and this is detailed in the next section.

§2.3. <u>The formulae for R and T</u>

The condition (2.3) is first expressed in terms of polar coordinates R and θ where the polar angle is measured from the Y-axis so that $X = R\sin\theta$, $Y = R\cos\theta$. This condition then takes the form

$$\frac{\partial}{\partial R}[\mathrm{Re}_j(F(jRe^{-j\theta}))] = -\frac{\partial}{\partial R}[\exp(-Re^{i\theta})] \text{ on } R = N$$
$$\text{for } -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2} \; .$$

Use of the Cauchy-Riemann equations in polar form (with θ replaced by $\frac{\pi}{2} - \theta$) gives the alternative form

$$-\frac{1}{N}\frac{\partial}{\partial\theta}[\mathrm{Im}_j(F(jNe^{-j\theta}))] = e^{i\theta}\exp(-Ne^{i\theta}),$$

and integration with respect to θ leads to the form of the stream function on the cylinder, viz.

$$\mathrm{Im}_j[F(jNe^{-j\theta})] = -\exp(-N\cos\theta)\sin(N\sin\theta) - i\exp(-N\cos\theta)\cos(N\ \mathrm{si}$$
$$+ c \tag{2.20}$$

where c is constant on $R = N$ and $-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$.

The right-hand side of this equation is clearly exhibited as the sum of two parts, one even and the other odd in θ. These symmetric and anti-symmetric parts are therefore equated with the corresponding parts on the left-hand side for $0 \leqslant \theta \leqslant \frac{\pi}{2}$.

Since $F_E(z)$ and $F_0(z)$ have real parts which are even and odd in X respectively (see (2.13), (2.14)), it follows (by use of the Cauchy-Riemann equations) that their imaginary parts will be odd and even in X. Hence, equating of the odd parts in (2.20) requires that

$$\mathrm{Im}_j\,[F_E(jNe^{-j\theta})] = -\exp(-N\cos\theta)\sin(N\sin\theta) \qquad (0 \leqslant \theta \leqslant \tfrac{\pi}{2}).$$

Reference to (2.17) and (2.9)-(2.11) together with the fact that α and the $\mathfrak{a}_m$ are independent of j shows that

$$\mathrm{Im}_j\,[F_E(jNe^{-j\theta})] = \alpha[\,\mathrm{Im}_j(s(jNe^{-j\theta})) + i\pi\exp(-N\cos\theta)\sin(N\sin\theta)]$$
$$+ \sum_{m=1}^{\infty} \alpha_m\left(\sin 2m\theta + \frac{N\sin(2m-1)\theta}{2m-1}\right)$$

where $\alpha_m = (-1)^{m+1}\mathfrak{a}_m/N^{2m}$.

Thus equating odd parts gives finally the condition

$$\exp(-N\cos\theta)\sin(N\sin\theta) = \alpha f_0(N;\theta) - \sum_{m=1}^{\infty} \alpha_m f_m(N;\theta) \tag{2.21}$$

for $0 \leqslant \theta \leqslant \frac{\pi}{2}$ where

$$f_0(N;\theta) = -\mathrm{Im}_j\,[s(jNe^{-j\theta})] - i\pi\exp(-N\cos\theta)\sin(N\sin\theta)$$

and $f_m(N;\theta) = \sin 2m\theta + \dfrac{N\sin(2m-1)\theta}{2m-1}$.

The equation (2.21) may be simplified somewhat using a uniqueness theorem concerning the $f_i(\theta)$ $(i \geqslant 0)$ which is implied by Ursell (1949). The theorem is used in the form

$$\sum_{i=0}^{\infty} A_i f_i(N;\theta) = 0 \qquad (0 \leqslant \theta \leqslant \tfrac{\pi}{2}) \Rightarrow A_i = 0.$$

This result is applied now as follows.

First (2.21) is divided by α (assumed non-zero) to obtain

$$\frac{1}{\alpha}\exp(-N\cos\theta)\sin(N\sin\theta) = f_0(N;\theta) - \sum_{m=1}^{\infty} \frac{\alpha_m}{\alpha} f_m(N;\theta) \tag{2.22}$$

Equating imaginary parts with respect to i gives the result

$$\text{Im}_i\left(\frac{1}{\alpha}\right) \exp(-N\cos\theta)\sin(N\sin\theta) = -\pi\exp(-N\cos\theta)\sin(N\sin\theta) - \sum_{m=1}^{\infty} \text{Im}_i\left(\frac{\alpha m}{\alpha}\right).f_m(N;\theta)$$

(see definition of $f_0(N;\theta)$ below (2.21))

or

$$[\,\text{Im}_i\left(\frac{1}{\alpha}\right) + \pi]\ \exp(-N\cos\theta)\sin(N\sin\theta) = -\sum_{m=1}^{\infty}\text{Im}_i\left(\frac{\alpha m}{\alpha}\right)f_m(N;\theta)\,. \quad (2.23)$$

The terms on the left-hand sides of (2.22) and (2.23) are now eliminated by multiplying (2.22) by $\text{Im}_i\left(\frac{1}{\alpha}\right) + \pi$, (2.23) by $\frac{1}{\alpha}$ and subtracting. This leads to the equation

$$[\,\text{Im}_i\left(\frac{1}{\alpha}\right) + \pi\,]\,f_0(N;\theta) + \sum_{m=1}^{\infty}\{\frac{1}{\alpha}\,\text{Im}_i\left(\frac{\alpha m}{\alpha}\right) - \frac{\alpha m}{\alpha}[\,\text{Im}_i\left(\frac{1}{\alpha}\right) + \pi]\,\}f_m(N;\theta) = 0$$

whence by the uniqueness theorem

$$\text{Im}_i\left(\frac{1}{\alpha}\right) + \pi = 0 \quad \text{and} \quad \frac{1}{\alpha}\,\text{Im}_i\left(\frac{\alpha m}{\alpha}\right) - \frac{\alpha m}{\alpha}[\,\text{Im}_i\left(\frac{1}{\alpha}\right) + \pi] = 0$$

$$\text{i.e.}\quad \text{Im}_i\left(\frac{1}{\alpha}\right) = -\pi \quad \text{and} \quad \text{Im}_i\left(\frac{\alpha m}{\alpha}\right) = 0 \quad (\text{since } \frac{1}{\alpha} \neq 0)\ . \qquad (2.24)$$

It now only remains to deal with the real part (with respect to i) of (2.22), i.e. the equation

$$\text{Re}_i\left(\frac{1}{\alpha}\right)\exp(-N\cos\theta)\sin(N\sin\theta) = \text{Re}_i\,[\,f_0(N;\theta)] - \sum_{m=1}^{\infty}\frac{\alpha m}{\alpha}\,f_m(N;\theta)$$

(where the fact that $\text{Im}_i\left(\frac{\alpha m}{\alpha}\right) = 0$ has been used).

This can be rearranged in the form

$$A_1\exp(-N\cos\theta)\sin(N\sin\theta) + \sum_{m=1}^{\infty} t_m f_m(N;\theta) = -\text{Im}_j\,[\,s(jNe^{-j\theta})] \qquad (2.25)$$

where $A_1 = \text{Re}_i\left(\frac{1}{\alpha}\right)$, $t_m = \frac{\alpha m}{\alpha}$ and reference is again made to the definition of $f_0(N;\theta)$.

A_1 is found from this equation by numerical methods to be detailed in the following section and, for the moment, attention is turned to the even parts in (2.20) whose correspondence requires that

$$\text{Im}_j\,[\,F_0(jNe^{-j\theta})] = -i\exp(-N\cos\theta)\cos(N\sin\theta) + c\ . \qquad (2.26)$$

From (2.14) (and the fact that $B = i\pi\beta$ proved before)

$$F_0(z) = \beta[\,s'(z) + ij\pi e^{jz}] + \sum_{m=1}^{\infty} b_m\left(-\frac{j}{z^{2m}} + \frac{2m}{z^{2m+1}}\right)$$

(where (2.9) and (2.11) have also been used).

Hence the value of F_0 on the cylinder is

$$F_0(jNe^{-j\theta}) = \beta[s'(jNe^{-j\theta}) + ij\pi\exp(-Ne^{-j\theta})] + \sum_{m=1}^{\infty}\beta_m j[e^{(2m+1)j\theta} + \frac{N}{2m}e^{2mj\theta}]$$

$$\text{where } \beta_m = \frac{b_m(-1)^{m+1}\,2m}{N^{2m+1}} \quad .$$

Imaginary parts (with respect to j) are now equated giving (after use of (2.26) and the fact that β and the b_m are independent of j)

$$i\exp(-N\cos\theta)\cos(N\sin\theta) = c + \beta[g_0(N;\theta)] - \sum_{m=1}^{\infty}\beta_m g_m(N;\theta) \qquad (2.27)$$

where

$$g_0(N;\theta) = -\mathrm{Im}_j[s'(jNe^{-j\theta})] - i\pi\exp(-N\cos\theta)\cos(N\sin\theta)$$

and $g_m(N;\theta) = \cos(2m+1)\theta + \frac{N}{2m}\cos 2m\theta$.

Equation (2.27) is treated in the same manner as was (2.21) i.e. it is divided by β, imaginary and real parts are taken with respect to i and an appropriate uniqueness theorem is used. From this process it follows that

$$\mathrm{Re}_i\left(\frac{1}{\beta}\right) = -\pi \ , \ \mathrm{Im}_i\left(\frac{c}{\beta}\right) = 0 \ \text{ and } \ \mathrm{Im}_i\left(\frac{\beta_m}{\beta}\right) = 0 \qquad (2.28)$$

while $\mathrm{Im}_i\left(\frac{1}{\beta}\right)$ is given by

$$B_1\exp(-N\cos\theta)\cos(N\sin\theta) + \sum_{m=1}^{\infty}u_m g_m(N;\theta) = \Gamma - \mathrm{Im}_j[s'(jNe^{-j\theta})] \qquad (2.29$$

where $B_1 = -\mathrm{Im}_i\left(\frac{1}{\beta}\right)$, $u_m = \frac{\beta_m}{\beta}$ (which is real by (2.28)) and $\Gamma = \frac{c}{\beta}$ (which is also real).

Again B_1 is determined from here numerically as detailed in the following section. Meanwhile the forms of R and T (in terms of A_1 and B_1) are now derived.

From (2.24) and the definition of A_1 it is easily seen that

$$\alpha = \frac{A_1 + i\pi}{A_1{}^2 + \pi^2}$$

while, similarly, from (2.28) and the definition of B_1

$$\beta = \frac{-\pi + iB_1}{\pi^2 + B_1{}^2} \quad .$$

Hence (by use of (2.18) and (2.19) it follows that

$$R = \left(\frac{\pi^2}{\pi^2 + B_1^2} - \frac{\pi^2}{\pi^2 + A_1^2}\right) + i\pi\left(\frac{A_1}{A_1^2 + \pi^2} - \frac{B_1}{B_1^2 + \pi^2}\right)$$

$$\text{and } T = 1 - \left(\frac{\pi^2}{\pi^2 + A_1^2} + \frac{\pi^2}{\pi^2 + B_1^2}\right) + i\pi\left(\frac{A_1}{A_1^2 + \pi^2} + \frac{B_1}{B_1^2 + \pi^2}\right) .$$

Finally, in terms of scaled parameters,

$$A_2 = \frac{A_1}{\pi} \quad \text{and } B_2 = \frac{B_1}{\pi} ,$$

the results are

$$\left.\begin{aligned} R &= \frac{A_2 - B_2}{(1 + A_2^2)(1 + B_2^2)}[(A_2 + B_2) + i(1 - A_2B_2)] \\ \text{and } T &= \frac{(1 + A_2B_2)}{(1 + A_2^2)(1 + B_2^2)}[(A_2B_2 - 1) + i(A_2 + B_2)] \end{aligned}\right\} . \qquad (2.30)$$

Clearly $i(1 + A_2B_2)R = (A_2 - B_2)T$ so that the phase of T differs from that of R by $\pm\frac{\pi}{2}$ (in agreement with Newman (1975), p.279) while the relationship $|R|^2 + |T|^2 = 1$ is also easily proved. Hence, if, at a given instant, a trough or crest occurs at a certain distance from the plane of symmetry for the transmitted wave, then at the same distance on the opposite side of this plane there will occur a point of zero displacement for the reflected wave; additionally the total energy in the transmitted and reflected wave trains is equal to the energy in the incident train as should be the case for non-viscous flows.

§2.4. Description of the numerical calculation of T and R

The notation

$$\psi_S(N;\theta) = -\mathrm{Im}_j\,[s(jNe^{-j\theta})] \quad \text{and}$$

$$\psi_D(N;\theta) = -\mathrm{Im}_j\,[s'(jNe^{-j\theta})]$$

is first introduced. θ is put equal to $\frac{\pi}{2}$ in equations (2.25) and (2.29) and the resulting equations are subtracted from the originals to give two modified equations:

$$A_1[\exp(-N\cos\theta)\sin(N\sin\theta) - \sin N] + \sum_{m=1}^{\infty} t_m[f_m(N;\theta) - f_m(N;\tfrac{\pi}{2})]$$
$$= \psi_s(N;\theta) - \psi_s(N;\tfrac{\pi}{2}) \tag{2.31}$$

and

$$B_1[\exp(-N\cos\theta)\cos(N\sin\theta) - \cos N] + \sum_{m=1}^{\infty} u_m[g_m(N;\theta) - g_m(N;\tfrac{\pi}{2})]$$
$$= \psi_D(N;\theta) - \psi_D(N;\tfrac{\pi}{2})\ . \tag{2.32}$$

The procedure now is to truncate each of the infinite series after $(M - 1)$ terms, then substitute M different values of θ between 0 and $\frac{\pi}{2}$ in each equation. The values chosen were

$$\theta = ((I - 1)/M)\ \frac{\pi}{2} \quad \text{with } 1 \leqslant I \leqslant M.$$

The resulting two sets of M simultaneous equations in M unknowns are solved for given N and various values of M to obtain the two basic sequences required consisting of values of A_1 and B_1, $\{A_1(M;N)\}$ and $\{B_1(M;N)\}$. From these, use of the formula (2.30) enables two further sequences $S_1(M;N) = \mathrm{Re}(T(M;N))$ and $S_2(M;N) = \mathrm{Im}(T(M;N)$ to be generated. Sequences for $|T(M;N)|$ and $\arg[T(M;N)]$ were also formed. The computer programme which performed the calculations gave print-outs of the terms of these 6 sequences to 10 decimal places for the following ranges of values of N and M:

$N = 0.01,\ (0.01)\ 0.09$	$5 \leqslant M \leqslant 30$
$N = 0.1\ (0.1)\ 0.9$	$5 \leqslant M \leqslant 30$
$N = 1\ (0.5)\ 5$	$5 \leqslant M \leqslant 50$
$N = 6\ (1)\ 20$	$5 \leqslant M \leqslant 80$.

The sequences obtained were observed to be monotonic (ultimately) and bounded and hence convergent. Indeed, provided a sufficient number of terms were taken, the differences between successive terms in the sequences were found ultimately to be monotonic and decreasing in magnitude as M increased. Careful observation of the magnitude and direction of these variations towards the latter part of the sequences enables values of the limits to be predicted to a meaningful number of significant figures and the results for the transmission coefficient are given in Tables 1-4 at the end of this Chapter (see also §2.6 for a fuller discussion).

The small values of T which occur as N increases necessitate keeping accuracy in intermediate calculations at a maximum. In this context, it should be noted that the equations (2.31) and (2.32) (which had to be solved numerically) contain terms which are readily evaluated to machine accuracy with the exception of ψ_s and ψ_D which involve quadratures. A discussion of how similar accuracy was achieved in the calculation of these functions appears in the next section. Once this had been attained, the linear equations were solved using the NAG routine F04ATF which produces a solution vector with a residual which is zero to machine accuracy. The routine contains 2 error messages, the first indicating that the matrix of coefficients is singular (possibly due to rounding errors) and the other that the matrix is too ill-conditioned to produce a correctly rounded solution. With double precision arithmetic no error messages were obtained in any of the cases considered.

§2.5. The forms of ψ_s and ψ_D used in the numerics

By definition

$$\psi_S(N;\theta) = -\mathrm{Im}_j\,[\,s(jNe^{-j\theta})] \tag{2.33}$$

and

$$\psi_D(N;\theta) = -\mathrm{Im}_j\,[\,s'(jNe^{-j\theta})] \tag{2.34}$$

where for Re(z) > 0

$$s(z) = e^{jz}\int_z^\infty \frac{e^{-jt}}{t}\,dt + j\pi e^{jz} \;. \tag{2.35}$$

Hence

$$-\mathrm{Im}_j[\,s(z)] = J_1(z) - \pi e^{-Y}\cos X \tag{2.36}$$

where the notation

$$J_1(z) = -\mathrm{Im}_j\,[e^{jz}\int_z^{\infty}\frac{e^{-jt}}{t}\,dt] \quad \text{is used.}$$

By rotation of the contour of integration through the fourth quadrant (so that the upper limit becomes $-j\infty$) and the substitution $u = jt$, it follows that

$$J_1(z) = -\mathrm{Im}_j\,[e^{jz}\,E_1(jz)]\,. \tag{2.37}$$

$$(E_1(w) = \int_w^{\infty}\frac{e^{-u}}{u}\,du)\,.$$

In order to separate $e^{jz}\,E_1(jz)$ into real and imaginary parts, it is necessary to replace z by $X + jY$ so that

$$e^{jz}\,E_1(jz) = e^{-Y}\,e^{jX}\int_{-Y+jX}^{\infty}\frac{e^{-u}}{u}\,du$$

whence, by rotation of the contour of integration so that the upper limit becomes $\infty + jX$ and the substitution $v = u + jX$, the form

$$e^{jz}\,E_1(jz) = e^{-Y}\int_{-Y}^{\infty}\frac{e^{-u}}{u + jX}\,du$$

is obtained. By splitting the range of integration into $[-Y,0]$ and $[0,\infty]$ and using (2.37), it can be seen that $J_1(z)$ may be expressed in the form

$$J_1(z) = J_{11}(z) + J_{12}(z) \tag{2.38}$$

where

$$J_{11}(z) = Xe^{-Y}\int_0^{\infty}\frac{e^{-u}}{u^2 + X^2}\,du$$

and

$$J_{12}(z) = Xe^{-Y}\int_{-Y}^{0}\frac{e^{-u}}{u^2 + X^2}\,du\,.$$

Tables of Laplace transforms (e.g. Bateman Manuscript Project (1954)) give immediately

$$J_{11}(z) = e^{-Y}\,[\,\mathrm{Ci}(X)\sin X - (\mathrm{Si}(X) - \tfrac{\pi}{2})\cos X] \tag{2.39}$$

where

$$\mathrm{Ci}(X) = \gamma + \ln X + \int_0^X\frac{\cos u - 1}{u}\,du \qquad (\text{for } X > 0)$$

and

$$\mathrm{Si}(X) = \int_0^X\frac{\sin u}{u}\,du\,.$$

The NAG routines S13ADF and S13ACF (whose accuracy is only limited by machine precision in the argument X) were used to evaluate Si(X) and Ci(X) respectively and hence to obtain $J_{11}(\mathbf{Z})$ for any X > 0.

In $J_{12}(\mathbf{Z})$, X and Y are replaced by N sin θ, N cos θ respectively and the variable rescaled by substituting u = -v N cos θ. This yields the form

$$J_{12}(jNe^{-j\theta}) = \sin\theta\cos\theta \int_0^1 \frac{\exp[N\cos\theta(u-1)]}{\cos^2\theta\, u^2 + \sin^2\theta}\, du \ . \qquad (2.40)$$

This form was presented to the NAG routine D01AJF for numerical integration, this being the recommended routine when considerations of time are not of over-riding importance. The important aspects of the routine are that

(a) it can deal with algebraic singularities in the integrand;

(b) it provides estimates of the accuracy actually achieved;

(c) it detects six different types of errors.

With the absolute error set at 10^{-15} and the relative error at 10^{-10} , no error messages were received. The results were checked by passing the same routine the alternative but equivalent form

$$J_{12}(jNe^{-j\theta}) = N\sin\theta \int_0^{N\cos\theta} \frac{\exp[(u - N\cos\theta)]}{u^2 + N^2\sin^2\theta}\, du$$

under the same conditions as above. Again, no errors were indicated and print-outs of the values showed that the integrals agreed to about 15 decimal places, this being near the limit of machine accuracy for double precision arithmetic.

Combination of equations (2.33), (2.36) and (2.38)-(2.40) gives the form of $\psi_s(N;\theta)$ used in the computations, namely

$$\psi_s(N;\theta) = \exp(-N\cos\theta)[\,\mathrm{Ci}(N\sin\theta)\sin(N\sin\theta) - (\mathrm{Si}(N\sin\theta) - \tfrac{\pi}{2})\cos(N\,s$$

$$+ \sin\theta\cos\theta \int_0^1 \frac{\exp[N\cos\theta(u-1)]}{\cos^2\theta u^2 + \sin^2\theta}\, du - \pi\exp(-N\cos\theta)\cos(N\,si$$

By use of the result

$$s'(z) = je^{jz} \int_z^{\infty} \frac{e^{-jt}}{t}\, dt - \frac{1}{z} - \pi e^{jz} \tag{2.41}$$

and an exactly similar development, it can be shown that

$$\psi_D(N;\theta) = -\frac{\cos\theta}{N} + \exp(-N\cos\theta)[\mathrm{Ci}(N\sin\theta)\cos(N\sin\theta)$$
$$+ (\mathrm{Si}(N\sin\theta) - \frac{\pi}{2})\sin(N\sin\theta)] + \cos^2\theta \int_0^1 \frac{u\exp[N\cos\theta(u-1)]}{\cos^2\theta u^2 + \sin^2\theta}\, du$$
$$+ \pi\exp(-N\cos\theta)\sin(N\sin\theta)\,. \tag{2.42}$$

<u>Note</u>: The above expressions are valid for $\theta > 0$. For $\theta = 0$, $\psi_s = 0$ (being an odd function) while ψ_D is found as follows.

By using the definitions of Ci and Si and integrating by parts it is seen from (2.42) that as $\theta \to 0+$,

$$\psi_D(N;\theta) = -\frac{1}{N} + \exp(-N)(\gamma + \log N + \log\sin\theta + o(1))$$
$$+ \{[\tfrac{1}{2}\log(\cos^2\theta u^2 + \sin^2\theta)\exp(N\cos\theta(u-1))]_0^1$$
$$- N\cos\theta\int_0^1 \tfrac{1}{2}\log(\cos^2\theta u^2 + \sin^2\theta)\exp[N\cos\theta(u-1)]\, du\}$$

i.e. $$\psi_D(N;\theta) = -\frac{1}{N} + \exp(-N)(\gamma + \log N + o(1))$$
$$- N\cos\theta\int_0^1 \tfrac{1}{2}\log(\cos^2\theta u^2 + \sin^2\theta)e^{N\cos\theta(u-1)}\, du$$
$$\to -\frac{1}{N} + \exp(-N)(\gamma + \log N) - N\int_0^1 \log u\, e^{N(u-1)}\, du \quad \text{as } \theta \to 0+\,.$$

Hence $$\psi_D(N;0) = -\frac{1}{N} + \exp(-N)(\gamma + \log N) - N\int_0^1 \log u\, e^{N(u-1)}\, du\,.$$

This expression is readily evaluated using again the NAG routine D01AJF.

§2.6. Discussion of the data obtained

(a) The sequences A_1 (M;N) and B_1 (M;N)

Up to about N = 7, the sequences are strictly monotonic decreasing as M increases for fixed N, the magnitudes of the differences between successive terms being themselves monotonic decreasing. For larger values of N, the sequences initially increase up to a certain value of M (increasing with N) and, thereafter, the decelerating monotonic decrease characteristic of the earlier cases sets in. (See Graphs 1-4 in §2.9).

At the long-wave end of the spectrum the values of A_1 and B_1 are both large in magnitude with $A_1 < 0$, $B_1 > 0$ and $|B_1| \gg |A_1|$.

At the short-wave end no particular pattern of this kind is observed but it may be noted that the product $A_1.B_1$ (for a wide range of values of M) is near to the value $-\pi^2$ (table 5 indicates the trend for M = 80). When it is recalled that $A_2 = {}^{A_1}/\pi$ and $B_2 = {}^{B_1}/\pi$ it follows that $A_2.B_2$ will be near -1 and reference to (2.30) shows that subtractive cancellation of significant figures will occur due to the presence of the factor $1 + A_2 B_2$ in the formula for T. It is this which necessitates the use of as great an accuracy as possible for large values of N.

(b) The sequences Re (T(M,N)) and Im (T(M,N)) ($M \geqslant 5$)

Up to about N = 3 these sequences are monotonic (sometimes increasing and sometimes decreasing) as M increases for fixed N. Again the magnitudes of the changes between successive terms decreases for fixed N as M increases and convergence is fairly rapid. The behaviour for larger values of N is similar to that of the sequences A_1 (M;N) and B_1 (M;N).

Thus (as typical)

$\text{Re}\ (T(5;0.5)) \triangleq 0.52023$ $\qquad$ $\text{Im}\ (T(10;2)) \triangleq -0.012519$

$\text{Re}\ (T(10;0.5)) \triangleq 0.52030$ $\qquad$ $\text{Im}\ (T(15;2)) \triangleq -0.012527$

$\text{Re}\ (T(15;0.5)) \triangleq 0.52031$ $\qquad$ $\text{Im}\ (T(20;2)) \triangleq -0.012529$.

Tables 1-4 give the values of the limits of the sequences as predicted from the multipole expansions for values of N up to 20 (together with the values of the modulus and principal values of the argument of the predicted values of T). The maximum number of terms used in the multipole expansions is also indicated.

The significant figures quoted were those obtained by examining the last two terms in the sequences and truncating their values 3 decimal places before the figure in which variations were still occurring. Thus, if the sequence Re (T(M;3)) is examined (where M was taken up to 50), it is found that

$$\text{Re}\ (T(49;3)) = -0.0009353058 \text{ and}$$

$$\text{Re}\ (T(50;3)) = -0.0009353129$$

so that variations are still occurring in the eighth decimal place. The values were accordingly truncated at the fifth decimal place giving the prediction -0.00094 for Re (T(3)) quoted in Table 3. The values of $|T(N)|$ and arg (T(N)) are in agreement with those given by Martin and Dixon (1983) who consider values of N up to 10 and use a different numerical scheme for their computations.

Two graphs of $|T(N)|$ against N are given in §2.9 (for N = 0.1 (0.1) 0.9 and N = 1(0.5) 5) to indicate the general behaviour. Examination of the values of arg (T(N) shows that the point representing T(N) in an Argand diagram spirals in towards the origin in a clockwise direction as N increases. A comparison of the results with Ursell's and Leppington's asymptotic forms for T(N) is given in §2.6 (d) below.

(c) Discussion of the accuracy of multipole expansions of less than ten terms

M = 6 is taken as typical and the values of T(6;N) are compared with the limiting values of the sequences as given in tables 1-4 for various values of N. The approximate absolute and relative errors are displayed in tabular form in Table 6.

It is seen that at the long-wave end of the spectrum these abbreviated multipole series work well and three significant figures of accuracy are maintained up to values of N of about 1. Thereafter, however, the loss of significant figures is rapid until when N is about 10 the order of magnitude of T(6;N) is in error. It may be noted that a 20 term multipole expansion maintains about 2 significant figures of accuracy up to N = 10 but subsequently an increasing number of terms must be employed to maintain accuracy in the short-wave range.

(d) Comparison of the data with the asymptotic formulae of Ursell and Leppington

Tables 7 and 8 compare the values of the real and imaginary parts of T as obtained from multipole expansions with the values given by Ursell's and Leppington's asymptotic formulae viz

$$T = \frac{2i}{\pi N^4} \quad \exp(-2iN) + O\left(\frac{\log N}{N^5}\right) \quad \text{(Ursell)}$$

and $$T = \frac{2i}{\pi N^4} \quad \exp(-2iN) \quad \left(1 + \frac{4}{\pi N}\log N\right) + O\left(\frac{1}{N^5}\right) \quad \text{(Leppington)}$$

for N = 8(1)15. The values of the error estimates are also given.

As can be seen, the differences between the computed and the asymptotic values are within the order of the error term in each case but this is far from conclusive evidence for a region of overlap since such an occurrence can take place when the computed and asymptotic values are of different orders of magnitude (see the case N = 11). Indeed in most cases there is no agreement of significant figures at all. (See also Graphs 7 and 8).

Clearly, to proceed it will be necessary to improve the asymptotic formula for T by completing the fifth order asymptotics. The next two chapters are concerned with laying the groundwork for the achievement of this in Chapter 5.

§2.7. The Convergence of the multipole form of the solution

When it is recalled that $e_m(z) = \frac{j}{(2m-1)z^{2m-1}} - \frac{1}{z^{2m}}$, the real parts (with respect to j) of the infinite series in (2.17) are seen to take the forms

$$\sum_{m=1}^{\infty} a_m \frac{(-1)^{m+1}}{R^{2m}} \left[\cos 2m\theta + \frac{R}{2m-1} \cos(2m-1)\theta\right]$$

and $$\sum_{m=1}^{\infty} b_m (-1)^m \frac{2m}{R^{2m+1}} \left[\sin(2m+1)\theta + \frac{R\sin 2m\theta}{2m}\right]$$

(since $z = jRe^{-j\theta}$).

Additionally the definition of α_m (above 2.21)) and of β_m (above (2.27)) show that $a_m = (-1)^{m+1} N^{2m} \alpha t_m$ (since $t_m = \frac{\alpha m}{\alpha}$)

$$b_m = \frac{(-1)^{m+1} N^{2m+1} \beta m}{2m} = \frac{(-1)^{m+1} N^{2m+1} \beta\, u_m}{2m} \quad (\text{since } u_m = \frac{\beta m}{\beta}).$$

Hence the series may be written in the forms

$$\alpha \sum_{m=1}^{\infty} t_m \left(\frac{N}{R}\right)^{2m} \left[\cos 2m\theta + \frac{R}{2m-1} \cos(2m-1)\theta\right] \quad \text{and}$$

$$-\beta \sum_{m=1}^{\infty} u_m \left(\frac{N}{R}\right)^{2m+1} \left[\sin(2m+1)\theta + \frac{R}{2m} \sin 2m\theta\right]$$

where the t_m and u_m (m=1,2,--) are to be found from (2.31) and (2.32).

In appendix 1 it is shown that t_m and u_m are $O\left(\frac{1}{m^3}\right)$ as $m \to \infty$ so that the two infinite series are uniformly convergent in $N \leqslant R < \infty$ for $-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$ and certainly twice differentiable term by term there. Hence, since the real parts of the individual terms of the multipole form (2.17) satisfy (2.1), (2.2), (2.4), (2.7) and (2.8) it follows also that these equations are satisfied by $\mathrm{Re}\,j(F(z))$. The conditions (2.5) and (2.6) are certainly true, so the real part of (2.17) does provide the unique solution to the problem.

It remains to show that the sequences of approximations to the coefficients A_l, B_l, t_m, u_m provided by the numerical scheme detailed in (2.4) do indeed converge (as $M \to \infty$) to the exact values. This has not been attempted in detail here, but Martin (1971) has proved the analogous results for the multipole expansions in the heaving and swaying problems for a semi-circular cylinder. The two equations in these cases are similar to (2.31) and (2.32) being of the forms $F(\theta) = \sum_{m=1}^{\infty} p_m f_m(\theta)$, $0 \leqslant \theta \leqslant \frac{\pi}{2}$ where F, f_m are even or odd according to which mode is being considered. By the same procedure as described in §2.4, a comparison problem is set up in the form

$$F(\theta_k) = \sum_{m=1}^{M} p_m^M f_m(\theta_k) \qquad (k = 1, 2, \dots, M)$$

and it is shown that the approximations p_m^M converge to the coefficients p_m in a strong metric.
Specifically $\sum_{r=1}^{\infty} |u_r^M p_r^M - r^2 p_r|^2 \to 0$ as $M \to \infty$ where u_r^M is a function which tends to r^2 as $M \to \infty$. Similar methods should be applicable to the transmission problem also.

§2.8. TABLES

TABLE 1

Values of Re (T(N)), Im (T(N)), |T(N)|, ARG (T(N)) for N = 0.01(0.01)0.09 from Multipole expansions of up to 30 terms

N	Re(T(N))	Im(T(N))	|T(N)|	ARG(T(N))
0.01	0.99962604	-0.0190202	0.99980698	-0.0190251
0.02	0.99857499	-0.0364655	0.9992406	-0.0365014
0.03	0.9969227	-0.052558	0.9983072	-0.0526719
0.04	0.994723	-0.06742	0.9970049	-0.067671
0.05	0.9920162	-0.081122	0.9953276	-0.081594
0.06	0.9888334	-0.093738	0.9932665	-0.094515
0.07	0.985198	-0.105318	0.9908111	-0.106496
0.08	0.981127	-0.115907	0.987950	-0.11759
0.09	0.976635	-0.125547	0.984672	-0.12785

TABLE 2

Vales of Re (T(N)), Im (T(N)), |T(N)|, ARG (T(N)) for N = 0.1(0.1)0.9 from multiple expansions of up to 30 terms

N	Re(T(N))	Im(T(N))	|T(N)|	ARG(T(N))
0.1	0.971730	-0.13428	0.980964	-0.13731
0.2	0.90057	-0.1798	0.91834	-0.19701
0.3	0.79102	-0.1767	0.81051	-0.21977
0.4	0.65648	-0.1614	0.67602	-0.2410
0.5	0.5203	-0.1520	0.54206	-0.2842
0.6	0.3995	-0.1489	0.42631	-0.3569
0.7	0.2995	-0.1472	0.33369	-0.4568
0.8	0.219	-0.1433	0.2621	-0.579
0.9	0.156	-0.1363	0.2073	-0.717

TABLE 3

Values of Re (T(N)), Im (T(N)), |T(N)|, ARG (T(N)) for N = 1(0.5)5 from multiple expansions of up to 50 terms

N	Re(T(N))	Im(T(N))	|T(N)|	ARG(T(N))
1.0	0.10700	-0.12624	0.16548	-0.8677
1.5	-0.00920	-0.06023	0.06093	-1.7224
2.0	-0.0235	-0.01253	0.02663	-2.6516
2.5	-0.01175	0.00593	0.0132	2.6739
3.0	-0.00094	0.00707	0.00713	1.7023
3.5	0.00312	0.00275	0.00416	0.7228
4.0	0.00248	-0.00066	0.00257	-0.2617
4.5	0.000524	-0.00158	0.00166	-1.250
5.0	-0.00069	-0.00088	0.00112	-2.240

TABLE 4

Values of Re(T(N)), Im(T(N)), |T(N)|, ARG(T(N)) for N = 6(1) 20 from multipole expansions of up to 80 terms

N	Re(T(N))	Im(T(N))	|T(N)|	ARG(T(N))
6	-0.000261	0.000492	0.000556	2.059
7	0.000305	0.0000211	0.000305	0.069
8	-0.0000622	-0.000170	0.000181	-1.923
9	-0.0000809	0.0000792	0.000113	2.367
10	0.0000693	0.0000270	0.0000743	0.372
11	-0.0000027	-0.0000507	0.0000508	-1.62
12	-0.0000318	0.0000165	0.0000358	2.66
13	0.0000204	0.0000160	0.0000259	0.666
14	0.00000456	-0.0000187	0.0000192	-1.331
15	-0.0000143	0.00000271	0.0000145	2.954
16	0.00000646	0.00000915	0.0000112	0.957
17	0.00000442	-0.00000757	0.00000876	-1.042
18	-0.00000692	-0.00000070	0.00000695	-3.041
19	0.00000179	0.00000529	0.00000558	1.244
20	0.00000330	-0.00000310	0.00000453	-0.755

TABLE 5

Values of $A_1(80;N).B_1(80;N)$ (to 4D) for N = 6(1) 20, showing the trend towards the value $-\pi^2$ ($\hat{=}$ - 9.8696)

N	$A_1(80;N).B_1(80;N)$
6	-9.8462
7	-9.8756
8	-9.8593
9	-9.8665
10	-9.8711
11	-9.8506
12	-9.8688
13	-9.8703
14	-9.8712
15	-9.8693
16	-9.8700
17	-9.8699
18	-9.8694
19	-9.8699
20	-9.8697

TABLE 6

Approximate absolute and relative errors in the values of T(6;N) for various values of N

	Re(T(6;N))		Im(T(6;N))	
N	|Absolute error|	|Relative error|	|Absolute error|	|Relative error|
0.05	3×10^{-7}	3×10^{-7}	2×10^{-6}	2×10^{-5}
0.09	1×10^{-6}	1×10^{-6}	5×10^{-6}	4×10^{-5}
0.60	5×10^{-5}	1×10^{-4}	2×10^{-4}	1×10^{-3}
1.0	1×10^{-4}	1×10^{-3}	2×10^{-4}	1.6×10^{-3}
3.0	7×10^{-5}	7×10^{-2}	5×10^{-5}	7×10^{-3}
4.5	3×10^{-5}	6×10^{-2}	2×10^{-5}	1×10^{-2}
7.0	5×10^{-5}	2×10^{-1}	4×10^{-6}	2×10^{-1}
10.0	2×10^{-4}	3	1×10^{-4}	4

TABLES 7 and 8

Comparison of values of the real and imaginary parts of T(N) as obtained from multiple expansions (A), Ursell's asymptotics (B), Leppington's asymptotics (C), with error estimates $\frac{\log N}{N5}$ (D) and $\frac{1}{N5}$ (E) and scale factor 10^6.

RE(T(N))

N	A	B	C	D	E
8	-62.2	-44.7	-59.6	63.5	30.5
9	-80.9	-72.9	-95.5	37.2	16.9
10	69.3	58.1	75.1	23.0	10.0
11	- 2.7	- 0.4	- 0.5	14.9	6.2
12	-31.8	-27.8	-35.1	10.0	4.0
13	20.4	17.0	21.3	6.9	2.7
14	4.56	4.49	5.57	4.9	1.9
15	-14.3	-12.4	-15.3	3.6	1.3

IM(T(N))

N	A	B	C
8	-170	-149	-198
9	79.2	64.1	84.0
10	27.0	26.0	33.6
11	- 50.7	- 43.4	- 55.5
12	16.5	13.0	16.5
13	16.0	14.4	18.0
14	- 18.7	- 16.0	- 19.8
15	2.7	1.9	2.4

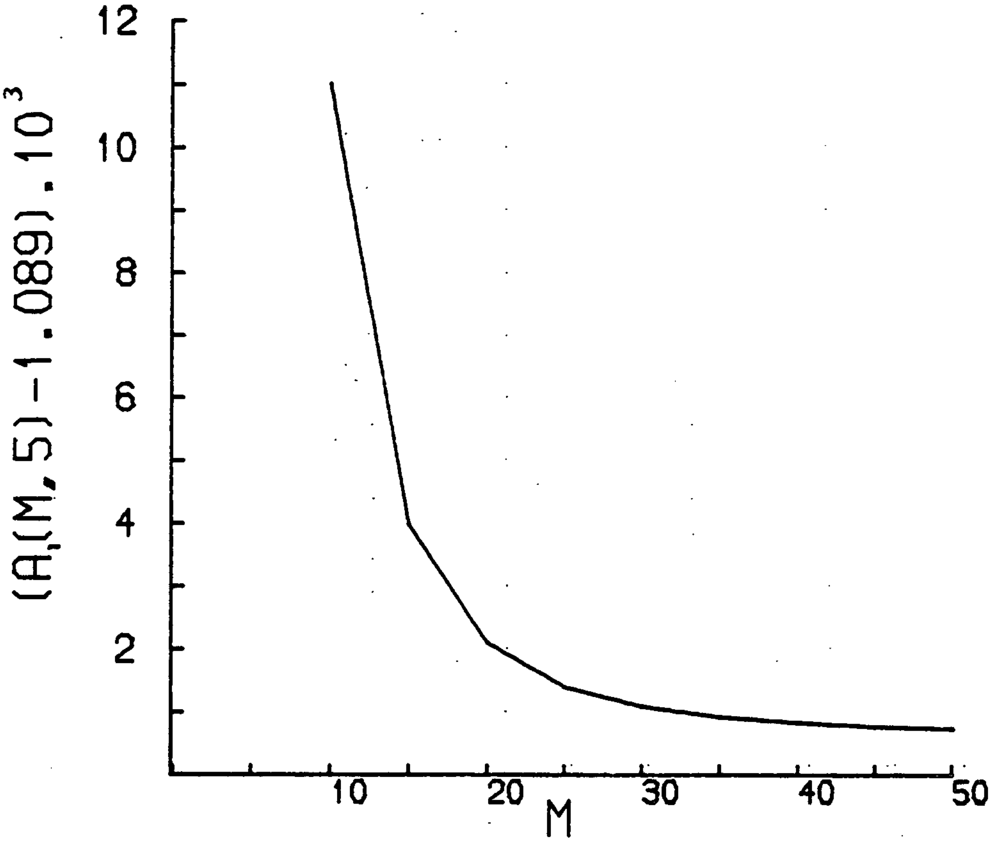
§2.9. Graphs (overleaf)

(a) Graphs 1-4 illustrate typical behaviour of the sequences $\{A_1(M;N); M = 1, 2, ...\}$ for various values of N, showing their ultimate monotonic nature. This is typical also of the other sequences computed viz. $\{B_1(M;N)\}$, $\{Re(T(M;N)\}$, $\{Im(T(M;N))\}$, $\{|T(M;N)|\}$ and $\{arg(T(M;N)\}$.

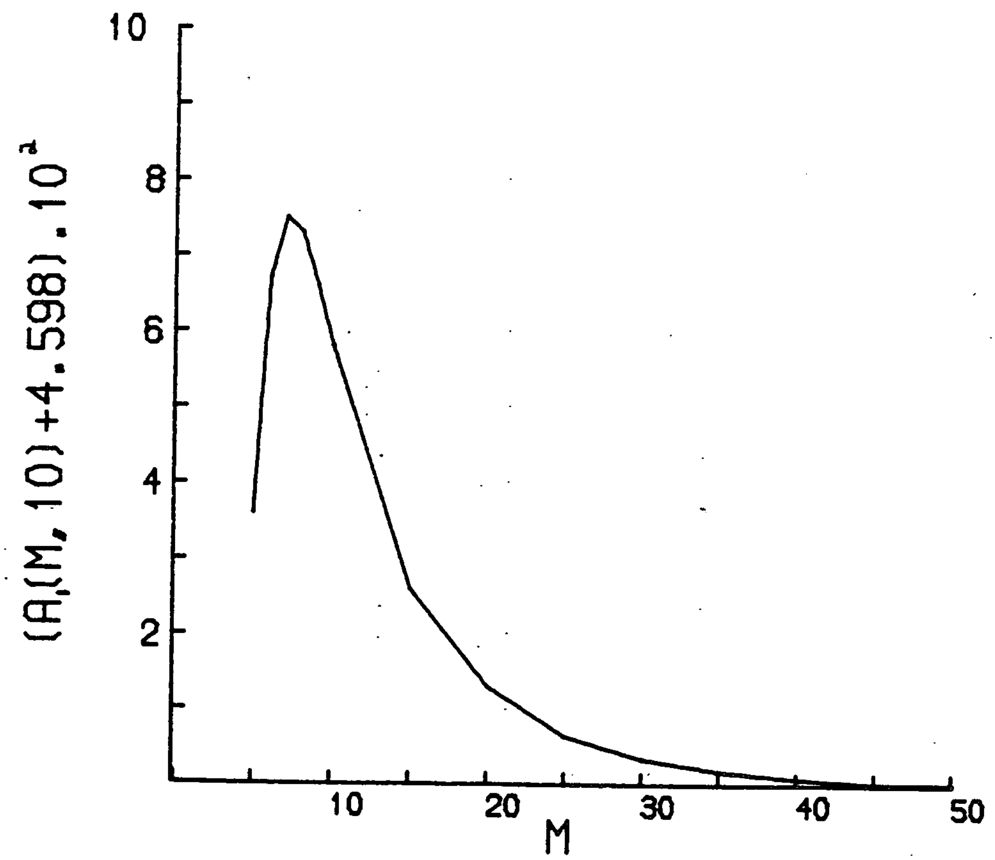
(b) Graphs 5, 6 illustrate the behaviour of $|T(N)|$ for N = 0.1 (0.1) 0.9 and 1(0.5)5 respectively.

(c) Graphs 7,8 compare the multipole values of the real and imaginary parts of T(N) with those obtained using Ursell's and Leppington's asymptotic forms. The values are normalised by Ursell's real and imaginary parts (denoted by Re(U(N)) and Im(U(N))). Hence Ursell's values are represented by the horizontal line through 1 on the vertical axis.

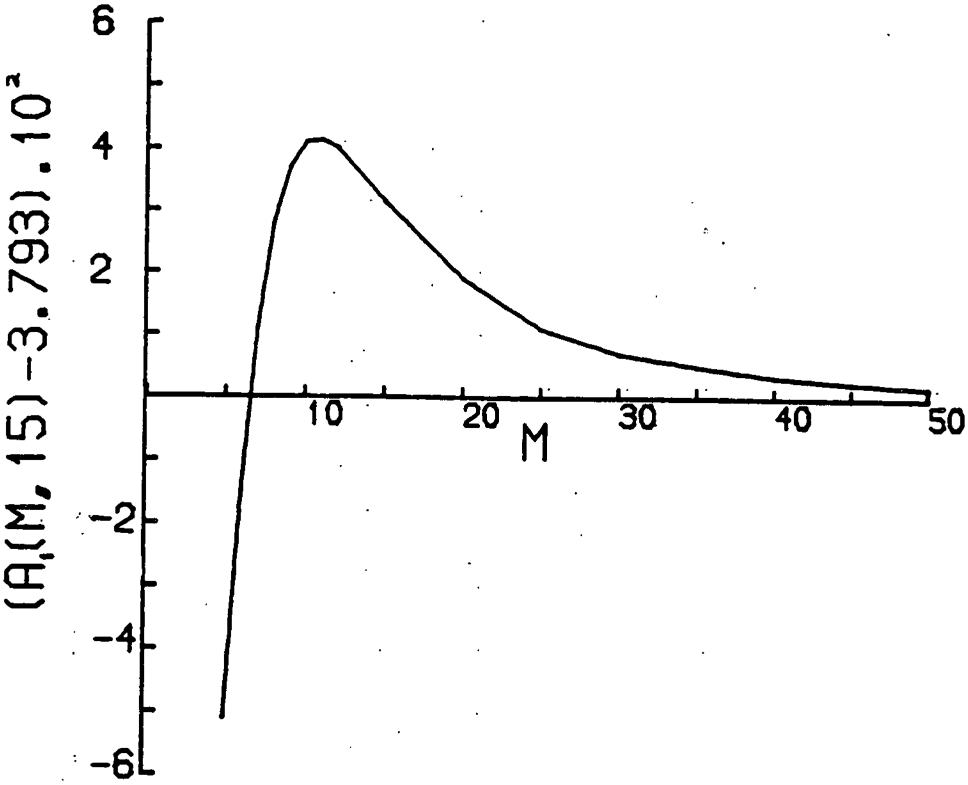
GRAPH 1



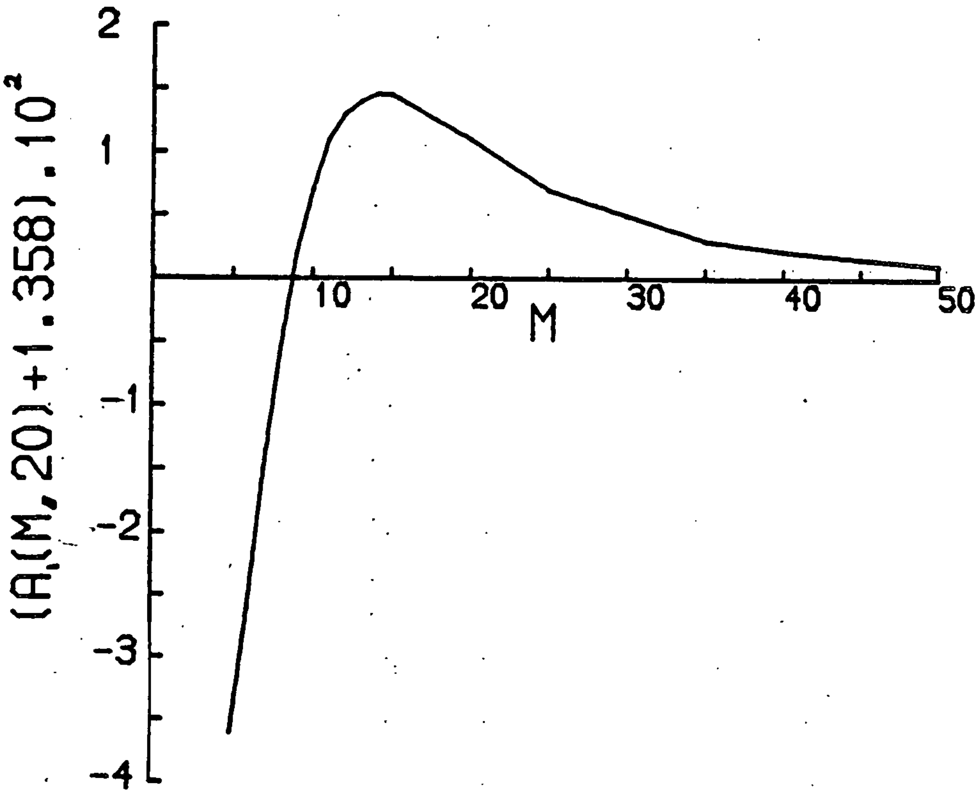
GRAPH 2



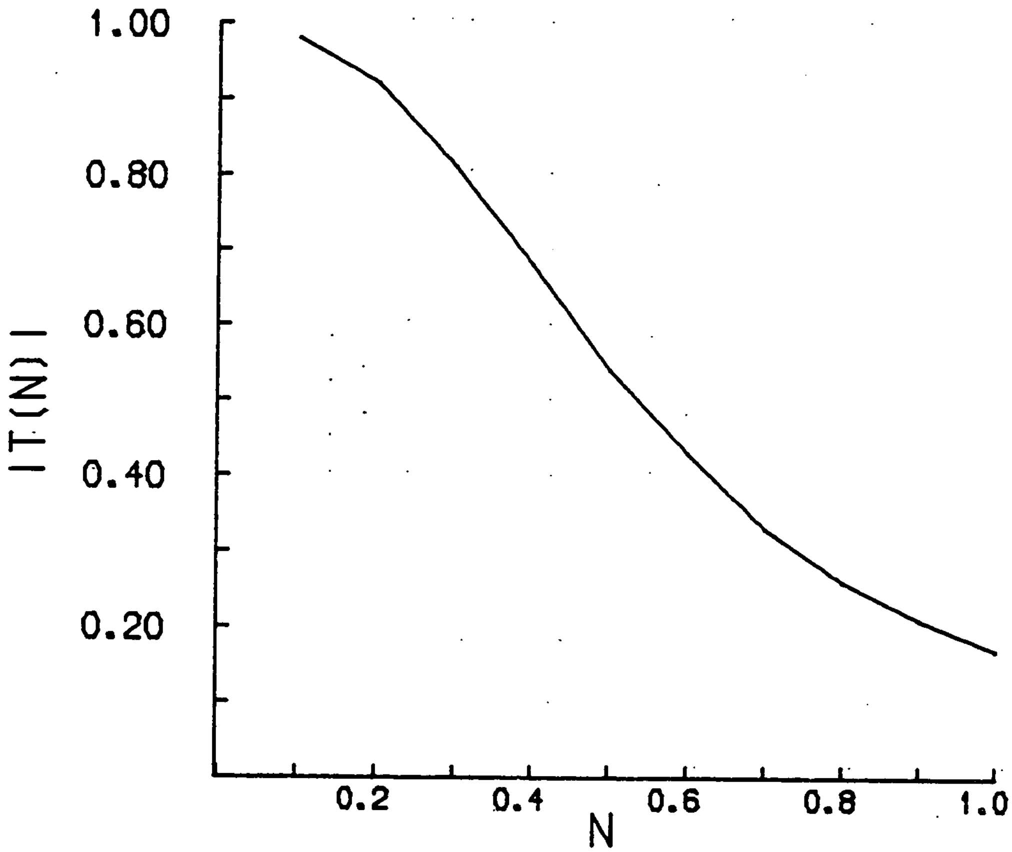
GRAPH 3



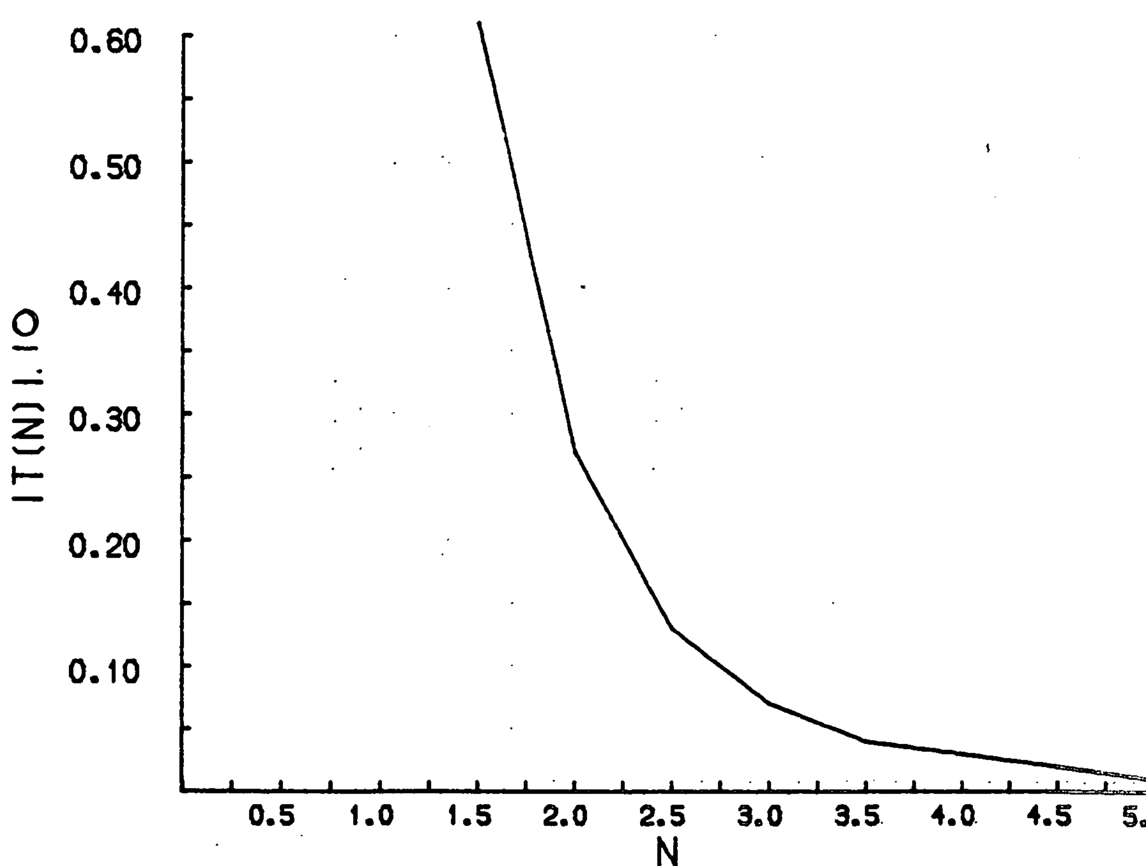
GRAPH 4



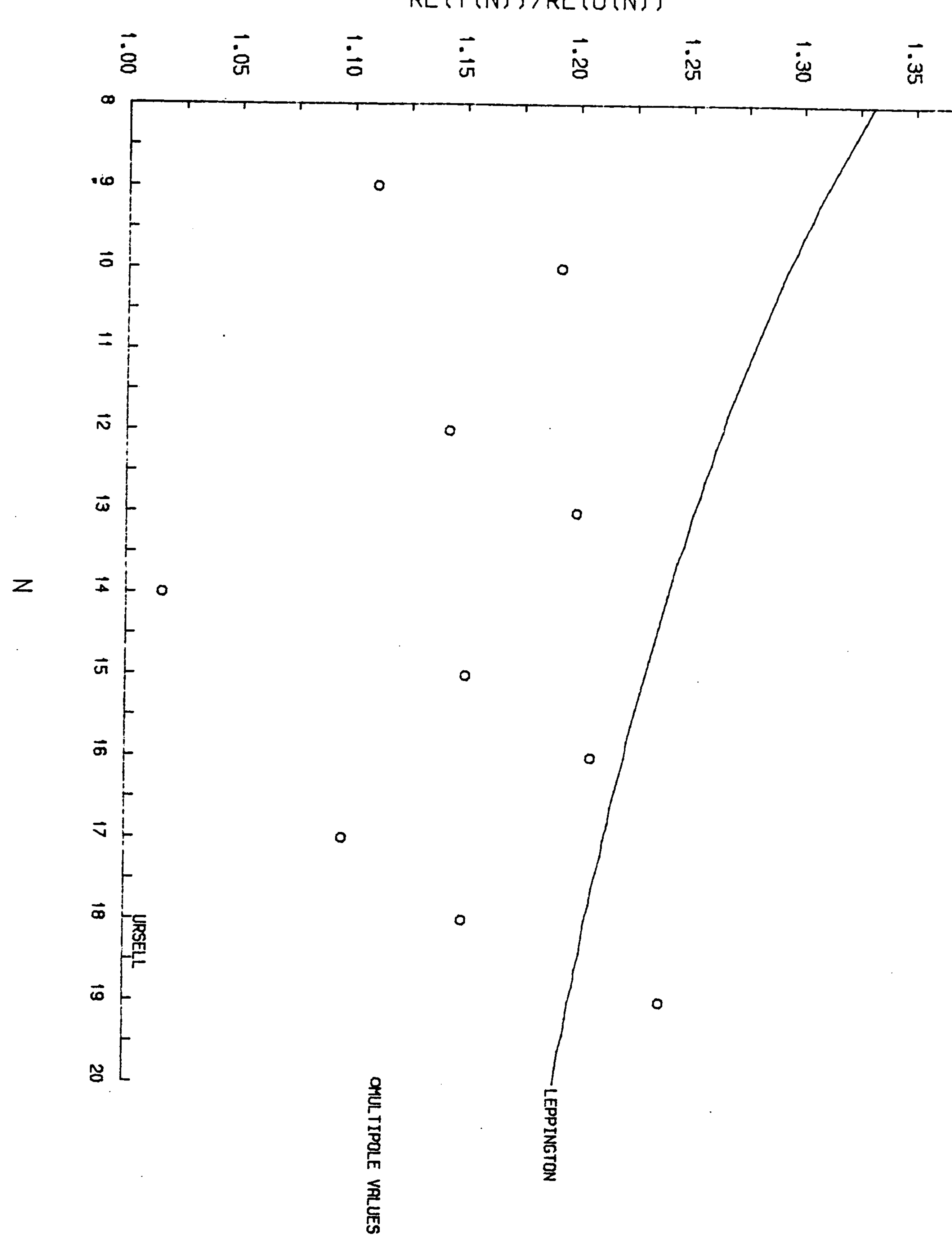
GRAPH 5



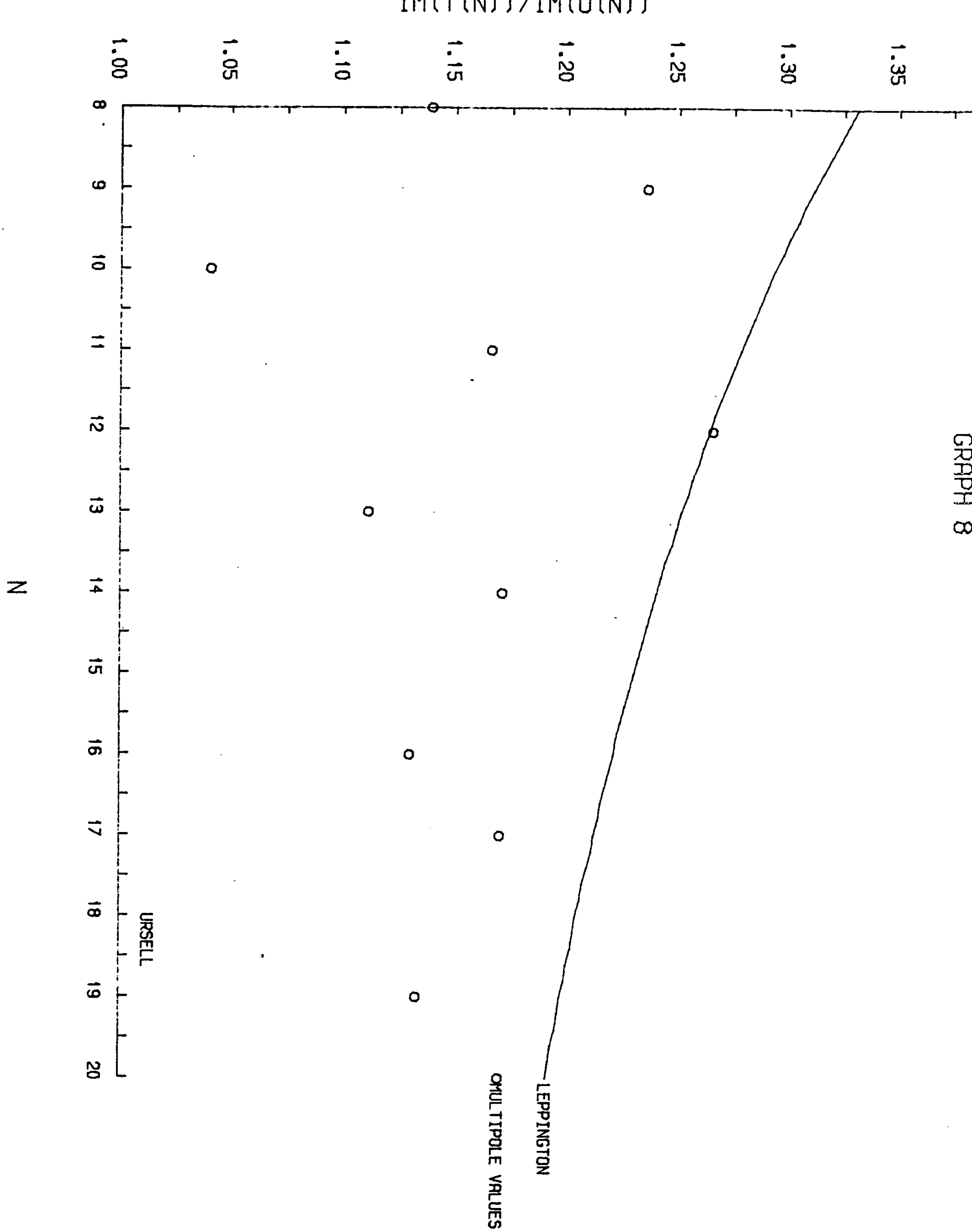
GRAPH 6



GRAPH 7



Comparison of the real parts of the transmission coefficient as given by multipole expansions and Ursell's and Leppington's asymptotic formulae. The values are normalised by Ursell's values so that his results are represented by the horizontal line through 1 on the vertical axis. At N = 11 the multipole value is about seven times bigger than Ursell's value (see Table 7 in §2.8) and cannot be shown (in scaled form) on the graph.



