Normal Products of Self-adjoint Operators and Self-adjointness of the Perturbed Wave Operator on $L^2(\mathrm{R}^n)$

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Abstract

This thesis contains five chapters. The first two are devoted to the background which consists of integration, Fourier analysis, distributions and linear operators in Hilbert spaces.

The third chapter is a generalization of a work done by Albrecht-Spain in 2000. We give a shorter proof of the main theorem they proved for bounded operators and we generalize it to unbounded operators. We give a counterexample that shows that the result fails to be true for another class of operators. We also say why it does not hold.

In chapters four and five, the idea is the same, that is to find classes of unbounded real-valued Vs for which $\Box + V$ is self-adjoint on $D(\Box)$ where $\Box$ is the wave operator.

In chapter four we consider the wave operator defined on $L^2(\mathbf{R}^2)$ while in chapter five we do the case $L^2(\mathbf{R}^\mathbf{n})$, $n \geq 3$. Throughout these two chapters we will see how different the Laplacian and the wave operator can be.

Declaration of originality

I hereby declare that the research recorded in this thesis and the thesis itself was composed and originated entirely by myself in the School of Mathematics at The University of Edinburgh.

MOHAMMED HICHEM MORTAD

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Contents

Chapter 0
Introduction

The main subject treated in this thesis is linear operators on Hilbert spaces (especially unbounded ones). We devote two chapters to the background that consists of different subjects such as L^p- spaces, distributions, Fourier analysis, interpolation theory of operators, linear bounded and unbounded operators and perturbation theory.

In Chapter three we generalize work done by Albrecht and Spain [1] who gave a condition that forced a product of two self-adjoint operators to be self-adjoint whenever it was normal. The generalization we make here is that the same condition allows us to prove the same thing for unbounded operators. We also give a shorter proof than theirs in the bounded case and a counterexample showing that the condition may fail to make a product of two self-adjoint operators, when it has a normal closure, essentially self-adjoint. In the last section we say why the proof may fail to work if we want to adapt it to the counterexample cited above.

The generalization and the counterexample form a paper by the author [2] which is due to be published in the October 2003 issue of the *Proceedings of the American Mathematical Society*

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In Chapters four and five we study the self-adjointness of the perturbed wave operator $\Box + V$ (the wave is a hyperbolic operator). We emphasize the word hyperbolic inasmuch as a lot of work has been done in the case of the perturbed elliptic operator mainly the perturbed Laplacian which is important in quantum mechanics (for a more detailed treatment of the subject we recommend [3]).

Since it may be quite hard to solve

$$(\Box + V)f = \pm if ...(E)$$

in $L^2(\mathbf{R}^n)$ and see whether it has a non-zero solution, we will be using the Kato-Rellich theorem to get round solving (E) explicitly. So the whole idea will be to prove estimates of the form

$$\|f\| \leq a\|\Box f\|_2 + b\|f\|_2$$

where $\|\cdot\|$ is a norm to be determined. All that with some interesting counterexamples.

Chapter four is devoted to the case $L^2(\mathbf{R^2})$ and Chapter five is devoted to the case $L^2(\mathbf{R^n})$, $n \geq 3$.

Chapter 1
Integration, Fourier analysis and distributions

1.1 Integration

1.1.1 L^p spaces

We cite [4], [5] or [6] as references where one can find detailed proofs of the well-known results stated in this section.

We start with L^p spaces as they will be used often in this thesis. We will only consider L^p spaces on $\mathbf{R}^n$. We have:

Definition 1. *Let $1 \leq p < \infty$. We define:*

$$L^p(\mathbf{R}^n) = \{f : \mathbf{R}^n \to \mathbf{C} \text{ measurable } : \|f\|_p := \left[\int_{\mathbf{R}^n} |f(x)|^p dx\right]^{\frac{1}{p}} < \infty.\}.$$

For $p = \infty$ we say that a measurable function f is in $L^\infty(\mathbf{R}^n)$ if:

$$\|f\|_\infty := \inf\{K : |f(x)| \leq K \text{ for almost every } x \in \mathbf{R}^n\} \text{ is finite }.$$

Remark 1. *We usually define the elements of L^p spaces as classes of equivalence rather than functions where we say f is equivalent to g if $f - g = 0$ a.e. Note that $\|f\|_p = 0$ if and only if $f = 0$ a.e. Also, $\|\cdot\|_p$ is a norm and $L^p(\mathbf{R}^n)$, equipped with this norm, is a Banach space.*

Finally, it will sometimes be convenient to refer to locally integrable functions $L^1_{loc}(\mathbf{R}^n)$; $f \in L^1_{loc}(\mathbf{R}^n)$ if and only if $\int_K |f| < \infty$ for each compact set K in $\mathbf{R}^n$.

We now collect together some well-known inequalities in the theory of L^p spaces which we will use throughout the thesis. We begin with Hölder's inequality.

Theorem 1 (Hölder's inequality). *Let* $1 \leq p \leq \infty, \frac{1}{p}+\frac{1}{q} = 1$. *Let* $f \in L^p(\mathbf{R}^n), g \in L^q(\mathbf{R}^n)$. *Then* $fg \in L^1(\mathbf{R}^n)$ *and*

$$\|fg\|_1 \leq \|f\|_p\|g\|_q.$$

Hölder's inequality can be deduced from Young's inequality which we shall use independently on several occasions.

Lemma 1 (Young's inequality). *For all* $a, b \geq 0$, *if* $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

The case $p = q = 2$ in Hölder's inequality is the classical Cauchy-Schwarz inequality.

Corollary 1 (Cauchy-Schwarz inequality).

$$\|fg\|_1 \leq \|f\|_2\|g\|_2.$$

The following lemma is usually called the converse of Hölder's inequality (for a proof one may consult [4], pp. 128):

Lemma 2. *Let* f *be a real-valued and measurable function. Let* $1 \leq p < \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$\|f\|_p = \sup_{\|g\|_q=1} \|fg\|_1$$

and the supremum on the right hand side is attained.

Observe that in the previous lemma, the function f is not assumed to be in L^p.

Remark 2. *The function* g *(in Lemma 2) which attains the supremum can be taken to be non-negative.*

As a consequence of Lemma 2 we prove Minkowski's inequality.

Corollary 2. *Let* $1 \leq p \leq \infty$. *Let* $f, g \in L^p$. *Then we have:*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

Proof. By Lemma 2 and the triangle inequality for $\|\cdot\|_1$,

$$\|f+g\|_p = \sup\{\|(f+g)h\|_1 : \|h\|_q = 1\}$$

$$\leq \sup\{\|fh\|_1 : \|h\|_q = 1\} + \sup\{\|gh\|_1 : \|h\|_q = 1\}.$$

Thus

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p.$$

□

We will need the generalized Hölder's inequality which is an immediate consequence of the classical case.

Proposition 1 (Generalized Hölder's inequality). *Let* $1 \leq p, q \leq \infty$ *and* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. *Let* $f \in L^p$ *and let* $g \in L^q$. *Then* $fg \in L^r$ *and*

$$\|fg\|_r \leq \|f\|_p \|g\|_q.$$

Definition 2. *The space of infinitely differentiable functions on* $\mathbf{R^n}$ *with compact support will be denoted by* $C_0^\infty(\mathbf{R^n})$.

Definition 3. *Let* f *and* g *be two functions in* $L^1(\mathbf{R^n})$. *Then we define the convolution of* f *and* g, *and we write* $f * g$, *by*

$$(f * g)(x) = \int_{\mathbf{R^n}} f(x-y)g(y)dy.$$

This integral exists almost everywhere.

Convolutions are often used in approximations. The following theorem is a well-known instance of this.

Theorem 2. *Let* k *be in* $L^1(\mathbf{R^n})$, $k \geq 0$ *and* $\int_{\mathbf{R^n}} k = 1$. *For* $\epsilon > 0$, *define* $k_\epsilon(x) = \epsilon^{-n} k(\frac{x}{\epsilon})$, *so that* $\int_{\mathbf{R^n}} k_\epsilon = 1$ *and* $\|k_\epsilon\|_1 = \|k\|_1$. *Let* $f \in L^p(\mathbf{R^n})$ *for some* $1 \leq p < \infty$ *and define* $f_\epsilon := k_\epsilon * f$. *Then*

$$f_\epsilon \in L^p(\mathbf{R^n}), \|f_\epsilon\|_p \leq \|k\|_1 \|f\|_p \text{ and } \lim_{\epsilon \to 0} \|f_\epsilon - f\|_p = 0.$$

$$\text{If } k \in C_0^\infty(\mathbf{R^n}), \text{ then } f_\epsilon \in C^\infty(\mathbf{R^n}) \text{ and } D^\alpha f_\epsilon = (D^\alpha k_\epsilon) * f = k_\epsilon * D^\alpha f.$$

The last equality in the previous line is meant to be in the distributional sense (see Section 1.4.1 below).

One can easily derive from Theorem 2 the following density results.

Corollary 3. *The space* $C_0^\infty(\mathbf{R^n})$ *is dense in* $L^p(\mathbf{R^n})$ *for* $1 \leq p < \infty$, *and hence in particular,* $L^1(\mathbf{R^n}) \cap L^p(\mathbf{R^n})$ *is dense in* $L^p(\mathbf{R^n})$.

Definition 4. *Let* $\lambda > 0$ *and* E_λ *denote the distribution function of* f, *i.e.,*

$$E_\lambda = \{x \in \mathbf{R^n} : |f(x)| \geq \lambda\}.$$

Proposition 2. *Let* $f \in L^p(\mathbf{R^n})$. *Then we have*

$$\|f\|_p^p = p \int_0^\infty \lambda^{p-1} |E_\lambda| d\lambda.$$

Proof. Using

$$|E_\lambda| = \int_{E_\lambda} dx,$$

we have

$$p \int_0^\infty \lambda^{p-1} |E_\lambda| d\lambda = p \int_0^\infty \lambda^{p-1} \int_{E_\lambda} dx d\lambda.$$

Since everything is positive one obtains by using Fubini's Theorem,

$$p \int_0^\infty \lambda^{p-1} |E_\lambda| d\lambda = p \int_{\mathbf{R^n}} \left(\int_0^{|f(x)|} \lambda^{p-1} d\lambda \right) dx = \int_{\mathbf{R^n}} |f(x)|^p dx = \|f\|_p^p.$$

(Here we have used $|\cdot|$ to denote Lebesgue measure but we will also use it to denote the usual norm in $\mathbf{R^n}$. The context will always be clear.) □

1.1.2 L^p_w Spaces

For references we cite [3] or [7].

Weak L^p spaces L^p_w, being larger than the L^p spaces, are often used when a particular object fails to be in L^p.

Definition 5. *A function f on $\mathbf{R^n}$ is said to be in* **weak-**L^p*, written $f \in L^p_w$, if there is a constant $C < \infty$ such that*

$$|\{x : |f(x)| > t\}| \leq Ct^{-p} \textit{ for all } t > 0.$$

If $f \in L^p_w$, we write

$$\|f\|_{p,w} = \sup_t (t^p |\{x : |f(x)| > t\}|)^{\frac{1}{p}}.$$

Notice that $\|\cdot\|_{p,w}$ is not a norm since it does not satisfy the triangle inequality. However, when $p > 1$, L^p_w carries the structure of a Banach space with a norm which is equivalent to $\|\cdot\|_{p,w}$.

Remark 3. *Any function in L^p is in L^p_w and we have:*

$$\|f\|_{p,w} \leq \|f\|_p.$$

In fact for any $t > 0$,

$$\|f\|_p^p \geq \int_{|f|>t} |f(x)|^p dx \geq |\{x : |f(x)| > t\}| t^p.$$

The inequality $t^p |\{x : |f(x)| > t\}| \leq \|f\|_p^p$ is called Chebyshev's inequality.

Example 1. *A typical example is the function $|x|^{-\frac{n}{p}}$. Then $|\{x : |f(x)| > t\}| = c_n t^{-p}$ where c_n is the volume of the unit ball in $\mathbf{R^n}$. Thus $f \in L^p_w(\mathbf{R^n})$ but f is not in $L^p(\mathbf{R^n})$.*

We come now to a result which will be important for us.

Theorem 3. *Let $r \geq 1$. If $r < p < s$ and $f \in L^r_w \cap L^s_w$, then $f \in L^p$ and*

$$\|f\|_p \leq a\|f\|_{r,w} + b\|f\|_{s,w} \tag{1.1}$$

where the constants a and b depend on p, r and s.

Proof. Let $f \in L^r_w(\mathbf{R^n})$. By definition

$$|E_\lambda| = |\{x \in \mathbf{R^n} : |f(x)| \geq \lambda\}| \leq c_r \lambda^{-r}.$$

Also when $f \in L^s_w(\mathbf{R^n})$

$$|E_\lambda| = |\{x \in \mathbf{R^n} : |f(x)| \geq \lambda\}| \leq c_s \lambda^{-s}$$

(here c_s and c_r denote $\|\cdot\|_{s,w}$ and $\|\cdot\|_{r,w}$ respectively). So

$$\begin{aligned}\|f\|^p_{L^p(\mathbf{R^n})} = p\int_0^\infty \lambda^{p-1}|E_\lambda| d\lambda &= p\int_0^1 \lambda^{p-1}|E_\lambda| d\lambda + p\int_1^\infty \lambda^{p-1}|E_\lambda| d\lambda \\ &\leq pc_r \int_0^1 \lambda^{p-r-1} d\lambda + pc_s \int_1^\infty \lambda^{p-s-1} d\lambda.\end{aligned}$$

Hence

$$\|f\|^p_{L^p(\mathbf{R^n})} \leq pc_r \left[\frac{\lambda^{p-r}}{p-r}\right]_0^1 + pc_s \left[\frac{\lambda^{p-s}}{p-s}\right]_1^\infty$$

which is finite if $r < p < s$. Therefore

$$\|f\|^p_{L^p(\mathbf{R^n})} \leq \frac{p}{p-r}\|f\|^r_{r,w} + \frac{p}{s-p}\|f\|^s_{s,w}.$$

Thus

$$\|f\|_{L^p(\mathbf{R^n})} \leq \tilde{c}\|f\|^{\frac{r}{p}}_{r,w} + \tilde{c}\|f\|^{\frac{s}{p}}_{s,w} \tag{1.2}$$

for some constant $\tilde{c}$ depending on p, r and s.

Now we proceed to make all the powers in (1.2) equal to one. We replace f by cf where c is a constant to be determined. We then have

$$\|f\|_{L^p(\mathbf{R^n})} \leq \tilde{c}c^{\frac{r-p}{p}}\|f\|^{\frac{r}{p}}_{r,w} + \tilde{c}c^{\frac{s-p}{p}}\|f\|^{\frac{s}{p}}_{s,w}.$$

Minimizing the quantity on the right hand side with respect to c shows that

$$\|f\|_{L^p(\mathbf{R^n})} \leq \tilde{c}\left(\|f\|^{\frac{s(r-p)}{p(r-s)}}_{s,w}\|f\|^{\frac{r(r-p)}{(s-r)p}+\frac{r}{p}}_{r,w}\right) + \tilde{c}\left(\|f\|^{\frac{s(s-p)}{(r-s)p}+\frac{s}{p}}_{s,w}\|f\|^{\frac{r(s-p)}{p(s-r)}}_{r,w}\right).$$

Now since the sum of the powers in each part of the right hand side is one, Young's inequality (Lemma 1) shows

$$\|f\|_p \leq a\|f\|_{r,w} + b\|f\|_{s,w},$$

establishing (1.1).

□

Finally we recall without proof the dominated convergence theorem.

Theorem 4. *Let $(f_k)_k$ be a sequence of measurable functions on $\mathbf{R^n}$ such that:*
1) each f_k is in L^1;
2) $f_k \longrightarrow f$ a.e. for some f;
3) there exists a function $G \in L^1(\mathbf{R^n})$ independent of k such that $|f_k| \leq G$ a.e. for all k.
Then $f \in L^1(\mathbf{R^n})$ and

$$\int_{\mathbf{R^n}} f(x)dx = \int_{\mathbf{R^n}} \lim_{k\to\infty} f_k(x)dx = \lim_{k\to\infty} \int_{\mathbf{R^n}} f_k(x)dx.$$

1.2 The Fourier transform

We mention [8], [9], [7] or [3] as references in the literature for this sections.

1.2.1 The L^1-Fourier transform

Definition 6. *The Fourier transform of $f \in L^1(\mathbf{R^n})$ is denoted by $\hat{f}$ or $\mathcal{F}f$ and defined by*

$$\mathcal{F}f(x) = \hat{f}(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbf{R^n}} f(t)e^{-ix.t}dt$$

for all x in $\mathbf{R^n}$.

The inverse Fourier transform, $\mathcal{F}^{-1}$, is defined on $L^1(\mathbf{R^n})$ functions by

$$\mathcal{F}^{-1}g(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbf{R^n}} g(x)e^{+ix.t}dx.$$

Proposition 3. *Let $f \in L^1$, then*
a) the mapping $f \longrightarrow \hat{f}$ is linear and if $\hat{f} \in L^1$, then $\mathcal{F}^{-1}\hat{f} = f$ a.e. ;
b) $\hat{f}$ is a bounded function and $\|\hat{f}\|_\infty \leq \frac{1}{(2\pi)^{\frac{n}{2}}}\|f\|_1$;
c) if $f \geq 0$ then $\|\hat{f}\|_\infty = \|f\|_1 = \hat{f}(0)$.

Proposition 4. *Let $f \in L^1(\mathbf{R})$ and $xf \in L^1(\mathbf{R})$. Then $\hat{f}$ is differentiable and*

$$\frac{d}{dt}\hat{f}(t) = \widehat{-ixf}(t).$$

The proofs of Proposition 3 can be found in [7] and that of Proposition 4 can be found in [9] on page 123.

1.2.2 The L^2-Fourier transform

The Fourier transform has a natural definition on L^2 and its theory is particularly elegant on this space. It is also important in quantum mechanics to define $\hat{f}$ for $f \in L^2(\mathbf{R}^n)$. In our work here we will be dealing with operators that are defined on the Hilbert space L^2.