Comparison of the imaginary parts of the transmission coefficient as given by multipole expansions and Ursell's and Leppington's asymptotic formulae. The values are normalised by Ursell's values so that his results are represented by the horizontal line through 1 on the vertical axis. At N = 18, the multipole value is approximately 0.9 times Ursell's value, so the scaled form of the multipole value (≏0.9) cannot be shown on the graph.

CHAPTER 3

§3.1. Introduction

In this Chapter a more detailed description is given of the method of matched asymptotic expansions as applied to bodies whose tangents at E_+ and E_- (see Fig. 2) are vertical. It will be seen that the perturbation expansions in the right and left inner regions have potential coefficients which are solutions of wave maker type problems, while the coefficients in the outer expansion are solutions of boundary value problems of known types involving use of complex variable methods.

§3.2. The right inner expansion

In the right inner region Δ_+, new coordinates (X,Y) are introduced (relative to axes E_+X and E_+Y as shown in Fig. 2) and are scaled so that Laplace's equation and the free surface condition do not contain ε explicitly. Reference to equation (1.2) indicates that the appropriate scale factor for the ordinates is ε so that $y = \varepsilon Y$ while the harmonic nature of the potential subsequently dictates that the abscissae be scaled in the same way by setting $x = a + \varepsilon X$. It follows that

$$\delta_+ = \varepsilon R \quad \text{where } R = \sqrt{X^2 + Y^2}$$

and that in Δ_+

$$\phi(x,y) = \phi(a + \varepsilon X, \varepsilon Y) \overset{D}{=} \Phi(X,Y,\varepsilon)$$ (the dependence on a will not be stated explicitly).

Equations (1.1) and (1.2) now become

$$\Phi_{XX} + \Phi_{YY} = 0 \quad \text{in } \Delta_+ \tag{3.1}$$

$$\Phi + \Phi_Y = 0 \quad \text{on } \Delta_+ \cap S \; , \tag{3.2}$$

while (1.3) is recast in the form

$$\Phi_X - \frac{dX}{dY}\Phi_Y = -(W_X{}^I - \frac{dX}{dY}W_Y{}^I) \quad \text{on } \Gamma \cap \Delta_+ \tag{3.3}$$

where W^I is the potential of the incoming wave

i.e. $W^I = e^{-iX-Y}$.

The equation of Γ near E_+ is now written in the form

$$x - a = f(y)$$

where it is assumed that the function f can be expanded in the form

$$f(y) = \sum_{k=2}^{\infty} a_k \, y^k \, . \tag{3.4}$$

In terms of scaled coordinates this becomes

$$X = \frac{f(\varepsilon Y)}{\varepsilon} = \sum_{k=2}^{\infty} a_k \, \varepsilon^{k-1} \, Y^k \tag{3.5}$$

so that $$\frac{dX}{dY} = f'(\varepsilon Y) = \sum_{k=2}^{\infty} k a_k \varepsilon^{k-1} \, Y^{k-1} \, . \tag{3.6}$$

Next Φ_X, Φ_Y are expanded in Taylor series about X = 0 and these series are substituted in (3.3) using the equation (3.5) to express powers of X in terms of Y. The same process is applied to the right-hand side of (3.3) whence this equation takes the form

$$\sum_{r=0}^{\infty} g_r(\varepsilon,Y) \left[\frac{\partial^{r+1}}{\partial X^{r+1}} \Phi(0,Y;\varepsilon) - f'(\varepsilon Y) \frac{\partial^{r+1}}{\partial X^r \partial Y} \Phi(0,Y;\varepsilon) \right]$$

$$= \sum_{r=0}^{\infty} g_r(\varepsilon,Y)\,(-i)^r e^{-Y} \, [\, i - f'(\varepsilon Y)] \tag{3.7}$$

where $g_r(\varepsilon,Y) = \dfrac{1}{r!}\left[\dfrac{f(\varepsilon Y)}{\varepsilon}\right]^r$ and it should be noted from (3.5) and (3.6) that

$g_r(\varepsilon,Y) = O(\varepsilon^r)$ and

$f'(\varepsilon Y) = O(\varepsilon)$ under the inner limiting process $\varepsilon \to 0$ with (X,Y) fixed.

On the assumption that Φ and its derivatives are bounded on X = 0 as $\varepsilon \to 0$, the equation (3.7) can be written in the form

$\Phi_X(0,Y;\varepsilon) + O(\varepsilon) = i\, e^{-Y} + O(\varepsilon)$ so that as $\varepsilon \to 0$

$\Phi_X(0,Y;\varepsilon)$ tends to the limit $i\, e^{-Y}$.

Hence $\Phi(X,Y;\varepsilon)$ tends to a limit $\Phi_0(X,Y)$ where Φ_0 is harmonic, satisfies the free surface condition on an arbitrarily large (but bounded) portion of the positive X axis and has a normal velocity $i\ e^{-Y}$ on an arbitrarily large (but again bounded) portion of the positive Y axis, i.e. Φ_0 is harmonic in Δ_+

$$\Phi_0 + \Phi_{0_Y} = 0 \text{ on } Y = 0 \text{ for } 0 < X < X_0 < \infty$$

$$\Phi_{0_X}(0,Y) = i\ e^{-Y} \quad \text{for } 0 < Y < Y_0 < \infty\ ,$$

and, in addition, Φ_0 should satisfy the edge condition. At this point it may be remarked that the region in which the solution Φ_0 is of interest may be considered small compared to the region of validity of the above equations (the outer solution can take over once the region of overlap of inner and outer solutions is reached; see §3.5 on the matching principle). Hence, in practice, the conditions on X = 0 and Y = 0 are extended to infinity so that Havelock's wave maker solution can be applied to find Φ_0 (this will apply also to later terms in the inner perturbation series). Clearly to obtain meaningful results it is necessary for the velocity distribution on the wave maker to decay sufficiently rapidly as $Y \to +\infty$ and this will be discussed more fully in Chap. 4 where the behaviour of Havelock's solution for various forms of the prescribed normal velocity on the wave maker will be investigated.

Certainly when the decay is of negative exponential type no problems arise and the solution for Φ_0 is given by

$$\Phi_0(X,Y) = P_0(X,Y) + E_0(X,Y) \quad \text{where}$$

$P_0(X,Y)$ is Havelock's particular solution and $E_0(X,Y)$ is a function satisfying the homogeneous problem i.e. $E_0(X,Y)$ is harmonic, satisfies the surface condition and has zero normal velocity on the wave maker as well as satisfying the edge condition. Such functions will be termed eigensolutions and they form a subset of solutions of the vertical barrier problem (all possible solutions are obtained explicitly in Chap. 4). For the moment it is sufficient to remark that the eigensolutions are wave-free, a result of fundamental importance in extending the asymptotic form of the transmission coefficient.

If the results contained in Chap. 4 are anticipated, then $P_0(X,Y)$ is given by Havelock (1929) in the form

$$P_0(X,Y) = \int_0^\infty H(X,Y;s)\, i\, e^{-s}\, ds \quad \text{where}$$

$$H(X,Y;s) = -2i \exp[iX - (Y+s)] - \frac{2}{\pi}\int_0^\infty \frac{(u\cos uY - \sin uY)(u\cos us - \sin us)}{u(u^2+1)} e^{-uX} du \,.$$

$P_0(X,Y)$ falls naturally into two parts, a wave part which will be denoted by $W_0(X,Y)$ and a wave-free part which will be denoted by $F_0(X,Y)$. Thus

$$W_0(X,Y) = -2i \exp(iX - Y) \int_0^\infty i\, e^{-2s}\, ds$$

$$= \exp(iX - Y)$$

and $$F_0(X,Y) = -\frac{2i}{\pi}\int_0^\infty e^{-s}\int_0^\infty \frac{(u\cos uY - \sin uY)(u\cos us - \sin us)}{u(u^2+1)} e^{-uX} du$$

Reversal of the order of integration above gives

$$F_0(X,Y) = -\frac{2i}{\pi}\int_0^\infty \frac{u\cos uY - \sin uY}{u(u^2+1)}\int_0^\infty e^{-s}(u\cos us - \sin us)\,ds\,du$$

and an integration by parts shows the inner integral to be zero.

Hence

$$F_0(X,Y) = 0 \text{ and}$$

$$P_0(X,Y) = \exp(iX - Y)\,.$$

The eigensolution $E_0(X,Y)$ must be a certain linear combination of the functions

$$\frac{R^{2m+1}\sin(2m+1)\theta}{2m+1} - R^{2m}\cos 2m\,\theta \qquad \text{where } X = R\cos\theta,\ Y = R\sin\theta,\ m \geqslant 0. \quad \text{(see §4.5).}$$

Such functions are $O(R^{2m+1})$ as $R \to \infty$ for some integer m so that the matching principle (which does not involve wave terms) would require that the leading term in the perturbation series for the outer potential should be of the form $\dfrac{\phi_0}{\varepsilon^{2m+1}}$ where $\phi_0 \sim A_0\,\delta_+^{2m+1}\sin(2m+1)$ (A_0 being a constant) as $\delta_+ \to 0$. In addition ϕ_0 would be harmonic and would satisfy $\phi_0 = 0$ on $y = 0$ ($|x| > a$), $\dfrac{\partial\phi_0}{\partial r} = 0$ on $r = a$ and $\phi_0 \to 0$ as $r \to \infty$ ($r = \sqrt{x^2 + y^2}$).

Near E_-, ϕ_0 would have one of the properties

$$\phi_0 = O\left(\delta_-^{\pm(2k-1)}\right) \qquad \text{(k a positive integer)}$$

as $\delta_- \to 0$ ($\delta_- = \sqrt{(x+a)^2 + y^2}$) since, near this point, ϕ_0 will behave like a solution of the problem

$$\phi = 0 \quad \text{on } y = 0 \qquad (x < -a)$$

$$\frac{\partial\phi}{\partial x} = 0 \text{ on } x = -a \qquad (y > 0)$$

and such solutions are linear combinations of functions of the form $\mathcal{R}e_j\{\frac{1}{j}[A(z+a)^{2k-1} + \dfrac{B}{(z+a)^{2k-1}}]\}$ where $z = x + jy$, k is a positive integer and A, B are independent of j.

If the positive sign is taken in the order relation for ϕ_0 then ϕ_0 is non-singular at E_- and, since it has also been shown to be non-singular at E_+, it follows by using the result

$$\int_{\text{fluid domain}} (\nabla\phi)^2\, dA = \int_{\text{boundary}} \phi\,\frac{\partial\phi}{\partial n}\, ds$$

and the boundary conditions, ($\underline{n}$ being a unit vector normal to the boundary curves) that

$$\int_{\text{fluid domain}} (\nabla\phi)^2\, dA = 0$$

provided that the integral of $\phi\,\dfrac{\partial\phi}{\partial n}$ along a semi-circle at infinity is zero i.e. provided $\phi\,\dfrac{\partial\phi}{\partial r}$ decays more rapidly than $\dfrac{1}{r}$. With this stipulation it follows that $\nabla\phi = 0$ whence ϕ = a constant and this constant can only be zero because of the condition on $y = 0$. Hence, in the case where $\phi_0 = O(\delta_-^{2k-1})$, the only solution is $\phi_0 = 0$. If, on the other hand, $\phi_0 = O(\delta_-^{-(2k-1)})$ as $\delta_- \to 0$, then the leading term in the perturbation series for

the potential in the left inner region would be of the form $\frac{\psi_0}{\varepsilon^{2m+2k}}$
where $\psi_0 = O(R_1^{-(2k-1)})$ as $R_1 \to \infty$ (where $\delta_- = \varepsilon R_1$). The only non-trivial vertical barrier solutions which have this property are of the form (see §4

$c\left[\frac{\sin(2k-1)\theta}{(2k-1)R_1^{2k-1}} - \frac{\cos 2k\theta}{R_1^{2k}}\right]$ plus linear combinations of similar terms involving higher powers of $\frac{1}{R}$ $(c \neq 0)$ and ψ_0 would not then satisfy the edge condition $\psi_0 \frac{\partial\psi_0}{\partial R_1} \to 0$ as $R_1 \to 0$. It must be concluded, therefore, that ϕ_0 is non-singular at E_- whence it follows as above that ϕ_0 and E_0 are identically zero.

Hence $\Phi(X,Y;\varepsilon) = \Phi_0(X,Y) + o(1)$ as $\varepsilon \to 0$ where

$$\Phi_0(X,Y) = \exp(iX - Y) .$$

Let it now be assumed that, under the inner limiting process $\varepsilon \to 0$ with (X,Y) fixed, it is possible for some integer m to write

$$\Phi(X,Y;\varepsilon) = \sum_{s=0}^{m} \varepsilon^s \Phi_s(X,Y) + o(\varepsilon^m) \text{ where for } 0 \leqslant s \leqslant m$$

the Φ_s are harmonic, satisfy the surface condition $\Phi_s + \Phi_{sY} = 0$ for $X > 0$, have normal derivatives on $X = 0$ which are expressed in terms of earlier occurring potentials in the expansion and/or the incoming wave, and do not contain eigensolutions. Suppose also that substitution of the above expansion in (3.7) causes cancellation of all terms up to order ε^m. (This result is true for $m = 0$).

Postulate then that

$$\Phi(x,y;\varepsilon) = \sum_{s=0}^{m} \varepsilon^s\Phi_s(X,Y) + \ell(\varepsilon)\Phi_\ell(X,Y) + \varepsilon^{m+1}\Phi_{m+1}(X,Y) + o(\varepsilon^{m+1})$$

as $\varepsilon \to 0$ where $\varepsilon^m \prec \ell(\varepsilon) \prec \varepsilon^{m+1}$ as $\varepsilon \to 0$. Substitution of this expansion in Laplace's equation and the free surface condition reveals that Φ_ℓ, Φ_{m+1} must be harmonic and satisfy the free surface condition while substitution in (3.7) and neglect of all terms of orders higher than ε^{m+1} reveals that $\Phi_{\ell X}(0,Y) = 0$ while $\Phi_{(m+1)X}(0,Y)$ is a function of the Φ_s for $0 \leqslant s \leqslant m$ and the incoming wave. It follows, therefore, that all the potential coefficients in the right inner perturbation series are either eigensolutions of the vertical barrier problem or solutions of the wave-maker type of problem. Considerations of whether or not Φ_ℓ is zero and whether eigensolutions should be added to Φ_{m+1} depend on a careful step by step development of the outer and left inner expansions. The arguments used will be similar to those employed in proving $E_0(X,Y) = 0$ earlier in this section but further details will be left to Chap. 5 when the special case of the semi-circular geometry is considered.

Note on notation

As in the case of $\Phi_0(X,Y)$ each $\Phi_k(X,Y)$ will be written as the sum of two parts, viz.

$\Phi_k(X,Y) = P_k(X,Y) + E_k(X,Y)$ where $P_k(X,Y)$ is Havelock's particular solution,

i.e. $P_k(X,Y) = \int_0^\infty H(X,Y;s)\Phi_{ks}(0,s)\,ds$ and $E_k(X,Y)$ is an eigensolution.

In addition, $W_k(X,Y)$ will be used to denote the wave part of $P_k(X,Y)$ and $F_k(X,Y)$ to denote the wave-free part.

§3.3. The outer expansion

The development of the perturbation series for the potential $\phi(x,y;\varepsilon)$ in the outer region is begun by formally putting $\varepsilon = 0$ in the original boundary value problem and neglecting any wave terms appearing in the equations (the outer potential does not recognise waves). Hence, to begin the expansion, it is postulated that

$$\phi(x,y;\varepsilon) = c_0(\varepsilon)\phi_0(x,y) + o(c_0(\varepsilon))$$

under the outer limiting process $\varepsilon \to 0$ with (x,y) fixed, where $\phi_0(x,y)$ is harmonic and satisfies

$$\phi_0 = 0 \text{ on } y = 0 \qquad (|x| > a)$$

$$\frac{\partial\phi_0}{\partial r} = 0 \text{ on } \Gamma$$

$$\phi_0 \to 0 \text{ as } r \to \infty .$$

The solution to this problem is made unique by matching the outer solution to the right inner solution using the principle to be described in §3.5. If it is assumed that a perturbation series of the form

$$\phi(x,y;\varepsilon) = \sum_{k=0}^{p} c_k(\varepsilon)\phi_k(x,y) + o(c_p(\varepsilon)) \quad \text{as } \varepsilon \to 0$$

has been formed, then substitution of this form in the governing equations for $\phi(x,y;\varepsilon)$ reveals that each ϕ_k must be harmonic and satisfy $\frac{\partial\phi_k}{\partial r} = 0$ on Γ. Formal substitution of the asymptotic series in the surface condition $\phi + \varepsilon\phi_y = 0$ reveals further that either $\phi_k = 0$ or $\phi_k(x,0) = -\phi_{my}(x,0)$ $(|x| > a)$ where $0 \leqslant m < k$. (The latter case will occur when there is a term with a certain scaling $c_k(\varepsilon)$ and an earlier occurring term with scaling $\frac{c_k(\varepsilon)}{\varepsilon}$. The scale factors which occur will depend on the matching which takes place with the right inner expansion.) Finally, each ϕ_k is made to satisfy $\phi_k \to 0$ as $r = \sqrt{x^2 + y^2} \to \infty$. The problems for the ϕ_k are therefore classical type boundary value problems soluble either by complex variable methods or by use of an appropriate Green's function.

§3.4. The left inner expansion

The procedure here is almost identical with that for the right inner expansion. A new system of coordinates is introduced relative to axes E_-X_1 and E_-Y_1 through E_- as shown in Fig. 2 scaled so that $x = -a - \varepsilon X_1$, $y = \varepsilon Y_1$. Thus $\delta_- = \varepsilon R_1$ where $\delta_- = \sqrt{(x+a)^2 + y^2}$, $R_1 = \sqrt{X_1^2 + Y_1^2}$. The polar angle is denoted by θ_1 and the equation of the curve near E_- is written in the form $x + a = -f_1(y)$ where it is assumed that f_1 can be expanded in the form

$$f_1(y) = \sum_{k=2}^{\infty} b_k\, y^k \, .$$

Hence $X_1 = \dfrac{f_1(\varepsilon Y_1)}{\varepsilon} = \displaystyle\sum_{k=2}^{\infty} b_k\, \varepsilon^{k-1}\, Y_1^{k}$

and $\dfrac{dX_1}{dY_1} = f_1'(\varepsilon Y_1) = \displaystyle\sum_{k=2}^{\infty} k\, b_k\, \varepsilon^{k-1}\, Y_1^{k-1}$.

The form of ϕ in Δ_- is denoted by $\psi(X_1,Y_1;\varepsilon)$ so that by analogy with (3.7) the boundary condition on Γ is written in the form

$$\sum_{r=0}^{\infty} h_r(\varepsilon,Y_1)\left[\frac{\partial^{r+1}}{\partial X_1{}^{r+1}}\psi(0,Y_1;\varepsilon) - f_1'(\varepsilon Y_1)\frac{\partial^{r+1}}{\partial X_1{}^r\partial Y_1}\psi(0,Y_1;\varepsilon)\right] = 0$$

(the incoming wave is not subtracted in the left inner region)

where

$$h_r(\varepsilon,Y_1) = \frac{1}{r!}\left[\frac{f_1(\varepsilon Y_1)}{\varepsilon}\right]^r .$$

The scale factors for the perturbation series in the left inner region will be determined by matching with the outer expansion and it will be found that (as in the case of the right inner expansion) the potential coefficients in the series will be either eigensolutions of the vertical barrier problem or solutions of the wave-maker problem in which the prescribed velocity on the wave-maker is determined by potentials appearing earlier in the series.

Notation

By analogy with the note on notation at the end of §3.2 the various parts of the potential coefficients $\psi_k(X_1,Y_1)$ in the left inner expansion will be denoted by $P_k(X_1,Y_1)$, $E_k(X_1,Y_1)$, $W_k(X_1,Y_1)$ and $F_k(X_1,Y_1)$. The notation will be local to the left inner region so no confusion will arise with the right inner region.

Note

Once the wave parts of $\underline{\Phi}(X,Y;\varepsilon)$ and $\psi(X_1,Y_1;\varepsilon)$ have been found to a certain order in ε the outer solution (which as described in §3.3 has no wave terms) is modified (in order to satisfy the outgoing wave requirements at infinity) by adding the waves from the right inner region (expressed in outer coordinates) to the solution for ϕ in $x > 0$ and the waves from the left inner region (again expressed in outer coordinates) to the solution for ϕ in $x < 0$. This modified solution will then be assumed to extend up through the boundary layer to the free surface. Since the ϕ_k in the outer expansion will be made to die off to zero at infinity, this device will enable the outgoing wave conditions to be satisfied and hence provide asymptotic forms for the reflection and transmission coefficients. The fact that the potential coefficients in the inner expansions depend only on potentials appearing earlier in the series and that eigensolutions are wave-free enables the asymptotic form of the transmission coefficient to be obtained to an order higher than that for which detailed matching has taken place in the left inner expansion.

§3.5. The matching principle

It will be seen that, in the case which will be considered in detail in Chapter 5, the scale factors in the perturbation series will all be of the form $\varepsilon^s (\log \varepsilon)^t$ where s, t are integers $\geqslant 0$. Hence to respect condition (iii) of theorem I in Fraenkel (1969) p.223, it will be necessary to adopt the matching principle proposed by Crighton and Leppington (1973) in which, ~~for given~~ s, all terms with scalings of this form must be determined and grouped together before detailed matching takes place. Assuming that this has been done, the right inner expansion of $\Phi(X,Y;\varepsilon)$ up to terms of order ε^s will be denoted by $\Phi^{(s)}$. If the outer limiting process ($\varepsilon \to 0$ with (x,y) fixed) is applied to this inner expansion, the result will be equivalent to that obtained by letting $R \to \infty$ in the potential coefficients (since $R = \frac{\delta_+}{\varepsilon}$). Assuming that the asymptotics of these potentials have been obtained to a certain order, then the result of replacing R by $\frac{\delta_+}{\varepsilon}$ and truncating the resulting series after terms of order ε^r will be donoted by $\Phi^{(s,r)}$. Similarly, let $\phi^{(r)}$ denote the outer expansion up to terms of order ε^r. Application of the inner limiting process ($\varepsilon \to 0$ with (X,Y) fixed) is equivalent to letting $\delta_+ \to 0$ in the potential coefficients. If the asymptotics of these potentials as $\delta_+ \to 0$ are obtained up to a certain order and δ_+ is replaced by εR and the resulting series truncated after terms of order ε^s, then the series obtained is denoted by $\phi^{(r,s)}$. The matching principle is that

$$\Phi^{(s,r)} = \phi^{(r,s)} .$$

A similar principle will be applied in matching the left inner region with the outer potential.

Crucial to the success of the matching process is the existence of an "overlap" region where both inner and outer approximations apply. In this case the right inner approximation is assumed valid at points close to E_+ on the a-scale ($\delta_+ \ll a$) while the outer approximation is assumed valid for $\delta_+ \gg \varepsilon$. If $\varepsilon \ll \delta_+ \ll a$ there is evidently a common region of validity in which both approximations are equivalent in the sense of the matching principle. Similar considerations establish the existence of an overlap region for the left inner and outer approximations.

CHAPTER 4

§4.1. Introduction

In §3.2 and §3.4 it was seen that the right and left inner expansions lead to a sequence of problems consisting of Laplace's equation in the quadrant $x > 0$, $y > 0$, the free surface condition on $y = 0$ and a prescribed normal velocity condition on $x = 0$. Such problems make up what will be called the Classical Wave-Maker family. The physically inspired classical wave-maker problem is to represent the two dimensional wave motion generated by a vertical wave-maker, idealised to be of infinite depth, having a prescribed velocity profile depending on depth. In particular the waves far down the channel are to be found, assuming there is no agency to generate or reflect waves back towards the wave-maker. For a bounded solution to exist, it is necessary that the prescribed velocity on the wave-maker decays sufficiently rapidly with depth. This is a mathematical difficulty consequent on the infinite depth idealisation and has no direct experimental relevance at finite depth. In the context of matched expansions the outgoing wave condition is replaced by a matching condition on the wave-free part of the potential (which may not even entail boundedness) and the prescribed velocity distribution (arising from previous potentials in the left and right inner perturbation series) may not have decay properties permitting of the direct application of Havelock's (1929) solution to them. This consideration calls for a more general study of the classical wave-maker family of problems and this is the subject of the present chapter.

No uniqueness theorem exists without the outgoing wave condition and attention is first devoted to finding particular solutions. The methods, which are well established in the outgoing wave case, are reviewed and order of magnitude properties of the wave-free part of Havelock's solution in the far field are derived. As would be expected, velocity profiles which produce only outgoing waves as $x \to \infty$ coincide with those requiring finite energy input. Other solutions whose wave-free parts are unbounded in the far field have no physical relevance but may be acceptable in the general setting of matched expansions.

Next, particular solutions are derived (by means of Lewy's (1946) reduction method) for two special cases of unbounded velocity profiles arising in Chapter 5 to which Havelock's solution cannot be applied directly and, finally, the general solutions are investigated by studying the homogeneous problem in which the wave-maker is at rest (referred to as the vertical barrier problem). The nature of solutions of the vertical barrier problem is controlled by the behaviour at infinity and by the singularity, if any, permitted at the surface point. Various possibilities are catalogued and it is proved explicitly that solutions containing outgoing waves can exist if and only if a logarithmic singularity is allowed at the surface point. This result provides a criterion for selecting the terms from the inner expansion which are associated with outgoing waves.

§4.2. Mathematical statement of the problem and Havelock's solution

Axes are taken with Ox in the undisturbed water surface and Oy along the rest position of the oscillating barrier downwards into the fluid which is assumed to occupy the first quadrant $x > 0$, $y > 0$. The velocity potential is assumed to have the form Re $[\phi(x,y)\, e^{-i\sigma t}]$ and the prescribed velocity on the barrier at a distance y below the surface is taken to be Re$[f(y)e^{-i\sigma t}]$ (f being a continuous function on $[0,\infty]$). The problem is to find a function ϕ (continuous and twice differentiable in the fluid domain $x > 0$, $y > 0$) such that

$$\phi_{xx} + \phi_{yy} = 0 \qquad (x > 0,\ y > 0) \tag{4.1}$$

$$k\phi + \phi_y = 0 \qquad (y = 0,\ x > 0) \quad (k = \sigma^2/g) \tag{4.2}$$

$$\phi_x = f \qquad (x = 0,\ y \geqslant 0) \tag{4.3}$$

(The final condition here is stated on x = 0 because of linearisation; Wehausen and Laitone (1960), p.553-555, gives a general discussion of forced harmonic oscillations).

If account is taken of minor differences in notation, Havelock (1929) provides a particular solution to the problem (4.1)-(4.3) in the form

$$P(x,y;k) = \int_0^\infty f(s)H(x,y;k;s)\,ds \qquad \text{where} \tag{4.4}$$

$$H(x,y;k;s) = -2i\exp[ikx - k(y+s)]$$
$$- \frac{2}{\pi}\int_0^\infty \frac{(u\cos uy - k\sin uy)(u\cos us - k\sin us)}{u(u^2+k^2)} e^{-ux}\,du\,. \tag{4.5}$$

This form for H is used once in this thesis in Chapter 3, §3.1, but elsewhere an equivalent form

$$H(x,y;k;s) = -2i\exp[ikx - k(y+s)]$$
$$+ \frac{1}{2\pi}\log\left[\frac{x^2+(y-s)^2}{x^2+(y+s)^2}\right] - \frac{2}{\pi}\int_0^\infty \frac{u\cos u(y+s) - k\sin u(y+s)}{u^2+k^2} e^{-ux}\,du \quad (4$$

is employed.

It is convenient also to introduce here some notation which will be used subsequently throughout Chapter 5 in discussing the left and right inner expansions where the form (4.6) of H will be used. First, the three terms in H will be denoted by H_W, H_L and H_I respectively while the corresponding parts of P will be denoted by W, L and I and the wave-free part of P (i.e. L+I) by F.

Thus
$$\left.\begin{aligned} H_W(x,y;k;s) &\overset{D}{=} -2i\exp[ikx - k(y+s)]\,, \\ H_L(x,y;s) &\overset{D}{=} \frac{1}{2\pi}\log\left[\frac{x^2+(y-s)^2}{x^2+(y+s)^2}\right], \\ H_I(x,y;k;s) &\overset{D}{=} -\frac{2}{\pi}\int_0^\infty \frac{u\cos u(y+s) - k\sin u(y+s)}{u^2+k^2}e^{-ux}\,du\,, \end{aligned}\right) \tag{4.7}$$

while

$$
\left.
\begin{aligned}
W(x,y;k) &\overset{D}{=} -2i\exp(ikx - ky)\int_0^{\infty} f(s)e^{-s}\,ds\,, \\
L(x,y) &\overset{D}{=} \frac{1}{2\pi}\int_0^{\infty} f(s)\log\left[\frac{x^2 + (y-s)^2}{x^2 + (y+s)^2}\right] ds\,, \\
I(x,y;k) &\overset{D}{=} -\frac{2}{\pi}\int_0^{\infty}\!\!\int_0^{\infty} \frac{u\cos u(y+s) - k\sin u(y+s)}{u^2 + k^2}\, e^{-ux} f(s)\,du\,ds\,, \\
F(x,y;k) &\overset{D}{=} L(x,y) + I(x,y;k)\,.
\end{aligned}
\right\} \qquad (4.8)
$$

In addition the values H, P, W, I, F for k=1 will be abbreviated to $H(x,y;s)$, $P(x,y)$, $W(x,y)$, $I(x,y)$, $F(x,y)$ respectively.

Hence

$$
\left.
\begin{aligned}
H(x,y;s) &\overset{D}{=} H(x,y;1;s)\,, \\
\phi(x,y) &\overset{D}{=} \phi(x,y;1)\,, \\
W(x,y) &\overset{D}{=} W(x,y;1)\,, \\
I(x,y) &\overset{D}{=} I(x,y;1)\,, \\
F(x,y) &\overset{D}{=} F(x,y;1)\,.
\end{aligned}
\right\} \qquad (4.9)
$$

Finally, with a view to discussing the behaviour of the integrals appearing in the wave-free part of Havelock's solution, it is remarked that (for fixed (x,y))

$$H_L(x,y;s) = -\frac{2y}{\pi s} + O\left(\frac{1}{s^2}\right) \qquad (4.10)$$

and

$$H_I(x,y;k;s) = \frac{2}{\pi k s} + O\left(\frac{1}{s^2}\right) \qquad (4.11)$$

as $s \to \infty$, the first result being proved using logarithmic expansions and the second by writing

$$H_I(x,y;k;s) = -\frac{2}{\pi}\,\mathrm{Re}_j \int_0^{\infty} \frac{\exp(-\zeta u)}{u - jk}\,du$$

$(\zeta = x - j(y+s))$ and using Watson's Lemma or simple integration by parts.

§4.3. The convergence of Havelock's solution and the behaviour in the far field of the wave-free part

The wave-free part of $\phi(x,y;k)$ is given by

$$F(x,y;k) = \int_0^{\infty} f(s)[H_L(x,y;s) + H_I(x,y;k;s)]\,ds \quad .$$

Hence, by (4.10) and (4.11) it is clear that (for fixed (x,y)) the integral is convergent if

(a) $\int_b^\infty \frac{f(s)}{s}\, ds$ exists $\quad (b > 0)$

and (b) $f(t)$ is bounded as $t \to \infty$.

For the discussion of the behaviour of the wave-free part as $r = \sqrt{x^2 + y^2} \to \infty$ which follows the stricter condition that $f(s) = O(\frac{1}{s}\alpha)$ $(\alpha > 0)$ as $s \to \infty$ will be assumed. It will be seen that, to obtain a solution with only outgoing waves at infinity, it is sufficient to take $\alpha > 1$ and that in cases of this type finite input of energy is required to maintain the velocity profile. The two components of $F(x,y;k)$ i.e. $L(x,y)$ and $I(x,y;k)$ are examined separately, but, first, it is noted by analogy with (4.10), (4.11) that as $r \to \infty$ with s bounded

$$H_L(r\cos\theta, r\sin\theta; s) = \frac{-2s\sin\theta}{\pi r} + O(\frac{1}{r^2}) \tag{4.12}$$

and $$H_I(r\cos\theta, r\sin\theta; k; s) = \frac{2\sin\theta}{\pi k r} + O(\frac{1}{r^2}) \quad . \tag{4.13}$$

Since the asymptotic result (4.12) above for $H_L(r\cos\theta, r\sin\theta; s)$ applies only in the case when s is bounded, the range of integration must be split into two parts, in one of which H_L is small for large r and in the other of which $f(s)$ is small for large s. This is achieved using the fact that $f(s) = O(\frac{1}{s}\alpha)$ as $s \to \infty$, whence there exist s_0 and A (both constants independent of s) such that $|f(s)| \leqslant \frac{A}{s}\alpha$ for $s \geqslant s_0$. s_0 is fixed and the equation for L is written

$$L(r\cos\theta, r\sin\theta) = \left(\int_0^{s_0} + \int_{s_0}^\infty\right) H_L(r\cos\theta, r\sin\theta; s)\, ds \ . \tag{4.14}$$

By a mean value theorem of the integral calculus, the first integral is equal to

$s_0\, H_L(r\cos\theta, r\sin\theta; \eta) f(\eta)$, where $0 < \eta < s_0$;

whence (using (4.12) and the boundedness of f) it can be deduced that this part is $O(\frac{1}{r})$ as $r \to \infty$.

By use of the substitution $s = ru$, the second integral takes the form

$$r \int_{s_0/r}^{\infty} H_L(r \cos \theta, r \sin \theta; ru)\ f(ru)\ du$$

$$= r \int_{s_0/r} H_L(\cos \theta, \sin \theta; u)\ f(ru)\ du \quad .$$

By use of the order property of f, the modulus of this integral is less than or equal to

$$A . \frac{r^{1-\alpha}}{2\pi} \int_{s_0/r}^{\infty} \log \left[\frac{u^2 + 2u \sin \theta + 1}{u^2 - 2u \sin \theta + 1}\right] . \frac{1}{u^{\alpha}}\ du \quad . \tag{4.15}$$

If it is assumed now that $\theta \neq \frac{\pi}{2}$ (see note (1) after equation (4.16) for the case $\theta = \frac{\pi}{2}$), then for $\alpha > 0$ and α non-integral, the integral here can be integrated by parts till arrival at the first integer k such that $k-\alpha > 0$ i.e. k integrations by parts are performed. After this point, the final integral remaining will be of the form

$(\text{constant}) . \int_{s_0/r}^{\infty} u^{k-a} \frac{d^k}{du^k} H_L(\cos \theta, \sin \theta; u) du$. This remainder is $O(1)$ as $r \to \infty$ as is seen by writing $\int_{s_0/r}^{\infty} = \int_0^{\infty} - \int_0^{s_0/r}$.

The integrated contributions to these integrations by parts will be of the form

$$(\text{constant}) . u^{m-\alpha} \frac{d^{m-1} H_L}{du^{m-1}} \qquad (1 \leq m \leq k) \quad .$$

These vanish at ∞ since $\frac{d^{m-1} H_L}{du^{m-1}} = O(\frac{1}{u^m})$ as $u \to \infty$ so the only contributions will come from the lower limit s_0/r. Since H_L is an odd function of u, it follows that

$$\frac{d^{m-1} H_L}{du^{m-1}} = \begin{cases} O(u) & \text{if m is odd} \\ O(1) & \text{if m is even} \end{cases} \quad \text{as } u \to 0$$

whence the contributions from the lower limits will be

$$\begin{cases} O\left(\frac{1}{r^{m+1-\alpha}}\right) & \text{if m is odd} \\ O\left(\frac{1}{r^{m-\alpha}}\right) & \text{if m is even} \end{cases} \quad \text{as } r \to \infty \quad .$$

It is recalled that there was a factor $r^{1-\alpha}$ outside the original integral (see (4.15)) so that (when multiplied by $r^{1-\alpha}$) the integrated contributions will be $\begin{cases} O(1/r^m) & m \text{ odd} \\ O(1/r^{m-1}) & m \text{ even} \end{cases}$ while the remainder will be $O(r^{1-\alpha})$.

Hence, for α non-integral, $\alpha > 0$

$$\frac{r^{1-\alpha}}{2\pi} \int_0^\infty \log \left[\frac{u^2 + 2u \sin \theta + 1}{u^2 - 2u \sin \theta + 1}\right] \frac{1}{u^\alpha} \, du = \begin{cases} O(r^{1-\alpha}) & 0 < \alpha < 2 \\ O(1/r) & \alpha > 2 \end{cases} .$$

From (4.14) and the comment below there, it follows that as $r \to \infty$

$$L(r \cos \theta, r \sin \theta) = \begin{cases} O(r^{1-\alpha}) & 0 < \alpha < 2 \\ O(1/r) & \alpha > 2 \end{cases}$$

when α is non-integral.

When α is an integer, Lemma 1 (see Appendix B,§B.2) gives the corresponding results.

With $f(\theta,t) = \log \left[\frac{t^2 + 2t \sin \theta + 1}{t^2 - 2t \sin \theta + 1}\right]$, $x = s_0/r$, Lemma 1 gives, as $r \to \infty$ (with $f(o) = 0$)

$$\int_{s_0/r}^\infty \frac{f(\theta(t))}{t^\alpha} \, dt = \begin{cases} O(1) & \alpha = 1 \\ O(\log r) & \alpha = 2 \\ O(r^{\alpha-2}) & \alpha = 3,\ 4,\ 5,\ \ldots \end{cases} .$$

Hence

$$\frac{r^{1-\alpha}}{2\pi} \int_{s_0/r}^\infty \frac{f(\theta,t)}{t^\alpha} \, dt = \begin{cases} O(1) & \alpha = 1 \\ O(\frac{\log r}{r}) & \alpha = 2 \\ O(1/r) & \alpha = 3,\ 4,\ 5,\ \ldots \end{cases} .$$

Combining the results for α non-integral and α integral together gives

$$\frac{r^{1-\alpha}}{2\pi} \int_{s_0/r}^\infty \frac{f(\theta,t)}{t^\alpha} \, dt = \begin{cases} O(r^{1-\alpha}) & 0 < \alpha < 2 \\ O(\log r/r) & \alpha = 2 \\ O(1/r) & \alpha > 2 \end{cases} .$$

Hence from (4.14), as $r \to \infty$

$$L(r\cos\theta, r\sin\theta) = \begin{cases} O(r^{1-\alpha}) & 0 < \alpha < 2 \\ O(\log r/r) & \alpha = 2 \\ O(1/r) & \alpha > 2 \end{cases} \qquad (4.16)$$

[NOTE (1): In the proof which follows (4.15) it has been assumed that $\theta \neq \frac{\pi}{2}$. If, however, $\theta = \frac{\pi}{2}$, the integral in (4.15) takes the form

$$2\int_{s_0/r}^{\infty} \log\left|\frac{u+1}{u-1}\right| \cdot \frac{1}{u^{\alpha}}\, du$$

in which the log function has a singularity at $u = 1$ which becomes non-integrably singular (when r is large) after two differentiations of the log function. However, this difficulty can be overcome as follows. By integration by parts ($\alpha \neq 1$)

$$\int_{s_0/r}^{\infty} \log\left|\frac{u+1}{u-1}\right| \frac{1}{u^{\alpha}}\, du = \left[\frac{u^{1-\alpha}}{1-\alpha}\log\left|\frac{u+1}{u-1}\right|\right]_{s_0/r}^{\infty}$$

$$- \fint_{s_0/r}^{\infty} u^{1-\alpha}\left[\frac{1}{u+1} - \frac{1}{u-1}\right] \quad \text{(a Cauchy P.V. integ}$$

The second integral may be expressed as

$$\pi i + \int_{s_0/r}^{\infty} u^{1-\alpha}\left[\frac{1}{u+1} - \frac{1}{u-1}\right] du$$ where the contour is the real axis indented by a small semi-circular arc (centred on $u=1$ and lying in the fourth quadrant) and the previous arguments can now be applied to the indented integral. Since the integral under consideration is the real part of the indented integral, plus the integrated contribution to the first integration by parts above, its order properties will be the same as before.]

Next

$$I(r\cos\theta, r\sin\theta; k) = \int_0^{\infty} H_I(r\cos\theta, r\sin\theta; k; s) f(s)\, ds$$

where

$H_I(r\cos\theta, r\sin\theta; k; s)$

$$= -\frac{2}{\pi}\int_0^{\infty} \frac{u\cos[u(r\sin\theta + s)] - k\sin[u(r\sin\theta + s)]}{u^2 + k^2} e^{-u R\cos\theta}\, du$$

ie $H_I(r\cos\theta, r\sin\theta; k; s) = -\frac{2}{\pi}\mathrm{Re}_j \int_0^\infty \frac{e^{-u\zeta}}{u - jk}\, du$ with $\zeta = re^{-j\theta} - js$.

For large r, the dominant term in H_I is $-\frac{2}{\pi}\mathrm{Re}_j\left[\frac{j}{k\zeta}\right]$ (by Watson's Lemma or straightforward integration by parts).

But $\frac{1}{\zeta} = \frac{r\cos\theta + j(r\sin\theta + s)}{r^2 + 2sr\sin\theta + s^2}$ so that

$$-\frac{2}{\pi}\mathrm{Re}\left[\frac{j}{k\zeta}\right] = \frac{2}{\pi k}\;\frac{r\sin\theta + s}{r^2 + 2sr\sin\theta + s^2} .$$

The leading asymptotics of $I(r\cos\theta, r\sin\theta; k)$ will, therefore, arise from

$$I_A(r\cos\theta, r\sin\theta; k) = \frac{2}{\pi k}\int_0^\infty \frac{r\sin\theta + s}{r^2 + 2sr\sin\theta + s^2} f(s)\, ds, \quad \text{or}$$

$$I_A(r\cos\theta, r\sin\theta; k) = \frac{2}{\pi k}\left(\int_0^{s_0} + \int_{s_0}^\infty\right)\left(\frac{r\sin\theta + s}{r^2 + 2\,sr\sin\theta + s^2}\right) f(s)\, ds . \quad (4.17$$

The first integral is $O(\frac{1}{r})$ as $r \to \infty$ and the substitution $s = ru$ in the second gives

$$\int_{s_0/r}^\infty \frac{\sin\theta + u}{1 + 2u\sin\theta + u^2} f(ru)\, du$$ which, in modulus, is less or equal to

$$\frac{A}{r^\alpha}\int_{s_0/r}^\infty \frac{\sin\theta + u}{1 + 2u\sin\theta + u^2}\cdot\frac{1}{u^\alpha}\, du$$ (using the order property of f).

Let $g(\theta, u) = \dfrac{\sin\theta + u}{1 + 2u\sin\theta + u^2}$.

Clearly

$$\frac{d^{m-1}}{du^{m-1}} g(\theta, u) = O\left(\frac{1}{u^m}\right)$$ as $u \to \infty$, while g and its derivatives are all O(1) as $u \to 0$. (g has no special odd or even properties).

Hence it can be proved, as before, by successive integrations by parts, that for α non-integral

$$\frac{1}{r^\alpha}\int_{s_0/r}^\infty \frac{g(\theta,u)}{u^\alpha}\, du = \begin{cases} O(\frac{1}{r^\alpha}) & 0 < \alpha < 1 \\ O(\frac{1}{r}) & \alpha > 1 . \end{cases}$$

For integral values of α, lemma 1 again gives the results:

$$\frac{1}{r^\alpha}\int_{s_0/r}^\infty \frac{g(\theta,u)}{u^\alpha}\, du = \begin{cases} O(\log r/r^\alpha) & \alpha = 1 \\ O(1/r) & \alpha = 2, 3, 4, \ldots . \end{cases}$$

Combining the above gives

$$\frac{1}{r^{\alpha}}\int_{s_0/r}^{\infty}\frac{g(\theta,u)}{u^{\alpha}}\,du = \begin{cases} O(1/r^{\alpha}) & 0<\alpha<1 \\ O(\log r/r) & \alpha=1 \\ O(1/r) & \alpha>1 \end{cases}.$$

Use of the above results and (4.17) implies that

$$I_A(r\cos\theta, r\sin\theta;k) = \begin{cases} O(1/r^{\alpha}) & 0<\alpha<1 \\ O(\log r/r) & \alpha=1 \\ O(1/r) & \alpha>1 \end{cases}.$$

Since I_A contains the leading asymptotic terms of I it is deduced that I has the same properties.

$$I(r\cos\theta, r\sin\theta;k) = \begin{cases} O(1/r^{\alpha}) & 0<\alpha<1 \\ O(\log r/r) & \alpha=1 \\ O(1/r) & \alpha>1 \end{cases}. \tag{4.18}$$

Addition of (4.16) and (4.18) leads to the result that the wave-free part of Havelock's solution has the properties

$$F(r\cos\theta, r\sin\theta;k) = \begin{cases} O(r^{1-\alpha}) & 0<\alpha<2 \\ O(\log r/r) & \alpha=2 \\ O(1/r) & \alpha>2 \end{cases} \tag{4.19}$$

as $r \to \infty$.

Hence, in general, up to $\alpha = 2$ (inverse square decay of the prescribed velocity) the wave-free behaviour in the far field is determined crucially by the motion of the vibrating wave maker deep down in the fluid, while for $\alpha > 2$, the disturbances are like those due to a dipole at the top of the wavemaker. It may be seen also from (4.19) that the condition $f(s) = O\left(\frac{1}{s^{\alpha}}\right)$ $(\alpha > 1)$ is sufficient to ensure that a solution is obtained with only outgoing waves at infinity. For $\alpha \leqslant 1$ the wave-free part will not, in general, tend to zero.

[NOTE (2): On the energy input required to maintain the motion of the vertical wave-maker

The horizontal force per unit of surface area at depth y necessary to maintain the prescribed motion of the barrier is equal to p, where

p = pressure in the liquid at depth y on $x = 0$. Hence the rate of energy input per unit of surface area at depth y is

$pF_x(0,y)$ where $F(x,y;t)$ is the total potential

i.e. $F(x,y;t) = \text{Re}\,[\,\phi(x,y)e^{-i\sigma t}]$.

Hence it follows that the energy input per unit of surface area at depth y over one cycle will be

$$E(y) = \int_0^{2\pi/\sigma} pF_x(0,y)\,dt \; ,$$

The linearised Bernouilli equation gives

$$p = p_0 + \rho g y - \rho F_t(0,y)$$

where p_0 = atmospheric pressure

and ρ = density of liquid

so that use of the periodic property of $F_x(0,y)$ gives

$$E(y) = -\rho \int_0^{2\pi/\sigma} F_t(0,y)F_x(0,y)\,dt \; .$$

Since $F_t(0,y) = \text{Re}\,[-i\sigma\,\phi(0,y)e^{-i\sigma t}]$ and

$F_x(0,y) = \text{Re}\,[\,\phi_x(0,y)e^{-i\sigma t}]$ it can be seen that

$$|E(y)| \leqslant 2\pi\rho\,|\phi(0,y)|\,|f(y)| \qquad (\phi_x(0,y) = f(y))$$

so that, by the results of the previous section

$$E(y) = \begin{cases} O(y^{1-2\alpha}) & 0 < \alpha < 2 \\ O\left(\dfrac{\log y}{y^{1+\alpha}}\right) & \alpha = 2 \\ O\left(\dfrac{1}{y^{1+\alpha}}\right) & \alpha > 2 \end{cases} \qquad \text{as } y \to \infty \; .$$

Thus the total energy input in one cycle (which is equal to $\int_0^\infty E(y)\,dy$) will be finite provided $\alpha > 1$ since the integral is convergent only for this range of values.

NOTE (3): Havelock's particular solution is continuous at the origin and satisfies the edge condition $r \frac{\partial P}{\partial r} \to 0$ as $r \to 0$.]