There are different routes to define the Fourier transform on L^2. The one we will use here is via the denseness of $L^1(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$ in $L^2(\mathbf{R}^n)$, see Corollary 3.

We prefer to state various aspects of the Plancherel theorem in different propositions and then we will summarize all properties in what will be called the Plancherel theorem. First, we recall the following facts:

The Fourier transform is defined on $L^1 \cap L^2$ since $L^1 \cap L^2 \subset L^1$ and:
(i) $L^1 \cap L^2$ is a linear subspace of both L^1 and L^2.
(ii) $L^1 \cap L^2$ is a dense subspace of both L^1 and L^2.

The following is the basic result.

Proposition 5. *If $f \in L^1 \cap L^2$, then $\hat{f} \in L^2$ and $\|f\|_2 = \|\hat{f}\|_2$.*

Since $L^1 \cap L^2$ is dense in L^2, Proposition 5 allows us to extend the definition of the Fourier transform $\mathcal{F}$ to all L^2.

Proposition 6 (The Plancherel theorem). *$\mathcal{F}$ is an isometry of L^2, i.e. $\|\mathcal{F}f\|_2 = \|f\|_2$ for all $f \in L^2$.*

The proofs of Propositions 5 and 6 can also be found in [7], on page 118.

1.2.3 The L^p-Fourier transform for $1 \leq p \leq 2$.

For $f \in L^p(\mathbf{R^n})$, $1 \leq p \leq 2$, we can decompose $f = g + h$ where $g \in L^1(\mathbf{R^n})$ and $h \in L^2(\mathbf{R^n})$. Therefore we can define the Fourier transform of f by $\hat{f} = \hat{g} + \hat{h}$ and this is well-defined, i.e., $\hat{f}$ is independent of the decomposition $f = g + h$.

Theorem 5 (Hausdorff-Young inequality). *Suppose $1 \leq p \leq 2$, and $\frac{1}{p} + \frac{1}{q} = 1$. Then the Fourier transform is a bounded map from $L^p(\mathbf{R^n})$ to $L^q(\mathbf{R^n})$ and*

$$\|\hat{f}\|_q \leq c_{n,p}\|f\|_p$$

for some constant $c_{n,p}$.

The proof is an easy application of the Riesz-Thorin theorem (see e.g., [3] Theorem IX.17).

We now state a version of the well-known Sobolev embedding theorem for $\mathbf{R^n}$ (see e.g., [3] Theorem IX.28).

Theorem 6. *Let $f \in L^2(\mathbf{R^n})$ such that $\Delta f \in L^2(\mathbf{R^n})$ in the distributional sense (this will be introduced in Section 1.4.1 below). Then*
a) if $n \leq 3$, f is a bounded continuous function and for any $a > 0$, there is a b, independent of f, so that

$$\|f\|_\infty \leq a\|\Delta f\|_2 + b\|f\|_2$$

b) if $n = 4$ and $2 \leq q < \infty$, then $f \in L^q(\mathbf{R^n})$ and for any $a > 0$ there is a b (depending only on q, n, and a) so that

$$\|f\|_q \leq a\|\Delta f\|_2 + b\|f\|_2$$

Furthermore this estimate is false for $q = \infty$. In fact in this case, f may be unbounded in a neighborhood of every point (see e.g., [10] pp. 159).
c) if $n \geq 5$ and $2 \leq q \leq \frac{2n}{n-4}$, then $f \in L^q(\mathbf{R^n})$ and for any $a > 0$ there is a b (depending only on q, n, and a) so that

$$\|f\|_q \leq a\|\Delta f\|_2 + b\|f\|_2.$$

1.3 The space BMO

The details of the following may be found in [8].

We indicate by $Q \subset \mathbf{R}^n$ any cube with sides parallel to the coordinate axes and by $|Q|$ its Lebesgue measure. For every locally integrable f, let f_Q denote the average of f on Q,

$$f_Q = \frac{1}{|Q|}\int_Q f.$$

Definition 7. *For $f \in L^1_{loc}$, let $f^\sharp_Q$ denote the* **mean oscillation** *of f in Q,*

$$f^\sharp_Q = \frac{1}{|Q|}\int_Q |f(t) - f_Q| dt.$$

Definition 8. *For $f \in L^1_{loc}$, let*

$$M^\sharp f(x) = \sup_{r>0} f^\sharp_{Q(x,r)}$$

where $Q(x, r)$ is the cube of side length r centered at x. The operator $M^\sharp : f \longrightarrow M^\sharp f$ will be called the **sharp maximal operator**.

Definition 9. *A function $f \in L^1_{loc}$ has* **bounded mean oscillation** *(and we say $f \in BMO$) if $M^\sharp f \in L^\infty$ and we set*

$$\|f\|_{BMO} = \|M^\sharp f\|_\infty.$$

Remark 4. *The quantity $\|\cdot\|_{BMO}$ is only a semi-norm since $\|f\|_{BMO} = 0$ if and only if $f(t) = C$ a.e. t. We can make BMO a norm linear space (in fact a Banach space) by passing to equivalent classes modulo constants.*

Remark 5. *Every L^∞-function is in BMO. The converse is not true. In fact* $\log|x|$ *is known to be in BMO (see e.g., [8] pp. 213).*

In Proposition 27 below we will give another example of a function which is in BMO and not in L^∞.

Theorem 7 (Sharp maximal theorem). *Let $1 \leq q \leq p$, $1 < p < \infty$, and suppose $f \in L^q(\mathbf{R}^n)$. Then $f \in L^p(\mathbf{R}^n)$ if and only if $M^\sharp f \in L^p(\mathbf{R}^n)$ and*

$$C_p^{-1}\|M^\sharp f\|_p \leq \|f\|_p \leq C_p\|M^\sharp f\|_p$$

for some constant C_p.

A proof of the sharp maximal theorem can be found in [8] on page 220.

1.4 Distributions

Distributions is a huge subject and is treated in many textbooks from which we refer to [11], [6] and [7].

Definition 10. *Let Ω be an open subset of $\mathbf{R}^n$. A sequence $(f_n)_n$ in $C_0^\infty(\Omega)$ converges in $C_0^\infty(\Omega)$ to some function $f \in C_0^\infty(\Omega)$ if and only if there is some fixed, compact set $K \subseteq \Omega$ such that the support of $f_n - f$ lies in K for all n and for each choice of nonnegative integers $p_1, ..., p_n$,*

$$\left(\frac{\partial}{\partial x_1}\right)^{p_1} \cdots \left(\frac{\partial}{\partial x_m}\right)^{p_m} f_n \to \left(\frac{\partial}{\partial x_1}\right)^{p_1} \cdots \left(\frac{\partial}{\partial x_m}\right)^{p_m} f$$

as $n \to \infty$, uniformly on K.

Definition 11. *A linear form T on $C_0^\infty(\Omega)$ is a distribution if, for every sequence $(\varphi_n)_n$ that converges to 0 in $C_0^\infty(\Omega)$, the sequence $(T(\varphi_n))_n$ tends to 0 in $\mathbf{C}$.*

We denote by $(C_0^\infty)'$ the set of distributions on Ω.
Also the value of a distribution T on a test function $\varphi \in C_0^\infty$, $T(\varphi)$, is often denoted by

$$(T, \varphi) \text{ or } \int_\Omega T(x)\varphi(x)dx.$$

Example 2. *The Dirac distribution δ_x for $x \in \mathbf{R}^n$ is defined by*

$$\delta_x(\varphi) = \varphi(x).$$

If $f \in L^1_{loc}$, then for any $\varphi \in C_0^\infty(\Omega)$ it makes sense to consider

$$T_f(\varphi) = \int_\Omega f(x)\varphi(x)dx$$

which defines an element in $(C_0^\infty(\Omega))'$.

Since $L^p(\Omega) \subset L^1_{loc}(\Omega)$, every L^p function is a distribution.

1.4.1 Distributional derivatives

We now define the notion of a distributional or weak derivative. The differentiation operator of order $|p| = \sum_{i=1}^{m} p_i$ on $(C_0^\infty)'$ is defined as follows: If $T \in (C_0^\infty)'$, set

$$< D^p T, \varphi >= (-1)^{|p|} < T, D^p \varphi > \text{ for all } \varphi \in C_0^\infty.$$

where $D^p = \left(\frac{\partial}{\partial x_1}\right)^{p_1} \dots \left(\frac{\partial}{\partial x_m}\right)^{p_m}$. Since the map $D^p : \varphi \mapsto D^p\varphi$ from C_0^∞ to C_0^∞ is continuous, the linear form D^pT defined on C_0^∞ is indeed a distribution. Thus the derivative of a distribution always exists and is another distribution.

Example 3. *Let*

$$g(x) = \begin{cases} x, & x \geq 0 \\ 0, & x \leq 0. \end{cases}$$

Then g is continuous but not everywhere differentiable in the classical sense. Since $g \in L^1_{loc}(\mathbf{R})$ then g is a distribution and hence has a derivative in $(C_0^\infty)'$. By definition

$$< g', \varphi >= - < g, \varphi' >= - \int_0^\infty x\varphi'(x)dx = \int_0^\infty \varphi(x)dx.$$

Thus as distributions $g' = H$ where H is the Heaviside function

$$H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

H is not even continuous, but it too has a derivative in $(C_0^\infty)'$ given by

$$< H', \varphi >= - < H, \varphi' >= - \int_0^\infty \varphi'(x)dx = \varphi(0) = \delta_0(\varphi).$$

So $H' = \delta_0$ and δ_0 also has a derivative defined by $< \delta_0', \varphi >= -\varphi'(0)$.

1.4.2 Multiplication of distributions by C^∞-functions

Consider a distribution T and $\psi \in C^\infty$. Define the product by its action on $\varphi \in C_0^\infty$ as

$$< \psi T, \varphi >=< T, \psi\varphi > .$$

That ψT is a distribution is an easy consequence of the fact that the product $\psi\varphi \in C_0^\infty$ if $\varphi \in C_0^\infty$.

1.5 Sobolev spaces

1.5.1 The space $H^1(\mathbf{R}^n)$

Now we define Sobolev spaces and for a reference see [11] or [7].

Definition 12. *We define $H^1(\mathbf{R}^n)$ to be*

$$H^1(\mathbf{R}^n) = \{f \in L^2(\mathbf{R}^n) : \nabla f \in L^2(\mathbf{R}^n)^n\}.$$

Here $\nabla f = \left(\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n}\right)$ is the gradient of f and by saying $\nabla f \in L^2(\mathbf{R}^n)^n$, we mean each $\frac{\partial f}{\partial x_j}$ is in $L^2(\mathbf{R}^n)$.

Remark 6. *If $f \in L^2$ and f' exists a.e. in the classical sense and $f' \in L^1_{loc}$, then as a distribution, f' is the distributional derivative of f.*

Remark 7. *It is not hard to show that $C_0^\infty(\mathbf{R}^n)$ is dense in $H^1(\mathbf{R}^n)$ in the norm $\|\cdot\|_{H^1}^2 = \|\cdot\|_2 + \|\nabla(\cdot)\|_2$. For a proof see [7], Theorem 7.6 .*

By applying exactly the same method one may also show that $C_0^\infty(\mathbf{R}^n)$ is dense in $\{f \in L^2(\mathbf{R}^n) : \Box f \in L^2(\mathbf{R}^n)\}$ in the norm $\|\cdot\| = \sqrt{\|\cdot\|_2 + \|\Box(\cdot)\|_2}$ (here $\Box$ is the wave operator, i.e., $\Box f = \frac{\partial^2 f}{\partial x_1^2} - \frac{\partial^2 f}{\partial x_2^2} - ... - \frac{\partial^2 f}{\partial x_n^2}$).

In pretty much the same way one may show that $C_0^\infty(\mathbf{R}^2)$ is dense in $\{f \in L^2(\mathbf{R}^2) : \frac{\partial^2 f}{\partial x \partial y} \in L^2(\mathbf{R}^2)\}$ (this will be used in Chapter 4, Sections 4.2 and 4.3) with respect to the norm

$$\|\cdot\| = \sqrt{\left\|\frac{\partial^2(\cdot)}{\partial x \partial y}\right\|_2 + \|\cdot\|_2}.$$

1.5.2 Fourier characterization of $H^1(\mathbf{R})$

Theorem 8. *Let f be in $L^2(\mathbf{R})$ with Fourier transform $\hat{f}$. Then f is in $H^1(\mathbf{R})$ if and only if the function $k \mapsto k\hat{f}(k)$ is in $L^2(\mathbf{R})$ and when $f \in H^1(\mathbf{R})$,*

$$(f')\hat{}(k) = ik\hat{f}(k) \text{ where } f' = \frac{df}{dx}.$$

We say few words about the proof. One first easily verifies the theorem for $C_0^\infty(\mathbf{R})$ functions and then use a density argument to pass from $C_0^\infty(\mathbf{R})$ to $L^2(\mathbf{R})$ (for a detailed proof one may see [7], pp. 165).

Notation 1. *Throughout the thesis we will denote by $\tilde{c}$ an absolute constant whose exact value may change from line to line.*

Chapter 2
Linear operators in Hilbert spaces

We cite [12], [6], [13], [14] or [15] for references for this chapter where one can find detailed proofs of the basic results.

2.1 Hilbert spaces

Definition 13. *A complex vector space V is called an inner product space if there is a complex-valued function $< \cdot, \cdot >$ on $V \times V$ that satisfies the following four conditions for all $x, y, z \in V$ and $\alpha \in \mathbf{C}$:*
a) $< x, x > \geq 0$ and $< x, x >= 0$ if and only if $x = 0$
b) $< x, y + z >=< x, y > + < x, z >$
c) $< x, \alpha y >= \bar{\alpha} < x, y >$
d) $< x, y >= \overline{< y, x >}$.
The function $< \cdot, \cdot >$ is called an inner product.
A complete inner product is called a Hilbert space.

Example 4. *The main example of a Hilbert space is $L^2(\mathbf{R^n})$ with the inner product*

$$< f, g >= \int_{\mathbf{R^n}} f(x)\overline{g(x)}dx$$

which is well-defined by the Cauchy-Schwarz inequality, Corollary 1.

Theorem 9. *Let H be a Hilbert space and let M be a closed subspace of H. Then $H = M \oplus M^{\perp}$.*

We also recall Riesz's lemma.

Theorem 10 (Riesz's lemma). *Let H be a Hilbert space and let f be a continuous linear functional on H. Then there exists a unique vector a in H such that $f(x) =< x, a >, \forall x \in H$.*

2.2 Bounded linear operators on Hilbert spaces

Definition 14. *Let H be a Hilbert space. A linear operator A from H into H is said to be bounded if there exists an $M > 0$ such that for all $f \in H$ we have:*

$$\|Af\|_H \leq M\|f\|_H. \tag{2.1}$$

We denote by $\mathcal{L}(H)$ the set of all bounded linear operators on H which is a Banach algebra with norm given by

$$\|A\|_{\mathcal{L}(H)} = \sup_{\|x\|_H \leq 1} \|Ax\|_H.$$

Example 5. *Let $H = L^2(0,1)$ and let M be defined on H by $Mf(x) = xf(x)$. M is called a multiplication operator. It is certainly linear and bounded.*

Theorem 11. *Let $A \in \mathcal{L}(H)$. Then there exists a unique operator A^* in $\mathcal{L}(H)$ called the adjoint of A such that:*

$$< Af, g >=< f, A^*g > \forall f, g \in H \text{ and } \|A\|_{\mathcal{L}(H)} = \|A^*\|_{\mathcal{L}(H)}.$$

Proposition 7. *Let $A, B \in \mathcal{L}(H)$ and $\alpha \in \mathbf{C}$. Then*
1) $A^{**} = A$.
2) $(A+B)^* = A^* + B^*$.
3) $(\alpha A)^* = \bar{\alpha} A^*$.
4) $(AB)^* = B^* A^*$.
5) $\|A^*A\| = \|AA^*\| = \|A\|^2$.
7) $Ker(A^*) = (RanA)^{\perp}$.

The proofs of Theorem 11 and Proposition 7 can be found in [14] pp.311-312.

Definition 15. *Let $A \in \mathcal{L}(H)$. Then A is said to be*
a) normal if $AA^* = A^*A$,
b) self-adjoint (symmetric or hermitian) if $A = A^*$,
c) unitary if $AA^* = I = A^*A$, where I is the identity operator on H,
d) a projection if $A^2 = A$,
e) positive if $< Ax, x > \geq 0, \forall x \in H$.

Example 6. *The Fourier transform is an important example of a unitary operator on $L^2(\mathbf{R}^n)$.*

Definition 16. *A projection P is called an orthogonal projection if it is self-adjoint.*

Theorem 12. *Let P be an orthogonal projection. Then:*
1) $Py = y, \forall y \in RanP$.
2) $RanP$ is closed in H. Moreover

$$H = KerP \oplus RanP.$$

3) $\forall x \in H, (x - Px) \in (RanP)^{\perp}$.
4) $\|P\|_{\mathcal{L}(H)} = 1$ (if $P \neq 0$).

Proposition 8. *Let P and Q be two orthogonal projections on H. Then $RanP \perp RanQ$ if and only if $PQ = 0$.*

The proofs of Proposition 8 and Theorem 12 are standard and can be found in ([14], pp. 314).