§4.4. Particular solutions of the wave-maker problem in cases of unbounded velocity profiles

Consideration is now given to two cases of the above type which occur in Chapter 5. For easier comparison later, the problem is stated in terms of scaled coordinates $X = kx$ and $Y = ky$ as

$$U_{XX} + U_{YY} = 0 \qquad (X > 0,\ Y > 0)$$

$$U + U_Y = 0 \qquad (Y = 0)$$

$$U_X = V \qquad (X = 0)$$

together with the edge condition (required by the inner expansions)

$$R \frac{\partial U}{\partial R} \to 0 \quad \text{as } R = \sqrt{X^2 + Y^2} \to 0 \ .$$

The two cases considered here are

(a) $V(Y) = Y$

(b) $V(Y) = Y \log Y$

the method employed being essentially the reduction method due to Lewy (1946). The problem is first restated in terms of a new harmonic function $N(X,Y)$ defined by

$$N(X,Y) = \frac{\partial}{\partial X}\left(\frac{\partial}{\partial Y} + 1\right)U(X,Y) \text{ so that on the free surface } N = 0$$

and on the wave-maker $N(0,Y) = V(Y) + V'(Y)$. This is a simpler problem which in the cases considered here yielded a particular solution for N without difficulty, say $N_p(X,Y)$. Once this has been extracted, it is convenient to consider $U(X,Y)$ as the real part of a complex potential $w(z)$ $(z = X + jY)$ so that

$$\frac{\partial}{\partial X}\left(\frac{\partial}{\partial Y} + 1\right)U(X,Y) = \mathrm{Im}_j\,[-w''(z) + jw'(z)] \ .$$

If also $N_p(X,Y)$ is written as $\mathrm{Im}_j[g(z)]$ then a particular solution for w can be found by solving the ordinary differential equation

$$w''(z) - jw'(z) = -g(z) \ .$$

This general method will now be applied to the two particular cases (a) and (b) mentioned above.

(a) In this case the value of N on the wave-maker is 1 + Y so it is easily seen by inspection (or separation of variables) that, with $X = R\cos\theta$, $Y = R\sin\theta$, a particular solution $N_p(X,Y)$ is given by

$$N_p(X,Y) = \frac{2}{\pi}\theta + R\sin\theta \text{ i.e.}$$

$$N_p(X,Y) = \mathrm{Im}_j\left[\frac{2}{\pi}\log z + z\right] \ .$$

Hence, in this case, it is necessary to solve the differential equation

$$w''(z) - jw'(z) = -\left(\frac{2}{\pi}\log z + z\right) \ .$$

One integration gives immediately

$$w'(z) - jw(z) = -\left[\frac{2}{\pi}(z\log z - z) + \frac{z^2}{2}\right] + A \qquad (A = \text{constant})$$

where it is noted that A must be real with respect to j, in order that the free surface condition (which in complex form is $\mathrm{Im}_j[w'(z) - jw(z)] = 0$ on $Y = 0$) be satisfied.

It follows by use of the integrating factor e^{-jz} that

$$w(z) = e^{jz}\int_z^{-j\infty} e^{-js}\left[\frac{2}{\pi}(s\log s - s) + \frac{s^2}{2}\right]ds + jA + Be^{jz}$$

where B is a constant.

Two integrations by parts give the more manageable form

$$w(z) = -j\left[\frac{2}{\pi}(z\log z - z) + \frac{z^2}{2}\right] - \left(z + \frac{2}{\pi}\log z\right) + j(A+1)$$
$$- \frac{2}{\pi}e^{jz}E_1(jz) + Be^{jz} \qquad (4.20)$$

whence $w'(z) = -j\left(\frac{2}{\pi}\log z + z\right) - \left(1 + \frac{2}{\pi z}\right) - \frac{2}{\pi}\frac{d}{dz}\left[e^{jz}E_1(jz)\right] + jBe^{jz}$.

It is proved in the following section §4.5 (equation (4.36)) that on $X = 0$, $\mathrm{Re}_j\left\{\frac{d}{dz}\left[e^{jz}E_1(jz)\right]\right\} = \pi e^{-Y}$ so that, again on $X = 0$,

$$\mathrm{Re}_j[w'(z)] = Y - 2e^{-Y} + e^{-Y}\,\mathrm{Re}_j(jB)$$

Thus, to satisfy $U_X = Y$ on $X = 0$, B may be taken to be $-2j$.

It follows then from (4.20) (if A is taken to be -1) that a particular solution for U(X,Y) in this case will be the real part of the function

$$w(z) = -j[\frac{2}{\pi}(z \log z - z) + \frac{z^2}{2}] - (z + \frac{2}{\pi} \log z) - \frac{2}{\pi}e^{jz} E_1(jz) - 2je^{jz}$$

i.e.

$$U(X,Y) = (\frac{R^2\sin 2\theta}{2} - R \cos \theta) + \frac{2}{\pi}[R(\sin \theta \log R + \theta \cos \theta) - R \sin \theta - \log$$

$$- \frac{2}{\pi} \{Re_j[e^{jz} E_1(jz)] - \pi e^{-Y} \sin X\} .$$

To this particular solution may be added solutions of the vertical barrier problem as required. Thus, in the case which occurs in Chapter 5, §5.7, it is convenient for matching purposes to add the eigensolution $\frac{2}{\pi}(R \sin \theta - 1)$ while to obtain a progressive wave at infinity the standing wave solution $-2i\, e^{-Y}\cos X$ is also added. This gives a modified solution:

$$U(X,Y) = (\frac{R^2\sin 2\theta}{2} - R \cos \theta) + \frac{2}{\pi}[R(\sin \theta \log R + \theta \cos \theta) - 1 - \log R]$$

$$- \frac{2}{\pi}\{Re_j[e^{jz} E_1(jz)]\} - 2i\, e^{iX-Y} .$$

This also satisfies the edge condition, since

$Re_j[e^{jz} E_1(jz)] \sim -\log R$ as $R \to 0$, as may be seen from the result

$$E_1(jz) = -\gamma - \log jz - \sum_{n=1}^{\infty} \frac{(-1)^n (jz)^n}{n\, n!}$$ given in Abramowitz & Stegun (1965). Thus the log terms cancel and U(X,Y) is bounded as $R \to 0$.

In conclusion it is remarked that the function ψ_2 required in §5.7 has the property

$$\psi_{2X} = - \frac{1}{\pi a^4} Y \quad \text{on the barrier}$$

and that its wave-free part $F_2(X,Y)$ satisfies

$$F_2(X,Y) \sim - \frac{1}{\pi a^4} (\frac{R^2\sin 2\theta}{2} - R \cos \theta) - \frac{2R}{\pi^2 a^4}(\sin \theta \log R + \theta \cos \theta)$$

$$+ \frac{2R \sin \theta}{\pi^2 a^4} (2 \log 2a + \gamma - 2 - i \frac{\pi}{8}) \quad \text{as } R \to \infty .$$

Hence the solution satisfying the edge condition will be

$$- \frac{1}{\pi a^4} U(X,Y) + \frac{2}{\pi^2 a^4}(2 \log 2a + \gamma - 2 - i \frac{\pi}{8})(R \sin \theta - 1)$$

$$\text{i.e. } \psi_2(X,Y) = - \frac{1}{\pi a^4} (\frac{R^2 \sin 2\theta}{2} - R \cos \theta) - \frac{2}{\pi^2 a^4} [R(\sin \theta \log R + \theta \cos \theta) - 1 - \log R] + \frac{2}{\pi^2 a^4}(2 \log 2a + \gamma - 2 - i \frac{\pi}{8})(R \sin \theta - 1) + \frac{2}{\pi^2 a^4} \{Re_j[e^{jz} E_1(jz)]\} + \frac{2i}{\pi a^4} e^{iX-Y} .$$

(b) With $V(Y) = Y \log Y$ the value of N on the wave-maker is $Y \log Y+1+\log Y$

and, as before, a particular solution for N is

$$N_p(X,Y) = R(\sin \theta \log R + \theta \cos \theta) + \frac{2}{\pi}(\theta + \theta \log R) \text{ or}$$

$$N_p(X,Y) = Im_j\{z \log z + \frac{2}{\pi}[\log z + \frac{1}{2}(\log z)^2]\} .$$

Hence, it is required to solve the differential equation

$$w''(z) - jw'(z) = -\{z \log z + \frac{2}{\pi}[\log z + \frac{1}{2}(\log z)^2]\} .$$

The solution (obtained as in (a)) is

$$w(z) = e^{jz} \int_z^{-j\infty} e^{-js} [\frac{s^2}{2} \log s - \frac{s^2}{4} + \frac{1}{\pi} s(\log s)^2] ds + jA + Be^{jz}$$

where A, B are constants and A is real with respect to j.

An integration by parts now gives

$$w(z) = -j[\frac{z^2}{2} \log z - \frac{z^2}{4} + \frac{1}{\pi} z(\log z)^2] + e^{jz} \int_z^{-j\infty} -je^{-js}(s \log s + \frac{2}{\pi} \log s)ds - \frac{j}{\pi} e^{jz} \int_z^{-j\infty} e^{-js}(\log s)^2 ds + jA + Be^{jz} .$$

This first integral occurring here is integrated by parts again but the latter integral is left as it is to avoid introduction of an awkward singularity at the origin. This leads to the result

$$w(z) = -j[\frac{z^2}{2} \log z - \frac{z^2}{4} + \frac{1}{\pi} z(\log z)^2] - (z \log z + \frac{2}{\pi} \log z) - \frac{2}{\pi} e^{jz} E_1(jz) - e^{jz} \int_z^{-j\infty} e^{-js} \log s \, ds - \frac{j}{\pi} e^{jz} \int_z^{-j\infty} e^{-js}(\log s)^2 ds + j(A + 1) + Be^{jz} . \qquad (4.21)$$

Clearly w is non-singular at the origin (the integrands above can be integrated through O) and the edge condition is therefore satisfied. It remains to choose B so that

$$\mathrm{Re}_j\,[\,w'(z)] = Y \log Y \text{ on } X = 0.$$

It can be shown without difficulty that

$$\mathrm{Re}_j\,[\,w'(jY)] = Y \log Y - 2e^{-Y} + e^{-Y}\,\mathrm{Im}_j\,[\int_{jY}^{-j\infty} e^{-js} \log s\, ds]$$
$$+ \frac{1}{\pi} e^{-Y}\,\mathrm{Re}_j\,[\int_{jY}^{-j\infty} e^{-js} (\log s)^2\, ds] + e^{-Y}\,\mathrm{Re}_j\,(jB)$$

whence the substitution $u = js$ in the integrals gives

$$\mathrm{Re}_j\,[w'(jY)] = Y \log Y - 2e^{-Y} + e^{-Y}\,\mathrm{Im}_j\,[\int_{-Y}^{\infty} e^{-u} \log(-uj)\,\frac{du}{j}\,]$$
$$+ \frac{1}{\pi} e^{-Y}\,\mathrm{Re}_j\{\int_{-Y}^{\infty} e^{-u}[\log(-uj)]^2\,\frac{du}{j}\} + e^{-Y}\,\mathrm{Re}_j\,(jB)\,. \quad (4.22)$$

The contours in the above integrals are taken to be along the real axis (strictly speaking, along the real axis indented by a small semi-circle centred at the origin but the contribution from there will tend to zero with the radius) and $\int_{-Y}^{\infty}$ is written as $\int_{-Y}^{0-} + \int_{0+}^{\infty}$.

In $(-Y,0)$ $\log(-uj) = \log(-u) + j\,\frac{\pi}{2}$ while

in $(0,\infty)$ $\log(-uj) = \log u - j\,\frac{\pi}{2}$.

Hence the imaginary part on the right-hand side of (4.22) is equal to

$$-\int_{-Y}^{0} e^{-u} \log(-u)\,du - \int_{0}^{\infty} e^{-u} \log u\, du$$

and the real part is $\pi[\int_{-Y}^{0} e^{-u} \log(-u)\,du - \int_{0}^{\infty} e^{-u} \log u\, du]$.

Substitution of these values in (4.22) now gives

$\mathrm{Re}_j\,[w'(jY)] = Y \log Y + 2(\gamma - 1)e^{-Y} + e^{-Y}\,\mathrm{Re}_j\,(jB)$ where γ is Euler's constant (since $\int_0^{\infty} e^{-u} \log u\, du = -\gamma$) and if B is chosen to have the value $2j(\gamma - 1)$ then the real part of (4.21) will be a particular solution of the problem. To this may be added solutions of the vertical barrier problem as required.

In particular, as it stands, the real part of (4.21) contains the standing wave term

$$-2(\gamma - 1)e^{-Y} \sin X .$$

Thus, if a progressing wave were required, it would be necessary to add the term $2i(\gamma - 1)e^{-Y} \cos X$ whence it is seen that the velocity profile $Y \log Y$ produces a progressing wave $2i(\gamma - 1)e^{iX-Y}$. It follows that the term $\frac{2}{\pi^2 a^5}$ $Y \log Y$ which appears in the velocity profile for ψ_4 (see equation (5.64)) will produce the progressing wave $\frac{4i(\gamma - 1)}{\pi^2 a^5} e^{iX-Y}$ as stated there.

<u>NOTE</u>: The terms $T_k(z) = e^{jz} \int_z^{-j\infty} e^{-js}(\log s)^k \, ds$ $(k = 1,2)$ in (4.21) do not contain standing waves as may be seen by considering their behaviour on $Y = 0$ as $X \to \infty$.

Indeed $T_k(X) = e^{jX} \int_X^{-j\infty} (\log s)^k \, e^{-js} \, ds$ whence rotation of the contour of integration so that the upper limit becomes $X - j\infty$ followed by the substitution $w = \frac{j(s - X)}{X}$ gives the form

$$T_k(X) = -jX \int_0^{\infty} [\log X + \log(1 - jw)]^k \, e^{-Xw} \, dw .$$

The constituent parts of the $T_k(X)$ are either unbounded or tend to zero as $X \to \infty$ so clearly they cannot represent standing waves (which are $O(1)$ as $X \to \infty$).

§4.5. <u>The vertical barrier problem</u>

With reference to §4.4 this is simply the case $V(Y) = 0$. It will be shown that the general solution is precisely the family of linear combinations of the following:

(i) a standing wave solution, non-singular at O

$$e^{-Y} \cos X$$

(ii) a set of solutions unbounded at ∞ and non-singular at O of the form $\frac{R^{2m+1} \sin(2m + 1)\theta}{(2m + 1)} - R^{2m} \cos 2m\theta \quad (m=0,1,2,\ldots)$

(iii) a solution representing a standing wave at ∞ and having a logarithmic singularity at O

$$\mathrm{Re}_j\,[e^{jz}\,E_1(jz)] - \pi e^{-Y}\sin X$$

(where $E_1(z)$ is the exponential integral

$$E_1(z) = \int_z^\infty \frac{e^{-u}}{u}\,du\;)$$

(iv) a set of solutions tending to zero at ∞ and having algebraic singularities at O of the form

$$\frac{\sin(2m-1)\theta}{(2m-1)R^{2m-1}} - \frac{\cos 2m\theta}{R^{2m}} \qquad (m=1,2,\ldots)\;.$$

These functions are the real parts (with respect to j) of the complex valued functions

(i) e^{jz}

(ii) $\dfrac{z^{2m+1}}{j(2m+1)} - z^{2m}$

(iii) $e^{jz}\,E_1(jz) + j\pi e^{jz}$

(iv) $\dfrac{j}{(2m-1)z^{2m-1}} - \dfrac{1}{z^{2m}}$ where $z = X + jY = Re^{j\theta}$.

The solution will be obtained explicitly using the reduction method of §4.5 (and the same notation). In this case N(X,Y) vanishes on $\theta = 0$ and $\theta = \frac{\pi}{2}$ so that

$$N(X,Y) = \sum_{n=1}^{\infty}(A_{2n}R^{2n} + \frac{B_{2n}}{R^{2n}})\sin 2n\theta \tag{4.23}$$

or $$N(X,Y) = \mathrm{Im}_j \sum_{n=1}^{\infty}(A_{2n}z^{2n} - \frac{B_{2n}}{z^{2n}})\;.$$

Thus it is necessary to solve the differential equation

$$\mathrm{Im}_j[w''(z) - jw'(z)] = -\mathrm{Im}_j[\sum_{n=1}^{\infty}(A_{2n}z^{2n} - \frac{B_{2n}}{z^{2n}})]\;. \tag{4.24}$$

The equation $g_{2n}''(z) - jg_{2n}'(z) = z^{2n} \qquad (n \geqslant 1)$ (4.25)

is considered first. It is not difficult to show that the function

$$h(z) = \frac{g_{2n}(z) - [z^{2n} + \frac{jz^{2n+1}}{2n+1}]}{-2n(2n-1)}$$ satisfies the equation

$$h''(z) - jh'(z) = z^{2n-2} \text{ whence } h(z) = g_{2n-2}(z),$$

and $$g_{2n}(z) = z^{2n} + \frac{jz^{2n+1}}{2n+1} - 2n(2n-1)g_{2n-2}(z) \qquad (n \geqslant 1) . \tag{4.26}$$

In addition a particular solution of the equation

$$g_0''(z) - jg_0'(z) = 1 \text{ is}$$

$$g_0(z) = jz + 1 .$$

Hence, by induction, a particular solution of (4.25) can be obtained as a linear combination of the functions

$$E_m(z) = z^{2m} + \frac{jz^{2m+1}}{2m+1} \qquad (0 \leqslant m \leqslant n)$$

the coefficients of the $E_m(z)$ being well-defined non-zero integers i.e. there exists a solution

$$g_{2n}(z) = \sum_{m=0}^{n} \alpha_m(n) E_m(z) \tag{4.27}$$

where $\alpha_n(n) = 1$ and no $\alpha_m(n)$ is zero.

Next the equation

$$h_{2n}''(z) - jh_{2n}'(z) = \frac{1}{z^{2n}} \qquad (n \geqslant 1) \tag{4.28}$$

is considered.

Again it is easily proved that for $n \geqslant 2$, the function

$$F(z) \equiv -(2n-2)(2n-1)h_{2n}(z) + \left[\frac{1}{z^{2n-2}} - \frac{j}{(2n-3)z^{2n-3}}\right]$$

satisfies the equation $F''(z) - jF'(z) = \frac{1}{z^{2n-2}}$ so that, for $n \geqslant 2$,

$$h_{2n}(z) = \frac{1}{(2n-2)(2n-1)}\left\{\left[\frac{1}{z^{2n-2}} - \frac{j}{(2n-3)z^{2n-3}}\right] - h_{2n-2}(z)\right\} . \tag{4.29}$$

Also $h_2(z)$ satisfies the equation

$$h_2''(z) - jh_2'(z) = \frac{1}{z^2}$$

so that a particular solution is therefore

$$h_2(z) = e^{jz} \int_z^{-j\infty} \frac{e^{-jt}}{t}\, dt \qquad . \tag{4.30}$$

It follows as previously that a particular solution of (4.28) can be obtained in the form

$$h_{2n}(z) = \sum_{m=0}^{n-1} b_m(n) G_m(z) \tag{4.31}$$

where the $b_m(n)$ are real non-zero integers and

$$\begin{cases} G_0(z) = e^{jz} E_1(jz) \\ G_m(z) = \dfrac{j}{(2m-1)z^{2m-1}} - \dfrac{1}{z^{2m}} \qquad (m \geqslant 1) \end{cases} \tag{4.32}$$

After the derivation of these particular solutions attention is returned to (4.24). In practice the sum on the right is not infinite (the number of terms is limited by order properties of ϕ at infinity and prescribed behaviour at the origin) so that

$\sum_{n=1}^{\infty} (A_{2n} z^{2n} - \frac{B_{2n}}{z^{2n}})$ is an analytic function of z ($z \neq 0$). The Cauchy-Riemann equations imply, therefore, that w must be analytic ($z \neq 0$) and w must satisfy the linear differential equation

$$w''(z) - jw'(z) = \sum_{n=1}^{\infty} (-A_n z^{2n} + \frac{B_{2n}}{z^{2n}}) + C$$ (where C is a constant real with respect to j).

If the operator $\dfrac{1}{D^2 - jD}$ $(D = \frac{d}{dz})$ is applied to both sides of the above equation a particular solution may be obtained in the form

$$\sum_{n=1}^{\infty} [- A_{2n} (\sum_{m=0}^{n} a_m(n) E_m(z)) + B_{2n} (\sum_{m=0}^{n-1} b_m(n) G_m(z))] + C(jz + 1)$$

(where use has been made of (4.27) and (4.31)) so that the general solution will be

$$w(z) = \sum_{n=1}^{\infty} [-A_{2n} (\sum_{m=0}^{n} a_m(n) E_m(z)) + B_{2n} (\sum_{m=0}^{n-1} b_m(n) G_m(z))] + C(jz+1) + E + Fe^{jz} \tag{4.33}$$

where E, F are constants.

The boundary conditions at the free surface and on the barrier are now checked in their complex forms:

$$\mathrm{Im}_j [f'(z) - jf(z)] = 0 \quad \text{on } Y = 0$$

and $\mathrm{Re}_j [f'(z)] = 0$ on $X = 0$.

It is easy to prove that the $E_m(z)$ and the $G_m(z)$ satisfy the free surface condition separately so that, when it is recalled that A_{2n}, B_{2n}, $a_m(n)$, $b_m(n)$, C are real with respect to j, it can be seen from (4.33) that

$$\mathrm{Im}_j\,[w'(z) - jw(z)] = -\mathrm{Im}_j(jE) \text{ whence}$$

$$\mathrm{Re}_j(E) = 0 \quad . \tag{4.34}$$

From (4.33) again

$$w'(z) = \sum_{n=1}^{\infty} [-A_{2n}(\sum_{m=0}^{n} a_m(n)E_m'(z)) + B_{2n}(\sum_{m=0}^{n-1} b_m(n)G_m'(z))] + jC + jFe^{jz}. \tag{4.35}$$

On X = 0 the $E_m'(z)$ are purely imaginary as are the $G_m'(z)$ for $m \geq 1$. However, $G_0'(z)$ must be more carefully considered.

(4.32) shows that

$$G_0'(z) = je^{jz}\int_z^{-j\infty} \frac{e^{-jt}}{t}\,dt - \frac{1}{z} \quad \text{so that}$$

$$G_0'(jY) = je^{-Y}\int_{jY}^{-j\infty} \frac{e^{-jt}}{t}\,dt - \frac{1}{jY}$$ (where the contour must avoid the origin and does not cross the negative real axis).

The substitution u = jt gives further

$$G_0'(jY) = je^{-Y}\int_{-Y}^{\infty} \frac{e^{-u}}{u}\,du - \frac{1}{jY}$$ where the contour here is chosen to be the real axis from -Y to $+\infty$ indented by a small semi-circular arc round the origin in the half-plane Im(u) > 0. Hence

$$G_0'(jY) = je^{-Y}(\fint_{-Y}^{\infty} \frac{e^{-u}}{u}\,du - j\pi) - \frac{1}{jY} \quad \text{whence}$$

$$\mathrm{Re}_j[G_0'(jY)] = \pi e^{-Y} \quad \text{i.e. from (4.32) } \mathrm{Re}_j\{\frac{d}{dz}[e^{jz}E_1(jz)]\} = \pi e^{-Y}$$

$$\text{on } X = 0 \,. \tag{4.36}$$

[It may be noted that the above can also be proved by writing

$$G_0'(jY) = je^{-Y}[-\mathrm{Ci}(jY) + j\,\mathrm{si}(jY)] - \frac{1}{jY}$$

and using the results 5.2.5, 5.2.22 and 5.2.24 in Abramowitz and Stegun (1965, p.232).]

(4.35) now gives the result

$$\mathrm{Re}_j[w'(jY)] = \sum_{n=1}^{\infty} B_{2n}\, b_0(n)\, \pi e^{-Y} - \mathrm{Im}_j(F)e^{-Y}$$

so that

$$\mathrm{Im}_j\,(F) = \pi \sum_{n=1}^{\infty} B_{2n}\, b_0(n)\ .$$

This relation together with (4.34) and the definitions of $E_m(z)$, $G_m(z)$ imply from (4.33) that

$$\begin{aligned}\mathrm{Re}_j[w(z)] = {} & \sum_{n=1}^{\infty} A_{2n}\Big[\sum_{m=0}^{n} \alpha_m(n)\Big(\frac{R^{2m+1}\sin(2m+1)\theta}{2m+1} - R^{2m}\cos 2m\theta\Big)\Big] \\ & + \sum_{n=1}^{\infty} B_{2n}\Big[b_0(n)\mathrm{Re}_j(G_0(z)) + \sum_{m=1}^{n-1} b_m(n)\Big(\frac{\sin(2m-1)\theta}{(2m-1)R^{2m-1}} - \frac{\cos 2m\theta}{R^{2m}}\Big)\Big] \\ & + C(1 - R\sin\theta) + e^{-Y}\Big[\mathrm{Re}_j(F)\cos X - \pi\sum_{n=1}^{\infty} B_{2n}\, b_0(n)\sin X\Big]\ .\end{aligned}$$

If A_0 is defined to be equal to $-C$ and the last term above is drawn under the second summation sign, the solution for U is seen to be

$$\begin{aligned}U(X,Y) = {} & \sum_{n=0}^{\infty} A_{2n}\Big[\sum_{m=0}^{n} \alpha_m(n)\Big(\frac{R^{2m+1}\sin(2m+1)\theta}{2m+1} - R^{2m}\cos 2m\theta\Big)\Big] \\ & + \sum_{n=1}^{\infty} B_{2n}\Big[b_0(n)(\mathrm{Re}_j(G_0(z)) - \pi e^{-Y}\sin X) + \sum_{m=1}^{n-1} b_m(n)\Big(\frac{\sin(2m-1)\theta}{(2m-1)R^{2m-1}} - \frac{\cos 2m\theta}{R^{2m}}\Big)\Big] \\ & + De^{-Y}\cos X\end{aligned}$$

where D is an arbitrary constant (independent of j but possibly depending on i) and $\sum_{m=1}^{0}$ is defined to be 0.

Clearly this expression can be rearranged as a linear combination of the functions mentioned at the beginning of this section and conversely, since the coefficients $a_m(n)$, $b_m(n)$ are non-zero, any finite linear combination of these functions can be rearranged in the above form. Thus the complete solution of the vertical barrier problem may be taken as

$$\begin{aligned}U(X,Y) = {} & \sum_{m=0}^{\infty} c_m\Big(\frac{R^{2m+1}\sin(2m+1)\theta}{2m+1} - R^{2m}\cos 2m\theta\Big) \\ & + \sum_{m=1}^{\infty} d_m\Big(\frac{\sin(2m-1)\theta}{(2m-1)R^{2m-1}} - \frac{\cos 2m\theta}{R^{2m}}\Big) \\ & + c[\mathrm{Re}_j(e^{jz}E_1(jz) - \pi e^{-Y}\sin X)] + De^{-Y}\cos X\ . \end{aligned} \qquad (4.37)$$

Since $E_1(jz) = -\gamma - \log jz - \sum_{n=1}^{\infty} \frac{(-1)^n (jz)^n}{n.n!}$ (see Abramowitz and Stegun (1965)), it is clear that

$$Re_j(G_0(z)) \sim -\log R \text{ as } R \to 0$$

and since the term $e^{-Y} \sin X$ in the above expression for U occurs if and only if the term $Re_j(G_0(z))$ is present, it follows that a solution with a progressive wave (which could only be formed by combining the two standing waves $e^{-Y} \sin X$ and $e^{-Y} \cos X$) can exist if and only if the potential has a logarithmic singularity at the origin (cf. Alker (1975) p.203).

Some other types of solution are now considered.

(a) $U(X,Y)$ bounded at infinity

(4.37) shows that $c_m = 0$ $(m \geqslant 0)$ while if, in addition, there are to be no singularities at the origin $d_m = 0$, $c = 0$. Hence the solution which is bounded at infinity and has no singularities at the origin is a multiple of the standing wave $e^{-Y} \cos X$.

If, on the other hand, a logarithmic (but no higher) singularity is allowed at the origin, then $d_m = 0$ $(m \geqslant 1)$ and the solution

$$c[Re_j(e^{jz} E_1(jz) - \pi e^{-Y} \sin X)] + De^{-Y} \cos X$$

is obtained. c would be fixed by specifying the strength of the source at the origin. D is still arbitrary and could be chosen to furnish a progressive wave (incoming or outgoing) if required.

(b) If U is unbounded at infinity, then some information would be necessary about its more precise behaviour there. Thus if $U(X,Y) = O(R^{\alpha})$ where $\alpha > 0$, the following cases arise:

(i) $0 \leqslant \alpha < 1 \quad \Rightarrow \quad c_m = 0 \quad (m \geqslant 0)$

(ii) $1 \leqslant \alpha < 3 \quad \Rightarrow \quad c_m = 0 \quad (m \geqslant 1)$

(iii) $3 \leqslant \alpha < 5 \quad \Rightarrow \quad c_m = 0 \quad (m \geqslant 2)$ and so on.

From these remarks it is certainly clear that a unique solution can be obtained for the fixed vertical barrier problem (and hence also for the classical wave-maker problem) only by giving a very clear specification of the behaviour required at the origin and in the far field.

§5.1 Introduction

As mentioned in chapter 1, Leppington (1973a) has derived the formula (for a semi-submerged circular cylinder):

$$T = \exp(-2ia/\varepsilon)\ [\ (\varepsilon/a)^4 - \frac{4}{\pi}(\varepsilon/a)^5 \log(\varepsilon/a) + O(\varepsilon/a)^5\ ]$$

as $\varepsilon \to 0$, using the formal method of matched asymptotic expansions. However, in chapter 2, it was pointed out that this result is still not accurate enough to provide a suitable comparison with the values of T obtained by the multipole expansion method for intermediate values of $N = a/\varepsilon$. The purpose of this chapter is to derive the next term in the asymptotic development of T together with an estimate of the error term so that a better comparison can be made. It is found that incorporation of the extra term enables a significant region of overlap to be observed.

§5.2 The right inner expansion to order ε

(see §3.2 for the notation)

It has already been shown in §3.2 that in all cases where the tangents at E_+ and E_- are vertical,

$$\Phi(X,Y;\varepsilon) = \Phi_0(X,Y) + o(1) \quad \text{as} \quad \varepsilon \to 0 \quad \text{where}$$

$$\Phi_0(X,Y) = \exp(iX - Y). \tag{5.1}$$

Let it now be postulated that

$$\Phi(X,Y;\varepsilon) = \Phi_0(X,Y) + \ell(\varepsilon)\Phi_\ell(X,Y) + \varepsilon\Phi_1(X,Y) + o(\varepsilon) \tag{5.2}$$

where $\quad 1 \prec \ell(\varepsilon) \prec \varepsilon \quad$ as $\varepsilon \to 0$.

The equation of the semi-circle for $x > 0$ is

$$x = \sqrt{a^2 - y^2} \quad \text{so that (in the notation of §3.2)}$$

$$\left.\begin{aligned}
f(y) &= \sqrt{a^2 - y^2} - a \\
X = \frac{f(\varepsilon Y)}{\varepsilon} &= -\frac{1}{2}\frac{\varepsilon Y^2}{a} + 0(\varepsilon^3) \\
g_o(\varepsilon, Y) &= 1 \\
g_1(\varepsilon, Y) &= -\frac{1}{2}\frac{\varepsilon Y^2}{a} + 0(\varepsilon^3) \\
\text{and} \quad f'(\varepsilon Y) &= -\frac{\varepsilon Y}{a} + 0(\varepsilon^3)
\end{aligned}\right) \qquad (E)$$

as $\varepsilon \to 0$.

If the above equations together with (5.1) are substituted in (3.7) (which it is recalled was

$$\sum_{r=o}^{\infty} g_r(\varepsilon, Y)\left[\frac{\partial^{r+1}}{\partial X^{r+1}}\Phi(0,Y;\varepsilon) - f'(\varepsilon Y)\frac{\partial^{r+1}}{\partial X^r \partial Y}\Phi(0,Y;\varepsilon)\right]$$

$$= [i - f'(\varepsilon Y)]\, e^{-Y} \sum_{r=o}^{\infty} (-i)^r g_r(\varepsilon, Y))$$

and terms of orders higher than ε are neglected, the result is

$$\Phi_{oX}(0,Y) + \ell(\varepsilon)\,\Phi_{\ell X}(0,Y) + \varepsilon\left[\Phi_{1X}(0,Y) + \frac{Y}{a}\Phi_{oY}(0,Y) - \frac{Y^2}{2a}\Phi_{oXX}(0,Y)\right]$$

$$= i\, e^{-Y} + \varepsilon\,\frac{e^{-Y}}{a}\left(Y - \frac{1}{2}Y^2\right).$$

Equating terms of corresponding orders in ε and using (5.1) gives

$$\Phi_{\ell X}(0,Y) = 0 \qquad \text{and}$$

$$\Phi_{1X}(0,Y) = \frac{1}{a}\, e^{-Y}(2Y - Y^2).$$

The eigensolutions $\Phi_\ell(X,Y)$ will be of the form

$$\Phi_\ell(X,Y) = \sum_{K=0}^{m} B_K\left(\frac{R^{2K+1}\sin(2K+1)\theta}{2K+1} - R^{2K}\cos 2K\theta\right)$$

(where the B_K are constants and $X = R\cos\theta$, $Y = R\sin\theta$) while, in the notation mentioned at the end of §3.2 and in §4.2, equations (4.8), (4.9),

$$\Phi_1(X,Y) = W_1(X,Y) + F_1(X,Y) + E_1(X,Y) \quad . \tag{5.3}$$

W_1 (the wave part of Havelock's particular solution) is given by

$$W_1(X,Y) = -\frac{2i}{a} \exp(i X - Y) \int_0^\infty e^{-2s} (2s - s^2) ds$$

whence

$$W_1(X,Y) = -\frac{i}{2a} \exp(i X - Y). \tag{5.4}$$

$F_1(X,Y)$ (the wave free part of Havelock's particular solution) consists of the two terms $L_1(X,Y)$ and $I_1(X,Y)$ where

$$L_1(X,Y) = \frac{1}{2\pi a} \int_0^\infty \log\left[\frac{X^2 + (Y-s)^2}{X^2 + (Y+s)^2} \right] (2s - s^2) e^{-s} ds \tag{5.5}$$

and

$$I_1(X,Y) = -\frac{2}{\pi a} \int_0^\infty \int_0^\infty \frac{u\cos(Y+s)u - \sin(Y+s)u}{u^2 + 1} e^{-uX} (2s - s^2) e^{-s} du\, ds \tag{5.6}$$

Finally the eigensolution $E_1(X,Y)$ is given by

$$E_1(X,Y) = \sum_{K=0}^{n} C_K \left[\frac{R^{2K+1}\sin(2K+1)\theta}{2K+1} - R^{2K}\cos 2K\theta \right]. \tag{5.7}$$

From (5.2), therefore, (together with (5.1), (5.4)) it follows that $W^{(1)}(X,Y;\varepsilon)$ (the wave part of $\Phi(X,Y;\varepsilon)$ to order ε) is given by

$$W^{(1)}(X,Y;\varepsilon) = \left(1 - \frac{i\varepsilon}{2a}\right) \exp(i X - Y). \tag{5.8}$$

Expressed in outer coordinates (x,y) this takes the form

$$(1 - \frac{i\varepsilon}{2a}) \exp[\frac{i(x-a)}{\varepsilon} - \frac{y}{\varepsilon}]$$

so, since the incoming wave has the form

$$\exp[\frac{-i(x-a)}{\varepsilon} - \frac{y}{\varepsilon}],$$

it follows that $R^{(1)}$ (the reflection coefficient to order ε) is given by

$$R^{(1)} = (1 - \frac{i\varepsilon}{2a}) \exp(-\frac{2ia}{\varepsilon})$$

(in agreement with Leppington (1973(a) p.136, eqn.(3.10) with $N = 2$, $d_N = \frac{1}{a}$).

Next, $\Phi^{(1)}(X,Y;\varepsilon)$ (the wave free part of $\Phi(X,Y;\varepsilon)$ to order ε) is given by

$$\Phi^{(1)}(X,Y;\varepsilon) = \ell(\varepsilon) \sum_{K=0}^{m} B_K [\frac{R^{2K+1}\sin(2K+1)\theta}{2K+1} - R^{2K}\cos 2K\theta]$$

$$+\varepsilon[L_1(X,Y) + I_1(X,Y) + \sum_{K=0}^{n} C_K [\frac{R^{2K+1}\sin(2K+1)\theta}{2K+1} - R^{2K}\cos 2K\theta] . \tag{5.9}$$

In order to apply the matching principle (see §3.5) it is now necessary to obtain the leading terms in the asymptotics of $L_1(X,Y)$ and $I_1(X,Y)$ as $R \to \infty$.

By integration by parts, it can be shown that

$$L_1(X,Y) = \frac{1}{\pi a} \int_0^\infty s^2 [\frac{Y+s}{X^2+(Y+s)^2} + \frac{Y-s}{X^2+(Y-s)^2}] e^{-s} ds.$$

If X,Y are replaced by $R\cos\theta$, $R\sin\theta$ respectively and the substitution $s = Ru$ is employed then, further,

$$L_1(X,Y) = \frac{R^2}{\pi a} \int_0^\infty u^2 (\frac{\sin\theta + u}{1 + 2u\sin\theta + u^2} + \frac{\sin\theta - u}{1 - 2u\sin\theta + u^2}) e^{-Ru} du .$$

As $u \to 0$, the non-exponential part of the integrand has the form

$$2u^2 \sin\theta \; + O(u^4) \quad .$$

so, by Watson's Lemma

$$L_1(X,Y) = \frac{4s\,in\theta}{\pi aR} \; + O(1/R^3) \text{ as } R \to \infty \tag{5.10}$$

Next,

$$\frac{u\cos(Y + s)u - \sin(Y + s)u}{u^2 + 1} = \mathrm{Re}_j' \{ \frac{\exp[Y + s)ju]}{u - j} \} ,$$

so that (5.6) can be written in the form

$$- \frac{\pi a}{2} \; I_1(X,Y) = \mathrm{Re}_j \int_0^\infty \frac{\exp[-u(X - jY)]}{u - j} \int_0^\infty (2s-s^2)\exp[-s(1-ju)]\,ds\,du \, .$$

The inner integral is easily shown to have the value

$$\frac{-2ju}{(1-ju)^3}$$

whence

$$- \frac{\pi a}{4} \quad I_1(X,Y) = \mathrm{Re}_j \int_0^\infty \frac{u}{(1 + ju)(1 - ju)^3} \; e^{-\zeta u}\,du \; ,$$

where

$$\zeta = X - j\,Y = R\,e^{-j\theta} \; .$$

Since $\qquad \dfrac{u}{(1 + ju)(1 - ju)^3} = u + O(u^2) \qquad$ as $u \to 0$,

Watson's Lemma again gives

$$- \frac{\pi a}{4} \; I_1(X,Y) = \mathrm{Re}_j \; (\frac{1}{\zeta^2} + O\;(\frac{1}{\zeta^3})) \quad \text{as } R$$

(and hence ζ) $\to \infty$.

Thus $I_1\,(X,Y) = -\dfrac{4\cos 2\theta}{\pi a R^2} + O\left(\dfrac{1}{R^3}\right) \quad \text{as } R \to \infty .$ (5.11)

If (5.10), (5.11) are now substituted in (5.9) (with R replaced by δ/ε, where $\delta = \delta_+$ in this case) and the resulting series is truncated after terms of order ε^2, then it is found that

$$\phi^{(1,2)} = \ell(\varepsilon) \sum_{K=0}^{m} B_K \left[\frac{1}{\varepsilon^{2K+1}} \frac{\delta^{2K+1}\sin(2K+1)\theta}{2K+1} - \frac{1}{\varepsilon^{2K}} \delta^{2K}\cos 2K\theta \right]$$

$$+ \varepsilon \sum_{K=0}^{n} C_K \left[\frac{1}{\varepsilon^{2K+1}} \frac{\delta^{2K+1}\sin(2K+1)\theta}{2K+1} - \frac{1}{\varepsilon^{2K}} \delta^{2K}\cos 2K\theta \right]$$

$$+ \varepsilon^2 \frac{4\sin\theta}{\pi a \delta} . \qquad (5.12)$$

(It is noted here that, in the next section, the coefficients B_K, C_K are shown to be zero so that the expansion of Φ to order ε is

$$\left.\begin{aligned} \Phi(X,Y;\varepsilon) &= \Phi_0(X,Y) + \varepsilon\Phi_1(X,Y) \text{ where} \\ \Phi_1(X,Y) &= W_1(X,Y) + L_1(X,Y) + I_1(X,Y) \end{aligned}\right\} \qquad (5.13)$$

§5.3 The outer expansion to order ε^2

Without any assumption concerning the form of the asymptotic development of the outer expansion, (5.12) and the matching principle

$$\phi^{(1,2)} = \phi^{(2,1)}$$

indicate that the form of the outer expansion of $\phi(x,y;\varepsilon)$ up to terms of order ε^2 will be

$$\phi^{(2)}(x,y;\varepsilon) = \ell(\varepsilon) \sum_{K=0}^{m} \left[\frac{f_{2K+1}(x,y)}{\varepsilon^{2K+1}} - \frac{g_{2K}(x,y)}{\varepsilon^{2K}} \right]$$

$$+ \varepsilon \sum_{K=0}^{n} \left[\frac{p_{2K+1}(x,y)}{\varepsilon^{2K+1}} - \frac{q_{2K}(x,y)}{\varepsilon^{2K}} \right]$$

$$+ \varepsilon^2 \phi_0(x,y) , \qquad (5.14)$$

where, as $\delta \to 0$,

$$\left.\begin{array}{lcl} f_{2K+1}(x,y) & \sim & \dfrac{B_K\ \delta^{2K+1}\sin(2K+1)\theta}{2K+1} \\ g_{2K}(x,y) & \sim & B_K.\delta^{2K}\cos 2K\theta \end{array}\right) \quad 0 \leqslant K \leqslant m$$

$$\left.\begin{array}{lcl} p_{2K+1}(x,y) & \sim & \dfrac{C_K\ \delta^{2K+1}\sin(2K+1)\theta}{2K+1} \\ q_{2K}(x,y) & \sim & C_K\ \delta^{2K}\cos 2K\theta \end{array}\right) \quad 0 \leqslant K \leqslant n$$

$$\phi_0\ (x,y) \quad \sim \quad \frac{4\sin\theta}{\pi a\delta} \quad .$$

If now (5.14) is substituted in the equation

$$\phi \ + \ \varepsilon \ \phi y = 0 \qquad (y = 0, \ |x| > a)$$

and the coefficients of the various gauge factors (up to order ε^2) are equated to zero, the following sequence of equations for the functions f and g is obtained:

$$f_{2m+1}(x,0) \quad = \quad 0 \tag{5.15a}$$

$$g_{2K}(x,0) \quad = \quad \frac{\partial}{\partial y}\ [f_{2K+1}(x,0)] \qquad m \geqslant K \geqslant 0$$

$$f_{2K-1}(x,0) \quad = \quad \frac{\partial}{\partial y}\ [g_{2K}\ (x,0)] \qquad m \geqslant K \geqslant 1$$

$$\frac{\partial}{\partial y}[g_o(x,0)] \quad = \quad 0\ .$$

These are followed by a similar sequence for the functions p and q ($f \to p$, $g \to q$, $m \to n$) though the last equation in this case is

$$\phi_0\ (x,0) \quad = \quad \frac{\partial}{\partial y}[\,q_o(x,0)]\ . \tag{5.15b}$$

In addition, all the potential coefficients must be harmonic, have zero normal derivative on Γ and die off to zero at infinity. Those functions which, in addition, vanish on $y = 0$ for $|x|>a$ are of the form

$$e_r(z) = \mathrm{Re}_j\{\frac{1}{j}\ [A(\frac{z-a}{z+a})^{2r+1} + B(\frac{z+a}{z-a})^{2r+1}]\} \quad (r = 0,1,2\ldots), \tag{5.16}$$

or linear combinations thereof, where A, B are real with respect to j (this can be seen by means of the transformation

$$w = \frac{z+a}{z-a}\ , \quad z = x + jy).$$

(It is noted here that, in the next paragraph, q_o is proved to be identically zero so that from (5.15b) ϕ_0 will vanish on $y = 0$, $|x|> a$. Since also $\phi_0(x,y) \sim \frac{4\sin\theta}{\pi a\delta}$ as $\delta \to o$, it follows that

$$\phi_0(x,y) = \frac{2}{\pi a^2}\ \mathrm{Re}_j\ [\frac{j(z+a)}{(z-a)}].) \tag{5.17}$$

It is now shown that all the coefficients B_K, C_K in (5.9) must be zero and attention is confined initially to the B_K. It is assumed that the set of integers $K \geqslant 0$ for which $B_K \neq 0$ is non-empty and that M is the largest member of the set. It follows that $\phi^{(2)}(x,y;\varepsilon)$ contains a term of the form $\ell(\varepsilon)\ f_{2M+1}(x,y)/\varepsilon^{2M+1}$ where

$$f_{2M+1}(x,0) = 0 \quad \text{(by (5.15a))}.$$

Hence $f_{2M+1}(x,y)$ must be of the form (5.16) and, in addition,

$$f_{2M+1}(x,y) \sim B_M\ \delta^{2M+1}\sin(2M+1)\theta/(2M+1) \quad \text{as} \quad \delta \to 0.$$

Thus $$f_{2M+1}(x,y) = \mathrm{Re}_j\ [\frac{B_M(2a)^{2M+1}}{(2M+1)j}\ (\frac{z-a}{z+a})^{2M+1}].$$

If z is now set equal to $-a-\delta_1 e^{-j\theta_1}$ (where $\delta_1 = \delta$ and $\theta_1 = \pi-\arg(z+a)$), it is seen that

$$f_{2M+1}(x,y) \sim \frac{B_M(2a)^{4M+2}\sin(2M+1)\theta_1}{(2M+1)\delta_1{}^{2M+1}} \quad \text{as } \delta_1 \to 0.$$

When δ_1 is replaced by εR_1 and the matching principle applied, the left inner expansion of $\psi(X_1,Y_1;\varepsilon)$ will be seen to contain a term of the form

$$\ell(\varepsilon)\psi_h/\varepsilon^{4M+2} \qquad \text{where}$$

$$\psi_h \sim \frac{B_M(2a)^{4M+2}\sin(2M+1)\theta_1}{(2M+1)\ R_1^{\ 2M+1}} \qquad \text{as } R_1 \to \infty\ , \tag{5.18}$$

since no other terms with the same scaling could appear (see (5.14)). In addition (as shown in §3.4), ψ_h would be harmonic, satisfy $\psi + \psi_{Y_1} = 0$ on $Y_1 = 0$ and also have the property $\psi_{X_1} = 0$ on $X_1 = 0$ (since $\ell(\varepsilon)\ \psi_h/\varepsilon^{4M+2}$ will be the leading term with a scaling of this form).

Thus ψ_h is a solution of the vertical barrier problem and, from §4.5, the solution satisfying (5.18) is

$$B_M\ (2a)^{4M+2}\ [\frac{\sin(2M+1)\theta_1}{(2M+1)R_1^{\ 2M+1}} - \frac{\cos(2M+2)\theta_1}{R_1^{\ 2M+2}}]\ .$$

It follows (since $M \geqslant 0$) that ψ_h would not satisfy the edge condition since, by hypothesis, $B_M \neq 0$. The only conclusion is that the original hypothesis is untenable implying that $B_K = 0$ for all $K \geqslant 0$. By similar reasoning it can also be proved that $C_K = 0$ $(K \geqslant 0)$ whence $q_o \equiv 0$.

The statement (5.13) is thus verified while (5.14) implies that

$$\phi^{(2)}\ (x,y;\varepsilon) = \varepsilon^2\,\phi_o(x,y), \tag{5.19}$$

with ϕ_o given by (5.17).

(Note: The addition of functions $e_r(z)$ with $r \geqslant 0$ and $B = 0$ to the solution (5.17) for ϕ_o and of similar functions with $r > M$ to the solution given for f_{2M+1} can be discounted by reasoning as above. Such functions would again lead to violation of the edge condition in corresponding terms of the left-inner expansion).

Finally, in this section, the substitution $z = -a - \delta_1 e^{-j\theta_1}$ in (5.17) shows that, as $\delta_1 \to 0$,

$$\phi_0(x,y) = \frac{\delta_1 \sin\theta_1}{\pi a^3} - \frac{\delta_1^2 \sin 2\theta_1}{2\pi a^4} + o(\delta_1^2) , \tag{5.20}$$

and the replacement of δ_1 by εR_1 followed by expansion of (5.19) up to terms of order ε^3 gives (near E_-)

$$\phi^{(2,3)}(x,y;\varepsilon) = \varepsilon^3 R_1 \sin\theta_1 / \pi a^3 \cdot \tag{5.21}$$

§5.4 The left inner expansion to order ε^3

The matching principle

$$\psi^{(3,2)} = \phi^{(2,3)}$$

and (5.21) indicate that

$$\psi^{(3)}(X_1,Y_1;\varepsilon) = \varepsilon^3 \psi_0(X_1,Y_1) \tag{5.22}$$

where

$$\psi_0(X_1,Y_1) \sim R_1 \sin\theta_1/\pi a^3 \qquad \text{as } R_1 \to \infty . \tag{5.23}$$

Near E_-, the equation of Γ in this case is

$$x = -a - f(y)$$

where (as for the right inner region)

$$f(y) = \sqrt{a^2 - y^2} - a.$$

or, in terms of left inner coordinates (see §3.4),

$$X_1 = f(\varepsilon Y_1)/\varepsilon \quad .$$

The boundary condition on Γ is now replaced by

$$\sum_{r=o}^{\infty} g_r(\varepsilon,Y_1)\left[\frac{\partial^{r+1}}{\partial X_1^{\,r+1}}\psi(0,Y_1;\varepsilon) - f'(\varepsilon Y_1)\frac{\partial^{r+1}}{\partial X_1^r\partial Y_1}\psi(0,Y_1;\varepsilon)\right] = 0\,, \qquad (5.24)$$

where it is recalled that

$$g_r(\varepsilon,Y_1) = \frac{1}{r!}\left[\frac{f(\varepsilon Y_1)}{\varepsilon}\right]^r = 0(\varepsilon^r) \qquad \text{as } \varepsilon \to 0\,.$$

If (5.22) is substituted in (5.24) and terms of order higher than ε^3 are neglected, the result is

$$\varepsilon^3\,\psi_{oX_1}(0,Y_1) = 0\,, \qquad \text{whence}$$

$$\psi_{oX_1}(0,Y_1) = 0\,.$$

Thus ψ_o is an eigensolution of the vertical barrier problem satisfying (5.23) whence

$$\psi_o(X_1,Y_1) = (R_1\sin\theta_1 - 1)/\pi a^3\,. \qquad (5.25)$$

§ 5.5 The right inner expansion and reflection coefficient to order ε^2

Reference to § 3.2 and equation (5.13) leads us to pose the development:

$$\Phi(X,Y;\varepsilon) = \Phi_o(X,Y) + \varepsilon\Phi_1(X,Y) + \ell(\varepsilon)\Phi_\ell(X,Y) + \varepsilon^2\Phi_2(X,Y) + 0(\varepsilon^2) \qquad (5.26)$$

for the right inner expansion as $\varepsilon \to o$, $\ell(\varepsilon)$ being a gauge factor such that $\varepsilon \prec \ell(\varepsilon) \prec \varepsilon^2$ as $\varepsilon \to 0$ and $\ell(\varepsilon)\,\Phi_\ell(X,Y)$ standing for a typical term with a scaling of this type.

By substituting (5.26) into (3.7) and retaining terms of orders up to ε^2, it can be shown that

$$\ell(\varepsilon)\,\Phi_{\ell X}(0,Y) + \varepsilon^2\Big[\Phi_{2X}(0,Y) + \frac{Y}{a}\Phi_{1Y}(0,Y) - \frac{Y^2}{2a}\Phi_{1XX}(0,Y)$$

$$- \frac{Y^3}{2a^2}\Phi_{oX}(0,Y) + \frac{Y^4}{8a^2}\Phi_{oXXX}(0,Y)\Big] = i\,e^{-Y}\left(\frac{Y^3}{2a^2} - \frac{Y^4}{8a^2}\right)\varepsilon^2\,,$$

(since the $O(1)$ and $O(\varepsilon)$ terms cancel, as arranged before).

Equating coefficients of the two gauge factors above gives

$$\Phi_{\ell X}(0,Y) = 0 ,$$

and

$$\Phi_{2X}(0,Y) = -\frac{1}{2a} \frac{d}{dY} [Y^2 \Phi_{1Y}(0,Y)],$$

where (5.1) and the relationship $\Phi_{1XX}(0,Y) = -\Phi_{1YY}(0,Y)$ have been used.

It can now be shown, by using arguments similar to those employed in §5.2, that the eigensolution $\Phi_\ell(X,Y)$ is identically zero and that $\Phi_2(X,Y)$ will not contain eigensolutions. Hence, provided $\Phi_{2X}(0,Y)$ decays sufficiently rapidly as $Y \to \infty$, $\Phi_2(X,Y)$ will be given by Havelock's particular solution

$$\Phi_2 (X,Y) = -\frac{1}{2a} \int_0^\infty H(X,Y;s) \frac{d}{ds} [s^2 \Phi_{1s}(0,s)] \, ds . \tag{5.27}$$

To examine the decay properties of $\Phi_{2X}(0,Y)$ it is first noted that (apart from a term of negative exponential order)

$$\begin{aligned} \Phi_1(0,Y) &= L_1(0,Y) + I_1(0,Y) \\ &= \frac{4}{\pi a Y} + \frac{4}{\pi a Y^2} + 0 \left(\frac{1}{Y^3}\right) \text{ as } Y \to \infty , \end{aligned}$$

(from (5.10) and (5.11)). Certainly the derivatives of $L_1(0,Y)$ and $I_1(0,Y)$ will have asymptotic expansions in powers of $\frac{1}{Y}$, given by differentiation under the integral sign and application of Watson's lemma to the result, so that the asymptotic properites of $\frac{d}{dy}(Y^2\Phi_{1Y}(0,Y))$ can be found by differentiation of the above equation for $\Phi_1(0,Y)$. From this it follows that

$$\frac{d}{dY} [Y^2 \Phi_{1Y}(0,Y)] = 0 \left(\frac{1}{Y^2}\right) \qquad \text{as } Y \to \infty ,$$

whence the discussion in §4.3 shows that the integral in (5.27) is convergent and that the leading term in the far-field form of $\Phi_2(X,Y)$ will be $O\left(\frac{\log R}{R}\right)$ (see (4.19)).

From (5.13) it is seen that the full expression for $\Phi_1(0,s)$ is

$$\Phi_1(0,s) = W_1(0,s) + L_1(0,s) + I_1(0,s)$$

with W_1, L_1 and I_1 given by (5.4) - (5.6).

Hence

$$\Phi_1(0,s) = -\frac{i}{2a}e^{-s} + \frac{1}{\pi a}\int_0^\infty \log\left|\frac{s-u}{s+u}\right|(2u-u^2)e^{-u}\,du$$
$$-\frac{2}{\pi a}\int_0^\infty\int_0^\infty \frac{v\cos(s+u)v - \sin(s+u)v}{v^2+1}(2u-u^2)e^{-u}\,dv\,du\ .$$

In appendix D (D.1) the integrals occurring here are recast in forms which can be differentiated twice under the integral sign without difficulty viz.

$$\Phi_1(0,s) = -\frac{i}{2a}e^{-s} + \frac{2}{\pi a}\,\mathrm{Im}_j\left[\int_0^\infty h^{(2)}(u)e^{-jsu}du\right]$$
$$+\frac{4}{\pi a}\,\mathrm{Re}_j\left[\int_0^\infty F(u)e^{jsu}\,du\right], \tag{5.28}$$

where

$$h(u) = \frac{1}{u^2+1},$$

and

$$F(u) = \frac{u}{(u-j)(u+j)^3},$$

and it is also shown that

$$\frac{d}{ds}[s^2\Phi_{1\,s}(0,s)] = \frac{i}{2a}(2s-s^2)e^{-s} + \frac{2}{\pi a}\,\mathrm{Im}_j\left[\int_0^\infty \frac{d}{du}[u^2h^{(3)}(u)]e^{-jsu}du\right]$$
$$+\frac{4}{\pi a}\,\mathrm{Re}_j\left[\int_0^\infty \frac{d}{du}(u^2F'(u))e^{jsu}\,du\right]. \tag{5.29}$$

Attention is first directed to the wave part of $\Phi_2(X,Y)$ (denoted by $W_2(X,Y)$) which is given by

$$W_2(X,Y) = -\frac{1}{2a} \int_0^\infty H_w(X,Y;s) \frac{d}{ds} [s^2 \Phi_{1s}(0,s)] \, ds ,$$

or

$$W_2 (X,Y) = \frac{i}{a} \exp(i X-Y) \int_0^\infty e^{-s} \frac{d}{ds} [s^2 \Phi_{1s}(0,s)] \, ds \qquad (5.30)$$

(see 4.7) and 4.9) .