Definition 17. *Let A be a linear bounded operator. Let $\mathcal{M}$ be a subspace of H. Say that $\mathcal{M}$ is a reducing subspace for A if $A\mathcal{M} \subseteq \mathcal{M}$ and $A\mathcal{M}^{\perp} \subseteq \mathcal{M}^{\perp}$, that is, both $\mathcal{M}$ and $\mathcal{M}^{\perp}$ are invariant subspaces of A.*

Proposition 9. *Let A be an everywhere defined linear operator on a Hilbert space H with $< f, Ag >=< Af, g >$ for all f and g in H. Then A is bounded.*

Proof. We will prove that $G(A)$ is closed (here $G(A)$ is the graph of A, that is, the set $\{(f, Af) : f \in H\}$. More details will be introduced in Definition 20 below) and then A will be bounded by the Closed Graph Theorem. Suppose that $(f_n, Af_n) \to (f, g)$. We need to prove that $(f, g) \in G(A)$, that is, that $g = Af$. But for any $h \in H$,

$$< h, g >= \lim_{n\to\infty} < h, Af_n >= \lim_{n\to\infty} < Ah, f_n >=< Ah, f >=< h, Af > .$$

Thus $g = Af$ and hence $G(A)$ is closed. □

We also recall the Putnam-Fuglede theorem.

Theorem 13 (Putnam-Fuglede theorem). *Assume that M, N and A are all bounded operators on a Hilbert space, M and N are normal, and*

$$NA = AM$$

then $N^*A = AM^*$.

For a proof see [15] on page 285.

2.3 Unbounded linear operators on Hilbert spaces

2.3.1 Domains, graphs, extensions and adjoints

Definition 18. *We say that an operator* A *is unbounded if it is defined on a linear subspace ,* $\mathcal{D}(A)$*, of the Hilbert space and if it does not satisfy (2.1) for* $f \in \mathcal{D}(A)$.

The subspace $\mathcal{D}(A)$ *is called the domain of* A.

An operator with dense domain will be called a densely defined operator.

Example 7. *Let* $H = L^2(\mathbf{R})$ *and let* $\mathcal{D}(A) = \{\varphi \in L^2(\mathbf{R}) : x\varphi \in L^2(\mathbf{R})\}$. *For* $\varphi \in \mathcal{D}(A)$ *define* $(A\varphi)(x) = x\varphi(x)$. *It is clear that* A *is unbounded since if we choose* φ *to have support near plus or minus infinity, we can make* $\|A\varphi\|$ *as large as we like while keeping* $\|\varphi\| = 1$.

Theorem 14. *If* M *is a closed invariant subspace of the symmetric operator* A *(see Definition 23 below) and if the projection* P *onto* M *satisfies the relation* $P\mathcal{D}(A) \subset \mathcal{D}(A)$ *then the subspace* M *reduces the operator* A.

Proposition 10. *Let* P *be the orthogonal projection on a given closed subspace* M*. Then* M *reduces* A *if and only if:*
1) $Pf \in \mathcal{D}(A)$,
2) $PAf = APf$
for all $f \in \mathcal{D}(A)$*, i.e., if the operators* A *and* P *commute.*

We now introduce the notion of a closed operator. Although an operator may not bounded it may be bounded in a different norm, that is the graph norm.

Definition 19. *The graph of an operator* A *is the set of pairs* $\{(f, Af) : f \in \mathcal{D}(A)\} = G(A)$. A *is called a closed operator if* $G(A)$ *is a closed subspace of* $H \times H$*, i.e., if and only if*

$$\forall (f_n, Af_n) \in G(A), f_n \to f, Af_n \to g \Rightarrow f \in \mathcal{D}(A) \text{ and } g = Af.$$

Example 8. *Let* $Mf(x) = xf(x)$ *and* $\mathcal{D}(M) = \{f \in L^2(\mathbf{R}) : xf \in L^2(\mathbf{R})\}$. *Then* M *is closed. Suppose* $f_n \longrightarrow f$ *and* $xf_n \longrightarrow g$ *in* L^2. *There is then a subsequence* $(f_{n(k)})$ *such that* $f_{n(k)}(x) \longrightarrow f(x)$, *a.e. Hence* $xf_{n(k)}(x) \longrightarrow xf(x)$, *a.e. On the other hand since* $xf_n \longrightarrow g$ *in* L^2 *then every subsequence,* $xf_{n(k)}$, *of* (xf_n) *converges to* g *in* L^2. *Hence there is a subsequence of* $xf_{n(k)}$ *which converges to* g *a.e.. Since all subsequences of* $xf_{n(k)}$ *converges to* xf *a.e. we conclude that* $g = xf$ *a.e. and* $G(M)$ *is closed.*

We have the following proposition:

Proposition 11. *Let* A *be a densely defined operator on a Hilbert space* H. *We define*

$$< f, g >_A = < f, g >_H + < Af, Ag >_H, \forall f, g \in \mathcal{D}(A).$$

Then, A *is closed if and only if* $(\mathcal{D}(A), < \cdot, \cdot >_A)$ *is a Hilbert space.*

Proposition 11 gives rise to the graph norm. For a densely defined operator A on a Hilbert space H the graph norm is defined as

$$\|f\|_A = \sqrt{\|f\|_H^2 + \|Af\|_H^2}.$$

Definition 20. *Let* A *and* B *be two unbounded operators.* B *is said to be an extension of* A *if* $\mathcal{D}(A) \subset \mathcal{D}(B)$ *and on* $\mathcal{D}(A)$, A *and* B *coincide.*

Definition 21. *An operator* A *is said to be closable if it has a closed extension. Every closable operator has a smallest closed extension, called its closure, which we denote by* $\overline{A}$.

Proposition 12. *If* A *is closable, then* $G(\overline{A}) = \overline{G(A)}$.

Remark 8. *If* A *is closed then obviously* $\overline{A} = A$.

Definition 22. *Let* A *be a densely defined linear operator on a Hilbert space* H. *Let* $\mathcal{D}(A^*)$ *be the set of* $\varphi \in H$ *for which there is an* $\eta \in H$ *with*

$$< A\psi, \varphi > = < \psi, \eta > \text{ for all } \psi \in \mathcal{D}(A).$$

For each such $\varphi \in \mathcal{D}(A^*)$, *we define* $A^*\varphi = \eta$. A^* *is called the* **adjoint** *of* A. *By the Riesz lemma,* $\varphi \in \mathcal{D}(A^*)$ *if and only if there exists* $C > 0$ *such that* $| < A\psi, \varphi > | \leq C\|\psi\|$ *for all* $\psi \in \mathcal{D}(A)$.

Remark 9. *We note that $A \subset B$ implies $B^* \subset A^*$.*

Notice that in order that the adjoint is well-defined we need the fact that $\mathcal{D}(A)$ is dense. To see this let us assume that $\mathcal{D}(A)$ is not dense. So if $f_0 \in (\mathcal{D}(A))^\perp \neq \{0\}$, then

$$\forall f \in \mathcal{D}(A), < f, A^*g + f_0 >=< f, A^*g > + < f, f_0 >=< f, A^*g > .$$

So A^*g is not unique.

Definition 23. *If A, B are operators in H, then we denote by $A + B$ the operator defined on $\mathcal{D}(A + B) = \mathcal{D}(A) \cap \mathcal{D}(B)$ by $(A + B)(f) = Af + Bf$.*

Lemma 3. *Let A and B be two operators in a Hilbert space H. Then,*
1) if A is closed, B bounded, then $A + B$ is closed;
2) if $A + B$ is densely defined, then $A^* + B^* \subset (A + B)^*$;
3) if A is densely defined, B bounded, then $A^* + B^* = (A + B)^*$.

Definition 24. *Let A, B be operators in H. Denote by BA the operator defined on $\mathcal{D}(BA) = \{f \in \mathcal{D}(A) : Af \in \mathcal{D}(B)\}$ by $(BA)(f) = B(Af)$.*

Lemma 4. *Let A and B be two densely defined operators and let BA be densely defined. Then,*
1) $A^*B^* \subset (BA)^*$;
2) for B bounded, $A^*B^* = (BA)^*$.

The proofs of both Lemma 3 and Lemma 4 can be found in ([13] pp. 214-215).

2.3.2 Symmetric and self-adjoint operators

Definition 25. *A densely defined operator A on a Hilbert space is called* **symmetric** *(or hermitian) if $A \subset A^*$, that is, if $\mathcal{D}(A) \subset \mathcal{D}(A^*)$ and $A\varphi = A^*\varphi$ for all $\varphi \in \mathcal{D}(A)$. Equivalently, A is symmetric if and only if*

$$< A\varphi, \psi >=< \varphi, A\psi > \text{ for all } \varphi, \psi \in \mathcal{D}(A).$$

Definition 26. *The operator A is called* **self-adjoint** *if $A = A^*$, that is, if and only if A is symmetric and $\mathcal{D}(A) = \mathcal{D}(A^*)$.*

Remark 10. *A symmetric operator A is always closable.*

Definition 27. *A symmetric operator A is called* **essentially self-adjoint** *if its closure $\overline{A}$ is self-adjoint.*

Example 9. *Let M be the operator defined by $Mf(x) = xf(x)$ on $\mathcal{D}(M) = \{f \in L^2(\mathbf{R}) : xf \in L^2(\mathbf{R})\}$. Then $\mathcal{D}(M)$ is dense in $L^2(\mathbf{R})$ and M is self-adjoint. In fact M is symmetric since for all $f, g \in \mathcal{D}(M)$ we have*

$$< Mf, g >= \int_{\mathbf{R}} xf(x)\overline{g(x)}dx = \int_{\mathbf{R}} f(x)\overline{xg(x)}dx =< f, xg > .$$

Therefore, to prove M is self-adjoint we only need check that we have $\mathcal{D}(M^*) \subset \mathcal{D}(M)$. Let $\psi \in \mathcal{D}(M^*)$ then $\varphi \mapsto < M\varphi, \psi >$ is continuous on $\mathcal{D}(M)$. Thus there exists a unique $M^*\psi \in L^2(\mathbf{R})$ such that

$$< x\varphi, \psi >=< \varphi, M^*\psi >, \forall \varphi \in \mathcal{D}(M),$$

$$\text{i.e.,} \quad < \varphi, x\psi >=< \varphi, M^*\psi >, \forall \varphi \in \mathcal{D}(M).$$

Thus by the density of $\mathcal{D}(M)$ one gets $M^*\psi = x\psi$ and hence $\psi \in \mathcal{D}(M)$.

2.3.3 The basic criterion for self-adjointness

The following theorem gives us an alternative way to prove a symmetric operator is self-adjoint. A proof can be found in ([6] pp. 257).

Theorem 15 (basic criterion for self-adjointness). *Let A be a symmetric operator on a Hilbert space H. Then the following three statements are equivalent:*
a) A is self-adjoint.
b) A is closed and $Ker(A^* \pm i) = \{0\}$.
c) $Ran(A \pm i) = H$.

Corollary 4. *Let A be a symmetric operator on a Hilbert space. Then the following three are equivalent:*
a) A is essentially self-adjoint.
b) $Ker(A^* \pm i) = \{0\}$.
c) $Ran(A \pm i)$ are dense.

It is worth mentioning that condition b) in Corollary 4 means that

$$A^{*}f = \pm if$$

has a zero solution in the Hilbert space.

In Chapters 4 and 5, when we will be dealing with perturbed wave operators, that is, $\Box + V$ where V is real-valued, to say that $\Box + V$ is essentially self-adjoint means that following weak PDE (i.e., a PDE in the distributional sense)

$$(\Box + V)f = \pm if$$

has a unique solution in L^2, that is, $f = 0$.

Remark 11. *Corollary 4 holds with αi; $\alpha > 0$ instead of i.*

The theorem that follows says that every self-adjoint operator can be diagonalized via a unitary transformation, i.e., every self-adjoint operator is unitarily equivalent to the multiplication operator by a real-valued function.

Theorem 16 (spectral theorem-multiplication operator form). *Let A be a self-adjoint operator on a separable Hilbert space H with domain $\mathcal{D}(A)$. Then there is a measure space (M, μ) with μ a finite measure, a unitary operator $U : H \longrightarrow L^2(M, \mu)$, and a real-valued function f on M which is finite a.e. so that*
a) $\psi \in \mathcal{D}(A)$ if and only if $f(.)(U\psi)(.) \in L^2(M, d\mu)$.
b) If $\varphi \in U[\mathcal{D}(A)]$, then $(UAU^{-1}\varphi)(x) = f(x)\varphi(x)$.

For a proof we refer to ([6] pp. 261).

Example 10. *The Fourier transform $\mathcal{F}$ is an important example of a unitary operator on L^2. We consider the operator $H = i\frac{d}{dx}$. Then one can show that H is self-adjoint on $H^1(\mathbf{R})$ see ([15] pp.341). On the other hand by the Fourier characterization of $H^1(\mathbf{R})$ we have:*

$$\mathcal{F}^{-1}\left(i\frac{d}{dx}\right)\mathcal{F}f(t) = tf(t), \text{ the multiplication operator.}$$

Since $\mathcal{F}$ is unitary and the multiplication operator is self-adjoint we conclude that A is self-adjoint on $H^1(\mathbf{R})$.

Another example of a self-adjoint operator that will be used often in chapters four and five is:

Example 11. *The wave operator* $\Box = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$ *is self-adjoint on* $\mathcal{D}(\Box) = \{f \in L^2(\mathbf{R}^2) : \Box f \in L^2(\mathbf{R}^2)\}$. *By using the Fourier transform and the same idea as for the Fourier characterization of* $H^1(\mathbf{R})$ *we get:*

$$\mathcal{F}^{-1}(\Box)\mathcal{F}f(\eta, \xi) = (-\eta^2 + \xi^2)f(\eta, \xi) := Mf(\eta, \xi).$$

So $\Box$ *is unitarily equivalent to the multiplication operator that has domain* $\mathcal{D}(M) = \{f \in L^2(\mathbf{R}^2) : Mf \in L^2(\mathbf{R}^2)\}$. *So by using this domain and exploiting the unitary equivalence we obtain the domain of* $\Box$ *mentioned above.*

2.3.4 Normal operators

For a wider treatment of this subject we recommend [15] and [14] where most of the proofs for the results in the following section can be found.

Definition 28. *A densely defined closed operator* N *is said to be normal if* $NN^* = N^*N$.

Example 12. *Let* μ *be a finite measure on* $\mathbf{C}$ *such that every polynomial in* z *and* $\bar{z}$ *belongs to* $L^2(\mu)$. *Let* $M\varphi(z) = z\varphi(z)$ *be defined on* $\mathcal{D}(M) = \{\varphi \in L^2 : z\varphi \in L^2(\mu)\}$. *Then* M *is normal on* $\mathcal{D}(M)$.

2.3.5 Spectral theory of linear operators

The following definition applies to both bounded and unbounded operators.

Definition 29. *If* $A : H \to H$ *is a linear operator,* $\rho(A)$, *the* **resolvent set** *for* A, *is defined as*

$$\rho(A) = \{\lambda \in \mathbf{C} : \lambda I - A \text{ is boundedly invertible } \}.$$

The **spectrum** *of* A *is the set* $\sigma(A)$ *which is the complement of* $\rho(A)$ *in* $\mathbf{C}$.

Proposition 13. *Let* A *be a bounded linear operator on a Hilbert space. Then the spectrum of* A, $\sigma(A)$, *is a non-empty compact set in* $\mathbf{C}$ *included in the closed ball of center 0 and radius* $\|A\|$.

Proposition 14. *Let A be a linear operator with adjoint A^* and spectrum $\sigma(A)$. Then*

$$\sigma(A^*) = \{\overline{\lambda} : \lambda \in \sigma(A)\}.$$

Proposition 15. *Let A be a linear operator. Then*
1) if A is self-adjoint then $\sigma(A)$ lies in the real line.
2) if A is normal then it is self-adjoint if and only if $\sigma(A)$ lies in the real line.

Remark 12. *An unbounded self-adjoint operator has always a non-empty spectrum.*

2.3.6 The spectral theorem for normal operators

We start with introducing the notion of a spectral measure.

Definition 30. *If X is a set, Ω is a σ-algebra of subsets of X, and H is a Hilbert space, a spectral measure is for (X, Ω, H) is a function $P : \Omega \longrightarrow \mathcal{L}(H)$ such that*
a) for each Δ in Ω, $P(\Delta)$ is a self-adjoint projection;
b) $P(\emptyset) = 0$ and $P(X) = I$;
c) $P(\Delta_1 \cap \Delta_2) = P(\Delta_1)P(\Delta_2)$ for Δ_1 and Δ_2 in Ω;
d) if $(\Delta_n)_{n=1}^{\infty}$ are pairwise disjoint sets from Ω then:

$$P(\bigcup_{n=1}^{\infty} \Delta_n) = \sum_{n=1}^{\infty} P(\Delta_n)$$

Remark 13. *The convergence of the infinite series in d) is meant to be in the strong operator topology.*

Theorem 17 (The spectral theorem). *If N is a normal operator on H then there is a unique spectral measure P defined on the Borel subsets of $\mathbf{C}$ such that*

$$< Nf, g >= \int_{\sigma(N)} z dP_{f,g}(z) \tag{2.2}$$

where $P_{f,g}(\Delta) =< P(\Delta)f, g >$ defines a complex measure.

One writes

$$N = \int_{\sigma(N)} z dP(z).$$

By Theorem 17 we can define $f(N)$, where f is a Borel function, to be

$$f(N) = \int_{\sigma(N)} f(z)dP(z).$$

The spectral theorem is one the most important theorems in the theory of linear operators if not the most important. It has many applications e.g., Proposition 15 is an immediate consequence of it. A proof of the spectral theorem can be found in ([15] pp. 269).

We can apply Theorem 17 to the special case of a self-adjoint operator and obtain the following result:

Proposition 16 (Spectral mapping theorem). *Let A be a self-adjoint operator. Let f be a continuous function on $\sigma(A)$. Then $f(A)$ is well-defined as a bounded operator. Besides one has*

$$f(\sigma(T)) = \sigma(f(T)).$$

Example 13. *Let N is a multiplication operator by a complex-valued function. Then the spectral measure of N, is the multiplication operator by a characteristic function of a Borel set Δ in $\mathbf{C}$ (see [15] pp. 271).*

Like self-adjoint operators, normal ones too are unitarily equivalent to multiplication operators. The difference is that self-adjoint operators are unitarily equivalent to multiplication operators by a real-valued function while normal ones are unitarily equivalent to multiplication operators by a complex-valued function.