The integral here is evaluated in appendix D (D.2) where it is shown to have the value $-\frac{i}{a}(\frac{2i}{3\pi} - \frac{1}{8})$.

Thus

$$W_2(X,Y) = \frac{1}{a^2} (\frac{2i}{3\pi} - \frac{1}{8}) \exp (iX-Y),$$

and $W^{(2)}(X,Y)$ (the wave part of $\Phi(X,Y;\varepsilon)$ up to terms of order ε^2) is given by

$$W^{(2)}(X,Y) = W^{(1)}(X,Y) + \varepsilon^2 W_2(X,Y) ,$$

i.e.

$$W^{(2)}(X,Y) = [1 - \frac{i\varepsilon}{2a} + (\frac{2i}{3\pi} - \frac{1}{8}) \frac{\varepsilon^2}{a^2}] \exp(iX-Y)$$

(using 5.8)).

It follows that $R^{(2)}$ (the reflection coefficient to order ε^2) is given by

$$R^{(2)} = [1 - \frac{i\varepsilon}{2a} + (\frac{2i}{3\pi} - \frac{1}{8}) \frac{\varepsilon^2}{a^2}] \exp (-{}^{2ia}/\varepsilon) .$$

This result extends the asymptotics of the reflection coefficient to second order for the cylindrical geometry.

As in the case of $\Phi_1(X,Y)$, the wave free part of $\Phi_2(X,Y)$ is written as the sum of two terms, $L_2(X,Y)$ and $I_2(X,Y)$ where

$$L_2(X,Y) = -\frac{1}{4\pi a} \int_0^{\infty} \log\left[\frac{X^2 + (Y-s)^2}{X^2 + (Y-s)^2}\right] \frac{d}{ds}\left[s^2 \Phi_{1s}(0,s)\right] ds \; ; \tag{5.31}$$

and

$$I_2(X,Y) = \frac{1}{\pi a} \int_0^{\infty} \int_0^{\infty} \frac{u\cos(Y+s)u - \sin(Y+s)u}{u + 1} e^{-uX} \frac{d}{ds}\left[s^2 \Phi_{1s}(0,s)\right] du\, ds \, . \tag{5.32}$$

Examination of the form of $\frac{d}{ds}\left[s^2 \Phi_{1s}(0,s)\right]$, given in (5.29), shows that $L_2(X,Y)$ and $I_2(X,Y)$ will consist of three distinct terms which will be denoted by $L_{2i}(X,Y)$ and $I_{2i}(X,Y)$ $(1 \leqslant i \leqslant 3)$ respectively, where

$$L_{21}(X,Y) =$$

$$-\frac{i}{8\pi a^2} \int_0^{\infty} \log\left[\frac{X^2+(Y-s)^2}{X^2+(Y+s)^2}\right] (2s-s^2)e^{-s} ds \; , \tag{5.33}$$

$$L_{22}(X,Y) =$$

$$-\frac{1}{2\pi^2 a^2} \operatorname{Im}_j \left\{ \int_0^{\infty} \int_0^{\infty} \log\left[\frac{X^2+(Y-s)^2}{X^2+(Y+s)^2}\right] \frac{d}{du}\left[u^2 h^{(3)}(u)\right] e^{-jsu} du\, ds \right\}, \tag{5.34}$$

$$L_{23}(X,Y) =$$

$$-\frac{1}{\pi^2 a^2} \operatorname{Re}_j \left\{ \int_0^{\infty} \int_0^{\infty} \log\left[\frac{X^2+(Y-s)^2}{X^2+(Y+s)^2}\right] \frac{d}{du}\left[u^2 F'(u)\right] e^{jsu} du\, ds \right\}, \tag{5.35}$$

$$I_{21}(X,Y) =$$

$$\frac{i}{2\pi a^2} \int_0^{\infty} \int_0^{\infty} \frac{u\cos(Y+s)u - \sin(Y+s)u}{u^2 + 1} e^{-uX} (2s-s^2)e^{-s} du\, ds \; , \tag{5.36}$$

$$I_{22}(X,Y) =$$

$$\frac{2}{\pi^2 a^2} \operatorname{Im}_j \left\{ \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} \frac{v\cos(Y+s)v - \sin(Y+s)v}{v^2 + 1} e^{-vX} \frac{d}{du}\left[u^2 h^{(3)}(u)\right] e^{-jsu} dv\, du\, ds \right\} \tag{5.37}$$

$$I_{23}(X,Y) =$$

$$\frac{4}{\pi^2 a^2} \operatorname{Re}_j \left\{ \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} \frac{v\cos(Y+s)v - \sin(Y+s)v}{v^2 + 1} e^{-vX} \frac{d}{du}\left[u^2 F'(u)\right] e^{jsu} dv\, du\, ds \right\} \tag{5.38}$$

The leading terms in the asymptotics of these six expressions as $R \to \infty$ are now required.

Comparison of (5.33) and (5.5) shows that

$$L_{21}(X,Y) = -\frac{i}{4a} L_1(X,Y),$$

whence use of (5.10) gives

$$L_{21}(X,Y) = -\frac{i\sin\theta}{\pi a^2 R} + O\left(\frac{1}{R^3}\right) \quad \text{as } R \to \infty . \tag{5.39}$$

In (5.34) and (5.35), X and Y are now replaced by $R\cos\theta$ and $R\sin\theta$ respectively and the substitution $s = Rt$ is made. This gives

$$L_{22}(X,Y) = -\frac{R}{2\pi^2 a^2} \operatorname{Im}_j\left\{ \int_0^\infty \int_0^\infty \log\left(\frac{1-2t\sin\theta+t^2}{1+2t\sin\theta+t^2} \right) \frac{d}{du}[u^2 h^{(3)}(u)] e^{-jRtu} du\,dt \right\},$$

and

$$L_{23}(X,Y) = -\frac{R}{\pi^2 a^2} \operatorname{Re}_j\left\{ \int_0^\infty \int_0^\infty \log\left(\frac{1-2t\sin\theta+t^2}{1+2t\sin\theta+t^2} \right) \frac{d}{du}[u^2 F'(u)] e^{jRtu} du\,dt \right\},$$

whence the results from (C.3) and (C.9) in appendix C indicate that, as $R \to \infty$,

$$L_{22}(X,Y) = o\left(\frac{1}{R}\right) \tag{5.40}$$

and

$$L_{23}(X,Y) = -\frac{8}{\pi^2 a^2 R}(\theta\cos\theta - \sin\theta\log R) - \frac{4\sin\theta}{\pi^2 a^2 R}(5-2\gamma) + o\left(\frac{1}{R}\right). \tag{5.41}$$

Addition of (5.39), (5.40) and (5.41) gives

$$L_2(X,Y) = \frac{\sin\theta}{\pi^2 a^2 R}(8\gamma - 20 - \pi i) - \frac{8}{\pi^2 a^2 R}(\theta\cos\theta - \sin\theta\log R) + o\left(\frac{1}{R}\right) \tag{5.42}$$

(γ = Euler's constant).

Comparison now of (5.36) and (5.6) shows that

$$I_{21}(X,Y) = -\frac{i}{4a} I_1(X,Y),$$

whence use of (5.11) gives

$$I_{21}(X,Y) = o\left(\frac{1}{R}\right) \qquad \text{as } R \to \infty \, . \tag{5.43}$$

Finally the results from §C.10 and §C.11 in appendix C show that, as $R \to \infty$,

$$I_{22}(X,Y) = \frac{4\sin\theta}{\pi^2 a^2 R} + o\left(\frac{1}{R}\right) \tag{5.44}$$

and

$$I_{23}(X,Y) = o\left(\frac{1}{R}\right) . \tag{5.45}$$

Addition now of equations (5.42) - (5.45) gives the leading asymptotics of the wave-free part of $\Phi_2(X,Y)$ viz, as $R \to \infty$,

$$F_2(X,Y) = \frac{8\sin\theta}{\pi^2 a^2 R}\left(\gamma - 2 - \frac{i\pi}{8}\right) - \frac{8}{\pi^2 a^2 R}\left(\theta\cos\theta - \sin\theta\log R\right) + o\left(\frac{1}{R}\right). \tag{5.46}$$

From (5.26) and the fact that $\Phi_\ell(X,Y) \equiv 0$ it can be seen that $\Phi^{(2)}(X,Y;\varepsilon)$ (the wave free part of $\Phi(X,Y;\varepsilon)$ up to order ε^2) is given by

$$\Phi^{(2)}(X,Y;\varepsilon) = \varepsilon F_1(X,Y) + \varepsilon^2 F_2(X,Y)$$

while substitution here of (5.10), (5.11), (5.46) (with R replaced by δ/ε) and truncation of the resulting series after terms of order ε^3 gives

$$\Phi^{(2,3)} = \varepsilon^2\left(\frac{4\sin\theta}{\pi a\delta}\right) + \varepsilon^3\log\varepsilon\left(-\frac{8\sin\theta}{\pi^2 a^2\delta}\right) +$$
$$+ \varepsilon^3\left[\frac{8}{\pi^2 a^2}\left(\gamma - 2 - i\frac{\pi}{8}\right)\frac{\sin\theta}{\delta} - \frac{8}{\pi^2 a^2}\left(\frac{\theta\cos\theta - \sin\theta\log\delta}{\delta}\right) - \frac{4\cos 2\theta}{\pi a\delta^2}\right].$$

This result is required for the further development of the outer expansion in the next section.

§5.6 The outer expansion to order ε^3

The matching principle

$$\phi^{(3,2)} = \phi^{(2,3)}$$

shows that the outer expansion of $\phi(x,y;\varepsilon)$ up to order ε^3 will be given by

$$\phi^{(3)}(x,y;\varepsilon) = \varepsilon^2\phi_0(x,y) + \varepsilon^3\log\varepsilon\,\phi_1(x,y) + \varepsilon^3\phi_2(x,y), \tag{5.47}$$

where, as $\delta\ (= \delta_+) \to 0$,

$$\left.\begin{aligned} \phi_0(x,y) &\sim \frac{4\sin\theta}{\pi a\delta} \\ \phi_1(x,y) &\sim -\frac{8\sin\theta}{\pi^2 a^2\delta} \end{aligned}\right) \tag{5.48}$$

and

$$\phi_2(x,y) \sim -\frac{4\cos 2\theta}{\pi a\delta^2} - \frac{8}{\pi^2 a^2}\left(\frac{\theta\cos\theta - \sin\theta\log\delta}{\delta}\right) + \frac{8}{\pi^2 a^2}\left(\gamma - 2 - \frac{i\pi}{8}\right)\frac{\sin\theta}{\delta}\,. \tag{5.49}$$

The potentials must also be harmonic, satisfy $\frac{\partial\phi}{\partial r} = 0$ on $r = a$, and die off to zero at infinity, while formal substitution of $\phi = \phi^{(3)}$ in the equation $\phi + \varepsilon\phi_y = 0$ (neglecting terms of order higher than ε^3) gives the additional conditions (on $y = 0$, $|x| > a$)

$$\phi_0 = 0,$$

$$\phi_1 = 0,$$

$$\phi_2 = -\phi_{0y}\,.$$

It has already been shown that

$$\phi_o(x,y) = \frac{2}{\pi a^2} \mathrm{Re}_j \left[\frac{j(z+a)}{(z-a)} \right] \qquad (z = x + jy),$$

so that comparison of the asymptotics in (5.48) leads to the conclusion that

$$\phi_1(x,y) = -\frac{4}{\pi^2 a^3} \mathrm{Re}_j \left[\frac{j(z+a)}{(z-a)} \right],$$

the addition of eigensolutions non-singular at E_+ being excluded as in §5.2. Hence, use of (5.20) gives

$$\phi_1(x,y) = -\frac{2\delta_1 \sin\theta_1}{\pi^2 a^4} + o(\delta_1), \quad \text{as} \quad \delta_1 \to 0. \tag{5.50}$$

To begin the solution for ϕ_2 it is first noted that

$$\frac{\partial\phi_o}{\partial y} = \frac{4}{\pi a} \mathrm{Re}_j \left[\frac{1}{(z-a)^2} \right],$$

and also that, in terms of δ and θ,

$$\frac{4}{\pi a} \mathrm{Re}_j \left[\frac{1}{(z-a)^2} \right] = \frac{4}{\pi a} \frac{\cos 2\theta}{\delta^2}.$$

In addition

$$\frac{\theta\cos\theta - \sin\theta \log\delta}{\delta} = -\mathrm{Re}_j \left[\frac{j\log(z-a)}{z-a} \right],$$

the log being made single valued by a cut along the real axis from a to $-\infty$. Hence the function

$$\phi = \mathrm{Re}_j \left[\frac{-4}{\pi a(z-a)^2} - \frac{4j}{\pi^2 a^3} \left(\frac{z+a}{z-a} \right) \log \left(\frac{z+a}{z-a} \right) \right]$$

will certainly satisfy $\phi = -\phi_{oy}$ on $y = o$, $|x| > a$ (since $\frac{z+a}{z-a}$ and $\log(\frac{z+a}{z-a})$ are then real).

If the asymptotics of ϕ as $\delta \to 0$ are now examined, it is found that

$$\phi = -\frac{4\cos 2\theta}{\pi a \delta^2} - \frac{8}{\pi^2 a^2}\left(\frac{\theta\cos\theta - \sin\theta\log\delta}{\delta} + \frac{\sin\theta\log 2a}{\delta} + \frac{\theta}{2a}\right) + o(1),$$

the $O(1)$ term $-\dfrac{4\theta}{\pi^2 a^3}$ being exhibited explicitly since it will be referred to later from chapter 7, §7.1. Hence the function

$$\phi(x,y) = \mathrm{Re}_j\left[-\frac{4}{\pi a(z-a)^2} - \frac{4j}{\pi^2 a^3}\left(\frac{z+a}{z-a}\right)\log\left(\frac{z+a}{z-a}\right)\right] + \frac{2}{\pi a}\left(\gamma + \log 2a - 2 - \frac{i\pi}{8}\right)\phi_0$$

will have the value $-\phi_{oy}$ on $y = 0$ and also satisfy the asymptotic condition (5.49). Indeed the imaginary part of the complex potential above is seen to be zero when $z = a\,e^{ju}$ $(0 < u < \pi)$ (implying that $\frac{\partial\phi}{\partial r} = o$ on $r = a$) so that the solution for ϕ_2 is now complete. (If $\frac{\partial\phi}{\partial r}$ had assumed a non-zero value on $r = a$ then, by reflection, the problem would have been reduced at this stage to one of exterior Neumann type).

By putting $z = -a-\delta_1 e^{-j\theta_1}$ in the solution for ϕ_2 and using elementary expansions it may be proved that, as $\delta_1 \to 0$,

$$\phi_2(x,y) = -\frac{1}{\pi a^3} + \frac{\delta_1\cos\theta_1}{\pi a^4} + \frac{2}{\pi^2 a^4}\left[-\delta_1(\sin\theta_1\log\delta_1 + \theta_1\cos\theta_1)\right.$$
$$\left. + \delta_1\sin\theta_1\left(2\log 2a + \gamma - 2 - \frac{i\pi}{8}\right)\right] + o(\delta_1).$$

When this result and those in (5.50) and (5.20) are substituted in (5.47) (with δ_1 replaced by εR_1) and the resulting series is truncated after terms of order ε^4, it is seen that, (near E_-)

$$\phi^{(3,4)} = \varepsilon^3\left(\frac{R_1\sin\theta_1 - 1}{\pi a^3}\right) + \varepsilon^4\log\varepsilon\left(\frac{-4R_1\sin\theta_1}{\pi^2 a^4}\right)$$
$$+\varepsilon^4\left\{\left(\frac{2R_1\cos\theta_1 - R_1^2\sin 2\theta_1}{2\pi a^4}\right)\right.$$
$$\left. + \frac{2}{\pi^2 a^4}\left[-R_1(\sin\theta_1\log R_1 + \theta_1\cos\theta_1) + R_1\sin\theta_1\left(2\log 2a + \gamma - 2 - \frac{i\pi}{8}\right)\right]\right\}.$$

§5.7 The left inner expansion to order ε^4

The matching principle

$$\psi^{(4,3)} = \phi^{(3,4)}$$

indicates that the left inner potential $\psi(X_1,Y_1;\varepsilon)$ has an expansion to order ε^4 given by

$$\psi^{(4)}(X_1,Y_1;\varepsilon) = \varepsilon^3\psi_o(X_1,Y_1) + \varepsilon^4\log\varepsilon\psi_1(X_1,Y_1) + \varepsilon^4\psi_2(X_1,Y_1), \tag{5.51}$$

where, as $R_1 \ (=\sqrt{X_1^2+Y_1^2}) \to \infty$,

$$\psi_o(X_1,Y_1) \sim (R_1\sin\theta_1 - 1)/\pi a^3, \tag{5.52}$$

$$\psi_1(X_1,Y_1) \sim \frac{-4R_1\sin\theta_1}{\pi^2 a^4}, \tag{5.53}$$

$$\psi_2(X_1,Y_1) \sim \frac{-R_1^2\sin 2\theta_1 + 2R_1\cos\theta_1}{2\pi a^4} - \frac{2R_1}{\pi^2 a^4}(\sin\theta_1\log R_1 + \theta_1\cos\theta_1)$$

$$+ \frac{2R_1\sin\theta_1}{\pi^2 a^4}\left(2\log 2a + \gamma - 2 - \frac{i\pi}{8}\right). \tag{5.54}$$

(It is worth noting here that the extension of the perturbation series for the outer potential has resulted in the complete eigen-solution $R_1\sin\theta_1 - 1$ appearing as the asymptotic form for ψ_o as $R_1 \to \infty$, instead of only $R_1\sin\theta_1$ as was the case previously (see 5.23)).

If (5.51) is substituted in (5.24) and terms of order higher than ε^4 neglected then use of the equations (E) in §5.2 gives

$$\varepsilon^4\log\varepsilon\psi_{1X_1}(0,Y_1) + \varepsilon^4[\psi_{2X_1}(0,Y_1) + \frac{Y_1}{a}\psi_{oY_1}(0,Y_1) + \frac{Y_1^2}{2a}\psi_{oY_1Y_1}(0,Y_1)] = 0$$

(since it has already been arranged that lower order terms vanish). Equating coefficients of the gauge factors here to zero gives

$$\psi_{1X_1}(0,Y_1) = 0, \tag{5.55}$$

and (when 5.25) is used)

$$\psi_{2X_1}(0,Y_1) = -Y_1/\pi a^4 . \tag{5.56}$$

In addition ψ_1, ψ_2 must be harmonic and satisfy both the free surface condition and the edge condition $R_1 \frac{\partial\psi_1}{\partial R_1} \to 0$ as $R_1 \to 0$. It follows immediately from (5.53) that

$$\psi_1(X_1,Y_1) = -\frac{4}{\pi^2 a^4}(Y_1 - 1) , \tag{5.57}$$

while a problem similar to that for ψ_2 is considered in chapter 4 (see §4.4(a)). From the equation at the end of §4.4(a), it is seen that the solution satisfying (5.54) and (5.56) is (with $z_1 = X_1 + jY_1$)

$$\psi_2(X_1,Y_1) = -\frac{1}{\pi a^4}\left(\frac{R_1^2\sin2\theta_1-2R_1\cos\theta_1}{2}\right) - \frac{2}{\pi^2 a^4}(R_1\sin\theta_1+R_1\theta_1\cos\theta_1-1-\log R_1)$$

$$+ \frac{2}{\pi^2 a^4}\left(2\log2a+\gamma-2-\frac{i\pi}{8}\right)(Y_1-1) + \frac{2}{\pi^2 a^4}\,\mathrm{Re}_j\,[e^{jz_1}E_1(jz_1)]$$

$$+ \frac{2i}{\pi a^4}\exp(i X_1 - Y_1) \tag{5.58}$$

(since terms of lower orders (as $R_1 \to \infty$) than those required in the matching (5.54) can be added if required).

It follows now that the wave part of $\psi(X_1,Y_1;\varepsilon)$ to order ε^4 is given by

$$W^{(4)}(X_1,Y_1) = \frac{2i}{\pi}\left(\frac{\varepsilon}{a}\right)^4 \exp(i X_1 - Y_1) . \tag{5.59}$$

Expressed in outer coordinates this takes the form

$$W^{(4)}(X_1,Y_1) = \frac{2i}{\pi}\left(\frac{\varepsilon}{a}\right)^4 \exp\left[\frac{-i(x+a)}{\varepsilon} - \frac{y}{\varepsilon}\right]$$

so, since the incoming wave has the potential

$$\phi^I = \exp\left[\frac{-i(x-a)}{\varepsilon} - \frac{y}{\varepsilon}\right],$$

the transmission coefficient to order ε^4 is given by

$$T^{(4)} = \frac{2i}{\pi}\left(\frac{\varepsilon}{a}\right)^4 \exp\left(-\frac{2ia}{\varepsilon}\right)$$

(in agreement with Leppington (1973a) p.141 (4.5)).

It is now possible to proceed to the fifth order estimate for T without further matching as detailed in the next section.

§5.8 The transmission coefficient to order ε^5

Examination of the higher asymptotic forms of $\phi_0(x,y)$, $\phi_1(x,y)$ and $\phi_2(x,y)$ when z is replaced by $-a-\delta_1 e^{-j\theta_1}$ and $\delta_1 \to 0$ indicate that $\psi^{(5)}$ (which would be obtained from the matching principle $\psi^{(5,4)} = \phi^{(4,5)}$) will certainly contain terms with scalings $\varepsilon^5\log\varepsilon$ and ε^5. However, the presence of terms with other scalings lying between ε^4 and ε^5 as $\varepsilon \to 0$ can not be rejected so it is postulated that

$$\psi^{(5)}(X_1,Y_1\,;\varepsilon) = \varepsilon^3\psi_0(X_1,Y_1) + \varepsilon^4\log\varepsilon\psi_1(X_1,Y_1) + \varepsilon^4\psi_2(X_1,Y_1)$$

$$+ s(\varepsilon)\psi_s(X_1,Y_1) + \varepsilon^5\log\varepsilon\psi_3(X_1,Y_1) + \varepsilon^5\psi_4(X_1,Y_1)\,, \qquad (5.60)$$

where $\varepsilon^4 < s(\varepsilon) < \varepsilon^5$ as $\varepsilon \to 0$ and the term $s(\varepsilon)\,\psi_s(X_1,Y_1)$ is to be considered as a typical term of this type which may be present.

(5.60) is now substituted into (5.24) and the series truncated after terms of order ε^5. The result (using again the equations (E) and the fact that the coefficients of scale factors up to order ε^4 have been chosen to vanish previously) is

$$s(\varepsilon)\,\psi_{sX_1}(0,Y_1) + \varepsilon^5\log\varepsilon[\psi_{3X_1}(0,Y_1) + \frac{Y_1}{a}\psi_{1Y_1}(0,Y_1) + \frac{Y_1^2}{2a}\psi_{1Y_1Y_1}(0,Y_1)]$$

$$+ \varepsilon^5[\psi_{4X_1}(0,Y_1) + \frac{Y_1}{a}\psi_{2Y_1}(0,Y_1) + \frac{Y_1^2}{2a}\psi_{2Y_1Y_1}(0,Y_1) - \frac{Y_1^3}{2a^2}\psi_{oX_1Y_1}(0,Y_1)$$

$$- \frac{Y_1^4}{8a^2}\psi_{oX_1Y_1Y_1}(0,Y_1)] = 0\,.$$

Equating coefficients of the various gauge factors to zero and using (5.25), (5.57) gives

$$\psi_{sX_1}(0,Y_1) = 0 ,$$

$$\psi_{3X_1}(0,Y_1) = \frac{4}{\pi^2 a^5} Y_1 , \qquad (5.61)$$

and

$$\psi_{4X_1}(0,Y_1) = -\frac{1}{2a}\frac{d}{dY_1}[Y_1^2 \psi_{2Y_1}(0,Y_1)] . \qquad (5.62)$$

In addition, as usual, ψ_s, ψ_3, ψ_4 are harmonic and satisfy the free surface and edge conditions.

It follows immediately that $\psi_s(X_1,Y_1)$ is wave free, so that the transmission coefficient will be determined to order ε^5 once the wave parts of ψ_3 and ψ_4 have been obtained. In the case of ψ_3 this presents no difficulties since comparison of (5.61) and (5.56) indicates that

$$\psi_3(X_1,Y_1) = -\frac{4}{\pi a} \psi_2(X_1,Y_1) + \text{(eigensolutions)}$$

whence, by use of (5.58), $W_3(X_1,Y_1)$ (the wave part of $\psi_3(X_1,Y_1)$) is given by

$$W_3(X_1,Y_1) = -\frac{8i}{\pi^2 a^5} \exp (i X_1 - Y_1). \qquad (5.63)$$

It remains to derive the wave part of $\psi_4(X_1,Y_1)$.

Reference to (5.58) shows that

$$\frac{\pi^2 a^4}{2}\psi_2(0,Y_1) = -(Y_1 \log Y_1 - 1 - \log Y_1) + (2\log 2a + \gamma - 2 - \frac{i\pi}{8})(Y_1 - 1)$$

$$+ \mathrm{Re}_j [e^{-Y_1} \int_{jY_1}^{\infty} \frac{e^{-jv}}{v} dv] + \pi i e^{-Y},$$

(where the exponential integral

$$\int_{jz}^{\infty} \frac{e^{-ju}}{u} du$$

has been recast first in the form

$$\int_z^{-j\infty} \frac{e^{-jv}}{v}\, dv$$

by means of the transformation $u = j\,v$, then in the form

$$\int_z^{\infty} \frac{e^{-jv}}{v}\, dv$$

by rotation of the contour of integration through the fourth quadrant so that the upper limit becomes $+\infty$). It then follows (after two differentiations and use of (5.62) that

$$\psi_{4\,X_1}(0,Y_1) = \frac{2}{\pi^2 a^5}\,[Y_1 \log Y_1 + (3 - \gamma - 2\log 2a + \frac{i\pi}{8})Y_1]$$

$$+ \frac{1}{\pi^2 a^5}\,[\mathrm{Re}_j \,(\int_{jY_1}^{\infty} \frac{e^{-jv}}{v}\, dv) + \pi i]\, \frac{d}{dY_1}(Y_1^2\, e^{-Y_1}) \,. \tag{5.64}$$

The progressive wave generated by the unbounded term $Y_1 \log Y_1$ is proved in §4.4(b) to be $\frac{4i}{\pi^2 a^5}(\gamma-1)\exp(i\,X_1-Y_1)$ while that produced by the second unbounded term can be shown (in the same way as for W_3 above) to be $\frac{-4i}{\pi^2 a^5}(3-\gamma-2\log 2a + \frac{i\pi}{8})\exp(i\,X_1-Y_1)$. The complete Havelock wavemaker solution can be applied to the term

$$\frac{i}{\pi a^5}\,\frac{d}{dY_1}\,(Y_1^2\, e^{-Y_1})$$

giving a wave part

$$\frac{2}{\pi a^5}\exp(i\,X_1-Y_1) \int_0^{\infty} e^{-s}\,\frac{d}{ds}\,(s^2 e^{-s})\, ds \,,$$

and evaluation of the integral shows this contribution to W_4 to be $\frac{1}{2\pi a^5}\exp(i\,X_1-Y_1)$. It remains to find the progressive wave produced by the term $\frac{1}{\pi^2 a^5}\,\frac{d}{dY_1}\,(Y_1^2\, e^{-Y_1})\ \mathrm{Re}_j\,(\int_{jY_1}^{\infty} \frac{e^{-jv}}{v}\, dv)$ and this is achieved in Appendix D (§D.3) where it is shown in fact to give zero wave contribution. Hence, by combining the three non-zero wave contributions mentioned above, the wave part of $\psi_4(X_1 Y_1)$ is seen to be given by

$$W_4(X_1,Y_1) = \frac{8i}{\pi^2 a^5}\,(\gamma+\log 2a-2-\frac{i\pi}{8}) \quad \exp(i\,X_1-Y_1). \tag{5.65}$$

Reference to (5.60) show shows that $W^{(5)}(X_1,Y_1;\varepsilon)$ (the wave part of $\psi^{(5)}(X_1,Y_1;\varepsilon)$ to order ε^5) is given by

$$W^{(5)}(X_1,Y_1;\varepsilon) = W^{(4)}(X_1,Y_1;\varepsilon) + \varepsilon^5 \log\varepsilon\, W_3(X_1,Y_1) + \varepsilon^5 W_4(X_1,Y_1),$$

whence use of (5.59), (5.63) and (5.65) gives

$$W^{(5)}(X_1,Y_1;\varepsilon) = [\frac{2i}{\pi}(\frac{\varepsilon}{a})^4 - \frac{8i}{\pi^2}(\frac{\varepsilon}{a})^5 \log\varepsilon + \frac{8i}{\pi^2}(\frac{\varepsilon}{a})^5(\gamma+\log 2a-2-\frac{i\pi}{8})]\exp(i\ X_1-Y_1),$$

or

$$W^{(5)}(X_1,Y_1;\varepsilon) = \frac{2i}{\pi}(\frac{\varepsilon}{a})^4[1-\frac{4}{\pi}(\frac{\varepsilon}{a})\log(\frac{\varepsilon}{a}) - \frac{4}{\pi}(\frac{\varepsilon}{a})(2-\gamma-\log 2+\frac{i\pi}{8})]\exp(i\ X_1-Y_1).$$

When this is expressed in outer coordinates and compared with the incoming wave it is seen that the transmission coefficient to order ε^5 is given by

$$T^{(5)} = \frac{2i}{\pi}(\frac{\varepsilon}{a})^4 \exp(^{-2ia}/\varepsilon)[1-\frac{4}{\pi}(\frac{\varepsilon}{a})\log(\frac{\varepsilon}{a}) - \frac{4}{\pi}(\frac{\varepsilon}{a})(2-\gamma-\log 2+\frac{i\pi}{8})].$$

The second term in $T^{(5)}$ agrees with Leppington (1973a, p.142) while the third term is the one which completes the fifth order asymptotics.

§5.9 Estimate of the error term for $T^{(5)}$

The order of the error term in the formula

$$T \sim \frac{2i}{\pi}(\frac{\varepsilon}{a})^4 \exp(-2ia/\varepsilon)[1-\frac{4}{\pi}(\frac{\varepsilon}{a})\log(\frac{\varepsilon}{a})-\frac{4}{\pi}(\frac{\varepsilon}{a})(2-\gamma-\log 2+\frac{i\pi}{8})]$$

depends on the form of the sixth order terms in the perturbation series for the potential in the left inner region and this, in turn, is determined by the higher forms of the asymptotic developments of the potential in the outer and right inner regions. A full discussion of these higher forms is given in chapter 7, §7.3 and, in this section, some of the results appearing there are anticipated.

The next approximation for the right inner potential is

$$\Phi^{(3)} = \Phi_0 + \varepsilon\,\Phi_1 + \varepsilon^2\,\Phi_2 + \varepsilon^3\,\Phi_3\,.$$

Examination of the higher asymptotic forms of F_1 and F_2 (the wave free parts of Φ_1 and Φ_2 respectively) shows that F_1 contains higher terms of orders $\frac{1}{R^2}$ and $\frac{1}{R^3}$ while F_2 contains higher terms with orders $\frac{1}{R^2}$ and $\frac{\log R}{R^2}$. In addition the velocity distribution $\Phi_{3X}(0,Y)$ is of order $\frac{1}{Y}$ as $Y \to \infty$ so that, from the discussion in chapter 4, §4.3, the wave free part of Φ_3 in the far field will contain terms which, as $R \to \infty$, are $0(1)$, $0\left(\frac{\log R}{R}\right)$ and $0\left(\frac{1}{R}\right)$. (The $0(1)$ term matches with the term $-\frac{4}{\pi^2 a^3}\theta$ which has already been noted in the expansion of Φ_2 near E_+ in §5.6). Moreover, it also contains a term which is $0\left(\frac{(\log R)^2}{R}\right)$.

It follows then from the matching principle $\phi^{(3,4)} = \phi^{(4,3)}$ that $\phi^{(4)}$ will be of the form

$$\phi^{(4)} = \varepsilon^2 \phi_0 + \varepsilon^3 \log \varepsilon\, \phi_1 + \varepsilon^3 \phi_2 + \varepsilon^4 (\log\varepsilon)^2 \phi_3 + \varepsilon^4 \log\varepsilon\, \phi_4 + \varepsilon^4 \phi_5 .$$

The potentials ϕ_3, ϕ_4 and ϕ_5 will be harmonic, satisfy $\frac{\partial \phi}{\partial r} = 0$ on $r = a$ and also tend to zero as $r \to \infty$. Formal substitution of $\phi = \phi^{(4)}$ in the surface condition (neglecting terms of order higher than ε^4) gives the further conditions $\phi_3 = 0$, $\phi_4 = -\phi_{1y}$, $\phi_5 = -\phi_{2y}$ on $y = 0$, $|x|>a$. ϕ_3 can be found explicitly. It is equal to $\frac{1}{\pi^2 a^2}\phi_0$. Hence the matching principle indicates that $\psi^{(5)}$ will contain a term $\varepsilon^5 (\log\varepsilon)^2 \psi_E$ (where ψ_E is a multiple of the eigensolution $(Y_1 - 1)$) and that $\psi^{(6)}$ will contain a term $\varepsilon^6 (\log\varepsilon)^2 \psi_6$ where ψ_6 does have a wave term arising from ψ_E. Hence the next term in the development of the transmission coefficient will be of order $(\log N)^2/N^6$. (The exact value of this term is derived in chapter 7).

§5.10 Comparison of asymptotic values of T with those obtained using multiple expansions for N = 8(1)20

It has been proved that the fifth order asymptotic formula for the transmission coefficient is

$$T^{(5)} = \frac{2i}{\pi N^4} \exp(-2iN) \left[1 + \frac{4}{\pi N} \log N - \frac{4}{\pi N}\left(2 - \gamma - \log 2 + \frac{i\pi}{8}\right)\right]$$

where $N = {}^{a}/\varepsilon$.

The real and imaginary parts are given by

$$Re(T^{(5)}) = \frac{2}{\pi N^4}\{[1+\frac{4}{\pi N}(\log 2N+\gamma-2)]\sin 2N + \frac{\cos 2N}{2N}\} \quad (5.66)$$

$$Im(T^{(5)}) = \frac{2}{\pi N^4}\{[1+\frac{4}{\pi N}(\log 2N+\gamma-2)]\cos 2N - \frac{\sin 2N}{2N}\} \quad (5.67)$$

These formulae enable a comparison to be made between the values of T obtained numerically and the asymptotic values for an intermediate range of values of N (the range chosen was N = 8(1)20). The comparison is presented in tabular form and graphical form at the end of this section.

In every case the magnitude of the difference between the computed and fifth order asymptotic values is less than $(\log N)^2/N^6$ (the order of the error term). The occurrence of very small relative differences of less than 1% for the larger values of N provides strong evidence of a region of overlap.

<u>TABLES 9 and 10</u> (Overleaf)

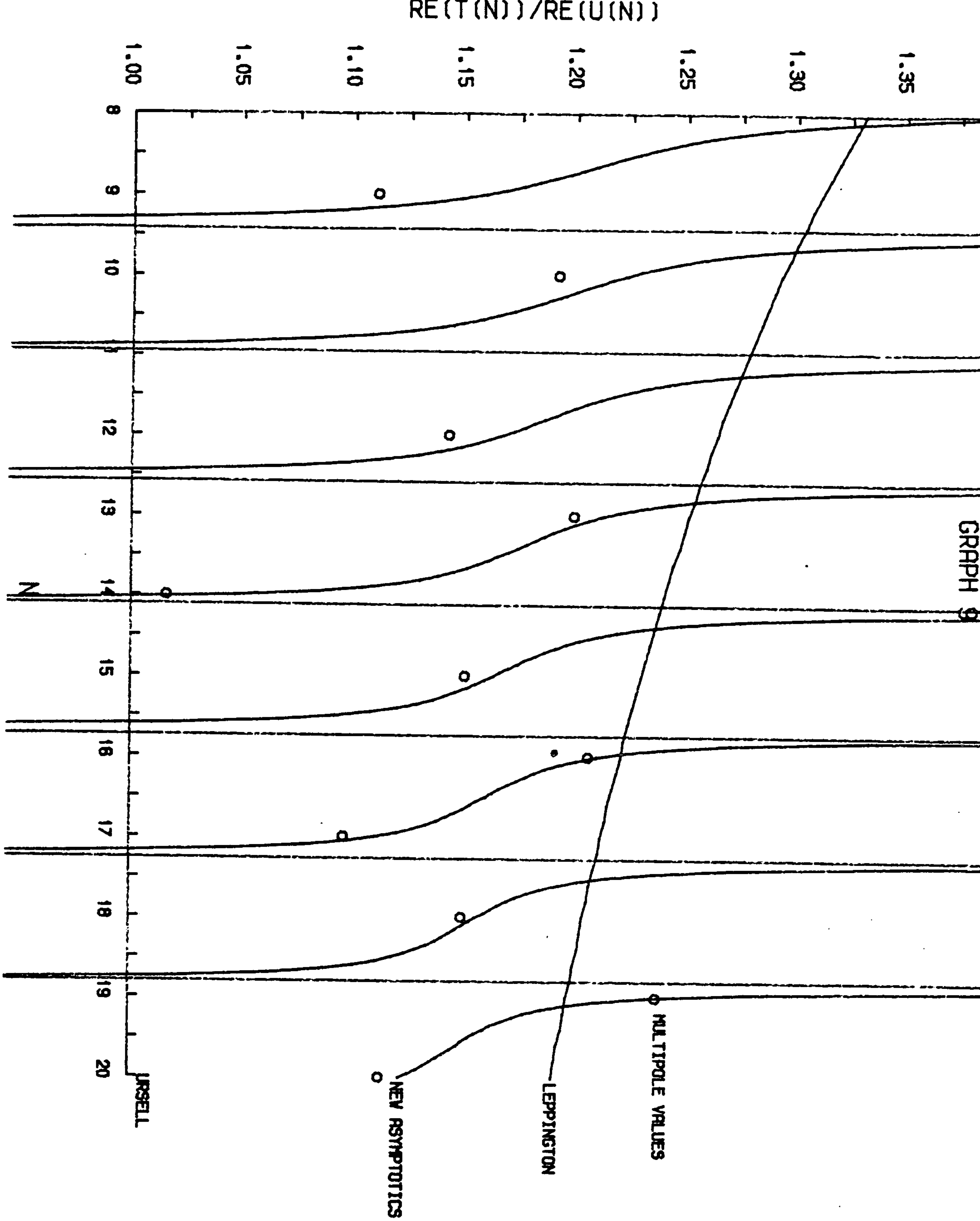
Comparison of the values of the real and imaginary parts of T(N) as obtained from (A) multipole expansions, (B) Leppington's asymptotics, and (C) the asymptotic formulae (5.66), (5.67). Column (D) contains the value of $(\log N)^2/N^6$. The values have been scaled by multiplying them by 10^6 for N = 8(1)15 and by 10^7 for N = 16(1)20.

TABLE 9 (Re(T(N))

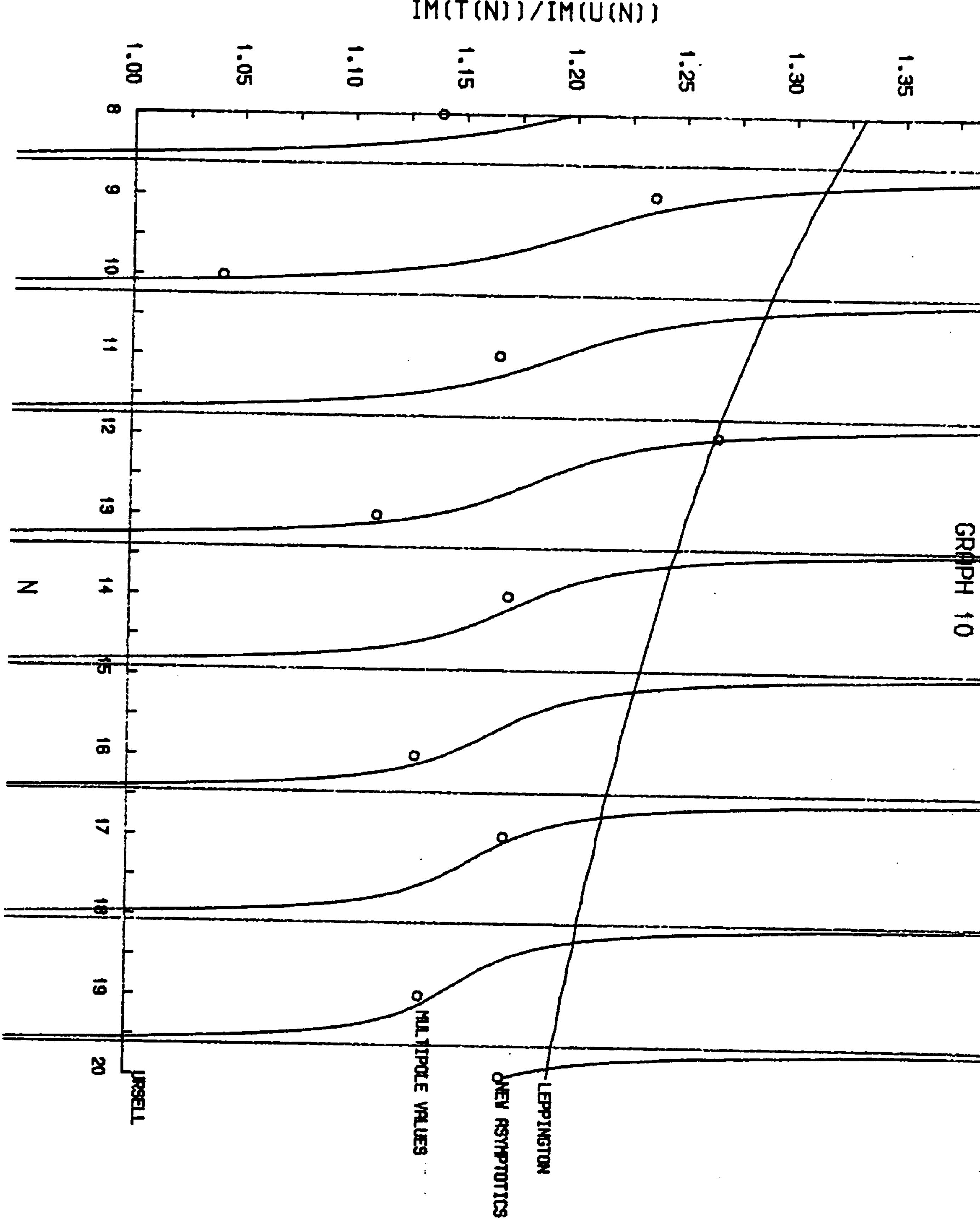
N	A	B	C	D	
8	-62.2	-59.6	-63.7	16.5	
9	-80.9	-95.5	-84.4	9.08	
10	69.3	75.1	71.1	5.30	
11	-2.68	-0.5	-2.44	3.25	x 10^{-6}
12	-31.8	-35.1	-32.4	2.07	
13	20.4	21.3	20.6	1.36	
14	4.56	5.57	4.70	0.925	
15	-14.3	-15.3	-14.4	0.644	
16	64.6	65.3	64.8	4.58	
17	44.2	48.9	44.8	3.33	x 10^{-7}
18	-69.2	-72.4	-69.6	2.46	
19	17.9	17.3	17.9	1.71	
20	33.0	35.3	33.3	1.40	

TABLE 10 (Im(T(N))

N	A	B	C	D	
8	-170	-198	-178	16.5	
9	79.2	84.0	81.4	9.08	
10	27.0	33.6	28.3	5.30	
11	-50.7	-55.5	-51.9	3.25	x 10^{-6}
12	16.5	16.5	16.6	2.07	
13	16.0	18.0	16.4	1.36	
14	-18.7	-19.8	-18.9	0.925	
15	2.71	2.4	2.68	0.644	
16	91.5	98.9	92.5	4.58	
17	-75.7	-78.4	-76.1	3.33	x 10^{-7}
18	-7.0	-9.35	-7.28	2.46	
19	52.9	55.9	53.2	1.71	
20	-31.0	-31.6	-31.1	1.40	



Comparison of the values of the real part of the transmission coefficient as given by multipole expansions, Ursell's asymptotic formula, Leppington's asymptotic formula and the new asymptotic formula (5.66). The values are normalised by Ursell's real part so that his results are illustrated by the horizontal line through 1 on the vertical axis. At N = 11 the multipole value is about seven times bigger than Ursell's value (see Table 7 in §2.8) and cannot be shown (in scaled form) on the graph.



Comparison of the values of the imaginary part of the transmission coefficient as given by multipole expansions, Ursell's asymptotic formula, Leppington's asymptotic formula and the new asymptotic formula (5.67). The values are normalised by Ursell's imaginary part so that his results are illustrated by the horizontal line through 1 on the vertical axis. At N = 18, the multipole value is approximately 0.9 times Ursell's value, so the scaled form of the multipole value ($\simeq$0.9) cannot be shown on the graph.

Chapter 6

§6.1 Introduction

The published work on the 2D radiation and scattering problems for bodies whose surface piercing tangents are non-vertical is much scarcer then in the vertical tangent case although the associated "sloping beach" problem (the generalisation of the vertical barrier problem discussed in detail in Chap.4) has received considerable attention (this will be considered in more detail in §6.2).

With regard to the radiation problem, Holford (1965) has presented a heuristic method for the determination of the leading term in the asymptotic form of the radiated wave amplitude in high frequency heaving of a cylinder of arbitrary cross-section although he considers only cases for which α, the angle at the intersection between the cross section and the free surface, measured in the fluid is not acute. His result agrees with those derived rigorously by Ursell(1953) (for a semi-circular cylinder) and Holford (1964) for a dock. Szu-Hsiung (1984) has extended Holford's work to the case of arbitrary high-frequency oscillatory motions giving the first two terms in the wave-amplitude asymptotics. The first of these agrees with Holford's term in the case of vertical oscillations and vanishes when the tangent at the waterline is vertical. This explains the step from $0\left(\frac{1}{N}\right)$ to $0\left(\frac{1}{N^2}\right)$ in the order of the amplitude of the radiated wave as $N \to \infty$. No discontinuity is involved. The first term simply contains a factor $\sin \mu\pi (\mu \doteq \pi/2\alpha)$, which tends continuously to zero as $\mu \to 1$. (Recall that $N = \frac{a}{\varepsilon}$, where a = semi-beam, $2\pi\varepsilon$ = wavelength).

Alker (1977) notes a similar "jump" from $0\left(\frac{1}{N^2}\right)$ to $0\left(\frac{1}{N^4}\right)$ in the order of magnitude of the transmission coefficient in the scattering problem for a partly submerged circular cylinder (for the same reason) although in this case the three leading terms in the general result tend continuously to zero as $\mu \to 1$. Alker's results apply formally to all surface angles of intersection so a drop in the order of magnitude in the transmission coefficient may be expected to occur at all angles of the form $\pi/2n$, where n is an integer, although clearly this will be difficult to observe numerically for acute angles since α then can never be more than $22\frac{1}{2}°$ different from an angle of this form.

The most important development from a numerical point of view has been the application of the method of null field equations to water wave problems by Martin.P.A. (1981),(1984). The method is applicable to general cross sections although, in his numerical work, Martin considers only normal intersection at the water surface (heaving and scattering in the case of a semi-submerged elliptic cylinder). His results agree well, (in the special case of a circular cylinder) with those obtained using Ursell's multipole method.

The purpose of this chapter is to apply the null field equations to the scattering problem for circular cylinders passing through two fixed points in the water surface at various angles, α. The range of values discussed is $45° \leq \alpha \leq 165°$ so that a regime not covered by John's (1950) uniqueness theorem is also considered. Results conforming to the traditional numerical tests, $|R|^2 + |T|^2 = 1$ and $|\arg R - \arg T| = \pi/2$ are obtained in this case also. In addition, in cases where α is acute, "interference" effects seem to occur at certain diameter/wavelength ratios in the sense that as $\frac{d}{\varepsilon}$ (d = diameter, $2\pi\varepsilon$ = wavelength) is increased with α fixed values occur (depending on α) at which the magnitude of the transmission coefficient suddenly drops by an above average amount. As $\frac{d}{\varepsilon}$ is increased beyond this value the magnitude of T gradually increases back to the previous orders of magnitude before the expected decrease sets in again. (This is illustrated in Graphs 11, 12). The values of $\frac{d}{\varepsilon}$ at which this effect was observed initially were seen to increase with α . For $\alpha = 45°, 60°, 75°$ the values of $\frac{d}{\varepsilon}$ at which T starts to increase are approximately 0.3, 0.6, 1.2 respectively or, with ref. to table 11a, at values of N between 0.7 and 0.8, 1.5 and 2.0, 3.5 and 4.0 respectively. There is however, no sign of the phenomenon in the case of the semi-submerged circular cylinder, $\alpha = 90°$ (up to $N = 20$) nor in the obtuse angle cases (up to $N = 10$).

For α in the range $135° - 160°$ the values of the real and imaginary parts of T are relatively large and accurate for $7 \leq N \leq 10$ so that comparison with the values obtained from the first two terms of Alker's asymptotic result become feasible. However, the magnitude of the error term (which is $O(1/N^{2\mu+1})$) increases with α and, in general, only one significant figure of agreement was obtainable, the relative differences being mainly of between 10% and 20% (see table 12). To produce stronger evidence of an overlap region will require numerical values of the third term and it is anticipated that such values could be obtained, without great difficulty, for a special angle of the form $p\pi/2n$ (p odd, n an integer). The simplest case for which suitably accurate values are given by the null field method for $7 \leq N \leq 10$ is $^{3\pi}/4$ for then the functions appearing in the third term have special forms which are more easily programmable than in the general case. This has not, however, been attempted here.

Finally, in this section, it may be remarked that since the sloping beach problem plays a crucial role in the application of the method of matched asymptotic expansions to the general radiation and transmission problem it seems worthwhile to give a brief outline of the developments leading to the solution for a general angle and then a short discussion of the behaviour of those particular solutions which satisfy the edge condition. This is the purpose of the next section.

§6.2 The sloping beach problem

The pioneering work of Hanson (1926) who considered angles of the form $\frac{\pi}{2n}$ (n an integer) and derived a standing wave solution bounded at the shore line does not seem to have been added to till Bondi (1943) derived a solution (for the same form of angle) with source like behaviour on the shore. Lewy (1946) then extended this work to obtain two analogous standing wave solutions for angles of the form $\frac{p\pi}{2n}$ (p and n integers such that p is odd and $2n > p$) and was thus able to derive progressive wave solutions for such angles by suitable combination of the standing waves. Stoker (1947) returned to angles of the form $\frac{\pi}{2n}$ but, unlike Lewy, did not assume from the outset that the solutions behave at ∞ like those on an ocean of infinite depth. Both writers use Lewy's method of reducing the problem to that of solving an ordinary non-homogeneous differential equation with constant coefficients for the complex potential (the method used in Chap.4 is a special case of this where no assumptions have been made about the behaviour at the shore line and at infinity). Lewy's solutions do not include the dock problem ($p = 2$, $n = 1$) but this was solved by him in conjunction with Friedrichs (1948) by assuming a solution for the complex potential in the form of a Laplace type integral $\frac{1}{2\pi} \int_P e^{z\zeta} f(\zeta) d\zeta$ and showing that f must satisfy a certain difference equation. By choosing the path P suitably they again constructed two standing-wave type solutions. In essence this was the method used by Peters (1950) (working on a more general problem) to derive the two different standing wave type solutions for any angle including π) (in the general case the function f is the solution of a differential difference equation). He had, however, been preceded by Isaacson (1950) who adapted Lewy's idea of continuous dependence of the solution on the angle of slope to deduce a form of solution which would be applicable for any angle and included Lewy's solutions as special cases. He then checks that the solution so derived does solve the problem in the general case. More recently, Morris (1974) has derived a solution having source like behaviour at a general point of the sector formed by the beach and the water surface. This solution is bounded at the shore line and gives an outgoing wave train at infinity.

A fundamental difference between solutions of the general sloping beach problem and the vertical barrier case is that solutions satisfying the edge condition $\delta \frac{\partial\phi}{\partial\delta} \to 0$ as $\delta \to 0$ and having progressive waves at ∞ do exist. Indeed the only cases in which such solutions do not exist (unless there is an incoming wave) are when the angle of slope is of the form $\frac{\pi}{2n}$. A short discussion is now given of the precise behaviour at the shore-line of the solutions which satisfy the edge condition (termed eigensolutions by analogy with the vertical barrier case) and have progressive waves at infinity.

§6.3 Behaviour of eigensolutions at the shore line.

(Note that scaled coordinates $X = Kx$, $Y = ky$ are used and that Peters' y-axis is upwards whereas throughout this work, the y-axis is taken downwards. In addition, the letter μ in Peters' work is used to denote π/α where α is the angle of slope. Here it denotes $\frac{\pi}{2\alpha}$.)

Reference, then, to Peters', p.333 and p.336, shows that the solutions satisfying an edge condition are the real parts of the complex functions

$$W_q(Z) = i\lambda \int_P \frac{e^{Z\zeta} g(\zeta)\, d\zeta}{\zeta^{2q\mu}(\zeta - i)} \qquad (Z = X + iY,\ q = 0,1,2,\ldots)$$

where λ is an arbitrary real constant and

$$g(\zeta) = \exp\left[-\frac{1}{\pi}\int_0^\infty \log\left(\frac{1-t^{-2\mu}}{1-t^{-2}}\right) \frac{\zeta}{t^2+\zeta^2}\, dt\right] .$$

The path P is defined as follows. If $\frac{g(\zeta)}{\zeta^{2\mu q}}$ has no branch points at the origin, then P is described in the positive direction on a circle of radius > 1 centre the origin. Otherwise a cut along $\arg \zeta = \pm\pi - \frac{1}{2}\alpha$ (to ensure convergence of the integral when $0 < \arg z < \alpha$) and linear paths extending to infinity on each side of the cut are added to the circle.

Peters proves (p.335) that as $Z \to \infty$

$$W_q(Z) \sim -2\pi\lambda \exp(-i\pi q\mu) g\,(e^{i\pi/2}) e^{iZ} \quad \text{and}$$

it can be shown that $g(e^{i\pi/2}) = \exp(-i\chi)/\mu^{\frac{1}{2}}$

where $\chi = \frac{1}{4}(1-\mu)\pi$.

Thus, if $E_q(X,Y)$ denotes the real part of $W_q(Z)$,

$$E_q(X,Y) \sim -\frac{2\pi\lambda}{\mu^{\frac{1}{2}}} \cos(X - \chi - \pi q\mu) e^{-Y} \quad \text{as } X \to \infty$$

and, if λ is chosen to be $-\frac{\mu^{\frac{1}{2}}}{\pi}$, then

$$E_q(X,Y) \sim 2\cos(X - \chi - \pi q\mu) e^{-Y} \quad \text{as } X \to \infty .$$

When the incoming waves are eliminated from two such solutions $E_m(X,Y)$ and $E_n(X,Y)$ $(m \neq n)$ it is seen that the solutions

$$e_{m,n}(X,Y) \overset{D}{=} \exp(i\pi n\mu) E_m(X,Y) - \exp(i\pi m\mu) E_n(X,Y)$$

$(m \neq n,\ m,n = 0,1,2,\ldots)$

have the property

$$e_{m,n}(X,Y) \sim -2i \sin[(m-n)\pi\mu] \exp[i(X-\chi)-Y] \quad \text{as } X \to \infty$$

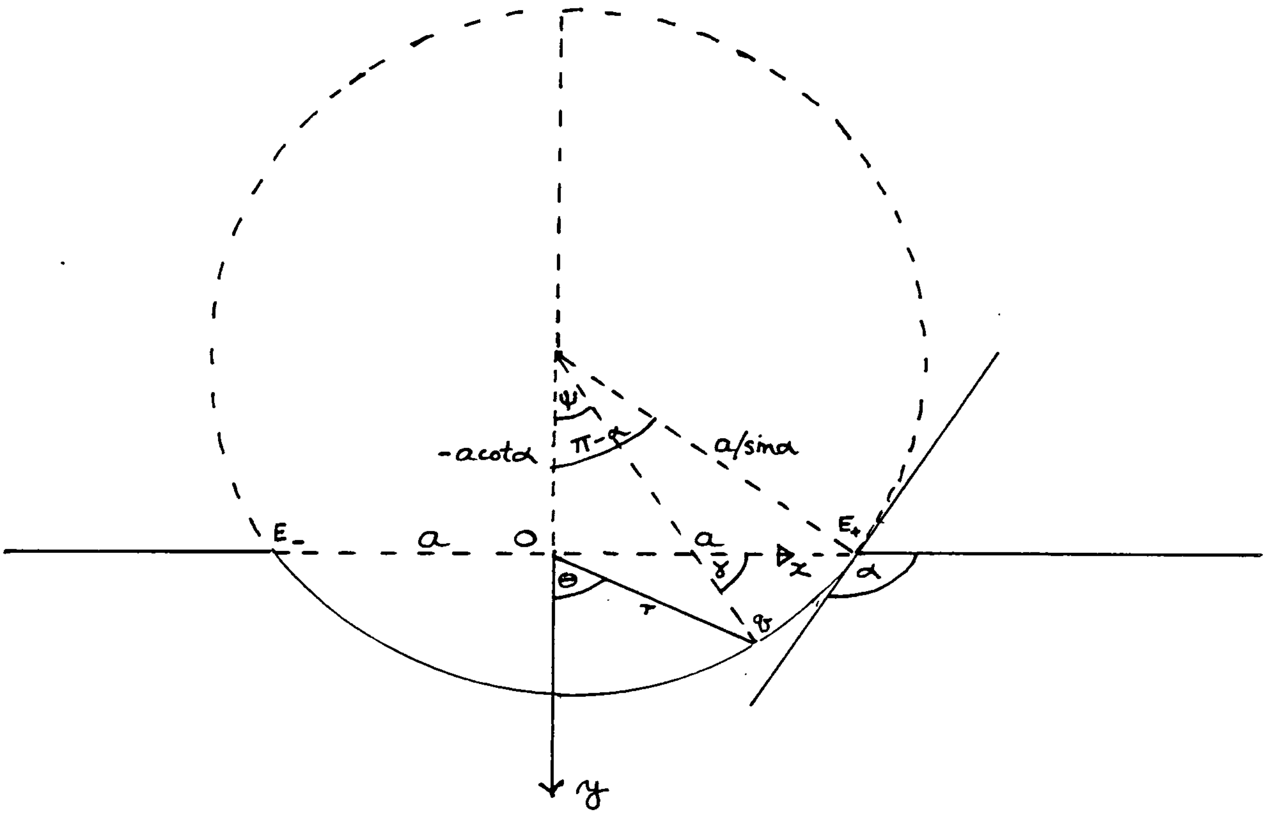
and (from Peters p.336 again)

$$e_{m,n}(X,Y) \sim$$

$$\begin{cases} 2\exp(i\pi n\mu)\mu^{\frac{1}{2}} R^{2m\mu} \cos(2m\mu\theta)/\Gamma(1+2m\mu) & \text{if } m = \min(m,n) \\ -2\exp(i\pi m\mu)\mu^{\frac{1}{2}} R^{2n\mu} \cos(2n\mu\theta)/\Gamma(1+2n\mu) & \text{if } n = \min(m,n) \end{cases}$$

Fig. 3

Notation used in Application of Null Field Equations



as $(X,Y) \to (0,0)$, where $X = R\cos\theta$, $Y = R\sin\theta$.

Clearly if μ is an integer i.e. if the angle of slope is of the form $\frac{\pi}{2r}$ (r an integer) then these solutions are wave-free.

§6.4 Application of the Null Field Equations to the scattering problem for a partly immersed circular cylinder.

Martin (1981) has described the application in detail for the radiation problem so only a brief summary will be given of the main results needed for the scattering problem. Reference should also be made to the discussion and notation mentioned in §1.4 in connection with the integral equation method and to the statement of the general problem for the diffraction potential in §1.2 eq.(1.1)-(1.6).

As in the case of the semi-circular cylinder, new coordinates X,Y will be introduced such that $X = Kx$, $Y = Ky$. The equations (1.1) - (1.6) then become :

$$\frac{\partial^2\phi}{\partial X^2} + \frac{\partial^2\phi}{\partial Y} = 0 \quad \text{in } D$$

$$\phi + \frac{\partial\phi}{\partial Y} = 0 \quad \text{on } S$$

$$\frac{\partial\phi}{\partial n} = -\frac{\partial}{\partial n}[\exp(-iX-Y)] \quad \text{on } \Gamma \tag{6.1}$$

$$|\nabla\phi| \to 0 \quad \text{as} \quad Y \to +\infty .$$

$$\phi(X,Y) \sim R\exp(i\,X-Y) \quad \text{as} \quad X \to +\infty$$

$$\phi(X,Y) \sim (T-1)\exp(-i\,X-Y) \quad \text{as} \quad X \to -\infty .$$

Thus (with reference to fig.3) a wave of fixed length 2π is considered to be incident on a circular cylinder passing through the points $(\pm N,0)$ and making an angle α (measured through the fluid) with the water surface at $E\pm$. S is $Y = 0$, $|X| > N$ and Γ is parameterised by the angle, ψ, between the downward vertical through the centre of the cylinder and the radius to a general point q of Γ $(-(\pi-\alpha) \leq \psi \leq \pi-\alpha)$.

(ψ seems to be the most convenient parameter having a natural transition from the cases when α is obtuse to those when α is acute).

It follows that the radius of the cylinder is $\frac{N}{\sin\alpha}$ and that the arc length s_q measured from the lowest point of Γ to a general point q of Γ is $\frac{N\psi}{\sin\alpha}$.

Thus on Γ

$$\left.\begin{aligned}
S_q &= \frac{N\psi}{\sin\alpha}\,, \\
X_q &= \frac{N\sin\psi}{\sin\alpha} \\
Y_q &= \frac{N(\cos\psi+\cos\alpha)}{\sin\alpha} \\
R_q &\overset{D}{=} \sqrt{X_q^2+Y_q^2} = \frac{N}{\sin\alpha}\sqrt{1+\cos^2\alpha+2\cos\alpha\cos\psi} \\
\sin\theta_q &= \frac{\sin\psi}{\sqrt{1+\cos^2\alpha+2\cos\alpha\cos\psi}} \\
\gamma &= \frac{\pi}{2}-\psi
\end{aligned}\right\} \qquad (6.2)$$

where γ is the angle which the normal at q into the fluid makes with the positive direction of the x-axis.

In the case $K=1$, the Green's function due to John (1950) has the form

$$G(P,Q) = \tfrac{1}{2}\log\frac{(X-\xi)^2+(Y-\eta)^2}{(X-\xi)^2+(Y+\eta)^2} - 2\int_0^\infty e^{-(Y+\eta)\lambda}\cos\lambda(X-\xi)\,\frac{d\lambda}{\lambda-1}$$

where the path of integration passes below the pole of the integrand at $\lambda = 1$ and it is noted that G is harmonic in the whole half-plane $Y > 0$ and satisfies the condition $G + G_Y = 0$ on the whole of $Y = 0$.

In addition to the equations

$$2\pi\phi(P) = \int_\Gamma [G(P,q)\,\frac{\partial}{\partial n_q}\phi(q) - \phi(q)\,\frac{\partial}{\partial n_q}G(P,q)]ds_q \qquad (6.3)$$

and $$\pi\phi(p) = \int_\Gamma [G(p,q)\,\frac{\partial}{\partial n_q}\phi(q) - \phi(q)\,\frac{\partial}{\partial n_q}G(p,q)]ds_q$$

(mentioned in §1.4) for the values of the potential in D and on Γ , Martin (1981) uses a third integral relation

$$0 = \int_\Gamma [G(P_-,q) \frac{\partial}{\partial n_q} \phi(q) - \phi(q) \frac{\partial}{\partial n_q} G(P_-,q)] ds_q \quad . \tag{6.4}$$

Here P_- is a point between the segment E_-E_+ and Γ (the interior region) and this last equation asserts that the potential generated by the source and dipole distributions over Γ, which are used to represent the actual potential in D, vanishes in this interior region. (This is not to say that the continuation of the actual exterior field vanishes there). This last equation leads to what Martin calls 'the null field equations for water-waves' as follows. Firstly, Ursell (1981) has derived the bilinear expansion of the Green's function

$$G(P,Q) = \sum_{m=o}^{\infty} \sum_{\sigma=1}^{2} \alpha_m^{\sigma}(P)\Phi_m^{\sigma}(Q) \tag{6.5}$$

when $r_Q > r_P$, where (for $K = 1$)

$$\alpha_o^1(P) = -2e^{-Y}\cos X \,, \qquad \alpha_o^2(P) = 2e^{-Y}\sin X \tag{6.6}$$

$$\Phi_o^1(P) = \int_o^\infty e^{-\lambda Y} \cos\lambda X \frac{d\lambda}{\lambda-1} \,, \qquad \Phi_o^2(P) = -\frac{\partial}{\partial X} \Phi_o^1(P) \tag{6.7}$$

$$\Phi_m^1(P) = \frac{\cos 2m\theta}{R^{2m}} + \frac{1}{2m-1} \frac{\cos(2m-1)\theta}{R^{2m-1}} \,, \qquad \Phi_m^2(P) = \frac{\sin(2m+1)\theta}{R^{2m+1}} + \frac{1}{2m} \frac{\sin 2m\theta}{R^{2m}} \tag{6.8}$$

for $m \geq 1$.

(The other $\alpha_m^\sigma(P)$ $(m \geq 1)$ are given in Martin (1981) p.328 but will not be needed explicitly here. It is sufficient to note that they are regular and satisfy the free surface condition.)

Clearly the Φ_m^1 are even in X and the Φ_m^2 odd. (6.9)

Suppose now that a semi-circle, centre O, is drawn in the interior region with diameter lying on the free surface. If P_- is any point within the circle then certainly $r_{P_-} < r_q$ so that (6.5) can be substituted into (6.4) giving

$$0 = \sum_{m=0}^{\infty} \sum_{\sigma=1}^{2} \alpha_m^{\sigma}(P\_) \int_\Gamma [\phi(q) \frac{\partial}{\partial n_q} \Phi_m^{\sigma}(q) - \frac{\partial}{\partial \bar{n}_q} \phi(q) \ \Phi_m^{\sigma}(q)] ds_q$$

whence

$$\int_\Gamma \phi(q) \frac{\partial}{\partial n_q} \Phi_m^{\sigma}(q) ds_q = - \int_\Gamma \Phi_m^{\sigma}(q) \frac{\partial}{\partial n_q} [\exp(-iX_q - Y_q)] ds_q \qquad (6.10)$$

(using (6.1)) where $\sigma = 1,2$: $m = 0,1,2,\ldots$). (6.10) are the null field equations for water waves and their numerical solution will be the subject of the next section.

Before proceeding to this, however, the expression for $\phi(P)$ in terms of $\Phi_m^{\sigma}(P)$ is derived. This is possible when P lies outside a semi-circle, with centre 0 and diameter along the free surface, and containing Γ . Then $r_P > r_Q$ so that from (6.5) and the symmetry of G ,

$$G(P,q) = \sum_{m=0}^{\infty} \sum_{\sigma=1}^{2} \alpha_m^{\sigma}(q) \Phi_m^{\sigma}(P) \quad .$$

Substitution of this in (6.3) gives

$$\phi(P) = \sum_{m=0}^{\infty} \sum_{\sigma=1}^{2} c_m^{\sigma} \Phi_m^{\sigma}(P) \qquad (6.11)$$

where $$c_m^{\sigma} = \frac{1}{2\pi} \int_\Gamma [\alpha_m^{\sigma}(q) \frac{\partial}{\partial n_q} \phi(q) - \phi(q) \frac{\partial}{\partial n_q} \alpha_m^{\sigma}(q)] ds_q$$

or when (6.1) is used again .

$$c_m^{\sigma} = - \frac{1}{2\pi} \int_\Gamma \{\alpha_m^{\sigma}(q) \frac{\partial}{\partial n_q} [\exp(-iX_q - Y_q)] + \phi(q) \frac{\partial}{\partial n_q} \alpha_m^{\sigma}(q)\} ds_q \quad (6.1$$

$$(\sigma = 1,2; \quad m = 0,1,2,\ldots).$$

(6.11) is exactly the form used for the potential in the case of the semi-submerged circular cylinder in Chap.2, (2.17). Indeed reference to (2.9) - (2.11) (with $Z = R \sin\theta + jR\cos\theta = jRe^{-j\theta}$) shows that, for $m \geq 1$,

$$Re_j(e_m(Z) = (-1)^{m+1} \Phi_m^{1}(P) \ , \quad Re_j(e_m'(Z)) = (-1)^m 2m\Phi_m^{2}(P) \ .$$

In addition $\Phi_0^1(P)$ (for $X > 0$) can be written in the alternative form $\int_0^\infty \frac{u \cos uY - \sin uY}{u^2+1} e^{-uX} du + \pi i \exp(iX-Y)$, (by writing $\cos\lambda X = \frac{1}{2}(e^{i\lambda X} + e^{-i\lambda X})$ expressing the integral as the sum of two integrals and closing in the third or fourth quadrants as appropriate). Alternatively

$$\Phi_o^1(P) = \mathrm{Re}_j(e^{jZ} E_1(jZ)) + \pi i \exp(iX-Y) \text{ and },$$

when the definitions of $s(Z)$ and $w(Z)$ in (2.17) are recalled from (2.10) and (2.11) it is clear that for $X > 0$

$$\mathrm{Re}_j(s(Z) + i\pi w(Z)) = \Phi_o^1(P)$$

and hence also for all X since both functions are even in X.

Finally

$$\mathrm{Re}_j(s'(z)+i\pi w'(z)) = \frac{\partial}{\partial x}\mathrm{Re}_j(s(z)+i\pi w(z)) = \frac{\partial}{\partial x}\Phi_o^1(P) = -\Phi_o^2(P) .$$

Thus the form (2.17) for the complex potential for the semi-cylindrical scatterer is just a special case of (6.11) and therefore fully justified. Comparison also with (2.18) and (2.19) shows that, in terms of the c_m^σ

$$R = \pi(ic_o^1 + c_o^2)$$
$$T = 1 + \pi(ic_o^1 - c_o^2) \qquad (6.13)$$

§6.5. The Numerical Solution of the Null Field Equations

The potential $\phi(q)$ on the cylinder is first written in the form

$$\phi(q) = \sum_{n=o}^{\infty}\sum_{\ell=1}^{2} d_n^\ell \phi_n^\ell(q)$$ where the ϕ_n^1 are even in X

and the ϕ_n^2 are odd and both are bases on the part of Γ for which $X > 0$ (which will be denoted by Γ_+).

This form is substituted in (6.10) to give

$$\sum_{n=o}^{\infty}\sum_{\ell=1}^{2} d_n^\ell \int_\Gamma \phi_n^\ell(q)\frac{\partial}{\partial n_q}\Phi_m^\sigma(q)ds_q = \int_\Gamma V(q)\,\Phi_m^\sigma(q)ds_q \qquad (6.14)$$

where $V(q) = -\frac{\partial}{\partial n_q}[\exp(-i X_q - Y_q)]$ $\qquad (\sigma = 1,2;\ m = 0,1,2,\ldots).$

It is now noted that

$$\frac{\partial}{\partial n_q} \Phi_m^{\sigma}(q) = (\cos\gamma \frac{\partial}{\partial X} + \sin\gamma \frac{\partial}{\partial Y}) \Phi_m^{\sigma}(q)$$ and that

$\cos\gamma$ is odd in X while $\sin\gamma$ is even $(X \to -X \Rightarrow \gamma \to \pi - \gamma)$.

It follows that $\frac{\partial}{\partial n_q} \Phi_m^{\sigma}(q)$ has the same parity as $\Phi_m^{\sigma}(q)$ and that the integral on the left in (6.14) will vanish if $\ell \neq \sigma$ since the integrand in this case will be an odd function (this will be true in general for bodies symmetric about $Y = 0$). Hence the equations (6.14) will decouple into two systems

$$\sum_{n=0}^{\infty} d_n^1 \int_\Gamma \phi_n^1(q) \frac{\partial}{\partial n_q} \Phi_m^1(q)\,ds_q = \int_\Gamma V(q)\Phi_m^1(q)\,ds_q \tag{6.15}$$

and
$$\sum_{n=0}^{\infty} d_n^2 \int_\Gamma \phi_n^2(q) \frac{\partial}{\partial n_q} \Phi_m^2(q)\,ds_q = \int_\Gamma V(q)\Phi_m^2(q)\,ds_q . \tag{6.16}$$

Since $V(q) = -\frac{\partial}{\partial n_q}[\exp(-i X_q - Y_q)]$ it is easily shown that $V(q) = \exp(-Y_q)\,[\sin(X_q + \gamma) + i\cos(X_q + \gamma)]$ in which the real part is even and the imaginary part odd or, since $\gamma = \frac{\pi}{2} - \psi$,

$$V(q) = \exp(-Y_q)\,[\cos(X_q - \psi) - i\sin(X_q - \psi)] .$$

When it is recalled that $s_q = \frac{N\psi}{\sin\alpha}$ and that the Φ_m^1 are even while the Φ_m^2 are odd, the equations (6.15), (6.16) can be further simplified to the forms

$$\sum_{n=0}^{\infty} A_{mn}\, d_n^1 = a_m \qquad (m = 0,1,2,\ldots)$$

and
$$\sum_{n=0}^{\infty} B_{mn}\, d_n^2 = b_m$$ where

$$A_{mn} = \int_0^{\pi-\alpha} \phi_n^1(q) \frac{\partial}{\partial n_q} \Phi_m^1(q)\,d\psi, \quad a_m = \int_0^{\pi-\alpha} \exp(-Y_q)\cos(X_q - \psi)\Phi_m^1(q)\,d\psi .$$

$$B_{mn} = \int_0^{\pi-\alpha} \phi_n^2(q) \frac{\partial}{\partial n_q} \Phi_m^2(q)\,d\psi, \quad b_m = -i\int_0^{\pi-\alpha} \exp(-Y_q)\sin(X_q - \psi)\Phi_m^2(q)\,d\psi .$$

The procedure now is to truncate each set of equations at $n = M$, substitute $m = 0,1,2,\ldots, M$ and then solve the resulting set of $(M+1)$ by $(M+1)$ equations to obtain numerical approximations for the expansion coefficients d_n^ℓ. $(\ell = 1,2)$.

The following forms gave accurate values for the values of the $\Phi_m^{\sigma}(P)$ and their derivatives.

(1) It is recalled that

$$\Phi_o^{1}(P) = \mathrm{Re}_j\,[e^{jZ}E_1(jZ)] + \pi i \exp(iX-Y) \qquad (Z = X + jY)\ .$$

By the methods used in §2.5 this can be written in the form

$$\Phi_o^{1}(P) = -Y^2\int_0^1 \frac{u\exp[Y(u-1)]}{Y^2u^2 + X^2}\,du - e^{-Y}[\mathrm{Ci}(X)\cos X + (\mathrm{Si}(X)-\pi/2)\sin X]$$

$$+ \pi i\, e^{iX-Y}\ . \qquad \text{for } X > 0\ .$$

For $X = 0$ i.e. at the lowest point of the cylinder the value of Φ_o^1 is obtained by letting $X \to o+$ above.

The value is found to be

$$h\int_0^1 \log u \exp[h(u-1)]du - \exp(-h)(\gamma + \log h - \pi i)$$

where $h = N\cot\frac{\alpha}{2}$. (The proof is similar to that at the end of §2.5).

(2) $$\frac{\partial\Phi_o^1}{\partial X} = \mathrm{Re}_j\,\frac{d}{dZ}\,[e^{jZ}\,E_1(jZ)] - \pi\exp(i\,X-Y)$$

$$= -\frac{X}{X^2+Y^2} - \mathrm{Im}_j\,[e^{jZ}\,E_1(jZ)] - \pi\exp(i\,X-Y)$$

$$= -\frac{X}{X^2+Y^2} + XY\int_0^1 \frac{\exp[Y(u-1)]}{Y^2u^2 + X^2}\,du + e^{-Y}[\mathrm{Ci}(X)\sin X - (\mathrm{Si}(X) - \tfrac{\pi}{2})\cos X]$$

$$-\pi\exp(iX-Y)\ . \qquad \text{(from §2.5)}\ .$$

(3) In a similar way

$$\frac{\partial\Phi_o^1}{\partial Y} = -\frac{Y}{X^2 + Y^2} - \Phi_o^1(X,Y)\ .$$

(4) $$\Phi_o^2(X,Y) = -\frac{\partial\Phi_o^1}{\partial X} \qquad \text{(by definition)}$$

(5) $$\frac{\partial\Phi_o^2}{\partial X} = -\frac{\partial^2\Phi_o^1}{\partial X^2} = \frac{\partial^2\Phi_o^1}{\partial Y^2} = \frac{\partial}{\partial Y}\left(-\frac{Y}{X^2+Y^2} - \Phi_o^1(X,Y)\right), \text{ from (3)}$$

or $$\frac{\partial\Phi_o^2}{\partial X} = \frac{Y^2 - X^2}{(X^2+Y^2)^2} + \frac{Y}{X^2+Y^2} + \Phi_o^1(X,Y)\ .$$

(6) $$\frac{\partial\Phi_o^2}{\partial Y} = -\frac{\partial}{\partial X}\left(\frac{\partial\Phi_o^1}{\partial Y}\right) = \frac{\partial}{\partial X}\left(\frac{Y}{X^2+Y^2} + \Phi_o^1(X,Y)\right)$$

or $$\frac{\partial\Phi_o^2}{\partial Y} = -\frac{2XY}{(X^2+Y^2)^2} - \Phi_o^2(X,Y)\ .$$

(7) For $m \geq 1$, $\Phi_m^{\,1} = \dfrac{\cos 2m\theta}{R^{2m}} + \dfrac{1}{2m-1} \cdot \dfrac{\cos(2m-1)\theta}{R^{2m-1}}$.

It can be proved without difficulty that

$$\frac{\partial\Phi_m^{\,1}}{\partial X} = -2m\left[\frac{\sin(2m+1)\theta}{R^{2m+1}} + \frac{1}{2m}\,\frac{\sin 2m\theta}{R^{2m}}\right]$$

$$\frac{\partial\Phi_m^{\,1}}{\partial Y} = -2m\left[\frac{\cos(2m+1)\theta}{R^{2m+1}} + \frac{1}{2m}\,\frac{\cos 2m\theta}{R^{2m}}\right] .$$

(8) For $m \geq 1$. $\Phi_m^{\,2} = \dfrac{\sin(2m+1)\theta}{R^{2m+1}} + \dfrac{1}{2m}\,\dfrac{\sin 2m\theta}{R^{2m}}$ so that

$$\frac{\partial\Phi_m^{\,2}}{\partial X} = (2m+1)\left[\frac{\cos(2m+2)\theta}{R^{2m+2}} + \frac{1}{2m+1}\,\frac{\cos(2m+1)\theta}{R^{2m+1}}\right]$$

$$\frac{\partial\Phi_m^{\,2}}{\partial Y} = -(2m+1)\left[\frac{\sin(2m+2)\theta}{R^{2m+2}} + \frac{1}{2m+1}\,\frac{\sin(2m+1)\theta}{R^{2m+1}}\right] .$$

By use of the results (1) - (8) (together with $\dfrac{\partial}{\partial n_q} = \sin\psi\dfrac{\partial}{\partial X} + \cos\psi\dfrac{\partial}{\partial Y}$) the values of the integrands in the null-field equations can be evaluated very accurately for any value of ψ when (6.2) is referred to also. The integrations can then be carried out using Simpson's rule.

§6.6 The formula for the transmission coefficient

From (6.13),

$$T = 1 + \pi(ic_o^{\,1} - c_o^{\,2})$$

where $\pi c_o^{\,\sigma} = \frac{1}{2}\int_\Gamma [\alpha_o^{\,\sigma}(q)V(q) - \phi(q)\dfrac{\partial}{\partial n_q}\alpha_o^{\,\sigma}(q)]ds_q$ ($\sigma=1,2$, $\alpha_o^{\,\sigma}(q)$ given by (6.5)).

Since $i\alpha_o^1 - \alpha_o^{\,2} = -2i\exp(iX-Y)$

$$\pi(ic_o^1 - c_o^{\,2}) = -i\int_\Gamma \{-\exp(iX_q - Y_q)\frac{\partial}{\partial n_q}[\exp(-iX_q - Y_q)] - \phi(q)\frac{\partial}{\partial n_q}[\exp(iX_q - Y_q)]\}ds_q .$$

It is easily shown that

$$\frac{\partial}{\partial n_q}[\exp(-i\,X_q - Y_q)] = -\exp(-i\,X_q - Y_q + i\psi) \quad \text{and}$$

$$\frac{\partial}{\partial n_q}[\exp(iX_q - Y_q)] = -\exp(i\,X_q - Y_q - i\psi) .$$

Hence

$$T = 1-i\int_T\{\exp(-2Y_q+i\psi) + \phi(q)\exp(-Y_q)[\cos(X_q-\psi)+ i\sin(X_q-\psi)]\}ds_q$$

or (when the expansion of $\phi(q)$ is used)

$$T = 1 - \frac{2iN}{\sin\alpha}\int_0^{\pi-\alpha}\exp(-2Y_q)\cos\psi - \frac{2iN}{\sin\alpha}\sum_{n=0}^{\infty} d_n^{\,1}\int_0^{\pi-\alpha}\phi_n^{\,1}(q)\exp(-Y_q)\cos(X_q-\psi)d\psi$$

$$+ \frac{2N}{\sin\alpha}\sum_{n=0}^{\infty} d_n^{\,2}\int_0^{\pi-\alpha}\phi_n^{\,2}(q)\exp(-Y_q)\sin(X_q-\psi)d\psi \, . \qquad (6.16)$$

It can be shown similarly that

$$R = -\frac{2iN}{\sin\alpha}\int_0^{\pi-\alpha}\exp(-2Y_q)\cos(2X_q-\psi)d\psi - \frac{2iN}{\sin\alpha}\sum_{n=0}^{\infty} d_n^{\,1}\int_0^{\pi-\alpha}\phi_n^{\,1}(q)\exp(-Y_q)\cos(X_q-\psi)d\psi$$

$$- \frac{2N}{\sin\alpha}\sum_{n=0}^{\infty} d_n^{\,2}\int_0^{\pi-\alpha}\phi_n^{\,2}(q)\exp(-Y_q)\sin(X_q-\psi)d\psi \; .$$

§6.7 Discussion of the numerical data

Three different pairs of bases were used for the potential ϕ_q on Γ_+

(a) Fourier bases: $\phi_n^{\,1} = \cos 2n\theta$, $\phi_n^{\,2} = \sin(2n+1)\theta$ $(n=0,1,2,...)$

(b) Chelyshev bases: $\phi_n^{\,1} = \cos\left[2n \arccos\left(\frac{\sin\psi}{\sin\alpha}\right)\right]$; $\phi_n^{\,2} = \cos\left[(2n+1)\arccos\left(\frac{\sin\psi}{\sin\alpha}\right)\right]$ $(n=0,1,2,...)$

(c) Multipole bases: $\phi_n^{\,1} = f_n(N;\theta)$; $\begin{cases}\phi_0^2 = 1 \\ \phi_n^2 = g_{n-1}(N;\theta) & (n=1,2,....)\end{cases}$

(see §2.3 for the definitions; after (2.21) for the $f_n(N;\theta)$ and after (2.27) for the $g_n(N;\theta)$) .

Use of $M+1$ terms of these bases in the representation for ϕ_q leads to a sequence of values T^M of the transmission coefficient (expansions of up to 30 terms were considered and a sequence for the reflection coefficient was also obtained). The "best estimate" of T from the set $\{T^M: M=1,...30\}$ will mean in this context the value of T^M such that $|T|^2+|R|^2$ is nearest to 1 and $|\arg T-\arg R|$ is nearest to $\pi/2$. With few exceptions this occurred for the same value of M . In any case the values of M at which this occurred never differed by more than 1 .

For α non-acute all the bases worked well, except near $180°$, and gave best estimates of T which were in agreement with each other and, in the case of $\alpha = 90°$, with those obtained by using multipole expansions. However, the Chebyshev bases gave sequences which converged more rapidly to the best estimate of T than the other two. For all α non-acute and values of N up to about 2.5 the best estimate occurred in the range $M \leq 15$ (very quickly for the smaller values). As N increased beyond this value the value of M giving the best estimate increased also (for α up to about 150°) and for values of N bigger than 5 expansions of more than 30 terms would be required to increase the accuracy of the estimates in Table 1. This did not apply to the cases when α was near 180°. The best estimate occurred early on in the sequence (round about $M=5$) but the accuracy was poor except for the lowest values of N.

Interestingly the Chebyshev bases were not so appropriate, for the acute angled case, except for small N, and gave unstable sequences of values for $|R|^2 + |T|^2$. The best base for these cases was found to be the multipole base which worked well down to $\alpha = 45°$. Thereafter the numerical stability of the whole system degenerated rapidly for all three bases.

Hence, to summarise, for $90° \leq \alpha \leq 150°$ and $0 < N \leq 2.5$ the Chebyshev base gives good estimates of T (to 4 significant figures generally) with series of less than 15 terms (only five or six terms are needed until N approaches 1) but for $N > 2.5$, in general, series of more than 30 terms would be needed to maintain this kind of accuracy. For $\alpha \geq 165°$ the best estimate occurs for $M \simeq 6$ but the accuracy is poor for the larger values of N and for $45° \leq \alpha < 90°$, the multipole base gives good estimates up to $N=3$ (usually 4 sig.figs.).

Examination of the sequences of real and imaginary parts of T shows that for $135° \leq \alpha \leq 160°$, two significant figures of accuracy can be maintained up to about $N = 10$ so that a comparison with Alker's (1977) asymptotic formula is possible. Table 12 contains the two sets of values together with the estimate of the error term obtained by taking the first two terms of Alker's result. Clearly there are positive signs of an overlap region but to obtain evidence of the same potency as in the semi-submerged case the third term of Alker's formula would have to be included in the comparison.

It may be noted also that the drop of two orders of magnitude in the value of $|T|$ for angles of the form $\frac{\pi}{2n}$ (predicted by theory), can be observed by comparing the values for $\alpha = 90°, 120°$ at $N = 6,7$. The acute angles $60°$ and $75°$ are themselves too near an angle of the form $\frac{\pi}{2n}$ for such a drop to be observed.

(Tables and graphs follow)

§6.8 TABLES AND GRAPHS

TABLE 11(a)

Values of $|T(N)|$ for α = 45° (15°) 90°

N \ α	45°	60°	75°	90°
0.01	9.998(-1)	9.998(-1)	9.998(-1)	9.998(-1)
0.02	9.991(-1)	9.992(-1)	9.992(-1)	9.992(-1)
0.03	9.978(-1)	9.981(-1)	9.982(-1)	9.983(-1)
0.04	9.958(-1)	9.965(-1)	9.968(-1)	9.970(-1)
0.05	9.928(-1)	9.943(-1)	9.950(-1)	9.953(-1)
0.06	9.886(-1)	9.915(-1)	9.927(-1)	9.933(-1)
0.07	9.830(-1)	9.879(-1)	9.898(-1)	9.908(-1)
0.08	9.755(-1)	9.835(-1)	9.865(-1)	9.880(-1)
0.09	9.659(-1)	9.781(-1)	9.825(-1)	9.847(-1)
0.1	9.538(-1)	9.717(-1)	9.780(-1)	9.810(-1)
0.2	6.453(-1)	8.327(-1)	8.929(-1)	9.183(-1)
0.3	2.611(-1)	5.754(-1)	7.359(-1)	8.105(-1)
0.4	8.625(-2)	3.397(-1)	5.513(-1)	6.760(-1)
0.5	2.488(-2)	1.902(-1)	3.915(-1)	5.421(-1)
0.6	5.943(-3)	1.054(-1)	2.735(-1)	4.263(-1)
0.7	2.079(-3)	5.824(-2)	1.914(-1)	3.337(-1)
0.8	3.044(-3)	3.183(-2)	1.351(-1)	2.621(-1)
0.9	5.047(-3)	1.703(-2)	9.637(-2)	2.073(-1)
1.0	6.792(-3)	8.797(-3)	6.943(-2)	1.655(-1)
1.5	8.038(-3)	3.313(-4)	1.499(-2)	6.093(-2)
2.0	5.16 (-3)	1.27 (-3)	3.48 (-3)	2.66 (-2)
2.5	2.96 (-3)	1.73 (-3)	7.5 (-4)	1.32 (-2)
3.0	1.67 (-3)	1.69 (-3)	1.2 (-4)	7.13 (-3)
3.5	9.6 (-4)	1.46 (-3)	1 (-5)	4.2 (-3)
4.0	5.7 (-4)	1.21 (-3)	3 (-5)	2.6 (-3)
4.5	3.5 (-4)	9.8 (-4)	6 (-5)	1.6 (-3)
5.0	2.1 (-4)	7.9 (-4)	1 (-4)	1.1 (-3)
6.0	9 (-5)	5.2 (-4)	1 (-4)	6 (-4)
7.0	7 (-5)	3.6 (-4)	1 (-4)	3 (-4)
8.0	4 (-5)	2.5 (-4)	–	–
9.0	–	1 (-4)	–	–
10.0	–	–	–	–

and

TABLE 11(b)

Values of $|T(N)|$ for = 105° (15°) 165°

N \ α	105°	120°	135°	150°	16
0.01	9.998(-1)	9.998(-1)	9.998(-1)	9.998(-1)	9.998(
0.02	9.993(-1)	9.993(-1)	9.993(-1)	9.993(-1)	9.993(
0.03	9.984(-1)	9.984(-1)	9.984(-1)	9.984(-1)	9.984(
0.04	9.971(-1)	9.972(-1)	9.972(-1)	9.973(-1)	9.973(
0.05	9.955(-1)	9.957(-1)	9.958(-1)	9.958(-1)	9.959(
0.06	9.936(-1)	9.939(-1)	9.940(-1)	9.941(-1)	9.942(
0.07	9.914(-1)	9.917(-1)	9.920(-1)	9.922(-1)	9.923(
0.08	9.888(-1)	9.893(-1)	9.897(-1)	9.900(-1)	9.902(
0.09	9.859(-1)	9.867(-1)	9.872(-1)	9.875(-1)	9.878(
0.1	9.826(-1)	9.837(-1)	9.844(-1)	9.849(-1)	9.852(
0.2	9.314(-1)	9.391(-1)	9.441(-1)	9.474(-1)	9.499(
0.3	8.491(-1)	8.714(-1)	8.854(-1)	8.949(-1)	9.016(
0.4	7.466(-1)	7.886(-1)	8.153(-1)	8.332(-1)	8.460(
0.5	6.389(-1)	7.003(-1)	7.405(-1)	7.679(-1)	7.875(
0.6	5.380(-1)	6.144(-1)	6.666(-1)	7.031(-1)	7.295(
0.7	4.501(-1)	5.359(-1)	5.973(-1)	6.415(-1)	6.741(
0.8	3.764(-1)	4.666(-1)	5.343(-1)	5.845(-1)	6.223(
0.9	3.159(-1)	4.070(-1)	4.782(-1)	5.328(-1)	5.748(
1.0	2.666(-1)	3.561(-1)	4.290(-1)	4.864(-1)	5.314(
1.5	1.257(-1)	1.957(-1)	2.621(-1)	3.209(-1)	3.72 (
2.0	6.833(-2)	1.202(-1)	1.748(-1)	2.27 (-1)	2.76 (
2.5	4.124(-2)	8.042(-2)	1.250(-1)	1.705(-1)	2.16 (
3.0	2.690(-2)	5.731(-2)	9.417(-2)	1.34 (-1)	1.74 (
3.5	1.862(-2)	4.285(-2)	7.386(-2)	1.09 (-1)	1.5 (
4.0	1.350(-2)	3.325(-2)	5.974(-2)	9.05 (-2)	1.3 (
4.5	1.016(-2)	2.658(-2)	4.95 (-2)	7.7 (-2)	1.1 (
5.0	7.88 (-3)	2.177(-2)	4.19 (-2)	6.7 (-2)	9.7 (
6.0	5.10 (-3)	1.543(-2)	3.14 (-2)	5.2 (-2)	8 (
7.0	3.54 (-3)	1.16 (-2)	2.5 (-2)	4.3 (-2)	6 (
8.0	2.59 (-3)	9.04 (-3)	2.0 (-2)	3.6 (-2)	6 (
9.0	1.97 (-3)	7.30 (-3)	1.7 (-2)	3.1 (-2)	5 (
10.0	1.56 (-3)	6.1 (-3)	1.4 (-2)	2.7 (-2)	4 (

TABLE 12

Comparison of values of Re(T(N)) and Im(T(N)) as obtained (a) by use of th
Null Field equations (b) by use of the first two terms of Alker's asymptoti
formula.*

(ER = order of magnitude of asymptotic error term)

	N \ α	135°	ER	140°	ER	145°	ER	150°	ER	155°	ER	160°	
(a)		0.0224		0.0269		0.032		0.037		0.042		0.048	
	7		0.011		0.012		0.013		0.014		0.015		0.
(b)		0.0185		0.0228		0.027		0.032		0.037		0.042	
(a)		-0.015		-0.0192		-0.024		-0.029		-0.035		-0.041	
	8		0.008		0.009		0.009		0.010		0.011		0.
(b)		-0.0121		-0.0157		-0.020		-0.024		-0.030		-0.035	
(a)		-0.0048		-0.0053		-0.0057		-0.0062		-0.0066		-0.007	
	9		0.006		0.007		0.007		0.008		0.009		0.
(b)		-0.0045		-0.0051		-0.0055		-0.0058		-0.0060		-0.006	
(a)		0.0142		0.0176		0.022		0.027		0.032		0.037	
	10		0.005		0.005		0.006		0.006		0.007		0.
(b)		0.0121		0.0153		0.019		0.023		0.027		0.031	

Im(T(N))

	N \ α	135°	ER	140°	ER	145°	ER	150°	ER	155°	ER	160°	
(a)		-0.0103		-0.0136		-0.017		-0.021		-0.026		-0.031	
	7		0.011		0.012		0.013		0.014		0.015		0.(
(b)		-0.0075		-0.0103		-0.014		-0.017		-0.021		-0.026	
(a)		-0.0132		-0.0156		-0.018		-0.021		-0.024		-0.027	
	8		0.008		0.009		0.009		0.010		0.011		0.(
(b)		-0.0114		-0.0138		-0.016		-0.019		-0.021		-0.023	
(a)		0.0160		0.020		0.025		0.030		0.036		0.043	
	9		0.006		0.007		0.007		0.008		0.009		0.(
(b)		0.0133		0.017		0.021		0.026		0.031		0.036	
(a)		-0.0019		-0.0030		-0.0044		-0.0060		-0.008		-0.010	
	10		0.005		0.005		0.006		0.006		0.007		0.(
(b)		-0.0013		-0.0022		-0.0033		-0.0048		-0.007		-0.009	

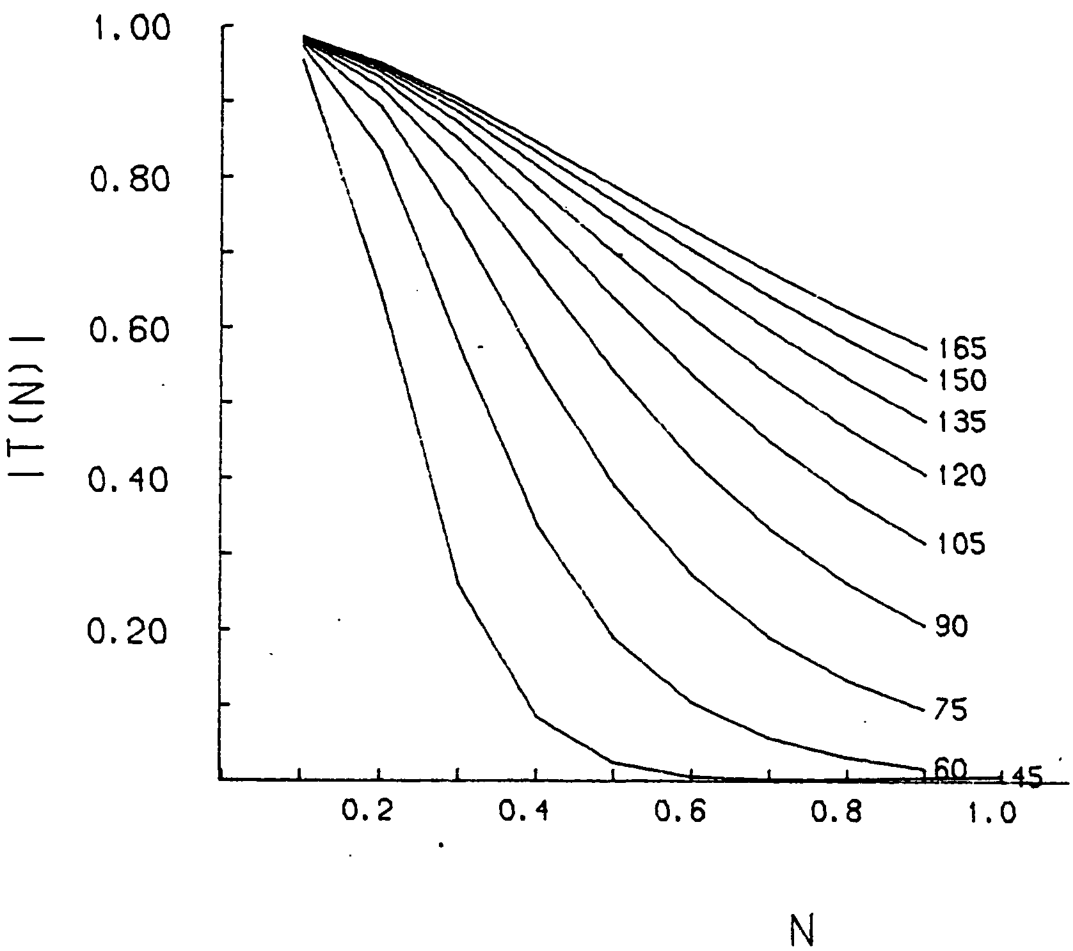
* $$\mathrm{Re}(T(N)) = F(N)\sin(2\chi + 2N) + O\left(\frac{1}{N^{2\mu+1}}\right)$$

$$\mathrm{Im}(T(N)) = F(N)\cos(2\chi + 2N) + O\left(\frac{1}{N^{2\mu+1}}\right)$$

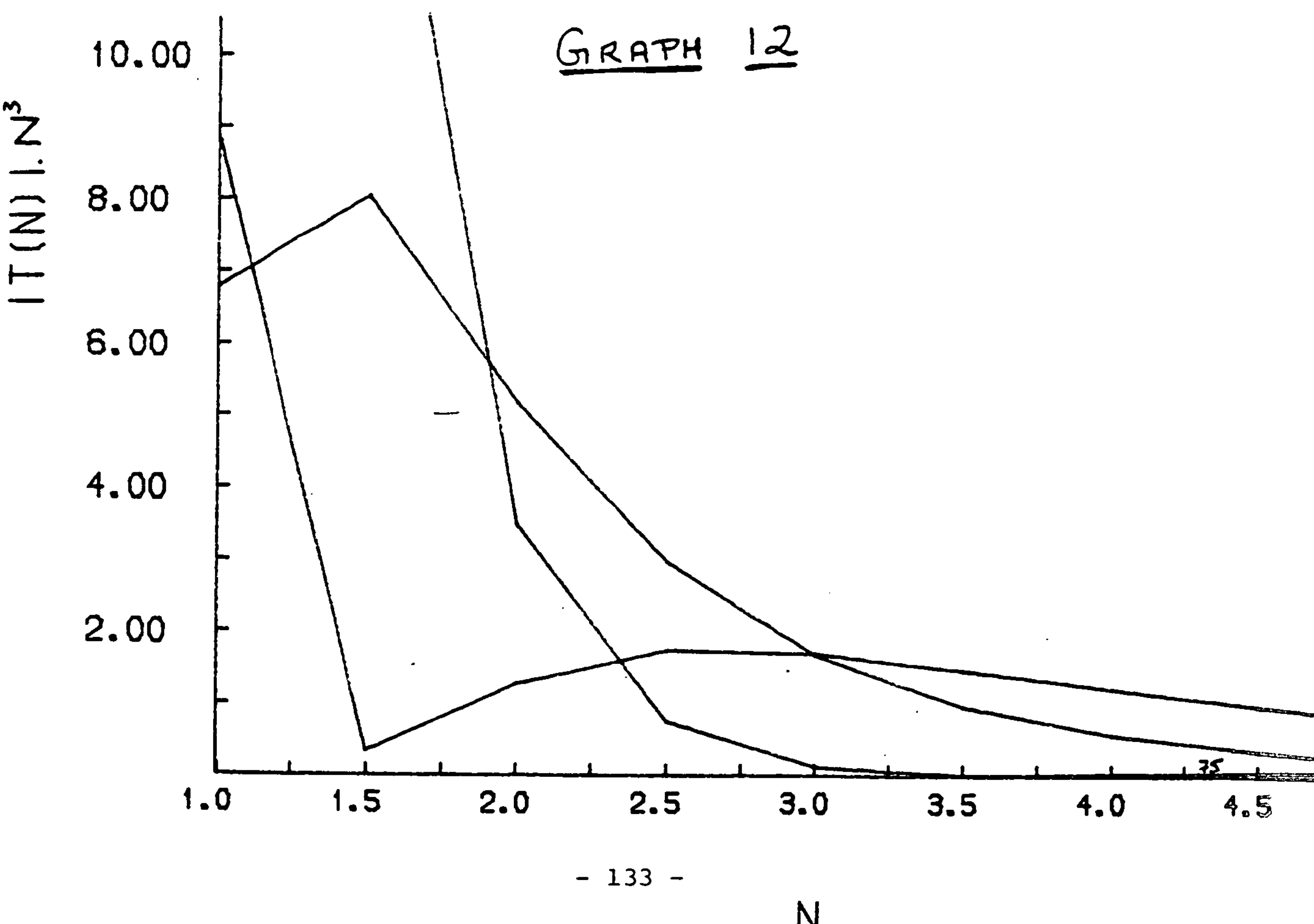
where

$$F(N) = \frac{\mu(\Gamma(\mu)\sin\mu\pi)^2}{\pi . 2^{2\mu-1}}\left[\frac{1}{N^{2\mu}} + \frac{4\mu^2}{\pi}\frac{\log N}{N^{2\mu+1}}\right] \quad \text{and} \quad \mu = \frac{\pi}{2\alpha}$$

GRAPH 11



GRAPH 12



CHAPTER 7

§7.1. Introduction

The asymptotic expansions of Chapter 5 are extended and further light is thereby thrown on the matching process as it is revealed that the asymptotics of the higher order terms in the outer potential near the left edge of the cylinder "fill in" the parts of solutions for terms in the left inner expansion whose appearance was previously justified by appeal to their being of lower order in the far field than demanded by the matching process at that stage. One simple case of this has already been noted in Chapter 5 (after eq. (5.54)) and a more striking example occurs in the case of ψ_2 (see §7.2 below).

The first two sixth order terms in the transmission coefficient are also found leading to the result

$$T = \frac{2i}{\pi N^4} \exp(-2iN)\left[1 + \frac{4}{\pi N}\log N - \frac{4}{\pi N}\left(2 - \gamma - \log 2 + \frac{i\pi}{8}\right) + \frac{8(\log N)^2}{\pi^2 N^2}\right.$$

$$\left. + \frac{8\log N}{\pi^2 N^2}\left(2\gamma + \log 4 - 5 - \frac{i\pi}{4}\right)\right] + O\left(\frac{1}{N^6}\right)$$

as $N \to \infty$ where $N = \frac{a}{\varepsilon}$.

§7.2. Summary of the calculations

The asymptotic series of Chapter 5 can be extended as follows:

(a) RIGHT INNER REGION

$$\Phi^{(3)} = \Phi_0 + \varepsilon\Phi_1 + \varepsilon^2\Phi_2 + \varepsilon^3\Phi_3$$

(b) OUTER REGION

$$\phi^{(4)} = \varepsilon^2\phi_0 + \varepsilon^3\log\varepsilon\phi_1 + \varepsilon^3\phi_2 + \varepsilon^4(\log\varepsilon)^2\phi_3 + \varepsilon^4\log\varepsilon\,\phi_4 + \varepsilon^4\phi_5$$

(c) LEFT INNER REGION

$$\psi^{(6)} = \varepsilon^3\psi_0 + \varepsilon^4\log\varepsilon\psi_1 + \varepsilon^4\psi_2 + \varepsilon^5(\log\varepsilon)^2\psi_5 + \varepsilon^5\log\varepsilon\psi_3 + \varepsilon^5\psi_4$$

$$+ s(\varepsilon)\psi_s + \varepsilon^6(\log\varepsilon)^2\psi_6 + \varepsilon^6\log\varepsilon\,\psi_7 + \varepsilon^6\psi_8$$

where $\varepsilon^5 \prec s(\varepsilon) \prec \varepsilon^6$ as $\varepsilon \to 0$ and $s(\varepsilon)$ is not of the same form as any of the other sixth order gauge factors which appear.

The full expansions of the wave-free parts of Φ_1 and Φ_2 to orders $\frac{1}{R^3}$ and $\frac{1}{R^2}$ respectively are detailed in Appendix C while the leading far field terms in the wave-free part of Φ_3 which are $O(1)$, $O[\frac{(\log R)^2}{R}]$ and $O\left\{\frac{\log R}{R}\right\}$ can be found explicitly. Part of the $O(\frac{1}{R})$ term is a multiple of $\frac{\sin\theta}{R}$ which is difficult to find and its extraction has not been attempted here.

The $O(1)$ term (which is just a multiple of θ) matches with the vortex singularity which has already been noted as occurring in ϕ_2, (see §5.6), while the forms of the $O[\frac{(\log R)^2}{R}]$ and $O\left\{\frac{\log R}{R}\right\}$ terms enable ϕ_3 and ϕ_4 to be found explicitly (ϕ_3 is a multiple of ϕ_1 while ϕ_4 is a linear combination of ϕ_2 and ϕ_1). ϕ_5 has been found to within an outer eigensolution (a multiple of ϕ_0) whose coefficient depends on the coefficient of the $\frac{\sin\theta}{R}$ term in Φ_3. Hence, when the matching principle

$$\psi^{(5,4)} = \phi^{(4,5)}$$

is applied, ψ_5 and ψ_3 can be found explicitly. ($\psi_5 = \frac{8}{\pi^2 a^2}\psi_0$ while $\psi_3 = -\frac{4}{\pi a}\psi_2 - \frac{2}{\pi a}\psi_1$).

In addition a term $\frac{2}{\pi^2 a^4}\log R_1$ (arising from a term $\frac{2}{\pi^2 a^4}\log \delta_1$ in the left edge asymptotics of ϕ_5) and a term

$\frac{2}{\pi^2 a^4}(3 - \gamma - 2\log 2a + i\frac{\pi}{8})$ (arising from a constant term in these asymptotics) are added to the asymptotic form of ψ_2 in the far field. Thus, the addition of these terms, which were not explicit at the previous matching stage $\psi^{(4,3)} = \phi^{(3,4)}$ (see §5.7, eq. (5.54)), in the solution for ψ_2 given in eq. (5.58) is fully justified. (It may be noted also that a term $\frac{4}{\pi^2 a^4}$ is added to the asymptotic form of ψ_1 thus "filling in" the part of its solution which was missing at the previous matching stage, see (5.53)).

Finally, since ψ_5 and ψ_3 are known explicitly, the waves generated by them in ψ_6 and ψ_7 respectively can be calculated as detailed in the next section.

Before proceeding to this, however, it will be convenient to make a slight change in the formulation of the problem for the right-inner region and introduce some convenient new notation. In the first instance, the problem is restated in terms of the total potential Φ^T i.e. the incoming wave e^{-iX-Y} is added on to the Φ of Chapters 3 and 5. This means that (to order ε^2):

$\Phi^T = \Phi_0{}^T + \varepsilon\Phi_1 + \varepsilon^2\Phi_2$ where $\Phi_0{}^T = 2e^{-Y}\cos X$ and Φ_1, Φ_2 are as before. The condition for Φ^T on $X = 0$ now becomes (cf. eq. (3.7)):

$$\sum_{r=0}^{\infty} g_r(\varepsilon,Y)\left[\frac{\partial^{r+1}}{\partial X^{r+1}}\Phi^T(0,Y;\varepsilon) - f'(\varepsilon Y)\frac{\partial^{r+1}}{\partial X^r\partial Y}\Phi^T(0,Y;\varepsilon)\right] = 0 \qquad (7.1)$$

with $f(y) = \sqrt{a^2 - y^2} - a$ and $g_r(\varepsilon,Y) = \frac{1}{r!}\left[\frac{f(\varepsilon Y)}{\varepsilon}\right]^r$ as before.

The distinguishing letter T will be dropped from now on in this discussion since no confusion will arise with the previous development.

Secondly the operator $\left(\frac{d}{dY}Y^K\frac{d}{dY}\right)$ will be denoted by M_K^Y and the Havelock integral $\int_0^\infty H(X,Y;s)\,f(s)\,ds$ (see §4.2) will be abbreviated to $\langle H,f\rangle$. If f is obtained from a function F(X,Y) of two variables by putting $X = 0$, this will be denoted by F^0.

§7.3. It can first be shown (by the same methods as for $\Phi^{(2)}$ in Chapter 5) that

$$\Phi^{(3)} = \Phi_0 + \varepsilon\Phi_1 + \varepsilon^2\Phi_2 + \varepsilon^3\Phi_3 \qquad (7.2)$$

where Φ_3 does not contain eigensolutions, since their presence would lead to violation of the edge condition by terms in the left-inner expansion and, possibly, to changes in terms which have already been determined in the outer expansion.