Proposition 17. *If N is a normal operator on the separable Hilbert space H, then there is a σ-finite measure space (X, Ω, μ) and an Ω-measurable function φ such that N is unitarily equivalent to the multiplication operator by φ.*

Let us consider the ball $B_R = \{z \in \mathbf{C} : |z| \leq R\}$. Let P_{B_R} be the spectral projection for N defined on the Borel set B_R. We have

Proposition 18. *Let N be a normal operator with domain $\mathcal{D}(N)$ and spectral projection P_{B_R}. Then we have*

$$f \in RanP_{B_R} \Leftrightarrow f \in \mathcal{D}(N^k), \forall k = 1, 2, ... \exists c > 0 \text{ such that } \|N^k f\| \leq cR^k. \qquad (2.3)$$

This last proposition was taken from [15] on page 330.

As a consequence of the spectral theorem we have

Proposition 19. *Let N be a normal operator with spectral projection P_{B_R}. Then the subspace $H_R = P_{B_R}H$ reduces N.*

The Fuglede-Putnam theorem is valid for unbounded operators.

Theorem 18 (Fuglede-Putnam theorem:the unbounded case). *If N, M are two unbounded normal operators and A is a bounded operator such that $AN \subset MA$, then $AN^* \subset M^*A$.*

A proof can be found in [16] and [17].
For more details about unbounded normal operators see [15] or [14].

2.4 Perturbation of unbounded linear operators

For a reference for this section and for Section 2.5 the reader may consult [3].
In this section we will state a theorem which says that if A is unbounded and self-adjoint and if B is symmetric and not too large compared to A, then $A + B$ is self-adjoint.

Definition 31. *Let A and B be densely defined linear operators on a Hilbert space H. Suppose that*
i) $\mathcal{D}(A) \subset \mathcal{D}(B)$
ii) for some a and b in $\mathbf{R}$ and all $\varphi \in \mathcal{D}(A)$,

$$\|B\varphi\| \leq a\|A\varphi\| + b\|\varphi\|.$$

*Then B is said to be A-**bounded**. The infimum of such a is called the **relative bound** of B with respect to A.*

Sometimes it is convenient to replace (ii) in the above definition by
iii) for some $\tilde{a}, \tilde{b} \in \mathbf{R}$ and all $\varphi \in \mathcal{D}(A)$,

$$\|B\varphi\|^2 \leq \tilde{a}^2\|A\varphi\|^2 + \tilde{b}^2\|\varphi\|^2.$$

A fundamental perturbation result that we will be using often is the Kato-Rellich perturbation theorem, that is

Theorem 19 (Kato-Rellich theorem). *Suppose that A is self-adjoint, B is symmetric, and B is A-bounded with relative bound $a < 1$. Then $A + B$ is self-adjoint on $\mathcal{D}(A)$.*

The reader can find a proof in [3], Theorem X.12 .

Example 14. *Let $-\Delta = H_0$ be the Laplacian defined on the domain $\mathcal{D}(H_0) = \{f \in L^2(\mathbf{R}^3) : \Delta f \in L^2(\mathbf{R}^3)\}$. If V is real-valued such that $V \in L^2 + L^\infty$ then $H_0 + V$ is self-adjoint on $D(H_0)$.*

Proof. First write $V = V_1 + V_2$ where $V_1 \in L^2(\mathbf{R}^3)$ and $V_2 \in L^\infty(\mathbf{R}^3)$. We have by applying Theorem 6 a), for $f \in \mathcal{D}(H_0)$,

$$\|Vf\|_{L^2(\mathbf{R}^3)} = \|(V_1 + V_2)f\|_{L^2(\mathbf{R}^3)} \leq \|V_1 f\|_2 + \|V_2 f\|_2 \leq \|V_1\|_2\|f\|_\infty + \|V_2\|_\infty\|f\|_2$$

$$\leq \|V_1\|_2(a\|\Delta f\|_2 + b\|f\|_2) + \|V_2\|_\infty\|f\|_2 \leq a\|V_1\|_2\|\Delta f\|_2 + (\|V_2\|_\infty + b\|V_1\|_2)\|f\|_2.$$

This implies that $\mathcal{D}(H_0) \subset \mathcal{D}(V) := \{f \in L^2 : Vf \in L^2\}$ and since we can make a small enough such that $a\|V_1\|_2 < 1$ (again by Theorem 6) we conclude by the Kato-Rellich theorem that $H_0 + V$ is self-adjoint on $\mathcal{D}(H_0)$. □

2.5 Limit point-limit circle case

This section deals with the one-dimensional Schrödinger operator, that is $-\frac{d^2}{dx^2} + V$ where V is a real-valued function that is usually called a potential. We give a criterion that tells us when the Schrödinger operator is essentially self-adjoint on $C_0^\infty(\mathbf{R})$. For a reference consult [3].

Definition 32. *We will say that $V(x)$ is in the limit circle case at ∞ (respectively at 0) if for some $\lambda \in \mathbf{C}$, and therefore all λ,* [1] *all solutions of*

$$-\varphi''(x) + V(x)\varphi(x) = \lambda\varphi(x)$$

[1] *In [3], Theorem X.6 says that if, for some λ, both solutions are square integrable at ∞ (at 0), then all solutions are so for all λ.*

are square integrable at ∞ *(respectively at 0). If* $V(x)$ *is not in the limit circle case at* ∞ *(respectively at 0), it is said to be in the limit point case.*

In the previous definition there are always exactly two independent solutions of the equation (see [3]).

A proof of the following theorem is in ([3] pp. 153).

Theorem 20 (Weyl's limit point-limit circle criterion). *Let* $V(x)$ *be a continuous real-valued function on* $(0, \infty)$. *Then* $-\frac{d^2}{dx^2} + V(x)$ *is essentially self-adjoint on* $C_0^\infty(0, \infty)$ *if and only if* $V(x)$ *is in the limit point case at both zero and infinity.*

Remark 14. *The previous theorem has an analogue for more general intervals than* $(0, \infty)$*; namely, if* $V(x)$ *is continuous on* (a, b) *with* $-\infty \leq a < b \leq \infty$*, then* $-\frac{d^2}{dx^2} + V(x)$ *is essentially self-adjoint on* $C_0^\infty(a, b)$ *if and only if* $V(x)$ *is in the limit point case at both a and b, with the obvious modifications in the definition of V being in the limit point case at any real number a.*

The next theorem allows us to say when V is or is not in the limit point case. This theorem is due to A. Wintner, see [18].

Theorem 21. *Let V be a twice continuously differentiable real-valued function on* $(0, \infty)$ *and suppose that* $V(x) \to -\infty$ *as* $x \to \infty$*. Suppose further that*

$$\int_c^\infty \left(\frac{[(-V)^{\frac{1}{2}}]'}{(-V)^{\frac{3}{2}}} \right)' (-V)^{-\frac{1}{4}} dx < \infty$$

for some c. Then V is in the limit point case at infinity if and only if $\int_c^\infty (-V(x))^{-\frac{1}{2}} dx = \infty$.

Example 15. *One easily concludes from Theorem 21 that* $-\frac{d^2}{dx^2} - x^\alpha$ *is in the limit point case at infinity if and only if* $\alpha \leq 2$.

By a change of variable we have the same theorem on $(-\infty, 0)$. In fact,

Proposition 20. *Let V be a twice continuously differentiable real-valued function on* $(-\infty, 0)$ *and suppose that* $V(x) \to -\infty$ *as* $x \to -\infty$*. Suppose further that*

$$\int_{-\infty}^d \left(\frac{[(-V)^{\frac{1}{2}}]'}{(-V)^{\frac{3}{2}}} \right)' (-V)^{-\frac{1}{4}} dx < \infty$$

for some d. Then V is in the limit point case at $-\infty$ *if and only if* $\int_{-\infty}^{d}(-V(x))^{-\frac{1}{2}}dx = \infty$.

Example 16. *By Theorem 21 and Proposition 20 we can say that* $V(x) = -x^4$ *is not in the limit point case at both* $+\infty$ *and* $-\infty$. *Hence by Remark 14,* $-\frac{d^2}{dx^2} - x^4$ *is not essentially self-adjoint on* $C_0^\infty(\mathbf{R})$.

Chapter 3
An application of the Putnam-Fuglede theorem to normal products of self-adjoint operators

3.1 Introduction

In 2000, E. Albrecht and P. G. Spain [1] proved that if we have two bounded self-adjoint operators K, H and if K satisfies $\sigma(K) \cap \sigma(-K) \subseteq \{0\}$ (we shall call this condition on the spectrum of K condition C.), then HK normal implies HK self-adjoint. The proof was given in a more general context of Banach algebras hence the result in $\mathcal{L}(H)$ was just a consequence of the main theorem in that paper. However, nothing was said about the case when at least one of the operators is unbounded. In this chapter we answer this question positively, i.e., if K is a bounded self-adjoint operator satisfying the condition C and if H is any unbounded self-adjoint operator then the result holds. Even when both K and H are unbounded self-adjoint operators such that K satisfies the condition C, the result also holds.

In the end we give a counterexample that shows that the product of two unbounded self-adjoint operators, when it has a normal closure, is not necessarily essentially self-adjoint even when the condition C is satisfied.

Most of this chapter (Sections 3.2 and 3.3) is a paper by myself [2] that has been accepted for publication in the "*Proceedings of the American Mathematical Society*" and that will appear in the October 2003 issue.

3.2 Normal products of self-adjoint operators

3.2.1 Bounded normal products of self-adjoint operators

We recall the Albrecht-Spain theorem:

Theorem 22. *Let H and K be two bounded self-adjoint operators. Let K satisfy the condition C. If HK is normal, then it is self-adjoint.*

We note that one can prove the result of Albrecht-Spain without calling on the theory of Banach algebras. The proof is given below.

Proof. Set $N = HK$. We have $KHK = KN = N^*K$ then using the Putnam-Fuglede theorem (Theorem 13) we obtain

$$KN^* = NK \text{ or } K^2H = HK^2$$

and by condition C, we have that

$$f : \sigma(K^2) \to \sigma(K) : \lambda^2 \mapsto \lambda$$

is well-defined and continuous then

$$f(K^2)H = Hf(K^2) \text{ or } KH = HK$$

which implies that HK is self-adjoint. □

Remark 15. *It is easy to construct noncommuting self-adjoint operators H and K with $H^2 = K^2 = I$, so some additional condition is required to get that $HK = KH$ from the fact that $HK^2 = K^2H$. Condition C does the job.*

3.2.2 Unbounded normal products of self-adjoint operators

Definition 33. *Let K be a bounded operator and H an unbounded one. Then K and H are said to commute if $KH \subset HK$.*

Proposition 21. *Let K be a bounded self-adjoint operator and let H be an unbounded self-adjoint one such that K and H commute. Then for any continuous function f defined on the compact set $\sigma(K)$ we also have*

$$f(K)H \subset Hf(K).$$

Before we start the proof we need the following lemma:

Lemma 5. *If K and H commute where K is self-adjoint then for any real polynomial P, $P(K)$ and H also commute.*

Proof. Set $P(\lambda) = a_0 + a_1\lambda + ... + a_n\lambda^n$ (the coefficients being real).
Let $x \in \mathcal{D}(H) = \mathcal{D}(P(K)H) = \mathcal{D}(KH) = \mathcal{D}(K^2H) = ... = \mathcal{D}(K^nH)$. K,H commute so $KH \subset HK$ i.e. $KHx = HKx$ for all $x \in \mathcal{D}(KH)$ and $\mathcal{D}(KH) \subset \mathcal{D}(HK)$. Also

$$K^2H = K(KH) \subset K(HK) = (KH)K \subset HK^2,$$

i.e.,

$$K^2Hx = HK^2x \text{ for all } x \text{ in } \mathcal{D}(K^2H) = \mathcal{D}(H) \text{ and } \mathcal{D}(K^2H) \subset \mathcal{D}(HK^2).$$

We do the same to the powers of K until we get $K^nH \subset HK^n$, i.e.,

$$K^nHx = HK^nx\,, \forall x \in \mathcal{D}(K^nH) \text{ and } \mathcal{D}(K^nH) \subset \mathcal{D}(HK^n).$$

Hence $\forall x \in \mathcal{D}(P(K)H) = \mathcal{D}(H)$ we have $(a_0IH + a_1KH + a_2K^2H + ... + a_nK^nH)x = (Ha_0I + Ha_1K + Ha_2K^2 + ... + Ha_nK^n)x$ and $\mathcal{D}(P(K)H) \subset \mathcal{D}(HP(K))$. This shows that $P(K)$ and H commute, i.e.,

$$P(K)H \subset HP(K).$$

□

Now we prove Proposition 21.

Proof. As the set of polynomials (that are defined on a compact set, here it is $\sigma(K)$) is dense in the set of continuous functions we can say that there is a sequence of polynomials P_n s.t. $P_n \to f$ in the supremum norm on $\sigma(K)$.

This implies that $P_n(K) \to f(K)$ in $\mathcal{L}(H)$. Let $y \in \mathcal{D}(H)$. Set $x_n = P_n(K)y$ and $x = f(K)y$. We have

$$Hx_n = HP_n(K)y = P_n(K)Hy \to f(K)Hy.$$

The closedness of H and $x_n \longrightarrow x$ imply that

$$f(K)y \in \mathcal{D}(H) \text{ and } Hx = f(K)Hy,$$

i.e., $f(K)H \subset Hf(K)$. □

Remark 16. *One only needs the closedness of H in this lemma.*

Theorem 23. *Let H be a densely defined self-adjoint operator and let K be a bounded self-adjoint operator such that $\sigma(K) \cap \sigma(-K) \subseteq \{0\}$. If HK is normal then it is self-adjoint.*

Proof. $N = HK$ is normal. We know that $N^* = (HK)^* \supset K^*H^* = KH$. We have

$$KHK = (KH)K = K(HK) \Rightarrow KN = (KH)K \subset N^*K.$$

But N and N^* are both normal so by means of the Fuglede-Putnam theorem (Theorem 18) we get

$$KN^* \subset N^{**}K = \overline{N}K = NK$$

since N is closed. It follows that

$$K^2H = K(KH) \subset KN^* \subset NK = (HK)K = HK^2,$$

i.e., K^2 and H commute in the sense of the definition given above (Definition 34). Now the function

$$f : \sigma(K^2) \longrightarrow \sigma(K)\,,\, \lambda^2 \longmapsto \lambda$$

is well-defined thanks to the condition C. Besides f is continuous. This implies that $f(K^2)$ and H commute or K and H commute i.e. $KH \subset HK$.

$$KH \subset HK \Rightarrow (HK)^* \subset (KH)^* = H^*K^* = HK.$$

Since HK is normal then $\mathcal{D}(HK) = \mathcal{D}((HK)^*)$ and on $D((HK)^*)$ we have $(HK)^* = HK$ which shows that HK is self-adjoint. □

Theorem 24. *Under the same assumptions as Theorem 23 and instead of assuming that HK is normal we assume that KH is normal. Then KH is self-adjoint.*

Proof. KH is normal then so is $(KH)^*$. But $(KH)^* = HK$ i.e. HK is normal. So as a consequence of Theorem 23 we know that HK is self-adjoint, i.e., $(HK)^* = HK$. On the other hand

$$(KH)^* = HK \text{ so that } (KH)^* \text{ is self-adjoint},$$

i.e., $(KH)^{**} = (KH)^*$ but

$$(KH)^{**} = \overline{KH} = KH \text{ since } KH \text{ is closed (it is normal).}$$

Thus $KH = (KH)^*$, i.e., KH is self-adjoint. □

Corollary 5. *Let K be a bounded positive self-adjoint operator and let H be any unbounded self-adjoint operator. Then if HK is normal (resp. KH is normal), then it is self-adjoint (resp. it is self-adjoint).*

Now we turn to the case where both K and H are unbounded. The result is also true. Besides one has a generalization of the Fuglede-Putnam theorem with rather stronger conditions.

Theorem 25. *If N is an unbounded normal operator and if K is self-adjoint such that $D(N) \subset D(K)$. Then $KN \subset N^*K$ implies $KN^* \subset NK$.*

Proof. Let P_{B_R} be the spectral projection for N. For convenience we set $H_R = RanP_{B_R}$. Let us restrict K to the Hilbert space H_R. We claim that $K : H_R \to H_R$ and that K is bounded. H_R is a subset of $\mathcal{D}(K)$ since $H_R \subset \mathcal{D}(N)$ by the spectral theorem and $\mathcal{D}(N) \subset \mathcal{D}(K)$. On the other hand since K/H_R is symmetric and defined everywhere then it is bounded on H_R by Proposition 9. Let us show now that $K\varphi \in H_R$ for $\varphi \in H_R$. Let $\varphi \in H_R$. By Proposition 18 we have

$$K\varphi \in H_R \text{ if and only if } \|(N^*)^k K\varphi\| \le \alpha R^k.$$

We also have $\|N^k\varphi\| \le cR^k$ and since K is bounded: $\|KN^k\varphi\| \le \alpha R^k$ but for such φ we have $\|KN^k\varphi\| = \|(N^*)^k K\varphi\|$ as a consequence of the hypothesis in the theorem and hence $K\varphi \in H_R$.

Now we need to show that $KN^* \subset NK$, i.e.,

$$D(KN^*) \subset D(NK) \text{ and on } D(KN^*) : KN^* = NK.$$

Let $\varphi \in D(KN^*)$. Define $\varphi_n = P_{B_n}\varphi$. Since $P_{B_n} \to I$ in the strong operator topology we deduce that $\varphi_n \to \varphi$.

Also $\varphi_n \in D(KN^*)$ since both K and N^* are bounded on H_n. Let us now show that $KN^*\varphi_n \to KN^*\varphi$. Since K is symmetric and maps H_R into itself by Theorem 14 H_R reduces K and hence we have by Proposition 10, $P_{B_R}K \subset KP_{B_R}$. It also reduces N by the spectral theorem so that we get:

$$KN^*\varphi_n = KN^*P_{B_n}\varphi = P_{B_n}KN^*\varphi \to KN^*\varphi. \tag{3.1}$$

Let us show now that $\varphi \in D(NK)$. Both K and N are bounded on H_n then by the Fuglede-Putnam for bounded operators we have that $KN\varphi_n = N^*K\varphi_n$ implies that $KN^*\varphi_n = NK\varphi_n$. This gives us with equation (3.1): $NK\varphi_n \to KN^*\varphi$.

N maps $H_R{}^\perp = RanP_{B_R^c}$ to $H_R{}^\perp$ (H_R is a reducing space for N) and N^{-1} is bounded on $H_R{}^\perp$ since in this case $N^{-1} = \int_{B_R^c} \frac{1}{\lambda} dP_\lambda$ and hence $|\frac{1}{\lambda}| \leq \frac{1}{R}$.

We also have

$$NK\varphi_n - KN^*\varphi = KN^*\varphi_n - KN^*\varphi \in H_R^\perp \text{ for } n > R$$

so that if we apply the inverse of N we get $K\varphi_n \to N^{-1}KN^*\varphi$. By the closedness of K we obtain $\varphi \in \mathcal{D}(K)$ and $K\varphi_n \to K\varphi$. But N is closed and $(NK\varphi_n)_n$ convergent together with $K\varphi_n \to K\varphi$ imply that

$$K\varphi \in D(N) \text{ (i.e, } \varphi \in D(NK)) \text{ and } KN^*\varphi = NK\varphi,$$

establishing Theorem 25. □

Corollary 6. *Let K, H be two unbounded self-adjoint operators. If $N = HK$ is normal then $KN \subset N^*K$ implies $KN^* \subset NK$.*

Proof. Obvious since $\mathcal{D}(N) = \mathcal{D}(HK) \subset \mathcal{D}(K)$. □

Theorem 26. *Let K, H be two unbounded self-adjoint operators such that $\sigma(K) \cap \sigma(-K) \subseteq \{0\}$. If HK is normal then it is self-adjoint.*

Proof. Set $N = HK$. We have

$$KHK = K(HK) = (KH)K \subset (HK)^*K$$

which implies that $KN \subset N^*K$. But $\mathcal{D}(N) \subset \mathcal{D}(K)$ so by Corollary 6 we can say that

$KN^* \subset NK$ or

$$K^2H \subset K(HK)^* \subset HK^2.$$

So we have

$$K^2H\varphi = HK^2\varphi \text{ for } \varphi \in \mathcal{D}(K^2H).$$

Using the same arguments as in the proof of Theorem 25 we can say that for $\varphi \in RanP_{B_n}$ we have: $K^2HK\varphi = HK^2K\varphi$ as $K\varphi \in \mathcal{D}(K^2H)$ since K^2N is bounded in this case. We have $K^2N\varphi = NK^2\varphi$.

Now take the same function f taken in the proof of Theorem 23 to get: $f(K^2)N\varphi = Nf(K^2)\varphi$ and hence $KN\varphi = NK\varphi$. But $KN\varphi = N^*K\varphi$ on H_n. Hence $N^*K\varphi = NK\varphi$.

We now use the orthogonal decomposition $H_n = \overline{RanK} \bigoplus KerK$ for the K restricted to H_n. We have

$$N = N^* \text{ on } \overline{RanK} \text{ and both are 0 on } KerK.$$

Hence $N = N^*$ on H_n. This shows that N_n (N_n is just N restricted to H_n) is self-adjoint. Hence $\sigma(N_n) \subseteq \mathbf{R}$ for all n and then $\sigma(N) \subseteq \mathbf{R}$ and a normal operator with a real spectrum is self-adjoint (Proposition 15). Thus HK is self-adjoint. □

Corollary 7. *Let K,H be two densely defined self-adjoint operators such that K is positive. If HK is normal then it is self-adjoint.*

Remark 17. *We have seen that the result is true for any couple of self-adjoint operators regardless of their boundedness and provided the condition C is satisfied. However, the hypothesis "HK normal" cannot be replaced by "HK having a normal closure". Here we give a counter example.*

3.3 A counterexample

Let us consider the operators K and H defined as:

$$H = -i\frac{d}{dx} : H^1(\mathbf{R}) \to L^2(\mathbf{R}), K = |x| : \mathcal{D}(K) \to L^2(\mathbf{R})$$

where $\mathcal{D}(K) = \{f \in L^2(\mathbf{R}) : |x|f \in L^2(\mathbf{R})\}$. K is obviously positive so that it does satisfy the condition C. We also know that those two operators are self-adjoint on the given domains.

$N = HK$ is defined on $\mathcal{D}(HK)$ that is

$$\{f \in \mathcal{D}(K) : Kf \in \mathcal{D}(H)\} = \{f \in L^2(\mathbf{R}) : |x|f, -i(|x|f)' \in L^2(\mathbf{R})\}$$

such that: $Nf = -i(|x|f)'$ where the derivative is taken in the distributional sense.

The operator N is densely defined since it contains $C_0^\infty(\mathbf{R})$. It is not closed but it has a closed extension $\overline{N}$ defined on $\mathcal{D}(\overline{N})$, which consists of the L^2-functions s.t. $|x|f'$ is in $L^2(\mathbf{R})$ where $|x|f'$ is a distribution on $\mathbf{R}\backslash\{0\}$, by $\overline{N}f = -i|x|f' - isignf$.

We need to check that $\overline{N}$ is closed on this domain with respect to the graph norm of $\overline{N}$. Take $(f_n, \overline{N}f_n) \in \mathcal{G}(\overline{N})$ such that $(f_n, \overline{N}f_n) \longrightarrow (f, g)$. Since $f_n \longrightarrow f$ in L^2 then in the distributional sense we have $f_n' \longrightarrow f'$. On $\mathbf{R}\backslash\{0\}$ we have $|x|f_n' \longrightarrow |x|f'$ again in the distributional sense. By uniqueness of the limit one gets that $\overline{N}f = |x|f'$ for almost every x hence we have the equality in $L^2(\mathbf{R})$. This tells us that $\overline{N}$ is closed in this domain.

The operator $\overline{N}$ is a closed extension of N. It is in fact the closure of N and this will be shown once we have shown that $C_0^\infty(\mathbf{R}\backslash\{0\})$ is dense in $\mathcal{D}(\overline{N})$ with respect to the graph norm of $\overline{N}$.

Definition 34. *The set of the functions in $\mathcal{D}(\overline{N})$ that have compact support away from the origin will be denoted by $\mathcal{D}(\overline{N})^*$.*

Lemma 6. *$C_0^\infty(\mathbf{R}\backslash\{0\})$ is dense in $\mathcal{D}(\overline{N})^*$ with respect to the graph norm of $\overline{N}$.*

Proof. Let f be in $\mathcal{D}(\overline{N})^*$. Let us find a sequence f_n in $C_0^\infty(\mathbf{R}\backslash\{0\})$ such that $f_n \longrightarrow f$ in the graph norm of $\overline{N}$ that is,

$$\|f_n - f\|^2_{\mathcal{D}(\overline{N})} = \|f_n - f\|_2^2 + \|xf_n' - xf'\|_2^2 \longrightarrow .$$

It suffices to show that the right hand side converges to zero as n tends to infinity.
Take k_n as in Theorem 2 (take $n = \frac{1}{\epsilon}$) such that k has compact support so that $k_n * f \in C_0^\infty(\mathbf{R}\backslash\{0\})$ for large n. Then by Theorem 2 (for $p = 2$) we have

$$\lim_{n\to\infty} \|f - k_n * f\|_2 = 0. \tag{3.2}$$

Now take $f_n = k_n * f$. The convergence of f_n to f follows from (3.2). At the same time we have $xf' \in L^2$ with support away from the origin. This implies that $f' \in L^2$.
Also, in the distributional sense, $f_n' = k_n * f'$. So, f_n' is in L^2 and has compact support away

from the origin. Thus,

$$\|x(k_n * f') - xf'\|_2 \le \tilde{c}\|(k_n * f') - f'\|_2 \to 0 \text{ by } (3.2).$$

Therefore,

$$\|f_n - f\|_{\mathcal{D}(\overline{N})} \to 0 \text{ as } n \to \infty,$$

establishing Lemma 6. □

Lemma 7. *$\mathcal{D}(\overline{N})^*$ is dense in $\mathcal{D}(\overline{N})$.*

Proof. Let $f \in \mathcal{D}(\overline{N})$. Let us find a sequence f_n in $\mathcal{D}(\overline{N})^*$ such that $f_n \to f$ in the graph norm of $\overline{N}$. Define the even function φ_n on $[\frac{1}{n}, 2n]$ by

$$\varphi_n(x) = \begin{cases} n(x - \frac{1}{n}) & if \;\; \frac{1}{n} \le x < \frac{2}{n} \\ 1 & if \;\; \frac{2}{n} \le x < n \\ -\frac{1}{n}(x-n) + 1 & if \;\; n \le x \le 2n. \end{cases}$$

Now take $f_n = f\varphi_n$. We have $supp f_n \subseteq supp f \cap supp \varphi_n \subseteq supp \varphi_n$ where $0 \notin supp \varphi_n$. One can show that φ_n tends to 1 pointwise. Also φ_n' exists almost everywhere. We need to show that $f_n \to f$ in the graph norm of $\overline{N}$. First, we have

$$\|f_n - f\|^2_{L^2(\mathbf{R})} = \int_{\mathbf{R}} |f_n(x) - f(x)|^2 dx = \int_{\mathbf{R}} |f(x)|^2 (\varphi_n(x) - 1)^2 dx \to 0$$

by the D.C.T. (dominated convergence theorem).
We also have $f_n'(x) = f'(x)\varphi_n(x) + f(x)\varphi_n'(x)$ then

$$\|xf_n' - xf'\|^2_{L^2(\mathbf{R})} = \int_{\mathbf{R}} |xf_n'(x) - xf'(x)|^2 dx$$
$$\le 2\int_{\mathbf{R}} |xf'(x)|^2(\varphi_n(x) - 1)^2 dx + 2\int_{\mathbf{R}} |xf(x)\varphi_n'(x)|^2 dx.$$

The first bit of the integral tends to zero again by the D.C.T. (the dominating function being $(xf')^2 \in L^1(\mathbf{R})$). For the second bit one has

$$\int_{\mathbf{R}} x^2 |f(x)|^2 |\varphi_n'(x)|^2 dx = \int_{\frac{1}{n}}^{\frac{2}{n}} x^2 n^2 |f(x)|^2 dx + \int_n^{2n} \frac{x^2}{n^2} |f(x)|^2 dx.$$

We have

$$\int_{\frac{1}{n}}^{\frac{2}{n}} x^2n^2|f(x)|^2dx \le 4\int_{\frac{1}{n}}^{\frac{2}{n}} |f(x)|^2dx = 4\int_{\mathbf{R}} |f(x)|^2\mathbf{1}_{[\frac{1}{n},\frac{2}{n}]}(x)dx \longrightarrow 0$$

by the D.C.T. since $\lim_{n\longrightarrow\infty} \mathbf{1}_{[\frac{1}{n},\frac{2}{n}]}(x) = 0$.

We also have

$$\int_{n}^{2n} \frac{x^2}{n^2}|f(x)|^2dx \le 4\int_{n}^{2n} |f(x)|^2dx = 4\int_{\mathbf{R}} |f(x)|^2\mathbf{1}_{[n,2n]}(x)dx$$

which tends to 0 by the D.C.T.. Thus $\|xf_n' - xf'\|^2_{L^2(\mathbf{R})} \longrightarrow 0$.
This tells us that

$$\|f_n - f\|_{\mathcal{D}(\overline{N})} \longrightarrow 0,$$

establishing Lemma 7. □

$C_0^\infty(\mathbf{R}\backslash\{0\})$ is dense in $\mathcal{D}(\overline{N})^*$ and the latter is dense in $\mathcal{D}(\overline{N})$. Thus $C_0^\infty(\mathbf{R}\backslash\{0\})$ is dense in $\mathcal{D}(\overline{N})$ with respect to the graph norm of $\overline{N}$.

Corollary 8. *The operator $\overline{N}$ is the closure of N.*

Proof. This follows from $C_0^\infty(\mathbf{R}\backslash\{0\}) \subset \mathcal{D}(N) \subset \mathcal{D}(\overline{N})$. Hence $\mathcal{D}(N)$ is dense in $\mathcal{D}(\overline{N})$ with respect to the graph norm of $\overline{N}$. □

In order to find the adjoint of N on $\mathcal{D}(N)$ it suffices to find it on $C_0^\infty(\mathbf{R}\backslash\{0\})$. Since if we restrict N to $C_0^\infty(\mathbf{R}\backslash\{0\})$ and we denote it by N_0 then $N^* = N_0^*$ (since $\overline{N_0} = \overline{N}$ then, $\overline{N_0}^* = \overline{N}^*$ and hence $N_0^{***} = N^{***}$. Therefore, $N_0^* = N^*$ [1] because N^* is closed for any densely defined operator N, see [14] (Theorem 13.9)).
The domain of N^* is defined as

$$\mathcal{D}(N^*) = \{g \in L^2(\mathbf{R}) | \exists h \in L^2(\mathbf{R}) \text{ s.t. } < Nf, g > = < f, h > \forall f \in C_0^\infty(\mathbf{R}\backslash\{0\})\}.$$

And we have

Lemma 8. $\mathcal{D}(N^*) = \{f \in L^2(\mathbf{R}) | |x|f' \in L^2(\mathbf{R})\}$.

[1]We have used the fact that $N^{**} = \overline{N}$ see [6], Theorem VIII.1.

Remark 18. *Recall that we denote the action of a distribution T on a test function φ by (T, φ).*

Proof. Let $f \in C_0^\infty(\mathbf{R}\backslash\{0\})$ and $g \in L^2(\mathbf{R})$. We have

$$< Nf, g >= \int_{\mathbf{R}} (|x|f(x))' \overline{ig(x)} dx = ((|x|f)', \overline{ig}) \text{ since } (|x|f)' \in C_0^\infty(\mathbf{R}\backslash\{0\}).$$

By definitions of the distributional derivative and the product of distributions (c.f. Sections 1.4.1. and 1.4.2.) since $|x|$ is C^∞ on $\mathbf{R}\backslash\{0\}$ one has

$$((|x|f)', \overline{ig}) = -(|x|f, -i\overline{g}') = (f, i|x|\overline{g}').$$

We also have $< f, h >= (f, \overline{h})$ where $h \in L^2$. Hence $h = -i|x|g'$ as a distribution but h is in L^2 then $|x|g' \in L^2$ and then $\mathcal{D}(N^*) = \{g \in L^2 : |x|g' \in L^2\}$ and $N^*g = -i|x|g'$. □

Now let us show that $\overline{N}$ is normal. First, we have that $\mathcal{D}(N^*) = \mathcal{D}(\overline{N}^*)$.

Clearly $\overline{N}$ is not self-adjoint (it is not even symmetric as $\overline{N} - \overline{N}^* \subseteq \pm i$). However, it is normal as

$$\overline{N}.\overline{N}^* f(x) = \overline{N}(-i|x|f'(x)) = -i(-i|x||x|f'(x))' = -x^2 f''(x) - 2xf'(x)$$

and

$$\overline{N}^*.\overline{N} f(x) = \overline{N}^*[-i(|x|f(x))'] = -x^2 f''(x) - 2xf'(x)$$

We also have

$$\mathcal{D}(\overline{N}.\overline{N}^*) = \{f \in \mathcal{D}(\overline{N}^*) | \overline{N}^* f \in \mathcal{D}(\overline{N})\} = \{f \in L^2(\mathbf{R}) |, |x|f', x^2 f'' \in L^2(\mathbf{R})\}$$

and $\mathcal{D}(\overline{N}^*.\overline{N})$ is exactly the same.

Thus, we have found two unbounded self-adjoint operators H, K such that $\sigma(K) \cap \sigma(-K) \subseteq \{0\}$ for which $N = HK$ has a normal closure without being essentially self-adjoint.

3.4 What went wrong?

In the Counterexample above N (actually, it is $\overline{N}$ which is normal but we keep on denoting it by N) is a normal operator and so according to Proposition 17 there is a unitary transformation, say U, that diagonalizes N. In other words via U, N will be unitarily equivalent to a multiplication operator by a complex-valued function. So here we find U explicitly and use the whole machinery to investigate what goes wrong in the proof of Theorem 25.

Proposition 22. *Let N be the normal operator defined on $D(N) = \{f \in L^2(\mathbf{R}) : xf' \in L^2(\mathbf{R})\}$ by $Nf = -i(|x|f)'$. Then N is unitarily equivalent to $M = M_+ \oplus M_-$ where M_+ is defined on $L^2(\mathbf{R})$ by $M_+f(s) = (s - \frac{1}{2}i)f(s)$ and M_- is defined on $L^2(\mathbf{R})$ by $M_-f(s) = (s + \frac{1}{2}i)f(s)$. The required unitary transformation is given by*

$$Uf = U_+f_+ \oplus U_-f_-$$

where f_+ is the restriction of f to $\mathbf{R}^+$, f_- is the restriction of f to $\mathbf{R}^-$. The operator U_+ is defined by $U_+ = \mathcal{F}^{-1}V$ where $\mathcal{F}^{-1}$ is the inverse L^2-Fourier transform and $V : L^2(\mathbf{R}^+) \to L^2(\mathbf{R})$ is the unitary operator defined by

$$(Vf)(t) = e^{\frac{t}{2}}f(e^t)$$

and U_- is defined by $U_- = \mathcal{F}^{-1}W$ where $W : L^2(\mathbf{R}^-) \to L^2(\mathbf{R})$ defined by

$$(Wf)(t) = e^{-\frac{t}{2}}f(e^{-t}).$$

Proof. Since we have the decomposition $L^2(\mathbf{R}) = L^2(\mathbf{R}^+) \oplus L^2(\mathbf{R}^-)$ then N may be written as $N_+ \oplus N_-$ where N_+ satisfies $N_+h = N_+^*h - ih$ and N_- satisfies $N_-h = N_-^*h + ih$. Let $\lambda \in \sigma(N_+)$ then $\lambda = \overline{\lambda} - i$ which gives $\Im\lambda = -\frac{1}{2}$ i.e. $\sigma(N_+) \subseteq \{\alpha - \frac{1}{2}i | \alpha \in \mathbf{R}\}$ (it is actually equal to this set as we will see later).