When (7.2) is substituted in (7.1) and terms of orders higher than ε^3 are neglected, the following equations are obtained (after noting that

$$\frac{f(\varepsilon Y)}{\varepsilon} = -\frac{1}{2}\frac{\varepsilon Y^2}{a} - \frac{1}{8}\frac{\varepsilon^3 Y^4}{a^3} + O(\varepsilon^5)$$

$$f'(\varepsilon Y) = -\frac{\varepsilon Y}{a} - \frac{1}{2}\frac{\varepsilon^3 Y^3}{a^3} + O(\varepsilon^5)$$

$$g_0(\varepsilon Y) = 1$$

$$g_1(\varepsilon Y) = -\frac{1}{2}\frac{\varepsilon Y^2}{a} - \frac{1}{8}\frac{\varepsilon^3 Y^4}{a^3} + O(\varepsilon^5)$$

$$g_2(\varepsilon Y) = \frac{1}{8}\frac{\varepsilon^2 Y^4}{a^2} + O(\varepsilon^4)$$

$$g_3(\varepsilon Y) = -\frac{1}{48}\frac{\varepsilon^3 Y^6}{a^3} + O(\varepsilon^5)\ \Big)$$

$$\Phi_{1X}{}^0 = -\frac{1}{2a} M_2(\Phi_0{}^0) \qquad \text{where } \Phi_0{}^0 = 2e^{-Y} \tag{7.3}$$

$$\Phi_{2X}{}^0 = -\frac{1}{2a} M_2(\Phi_1{}^0) \tag{7.4}$$

(these are just repetitions of previous equations in the new notation) and

$$\Phi_{3X}{}^0 = -\frac{1}{2a} M_2(\Phi_2{}^0) + \frac{1}{8a^2} M_4(\Phi_{1X}^{\ 0}) - \frac{1}{8a^3} M_4(\Phi_0{}^0) + \frac{1}{48a^3} M_6(\Phi_0{}^0).$$

When (7.3) and (7.4) are used and it is recalled that the wave part of Φ_2 is $\frac{1}{a^2}(\frac{2i}{3\pi} - \frac{1}{8})\exp(iX - Y)$ this becomes

$$\Phi_{3X}{}^0 = -\frac{1}{2a} M_2(F_2{}^0) - \frac{1}{2a^3}\left(\frac{2i}{3\pi} - \frac{1}{8}\right)M_2(e^{-Y}) - \frac{1}{8a^3}M_4M_2(e^{-Y})$$
$$- \frac{1}{4a^3}M_4(e^{-Y}) + \frac{1}{24a^3}M_6(e^{-Y})$$

where F_2 is the wave-free part of Φ_2 .

It is easily shown that $M_4M_2(e^{-Y}) = (2M_4 - 4M_5 + M_6)(e^{-Y})$ so that, consequently,

$$\Phi_{3X}{}^0 = -\frac{1}{2a} M_2(F_2{}^0) - \frac{1}{2a^3}\left(\frac{2i}{3\pi} - \frac{1}{8}\right)M_2(e^{-Y}) - \frac{1}{2a^3}M_4(e^{-Y}) + \frac{1}{2a^3}M_5(e^{-Y}) - \frac{1}{12a^3}M_6(e$$

and F_3 (the wave-free part of Φ_3) is given by $F_3 = \langle H_F, \Phi_{3X}{}^0\rangle$ where H_F

is the wave-free part of H.

The asymptotics of expressions of the form $\langle H_F, M_n(e^{-Y})\rangle$ $(n = 1, 2, \ldots)$ present no difficulties and can be found by the same methods as were employed in the case of $\langle H_F, M_2(e^{-Y})\rangle$ (for F_1, see §5.2). They will involve inverse powers of R only, the first term being a multiple of $\frac{\sin\theta}{R}$. Any higher order terms will, therefore, arise from the expression

$$E \overset{D}{=} -\frac{1}{2a}\langle H_F, M_2(F_2{}^0)\rangle \tag{7.5}$$

This is difficult to tackle directly as the explicit expression for E involves three and four dimensional integrals. Progress can be made, however, by using the known asymptotic form of $F_2(O,Y)$ as $Y \to \infty$ and the fact (proved in Appendix C, §C.3) that this can be differentiated twice to give the asymptotic form of $M_2(F_2{}^0)$.

From (C.12) in §C.2 of Appendix C it is seen that

$$F_2(O,Y) = \frac{8\log Y}{\pi^2a^2Y} - \frac{8}{\pi^2a^2Y}(\gamma - 2 + i\frac{\pi}{8}) + \frac{8\log Y}{\pi^2a^2Y^2} - \frac{8}{\pi^2a^2Y^2}(3 - \gamma + i\frac{\pi}{8}) + O(\frac{\log Y}{Y^3}$$

as $Y \to \infty$, the order term being deduced from theorem A in Appendix B which was used in finding the asymptotics of F_2. Hence, by differentiation,

$$-\frac{1}{2a}M_2(F_2{}^0) = \frac{4}{\pi^2a^3}[\frac{1}{Y} - \frac{2\log Y}{Y^2} + \frac{1}{Y^2}(9 - 2\gamma + i\frac{\pi}{4}) + O(\frac{\log Y}{Y^3})] \text{ as } Y \to \infty .$$

It follows that there exists a number $Y_0\,(>1)$ such that, for $Y \geqslant Y_0$

$$-\frac{1}{2a}M_2(F_2{}^0) = \frac{4}{\pi^2a^3}[\frac{1}{Y} - \frac{2\log Y}{Y^2} + \frac{1}{Y^2}(9 - 2\gamma + i\frac{\pi}{4}) + R(Y)] \text{ where}$$

$|R(Y)| \leqslant \frac{A\log Y}{Y^3}$ (A being a constant).

The expression E in (7.5) is now written as the sum of two integrals viz. $-\frac{1}{2a}(\int_0^{Y_0} + \int_{Y_0}^{\infty})H_F.M_2(F_2{}^0)$. The first integral is of order $\frac{1}{R}$ as $R \to \infty$ (because H_F is of this order in a bounded interval of integration) and its leading term will be a multiple of $\frac{\sin\theta}{R}$ (see §4.2, eqs. (4.12), (4.13)) while the leading terms in the asymptotics of the second integral are found in Appendix C §C.4 (10)). Use of this result gives

$$F_3(R\cos\theta, R\sin\theta) = -\frac{4\theta}{\pi^2 a^3} + \frac{8\sin\theta}{\pi^3 a^3 R}[(\log R)^2 - 2\log R(3 - \gamma - \frac{i\pi}{8})]$$

$$-\frac{16\log R\,\theta\cos\theta}{\pi^3 a^3 R} + o(\frac{1}{R})$$

(the parts of the $o(\frac{1}{R})$ term which combine with the logs to produce harmonic functions have not been written out explicitly).

It is recalled also that

$$F_2(R\cos\theta, R\sin\theta) = -\frac{8}{\pi^2 a^2 R}(\theta\cos\theta - \sin\theta\log R) - \frac{8\sin\theta}{\pi^2 a^2 R}(2 - \gamma + \frac{i\pi}{8})$$

$$-\frac{8}{\pi^2 a^2 R^2}(\log R.\cos 2\theta + \theta\sin 2\theta) + \frac{8\cos 2\theta}{\pi^2 a^2 R^2}(3 - \gamma + i\frac{\pi}{8}) - \frac{44\sin 2\theta}{\pi a^2 R^2} + o(\frac{1}{R}$$

(from(C.12)) and

$$F_1(R\cos\theta, R\sin\theta) = \frac{4}{\pi a}(\frac{\sin\theta}{R} - \frac{\cos 2\theta}{R^2} - \frac{8\sin 3\theta}{R^3}) + o(\frac{1}{R^3}) \quad \text{(from (C.1))}$$

The matching principle

$$\phi^{(3,4)} = \phi^{(4,3)}$$

indicates, therefore, that $\phi^{(4)}$ will be of the form

$$\phi^{(4)} = \varepsilon^2\phi_0 + \varepsilon^3\log\varepsilon\phi_1 + \varepsilon^3\phi_2 + \varepsilon^4(\log\varepsilon)^2\phi_3 + \varepsilon^4\log\varepsilon\phi_4 + \varepsilon^4\phi_5$$

where, as $\delta \to 0$ (recall that $\delta = |z - a| = \varepsilon R$ with $z = x + jy$)

$$\phi_3 \sim \frac{8\sin\theta}{\pi^3 a^3 \delta}$$

$$\phi_4 \sim \frac{8\cos 2\theta}{\pi^2 a^2 \delta^2} + \frac{16}{\pi^3 a^3}(\frac{\theta\cos\theta - \sin\theta\log\delta}{\delta}) - \frac{16}{\pi^3 a^3}(\gamma - 2 - i\frac{\pi}{8})\frac{\sin\theta}{\delta}$$

$$+ \frac{16}{\pi^3 a^3}\frac{\sin\theta}{\delta}$$

and

$$\phi_5 = -\frac{32}{\pi a}\frac{\sin 3\theta}{\delta^3} - \frac{8}{\pi^2 a^2 \delta^2}(\log\delta\cos 2\theta + \theta\sin 2\theta) + \frac{8\cos 2\theta}{\pi^2 a^2 \delta^2}(3 - \gamma + \frac{i\pi}{8})$$

$$-\frac{44\sin 2\theta}{\pi a^2 \delta^2} + \frac{8\sin\theta}{\pi^3 a^3 \delta}[(\log\delta)^2 - 2\log\delta(3 - \gamma + i\frac{\pi}{8}) - \frac{16\log\delta.\theta\cos\theta}{\pi^3 a^3 \delta} +$$

The potentials must also be harmonic, satisfy $\frac{\partial\phi}{\partial r} = 0$ on $r = a$, and die off to zero at infinity. While formal substitution of $\phi = \phi^{(4)}$ in the equation $\phi + \varepsilon\phi_y = 0$ (neglecting terms of order higher than ε^4) gives the additional equations, on $y = 0$, $|x| > a$, $\phi_3 = 0$, $\phi_4 = -\phi_{1y}$, $\phi_5 = -\phi_{2y}$.

Comparison of the problem for ϕ_3 with that for ϕ_1 (see §5.6) shows that $\phi_3 = -\frac{1}{\pi a}\phi_1$. In addition, when it is recalled that $\phi_1 = -\frac{2}{\pi a}\phi_0$, the problem for ϕ_4 is seen to resemble closely that for ϕ_2 in §5.6 again. Indeed, examination of the asymptotic conditions and the condition on $y = 0$ shows that $\phi_4 = -\frac{2}{\pi a}(\phi_2 + \phi_1)$.

It is now verified that the real part $U(x,y)$ (with respect to j) of the complex potential

$$w(z) = -\frac{4j}{\pi a^4}\left(\frac{z+a}{z-a}\right)^3 + \frac{8}{\pi^2 a^2}\frac{1}{(z-a)^2}\log\left(\frac{z+a}{z-a}\right) + \frac{4j}{\pi a^2 (z-a)^2}$$

$$+ \frac{8}{\pi^2 a^2}(3-\gamma - \log 2a + i\tfrac{\pi}{8})/(z-a)^2 + \frac{8j}{\pi^3 a^4}(3-\gamma - \log 2a + i\tfrac{\pi}{8})\left(\frac{z+a}{z-a}\right)\log\left(\frac{z+a}{z-a}\right)$$

$$+ \frac{4j}{\pi^3 a^4}\left(\frac{z+a}{z-a}\right)\left[\log\left(\frac{z+a}{z-a}\right)\right]^2$$

(the logs being made single valued by a cut from a to $-\infty$) satisfies the boundary conditions required of ϕ_5 and also the asymptotic condition as stated. The six terms occurring in $w(z)$ will be called $T_1, \ldots, T_6$ consecutively and the conditions are verified in turn.

(a) <u>The condition on $y = 0$</u>

Recall that ϕ_2 is the real part of the complex potential

$$w(z) = -\frac{4}{\pi a (z-a)^2} - \frac{4j}{\pi^2 a^3}\left(\frac{z+a}{z-a}\right)\log\left(\frac{z+a}{z-a}\right) + \frac{4j}{\pi^2 a^3}(\gamma + \log 2a - 2 - \tfrac{i\pi}{8})\left(\frac{z+a}{z-a}\right).$$

Then $-\phi_{2y} = \mathrm{Im}_j(w'(z))$ and on $y = 0$, $|x| > a$

$$-\phi_{2y} = \frac{8}{\pi^2 a^2 (x-a)^2}\log\left(\frac{x+a}{x-a}\right) + \frac{8}{\pi^2 a^2}(3 - \gamma - \log 2a + i\tfrac{\pi}{8})/(x-a)^2.$$

U is seen to have this value on $y = 0$ from the terms T_2 and T_4; the other terms give zero contribution being imaginary on $y = 0$.

(b) The condition on $r = a$

z is put equal to ae^{ju} $(0 < u < \pi)$ and it is noted that

$\frac{z+a}{z-a} = -j \cot \frac{u}{2}$ and $(z-a)^2 = -4a^2 \sin^2 \frac{u}{2} e^{ju}$. When the terms of $w(z)$ are grouped as $T_1 + (T_2 + T_3 + T_6) + (T_4 + T_5)$ it can be seen that the imaginary part of each group of terms is zero. Hence the imaginary part of $w(z)$ vanishes on $r = a$ whence $\frac{\partial U}{\partial r} = 0$ on $r = a$.

(c) The condition at ∞

As $z \to \infty$, the first term tends to an imaginary number and all the other terms tend to zero. Hence, $U \to 0$ as $r \to \infty$.

(d) The asymptotic condition

z is put equal to $a + \delta e^{j\theta}$ and $\delta \to 0$. It is seen that

$$Re_j (T_1 + T_3) = - \frac{32 \sin 3\theta}{\pi a \delta^3} - \frac{44 \sin 2\theta}{\pi a^2 \delta^2} + O(\frac{1}{\delta})$$

$$Re_j (T_2 + T_4) = - \frac{8}{\pi^2 a^2 \delta^2}(\log \delta \cos 2\theta + \theta \sin 2\theta) + \frac{8 \cos 2\theta}{\pi^2 a^2 \delta^2}(3 - \gamma + \frac{i\pi}{8}) + O(\frac{1}{\delta})$$

$$Re_j (T_5) = - \frac{16}{\pi^3 a^3}(3 - \gamma - \log 2a + \frac{i\pi}{8}) \frac{\sin \theta \log \delta}{\delta} + O(\frac{1}{\delta})$$

$$Re_j (T_6) = \frac{8}{\pi^3 a^3 \delta}[\sin \theta (\log \delta)^2 - 2 \sin \theta \log 2a \log \delta - 2\theta \cos \theta \log \delta] + O(\frac{1}{\delta})$$

(it may be noted that this last term also contains expressions of order $(\log \delta)^2$, $\log \delta$ which are of lower orders than is required by the matching. These terms would be "filled in" by terms of order $(\log R)^2$ and $\log R$ in the far field of Φ_4, the coefficient of the subsequent term $\varepsilon^4 \Phi_4$ of the right inner expansion; see note at end of Chapter).

By adding the above equations it is now seen that $Re_j (w(z))$ satisfies the asymptotic condition on ϕ_5. It follows that $\phi_5 = U + U'$ where U' vanishes on $y = 0$, has zero normal derivative on $r = a$, tends to zero at infinity and is of order $\frac{1}{\delta}$ as $\delta \to 0$ i.e. U' is an eigensolution of the outer problem with a dipole singularity at the right-hand edge and can, therefore, only be a multiple of ϕ_0. This multiple can only be determined by finding the $\frac{\sin\theta}{R}$ term in the far field asymptotics of F_3.

Next z is set equal to $-a - \delta_1 e^{j\theta_1}$ and $\delta_1 \to 0$. The asymptotics of ϕ_0, ϕ_1, ϕ_3 proceed in powers of δ_1 while those of ϕ_2, ϕ_4 contain also terms $\delta_1^{\,n} \log \delta_1$ $(n \geqslant 1)$. ϕ_5 contains, in addition to such terms, a term in $\delta_1(\log \delta_1)^2$. Hence, when the matching principle $\phi^{(4,5)} = \psi^{(5,4)}$ is applied, it is seen that $\psi^{(5)}$ will be of the form

$$\psi^{(5)} = \varepsilon^3\psi_0 + \varepsilon^4 \log \varepsilon\ \psi_1 + \varepsilon^4\psi_2 + \varepsilon^5(\log \varepsilon)^2\psi_5 + \varepsilon^5 \log \varepsilon\ \psi_3 + \varepsilon^5\psi_4$$

and that the asymptotics of ψ_5 as $R_1 \to \infty$ will be affected only by the potentials ϕ_3, ϕ_4, ϕ_5. Attention is now turned to determining ψ_5.

It was found previously that

$$\phi_3 = -\frac{1}{\pi a}\phi_1 = \frac{2}{\pi^3 a^5}\delta_1 \sin\theta_1 + o(\delta_1) \text{ as } \delta_1 \to 0 \text{ (from (5.50)) and that}$$

$$\phi_4 = -\frac{2}{\pi a}(\phi_2 + \phi_1) = \frac{2}{\pi^2 a^4} + \frac{4}{\pi^3 a^5}\delta_1 \sin\theta_1 \log\delta_1 + O(\delta_1)$$

(from the asymptotics of ϕ_2 near the end of §5.6).

If the terms of $\mathrm{Re}_j(w(z))$ are examined in turn, it is seen that

$$\mathrm{Re}(T_1) = O(\delta_1^{\,3})$$

$$\mathrm{Re}(T_2) = \frac{2}{\pi^2 a^4}(\log\delta_1 - \log 2a) + O(\delta_1 \log \delta_1)$$

$$\mathrm{Re}(T_3) = O(\delta_1)$$

$$\mathrm{Re}(T_4) = \frac{2}{\pi^2 a^4}(3 - \gamma - \log 2a + i\frac{\pi}{8}) + O(\delta_1)$$

$$\mathrm{Re}(T_5) = O(\delta_1 \log \delta_1)$$

$$\mathrm{Re}(T_6) = \frac{2 \sin \theta_1}{\pi^3 a^5} \delta_1 (\log \delta_1)^2 + O(\delta_1 \log \delta_1) .$$

When δ_1 is replaced by ϵR_1 and the matching principle $\phi^{(4,5)} = \psi^{(5,4)}$ is applied it is seen, therefore, that

(a) a term $\frac{4}{\pi^2 a^4}$ (from ϕ_4 and $\mathrm{Re}(T_2)$) will be added to the asymptotics of ψ_1 as $R_1 \to \infty$ so that, from (5.53), the asymptotic form becomes $-\frac{4}{\pi^2 a^4}(R_1 \sin \theta_1 - 1)$, thus verifying the solution (5.57) for ψ_1;

(b) the terms $\frac{2}{\pi^2 a^4}(\log R_1 - \log 2a)$ (from $\mathrm{Re}(T_2)$) and $\frac{2}{\pi^2 a^4}(3 - \gamma - \log 2a + i\frac{\pi}{8})$ (from $\mathrm{Re}(T_4)$) will be added to the asymptotics of ψ_2 which now become (see (5.54))

$$\psi_2 \sim \frac{-R_1^2 \sin 2\theta_1 + R_1 \cos \theta_1}{2\pi a^4} - \frac{2}{\pi^2 a^4}[R_1(\sin \theta_1 \log R_1 + \theta_1 \cos \theta_1) - \log R_1 -$$

$$+ \frac{2}{\pi^2 a^4}(2 \log 2a + \gamma - 2 - \frac{i\pi}{8})(R_1 \sin \theta_1 - 1) ,$$

thus justifying the addition of these lower order terms (not explicit in the previous matching condition) in the solution (5.58) for ψ_2;

(c) the asymptotic relation for ψ_5 is

$$\psi_5 \sim \frac{8}{\pi^3 a^5} R_1 \sin \theta_1 \quad .$$

In addition, substitution of $\psi^{(5)}$ in (7.1) (which has the same form for the left inner region, see (5.24)) shows that

$$\psi_{5X_1^0} = 0 .$$

Hence ψ_5 must be the eigensolution $\frac{8}{\pi^3 a^5}(Y_1 - 1)$.

The wave term generated by ψ_5 can now be found without further detailed matching by noting that $\psi^{(6)}$ will certainly contain terms with scaling

$\varepsilon^6(\log\varepsilon)^2$, $\varepsilon^6\log\varepsilon$ and ε^6 (because of the extended asymptotic forms of the potentials in the outer region) and that a term with a sixth order scaling other than these must be an eigensolution which is wave-free. Substitution, now, of the form (c) in the summary in §6.2 into (7.1) shows that

$$\psi_{6X_1^0} = \frac{-1}{2a} M_2(\psi_5{}^0) = -\frac{8}{\pi^3 a^6} Y_1 .$$

Comparison of this equation with the corresponding one for ψ_{2X_1} (see (5.56)) shows that $W_6(X_1,Y_1)$ (the wave part of ψ_6) will be $\frac{8}{\pi^2 a^2}$ (wave part of ψ_2) i.e.

$$W_6(X_1,Y_1) = \frac{16i}{\pi^3 a^6} \exp\,(iX_1 - Y_1) .$$

Hence the first sixth order wave term will be

$$\frac{16i}{\pi^3}\left(\frac{\varepsilon}{a}\right)^6 (\log\varepsilon)^2 \exp\left(-\frac{ix}{\varepsilon} - \frac{y}{\varepsilon} - \frac{ia}{\varepsilon}\right) .$$

[<u>Note</u>: The occurrence of the factor $\frac{16}{\pi^3}$ in the $\frac{(\log N)^2}{N^6}$ term of the transmission coefficient means that the error bounds given in Tables 9 and 10 in Chapter 5 should be roughly halved. It is seen that the differences between the multipole and asymptotic values are still within this tighter bound.]

It was also found possible to find ψ_3 explicitly and hence to determine the term in the transmission coefficient of order $\log N/N^6$ as follows. The contributions to the asymptotics of ψ_3 through the matching principle $\phi^{(4,5)} = \psi^{(5,4)}$ will arise as follows:

(1) from $\varepsilon^2\phi_0$ - no contribution since the asymptotics of ϕ_0 for $\delta_1 \to 0$ involve only powers of δ_1;

(2) from $\varepsilon^3 \log\varepsilon\, \phi_1$ - the term in ϕ_1 of order $\delta_1{}^2$.

$$\phi_1 = -\frac{2}{\pi a}\phi_0 = -\frac{2\delta_1 \sin\theta_1}{\pi^2 a^4} + \frac{\delta_1{}^2 \sin 2\theta_1}{\pi^2 a^5} + o(\delta_1{}^2) \quad \text{(from (5.20))};$$

contribution to ψ_3 is $\dfrac{R_1^2 \sin 2\theta_1}{\pi^2 a^5}$;

(3) from $\varepsilon^3\ \phi_2$ - the term in ϕ_2 of order $\delta_1^2 \log \delta_1$;

$$\phi_2 = \mathrm{Re}_j[-\frac{4}{\pi a(z-a)^2} - \frac{4j}{\pi^2 a^3}\left(\frac{z+a}{z-a}\right)\log\left(\frac{z+a}{z-a}\right) + \frac{2}{\pi a}(\gamma + \log 2a - 2 - i\frac{\pi}{8})\phi_0];$$

the term in $\delta_1^2 \log \delta_1$ must come from the second term here and is seen to be $\delta_1^2 \sin 2\theta_1 \log \delta_1/\pi^2 a^5$;

contribution to ψ_3 is $R_1^2 \sin 2\theta_1/\pi^2 a^5$;

(4) $\varepsilon^4(\log \varepsilon)^2\phi_3$ - no contribution since ϕ_3 is a multiple of ϕ_1 ;

(5) $\varepsilon^4 \log \varepsilon\ \phi_4$ - the terms in ϕ_4 of orders δ_1, $\delta_1 \log \delta_1$;

$\phi_4 = \dfrac{-2}{\pi a}(\phi_2 + \phi_1)$ (the asymptotic form of ϕ_2 is in §5.6);

it is seen that the required terms are

$$-\frac{2\delta_1 \cos \theta_1}{\pi^2 a^5} + \frac{4}{\pi^3 a^5}\delta_1(\sin \theta_1 \log \delta_1 + \theta_1 \cos \theta_1)$$

$$-\frac{4}{\pi^3 a^5}\delta_1 \sin \theta_1(2 \log 2a + \gamma - 3 - \frac{i\pi}{8}) ;$$

contribution to ψ_3 is

$$-\frac{2R_1 \cos \theta_1}{\pi^2 a^5} + \frac{4}{\pi^3 a^5}(R_1 \sin \theta_1 \log R_1 + R_1\theta_1 \cos \theta_1)$$

$$-\frac{4}{\pi^3 a^5}R_1 \sin \theta_1(2 \log 2a + \gamma - 3 - i\frac{\pi}{8}) ;$$

(6) from $\varepsilon^4\ \phi_5$ - terms of order $\delta_1 \log \delta_1$ and $\delta_1(\log \delta_1)^2$;

these arise as follows:

from $\mathrm{Re}(T_1)$: zero contribution ;

from $\mathrm{Re}(T_2)$: $-\dfrac{2}{\pi^2 a^5}\delta_1 \cos \theta_1 \log \delta_1$;

from $\mathrm{Re}(T_3)$: zero contribution ;

from $\mathrm{Re}(T_4)$: zero contribution ;

from $\mathrm{Re}(T_5)$: $-\dfrac{4}{\pi^3 a^5}(\log 2a + \gamma - 3 - i\frac{\pi}{8})\ \delta_1 \sin \theta_1 \log \delta_1$;

from $\mathrm{Re}(T_6)$: $\dfrac{2\delta_1 \sin\theta_1}{\pi^3 a^5}(\log\delta_1)^2 + \dfrac{4\theta_1\cos\theta_1\delta_1\log\delta_1}{\pi^3 a^5}$

$$- \frac{4\sin\theta_1}{\pi^3 a^5}\,\delta_1 \log\delta_1 \log 2a\ ;$$

contribution to ψ_3 is:

$$-\frac{2R_1\cos\theta_1}{\pi^2 a^5} + \frac{4}{\pi^3 a^5}(R_1\sin\theta_1\log R_1 + \theta_1\cos\theta_1)$$

$$-\frac{4}{\pi^3 a^5}(2\log 2a + \gamma - 3 - \frac{i\pi}{8})R_1\sin\theta_1\ .$$

When the above are combined it is seen that, as $R_1 \to \infty$,

$$\psi_3 \sim \frac{2R_1^2\sin 2\theta_1 - 4R_1\cos\theta_1}{\pi^2 a^5} + \frac{8}{\pi^3 a^5}(R_1\sin\theta_1\log R_1 + R_1\theta_1\cos\theta_1)$$

$$-\frac{8}{\pi^3 a^5}(2\log 2a + \gamma - 2 - i\frac{\pi}{8}) + \frac{8}{\pi^3 a^5}R_1\sin\theta_1\ .$$

Comparison with (5.53), (5.54) shows that

$$\psi_3 \sim -\frac{4}{\pi a}(\text{asymptotic form of } \psi_2) - \frac{2}{\pi a}(\text{asymptotic form of } \psi_1)\ .$$

Also (in the usual way)

$$\psi_{3X_1}^{\ 0} = -\frac{1}{2a}M_2\ (\psi_1^{\ 0}) = -\frac{4}{\pi a}\psi_{2X_1}^{\ 0} \qquad (\psi_1^{\ 0} = -\frac{4}{\pi a}\psi_0^{\ 0}).$$

Hence $\psi_3 = -\dfrac{4}{\pi a}\psi_2 - \dfrac{2}{\pi a}\psi_1$. (The second term fills in the missing eigensolution mentioned in §5.8).

Substitution of the form (c) in the boundary condition on $X = 0$ gives

$$\psi_{7X_1}^{\ 0} = -\frac{1}{2a}M_2\ \psi_3^{\ 0}$$

$$= -\frac{4}{\pi a}(-\frac{1}{2a}M_2\ \psi_2^{\ 0}) + \frac{8}{\pi^2 a^2}(-\frac{1}{2a}M_2\ \psi_0^{\ 0})$$

$$= -\frac{4}{\pi a}\psi_{4X_1}^{\ 0} + \frac{8}{\pi^2 a^2}\psi_{2X_1}^{\ 0} \qquad (\psi_0 \text{ generates waves in } \psi_2).$$

Thus $W_7 = -\dfrac{4}{\pi a}W_4 + \dfrac{8}{\pi^2 a^2}W_2$

$$= [-\frac{32i}{\pi^3 a^6}(\gamma + \log 2a - 2 - i\frac{\pi}{8}) + \frac{16i}{\pi^3 a^6}]\ \exp(iX_1 - Y_1)$$

(from (5.65) and (5.58)).

Hence $W_6 + W_7$

$$= \frac{16i}{\pi^3}\left(\frac{\varepsilon}{a}\right)^6 [((\log \varepsilon)^2 - 2 \log a \log \varepsilon) + \log \varepsilon (5 - 2\gamma - \log 4 + \frac{i\pi}{4})] \exp(iX_1 - Y_1$$

and hence

$$T = T^{(5)} + \frac{16i \log N}{\pi^3 N^6}[(\log N + 2\gamma + \log 4 - 5) - \frac{i\pi}{4}] \exp(-2iN) + O(\frac{1}{N^6})$$

as $N \to \infty$ (on appeal to the dependence of T on $\frac{\varepsilon}{a}$ only),

i.e. $$T = T^{(5)} + \frac{2i}{\pi N^4} \exp(-2iN)[\frac{8(\log N)^2}{\pi^2 N^2} + \frac{8 \log N}{\pi^2 N^2}(2\gamma + \log 4 - 5 - \frac{i\pi}{4})] + O(\frac{1}{N$$

as $N \to \infty$.

[Note: ϕ^0_{4X} contains the term $-\frac{1}{2a} M_2(F_3{}^0)$ so that if this is the dominant term again as $Y \to \infty$ and, if differentiation of the asymptotics of $F_3{}^0$ is allowed, then the dominant term in ϕ^0_{4X} will be of order $\frac{\log Y}{Y}$ at infinity (since that in $F_3{}^0$ is of order $\frac{(\log Y)^2}{Y}$). Such a term would produce $O(\log R)^2$ and $O(\log R)$ terms in the far field form of F_4 as was suggested by the lower asymptotic form of $\mathrm{Re}(T_6)$ in ϕ_5 as $\delta \to 0$.]

Summary and Conclusions

A comparative study has been carried out of three different methods for determining the transmission coefficient (T) for bodies partly submerged in deep water and lying in the path of a sinusoidal wave train. The available methods divide into two groups: direct computational methods for moderate values of the ratio beam/wavelength and asymptotic methods for large values of this ratio. In the latter case, T is known to be very small and the computational methods have been unable to produce these small values with adequate accuracy. An outstanding problem, therefore, was suitably to refine the methods so that significant agreement between them would be obtained over some range of values of the ratio beam/wavelength. This has been achieved in the present work.

The three methods employed in this thesis have been

(a) the method of multipole expansions (Ursell (1949))

(b) the method of matched asymptotic expansions (Leppington (1973))

(c) the method of null field equations (Martin (1981)).

(The rigorous integral equation method due to Ursell (1953, 1964) does not lend itself to obtaining more than the leading term in the asymptotics of the transmission coefficient and has not been used here.)

All these use as a model the linear theory of water waves on an irrotational ocean and surface tension is neglected. In this work the bodies considered have been circular cylinders with axes parallel to the wave crests and lengths which are long enough in comparison to their diameters for end effects to be neglected at all beam/wavelength ratios. Attention, therefore, has been confined to the two dimensional scattering problem with particular regard to the case when the incident waves are short compared to the cylinder beam.

Acute, normal and obtuse values of the angle of intersection (α) of the cylindrical cross-section with the mean water surface have been considered for beam/wavelength ratios ranging from about $\frac{1}{300}$ to 7 in the case of normal intersection (using multipole expansions) and from about $\frac{1}{300}$ to 3.5 in the other cases (using the null field equations),

i.e. $0.01 \leqslant N \leqslant 20$ for normal intersection

$0.01 \leqslant N \leqslant 10$ for acute or obtuse intersection

(where $\frac{2\pi}{K}$ = wavelength, a = semi-beam, N = Ka).

At the short wave end this has extended the data available well beyond the range of previously published results for the semi-circular cylinder (Martin and Dixon (1983) compute values of T up to Ka = 10) while, in the case of non-vertical intersection, no published results appear to have previously existed.

The main purpose of these extended calculations is to supply data for comparison with that provided by asymptotic formulae for T derived by Ursell (1964) and Leppington (1973(a)) for the semi-circular geometry, and by Alker (1977) for the obtuse angle case, with a view to establishing the existence of a region of overlap. The comparison with Ursell's and Leppington's values is inconclusive (see Tables 7, 8, and graphs 7, 8, at the end of Chapter 2). The differences between the values are within the order of the asymptotic error term but the magnitude of this error is such that this can occur even when there are no significant figures of agreement between the values (e.g. Table 7, page 40, for N = 14). It thus becomes necessary to reduce the size of the error term by completing the fifth order asymptotics and this has been achieved in Chapter 5 (via Chapters 3, 4). Subsequent comparison indicates excellent agreement between the two sets of values (see Tables 9, 10, and graphs 9, 10, at the end of Chapter 5). The first two sixth order terms have also been derived (Chapter 7) with the aim, first, of providing an exact error term for the fifth order asymptotics and, secondly, of attempting to improve the comparison even further. This leads to the improved formula

$$T = \frac{2i}{\pi N^4} \exp(-2iN)[1 + \frac{4}{\pi N} \log N - \frac{4}{\pi N}(2 - \gamma - \log 2 + \frac{i\pi}{8}) + \frac{8(\log N)^2}{\pi^2 N^2}$$

$$+ \frac{8 \log N}{\pi^2 N^2}(2\gamma + \log 4 - 5 - \frac{i\pi}{4})] + O(\frac{1}{N^6}) \text{ as } N \to \infty \quad (\gamma = \text{Euler's constant})$$

in which the first term agrees with Ursell's result, the first two terms with Leppington's result, and the remaining three terms are the additional ones derived in this work. However, the additional sixth order terms (when combined) contain a factor $(\log 4N + 2\gamma - 5 - \frac{i\pi}{4})$, the real part of which is negative until $N \doteq 12$. Indeed, at $N = 20$, $\log 4N \doteq 4.4$, while $5 - 2\gamma \doteq 3.8$ so that the log term is not yet big enough (in comparison with $5 - 2\gamma$) to give the sin 2N and cos 2N terms which it multiplies in the real and imaginary parts of T their due "asymptotic weight". In fact, to obtain a value of log 4N more than twice that of $5 - 2\gamma$ would require taking $N \doteq 550$. It would seem, therefore, that for comparison with computations the fifth order asymptotics meet precisely the limits of numerical practicality at the moment.

The calculation of the two sixth order terms involves extending the asymptotic approximations to the potential in the various fluid regions (see Fig. 2, p. 15a). This depends crucially on theorem A in Appendix B, §B.3, and the results in the first part of Chapter 4, §4.3.

During the course of the calculations, interesting evidence of the cohesion of the matching principle is observed. It is recalled first that the potentials in the right inner expansion are solutions of classical wave-maker type problems and the leading terms in the far field asymptotics of their wave-free parts depend on the decay properties of the velocity profile on the wave-maker (this is discussed in the first part of Chapter 4). As the expansion is extended, the decay properties of the velocity profiles generating the potentials in the lower order terms weaken till a point is reached where the potentials have non-vanishing far field terms. Initially this seems disturbing since an application of the matching principle which includes such potentials appears to indicate that these high order terms occurring in later terms of the right inner expansion will affect the near field forms (near E_+, that is) of potentials in the outer expansion which have already been determined at an earlier matching stage. However, closer investigation reveals, in contrast, that these high order terms in the far field of later occurring potentials in the right inner expansion "fill in" lower order near field terms in earlier occurring potentials in the outer expansion, it being noted that these lower order terms had not been demanded at the earlier matching stage but had appeared because of the nature of the solution demanded by other conditions imposed on the potentials in question. Similarly high order near field terms (near E_- this time) from later occurring potentials in the outer expansion are observed to "fill in" lower order far field terms (not demanded by earlier matching) appearing in earlier occurring potentials in the left inner expansion. Specific instances of this "filling in" process are mentioned in Chapter 7, p. 135, and Chapter 5, p. 102, and it may be anticipated that this is a characteristic process generally when the method of matching asymptotic expansions is used, although its occurrence may not be noticeable till fairly advanced matching stages are reached.

In the cases where α is acute or obtuse, comparison is made in Chapter 6 of the values of T as obtained using the null field equations and those obtained using the first two terms of Alker's (1977) result, viz.

$$T = \frac{\mu(\Gamma(\mu)\ \sin\mu\pi)^2}{\pi.2^{2\mu-1}}\ \exp(-2i\chi - 2iN).(\frac{1}{N^{2\mu}} + \frac{4\mu^2}{\pi}\frac{\log N}{N^{2\mu+1}} + O(\frac{1}{N^{2\mu+1}})$$

as $N \to \infty$, where $\chi = \frac{1}{4}(1 - \mu)\pi$ and $\mu = \frac{\pi}{2\alpha}$.

(Note that some small mistakes in Alker's paper have been taken account of). The comparison is by no means as clear-cut as in the case $\alpha = 90^\circ$ and the need for numerical calculation of the third term in Alker's asymptotics is indicated. Use of the first two terms only leads to a situation similar to that in relation to Leppington's formula in the semi-submerged circular cylinder case. The differences between the null field values and the asymptotic values are within the order of the asymptotic error term even although the relative differences may be as high as 35%-40% (for the smaller values). In most cases, one significant figure of agreement is observed and the relative differences are between 10% and 20% (see Table 12 at the end of Chapter 6, p. 132).

Three different bases have been used in the null field calculations (see Chapter 6, §6.7, p. 127) and it is found that the Chebyshev bases are most efficient for values of $\alpha > 90^\circ$ while the multipole type bases are best for acute values of α. In the latter case (not covered by John's uniqueness theorem) values of T (and R, the reflection coefficient) are obtained, satisfying the usual numerical tests $|R|^2 + |T|^2 = 1$, $|\arg T - \arg R| = \frac{\pi}{2}$.

In performing the calculations for T using multipole expansions and the collocation method of Chapter 2 for the calculation of the coefficients in the multipole expansions, it is observed that the nature of the collocation points is of critical importance (especially for the larger values of N) in determining whether the sequence of values obtained for T is monotonic or oscillatory. The dissection D_M (used in Chapter 2) by means of M equally spaced points (including the lowest point) produces sequen of approximations to the real and imaginary parts of the transmission coefficient (denoted by Re(T(M;N)) and Im(T(M;N)) respectively which are monoto for the lower values of N and ultimately monotonic for the larger values (the behaviour is similar to the illustrations in Graphs 1-4 at the end of Chapter

However, slight variations in the collocation can lead to oscillatory sequenc which are very slowly convergent and two specific cases are now discussed.

In the first of these the collocation D_M was modified by removing one or more of the collocation points nearest to the lowest point (the lowest point remaining) and reducing the dissection interval over the remainder of the cylindrical surface to maintain an M-point collocation; and, in the second case, the lowest point was not used and a collocation by means of points with polar coordinates $r = a$, $\theta_k = \frac{k}{M+1}\frac{\pi}{2}$ $(k=1,\ldots,M)$ was employed (note that the equations (2.31) and (2.32) in Chapter 2 are automatically satisfied when $\theta = \frac{\pi}{2}$). In the first case, the removal of two or more points produces sequences $\mathrm{Re}(T(M;N))$ and $\mathrm{Im}(T(M;N))$ which are oscillatory and which (after a certain value of M) consist of two subsequences which are monotonic in opposite senses, the one increasing and the other decreasing Application of the Shanks' (1955) first order accelerated convergence process (a special case of Aitken's (1937) δ^2 process) to the terms of the complete sequences and of Richardson extrapolation (Bender and Orszag (1978)) to the terms of the monotonic subsequences produces modified sequences whose limits agree, in their first two or three significant figures, with those which are obtained using the dissection D_M. The same behaviour is also observed in the second case. Thus, while the efficacy of such accelerated convergence processes is of interest in itself, a more important observation to be made from these numerical experiments, in the case of short waves, is that a collocation consisting of a dense set of points on the cylinder near the water surface (where the wave effects are most apparent), together with a scattering of points over the lower part of the cylinder, does certainly not produce accurate sequences of approximations to T. In contrast, it seems clear that too wide a spacing of points <u>anywhere</u> on the cylinder will result in the occurrence of considerable distortion in the sequences.

Finally, comparison of the multipole and null field calculations in the case $\alpha = 90^\circ$ shows that, to obtain a given accuracy, the former are much less costly in terms of computer time; in addition, the easy decoupling of the multipole equations into real and imaginary parts (see §2.3) results in considerable savings in terms of computer space needed.

APPENDIX A

It is proved that the coefficients in the multipole series (2.31) and (2.32) have the properties $|t_r(N)| \le \frac{\lambda(N)}{r^3}$ and $|u_r(N)| \le \frac{\mu(N)}{r^3}$ for all r and suitable functions λ, μ. The method was originated by Ursell (1949, 1953) and used also by Martin (1971).

Note The usual notation ℓ^2 is used to denote the Hilbert space $\{f : f = \{x_n\}_1^\infty , \sum_1^\infty |x_n|^2 < \infty\}$ which is complete and separable while $L^2(0,\frac{\pi}{2})$ denotes the space of functions square integrable on $(0,\frac{\pi}{2})$.

Proofs:

(a) In the anti-symmetric part of the problem, it is required to find real numbers $A_1(N)$, $t_m(N)$ $(m = 1,2,..)$ such that (2.31) holds i.e.

$$A_1[\exp(-N\cos\theta)\sin(N\sin\theta)-\sin N]+\sum_{m=1}^{\infty} t_m\{\sin 2m\theta+\frac{N}{2m-1}[\sin(2m-1)\theta+(-1)^m]\}$$

$$= \psi_s(N,\theta) - \psi_s(N,\tfrac{\pi}{2}) \qquad (0 \le \theta \le \tfrac{\pi}{2}) \qquad (1)$$

where $\psi_s(N,\theta) = -\operatorname{Im}_j (s(jNe^{-j\theta}))$

and $$s(Z) = e^{jZ}\int_Z^\infty \frac{e^{-jt}}{t}\,dt + j\pi e^{jZ} \qquad (\operatorname{Re}(Z) \ge 0).$$

First set $\theta = 0$ in (1). Then

$$-A_1 \sin N + \sum_{m=1}^{\infty} t_m \frac{(-1)^m N}{2m-1} = \psi_s(N,0) - \psi_s(N,\tfrac{\pi}{2}).$$

Elimination of A_1 between this equation and (1) gives

$$\sum_{m=1}^{\infty} t_m\, e_m(N,\theta) = E(N,\theta) \qquad (2)$$

where $$e_m(N,\theta) = \sin 2m\theta + \frac{N}{2m-1}\left[\sin(2m-1)\theta+\frac{(-1)^m \exp(-N\cos\theta)\sin(N\sin\theta)}{\sin N}\right]$$

and $$E(N,\theta)=\psi_s(N,\theta)-\psi_s(N,\tfrac{\pi}{2})+\frac{\exp(-N\cos\theta)\sin(N\sin\theta)-\sin N}{\sin N}[\psi_s(N,0)-\psi_s(N,\tfrac{\pi}{2})].$$

In (2), E is a continuous function of θ (for each N) for $0 \le \theta \le \frac{\pi}{2}$, $E(N,0) = 0$, $E(N,\frac{\pi}{2}) = 0$ and $E(N,\theta)$ can be differentiated repeatedly without restriction (since $s(\tilde{Z})$ is analytic). It should be noted also that, as $N \to o$, the $e_m(N,\theta)$ tend to the orthogonal set $\{\sin 2m\theta\}$. If the operator $\frac{4}{\pi}\int_0^{\pi/2} . \sin 2r\theta \, d\theta$ is applied to (2) this gives an equation of the form

$$t_r + N \sum_{m=1}^{\infty} d_{rm} t_m = E_r(N) \tag{3}$$

where

$$E_r(N) = \frac{4}{\pi} \langle E(N,\theta), \sin 2r\theta \rangle , \tag{4}$$

$$d_{rm} = \frac{4}{\pi} \cdot \frac{1}{2m-1} \left\{ \frac{(-1)^{m+r+1}\, 2r}{(2m-1-2r)(2m-1+2r)} + \frac{(-1)^m}{\sin N} \langle \exp(-N\cos\theta)\sin(N\sin\theta), \sin 2r\theta \rangle \right\}$$

and $\langle \; , \; \rangle$ denotes the usual scalar product for $L^2(0,\pi/2)$.

The scalar product term in the braces above can be written as $(-1)^m[\langle \sin\theta, \sin 2r\theta\rangle + \langle h(\theta), \sin 2r\theta\rangle]$ where

$$h(\theta) = \frac{\exp(-N\cos\theta)\sin(N\sin\theta) - \sin N \sin\theta}{\sin N} \quad \text{so that}$$

$$d_{rm} = \frac{4}{\pi} \cdot \frac{1}{(2m-1)} \left[\frac{2r}{4r^2-1} \frac{(-1)^{m+r+1}(4m^2-4m)}{(2m-1-2r)(2m-1+2r)} + h_r(N) \right] \tag{5}$$

where $h_r(N) = (-1)^m \langle h(\theta), \sin 2r\theta \rangle$.

It is easily seen that $h(0) = h\left(\frac{\pi}{2}\right) = 0$ and that $h(\theta)$ can be repeatedly differentiated (as also noted for $E(N,\theta)$ above). Hence 3 integrations by parts in each case will give the results

$$E_r(N) = 0\left(\frac{1}{r^3}\right) \quad \text{as} \quad r \to \infty \tag{6}$$

$$h_r(N) = 0\left(\frac{1}{r^3}\right) \quad \text{as} \quad r \to \infty . \tag{7}$$

Next the coefficients t_K $(K=1,2,..)$ are rescaled by writing

$$x_K = K^2 t_K$$

(this is crucial at a later stage in the proof) so that (3) now takes the form

$$x_r + N \sum_{m=1}^{\infty} a_{rm} x_m = c_r \tag{8}$$

where $$c_r = r^2 E_r$$

and $$a_{rm} = \frac{r^2}{m^2} d_{rm}$$

i.e. $$a_{rm} = \frac{4}{\pi}\frac{r^2}{m^2} \cdot \frac{1}{2m-1}\left[\frac{2r}{4r^2-1}\frac{(-1)^{m+r+1}(4m^2-4m)}{(2m-1-2r)(2m-1+2r)} + h_r(N)\right] .$$

It will now be proved that $\sum_{m=1}^{\infty} a_{rm}^2 = 0\left(\frac{1}{r^2}\right)$ as $r \to \infty$.

a_{rm}^2 can be written as the sum of three terms

$\alpha_{rm} + \beta_{rm} + \gamma_{rm}$ where

$$\alpha_{rm} = \left(\frac{8}{\pi}\right)^2\left(\frac{4r^3}{4r^2-1}\right)^2\left[\frac{4m^2 - 4m}{4m^2(2m-1)(2m-1-2r)(2m-1+2r)}\right]^2$$

$$\beta_{rm} = \left(\frac{4}{\pi}\right)^2\left(\frac{4r^5}{4r^2-1}\right)h_r(N) \cdot \frac{(-1)^{m+r+1}(4m^2-4m)}{m^4(2m-1)^2(2m-1-2r)(2m-1+2r)}$$

$$\gamma_{rm} = \left(\frac{4}{\pi}\right)^2 r^4 [h_r(N)]^2 \frac{1}{m^4(2m-1)^2} .$$

Clearly by (7) , $\sum_{m=1}^{\infty} \gamma_{rm} = 0\left(\frac{1}{r^2}\right)$.

Next consider $$\left|\sum_{m=1}^{\infty} \beta_{rm}\right| \leq \left(\frac{4}{\pi}\right)^2\left(\frac{4r^5}{4r^2-1}\right)|h_r(N)| \sum_{m=1}^{\infty}\frac{1}{m^4|2m-1-2r|(2m-1+2r)} .$$

Hence $$\left|\sum_{m=1}^{\infty} \beta_{rm}\right| \leq \left(\frac{4}{\pi}\right)^2\left(\frac{4r^5}{4r^2-1}\right)\frac{|h_r(N)|}{(2r-1)}\sum_{m=1}^{\infty}\frac{1}{m^4|2m-1-2r|} . \tag{10}$$

The sum here can be written as $\left(\sum_{m=1}^{[\frac{r}{2}]} + \sum_{m=[\frac{r}{2}]+1}^{\infty}\right)\frac{1}{m^4|2m-1-2r|}$.

In the first sum $|2m-1-2r| = 2r+1-2m \geq 2r+1-r = r+1$ so that this sum is less or equal to $\frac{1}{r+1}\sum_{m=1}^{\infty}\frac{1}{m^4} = 0\left(\frac{1}{r}\right)$ as $r \to \infty$.

In the second sum $|2m-1-2r| \geq 1$ and $m^4 \geq ([\frac{r}{2}]+1)m^3$.

Hence this sum is less or equal to $\frac{2}{r+1}\sum_{m=[\frac{r}{2}]+1}^{\infty}\frac{1}{m^3} \leq \frac{2}{r+1}\sum_{m=1}^{\infty}\frac{1}{m^3}$

which is also $0\left(\frac{1}{r}\right)$ as $r \to \infty$. It follows then that

$\sum_{m=1}^{\infty}\frac{1}{m^4|2m-1-2r|} = 0\left(\frac{1}{r}\right)$ as $r \to \infty$ and hence from (10) and (7) that

$$\sum_{m=1}^{\infty} \beta_{rm} = 0\left(\frac{1}{r^2}\right) \quad \text{as } r \to \infty .$$

Finally,

$$\sum_{m=1}^{\infty} \alpha_{rm} \leq \left(\frac{8}{\pi}\right)^2 \left(\frac{4r^3}{4r^2-1}\right)^2 \sum_{m=1}^{\infty} \left[\frac{1}{(2m-1)(2m-1-2r)(2m-1+2r)}\right]^2$$

$$\leq \left(\frac{8}{\pi}\right)^2 \left(\frac{4r^3}{4r^2-1}\right)^2 \left(\frac{1}{2r-1}\right)^2 \sum_{m=1}^{\infty} \frac{1}{(2m-1)^2(2m-1-2r)^2} \quad .$$

Again the sum here is written as

$$\left(\sum_{m=1}^{[r/2]} + \sum_{m=[r/2]+1}^{\infty}\right) \frac{1}{(2m-1)^2(2m-1-2r)^2} \quad .$$

As before the first sum is easily proved to be $\mathcal{O}\left(\frac{1}{r^2}\right)$ as $r \to \infty$ while the second sum is less than or equal to

$$\frac{1}{r^2} \sum_{m=[r/2]+1}^{\infty} \frac{1}{(2m-1-2r)^2}$$

$$= \frac{1}{r^2} \sum_{m=1+[r/2]-r}^{\infty} \frac{1}{(2m-1)^2}$$

$$\leq \frac{2}{r^2} \sum_{m=1}^{\infty} \frac{1}{(2m-1)^2} = 0\left(\frac{1}{r^2}\right) \quad \text{as} \quad r \to \infty \; .$$

It follows immediately that

$$\sum_{m=1}^{\infty} a_{rm}^2 = \sum_{m=1}^{\infty} \alpha_{rm} + \sum_{m=1}^{\infty} \beta_{rm} + \sum_{m=1}^{\infty} \gamma_{rm} = 0\left(\frac{1}{r^2}\right) \tag{11}$$

as $r \to \infty$ and hence that $\sum_{r=1}^{\infty} \sum_{m=1}^{\infty} a_{rm}^2 < \infty$. (12)

The equation (8) is now written in the form

$x + Tx = c$ where T is the operator (on ℓ^2)

$$N \sum_{m=1}^{\infty} a_{rm} \; , \quad x = \{x_r\}_1^{\infty} \; , \quad c = \{c_r\}_1^{\infty} \quad .$$

By (12) T is bounded and hence completely continuous on ℓ^2 (Akhiezer and Glazman p.92,93) so that by Hilbert's generalisation of Fredholm's first theorem (Schmeidler p.53) the equation $x + Tx = c$ has a unique solution $x \in \ell^2$, since $c \in \ell^2$ (by (6) and (8)). (Note that the existence of linearly independent solutions of the homogeneous equation would imply non-uniqueness of the original boundary value problem). It follows then from (8) that

$$|x_r - c_r|^2 \le N^2 \, \Sigma a_{rm}^2 \, \Sigma x_m^2 = 0\left(\frac{1}{r^2}\right) \quad \text{(by (11) and the}$$

discussion above) whence

$$x_r = c_r + 0\left(\frac{1}{r}\right) = 0\left(\frac{1}{r}\right) \qquad \left(\text{since } c_r = 0\left(\frac{1}{r}\right) \text{ from (6),(8)}\right).$$

Thus $\quad t_r \overset{D}{=} \dfrac{x_r}{r^2} = 0\left(\dfrac{1}{r^3}\right) \quad$ so that

$$|t_r(N)| \le \frac{\lambda(N)}{r^3} \qquad \text{for all } r \text{ and a suitable function } \lambda .$$

(b) In the symmetric problem it is required to find real numbers $B_1(N)$, $u_m(N)$ $(m=1,2,3,..)$ such that (2.32) holds i.e.

$$B_1[\exp(-N\cos\theta)\cos(N\sin\theta)-\cos N] + \sum_{m=1}^{\infty} u_m\{\cos(2m+1)\theta + \frac{N}{2m}[\cos 2m\theta + (-1)^{m+1}]\}$$

$$= \psi_D(N,\theta) - \psi_D(N,\tfrac{\pi}{2}) \qquad (0 \le \theta \le \tfrac{\pi}{2}) \tag{13}$$

where $\psi_D(N,\theta) = -\mathrm{Im}_j[s'(jNe^{-j\theta})]$

and, as before, $\quad s(Z) = e^{jZ}\displaystyle\int_Z^\infty \frac{e^{-jt}}{t}\,dt + j\pi e^{jZ} \qquad (\mathrm{Re}(Z) \ge 0)$.

It is noted first that, as $N \to 0$, the coefficient of B_1 is equal to $-N\cos\theta + 0(N^2)$ so that if a new set of coefficients v_K is defined by

$$v_1 = -NB_1$$
$$v_K = u_{K-1} \qquad (K \ge 2)$$

the equation (13) can be written in the form

$$\sum_{m=1}^{\infty} v_m[\cos(2m-1)\theta + N\,h_m(N,\theta)] = V(N,\theta) \tag{14}$$

where

$$h_1(N,\theta) = -\frac{\exp(-N\cos\theta)\cos(N\sin\theta) - \cos N + N\cos\theta}{N^2} \tag{15}$$

$$h_m(N,\theta) = \frac{1}{2m-2}[\cos(2m-2)\theta + (-1)^m] \qquad (m \ge 2)$$

$$V(N,\theta) = \psi_D(N,\theta) - \psi_D(N,\tfrac{\pi}{2}) \quad .$$

It should be noted that, as $N \to 0$, the terms multiplying the v_m in (14) tend to the orthogonal set $\{\cos(2m-1)\theta\}$. The operator $\frac{4}{\pi} < , \cos(2r-1)\theta>$ is now applied to (14). This gives an equation of the form

$$v_r + N \sum_{m=1}^{\infty} d_{rm} v_m = V_r(N) \tag{16}$$

where $$V_r(N) = \frac{4}{\pi} < V(N,\theta), \cos(2r-1)\theta> \tag{17}$$

$$d_{r1} = \frac{4}{\pi} <h_1(N,\theta), \cos(2r-1)\theta> \tag{18}$$

and for $m \geq 2$

$$d_{rm} = \frac{8}{\pi} \frac{(-1)^{m+r+1}}{2r-1} \frac{(m-1)}{(2m-2)^2-(2r-1)^2} .$$

It is noted first that the functions $V(N,\theta)$ and $h_1(N,\theta)$ vanish at $\theta = \frac{\pi}{2}$, have derivatives which vanish at $\theta = 0$ and can be differentiated repeatedly without restriction. Thus three integrations by parts in (17) and (18) show that

$$V_r(N) = 0\left(\frac{1}{r^3}\right) \quad \text{as} \quad r \to \infty \tag{20}$$

$$d_{r1} = 0\left(\frac{1}{r^3}\right) \quad \text{as} \quad r \to \infty . \tag{21}$$

Next the coefficients v_K $(K=1,2,\ldots)$ are rescaled by writing $x_K = K^2 v_K$ so that (16) now takes the form

$$x_r + N \sum_{m=1}^{\infty} a_{rm} x_m = c_r \tag{22}$$

where $$c_r = r^2 V_r \tag{23}$$

and $$a_{rm} = \frac{r^2}{m^2} d_{rm} \quad \text{i.e.}$$

$$a_{r1} = r^2 d_{r1} \tag{24}$$

and $$a_{rm} = \frac{8}{\pi} \frac{(-1)^{m+r+1} r^2}{2r-1} \frac{m-1}{m^2(2m+2r-3)(2m-2r-1)} \quad \text{for } m \geq 2 , \tag{25}$$

(21) amd (24) show that $a_{r1} = 0\left(\frac{1}{r}\right)$ and by methods similar to those used in (a) it is not difficult to prove that $\sum_{m=2}^{\infty} a_{rm}{}^2 = 0\left(\frac{1}{r^2}\right)$ whence $\sum_{m=1}^{\infty} a_{rm}{}^2 = 0\left(\frac{1}{r^2}\right)$.

It is then proved as before that $x_r = 0\left(\frac{1}{r}\right)$ so that $v_r\left(= \frac{x_r}{r^2}\right) = 0\left(\frac{1}{r^3}\right)$.

Thus $u_r = v_{r+1}$ is also $0\left(\frac{1}{r^3}\right)$ as $r \to \infty$.

APPENDIX B ASYMPTOTICS I

§B.1 In this appendix a theorem on the asymptotics of a repeated integral of the form

$$\int_0^\infty \int_0^\infty f(t)g(u)e^{-ztu}\,du\,dt \quad (\mathrm{Re}(z)\geqslant 0) \text{ as } |z|\to\infty$$

is proved.

Such repeated integrals occur several times in chapter 5 as coefficients of higher order terms in the right inner perturbation series. Indeed they arise when Havelock's classical wave maker solution is applied to a velocity distribution which itself results from a previous Havelock type solution.

The cases which appear in chapter 5 all have Re(z) = 0. This necessitates the placing of rather more stringent conditons on the functions f and g than would be required for the case Re(z)>0. Although only one or two terms in the asymptotic expansion are required (for the purposes of matching with the outer expansion) a full asymptotic expansion is obtained viz

$$\int_0^\infty \int_0^\infty f(t)g(u)e^{-ztu}\,du\,dt \sim$$

$$\sim \sum_{r=0}^{\infty} \frac{1}{r!z^{r+1}} \{[\log z + \gamma + S_r]f^r(o)g^r(o)$$
$$- f^r(o)\int_0^\infty g^{r+1}(t)\log t\,dt - g^r(o)\int_0^\infty f^{r+1}(t)\log t\,dt\}$$

where $S_r = \begin{cases} 0 & \text{if } r = 0 \\ \sum_{K=1}^{r} \frac{1}{K} & \text{if } r \geqslant 1 \end{cases}$ and γ is Euler's constant.

The theorem is preceded by several lemmas of which lemma 1 is of fundamental importance and is used in several other parts of the text (chapter 4 particularly).

§B.2 Lemma 1

Statement

Let f be a function (possibly complex valued) defined on $[0,\infty)$ such that

a) $f \in C^{\infty}[0,\infty)$ (being continuous and differentiable on the right at 0).

Suppose also f is such that, for some integer $r \geqslant 0$.

b) $\int_x^\infty \frac{f^r(t)}{t}\,dt$ exists for all $x > 0$.

c) $f^r(t)\log t \to 0$ as $t \to \infty$.

d) For $r \geqslant 1$ and $0 \leqslant K \leqslant r-1$

$$f^K(t) = o(t^{r-k}) \text{ as } t \to \infty .$$

Then

i) $I_r(x) \overset{D}{=} \int_x^\infty \frac{f(t)}{t^{r+1}}\,dt$ exists for $x > 0$.

and, as $x \to 0+$

ii) $I_r(x) \sim \sum_{K=0}^{\infty} a_{Kr}(x)\, x^{K-r}$ where

$$a_{Kr}(x) = \begin{cases} \dfrac{f^K(0)}{K!\,(r-K)} & (K \neq r) \\ C_r - \dfrac{f^r(0)}{r!}\log x & (K = r) \end{cases}$$

with

$$C_r = \begin{cases} \dfrac{f^r(0)}{r!} \sum_{m=1}^{r} \dfrac{1}{m} - \dfrac{1}{r!} \displaystyle\int_0^\infty f^{r+1}(t)\log t\,dt & (r>0) \\ \\ -\displaystyle\int_0^\infty f'(t)\log t\,dt & (r = 0) \end{cases}$$

Preliminary Remarks

The obvious method of integrating $I_r(x)$ by parts immediately leads to cumbersome coefficients in the asymptotic expansion which are not obviously simplifiable.

The form obtained in the lemma is essential in deriving a result which can be successfully applied later in the proof of the main theorem.

Extensions to the case of x complex and cases where the asymptotic series involved are in fact convergent power series are also detailed in the notes after the lemma.

Proof of Lemma 1

The result

$$I_r(x) = \sum_{K=o}^{n+r} a_{Kr}(x)\, x^{K-r} + O(x^{n+1}) \quad \text{as } x \to 0+$$

(for any integer $n \geqslant 0$)

will be established whence the lemmas as stated before will follow.

It is first noted that the formal Maclaurin expansion

$$\sum_{K=o}^{\infty} \frac{f^K(o)}{K!}\, x^K$$

(even if it has zero radius of convergence) is, under condition (a) the asymptotic expansion of f(x) as $x \to 0+$.

Indeed it can be seen by induction that

$$f(x) = \sum_{K=o}^{n} \frac{f^K(o)}{K!}\, x^K + R_n(x) \qquad (1)$$

where $R_n(x) = \frac{1}{n!} \int_0^x (x - t)^n f^{n+1}(t)dt$ whence

the substitution $t = ux$ in the integral makes it clear that $R_n(x) = O(x^{n+1})$. This result will be used to establish the lemma without any assumptions being made concerning the radius of convergence of the Maclaurin series for f.

Part (i) of the lemma (the existence of $I_r(x)$) is easily established using (b) and (d) and repeated integration by parts.

If $r > 0$ it can be shown that

$$\int_x^\infty \frac{f^r(t)}{t}\,dt = -\frac{f^{r-1}(x)}{x} - \frac{f^{r-2}(x)}{x^2} - \cdots - (r-1)!\frac{f(x)}{x^r} + r!\int_x^\infty \frac{f(t)}{t^{r+1}}\,dt$$

whence the existence of $I_r(x)$ is immediately verified while the existence for $r = 0$ follows immediately from (b).

ii) The integral $I_r(x)$ is first written in the form

$$I_r(x) = \int_x^\infty \frac{f(t) - \sum_{K=o}^{r-1} \frac{f^K(o)}{K!} t^K}{t^{r+1}}\,dt + \sum_{K=o}^{r-1} \frac{f^K(o)}{K!} \int_x^\infty t^{K-r-1}\,dt$$

(where a reversal of integration and summation operators has taken place in the second term and the sums are defined as 0 for $r = 0$).

The integration is performed and a function F defined by

$$F(t) \stackrel{D}{=} f(t) - \sum_{K=o}^{r-1} \frac{f^K(o)}{K!}\, t^K \tag{D1}$$

whence

$$I_r(x) = \int_x^\infty \frac{F(t)}{t^{r+1}}\,dt + \sum_{K=o}^{r-1} \frac{f^K(o)}{K!}\, \frac{x^{K-r}}{r-K} . \tag{2}$$

The function F (defined on $[0,\infty)$) has the properties

$F^K(0) = 0 \qquad (0 \leqslant K \leqslant r-1)$

and $F^K(t) = f^K(t) \qquad (K \geqslant r)$

whence it follows from (1) that

$$F(t) \sim \sum_{K=r}^{\infty} \frac{f^K(o)}{K!} t^K \text{ as } t \to 0+ . \tag{3}$$

It then follows from (3) that the function

$$g_r(t) \overset{D}{=} \frac{F(t)}{t^{r+1}} - \frac{1}{r!} \frac{f^r(t)}{t} \tag{D2}$$

has a removable singularity at the origin, a fact which will be of importance later. Meanwhile another function $h_r(x)$ is defined by the equation

$$h_r(x) \overset{D}{=} \int_x^{\infty} g_r(t)\, dt \tag{D3}$$

and the equation (2) for $I_r(x)$ is rewritten in the form

$$I_r(x) = h_r(x) + \frac{1}{r!} \int_x^{\infty} \frac{f^r(t)}{t} dt + \sum_{K=o}^{r-1} \frac{f^K(o)}{K!} \frac{x^{K-r}}{r-K} . \tag{4}$$

Attention is first concentrated on $\int_x^{\infty} \frac{f^r(t)}{t} dt$.

A preliminary integration by parts (using (c)) gives

$$\int_x^{\infty} \frac{f^r(t)}{t} dt = -f^r(x) \log x - \int_0^{\infty} f^{r+1}(t) \log t dt + \int_0^x f^{r+1}(t) \log t dt$$

whence the substitution $t = xu$ in the second integral above leads to the result

$$\int_x^{\infty} \frac{f^r(t)}{t} dt = -f^r(0) \log x - \int_0^{\infty} f^{r+1}(t) \log t dt + x \int_0^1 f^{r+1}(ux) \log u du . \tag{5}$$

The last term in (5) is now dealt with using (1) again whence for any fixed $u \geqslant 0$

$$f^{r+1}(ux) = \sum_{K=1}^{n} \frac{f^{r+K}(0)}{(K-1)!} x^{K-1} u^{K-1} + R_{n-1}(ux) .$$

The above equation is now multiplied by logu, integrated from 0 to 1 with respect to u and the result multiplied by x. When the result

$$\int_0^1 u^{K-1} \log u \, du = -\frac{1}{K^2} \qquad \text{is used this gives}$$

$$x \int_0^1 f^{r+1}(ux) \log u \, du = -\sum_{K=1}^{n} \frac{f^{r+K}(0)}{K.K!} x^K + \rho_{n-1}(ux) \text{ where}$$

$$\rho_{n-1}(ux) = \frac{x^{n+1}}{(n-1)!} \int_0^1 u^n \log u \int_0^1 (1-v)^{n-1} f^{r+n+1}(uxv)\, dv\, du$$

and is clearly $0(x^{n+1})$ as $x \to 0+$ since (by (a)) $f^{r+n+1}(uxv)$ is uniformly bounded as $x \to 0+$ for $0 \leqslant u \leqslant 1$, $0 \leqslant v \leqslant 1$.

It follows by substitution in (5) that

$$\int_x^\infty \frac{f^r(t)}{t} dt = -f^r(0) \log x - \int_0^\infty f^{r+1}(t) \log t \, dt - \sum_{K=1}^{n} \frac{f^{r+K}(0)}{K.K!} x^K + 0(x^{n+1}).$$

[It should be noted that the result still holds for n = 0 if the sum in (6) is defined to be zero in this case. This is obvious from (5) where the last term is $0(x)$ as $x \to 0+$] .

It remains now to attend to the term $h_r(x)$ in (4). From (D3) it is seen that $h_r' = -g_r$ and since g_r has a removable singularity at the origin this implies that

$$h_r(x) - h_r(0) = -\int_0^x g_r(t)\, dt$$

It is easily shown using the definition of g_r (D2) and the equations (3) and (1) that

$$g_r(t) = \sum_{K=1}^{n} \frac{f^{K+r}(0)}{(K+r)!} t^{K-1} - \frac{1}{r!} \sum_{k=1}^{n} \frac{f^{r+K}(0)}{K!} t^{K-1} + 0(t^n) \text{ as } t \to 0+$$

(the sums being defined as 0 if n = 0)

whence

$$h_r(x) - h_r(0) = \frac{1}{r!} \sum_{K=1}^{n} \frac{f^{r+K}(0)}{K!} \frac{x^K}{K} - \sum_{K=1}^{n} \frac{f^{r+K}(0)}{(K+r)!} \frac{x^K}{K} + 0(x^{n+1}) \text{ as } x \to 0+$$

or

$$h_r(x) = h_r(0) + \frac{1}{r!} \sum_{K=1}^{n} \frac{f^{r+K}(0)}{K!} \frac{x^K}{K} - \sum_{K=r+1}^{n+r} \frac{f^K(0)}{K!} \frac{x^{K-r}}{K-r} + 0(x^{n+1})$$

$$\text{as } x \to 0+ \qquad (7)$$

Substituting from (6) and (7) into (4) (noting that two summations terms cancel) gives the result

$$I_r(x) = h_r(o) - \frac{1}{r!} \int_0^\infty f^{r+1}(t)\log t \, dt - \frac{f^r(o)}{r!}\log x + \sum_{\substack{K=o \\ K\neq r}}^{n+r} \frac{f^K(o)}{K!}\frac{x^{K-r}}{r-K} + O(x^{n+1}) \text{ as } x \to 0+ \quad . \qquad (8)$$

The final step is to find $h_r(o)$. This is achieved by returning to the definition of $h_r(x)$ whence

$$h_r(x) + \frac{1}{r!}\int_x^\infty \frac{f^r(t)}{t}\,dt = \int_x^\infty \frac{F(t)}{t^{r+1}}\,dt \quad .$$

For $r = 0$, $h_r(x) = 0$ since the summation part of the definition of $F(t)$ does not exist in this case (See (D1)). Otherwise, for $r \geqslant 1$ repeated integration by parts of the right hand side gives

$$h_r(x) + \frac{1}{r!}\int_x^\infty \frac{f^r(t)}{t}\,dt =$$

$$\frac{1}{r}\frac{F(x)}{x^r} + \frac{1}{r(r-1)}\frac{F'(x)}{x^{r-1}} + \text{---} + \frac{1}{r!}\frac{F^{r-1}(x)}{x} + \frac{1}{r!}\int_x^\infty \frac{F^r(t)}{t}\,dt$$

the infinite limit giving zero contribution each time because of the definition of $F(t)$ and the property (d) of $f(t)$.

It follows (since $F^r(t) = f^r(t)$) that

$$h_r(x) = \frac{1}{r}\frac{F(x)}{x^r} + \frac{1}{r(r-1)}\frac{F'(x)}{x^{r-1}} + \text{---} + \frac{1}{r!}\frac{F^{r-1}(x)}{x}$$

and since $h_r(x)$ is continuous in the right at 0 (being dfferentiable on the right there) $h_r(0)$ can be evaluated as $\lim_{x\to 0+} h_r(x)$

$$= \frac{1}{r}\frac{f^r(0)}{r!} + \frac{1}{r(r-1)}\frac{f^r(0)}{(r-1)!} + \text{----} + \frac{1}{r!}\frac{f^r(0)}{1!} \qquad \text{(from (3))}$$

$$= \frac{f^r(0)}{r!}\sum_{m=1}^{r}\frac{1}{m} \quad .$$

Finally the substitution of $h_r(0)$ into (8) and use of the definitions of the $a_{Kr}(x)$ give the required result for $r \geqslant 0$ (since $h_o(0) = 0$).

Corollary

Let f be such that

a) $f \in C^1[0,\infty)$

b) $\int_x^\infty \frac{f(t)}{t}\, dt$ exists for all $x > 0$

c) $f(t) \log t \to 0$ as $t \to \infty$

then

$$\int_0^\infty f'(t) \log t \, dt = -\lim_{x \to 0+} \left(\int_x^\infty \frac{f(t)}{t}\, dt + f(0) \log x \right) .$$

Proof Let $x \to 0$ in equation (5) with $r = 0$

This result will be used several times in Appendix C.

Note that if the Maclaurin series for f is convergent for $|x|<R$ then the asymptotic relations in the proof of lemma 1 become equalities so that if $x \neq 0$ the result becomes

$$I_r(x) = \sum_{K=o}^{\infty} a_{Kr}(x)\, x^{K-r} \qquad \text{or}$$

$$I_r(x) = \sum_{\substack{K=o \\ K\neq r}}^{\infty} \frac{f^K(0)}{K!} \frac{x^{K-r}}{r-K} + C_r - \frac{f^r(0)}{r!} \log x \qquad \text{for } 0 < x < R .$$

The right hand side is analytic in the complex plane cut from 0 to ∞ for $|x|<R$ so that if x is a complex number z the corresponding result is

$$\int_z^\infty \frac{f(t)}{t^{r+1}}\, dt = \sum_{\substack{K=o \\ K\neq r}}^{\infty} \frac{f^K(0)}{K!} \frac{z^{K-r}}{r-K} + C_r - \frac{f^r(0)}{r!} \log z$$

$$(|z|<R, \ |\arg z|<\pi, z\neq 0) .$$

As an example consider the function

$$E_r(z) = \int_1^\infty \frac{e^{-zt}}{t^r}\,dt \qquad (r \text{ an integer} \geq 2).$$

The integral converges for $Re(z) \geq 0$ but not for $Re(z) < 0$.

However, the substitution $u = zt$ gives

$$E_r(z) = z^{r-1} \int_z^\infty \frac{e^{-u}}{u^r}\,du .$$

All the conditions of the lemma are satisfied by e^{-u} which in addition has infinite radius of convergence. The lemma therefore provides the analytic continuation of $E_r(z)$ into the cut plane so that for $z \neq 0$ and $|\arg z| < \pi$

$$E_r(z) = z^{r-1} \Big[\sum_{\substack{K=o\\ K\neq r-1}}^{\infty} \frac{(-1)^K}{K!} \frac{z^{K-r+1}}{(r-1-K)} + \frac{(-1)^{r-1}}{(r-1)!} \sum_{m=1}^{r-1} \frac{1}{m}$$

$$- \frac{1}{(r-1)!} \int_0^\infty (-1)^r e^{-t} \log t\,dt - \frac{(-1)^{r-1}}{(r-1)!} \log z \Big] ,$$

$$= \frac{(-z)^{r-1}}{(r-1)!} \Big(\sum_{m=1}^{r-1} \frac{1}{m} - \log z + \int_0^\infty e^{-t} \log t\,dt \Big)$$

$$- \sum_{\substack{K=o\\ K\neq r-1}}^{\infty} \frac{(-z)^K}{K!\,(K-r+1)} .$$

Since $\int_0^\infty e^{-t} \log t\,dt = -\gamma$ (γ = Euler's constant) this agrees with the result in Abramowitz and Stegun.* Similarly results could be obtained for

$$\int_1^\infty \frac{\cos zt}{t^r}\,dt \qquad \text{and} \qquad \int_1^\infty \frac{\sin zt}{t}\,dt$$

by considering the real and imaginary parts of

$$\int_1^\infty \frac{e^{-jzt}}{t^r}\,dt$$

and using the previous result with z replaced by jz.

* p. 229, eq. 5.1.12

In the lemmas which follow f and g are two (possibly complex-valued) functions defined on $[0,\infty)$ with the following properties

a) $f, g \in C^{\infty}[0,\infty)$.

For $K = 0, 1, 2, \ldots$.

b) $f^K(t)\log t,\ g^K(t)\log t \to 0$ as $t \to \infty$.

c) $\int_x^\infty \frac{f^K(t)}{t}\,dt$ exists for all $x > 0$.

d) $\int_0^\infty g^K(t)\,dt$ is absolutely convergent.

In addition z is a complex number such that $|z| = R$, $|\arg z| \leqslant \frac{\pi}{2}$ and α is a real number such that $0 < \alpha < 1$.

Lemma 2

i) $I_1(z,\alpha) \overset{D}{=} \int_{\frac{1}{R^\alpha}}^\infty f(t) \int_0^\infty e^{-ztu} g(u)\,du\,dt$ exists and

ii) for any integer $N \geqslant 1$

$$I_1(z,\alpha) = \sum_{r=0}^{N-1} \frac{g^r(o)}{z^{r+1}} \sum_{K=0}^{r} a_{Kr}\left(\frac{1}{R^\alpha}\right) \frac{1}{R^{\alpha(K-r)}}$$
$$- \sum_{r=0}^{N-1} \frac{g^r(o)}{z^{r+1}} F_r\left(\frac{1}{R^\alpha}\right) + O\left(\frac{1}{R^{\beta N+1}}\right)$$

as $R \to \infty$ where the a_{Kr} are as defined in lemma 1, $\beta = 1-\alpha$ and the functions F_r are such that

$$F_r\left(\frac{1}{R^\alpha}\right) \sim \sum_{K=1}^{\infty} \frac{f^{K+r}(0)}{(K+r)!\,K} \frac{1}{R^{\alpha K}} \quad \text{as } R \to \infty .$$

Proof

i) Existence of the repeated integral is first proved.

For $|\arg z| \leqslant \frac{\pi}{2}$, $\int_0^\infty e^{-ztu} g(u)\,du$

is absolutely and uniformly convergent (for $t \geqslant 0$) by the Weierstrass test and (d) with $K = 0$. This integral is therefore a continuous function of t for $t \geqslant 0$.

Also for any, fixed non-zero z with $|\arg z| \leqslant \frac{\pi}{z}$

$$\int_0^\infty e^{-ztu} g(u)du = \frac{g(0)}{zt} + 0\left(\frac{1}{t^2}\right) \quad \text{as } t \to \infty$$

by integration by parts, using (a) and (d) to prove boundedness of g at ∞.

Hence by (a), (c) and (d)

$$\int_{\frac{1}{R^\alpha}}^\infty f(t) \int_0^\infty e^{-ztu} g(u)\,du\,dt \text{ exists.}$$

ii) The inner integral above can be integrated by parts repeatedly to give

$$\int_0^\infty g(u)e^{-ztu}du = \sum_{r=o}^{N-1} \frac{g^r(0)}{z^{r+1}t^{r+1}} + R_N(zt) \tag{9}$$

where
$$R_N(zt) = \frac{1}{z^N t^N} \int_0^\infty e^{-ztu}\, g^N(u)du$$

(the upper limit gives zero contribution due to the boundedness of the functions g^r at ∞, this being a consequence of (a) and (d)).

Further

$$R_N(zt) = \frac{1}{z^{N+1}t^{N+1}}\left[g^N(0) + \int_0^\infty e^{-ztu}\, g^{N+1}(u)du\right] \text{ so that}$$

$|R_N(zt)| \leqslant \frac{K}{|z|^{N+1}t^{N+1}}$ (where K is a constant independent of z and t e.g. $K = |g^N(0)| + \int_0^\infty |g^N(u)|\,du$).

It follows (using (9)) that

$$I_1(z,\alpha) = \sum_{r=o}^{N-1} \frac{g^r(0)}{z^{r+1}} \int_{\frac{1}{R^\alpha}}^\infty \frac{f(t)}{t^{r+1}}dt + \int_{\frac{1}{R^\alpha}}^\infty f(t)\, R_N(zt)dt\ . \tag{10}$$

Since f is continuous and tends to zero as $t \to \infty$ by (b), $M = \sup_{t \geqslant 0} f(t)$ exists and the modulus of the second term is less than or equal to

$$MK/|z|^{N+1} \int_{\frac{1}{R^\alpha}}^\infty \frac{1}{t^{N+1}}dt$$

$$= \frac{MK}{N_R^{N+1}} R^{\alpha N} \qquad (\text{since } N \geqslant 1).$$

$$\text{Hence} \int_{1/R^\alpha}^{\infty} f(t) R_N(zt)\,dt = 0 \left(\frac{1}{R^{\beta N+1}}\right) \quad \text{as } R \to \infty . \tag{11}$$

To deal with the first term in (10), Lemma 1 is invoked. Clearly f satisfies all the conditions of Lemma 1, (the condition (d) there being a consequence of (c) in Lemma 2 which is supposed to hold for K = 0, 1, 2, -----). Hence, using Lemma 1,

$$\int_{1/R^\alpha}^{\infty} \frac{f(t)}{t^{r+1}}\,dt = \sum_{K=o}^{r} \left[a_{Kr}\left(\frac{1}{R^\alpha}\right)\right] + G_r\left(\frac{1}{R^\alpha}\right) \qquad \text{where}$$

$$G_r\left(\frac{1}{R^\alpha}\right) \sim \sum_{K=r+1}^{\infty} a_{Kr}\left(\frac{1}{R^\alpha}\right) \frac{1}{R^{\alpha(K-r)}} \qquad \text{as } R \to \infty ,$$

$$\text{i.e. } G_r\left(\frac{1}{R^\alpha}\right) \sim \sum_{K=r+1}^{\infty} \frac{f^K(0)}{K!\,(r-K)} \frac{1}{R^{\alpha(K-r)}}$$

$$= - \sum_{K=1}^{\infty} \frac{f^{K+r}(0)}{(K+r)!\,K} \frac{1}{R^{\alpha K}} .$$

Hence

$$\int_{1/R^\alpha}^{\infty} \frac{f(t)}{t^{r+1}}\,dt = \sum_{K=o}^{r} \left[a_{Kr}\left(\frac{1}{R^\alpha}\right) \frac{1}{R^{\alpha(K-r)}}\right] - F_r\left(\frac{1}{R^\alpha}\right)$$

(F_r as in the statement of Lemma 2).

Substitution of the above result in (10) and use of (11) gives the result of the lemma in the form stated.

<u>Lemma 3</u>

i) $I_2(z,\alpha) = \int_0^{1/R^\alpha} f(t) \int_0^\infty e^{-ztu} g(u)\,du\,dt$ exists and for any integer $N \geqslant 2$

$$\text{ii)} \quad I_2(z,\alpha) = \sum_{K=o}^{N-1} b_K(z,\alpha) \frac{f^K(0)}{K!\,z^{K+1}} + \sum_{r=o}^{N-2} \frac{g^r(0)}{z^{r+1}} \sum_{K=1}^{N-r-1} \frac{f^{K+r}(0)}{(K+r)!\,K} \frac{1}{R^{\alpha K}}$$

$$+ 0\left(\frac{1}{R^{\alpha N+\alpha}}\right) \qquad \text{as } R \to \infty, \text{ where}$$

$$b_K(z,\alpha) = (\gamma + \log\zeta) g^K(0) - \int_0^\infty g^{K+1}(u) \log u\,du - \int_0^\infty g^{K+1}(u) E_1(\zeta u)\,du$$

and $\zeta = {}^z/R^\alpha$.

Proof

(i) follows from the continuity in t of the inner integral as proved in Lemma 2 and the continuity of f.

ii) Equation (1) in this appendix gives

$$f(t) = \sum_{K=o}^{N-1} \frac{f^K(0)}{K!} \; t^K + R_N(t) \text{ where } R_N(t) = 0(t^N) \text{ as } t \to 0+ .$$

It follows that, as $R \to \infty$

$$\int_0^{1/R^\alpha} R_N(t) \int_0^\infty e^{-ztu} g(u)du$$

$$= 0 \left(\int_0^{1/R^\alpha} t^N \, dt \right) \qquad \text{(using (b) with } K = 0)$$

$$= 0 \left(\frac{1}{R^{\alpha N+\alpha}} \right).$$

Hence

$$I_2(z,\alpha) = \sum_{K=o}^{N-1} \frac{f^K(0)}{K!} \int_0^{1/R^\alpha} t^K \int_0^\infty e^{-ztu} g(u)\,du\,dt + 0 \left(\frac{1}{R^{\alpha N+\alpha}}\right) \text{ as } R \to \infty. \tag{12}$$

It is first noted that the expression $(zt)^K \int_0^\infty e^{-ztu} g(u)du$ is of the form $s^K \int_0^\infty e^{-su} g(u)du$ with $s = zt$. This latter integral is associated with the Laplace transform of $g^K(u)$. Indeed

$$\int_0^\infty g^K(u) \; e^{-su} du = s^K \int_0^\infty e^{-su} g(u)du$$

$$- \; s^{K-1} g(0) - s^{K-2} g'(0) ----- -g^{K-1}(0)$$

the result being true even when $Re(s) = 0$ provided g and all its derivatives up to order K-1 vanish at ∞ (the result is also true for $s = 0$ by (d)).

Hence for $|\arg z| \leqslant \frac{\pi}{z}$, $t \geqslant 0$ and $K \geqslant 0$

$$(zt)^K \int_0^\infty e^{-ztu} g(u)du = \int_0^\infty g^K(u)\, e^{-ztu} du + \sum_{r=1}^{K} (zt)^{K-r} g^{r-1}(0)$$

(the sum being defined as 0 when $K = 0$).

Thus

$$t^K \int_0^\infty e^{-ztu} g(u)du = \frac{1}{z^K} \int_0^\infty e^{-ztu} g^K(u)du + \sum_{r=1}^{K} \frac{1}{z^r} t^{K-r} g^{r-1}(0)$$

whence

$$\int_0^{1/R^\alpha} t^K \int_0^\infty e^{-ztu} g(u)dudt = \frac{1}{z^K} \int_0^{1/R^\alpha} \int_0^\infty e^{-ztu} g^K(u)dudt$$

$$+ \sum_{r=1}^{K} \frac{g^{r-1}(0)}{z^r} \frac{R^{-\alpha(K-r+1)}}{K-r+1} . \tag{13}$$

Since $\int_0^\infty g^K(u)e^{-ztu}du$ is uniformly convergent in t for $t \geqslant 0$, reversal of the order of integration is permitted above giving

$$\int_0^{1/R^\alpha} \int_0^\infty g^K(u)e^{-ztu}dudt = \frac{1}{z} \int_0^\infty g^K(u) \frac{1-e^{-\zeta u}}{u} du \quad \text{where } \zeta = \frac{z}{R^\alpha}$$

$$= \frac{1}{z} \int_0^\infty g^K(u) \frac{d}{du} [E_1(\zeta u) + \log(\zeta u) + \gamma]du$$

from Abramowitz and Stegun (p.230).

Integrating by parts gives an integrated part of zero since, at the upper limit, $g^K(u)E_1(\zeta u)$, $g^K(u)\log(\zeta u)$ and $\gamma g^K(u)$ are separately zero while the whole expression

$$E_1(\zeta u) + \log(\zeta u) + \gamma = O(u) \text{ as } u \to 0+$$

(from Abramowitz and Stegun, p.229).

Simple manipulations give subsequently the result:

$$\int_0^{1/R^\alpha} \int_0^\infty e^{-ztu} g^K(u)\,du\,dt = \frac{b_K(z,\alpha)}{z} \text{ and hence, from (13),}$$

$$\int_0^{1/R^\alpha} t^K \int_0^\infty e^{-ztu} g(u)\,du\,dt = \frac{b_K(z,\alpha)}{z^{K+1}} + \sum_{r=1}^{K} \frac{g^{r-1}(0)}{z^r} \frac{R^{-\alpha(K-r+1)}}{K-r+1} .$$

Substitution of this result in (12) now gives

$$I_2(z,\alpha) = \sum_{K=o}^{N-1} b_K(z,\alpha) \frac{f^K(0)}{K!\, z^{K+1}} + \sum_{K=o}^{N-1} \frac{f^K(0)}{K!} \sum_{r=1}^{K} \frac{g^{r-1}(0)}{z^r} \frac{R^{-\alpha(K-r+1)}}{K-r+1}$$

$$+ \quad 0 \quad \left(\frac{1}{R^{\alpha N+\alpha}}\right) \qquad \text{as } R \to \infty . \tag{14}$$

The double sum can be written

$$\sum_{K=1}^{N-1} \frac{f^K(0)}{K!} \sum_{r=1}^{K} \frac{g^{r-1}(0)}{z^r} \frac{R^{-\alpha(K-r+1)}}{K-r+1}$$

(since the inner sum is zero for K = 0)

$$= \sum_{K=1}^{N-1} \frac{f^K(0)}{K!} \sum_{r=o}^{K-1} \frac{g^r(0)}{z^{r+1}} \frac{R^{-\alpha(K-r)}}{K-r} .$$

Reversing the order of summation gives

$$\sum_{r=o}^{N-2} \frac{g^r(0)}{z^{r+1}} \sum_{K=r+1}^{N-1} \frac{f^K(0)}{K!} \frac{R^{-\alpha(K-r)}}{K-r} \qquad \text{or}$$

$$\sum_{r=o}^{N-2} \frac{g^r(0)}{z^{r+1}} \sum_{K=1}^{N-r-1} \frac{f^{K+r}(0)}{(K+r)!\,K} \frac{1}{R^{\alpha K}}$$

and substituion of this expression in (14) gives the result required by the lemma.

Lemma 4

i) $I(z) \overset{D}{=} \int_0^\infty g(t)\ E_1(zt)dt$ exists and

ii) $I(z) \sim \sum_{r=o}^{\infty} \frac{g^r(0)}{r+1} \frac{1}{z^{r+1}}$ as $z \to \infty$.

Proof

i) From properties of the exponential integral, it is known that for any non zero z such that $|\arg(z)| < \pi$ and for real t,

$$E_1(zt) \sim \frac{e^{-zt}}{zt} \quad \text{as } t \to \infty .$$

Thus

$$g(t)\ E_1(zt) \sim \frac{e^{-zt}}{z} \frac{g(t)}{t} \quad \text{as } t \to \infty \quad \text{for } |\arg z| \leq \frac{\pi}{2} .$$

But (d) and a comparison test imply that

$$\int_a^\infty \frac{g(t)}{t} dt$$

is absolutely convergent $(a > 0)$ and clearly

$$|g(t)E_1(zt)| = 0\left(\frac{|g(t)|}{t}\right) \quad \text{as } t \to \infty$$

by above if $z \neq 0$ for $|\arg z| \leq \frac{\pi}{2}$.

Hence the existence of

$$\int_a^\infty g(t)E_1(zt)dt$$

is ensured if $a > 0$.

At the origin, $E_1(zt) \sim \ell nt$ as $t \to 0+$ for fixed non zero z so that $g(t)E_1(zt) \sim g(0)\ \ell nt$ as $t \to 0+$. Hence $g(t)E_1(zt)$ is integrable over any interval $[0,a]$ being continuous (except at

0) and having an integrable singularity at 0. It follows that $I(z)$ exists under the conditions on g stated at the beginning of this section.

ii) By the definition of E_1

$$I(z) = \int_0^\infty g(t) \int_{zt}^\infty \frac{e^{-u}}{u} \, du dt .$$

The transformation $u = zv$ in the inner integral and reversal of the order of integration gives

$$I(z) = \int_0^\infty \frac{e^{-zv}}{v} \int_0^v g(t) dt dv$$

i.e. $I(z) = \int_0^\infty h(v) \, e^{-zv} dv$ where

$$h(v) \overset{D}{=} \frac{1}{v} \int_0^v g(t) dt.$$

The function $h(v)$ is continuous for $v \neq 0$ and has a removable singularity at the origin (define $h(0) = \lim_{v \to 0} h(v) = g(0)$).

Also by repeated differentiation and use of (b), $h(v)$ and all its derivatives vanish as $v \to \infty$.

Hence, by repeated integration by parts,

$$I(z) = \int_0^\infty h(v) \, e^{-zv} \, dv \sim \sum_{r=0}^{\infty} \frac{h^r(0)}{z^{r+1}} \text{ as } z \to \infty \qquad (15)$$

$$\text{in } |\arg z| \leqslant \frac{\pi}{2} .$$

It remains now to express $h^r(0)$ in terms of $g^r(0)$. Since

$g(t) \sim \sum_{r=0}^{\infty} \frac{g^r(0)}{r!} t^r$ as $t \to 0+$, it follows that,

as $v \to 0+$, $\int_0^v g(t) dt \sim \sum_{r=0}^{\infty} \frac{g^r(0)}{(r+1)!} v^{r+1}$ whence

$$h(v) \sim \sum_{r=0}^{\infty} \frac{g^r(0)}{(r+1)!} v^r .$$

But as $v \to 0+$, $h(v) \sim \sum_{r=0}^{\infty} \frac{h^r(0)}{r!} v^r$

so that, by the uniqueness of asymptotic expansions,

$$h^r(0) = \frac{g^r(0)}{r+1} .$$

Substituting this in (15) gives the required result.

§B.3 (The main theorem) THEOREM A

Statement

Under the conditions on f and g stated at the top of page 170.

i) $I(z) \overset{D}{=} \int_0^\infty \int_0^\infty f(t)g(u)e^{-ztu}dudt$ exists and

ii) $I(z) \sim \sum_{r=0}^{\infty} \frac{d_r(z)}{r!\, z^{r+1}}$

as $z \to \infty$ in $|\arg z| \leqslant \frac{\pi}{2}$ where

$$d_r(z) = [\log z+\gamma+S_r] f^r(0)g^r(0) - f^r(0)\int_0^\infty g^{r+1}(t)\log t\,dt - g^r(0)\int_0^\infty f^{r+1}(t)\log t\,dt$$

with $S_r = \begin{cases} 0 & \text{if } r = 0 \\ \sum_{m=1}^{r} \frac{1}{m} & \text{if } r \geqslant 1 \end{cases}$

Proof

i) $I(z) = I_1(z,\alpha) + I_2(z,\alpha)$ so the existence of $I(z)$ is ensured by lemmas 2 and 3.

ii) It will be proved that, for any integer $n \geqslant 1$,

$$I(z) = \sum_{r=0}^{n-1} \frac{1}{r!z^{r+1}}\{[\log z+\gamma+S_r]f^r(0)g^r(0) - f^r(0)\int_0^\infty g^{r+1}(t)\log t\,dt - g^r(0)\int_0^\infty f^{r+1}(t)\log t\,dt\} + o\left(\frac{1}{|z|^n}\right)$$

as $z \to \infty$ in $|\arg z| \leqslant \frac{\pi}{2}$ whence the result will follow.

Given any integer $n \geqslant 1$ and any α $(0 < \alpha < 1)$ first choose N (and keep it fixed) so that $\alpha N \geqslant n$, $\beta N \geqslant n$. Clearly this is always possible e.g. $N = \max\{[\frac{n}{\alpha}] + 1,\ [\frac{n}{\beta}] + 1\}$ and certainly $N > n$ so that also $N \geqslant 2$.

With this value of N the remainder terms in $I_1(z,\alpha)$ and $I_2(z,\alpha)$ are clearly $o\left(\frac{1}{R^n}\right)$ as $R \to \infty$.

The contribution of the second terms in $I_1(z,\alpha)$ and $I_2(z,\alpha)$ to $I(z)$ are

$$- \sum_{r=0}^{N-2} \frac{g^r(0)}{z^{r+1}} \left[F_r\left(\frac{1}{R^\alpha}\right) - \sum_{K=1}^{N-r-1} \frac{f^{K+r}(0)}{(K+r)!\,K} \frac{1}{R^{\alpha K}} \right] - \frac{g^{N-1}(0)}{z^N} F_{N-1}\left(\frac{1}{R^\alpha}\right).$$

Since $F_r\left(\frac{1}{R^\alpha}\right) \sim \sum_{K=1}^{\infty} \frac{f^{K+r}(0)}{(K+r)!\,K} \frac{1}{R^{\alpha K}}$ as $R \to \infty$

the expression in square brackets above will be

$$o\left(\frac{1}{R^{\alpha N - \alpha r - \alpha}}\right) \qquad \text{as } R \to \infty$$

whence for $r \geqslant 0$ the general term in the sum above will be

$$o\left(\frac{1}{R^{r+1+\alpha N - \alpha r - \alpha}}\right)$$

$$= \quad o\left(\frac{1}{R^{\alpha N + \beta r + \beta}}\right)$$

$$= \quad o\left(\frac{1}{R^n}\right) \qquad \text{as } R \to \infty \quad \text{since} \quad N > n, \beta > 0.$$

It follows that the sum itself (being finite) is $o\left(\frac{1}{R^n}\right)$ as clearly also is the other term

$$- \frac{g^{N-1}(0)}{z^N} F_{N-1}\left(\frac{1}{R^\alpha}\right).$$

Hence addition of $I_1(z,\alpha)$ and $I_2(z,\alpha)$ gives

$$I(z) = \sum_{r=0}^{N-1} \frac{g^r(0)}{z^{r+1}} \sum_{K=0}^{r} a_{Kr} \left(\frac{1}{R^\alpha}\right) \frac{1}{R^{\alpha(K-r)}} + \sum_{K=0}^{N-1} b_K(z,\alpha) \frac{f^K(0)}{K!\,z^{K+1}} + o\left(\frac{1}{R^n}\right) \quad \text{as } R \to \infty. \tag{18}$$

Using the definitions of the a_{Kr} (see lemma 1) the first sum, S_1 (say), can be written

$$S_1 = \sum_{r=o}^{N-1} \frac{g^r(0)}{z^{r+1}} \sum_{K=o}^{r-1} \frac{f^K(0)}{K!(r-K)} \frac{1}{R^{\alpha(K-r)}}$$

$$+ \sum_{r=o}^{N-1} \frac{g^r(0)}{z^{r+1}} \left[C_r - \frac{f^r(0)}{r!} \log\left(\frac{1}{R^\alpha}\right)\right] .$$

The double sum above can be written

$$\sum_{r=1}^{N-1} \frac{g^r(0)}{z^{r+1}} \sum_{K=o}^{r-1} \frac{f^K(0)}{K!\,(r-K)} \frac{1}{R^{\alpha(K-r)}}$$ (since the inner sum is zero for $r = 0$)

and by reversing the oder of summation this is equal to

$$\sum_{K=o}^{N-2} \frac{f^K(0)}{K!} \sum_{r=K+1}^{N-1} \frac{g^r(0)}{z^{r+1}} \frac{1}{r-K} \frac{1}{R^{\alpha(K-r)}}$$ or

$$\sum_{K=o}^{N-2} \frac{f^K(0)}{K!} \sum_{r=o}^{N-K-2} \frac{g^{r+K+1}(0)}{z^{r+K+2}} \frac{R^{\alpha(r+1)}}{r+1} .$$

Hence S_1 can be written in the form

$$S_1 = \sum_{K=o}^{N-2} \frac{f^K(0)}{K!\,z^{K+1}} \sum_{r=o}^{N-K-2} \frac{g^{r+K+1}(0)}{(r+1)\zeta^{r+1}} +$$

$$\sum_{r=o}^{N-1} \frac{g^r(0)}{z^{r+1}} \left[C_r - \frac{f^r(0)}{r!} \log\left(\frac{1}{R^\alpha}\right)\right] ; \left(\zeta = \frac{z}{R^\alpha}\right) . \tag{19}$$

The second sum in (18) (using the definition of the $b_K(z\rho)$), S_2 (say), is

$$S_2 = \sum_{K=o}^{N-1} \frac{f^K(0)}{K!\,z^{K+1}} \left[(\gamma+\log\zeta) g^K(0) - \int_0^\infty g^{K+1}(u) \log u\, du\right]$$
$$- \sum_{K=o}^{N-1} \frac{f^K(0)}{K!\,z^{K+1}} \int_0^\infty g^{K+1}(u)\, E_1(\zeta u)\, du . \tag{20}$$

Combining the first term in (19) with the second term in (20) gives the expression

$$E \stackrel{D}{=} - \sum_{K=o}^{N-2} \frac{f^K(0)}{K!\,z^{K+1}} \left[\int_0^\infty g^{K+1}(u) E_1(\zeta u)\, du - \sum_{r=o}^{N-K-2} \frac{g^{K+1+r}(0)}{(r+1)\zeta^{r+1}}\right]$$
$$- \frac{f^{N-1}(0)}{(N-1)!\,z^N} \int_0^\infty g^{K+1}(u) E_1(\zeta u)\, du .$$

By Lemma 4 (applied to $g^{K+1)}$ the term in square brackets above is

$$o\left(\frac{1}{|\zeta|^{N-K-1}}\right) \quad \text{as } R \to \infty, \qquad \text{whence}$$

each term of the sum is

$$o\left(\frac{1}{R^{K+1}} \frac{1}{R^{\beta N-\beta K-\beta}}\right) \qquad (|\zeta| = R^{\beta})$$

$$= o\left(\frac{1}{R^{\alpha K+\alpha+\beta N}}\right)$$

$$= o\left(\frac{1}{R^n}\right) \qquad \text{as } R \to \infty .$$

Clearly the sum itself (being finite) is also $o(\frac{1}{R^n})$ as is the other term in E whence E itself is

$$o\left(\frac{1}{R^n}\right) \quad \text{as } R \to \infty.$$

When it is recalled that

$$C_r = \frac{f^r(0)}{r!} \sum_{m=1}^{r} \frac{1}{m} - \frac{1}{r!} \int_0^\infty f^{r+1}(t)\log t dt \text{ and } \zeta = \frac{z}{R^\alpha},$$

addition of (19) and (20) gives

$$S_1 + S_2 = \sum_{r=o}^{N-1} \frac{d_r(z)}{r!\, z^{r+1}} + o\left(\frac{1}{R^n}\right) \qquad \text{as } R \to \infty,$$

The sum in the equation can be written

$$\sum_{r=o}^{n-1} \frac{d_r(z)}{r!\, z^{r+1}} + o\left(\frac{1}{R^n}\right) \qquad \text{as } R \to \infty,$$

since each of the terms between n and N-1 inclusive is $o\left(\frac{1}{R^n}\right)$. Hence, since S_1 and S_2 are the two sums appearing in equation (18) for I(z), it follows that

$$I(z) = \sum_{r=o}^{n-1} \frac{d_r(z)}{r!\, z^{r+1}} + o\left(\frac{1}{R^n}\right) \quad \text{as } R \to \infty \qquad \text{as required.}$$

The main part of this appendix (§C.2) employs theorem A in appendix B (§B.3) to derive the asymptotics of the wave free part of $\Phi_2(X,Y)$ in the right inner expansion up to terms of order $\frac{1}{R^2}$. §C.1 gives the full asymptotic expansion for the wave-free part of $\Phi_1(X,Y)$ and §C.3, §C.4 contain two results required in Chapter 7.

§C.1. Full Asymptotic series for $F_1(R\cos\theta, R\sin\theta)$

From §5.2, $F_1(R\cos\theta, R\sin\theta)$ (the wave free part of Φ_1) is given by

$$F_1(R\cos\theta, R\sin\theta) = L_1(R\cos\theta, R\sin\theta) + I_1(R\cos\theta, R\sin\theta)$$

where

$$L_1(R\cos\theta, R\sin\theta) = \frac{R^2}{2\pi a}\int_0^\infty u^2 \frac{d}{du}\left[\log\left(\frac{1+2u\sin\theta+u^2}{1-2u\sin\theta+u^2}\right)\right] e^{-Ru}\,du$$

and

$$I_1(R\cos\theta, R\sin\theta) = -\frac{4}{\pi a}\,\mathrm{Re}_j \int_0^\infty \frac{u}{(1+ju)(1-ju)^3}\, e^{-\zeta u}\,du \quad (\zeta = Re^{-j\theta})$$

(see section between equations (5.9) and (5.11).) .

In L_1, the log term in the integrand may be written in the form

$\log\left[\frac{(1-iue^{i\theta})(1+iue^{-i\theta})}{(1-iue^{-i\theta})(1+iue^{i\theta})}\right]$ whence, by logarithmic expansion,

$$u^2 \frac{d}{du}\left[\log\left(\frac{u^2+2u\sin\theta+1}{u^2-2u\sin\theta+1}\right)\right] \sim 4\sum_{r=0}^{\infty}(-1)^r u^{2r+2}\sin(2r+1)\theta \text{ as } u\to 0+ .$$

Hence, by Watson's Lemma,

$$L_1(R\cos\theta, R\sin\theta) \sim \frac{2}{\pi a}\sum_{r=0}^{\infty}(-1)^r (2r+2)!\,\frac{\sin(2r+1)\theta}{R^{2r+1}} \quad \text{as} \quad R\to\infty.$$

Similarly use of the result

$$\frac{u}{(1+ju)(1-ju)^3} \sim \frac{1}{8}\sum_{r=0}^{\infty}[1+(-1)^r-2(r+1)^2]\,j^{r+1}u^r \quad \text{as } u\to 0$$

and Watson's Lemma gives

$$\int_0^\infty \frac{u}{(1+ju)(1-ju)^3} e^{-\zeta u} du \sim \frac{1}{8} \sum_{r=o}^{\infty} [1+(-1)^r - 2(r+1)^2] j^{r+1} \frac{r!}{\zeta^{r+1}}$$

$$= \frac{1}{8} \sum_{r=o}^{\infty} [1+(-1)^r - 2(r+1)^2] j^{r+1} \frac{r! e^{j(r+1)\theta}}{R^{r+1}}$$

as $R \to \infty$, certainly for $-\frac{\pi}{2} \le \theta \le \pi/2$.

Hence

$$\mathrm{Re}_j \int_0^\infty \frac{u}{(1+ju)(1-ju)^3} e^{-\zeta u} du \sim \sum_{r=o}^{\infty} \frac{a_r(\theta)}{R^{r+1}} , \qquad \text{as } R \to \infty$$

where, for $K \ge 0$,

$$\begin{cases} a_{2K}(\theta) = (-1)^K K(K+1)(2K)! \sin(2K+1)\theta \\ a_{2K+1}(\theta) = (-1)^K (K+1)^2 (2K+1)! \cos(2K+2)\theta . \end{cases}$$

Thus

$$I_1(R\cos\theta, R\sin\theta) \sim -\frac{4}{\pi a} \sum_{r=o}^{\infty} \frac{a_r(\theta)}{R^{r+1}} \qquad \text{as } R \to \infty .$$

It follows immediately that

$$F_1(R\cos\theta, R\sin\theta) \sim \frac{4}{\pi a} \sum_{r=o}^{\infty} \frac{f_r(\theta)}{R^{r+1}} \qquad \text{as } R \to \infty \text{ where}$$

for $K \ge 0$,

$$\begin{cases} f_{2K}(\theta) = (-1)^K (K+1)^2 (2K)! \sin(2K+1)\theta \\ f_{2K+1}(\theta) = (-1)^{K+1} (K+1)^2 (2K+1)! \cos(2K+2)\theta . \end{cases}$$

In particular

$$F_1(R\cos\theta, R\sin\theta) = \frac{4}{\pi a} \left(\frac{\sin\theta}{R} - \frac{\cos 2\theta}{R^2} - \frac{8\sin 3\theta}{R^3} \right) + o\left(\frac{1}{R^3}\right) . \tag{C.1}$$

---

§C.2. Asymptotic series for $F_2(R\cos\theta, R\sin\theta)$ to order $\frac{1}{R^2}$

From §5.5, equations (5.33) - (5.38) it is seen that

$$F_2(R\cos\theta, R\sin\theta) = \sum_{i=1}^{3} L_{2i}(R\cos\theta, R\sin\theta) + \sum_{i=1}^{3} I_{2i}(R\cos\theta, R\sin\theta).$$

The terms are dealt with in turn .

(a) From comparison of (5.33) with (5.5), and (5.36) with (5.6) it is seen that

$$L_{21} + I_{21} = -\frac{i}{4a} F_1 = -\frac{i}{\pi a^2} \left(\frac{\sin\theta}{R} - \frac{\cos 2\theta}{R^2} \right) + o\left(\frac{1}{R^2}\right) . \tag{C.2}$$

(b) $L_{22}(R\cos\theta, R\sin\theta) =$

$$= -\frac{R}{2\pi^2 a^2}\,\mathrm{Im}_j\left\{\int_0^{\infty}\int_0^{\infty}\log\left(\frac{1-2t\sin\theta+t^2}{1+2t\sin\theta+t^2}\right)\frac{d}{du}\left[u^2 h^{(3)}(u)\right]e^{-jRtu}\,du\,dt\right.$$

(see after (5.39)) .

The double integral has the form of theorem A with

$$f(t) = \log\left[\frac{1-2t\sin\theta+t^2}{1+2t\sin\theta+t^2}\right],\quad g(u) = \frac{d}{du}\left[u^2 h^{(3)}(u)\right]\quad \left(\text{where } h(u) = \frac{1}{u^2+1}\right)$$

and $Z = jR$.

It is noted first that $f(t) = 0\left[\frac{1}{t}\right]$ as $t \to \infty$ and that $f'(t)$ is a rational algebraic fraction which is $0\left(\frac{1}{t^2}\right)$ as $t \to \infty$.

Hence the orders of all higher derivatives can be obtained by differentiation of this order term. In addition $g(u)$ itself is a rational algebraic fraction which is $0\left(\frac{1}{u^4}\right)$ as $u \to \infty$, While $\arg z = \pi/2$. Thus the conditions of the theorem are satisfied and, as $R \to \infty$,

$L_{22}(R\cos\theta, R\sin\theta) =$

$$-\frac{R}{2\pi^2 a^2}\,\mathrm{Im}_j\sum_{r=o}^{2}\frac{1}{r!(jR)^{r+1}}\{[\log R + j\frac{\pi}{2} + \gamma + S_r]f^r(0)g^r(0) - f^r(0)\int_0^{\infty} g^{r+1}(t)\log t$$

$$- g^r(0)\int_0^{\infty} f^{r+1}(t)\log t\,dt\} + o\left(\frac{1}{R^2}\right) \text{ as } R \to \infty .$$

But f and g are defined also for negative values of t and u and are real for such values, f being an odd function and g an even function (since h is even), so that

$$f^{2K}(0) = 0 \quad\text{and}\quad g^{2K+1}(0) = 0 \quad (K \geq 0) .$$ In addition $g(0)=0$.

Hence, $L_{22}(R\cos\theta, R\sin\theta) = -\dfrac{R}{2\pi^2 a^2}\,\mathrm{Im}_j\{\dfrac{j}{2.R^3}\left[-g^{(2)}(0)\int_0^{\infty} f^{(3)}(t)\log t\,dt\right]\}$

$$+ o\left(\frac{1}{R^2}\right) \quad\text{as } R \to \infty$$

$$= \frac{g^{(2)}(0)}{4\pi^2 a^2 R^2}\int_0^{\infty} f^{(3)}(t)\log t\,dt + o\left(\frac{1}{R^2}\right).$$

Also $g^{(2)}(0) = 144$ and it is proved in the next section (see eq.(C.8)) that $\int_0^{\infty} f^{(3)}(t)\log t\,dt = -2\pi\sin 2\theta$. Thus

$$L_{22}(R\cos\theta, R\sin\theta) = -\frac{72\sin 2\theta}{\pi a^2 R^2} + o\left(\frac{1}{R^2}\right) \text{ as } R \to \infty . \tag{C.3}$$

(c) $L_{23}(R\cos\theta, R\sin\theta) =$

$$-\frac{R}{\pi^2 a^2}\operatorname{Re}_j\{\int_0^\infty\int_0^\infty \log\left[\frac{1-2t\sin\theta+t^2}{1+2t\sin\theta+t^2}\right]\frac{d}{du}[u^2F'(u)]e^{jRtu}dudt$$

where $F(u) = \dfrac{u}{(u-j)(u+j)^3}$.