Now let us try to find the eigenvalues of the operator N_+. We have $-ixh' - ih = \lambda h$ or $\frac{h'(x)}{h(x)} = \frac{i(\lambda+i)}{x}$ hence $h(x) = cx^{-\frac{1}{2}+i\alpha}$ where c is arbitrary and where $\alpha = \lambda + \frac{1}{2}i$. This h is clearly not in $L^2(\mathbf{R}^+)$ hence we do not have any eigenvalues but this try will allow us to find the unitary equivalence of N. It is done as follows. Define

$$(U_+f)(u) = \frac{1}{\sqrt{2\pi}}\int_0^\infty x^{-\frac{1}{2}+iu}f(x)dx \text{ where } f \in L^2(\mathbf{R}^+) \; (*)$$

The previous equation is a well defined Fourier transform in $L^2(\mathbf{R})$ by making the change of variable $x = e^t$ in $(*)$. We then get:

$$(U_+f)(u) = \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} [e^{\frac{1}{2}t}f(e^t)]e^{iut}dt.$$

It is well-defined in $L^2(\mathbf{R})$ since

$$\int_{\mathbf{R}} |g(t)|^2 dt = \int_{\mathbf{R}} e^t |f(e^t)|^2 dt = \int_0^\infty |f(x)|^2 dx < \infty$$

where we have made the change of variable $t = \ln x$ and where we have set $g(t) = e^{\frac{1}{2}t} f(e^t)$. The inversion formula is then

$$g(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} (U_+ f)(u) e^{-iut} du.$$

Hence we obtain

$$F(t) = f(e^t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} (U_+ f)(u) e^{-\frac{1}{2}t - iut} du. \tag{3.3}$$

Let us check that via equation (3.3) N_+ is unitarily equivalent to M_+ that is in the proposition above. We have $F'(e^t) = e^t f'(e^t) = x f'(x)$ and at the same time

$$F'(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} (-\frac{1}{2} - iu)(U_+ f)(u) e^{-\frac{1}{2}t - iut} du.$$

Hence $-iF'(t) - iF(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} (-u - \frac{1}{2}i)(U_+ f)(u) e^{-\frac{1}{2}t - iut} du$. Then

$$N_+ f(x) = -ixf'(x) - if(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} (-u - \frac{1}{2}i)(U_+ f)(u) e^{-\frac{1}{2}t - iut} du.$$

Thus

$$U_+ N_+ f(s) = (s - \frac{1}{2}i)(U_+ f)(s) = (M_+ U_+ f)(s).$$

So N_+ is unitarily equivalent to M_+ and the unitary operator is given by (3.3) and hence $\sigma(N_+) = \{s - \frac{1}{2}i | s \in \mathbf{R}\}$. The proof for the case $L^2(\mathbf{R}^-)$ is very similar so we shall not do it. We just give the unitary operator in this case which is

$$f(e^{-t}) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} (U_- f)(u) e^{+\frac{1}{2}t - iut} du$$

and hence $\sigma(N_-) = \{s + \frac{1}{2} | s \in \mathbf{R}\}$.

In the end $N = N_+ \oplus N_-$ is unitarily equivalent to $M = M_+ \oplus M_-$ where $M_+ f(\alpha) =$

$(-\alpha - \frac{1}{2}i)f(\alpha)$ and $M_- f(\alpha) = (-\alpha + \frac{1}{2}i)f(\alpha)$. Thus

$$\sigma(N) = \sigma(N_+) \cup \sigma(N_-) = \{s - \frac{1}{2}i | s \in \mathbf{R}\} \cup \{s + \frac{1}{2}i | s \in \mathbf{R}\}.$$

□

We have constructed this unitary equivalence to use it to investigate what goes wrong in the proof of Theorem 25 if we want to prove the same result for operators that have normal closure and that are essentially self-adjoint.

The first thought is the closedness of the operator. Truly the closedness plays a role in making the result untrue but there is something else that is in the proof of Theorem 25 and that is we cannot restrict K to H_R since H_R is not a subset of $\mathcal{D}(K)$.

Lemma 9. *Let P_{B_R} be the spectral projection of the normal operator N that is defined in Section 3.3. Then $H_R = P_{B_R}H$ is not a subset of $D(K)$.*

Proof. We need to find an f that is in H_R and not in $D(K)$ i.e. $xf \notin L^2(\mathbf{R})$. It suffices to do this in $L^2(\mathbf{R}^+)$ and we also denote the spectral projection for N^+ by P_{B_R}. The operator M_+ has $\mathbf{R} \times \{-\frac{1}{2}\}$ as spectrum. So its spectrum lies in a line.

Also since the multiplication operator, M_+, has the multiplication by a characteristic function, say $\mathbf{1}_{I_m}$, as its spectral measure (Example 13) and since N_+ is unitarily equivalent to M_+ then it follows that P_{B_R} is unitarily equivalent to $\mathbf{1}_{I_m}$ (m and $-m$ represent the intersection of the disc of radius R and the line $y = -\frac{1}{2}$) via the transform defined in (3.3). Let us call that transform F. Then we have

$$FP_{B_R}F^{-1} = \mathbf{1}_{I_m} \text{ where } I_m = [-m, m].$$

Hence $P_{B_R}F^{-1} = F^{-1}\mathbf{1}_{I_m}$. So for $g \in L^2(\mathbf{R}^+)$ one has

$$f = P_{B_R}F^{-1}g = F^{-1}\mathbf{1}_{I_m}g.$$

We observe that to say that $f \in H_R$ or $Ff = \mathbf{1}_{I_m}g, g \in L^2(\mathbf{R}^+)$ is the same thing hence we seek an f such that $Ff(s) = 1$ on $[0, m]$ and zero otherwise (we have taken $g = \mathbf{1}_{[0,m]}$) such that $xf \notin L^2(\mathbf{R}^+)$ or $e^t f(e^t) \notin L^2(\mathbf{R})$. By (3.3) we have

$$f(e^t) = \frac{1}{\sqrt{2\pi}} \int_0^m e^{-\frac{1}{2}t - ist} ds = \frac{1}{it\sqrt{2\pi}} e^{-\frac{1}{2}t}(1 - e^{-imt}).$$

Of course

$$f \in L^2(\mathbf{R}^+) \text{ but } e^t f(e^t) \notin L^2(\mathbf{R})$$

since

$$\int_{\mathbf{R}} |\frac{1}{it} e^{\frac{1}{2}t}(1 - e^{-imt})|^2 dt = \int_{\mathbf{R}} \frac{e^t}{t}(2 - 2\cos(mt))dt \geq \int_{\mathbf{R}^+} \frac{e^t}{t}(2 - 2\cos(mt))dt = \infty.$$

□

Chapter 4
Self-adjointness of the perturbed wave operator on $L^2(\mathbf{R}^2)$

4.1 Introduction

There are many classes of unbounded real-valued Vs for which $-\Delta + V$ is self-adjoint (see, [3], Section X.1 to Section X.6) which is very important in quantum mechanics. Many of those results exploit the fact that $-\Delta$ is positive (see, e.g., [19]). What we will be doing in the next two chapters is to investigate the self-adjointness of $\Box + V$ (there is no need to say that it is easier to prove that something is self-adjoint than to prove that it is not). This work may not have any direct application to another science and for the moment it is only a mathematical curiosity.

We will also observe the difference between the wave operator and the Laplacian in the way they behave. There is also another difference that is worth mentioning that is: the Laplacian is a positive operator while the wave operator has no sign. In the end we will also give a counterexample showing another difference.

In this chapter we are only interested in the case $L^2(\mathbf{R}^2)$. We want to find a class of unbounded $V : \mathbf{R}^2 \longrightarrow \mathbf{R}$ such that $\Box + V$ is self-adjoint on $\mathcal{D}(\Box)$. For V essentially bounded the result is true either as a consequence of the Kato-Rellich perturbation theorem or as will be shown below. We also recall that $\Box + V$ is essentially self-adjoint on $C_0^\infty(\mathbf{R}^2)$ (see the discussion after Corollary 4 and Remark 11) if

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) U(x,t) + V(x,t)U(x,t) = \pm i\alpha U(x,t) \ (\alpha > 0)$$

has a unique solution in $L^2(\mathbf{R}^2)$ (that is $U = 0$) and eventually self-adjoint on $\mathcal{D}(\Box) \cap \mathcal{D}(V)$ if we also prove that $\Box + V$ is closed on $\mathcal{D}(\Box) \cap \mathcal{D}(V)$.

4.2 First class of self-adjoint $\Box + V$

Remark 19. *The natural domain of V is $\{f \in L^2 : Vf \in L^2\}$.*

Proposition 23. *Let $\Box$ be the wave operator on $L^2(\mathbf{R}^2)$. Let $V \in L^\infty(\mathbf{R}^2)$ be real-valued. Then $\Box + V$ is essentially self-adjoint on $C_0^\infty(\mathbf{R}^2)$.*

Proof. We shall attempt to solve the adjoint equation directly by using the Fourier transform. We need to show that the following PDE

$$\Box u \pm i\alpha u = Vu \tag{4.1}$$

has a unique solution in $L^2(\mathbf{R}^2)$ that is, $u = 0$. Put $M = \|V\|_\infty$. We also choose $\alpha > M$. Now take the Fourier transform in equation (4.1) and we get:

$$(-\eta^2 + \xi^2 \pm \alpha i)\hat{u} = \widehat{Vu}.$$

Then

$$\|(-\eta^2 + \xi^2 \pm \alpha i)\hat{u}\|_2 \geq \alpha\|\hat{u}\|_2 = \alpha\|u\|_2.$$

Also

$$\|(-\eta^2 + \xi^2 \pm \alpha i)\hat{u}\|_2 = \|\widehat{Vu}\|_2 \leq M\|u\|_2.$$

Hence

$$0 \leq \alpha\|u\|_2 \leq M\|u\|_2 \Rightarrow (M - \alpha)\|u\|_2 \geq 0 \Rightarrow u = 0.$$

$\Box$

Remark 20. *The result is true in any dimension $n \geq 2$ by the same method and for any constant coefficient symmetric partial differential operator. Also, it is known that a multiplication operator by a real-valued essentially bounded function, when added to a self-adjoint operator, does not destroy its self-adjointness. In fact, it is an immediate consequence of the Kato-Rellich Theorem (Theorem 19).*

First recall that $\Box$ is an unbounded self-adjoint operator on $D(\Box) = \{f \in L^2(\mathbf{R}^2) : \Box f \in L^2(\mathbf{R}^2)\}$ by means of the Fourier transform (c.f. Example 11).

We now we give the first class of unbounded Vs for which the operator $\Box + V$ is self-adjoint on $\mathcal{D}(\Box)$ but before that, we get classes of real-valued V for which $\frac{\partial^2}{\partial x \partial y} + V$ is be self-adjoint on

$$\mathcal{D}(\frac{\partial^2}{\partial x \partial y}) = \{\varphi \in L^2(\mathbf{R^2}) : \frac{\partial^2 \varphi}{\partial x \partial y} \in L^2(\mathbf{R^2})\}.$$

And then the results for $\Box$ will follow by a change of variables.

Definition 35. *Set*

$$M'^2 = \{\varphi \in L^2(\mathbf{R^2}) : \frac{\partial^2 \varphi}{\partial x \partial y} \in L^2(\mathbf{R^2})\}$$

and set

$$M^2 = \{\varphi \in L^2(\mathbf{R^2}) : \Box\varphi \in L^2(\mathbf{R^2})\}.$$

We also denote by ρ, the function $\frac{\partial^2 \varphi}{\partial x \partial y}$.

Proposition 24. *For all $a > 0$, there exists $b > 0$ such that*

$$\operatorname*{ess\,sup}_{x,y \in \mathbf{R}} \int_{\mathbf{R}} |\varphi(x+\lambda, y+\lambda)|^2 d\lambda \leq a \left\| \frac{\partial^2 \varphi}{\partial x \partial y} \right\|_2^2 + b\|\varphi\|_2^2 \tag{4.2}$$

for all $\varphi \in M'^2$.

Proof. We shall first prove the proposition for C_0^∞ functions then extend the result to functions in M'^2. Let $\varphi \in C_0^\infty$ then we have the following identity for φ

$$\varphi(x,y) = \int_s^x \int_t^y \rho + \varphi(x,t) + \varphi(s,y) - \varphi(s,t)$$

which is an easy consequence of the fundamental theorem of calculus. We also note that s and t are yet to be chosen.

We have

$$\varphi(x+\lambda, y+\lambda) = \int_s^{x+\lambda} \int_t^{y+\lambda} \rho + \varphi(x+\lambda, t) + \varphi(s, y+\lambda) - \varphi(s,t)$$

where λ is a real number.

Then

$$|\varphi(x+\lambda, y+\lambda)| \leq \int_s^{x+\lambda} \int_t^{y+\lambda} |\rho| + |\varphi(x+\lambda, t)| + |\varphi(s, y+\lambda)| + |\varphi(s,t)|. \tag{4.3}$$

But applying the Cauchy-Schwarz inequality, Corollary 1, gives us

$$\int_s^{x+\lambda}\int_t^{y+\lambda}|\rho| \leq \left(\int_s^{x+\lambda}\int_t^{y+\lambda}|\rho|^2\right)^{\frac{1}{2}}(|x+\lambda-s|)^{\frac{1}{2}}(|y+\lambda-t|)^{\frac{1}{2}}.$$

Squaring both sides of Equation (4.3) gives us

$$|\varphi(x+\lambda,y+\lambda)|^2 \leq \tilde{c}\int_s^{x+\lambda}\int_t^{y+\lambda}|\rho|^2(|x+\lambda-s|)(|y+\lambda-t|)+\tilde{c}|\varphi(x+\lambda,t)|^2$$
$$+\tilde{c}|\varphi(s,y+\lambda)|^2+\tilde{c}|\varphi(s,t)|^2.$$

Now we choose s and t such that $k+x \leq s \leq k+x+1$ and $k+y \leq t \leq k+y+1$ where $k \in \mathbf{Z}$ and take λ such that $k \leq \lambda \leq k+1$. Then $|x+\lambda-s| \leq 1$ and $|y+\lambda-t| \leq 1$. So

$$|\varphi(x+\lambda,y+\lambda)|^2 \leq \tilde{c}\left(\int_{x+k}^{x+k+1}\int_{y+k}^{y+k+1}|\rho|^2+|\varphi(x+\lambda,t)|^2+|\varphi(s,y+\lambda)|^2+|\varphi(s,t)|^2\right).$$

Integrating in λ, s and t in their respective ranges gives us

$$\int_k^{k+1}|\varphi(x+\lambda,y+\lambda)|^2d\lambda \leq \tilde{c}\int_{x+k}^{x+k+1}\int_{\mathbf{R}}|\rho|^2+\tilde{c}\int_k^{k+1}\int_{\mathbf{R}}|\varphi(x+\lambda,t)|^2dtd\lambda$$
$$+\tilde{c}\int_k^{k+1}\int_{\mathbf{R}}|\varphi(s,y+\lambda)|^2dsd\lambda+\tilde{c}\int_{x+k}^{x+k+1}\int_{\mathbf{R}}|\varphi(s,t)|^2dtds.$$

Now sum in k to see

$$\int_{\mathbf{R}}|\varphi(x+\lambda,y+\lambda)|^2d\lambda \leq \tilde{c}[\|\rho\|_2^2+\|\varphi\|_2^2]. \tag{4.4}$$

Taking the essential supremum of both sides in x and y in $\mathbf{R}$ establishes (4.2).

We now proceed to make the constant in front of $\|\frac{\partial^2\varphi}{\partial x\partial y}\|_2$, in (4.2), arbitrary.
Set $\varphi_r(x,y)=\varphi(rx,ry)$, $r>0$. Then one gets

$$\sup_{x,y\in\mathbf{R}}\int_{\mathbf{R}}|\varphi_r(x+\lambda,y+\lambda)|^2d\lambda = \frac{1}{r}\sup_{x,y\in\mathbf{R}}\int_{\mathbf{R}}|\varphi(rx+r\lambda,ry+r\lambda)|^2d(r\lambda).$$

Also

$$\iint_{\mathbf{R}^2} \left| \frac{\partial^2 \varphi_r(x,y)}{\partial x \partial y} \right|^2 dxdy = \iint_{\mathbf{R}^2} \left| \frac{\partial^2 \varphi(rx,ry)}{\partial x \partial y} \right|^2 dxdy = r^2 \iint_{\mathbf{R}^2} \left| \frac{\partial^2 \varphi(X,Y)}{\partial x \partial y} \right|^2 dXdY.$$

Finally,

$$\|\varphi_r\|_2^2 = \frac{1}{r^2}\|\varphi\|_2^2.$$

Applying (4.2) to φ_r gives us

$$\operatorname*{ess\ sup}_{x,y\in\mathbf{R}} \int_{\mathbf{R}} |\varphi(x+\lambda, y+\lambda)|^2 d\lambda \leq \bar{c}[r^3\|\rho\|_2^2 + \frac{1}{r}\|\varphi\|_2^2].$$

Since r is an arbitrary positive number we can take the constant in front of $\|\rho\|_2^2$ arbitrary.

We now show that using a density argument one can extend this result to functions in M'^2. Let $f \in M'^2$. Then there exists a sequence, φ_n, of functions in C_0^∞ (Remark 7) such that

$$\|\varphi_n - f\|_2 \to 0 \text{ and } \left\| \frac{\partial^2 \varphi_n}{\partial x \partial y} - \frac{\partial^2 f}{\partial x \partial y} \right\|_2 \to 0.$$

We can then extract a subsequence $\varphi_{n(k)}$ such that $\varphi_{n(k)}(x,y) \to f(x,y)$ a.e. . On the other hand

$$\|\varphi_n\|_2 \to \|f\|_2 \text{ and } \left\| \frac{\partial^2 \varphi_n}{\partial x \partial y} \right\| \to \left\| \frac{\partial^2 f}{\partial x \partial y} \right\|_2.$$

Now, for all x, y and k we apply (4.4) to $\varphi_{n(k)}$ to get

$$\int_{\mathbf{R}} |\varphi_{n(k)}(x+\lambda, y+\lambda)|^2 d\lambda \leq a \left\| \frac{\partial^2 \varphi_{n(k)}}{\partial x \partial y} \right\|_2^2 + b\|\varphi_{n(k)}\|_2^2.$$

But

$$\int_{\mathbf{R}} |f(x+\lambda, y+\lambda)|^2 d\lambda = \int_{\mathbf{R}} \liminf_{k\to\infty} |\varphi_{n(k)}(x+\lambda, y+\lambda)|^2 d\lambda.$$

Applying Fatou's Lemma tells us that for $a.e.$

$$\int_{\mathbf{R}} |f(x+\lambda, y+\lambda)|^2 d\lambda \leq \liminf_{k\to\infty} \int_{\mathbf{R}} |\varphi_{n(k)}(x+\lambda, y+\lambda)|^2 d\lambda$$

$$\leq \liminf_{k\to\infty} \left(a \left\| \frac{\partial^2 \varphi_{n(k)}}{\partial x \partial y} \right\|_2^2 + b\|\varphi_{n(k)}\|_2^2 \right).$$

Taking the essential supremum in x and y establishes (4.2) for functions in M'^2. □

We now give the first class of unbounded Vs.

Theorem 27. *Let □ be the wave operator in $L^2(\mathbf{R}^2)$. Suppose V is real-valued such that $\|V\|^2 = \int_{\mathbf{R}} \sup_{\eta\in\mathbf{R}} |V(\eta,\xi)|^2 d\xi < \infty$. Then $\Box + V$ is self-adjoint on $\mathcal{D}(\Box) = M^2 \subset L^2(\mathbf{R}^2)$.*

Proof. Set

$$\varphi(x,y) = \psi\left(\frac{x+y}{2}, \frac{x-y}{2}\right) = \psi(\eta,\xi)$$

(then $\frac{\partial^2\varphi}{\partial x \partial y} = \frac{1}{4}\Box\psi$ where □ is $\frac{\partial^2}{\partial\eta^2} - \frac{\partial^2}{\partial\xi^2}$).

We have

$$\|V\psi\|_2^2 = \iint_{\mathbf{R}^2} |V(\eta,\xi)\psi(\eta,\xi)|^2 d\eta d\xi.$$

We also have

$$\int_{\mathbf{R}} |V(\eta,\xi)\psi(\eta,\xi)|^2 d\eta \leq \sup_{\eta\in\mathbf{R}} |V(\eta,\xi)|^2 \int_{\mathbf{R}} |\psi(\eta,\xi)|^2 d\eta.$$

But

$$\int_{\mathbf{R}} |\psi(\eta,\xi)|^2 d\eta = \int_{\mathbf{R}} |\varphi(\eta+\xi,\eta-\xi)|^2 d\eta = \int_{\mathbf{R}} |\varphi(\lambda+\xi,\lambda-\xi)|^2 d\lambda.$$

Hence

$$\|V\psi\|_2^2 \leq \int_{\mathbf{R}} \left[\sup_{\eta\in\mathbf{R}} |V(\eta,\xi)|^2 \int_{\mathbf{R}} |\varphi(\lambda+\xi,\lambda-\xi)|^2 d\lambda\right] d\xi.$$

Therefore

$$\|V\psi\|_2^2 \leq \sup_{\xi\in\mathbf{R}} \int_{\mathbf{R}} |\varphi(\lambda+\xi,\lambda-\xi)|^2 d\lambda \left[\int_{\mathbf{R}} \sup_{\eta\in\mathbf{R}} |V(\eta,\xi)|^2 d\xi\right].$$

Now by Proposition 24 one obtains

$$\|V\psi\|_2^2 \leq \int_{\mathbf{R}} \sup_{\eta\in\mathbf{R}} |V(\eta,\xi)|^2 d\xi \left(a\left\|\frac{\partial^2\varphi}{\partial x\partial y}\right\|_2^2 + b\|\varphi\|_2^2\right).$$

Thus

$$\|V\psi\|_2^2 \leq \tilde{c}\left[\int_{\mathbf{R}} \sup_{\eta\in\mathbf{R}} |V(\eta,\xi)|^2 d\xi\right] \left(a\|\Box\psi\|_2^2 + b\|\psi\|_2^2\right). \tag{4.5}$$

By Equation (4.5) we can see that $\mathcal{D}(V) \subset \mathcal{D}(\Box)$. Hence V is □-bounded and since we can make the constant a in front of $||\Box\psi||_2$ as small as we like (Proposition 24), we conclude by the Kato-Rellich theorem (Theorem 19) that $\Box + V$ is self-adjoint on $\mathcal{D}(\Box)$. $\Box$

Remark 21. *In Chapter five we will give another method to get exactly the same norm of V by using Fourier transforms. We do not want to give this method here since it also works in higher dimensions so we prefer to leave it until then.*

4.3 M'^2 and the space BMO

Now we prove an important estimate that will allow us to say that $\Box + V$ is self-adjoint for any real-valued $V \in L^{2+\epsilon}(\mathbf{R}^2), \forall\epsilon > 0$.

Before that we show that $M'^2 \subset BMO(\mathbf{R}^2)$. Then it will follow by the sharp maximal theorem (Theorem 7) that $M'^2 \subset L^p$, $2 \leq p < \infty$. We will then deduce that $M^2 \subset L^p(\mathbf{R}^2)$; $2 \leq p < \infty$. We first have

Theorem 28. *Let* $\varphi \in M'^2$. *Then* $\varphi \in BMO(\mathbf{R}^2)$ *and*

$$||\varphi||_{BMO(\mathbf{R}^2)} \leq a\left\|\frac{\partial^2\varphi}{\partial x\partial y}\right\|_2 + b||\varphi||_2 \tag{4.6}$$

where a *and* b *are two constants.*

Theorem 28 will only be proved for C_0^∞-functions (to get the estimate (4.6) for functions in M'^2 we use the usual density argument c.f. the proof of Proposition 24). The following lemma will be needed in the proof of Theorem 28.

Lemma 10. *Fix* $y_1 \in \mathbf{R}$ *and put* $f_1(x) = \varphi(x, y_1)$ *where* $\varphi \in C_0^\infty$. *Then* $f_1 \in BMO(\mathbf{R})$ *with uniform* BMO *bound, i.e.,*

$$||f_1||_{BMO(\mathbf{R})} \leq a\left\|\frac{\partial^2\varphi}{\partial x\partial y}\right\|_2 + b||\varphi||_2$$

Also fix x_1 *then* $y \mapsto g_1(y) = \varphi(x_1, y)$ *is in* $BMO(\mathbf{R})$.

Proof. We have to prove $\frac{1}{|I|}\int_I |f_1(x) - \overline{f_1}|dx \leq c$ where c does not depend on y_1, and where $\overline{f_1} = \frac{1}{|I|}\int_I f_1(x)dx$. Then we have

$$(f_1 - f)'(x) = \int_y^{y_1} \rho(x, z)dz \text{ where } f(x) = \varphi(x, y)$$

and where we have the freedom to choose the y that is convenient for us. Then we have

$$\|(f_1 - f)'\|_2^2 = \int_{\mathbf{R}} |(f_1 - f)'(x)|^2 dx = \int_{\mathbf{R}} |\int_y^{y_1} \rho(x,z)dz|^2 dx.$$

Thus

$$\|(f_1 - f)'\|_2^2 \le (y_1 - y)\|\rho\|_2^2 < \infty \text{ with the assumption } y < y_1 .$$

So $(f_1 - f)' \in L^2(\mathbf{R})$ hence $(f_1 - f)' \in L^1_{loc}(\mathbf{R})$ and so $f_1 - f$ is absolutely continuous on every compact set of $\mathbf{R}$ hence continuous. So the average $\overline{f_1 - f}$ is $(f_1 - f)(c)$ where $c \in I$ and such a c exists by the intermediate value theorem. So

$$(f_1 - f)(x) - \overline{f_1 - f} = (f_1 - f)(x) - (f_1 - f)(c).$$

Since $f_1 - f$ is absolutely continuous on I, it is differentiable almost everywhere, $(f_1 - f)' \in L^1$ and

$$(f_1 - f)(x) - (f_1 - f)(c) = \int_c^x (f_1 - f)'(t)dt$$

and then

$$|(f_1 - f)(x) - (f_1 - f)(c)| \le \int_c^x |(f_1 - f)'(t)|dt \le (x - c)^{\frac{1}{2}} \|(f_1 - f)'\|_2.$$

Then

$$|(f_1 - f)(x) - (f_1 - f)(c)| \le (b - a)^{\frac{1}{2}} \|(f_1 - f)'\|_2 = |I|^{\frac{1}{2}} \|(f_1 - f)'\|_2$$

and hence

$$\frac{1}{|I|}\int_I |(f_1 - f)(x) - (f_1 - f)(c)|dx \le |I|^{\frac{1}{2}} \|(f_1 - f)'\|_2.$$

But we have $\|(f_1 - f)'\|_2 \le (y_1 - y)^{\frac{1}{2}} \|\rho\|_2$. So in order to find a uniform bound for $\|f_1 - f\|_{BMO}$ for this particular I it suffices to take y such that $(y_1 - y)^{\frac{1}{2}} \le \frac{1}{|I|^{\frac{1}{2}}}$ and in such case we will have:

$$\frac{1}{|I|}\int_I |(f_1 - f)(x) - \overline{(f_1 - f)}|dx \le \|\rho\|_2.$$

But our interest is in the function f_1 itself not in $f_1 - f$ so from $f_1(x) - \overline{f_1} = f_1(x) - f(x) - \overline{f_1} + \overline{f} + f(x) - \overline{f}$ (since $\overline{f - g} = \overline{f} - \overline{g}$) we can find a BMO bound for f_1 for this particular I if we come to show that f is BMO. We have

$$\frac{1}{|I|}\int_I |f(x) - \overline{f}|dx \le \frac{1}{|I|}\int_I |f(x)|dx + \frac{1}{|I|}\int_I (\frac{1}{|I|}\int_I |f(t)|dt)dx$$

Then

$$\frac{1}{|I|}\int_I |f(x)-\overline{f}|dx \leq \frac{2\|f\|_2}{|I|^{\frac{1}{2}}}.$$

So far we have not used the fact that $\varphi \in L^2(\mathbf{R^2})$! To have a uniform BMO bound for f we need for instance $\|f\|_2 \leq c|I|^{\frac{1}{2}}$ ($c>0$, to be determined). Let us assume that there is no y such that $|y_1-y| \leq \frac{1}{(b-a)}$ and such that the previous inequality holds i.e.

$$\forall y \in \mathbf{R}, |y_1-y| \leq \frac{1}{(b-a)}$$

and

$$\int_{\mathbf{R}} |\varphi(x,y)|^2 dx > c^2|I|.$$

Hence

$$\|\varphi\|_2^2 > \int_E\int_{\mathbf{R}} |\varphi(x,y)|^2 dxdy > c^2(b-a)\frac{1}{(b-a)} = c^2$$

where $E=\{y\in \mathbf{R}|y_1-\frac{1}{(b-a)} \leq y \leq \frac{1}{(b-a)}+y_1\}$. To obtain a contradiction it is enough to choose $c=2\|\varphi\|_2$. Hence $\exists y \in E$ such that $\frac{1}{|I|}\int_I |f(x)-\overline{f}|dx \leq \tilde{c}\|\varphi\|_2$. So we now have

$$\frac{1}{|I|}\int_I |f_1(x)-\overline{f_1}|dx \leq \tilde{c}\|\rho\|_2+\tilde{c}\|\varphi\|_2.$$

We do that for all intervals I.

The proof for g_1 is merely the same.

□

Now we prove Theorem 28.

Proof. We already have

$$\frac{1}{|I|}\int_I |\varphi(x,y)-\overline{\varphi}(y)|dx \leq \tilde{c}\|\rho\|_2+\tilde{c}\|\varphi\|_2 := K$$

where $\overline{\varphi}(y)=\frac{1}{|I|}\int_I \varphi(x,y)dx$. We only need to show the estimate:

$$\frac{1}{|J|}\int_J |\overline{\varphi}(y)-\overline{\varphi}|dy \leq K$$

where $\overline{\varphi}=\frac{1}{|I\times J|}\int_{I\times J}\varphi(x,y)dxdy$. Put $R=I\times J$ where R is a **rectangle**.

We have $\frac{1}{|I|}\int_I |\varphi(x,y) - \overline{\varphi}(y)|dx \leq K$ then

$$\frac{1}{|R|}\int_R |\varphi(x,y) - \overline{\varphi}(y)|dxdy \leq \frac{1}{|J|}\int_J Kdy = K.$$

To finish off the proof we need to bound $\frac{1}{|R|}\int_R |\overline{\varphi}(y) - \overline{\varphi}|dxdy$ by K. We have

$$\frac{1}{|J|}\int_J |\overline{\varphi}(y) - \overline{\varphi}|dy = \frac{1}{|J|}\int_J \left|\frac{1}{|I|}\int_I \varphi(x,y)dx - \frac{1}{|R|}\int_R \varphi(x,z)dxdz\right| dy$$

$$= \frac{1}{|R|}\int_J \left|\int_I \varphi(x,y)dx - \frac{1}{|J|}\int_R \varphi(x,z)dxdz\right| dy$$

$$= \frac{1}{|R|}\int_J \left|\int_I \varphi(x,y)dx - \frac{1}{|J|}\int_I \left(\int_J \varphi(x,z)dz\right) dx\right| dy$$

$$= \frac{1}{|R|}\int_J \left|\int_I \left[\varphi(x,y) - \frac{1}{|J|}\int_J \varphi(x,z)dz)\right] dx\right| dy$$

$$\leq \frac{1}{|R|}\int_J\int_I \left|\varphi(x,y) - \frac{1}{|J|}\int_J \varphi(x,z)dz)\right| dxdy$$

$$\leq \frac{1}{|I|}\int_I \left(\frac{1}{|J|}\int_J \left|\varphi(x,y) - \frac{1}{|J|}\int_J \varphi(x,z)dy\right| dz\right) dx$$

$$\leq \frac{1}{|I|}\int_I Kdx = K$$

by the BMO bound of g. Thus

$$\frac{1}{|R|}\int_R |\overline{\varphi}(y) - \overline{\varphi}|dy \leq \frac{1}{|R|}\int_R Kdx = K.$$

So for all rectangles R in $\mathbf{R^2}$ we have:

$$\frac{1}{|R|}\int_R |\varphi(x,y) - \overline{\varphi}|dxdy = \frac{1}{|R|}\int_R |\varphi(x,y) - \overline{\varphi}(y) + \overline{\varphi}(y) - \overline{\varphi}|dxdy$$

$$\leq \frac{1}{|R|}\int_R |\varphi(x,y) - \overline{\varphi}(y)|dxdy + \frac{1}{|R|}\int_R |\overline{\varphi}(y) - \overline{\varphi}|dxdy$$

that is

$$\|\varphi\|_{BMO} \leq a\|\rho\|_2 + b\|\varphi\|_2,$$

establishing (4.6). □

Proposition 25. *Let $\varphi \in M'^2$. Then for all $a > 0$, there exists a $b > 0$ such that*

$$\|\varphi\|_p \leq \|\rho\|_2 + b\|\varphi\|_2 \tag{4.7}$$

where $2 \leq p < \infty$.

Remark 22. *The case $p = \infty$ is false as will be shown in Proposition 27 below.*

Proof. Let $\varphi \in M'^2$ hence $\varphi \in BMO(\mathbf{R}^2)$ by Theorem 28.
We also know, by Theorem 7, that $\|\varphi\|_p \leq \tilde{c}\|M^\sharp\varphi\|_p$. So to prove this proposition we are only required to show that $M^\sharp\varphi \in L^p(\mathbf{R}^2)$. We have

$$\|M^\sharp\varphi\|_2 \leq \tilde{c}\|\varphi\|_2 \text{ so } M^\sharp\varphi \in L^2 \text{ since } \varphi \in L^2.$$

Also by definition of a BMO we have $M^\sharp\varphi \in L^\infty$ (since $\varphi \in BMO$). So one gets

$$\|M^\sharp\varphi\|_p \leq \|M^\sharp\varphi\|_2^{\frac{2}{p}} \|M^\sharp\varphi\|_\infty^{1-\frac{2}{p}}.$$

So $M^\sharp\varphi \in L^p$. But $\|M^\sharp\varphi\|_\infty = \|\varphi\|_{BMO}$ and $\|M^\sharp\varphi\|_2 \leq \tilde{c}\|\varphi\|_2$. Hence for $2 < p < \infty$

$$\|M^\sharp\varphi\|_p \leq \tilde{c}\|\varphi\|_2^{\frac{2}{p}} \|\varphi\|_{BMO}^{1-\frac{2}{p}}$$

and thus

$$\|\varphi\|_p \leq \tilde{c}\|\varphi\|_2^{\frac{2}{p}} \|\varphi\|_{BMO}^{1-\frac{2}{p}}.$$

We also have by Theorem 28 that $\|\varphi\|_{BMO} \leq a\|\rho\|_2 + b\|\varphi\|_2$ then

$$\|\varphi\|_p \leq \tilde{c}\|\varphi\|_2^{\frac{2}{p}} \|\varphi\|_{BMO}^{1-\frac{2}{p}} \leq \tilde{c}\|\varphi\|_2^{\frac{2}{p}} (a\|\rho\|_2 + b\|\varphi\|_2)^{1-\frac{2}{p}}.$$

Now take $\frac{1}{\alpha} = \frac{2}{p}; \frac{1}{\beta} = 1 - \frac{2}{p}$. Hence

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1 \text{ , } \alpha = \frac{p}{2} \text{ and } \beta = \frac{p}{p-2}$$

and for $2 < p < \infty$ we obtain by using Young's inequality (Lemma 1),

$$\|\varphi\|_p \leq \tilde{c}\|\varphi\|_2^{\frac{2}{p}} (\|\rho\|_2 + \|\varphi\|_2)^{1-\frac{2}{p}} \leq \tilde{c}\frac{\|\varphi\|_2^{\alpha\frac{2}{p}}}{\alpha} + \tilde{c}\frac{(\|\rho\|_2 + \|\varphi\|_2)^{\beta(1-\frac{2}{p})}}{\beta}$$

Then

$$\|\varphi\|_p \leq \tilde{c}\frac{\|\varphi\|_2^{\frac{p}{2}\frac{2}{p}}}{\frac{p}{2}} + \tilde{c}\frac{(a\|\rho\|_2 + b\|\varphi\|_2)^1}{\frac{p}{p-2}}.$$

So

$$\|\varphi\|_p \leq \tilde{c}\|\varphi\|_2 + \tilde{c}\|\rho\|_2 + \tilde{c}\frac{p-2}{p}\|\varphi\|_2.$$

Thus

$$\|\varphi\|_p \leq a\|\rho\|_2 + b\|\varphi\|_2.$$

□

Corollary 9. *Let $f \in M^2$. Then for all $a > 0$, there exists $b > 0$ such that*

$$\|f\|_p \leq a\|\Box f\|_2 + b\|f\|_2 \tag{4.8}$$

where $2 \leq p < \infty$.