Comparison with theorem A in this case shows that

$$f(t) = \log\left(\frac{1-2t\sin\theta+t^2}{1+2t\sin\theta+t^2}\right),\quad g(u) = \frac{d}{du}[u^2F'(u)],\quad Z = -jR .$$

f satisfies the requirements of the theorem (as before) as does g since $g(u) = 0\left(\frac{1}{u^3}\right)$ as $u\to\infty$, while $\arg z = -\frac{\pi}{2}$. In addition

$f(0) = 0, f'(0) = -4\sin\theta, f^{(2)}(0) = 0,\ g(0) = 0, g'(0) = -2, g^{(2)}(0) = -24j.$

Thus $L_{23}(R\cos\theta, R\sin\theta) =$

$$-\frac{R}{\pi^2 a^2}\operatorname{Re}_j\sum_{r=0}^{2}\frac{1}{r!(-jR)^{r+1}}\{[\log R - j\frac{\pi}{2}+\gamma+S_r]f^r(0)g^r(0)$$

$$-f^r(0)\int_0^\infty g^{r+1}(t)\log t\,dt - g^r(0)\int_0^\infty f^{r+1}(t)\log t\,dt\} + o\left(\frac{1}{R^2}\right)$$

$$= -\frac{R}{\pi^2 a^2}\operatorname{Re}_j\{-\frac{1}{R^2}[(\log R - j\frac{\pi}{2}+\gamma+1)(8\sin\theta) + 4\sin\theta\int_0^\infty g^{(2)}(t)\log t\,dt.$$

$$+ 2\int_0^\infty f^{(2)}(t)\log t\,dt]$$

$$-\frac{j}{2R^3}[24j\int_0^\infty f^{(3)}(t)\log t\,dt]\} + o\left(\frac{1}{R^2}\right)$$

$$= \frac{1}{\pi^2 a^2}\{\frac{1}{R}[(\log R+\gamma+1)8\sin\theta + 4\sin\theta\operatorname{Re}_j\int_0^\infty g^{(2)}(t)\log t\,dt$$

$$+ 2\int_0^\infty f^{(2)}(t)\log t\,dt] - \frac{12}{R^2}\int_0^\infty f^{(3)}(t)\log t\,dt.\} + o\left(\frac{1}{R^2}\right). \quad (C.4)$$

The integrals occurring here are now evaluated in turn

(i) $\quad I_1 \overset{D}{=} \operatorname{Re}_j\int_0^\infty g^{(2)}(t)\log t\,dt$.

By the corollary after lemma 1 in §B.2,

$$\int_0^\infty g^{(2)}(t)\log t\,dt = -\lim_{x\to 0+}\left\{\int_x^\infty \frac{g'(t)}{t}dt + g'(0)\log x\right\}. \quad (C.5)$$

From the definition of g, $g'(t) = t^2F^{(3)}(t) + 4tF^{(2)}(t) + 2F'(t)$

so that $\displaystyle\int_x^\infty \frac{g'(t)}{t}dt = \int_x^\infty tF^{(3)}(t)dt + 4\int_x^\infty F^{(2)}(t)dt + 2\int_x^\infty \frac{F'(t)}{t}dt$.

Since $\int_0^\infty tF^{(3)}(t)dt$ and $\int_0^\infty F^{(2)}(t)dt$ exist, this can be written

$$\int_x^\infty \frac{g'(t)}{t}dt = \int_0^\infty tF^{(3)}(t)dt + 4\int_0^\infty F^{(2)}(t)dt + 2\int_x^\infty\frac{F'(t)}{t}dt + o(1)$$

as $x\to 0+$.

Using $F'(0) = -1$ and integration by parts leads to the result

$$\int_x^\infty \frac{g'(t)}{t}\,dt = 5 + 2\int_x^\infty \frac{F(t)}{t^2} + o(1) \quad \left(\text{since } \frac{F(x)}{x} \to -1 \text{ as } x \to o+\right)$$

or $$\int_x^\infty \frac{g'(t)}{t}\,dt = 5 + 2\int_x^\infty \frac{1}{t(t-j)(t+j)^3}\,dt + o(1)$$ (when the definition of F is used).

Thus $$\mathrm{Re}_j \int_x^\infty \frac{g'(t)}{t}\,dt = 5 + 2\int_x^\infty \frac{t^2-1}{t(t^2+1)^3}\,dt + o(1)$$

$$= 5 + \int_{x^2}^\infty \frac{(u-1)\,du}{u(u+1)^3} + o(1) \quad (\text{where } u = t^2)\ .$$

The integral here is evaluated by elementary methods giving

$$\log\left(\frac{x^2}{1+x^2}\right) + \frac{1}{1+x^2} + \frac{1}{(1+x^2)^2} \qquad \text{whence}$$

$$\mathrm{Re}_j \int_x^\infty \frac{g'(t)}{t}\,dt = 7 + 2\log x + o(1) \quad \text{as } x \to o+\ .$$

Thus from (C.5) (when it is recalled that $g'(o) = -2$)

$$I_1 = \mathrm{Re}_j \int_0^\infty g^{(2)}(t)\log t\,dt = -7 \tag{C.6}$$

(ii) $I_2 \overset{D}{=} \int_0^\infty f^{(2)}(t)\log t\,dt$.

Again the result

$$\int_0^\infty f^{(2)}(t)\log t\,dt = -\lim_{x\to o+}\left[\int_x^\infty \frac{f'(t)}{t}\,dt + f'(o)\log x\right] \text{is used .}$$

It can be shown without difficulty that

$$\int_x^\infty \frac{f'(t)}{t}\,dt = 2\sin\theta \int_{x^2}^\infty \frac{u-1}{u(u^2+2u\cos 2\theta+1)}\,du$$ whence, by elementary methods of integration,

$$\int_x^\infty \frac{f'(t)}{t}\,dt = \sin\theta\log\left(\frac{x^4}{x^4+2x^2\cos 2\theta+1}\right) + 2\cos\theta\left[\frac{\pi}{2} - \arctan\left(\frac{x^2+\cos 2\theta}{\sin 2\theta}\right)\right]$$

or

$$\int_x^\infty \frac{f'(t)}{t}\,dt = 4\sin\theta\log x + 2\cos\theta\left[\frac{\pi}{2} - \left(\frac{\pi}{2} - 2\theta\right)\right] + o(1) \text{ as } x \to o+$$

$$\left(\text{since } -\frac{\pi}{2} \le \frac{\pi}{2} - 2\theta \le \frac{\pi}{2} \text{ when } 0 \le \theta \le \frac{\pi}{2}\right).$$

Hence $$\int_x^\infty \frac{f'(t)}{t}\,dt = 4\sin\theta\log x + 4\theta\cos\theta + o(1) \text{ as } x \to o+$$ so that (with $f'(o) = -4\sin\theta$)

$$I_2 = \int_0^\infty f^{(2)}(t)\log t\,dt = -4\theta\cos\theta\ . \qquad (C.7)$$

(iii) $I_3 \overset{D}{=} \int_0^\infty f^{(3)}(t)\log t\,dt$.

Since $f^{(2)}$ is an odd function

$$I_3 = -\int_0^\infty \frac{f^{(2)}(t)}{t}\,dt = 8\sin\theta \int_0^\infty \frac{t^4-2t^2-1-2\cos 2\theta}{(t^4+2t^2\cos 2\theta+1)^2}\,dt\ ;$$

also I_3 is odd in θ and $\frac{I_3}{R^2}$ must satisfy Laplace's equation.

It follows that I_3 must be a constant multiple of $\sin 2\theta$,

whence

$$\int_0^\infty \frac{t^4-2t^2-1-2\cos 2\theta}{(t^4+2t^2\cos 2\theta+1)^2} = A\cos\theta \qquad (A = \text{constant})\ .$$

A may be found by putting $\theta = 0$ whence

$$A = \int_0^\infty \left[\frac{1}{(t^2+1)^2} - \frac{4}{(t^2+1)^3}\right]dt\ ,$$

and the substitution $t = \tan\theta$ gives $A = -\pi/2$.

Thus $\quad I_3 = \int_0^\infty f^{(3)}(t)\log t\,dt = -2\pi\sin 2\theta\ . \qquad$ (C.8)

Substitution of (C.6), (C.7) and (C.8) in (C.4) now gives

$L_{23}(R\cos\theta, R\sin\theta)$

$$= -\frac{8}{\pi^2 a^2 R}(\theta\cos\theta - \sin\theta\log R) - \frac{4\sin\theta}{\pi^2 a^2 R}(5-2\gamma) + \frac{24\sin 2\theta}{\pi a^2 R^2} + o\left(\frac{1}{R^2}\right); \qquad (C.9)$$

(d) From (5.37)

$$I_{22}(X,Y) = \frac{2}{\pi^2 a^2}\,\mathrm{Im}_j\left\{\int_0^\infty\int_0^\infty\int_0^\infty \frac{v\cos(Y+s)v - \sin(Y+s)v}{v^2+1}\,e^{-vX}\,\frac{d}{du}[u^2 h^{(3)}(u)]\,e^{-jsu}dvdud\right.$$

With $X = R\cos\theta$, $Y = R\sin\theta$, the inner integral

$\int_0^\infty \frac{v\cos(Y+s)v - \sin(Y+s)v}{v^2+1}\,e^{-vX}dv$ can be written in the form

$\mathrm{Re}_i\int_0^\infty \frac{e^{-v\zeta}}{v-i}\,dv$ where $\zeta = X - iY - is = Re^{-i\theta} - is$. By

integration by parts this may be put in the alternative form

$\mathrm{Re}_i\left(\frac{i}{\zeta} + \frac{1}{\zeta^2} + e(\zeta)\right)$ where $e(\zeta) = \frac{2}{\zeta^2}\int_0^\infty \frac{e^{-v\zeta}}{(v-i)^3}\,dv$ and is

$O(1/\zeta^3)$ as $R \to \infty$ for $|\theta| \leq \pi/2$.

Hence $I_{22}(R\cos\theta, R\sin\theta) = \frac{2}{\pi^2 a^2} \operatorname{Im}_j \operatorname{Re}_i [J_1(R,\theta) + J_2(R,\theta) + J_3(R,\theta)]$

where

$$J_1(R,\theta) = \int_0^\infty \frac{i}{Re^{-i\theta} - is} \int_0^\infty \frac{d}{du}(u^2 h^{(3)}(u)) e^{-jsu}\, du\, ds$$

$$J_2(R,\theta) = \int_0^\infty \frac{1}{(Re^{-i\theta} - is)^2} \int_0^\infty \frac{d}{du}(u^2 h^{(3)}(u)) e^{-jsu}\, du\, ds\ .$$

and $J_3(R,\theta)$ is of the same order (as $R \to \infty$) as

$$\int_0^\infty \frac{1}{(Re^{-i\theta} - is)^3} \int_0^\infty \frac{d}{du}(u^2 h^{(3)}(u)) e^{-jsu}\, du\, ds\ .$$

The substitution $s = Rt$ gives forms for the integrals to which theorem A can be applied viz .

$$J_1(R,\theta) = \int_0^\infty \frac{i}{e^{-i\theta} - it} \int_0^\infty \frac{d}{du}(u^2 h^{(3)}(u)) e^{-jRtu}\, du\, dt\ ,$$

$$J_2(R,\theta) = \frac{1}{R}\int_0^\infty \frac{1}{(e^{-i\theta} - it)^2} \int_0^\infty \frac{d}{du}(u^2 h^{(3)}(u)) e^{-jRtu}\, du\, dt\ ,$$

and $J_3(R,\theta)$ is of the same order as

$$\frac{1}{R^2}\int_0^\infty \frac{1}{(e^{-i\theta} - it)^3} \int_0^\infty \frac{d}{du}(u^2 h^{(3)}(u)) e^{-jRtu}\, du\, dt\ .$$

Theorem A is now applied to the three integrals in turn

(i) In the case of $J_1(R,\theta)$

$$f(t) = \frac{i}{e^{-i\theta} - it}\ , \quad g(u) = \frac{d}{du}(u^2 h^{(3)}(u)) \quad \text{where} \quad h(u) = \frac{1}{u^2+1}$$

and $Z = jR$ so that

$$f(0) = i\, e^{i\theta}\ , \quad f'(0) = -e^{2i\theta}\ , \quad g(0) = 0\ , \quad g'(0) = 0\ .$$

Hence by theorem A

$\operatorname{Im}_j \operatorname{Re}_i [J_1(R,\theta)] =$

$$\operatorname{Im}_j \operatorname{Re}_i \{\tfrac{1}{jR}[-ie^{i\theta} \textstyle\int_0^\infty g'(t)\log t\, dt] - \tfrac{1}{R^2}[e^{2i\theta} \textstyle\int_0^\infty g''(t)\log t\, dt]\} + o\left(\frac{1}{R^2}\right).$$

By integrating by parts (using the definition of g) it is easily shown that $\int_0^\infty g'(t)\log t\, dt = h^{(2)}(o) = -2$.

Hence $\operatorname{Re}_i [J_1(R,\theta)] = \frac{1}{jR}(-2\sin\theta) - \frac{\cos 2\theta}{R^2} \int_0^\infty g^{(2)}(t)\log t\, dt + o\left(\frac{1}{R^2}\right)$

and $\quad \mathrm{Im}_j \; \mathrm{Re}_i [J_1(R,\theta)] = \dfrac{2\sin\theta}{R} + o\left(\dfrac{1}{R^2}\right) \quad$ as $\quad R \to \infty$.

since g is real in this case (with respect to j) .

(ii) For $J_2(R,\theta)$,

$$f(t) = \frac{1}{(e^{-i\theta}-it)^2} \quad \text{whence} \quad f(0) = e^{2i\theta} \quad \text{and } g, Z \text{ are as before .}$$

Hence

$$J_2(R,\theta) = \frac{1}{R}\left\{\frac{1}{jR}\left[-e^{2i\theta}\int_0^\infty g'(t)\log t\,dt\right]\right\} + O\left(\frac{1}{R^3}\right)$$

$$= \frac{-2je^{2i\theta}}{R^2} + o\left(\frac{1}{R^2}\right)$$

and

$$\mathrm{Im}_j \; \mathrm{Re}_i \; [J_2(R,\theta)] = -\frac{2\cos 2\theta}{R^2} + o\left(\frac{1}{R^2}\right) \quad \text{as} \quad R \to \infty \; .$$

(iii) Clearly $\; J_3(R,\theta) = O\left(\dfrac{\log R}{R^3}\right) = o\left(\dfrac{1}{R^2}\right)$ as $R \to \infty$.

Adding the results in (i,) (ii) and (iii) here gives

$$I_{22}(R\cos\theta, R\sin\theta) = \frac{4\sin\theta}{\pi^2 a^2 R} - \frac{4\cos 2\theta}{\pi^2 a^2 R^2} + o\left(\frac{1}{R^2}\right) . \tag{C.10}$$

(e) From (5.38)

$$I_{23}(X,Y) = \frac{4}{\pi^2 a^2}\,\mathrm{Re}_j\left\{\int_0^\infty\int_0^\infty\int_0^\infty \frac{v\cos(Y+s)v - \sin(Y+s)v}{v^2+1}\, e^{-vX}\frac{d}{du}[u^2F'(u)]e^{jsu}\,dv\,du\,ds\right\}$$

where $\quad F(u) = \dfrac{u}{(u-j)(u+j)^3}$.

The treatment here is similar to that in the previous case (d) .

Thus

$I_{23}(R\cos\theta, R\sin\theta) = \dfrac{4}{\pi^2 a^2}\,\mathrm{Re}_j \; \mathrm{Re}_i [K_1(R,\theta) + K_2(R,\theta) + K_3(R,\theta)]$ where

$$K_1(R,\theta) = \int_0^\infty \frac{i}{e^{-i\theta}-it}\int_0^\infty \frac{d}{du}[u^2F'(u)]e^{jRtu}\,du\,dt$$

$$K_2(R,\theta) = \frac{1}{R}\int_0^\infty \frac{1}{(e^{-i\theta}-it)^2}\int_0^\infty \frac{d}{du}[u^2F'(u)]e^{jRtu}\,du\,dt$$

and $\; K_3(R,\theta) = o\left(\dfrac{1}{R^2}\right) \quad$ as $\quad R \to \infty$.

(i) Theorem A is applied with $f(t) = \dfrac{i}{e^{-i\theta}-it}$, $\; g(u) = \dfrac{d}{du}[u^2F'(u)]$

and $Z = -jR$, so that $f(0) = ie^{i\theta}, f'(0) = -e^{2i\theta}, g(0) = 0,\; g'(0) = -2$.

Thus

$$\mathrm{Re}_j\mathrm{Re}_i\,[K_1(R,\theta)]$$

$$=\mathrm{Re}_j\mathrm{Re}_i\{\frac{1}{-jR}[-ie^{i\theta}\int_0^\infty g'(t)\log t\,dt]$$

$$-\frac{1}{R^2}[(\log R-j\frac{\pi}{2}+\gamma+1)2e^{2i\theta}+e^{2i\theta}\int_0^\infty g^{(2)}(t)\log t\,dt+2\int_0^\infty f^{(2)}(t)\log t\,dt]\}$$

$$+\ o\left(\frac{1}{R^2}\right).$$

The three integrals here are dealt with in turn.

(1) By integration by parts (using the definition of g) it is seen that

$$\int_0^\infty g'(t)\log t\ dt\ =\ F(0)\ =\ 0\ .$$

(2) $\mathrm{Re}_j\int_0^\infty g^{(2)}(t)\log t\ dt$ has been evaluated in (c)(i) of this section. Its value is $-\,7$.

(3) $\int_0^\infty f^{(2)}(t)\log t\ dt$ is evaluated (using the corollary after Lemma 1 again) as $-\lim\limits_{x\to 0+}\left(\int_x^\infty \frac{f'(t)}{t}\,dt\ +\ f'(0)\log x\right)$.

The integral here is easily found by elementary methods giving

$$\int_x^\infty \frac{f'(t)}{t}\,dt = e^{2i\theta}\log\left(\frac{x}{x+ie^{-i\theta}}\right) + \frac{ie^{i\theta}}{x+ie^{-i\theta}}$$

$$= e^{2i\theta}\log x - ie^{2i\theta}\left(\frac{\pi}{2}-\theta\right) + e^{2i\theta} + o(1)\ \text{as}\ x\to 0+\ .$$

Whence

$$\lim_{x\to 0+}\left(\int_x^\infty \frac{f'(t)}{t}\,dt + f'(0)\ \log x\right) = -\ ie^{2i\theta}\left(\frac{\pi}{2}-\theta\right)+e^{2i\theta}\ .$$

Thus

$$\int_0^\infty f^{(2)}(t)\log t\ dt = ie^{2i\theta}\left(\frac{\pi}{2}-\theta\right)-e^{2i\theta}\qquad \text{and}$$

$$\mathrm{Re}_i\int_0^\infty f^{(2)}(t)\log t\ dt = \left(\theta-\frac{\pi}{2}\right)\sin 2\theta - \cos 2\theta\ .$$

The results (1), (2), (3) now give

$$\mathrm{Re}_j\mathrm{Re}_i\,[K_1(R,\theta)] = -\frac{1}{R^2}[(\log R+\gamma+1)2\cos 2\theta - 7\cos 2\theta+(2\theta-\pi)\sin 2\theta-2\cos 2\theta]$$

i.e.

$$\mathrm{Re}_j\mathrm{Re}_i\,[K_1(R,\theta)] = -\frac{2\cos 2\theta\log R+2\theta\sin 2\theta}{R^2}+\frac{\pi\sin 2\theta}{R^2}+\frac{(7-2\gamma)\cos 2\theta}{R^2}+o\left(\frac{1}{R^2}\right).$$

(ii) In the case of $K_2(R,\theta)$

$$f(\theta) = \frac{1}{(e^{-i\theta}-it)^2} \quad \text{so} \quad f(0) = e^{2i\theta} \text{ and } g,Z \text{ are as in } e(i) .$$

Thus $Re_j\, Re_i\, [K_2(R,\theta)]$

$$= Re_j\, Re_i\, \{\frac{1}{R}\left[\frac{1}{-jR}\left(-e^{2i\theta}\int_0^\infty g'(t)\log t\, dt\right)\right]\} + o\left(\frac{1}{R^2}\right)$$

$$= o\left(\frac{1}{R^2}\right) \qquad \text{since} \quad \int_0^\infty g'(t)\log t\, dt = 0 .$$

It now follows that

$$I_{23}(R\cos\theta, R\sin\theta) = -\frac{8}{\pi^2 a^2 R^2}(\log R.\cos 2\theta + \theta\sin 2\theta) + \frac{4\sin 2\theta}{\pi\, a^2 R^2}$$

$$+ \frac{4(7-2\gamma)\cos 2\theta}{\pi^2 a^2 R^2} + o\left(\frac{1}{R^2}\right) . \qquad \text{(C.11)}$$

Finally, addition of (C.2), (C.3), (C.9), (C.10) and (C.11) gives the result

$$F_2(R\cos\theta, R\sin\theta) = -\frac{8}{\pi^2 a^2 R}(\theta\cos\theta - \sin\theta\log R) - \frac{8\sin\theta}{\pi^2 a^2 R}(2-\gamma+i\,\frac{\pi}{8})$$

$$-\frac{8}{\pi^2 a^2 R^2}(\log R.\cos 2\theta + \theta\sin 2\theta) + \frac{8\cos 2\theta}{\pi^2 a^2 R^2}(3-\gamma+i\,\frac{\pi}{8}) - \frac{44\sin 2\theta}{\pi a^2 R^2} + o\left(\frac{1}{R^2}\right). \quad \text{(C.12)}$$

---

§C.3 A result on the differentiation of an asymptotic form (required in chapter 7, § 7.3) is now derived. In particular it is required to show that the asymptotics of the expression $\frac{d}{dy}(Y^2 F_{2Y}(0,Y))$ can be obtained from those of $F_2(0,Y)$ by differentiation. The various parts of $F_2(0,Y)$ have asymptotic expansions of the form

$$\sum_{r=o}^{\infty} \frac{a_r}{Y^{r+1}} \quad \text{and} \quad \sum_{r=o}^{\infty} \frac{b_r + c_r \log Y}{Y^{r+1}}$$ so that the problem basically is to show that the differentiated forms of the parts have asymptotic expansions of the same forms.

It is first recalled that

$$F_2(0,Y) = \sum_{i=1}^{3} (L_{2i}(0,Y) + I_{2i}(0,Y)) \quad \text{where}$$

L_{2i}, I_{2i} are given by (5.33)-(5.38).

As previously mentioned $L_{21}(0,Y) = -\frac{i}{4a} L_1(0,Y)$ and $I_{21}(0,Y) = -\frac{i}{4a} I_1(0,Y)$ so that $L_{21}(0,Y) + I_{21}(0,Y) = -\frac{i}{4a} F_1(0,Y)$.

In §5.5 eq. (5.29) it has been shown that

$$\frac{d}{dy}(Y^2 F_{1Y}(0,Y) = \frac{2}{\pi a} \operatorname{Im} [\int_0^\infty \frac{d}{du}(u^2 h^{(3)}(u))e^{-iYu}\, du$$

$$+ \frac{4}{\pi a} \operatorname{Re} [\int_0^\infty \frac{d}{du}(u^2 F'(u))e^{iYu}\, du$$

where $h(u) = \frac{1}{u^2+1}$ and $F(u) = \frac{u}{(u-i)(u+i)^3}$.

Clearly these integrals have asymptotic expansions in inverse powers of Y (by Watson's Lemma) which can be obtained therefore from those of $F_1(0,Y)$ by differentiation.

The asymptotics of the other terms L_{2i}, I_{2i} $(i = 2,3)$ all arise from integrals of the form of Theorem A . Specifically they are the real or imaginary parts of integrals of the form
$I = \int_0^\infty \int_0^\infty f(t)g(u)e^{\pm iYtu}\, dudt$ where $\int_a^\infty \frac{f(t)}{t}\, dt$ exists $(a>0)$ and $g(u) = 0\left(\frac{1}{u^3}\right)$ (at least) as $u \to \infty$ (see §C.2).

Hence the operator M_2^Y $(= \frac{d}{dY} Y^2 \frac{d}{dY})$ can be applied under the integral sign to give

$$M_2^Y(I) = \int_0^\infty \int_0^\infty f(t)g(u)M_2^Y(e^{\pm iYtu})dudt$$

$$= \int_0^\infty \int_0^\infty f(t)g(u)M_2^u(e^{\pm iYtu})dudt \quad .$$

Two integrations by parts in the inner integral (using the order property of g at ∞) give

$$M_2^Y(I) = \int_0^\infty \int_0^\infty f(t)\ M_2^u(g(u))\ .\ e^{\pm iYtu}dudt\ .$$

$M_2{}^u(g(u))$ will also be of order $\frac{1}{u^3}$ at infinity (in the cases in §C.2, g is a rational algebraic fraction so the order properties of its derivatives can certainly be found by differentiation) and $f(t)$ is as before so that Theorem A can be applied. Hence the asymptotic series for $M_2^Y(I)$ will be of the same form as those for I and hence obtainable by differentiation.

---

§C.4 The leading asymptotics of the expression

$$E_1 \equiv \frac{4}{\pi^2 a^3}\int_{Y_O}^{\infty} H_F\left[\frac{1}{s}+\frac{1}{s^2}(9-2\gamma+i\frac{\pi}{4})-\frac{2\log s}{s^2}+R(s)\right]ds \quad \left(|R(s)|\leq \frac{A\log s}{s^3},\ Y_O>1\right)$$

are required in chapter 7, §7.3 where $H_F = H_L + H_I$ (see defs. (4.7) in §4.2). These are now derived via a series of lemmas but it is recalled first from §4.3 (after Note (1)) that (with $K = 1$)

(*) $$H_I(R\cos\theta,R\sin\theta;s) = \frac{2}{\pi}\frac{R\sin\theta+s}{R^2+2sR\sin\theta+s^2} + O\left(\frac{1}{R^2+2sR\sin\theta+s^2}\right) \text{ as } R\to\infty .$$

1) For $\alpha \geq 1$, $I(R\theta) \overset{D}{=} \int_{Y_O}^{\infty} \frac{1}{s^\alpha}\frac{1}{R^2+2sR\sin\theta+s^2}\,ds = o\left(\frac{1}{R}\right)$ as $R\to\infty$.

<u>Proof</u> Put $s = Rt$ so

$$I(R,\theta) = \frac{1}{R^{1+\alpha}}\int_{Y_O/R}^{\infty}\frac{1}{t^\alpha}\cdot\frac{1}{1+2t\sin\theta+t^2}\,dt .$$

For $\alpha > 1$, $I(R,\theta) \leq \frac{1}{R^{1+\alpha}}\int_{Y_O/R}^{\infty}\frac{dt}{t^\alpha}$

$$= \frac{1}{R^{1+\alpha}}\frac{1}{\alpha-1}\cdot\left(\frac{Y_O}{R}\right)^{1-\alpha} = o\left(\frac{1}{R}\right) \quad \text{as } R\to\infty .$$

For $\alpha = 1$, $I(R,\theta) = \frac{1}{R^2}\int_{Y_O/R}^{\infty}\frac{1}{t}\cdot\frac{1}{1+2t\sin\theta+t^2}\,dt = O\left(\frac{\log R}{R^2}\right)$ (by integration by parts)

$$= o\left(\frac{1}{R}\right) \quad \text{as } R\to\infty.$$

2) For $\alpha > 1$, $J(R,\theta) \overset{D}{=} \int_{Y_O}^{\infty}\frac{1}{s^\alpha}\cdot\frac{R\sin\theta+s}{R^2+2sR\sin\theta+s^2}\,ds = O\left(\frac{1}{R}\right)$ as $R\to\infty$.

<u>Proof</u> Put $s = Ru$ so that

$$J(R,\theta) = \frac{1}{R^\alpha}\int_{Y_O/R}^{\infty}\frac{1}{u^\alpha}\cdot\frac{\sin\theta+u}{1+2u\sin\theta+u^2}\,du$$

$$= \frac{1}{R^\alpha}\int_{Y_O/R}^{\infty}\frac{g(\theta,u)}{u^\alpha}\,du$$ (in the notation of §4.3 after equation (4.17)).

From the discussion given there the result follows.

3) For $\alpha > 1$, $\int_{Y_O}^{\infty}\frac{1}{s^\alpha}H_I(R\cos\theta,R\sin\theta;s)\,ds = O\left(\frac{1}{R}\right)$ as $R\to\infty$.

<u>Proof</u> This follows immediately by using (*), (1) and (2) .

4. $K(R,\theta) \overset{D}{=} \int_{Y_o}^{\infty} \frac{1}{s} H_I(R\cos\theta, R\sin\theta; s)\,ds = \frac{2\sin\theta\log R}{\pi R} + O\left(\frac{1}{R}\right)$ as $R \to \infty$.

Proof

$$K(R,\theta) = \frac{2}{\pi}\int_{Y_o}^{\infty} \frac{1}{s}\cdot\frac{R\sin\theta + s}{R^2 + 2sR\sin\theta + s^2}\,ds \quad + o\left(\frac{1}{R}\right) \quad \text{by } 1).$$

$$= \frac{2}{\pi R}\int_{Y_o/R}^{\infty} \frac{1}{t}\,\frac{\sin\theta + t}{1 + 2t\sin\theta + t^2}\,dt \qquad (s = Rt)$$

$$= \frac{2}{\pi R}\left[-\sin\theta\log\left(\frac{Y_o}{R}\right) + O(1)\right] \quad \text{as } R \to \infty \quad \text{(by Lemma 1, appendix B}$$

with $r = o$, $f(t) = \frac{\sin\theta + t}{1 + 2t\sin\theta + t^2}$, $x = \frac{Y_o}{R}$).

i.e. $K(R,\theta) = \frac{2\sin\theta\log R}{\pi R} + O\left(\frac{1}{R}\right)$ as $R \to \infty$.

5). $\int_{Y_o}^{\infty} H_I\left[\frac{1}{s} + \frac{1}{s^2}\left(9 - 2\gamma + i\frac{\pi}{4}\right) - \frac{2\log s}{s^2} + R(s)\right]ds = \frac{2\sin\theta\log R}{\pi R} + O\left(\frac{1}{R}\right)$ as $R \to \infty$.

Proof This follows immediately from (4) and (3) and the fact that

$\frac{\log s}{s^2}$ and $\frac{\log s}{s^3}$ are certainly $O\left(\frac{1}{s^\alpha}\right)$ for some $\alpha > 1$. e.g. $\alpha = 3/2$.

6) $L(R,\theta) \overset{D}{=} \int_{Y_o}^{\infty} \frac{1}{s} H_L(R\cos\theta, R\sin\theta; s)\,ds = -\theta + O\left(\frac{1}{R}\right)$ as $R \to \infty$.

Proof (Recall that $H_L(R\cos\theta, R\sin\theta; s) = \frac{1}{2\pi}\log\left(\frac{R^2 - 2sR\sin\theta + s^2}{R^2 + 2sR\sin\theta + s^2}\right)$).

$$L(R,\theta) = \frac{1}{2\pi}\int_{Y_o/R}^{\infty} \frac{1}{t}\log\left(\frac{1 - 2t\sin\theta + t^2}{1 + 2t\sin\theta + t^2}\right)dt \; .$$

Since the integrand has a removable singularity at the origin,

$$L(R,\theta) = \frac{1}{2\pi}\int_{0}^{\infty} \frac{1}{t}\log\left(\frac{1 - 2t\sin\theta + t^2}{1 + 2t\sin\theta + t^2}\right)dt \quad + O\left(\frac{1}{R}\right).$$

The integral here vanishes when $\theta = o$ and can be found by

differentiation under the integral sign with respect to θ .

Its value is $-2\pi\theta$.

Thus $L(R,\theta) = -\theta + O\left(\frac{1}{R}\right)$.

7) $M(R,\theta) \overset{D}{=} \int_{o}^{\infty} \frac{1}{s^2} H_L(R\cos\theta, R\sin\theta; s)\,ds = -\frac{2\sin\theta\log R}{\pi R} + O\left(\frac{1}{R}\right)$ as $R \to \infty$.

Proof

$$M(R,\theta) = \frac{1}{2\pi R}\int_{Y_o/R}^{\infty} \frac{1}{t^2}\log\left(\frac{1 - 2t\sin\theta + t^2}{1 + 2t\sin\theta + t^2}\right)dt$$

$$= \frac{1}{2\pi R}\ [-f'(0)\log\left(\frac{Y_O}{R}\right) + O(1)] \quad \text{as} \quad R \to \infty \quad \text{(by lemma 1, appendix B}$$

with $r = 1$, $f(t) = \log\left(\frac{1-2t\sin\theta+t^2}{1+2t\sin\theta+t^2}\right)$, $x = Y_O/R)$.

Hence $M(R,\theta) = -\frac{2\sin\theta\log R}{\pi R} + O\left(\frac{1}{R}\right)$ as $R \to \infty$.

8) $$N(R,\theta) \overset{D}{=} \int_{Y_O}^{\infty} \frac{\log s}{s^2}\ H_L(R\cos\theta, R\sin\theta; s)\,ds$$

$$= -\frac{\sin\theta(\log R)^2}{\pi R} + \frac{2\log R}{\pi R}(\theta\cos\theta - \sin\theta) + O\left(\frac{1}{R}\right) \quad \text{as} \quad R \to \infty \ .$$

Proof By putting $s = Rt$ as usual it may be seen that

$$N(R,\theta) = \frac{\log R}{2\pi R}\int_{Y_O/R}^{\infty} \frac{1}{t^2}\log\left(\frac{1-2t\sin\theta+t^2}{1+2t\sin\theta+t^2}\right)dt + \frac{1}{2\pi R}\int_{Y_O/R}^{\infty}\frac{\log t}{t^2}\log\left(\frac{1-2t\sin\theta+t^2}{1+2t\sin\theta+t^2}\right)dt.$$

The two integrals here will be called $N_1(R,\theta)$ and $N_2(R,\theta)$.

By lemma 1 (appendix B) with $r = 1$, $f(t) = \log\left(\frac{1-2t\sin\theta+t^2}{1+2t\sin\theta+t^2}\right)$, $x = Y_O/R$,

$$N_1(R,\theta) = -f'(o)\log\frac{Y_O}{R} + f'(o) - \int_O^{\infty} f^{(2)}(t)\log t\,dt + O\left(\frac{1}{R}\right)$$

i.e. $N_1(R,\theta) = 4\sin\theta\log\frac{Y_O}{R} - 4\sin\theta + 4\theta\cos\theta + O\left(\frac{1}{R}\right)$ (see eq.(C.7)).

By integration by parts.

$$N_2(R,\theta) = \left[-\frac{\log t}{t}\ f(t)\right]_{Y_O/R}^{\infty} + \int_{Y_O/R}^{\infty}\frac{1}{t}\left(\frac{f(t)}{t} + f'(t)\log t\right)dt$$

$$= \log\frac{Y_O}{R}\,.\,f'(o) + O\left(\frac{1}{R}\right) - f'(o)\log\frac{Y_O}{R} + O(1)$$

$$+ [\tfrac{1}{2}(\log t)^2 f'(t)]_{Y_O/R}^{\infty} - \int_{Y_O/R}^{\infty}\tfrac{1}{2}(\log t)^2 f^2(t)dt \text{ as } R\to\infty \ .$$

$$= -\tfrac{1}{2}(\log Y_O - \log R)^2 f'(0) + O(1)$$

$$= 2\sin\theta(\log R)^2 - 4\sin\theta\log Y_O\log R + O(1)\ .$$

Hence

$$N(R,\theta) = \frac{1}{2\pi R}\ [-4\sin\theta(\log R)^2 + 4\sin\theta\log Y_O\log R - 4\sin\theta\log R + 4\theta\cos\theta\log R$$

$$+ 2\sin\theta(\log R)^2 - 4\sin\theta\log Y_O\log R + O(1)]$$

i.e.

$$N(R,\theta) = -\frac{\sin\theta(\log R)^2}{\pi R} + \frac{2\log R}{\pi R}(\theta\cos\theta - \sin\theta) + 0\left(\frac{1}{R}\right) .$$

9) $\int_{Y_0}^{\infty} R(s)H_L(R\cos\theta, R\sin\theta; s)\,ds = 0\left(\frac{1}{R}\right)$ as $R \to \infty$.

<u>Proof</u> Since $R(s) = 0\left(\frac{\log s}{s^3}\right) = 0\left(\frac{1}{s^\alpha}\right)$ for some $\alpha > 2$ e.g. $5/2$ the discussion in §4.3 (before (4.16)) gives the result immediately.

10) $E_1 = -\frac{4\theta}{\pi^2 a^3} + \frac{8\sin\theta}{\pi^3 a^3 R}\left[(\log R)^2 - 2\log R\left(3-\gamma+i\frac{\pi}{8}\right)\right] - \frac{16\log R}{\pi^3 a^3 R}\theta\cos\theta + 0\left(\frac{1}{R}\right)$.

<u>Proof</u> This follows from the definition of E at the beginning of §C.4 and use of (5),(6),(7),(8),(9).

APPENDIX D

Several results quoted in Chapter 5 are now proved.

§D.1

Comparison of equation (5.28) with the equation preceding it shows that (5.28) is verified if it can be proved that

$$\text{(a)}\quad K_1(s) \overset{D}{=} \int_0^\infty \log\left|\frac{s-u}{s+u}\right| (2u - u^2)e^{-u}du = 2\mathrm{Im}_j[\int_0^\infty h^{(2)}(u)e^{-jsu}\,du]$$

$$\text{where } h(u) = \frac{1}{u^2+1} \quad ,$$

and

$$\text{(b)}\quad K_2(s) \overset{D}{=} \int_0^\infty\int_0^\infty \frac{v\cos(s+u)v - \sin(s+u)v}{v^2+1}(2u-u^2)e^{-u}dv\,du = -2\mathrm{Re}_j[\int_0^\infty F(u)e^{jsu}\,du$$

$$\text{where } F(u) = \frac{u}{(u-j)(u+j)^3} \quad .$$

These results enable $\Phi_1(o,s)$ to be differentiated twice under the integral sign without difficulty whence (5.29) is easily proved.

Proof of (a)

By integration by parts it is seen that

$$K_1(s) = 2⨍_0^\infty \frac{s}{s^2-u^2}u^2e^{-u}\,du \quad \text{(where } ⨍ \text{ denotes a Cauchy principal value integral)}$$

whence the result will be proved if

$$J_1(s) \overset{D}{=} ⨍_0^\infty \frac{s}{s^2-u^2}u^2e^{-u}\,du = \mathrm{Im}_j[\int_0^\infty h^{(2)}(u)e^{-jsu}\,du] \quad .$$

Clearly $J_1(s) = s[\int_0^\infty -e^{-u}\,du + s^2 ⨍_0^\infty \frac{e^{-u}}{s^2-u^2}\,du]$ whence

$$J_1(s) = -s - s^2 ⨍_0^\infty \frac{e^{-su}}{u^2-1}\,du \quad .$$

The integral here can be written as $\displaystyle\int_0^\infty \frac{e^{-su}}{u^2-1}du + \tfrac{1}{2}\pi je^{-s}$ where the contour of integration consists of the positive real axis indented by a small semi-circular arc in the first quadrant centred on $u = 1$. The form of the integrand now allows rotation of the contour of integration through

$\frac{\pi}{2}$ radians after which the substitution $u \to ju$ gives

$$J_1(s) = -s + js^2 \int_0^\infty \frac{e^{-jsu}\,du}{u^2+1} - \tfrac{1}{2}\pi j s^2 e^{-s} \quad \text{or}$$

$$J_1(s) = -s + js^2 \int_0^\infty h(u) e^{-jsu}\,du - \tfrac{1}{2}\pi j s^2 e^{-s} .$$

Two integrations by parts now give

$$J_1(s) = -j \int_0^\infty h^{(2)}(u) e^{-jsu}\,du - \tfrac{1}{2}\pi j s^2 e^{-s} .$$

Hence, since $J_1(s)$ is real,

$$J_1(s) = \mathrm{Im}_j\Big[\int_0^\infty h^{(2)}(u) e^{-jsu}\,du \Big] \text{ as was required.} \tag{D.1}$$

Proof of (b)

Comparison with (5.6) shows that

$K_2(s) = -\frac{\pi a}{2} I_1(0,s)$ whence use of the equation for $I_1(X,Y)$

which occurs 8 lines below (5.10) shows that

$$K_2(s) = +2\mathrm{Re}_j \int_0^\infty \frac{u}{(1+ju)(1-ju)^3} e^{jsu}\,du .$$

Thus $K_2(s) = -2\mathrm{Re}_j \int_0^\infty F(u) e^{jsu}\,du$.

(5.28) can now be written in the form

$$\Phi_1(0,s) = -\frac{i}{2a} e^{-s} + \frac{2}{\pi a} J_1(s) + \frac{4}{\pi a} J_2(s) \quad \text{where}$$

$$J_2(s) \stackrel{D}{=} \mathrm{Re}_j \Big[\int_0^\infty F(u) e^{jsu}\,du \Big] . \tag{D.2}$$

Hence

$$\frac{d}{ds}[s^2 \Phi_{1s}(0,s)] = \frac{i}{2a}\frac{d}{ds}(s^2 e^{-s}) + \frac{2}{\pi a}\frac{d}{ds}[s^2 J_1'(s)] + \frac{4}{\pi a}\frac{d}{ds}[s^2 J_2'(s)]. \tag{D.3}$$

From (D.1)

$$\frac{d}{ds}[s^2 J_1'(s)] = \mathrm{Im}_j \int_0^\infty h^{(2)}(u) \frac{\partial}{\partial s}\Big[s^2 \frac{\partial}{\partial s}(e^{-jsu})\Big]\,du$$

whence $\frac{d}{ds}[s^2 J_1'(s)] = \mathrm{Im}_j \int_0^\infty h^{(2)}(u) \frac{\partial}{\partial u}\Big[u^2 \frac{\partial}{\partial u}(e^{-jsu})\Big]\,du$.

Two integrations by parts now lead to the result

$$\frac{d}{ds}[s^2 J_1'(s)] = \mathrm{Im}_j \int_0^\infty \frac{d}{du}[u^2 h^{(3)}(u)]\ e^{-jsu}\, du \tag{D.4}$$

and by exactly the same method

$$\frac{d}{ds}[s^2 J_2'(s)] = \mathrm{Re}_j \int_0^\infty \frac{d}{du}[u^2 F'(u)]\ e^{jsu}\, du\ . \tag{D.5}$$

Substitution of (D.4) and (D.5) in (D.3) gives the verification of (5.29).

§D.2

It is now shown that

$$I \overset{D}{=} \int_0^\infty e^{-s} \frac{d}{ds}[s^2\ \Phi_{1s}(0,s)]\ ds = -\frac{i}{a}\left(\frac{2i}{3\pi} - \frac{1}{8}\right)$$

(see (5.30 et seq.).

From (D.2) and the definitions of J_1 and J_2 it is clear that $\Phi_1(0,s)$ and its derivatives are bounded for all s. Hence two integrations by parts give

$$I = -\int_0^\infty \Phi_1(0,s)\ \frac{d}{ds}\ (s^2 e^{-s})\ ds\ ,$$

and (D.2) then implies that

$$I = \frac{i}{2a}\int_0^\infty e^{-s}\ \frac{d}{ds}(s^2 e^{-s})ds - \frac{2}{\pi a}\int_0^\infty J_1(s)\ \frac{d}{ds}(s^2 e^{-s})ds - \frac{4}{\pi a}\int_0^\infty J_2(s)\frac{d}{ds}(s^2 e^{-s})ds \tag{D.6}$$

Let the three integrals here be called I_1, I_2, I_3 in turn. It is easily proved that $I_1 = \frac{1}{4}$. (D.7)

Next $I_2 = \mathrm{Im}_j \int_0^\infty\int_0^\infty h^{(2)}(u)e^{-jsu}(2s - s^2)e^{-s}\ du\ ds$. (from (D.1))

Reversal of the order of integration gives

$$I_2 = \mathrm{Re}_j \int_0^\infty \frac{2u\ h^{(2)}(u)}{(1+ju)^3}\ du\ .$$

By writing $h(u) = \frac{1}{2}\left(\frac{1}{1+ju} + \frac{1}{1-ju}\right)$ it is easily shown that

$$\int_0^\infty \frac{2u\ h^{(2)} u}{(1+ju)^3} = -2\ [\int_0^\infty \frac{u}{(1+ju)^6}\ du + \int_0^\infty \frac{u}{(1+u^2)^3}\ du\ ]\ .$$

These integrals can be evaluated by elementary methods to give

$$I_2 = -\frac{2}{5} . \qquad (D.8)$$

Finally,

$$I_3 = \mathrm{Re}_j \int_0^\infty\int_0^\infty \frac{u}{(u-j)(u+j)^3}(2s-s^2)e^{-(1-ju)s}\,du\,ds \qquad \text{(from (D.2))}.$$

As for I_2, it can be shown that

$$I_3 = \mathrm{Re}_j \int_0^\infty \frac{-2u^2 j}{(u-j)(u+j)^3(1-ju)^3}\,du$$

i.e. $I_3 = \mathrm{Re}_j \int_0^\infty \frac{-2u^2}{(u-j)(u+j)^6}\,du$ or

$$I_3 = -2\mathrm{Re}_j \int_0^\infty \frac{u^2}{(u-j)(u+j)^6}\,du = -2\mathrm{Re}_j\left[\int_0^\infty \frac{du}{(u+j)^5} - \int_0^\infty \frac{du}{(u-j)(u+j)^6}\right].$$

The first integral here has the value ¼ while the second can be evaluated by means of the reduction formula

$$T_n = \frac{(-1)^{n+1}j^n}{2(n-1)} - \frac{j}{2}T_{n-1} \qquad \left(\text{where } n > 1 \text{ and } T_n = \int_0^\infty \frac{du}{(u-j)(u+j)^n}\right)$$

whence $\mathrm{Re}_j(T_6) = \frac{4}{15}$.

Thus $I_3 = \frac{1}{30}$.

Substitution of (D.7), (D.8) and (D.9) in (D.6) shows that

$$I = \frac{i}{8a} + \frac{4}{5\pi a} - \frac{2}{15\pi a} = \frac{i}{8a} + \frac{2}{3\pi a} = -\frac{i}{a}\left(\frac{2i}{3\pi} - \frac{1}{8}\right).$$

This is the result required after (5.30).

§D.3

In Chapter 5, §5.8, the wave contribution from a velocity distribution $\mathrm{Re}\left(\int_{iY}^\infty \frac{e^{-iv}}{v}\,dv\right)\frac{d}{dY}(Y^2e^{-Y})$ on a vertical wavemaker ($X = 0$, $Y > 0$) is required. It is shown that this distribution produces no progressive waves in the far field.

It is first noted that the real part of the integral occurring here is equal to the Cauchy principal value integral $⨍_{-Y}^\infty \frac{e^{-u}}{u}\,du$ whence an integration by parts gives the equivalent form

$$-e^{Y}\log Y + \int_{-Y}^{\infty} e^{-u}\log|u|\ du \quad \text{or}$$

$$-e^{Y}\log Y - \gamma + \int_{0}^{Y} e^{u}\log u\ du . \qquad (\gamma = \text{Euler's constant}).$$

Hence, the given velocity distribution takes the form

$$(Y^2 - 2Y)\log Y - \gamma\frac{d}{dY}(Y^2e^{-Y}) + \frac{d}{dY}(Y^2e^{-Y})\int_0^Y e^u \log u\ du .$$

The wave part of Havelock's wavemaker solution shows therefore that the wave amplitude produced by this distribution will be proportional to

$$\int_0^\infty e^{-Y}(Y^2 - 2Y)\log Y\,dY - \gamma\int_0^\infty e^{-Y}\frac{d}{dY}(Y^2e^{-Y})dY + \int_0^\infty e^{-Y}\frac{d}{dY}(Y^2e^{-Y})\int_0^Y e^u\log u\ d$$

$$= -\int_0^\infty \frac{d}{dY}(Y^2e^{-Y})\log Y\,dY - \gamma\int_0^\infty e^{-Y}\frac{d}{dY}(Y^2e^{-Y})dY + \int_0^\infty e^{-Y}\frac{d}{dY}(Y^2e^{-Y})\int_0^Y e^u\log u\ du$$

The values of these three integrals can be found by integration by parts and the sum of the three terms here is seen to be

$$1 - \frac{\gamma}{4} + (-1 + \frac{\gamma}{4}) = 0 .$$

Hence this velocity distribution produces no contribution to the wave amplitude.

REFERENCES

ABRAMOWITZ, M. and STEGUN, I.A. (1965). Handbook of Mathematical Functions. Dover Publications, N. York.

AKHIEZER, N.I. and GLAZMAN, I.M. (1981). Theory of Linear Operators in Hilbert Space (I). Pitman-Monographs and Studies in Mathematics.

AITKEN, A. (1937). Studies in Practical Mathematics. Proc. Roy. Soc. Edinburgh 57, pp. 269-304.

ALKER, G. (1975). On the Radiation of Short Surface Waves by a Heaving Circular Cylinder. J. Eng. Maths. 9, pp. 197-205.

ALKER, G. (1977). The Scattering of Short Surface Waves by a Cylinder. J. Fluid Mech. 82, (part 4), pp. 673-686.

AYAD, A.M. and LEPPINGTON, F.G. (1977). Diffraction of Surface Waves by Plane Vertical Obstacles. J. Fluid Mech. 80, pp. 593-604.

BATEMAN MANUSCRIPT PROJECT (1954). Tables of Integral Transforms. (I) Cal. Inst. of Tech.

BENDER, C.M. and ORSZAG, S.A. (1978). Advanced Mathematical Methods for Scientists and Engineers. Int. Ser. Pure and Applied Maths.

BONDI, O. (1943). On the Problem of Breakers. Admiralty Computing Service (from Stoker's (1947) references).

CRIGHTON, D.G. and LEPPINGTON, F.G. (1973). Singular Perturbation Methods in Acoustics; Diffraction by a Plate of Finite Thickness. Proc. Roy. Soc. London, Ser. A. 335, pp. 313-339.

DEAN, W.R. (1945). On the Reflection of Surface Waves by a Submerged Plane Barrier. Proc. Camb. Phil. Soc. 41, pp. 231-238.

FRAENKEL, L.E. (1969). On the Method of Matched Asymptotic Expansions. (Part I). Proc. Camb. Phil. Soc. 65, pp. 209-231.

FRIEDRICHS, K.O. and LEWY, H. (1948). The Dock Problem. Comm. App. Maths. 1 (No. 2), pp. 135-148.

HANSON, E.T. (1926). The Theory of Ship Waves. Proc. Roy. Soc. Lond. Ser. A. 111, pp. 491-529.

HAVELOCK, T.H. (1929). Forced Surface Waves on Water. Phil. Mag. 8, pp. 569-576.

HOLFORD, R.L. (1964a, b). Short Surface Waves in the Presence of a Finite Dock. (Parts I, II). Proc. Camb. Phil. Soc. 60, pp. 957-1011.

HOLFORD, R.L. (1965). On the Generation of Short Two Dimensional Surface Waves by Oscillating Surface Cylinders of Arbitrary Cross-section. Tech. Report Dep. Math. Stanford Univ. (No. 4).

ISAACSON, E. (1950). Water Waves over a Sloping Bottom. Comm. Pure App. Maths. 3 (No. 1), pp. 11-31.

JOHN, F. (1948). Waves in the Presence of an Inclined Barrier. Comm. Appl. Maths. 1 (No. 2), pp. 149-200.

JOHN, F. (1950). On the Motion of Floating Bodies. (Part II). Comm. Pure Appl. Maths. 3, pp. 45-101.

LEPPINGTON, F.G. (1972). On the Radiation and Scattering of Short Surface Waves. (Part I). J. Fluid Mech. 56, pp. 101-119.

LEPPINGTON, F.G. (1973a). On the Radiation and Scattering of Short Surface Waves. (Part II). J. Fluid Mech. 59, pp. 129-146.

LEPPINGTON, F.G. (1973b). On the Radiation and Scattering of Short Surface Waves. (Part III). J. Fluid Mech. 59, pp. 147-157.

LEWY, H. (1946). Water Waves on Sloping Beaches. Bull. Amer. Math. Soc. 52 (No. 9), pp. 737-775.

MacCAMY, R.C. (1961). On the Heaving Motion of Cylinders of Shallow Draft. J. Ship. Res. 5, pp. 34-43.

MARTIN, J. (1971). Some Notes on the Two Dimensional Ship Wave Generation Problem. (Private communication by the author).

MARTIN, P.A. (1981). On the Null Field Equations for Water Wave Radiation Problems. J. Fluid Mech. 113, pp. 315-332.

MARTIN, P.A. and DIXON, A.G. (1983). The Scattering of Regular Surface Waves by a Fixed, Half-immersed Circular Cylinder. App. Ocean Res. 5 (No. 1), pp. 13-23.

MARTIN, P.A. (1984). On the Null Field Equations for Water Wave Scattering Problems. IMA J. App. Maths 33, pp. 1-15.

MORRIS, C.A.N. (1974). The Generation of Surface Waves over a Sloping Beach by an Oscillating Line Source. (Part I). Proc. Camb. Phil. Soc. 76, pp. 545-554.

NEWMAN, J.N. (1975). Interaction of Waves with Two Dimensional Obstacles: a Relation between the Radiation and Scattering Problems. J. Fluid Mech. 71 (part 2), pp. 273-282.

NEWMAN, J.N. (1977). Marine Hydrodynamics. M.I.T. Press.

PETERS, A.S. (1950). The Effect of a Floating Mat on Water Waves. Comm. Pure Appl. Maths. 3 (No. 4), pp. 319-354.

RHODES-ROBINSON, P.F. (1970a, b). On the Short Wave Asymptotic Motion due to a Cylinder Heaving on Water of Finite Depth. (Parts I, II). Proc. Camb. Phil. Soc. 67, pp. 423-468.

RHODES-ROBINSON, P.F. (1972). On the Short Wave Asymptotic Motion due to a Cylinder Heaving on Water of Finite Depth. (Part III). Proc. Camb. Phil. Soc. 72, pp. 83-94.

RHODES-ROBINSON, P.F. (1982). Technical Report. Dept. Maths, Stanford University (299).

RUBIN, H. (1954). The Dock of Finite Extent. Comm. Pure Appl. Maths. 7, pp. 317-344.

SCHMEIDLER, W. (1965). Linear Operators in Hilbert Space. Academic Press.

SHANKS, D. (1955). Non-linear Transformation of Divergent and Slowly Convergent Sequences. J. Math. and Phys. (Mass.) 34, pp. 1-42.

SPARENBERG, J.A. (1957). Applications of the Hilbert Problem to Problems of Mathematical Physics. (Ph.D. Thesis, Delft Univ.)

STOKER, J.J. (1947). Surface Waves in Water of Variable Depth. Quart. App. Maths. 5, pp. 1-54.

STOKER, J.J. (1957). Water Waves. Interscience, New York.

SZU-HSUING, G. (1984). Two Dimensional Short Surface Waves of an Oscillating Cylinder with Arbitrary Shape of Cross-section. Scientia Sinica Ser. A. 27 (No. 1), pp. 70-79.

URSELL, F. (1947). The Effect of a Fixed Vertical Barrier on Surface Waves in Deep Water. Proc. Camb. Phil. Soc. 43, pp. 374-382.

URSELL, F. (1949). On the Heaving Motion of a Circular Cylinder on the Surface of a Fluid. Quart. J. Mech. Appl. Math. 2, pp. 218-231.

URSELL, F. (1950). Surface Waves on Deep Water in the Presence of a Submerged Circular Cylinder. (Part II). Proc. Camb. Phil. Soc. 46, pp. 153-158.

URSELL, F. (1953). Short Surface Waves due to an Oscillating Immersed Body. Proc. Roy. Soc. Lond. Ser. A. 220, pp. 90-103.

URSELL, F. (1961). The Transmission of Surface Waves under Surface Obstacles. Proc. Camb. Phil. Soc. 57, pp. 638-668.

* VUGTS, J.H. (1970). The Hydrodynamic Forces and Ship Motions in Waves. (Ph.D. Thesis, Delft Univ.)

WEHAUSEN, J.V. and LAITONE, E.V. (1960). Surface Waves. Encyclopaedia of Physics 9. Fluid Dynamics 3, Berlin.

* URSELL. F. (1981). IRREGULAR FREQUENCIES AND THE MOTION OF FLOATING BODIES. J. FLUID MECH. 105 pp. 143-156.