Remark 23. *Corollary 9 could have been a corollary to Theorem 28 as this latter is true for $\varphi \in M^2$ since we have the elementary fact that the BMO norm (up to a constant) is invariant under the change of variables we made in the proof of Theorem 27.*

Before giving the second class of self-adjoint $\Box + V$ we give the following lemma:

Lemma 11. *The constant a in (4.8) may be made as small as we would like.*

Proof. Take $\varphi_\lambda(x, y) = \varphi(\lambda x, \lambda y) : \lambda > 0$. We get:

$$\|\Box\varphi_\lambda\|_2 = \lambda\|\Box\varphi\|_2,\ \|\varphi_\lambda\|_2 = \frac{1}{\lambda}\|\varphi\|_2 \text{ and } \|\varphi_\lambda\|_p = \frac{1}{\lambda^{\frac{2}{p}}}\|\varphi\|_p.$$

Thus the estimate (4.8) applied to φ_λ instead of φ becomes:

$$\|\varphi\|_p \leq a\lambda^{\frac{2}{p}+1}\|\Box\varphi\|_2 + b\lambda^{\frac{2}{p}-1}\|\varphi\|_2, \lambda > 0, p \geq 2.$$

Take λ small enough and the constant in front of $\|\Box\varphi\|_2$ will be arbitrarily small. □

Theorem 29. *Let $\Box$ be the wave operator on $L^2(\mathbf{R}^2)$. Let $\epsilon > 0$ and let $V : \mathbf{R}^2 \longrightarrow \mathbf{R}$ such that $V \in L^{2+\epsilon}(\mathbf{R}^2)$. Then $\Box + V$ is self-adjoint on $\mathcal{D}(\Box)$.*

Proof. We have by Corollary 9,

$$\|\varphi\|_p \leq a\|\Box\varphi\|_2 + b\|\varphi\|_2, \text{ for } 2 \leq p < \infty.$$

Then by the generalized Hölder's inequality:

$$\|V\varphi\|_2 \leq \|V\|_q\|\varphi\|_p \leq a\|V\|_q\|\Box\varphi\|_2 + b\|V\|_q\|\varphi\|_2, \text{ for } \frac{1}{2} = \frac{1}{p} + \frac{1}{q} \text{ or}$$

$q = \frac{2p}{p-2}$ (this actually means that $q \to 2$ as $p \to \infty$).
Since the constant in front of $\|\Box\varphi\|_2$ may be made arbitrarily small so that we have $a\|V\|_q < 1$, we conclude by the Kato-Rellich perturbation theorem that $\Box + V$ is self-adjoint on $\mathcal{D}(\Box) = M^2$. □

Remark 24. *Adding a bounded multiplication operator by a real-valued function does not destroy the self-adjointness and we have:*

Proposition 26. *Let $V \in L^{2+\epsilon}(\mathbf{R}^2) + L^\infty(\mathbf{R}^2)$ be a real valued function, $\epsilon > 0$. Then $\Box + V$ is self-adjoint on $\mathcal{D}(\Box)$.*

Proof. We have

$$\|\varphi\|_p \leq a\|\Box\varphi\|_2 + b\|\varphi\|_2.$$

Put $V = V_1 + V_2$ where $V_1 \in L^{2+\epsilon}$, $V_2 \in L^\infty$. Then $\|V\varphi\|_2 = \|(V_1 + V_2)\varphi\|_2$ and by the generalized Hölder's inequality (Proposition 1)

$$\|V\varphi\|_2 \leq \|V_1\|_{2+\epsilon}\|\varphi\|_p + \|V_2\|_\infty\|\varphi\|_2 \leq a\|V_1\|_{2+\epsilon}\|\Box\varphi\|_2 + (\|V_2\|_\infty + b)\|\varphi\|_2.$$

Since we can take a as small as we like, $\Box + V$ is self-adjoint on M^2 by the Kato-Rellich Theorem. □

Example 17. *Let $s > 1$. Take $V(x,t) = \frac{1}{(x^2+t^2)^{\frac{1}{2s}}}$. Choose $\epsilon > 0$ such that $\frac{2+\epsilon}{s} < 2$. Then $V \in L^{2+\epsilon} + L^\infty$ and hence $\Box + \frac{1}{(x^2+t^2)^{\frac{1}{2s}}}$ is self-adjoint on M^2.*

4.4 Counterexamples

It was mentioned in Remark 22 that a function in M'^2 needed not be in L^∞. So we have

Proposition 27. *Let $\varphi \in L^2(\mathbf{R}^2)$ such that $\frac{\partial^2\varphi}{\partial x\partial y} \in L^2(\mathbf{R}^2)$. Then φ need not be essentially bounded on $\mathbf{R}^2$.*

We give two methods of how to do this. First we give an explicit counterexample that we construct in the following proof:

Proof. We are going to build up the counterexample by using a linear interpolation. We define $(x, y) \mapsto \varphi(x, y)$ on $\mathbf{R} \times (y_n, y_{n+1}]$ by

$$\varphi(x,y) = \frac{1}{y_{n+1} - y_n}[(y - y_n)f_{n+1}(x) - (y - y_{n+1})f_n(x)] \text{ where } f_n(x) = \varphi(x, y_n)$$

and the f_n and y_n are to be defined below.

Observe that φ is only defined for $y > y_1 := 0$. At the end of this proof we will extend it to the case $y \leq 0$ by a symmetry.

Hence on $\mathbf{R} \times (0, \infty)$ we have

$$\|\varphi\|_2^2 = \iint\limits_{\mathbf{R}\times(0,\infty)} |\varphi(x,y)|^2 dxdy = \sum_1^\infty \iint\limits_{\mathbf{R}\times(y_n,y_{n+1}]} |\varphi(x,y)|^2 dxdy$$

In order to have φ in $L^2(\mathbf{R} \times (0, \infty))$, it is sufficient to have

$$\sum_1^\infty (y_{n+1} - y_n)(\|f_n\|_2^2 + \|f_{n+1}\|_2^2) < \infty.$$

We also have

$$\frac{\partial^2\varphi}{\partial x\partial y}(x,y) = \frac{1}{y_{n+1} - y_n}(f'_{n+1}(x) - f'_n(x)).$$

In order to have $\frac{\partial^2\varphi}{\partial x\partial y}$ in $L^2(\mathbf{R} \times (0, \infty))$, it is sufficient to have

$$\sum_1^\infty \frac{1}{y_{n+1} - y_n}\|\psi'_n\|_2^2 < \infty \text{ where } \psi_n(x) = f_{n+1}(x) - f_n(x).$$

We are going to to define f_n by first constructing ψ_n then putting $f_n(x) = -\sum_{k=n}^{\infty} \psi_k(x)$.

We also want $f_n \notin L^\infty(\mathbf{R})$ so that $\varphi \notin L^\infty(\mathbf{R}^2)$.

Take

$$\psi_n(x) = \begin{cases} \frac{e^n}{n}x + \frac{1}{n} & if \quad -e^{-n} \le x \le 0, \\ -\frac{e^n}{n}x + \frac{1}{n} & if \quad 0 \le x \le e^{-n}, \\ 0 & if \quad |x| \ge e^{-n}. \end{cases}$$

Hence $\|\psi_n'\|_2^2 \sim \frac{e^n}{n^2}$ and $\|\psi_n\|_2^2 \sim \frac{e^{-n}}{n^2}$. We also have

$$\|f_n\|_2 \le \sum_{k=n}^{\infty} \|\psi_k\|_2 = a\sum_{k=n}^{\infty} \frac{e^{-\frac{n}{2}}}{n} \simeq \int_n^{\infty} \frac{e^{-\frac{x}{2}}}{x} dx \le \frac{1}{n}\int_n^{\infty} e^{-\frac{x}{2}} dx \sim \frac{e^{-\frac{n}{2}}}{n}.$$

Now if we choose $y_{n+1} - y_n = e^n$ then the series

$$\sum_1^{\infty} \frac{1}{y_{n+1} - y_n} \|\psi_n'\|_2^2 = 2\sum_1^{\infty} \frac{1}{e^n} \times \frac{e^n}{n^2} = 2\sum_1^{\infty} \frac{1}{n^2} \text{ obviously converges .}$$

And so does the series

$$\sum_1^{\infty} (y_{n+1} - y_n)(\|f_n\|_2^2 + \|f_{n+1}\|_2^2) \le \sum_1^{\infty} e^n \times [\frac{e^{-(n+1)}}{(n+1)^2} + \frac{e^{-n}}{n^2}] \sim \sum_1^{\infty} \frac{1}{n^2}.$$

Now the φ defined on $\mathbf{R} \times (y_n, y_{n+1})$ is given by

$$\varphi(x,y) = e^{-n}\left[(y - y_n)\left(-\sum_{n+1}^{\infty} \psi_k(x)\right) - (y - y_{n+1})\left(-\sum_{n}^{\infty} \psi_k(x)\right)\right].$$

This φ is actually defined only for $x \in \mathbf{R}$ and $y \ge 0$. To extend it to the case $y < 0$ we define φ for $x \in \mathbf{R}$ and $-y_{n+1} < y < -y_n$ as follows:

$$\varphi(x,y) = \frac{1}{y_{n+1} - y_n}[(y - y_n)f_{n+1}(x) - (y - y_{n+1})f_n(x)].$$

This φ is clearly in M'^2. Now we need to show that φ is not in $L^\infty(\mathbf{R}^2)$. Let $x > 0$ and $x \le e^{-k}$ then $\ln x \le -k$ or $\ln \frac{1}{x} \ge k$. So

$$f_n(x) = -x\sum_{k=n}^{\lfloor \ln \frac{1}{x} \rfloor} \frac{e^k}{k} + \sum_{k=n}^{\lfloor \ln \frac{1}{x} \rfloor} \frac{1}{k} \le (-x^2 + 1)\sum_{k=n}^{\lfloor \ln \frac{1}{x} \rfloor} \frac{1}{k}.$$

But
$$\sum_{k=n}^{\lfloor \ln \frac{1}{x} \rfloor} \frac{1}{k} = \sum_{1}^{\lfloor \ln \frac{1}{x} \rfloor} \frac{1}{k} - \sum_{1}^{n-1} \frac{1}{k} = 1 + ... + \frac{1}{\lfloor \ln \frac{1}{x} \rfloor} - (1 + \frac{1}{2} + ... + \frac{1}{n-1}).$$

But from $1 + \frac{1}{2} + ... + \frac{1}{n} \sim \ln n + \gamma$ (here γ represents Euler's constant) we have

$$\sum_{n}^{\lfloor \ln \frac{1}{x} \rfloor} \frac{1}{k} \sim \ln \lfloor \ln \frac{1}{x} \rfloor + \gamma - (1 + \frac{1}{2} + ... + \frac{1}{n-1}).$$

Now as $x \to 0$ then $\lfloor \ln \frac{1}{x} \rfloor \to \infty$ hence $\ln \lfloor \ln \frac{1}{x} \rfloor \to \infty$. Thus $-f_n(x) \to \infty$ which implies that $\varphi(x, y) \to \infty$. So $\varphi \notin L^\infty(\mathbf{R}^2)$. □

Remark 25. *This counterexample found is actually a BMO function by Theorem 28.*

The second method is proving the existence of such a function without exhibiting an explicit one. It is done as follows:

Proof. First, consider $f : \mathbf{R}^2 \to \mathbf{R}$ such that

$$f(u, v) = \frac{1}{1 + |uv|}.$$

The function f is obviously positive. Besides, it does not belong to $L^2(\mathbf{R}^2)$ [1] since

$$\iint_{\mathbf{R}^2} \frac{1}{(1 + |uv|)^2} dudv \geq \int_0^\infty \int_0^\infty \frac{1}{(1 + uv)^2} dudv = \lim_{R \to \infty} \int_0^R \int_0^R \frac{1}{(1 + uv)^2} dudv$$

But
$$\int_0^R \frac{1}{(1 + uv)^2} du = [-\frac{1}{v}(1 + uv)^{-1}]_0^R = \frac{1}{v} - \frac{1}{v(1 + Rv)} = \frac{R}{1 + Rv}.$$

So

$$\|f\|_2^2 \geq \lim_{R \to \infty} \int_0^R \frac{R}{1 + Rv} dv = \lim_{R \to \infty} [\ln(1 + Rv)]_0^R = \lim_{R \to \infty} \ln(1 + R^2) = \infty.$$

Now by Lemma 2 and Remark 2 we know that there exists $\psi \geq 0, \psi \in L^2$ such that $\psi f \notin L^1$. Since $f \in L^\infty$, ψf belongs to L^2 and it legitimate to define $\varphi = \mathcal{F}^{-1}(\psi f)$ where $\mathcal{F}$ is the

[1] In fact f is not in $L^p(\mathbf{R}^2)$ for any $p \geq 1$.

L^2-Fourier transform. By the Plancherel theorem φ is in L^2. Also

$$\mathcal{F}\left(\frac{\partial^2\varphi}{\partial x\partial y}\right) = uv\mathcal{F}\varphi = uv\psi(u,v)f(u,v).$$

Since $(u,v) \mapsto \frac{uv}{1+|uv|} \in L^\infty(\mathbf{R^2})$ and since $\psi \in L^2(\mathbf{R^2})$ it follows that $\frac{uv}{1+|uv|}\psi \in L^2(\mathbf{R^2})$ and hence, as a consequence of the Plancherel theorem, one gets that $\frac{\partial^2\varphi}{\partial x\partial y} \in L^2(\mathbf{R^2})$.

Before carrying on the proof we give the following lemma:

Lemma 12. *Let $\varphi \in L^2(\mathbf{R^n})$. If $\hat{\varphi} \notin L^1(\mathbf{R^n})$ with $\hat{\varphi} \geq 0$, then $\varphi \notin L^\infty(\mathbf{R^n})$.*

Proof. Let $\varphi \in L^2(\mathbf{R^n})$. Suppose $\varphi \in L^\infty(\mathbf{R^n})$. Take

$$\hat{f}_{p,m}(x) = (\hat{\varphi} * \psi_p)g_m(x)$$

where ψ_p is a smoothing function like the one defined in Theorem 2 that satisfies $\|\hat{\psi}_p\|_\infty \leq 1$ (take $\frac{1}{p} = \epsilon$). We assume that $g_m \geq 0$ and it is a C_0^∞-function. We finally assume that g_m tends to one pointwise. Then we have $\hat{f}_{p,m} \in L^1(\mathbf{R^n})$ and $\hat{f}_{p,m} \geq 0$.

We now apply Proposition 3 to have

$$\|\hat{f}_{p,m}\|_1 = \|f_{p,m}\|_\infty.$$

Applying Young's inequality for convolution, taking $\tilde{\varphi}(x) = \varphi(-x)$ and since $\hat{\hat{\varphi}} = \tilde{\varphi}$ gives us

$$\|(\hat{\varphi} * \psi_p)g_m\|_1 = \frac{1}{(2\pi)^{\frac{n}{2}}}\|(\tilde{\varphi}\hat{\psi}_p) * g_m\|_\infty \leq \frac{1}{(2\pi)^{\frac{n}{2}}}\|\tilde{\varphi}\hat{\psi}_p\|_\infty\|g_m\|_1 < \infty.$$

Or

$$\|(\hat{\varphi} * \psi_p)g_m\|_1 \leq \frac{1}{(2\pi)^{\frac{n}{2}}}\|\varphi\|_\infty.$$

So as p tends to infinity we obtain

$$\|(\hat{\varphi} * \psi_p)g_m\|_1 \to \|\hat{\varphi}g_m\|_1.$$

In the end one has

$$\int_{\mathbf{R^n}} \hat{\varphi}(x)g_m(x)dx = \|\hat{\varphi}g_m\|_1 \leq \frac{1}{(2\pi)^{\frac{n}{2}}}\|\tilde{\varphi}\hat{\psi}_p\|_\infty \leq \frac{1}{(2\pi)^{\frac{n}{2}}}\|\varphi\|_\infty.$$

Taking the lim inf of each side and applying Fatou's lemma and since $\hat{\varphi} \geq 0$ give us that $\hat{\varphi} \in L^1(\mathbf{R}^n)$. □

Now we finish the proof. Since $\mathcal{F}(\varphi) \in L^2$, since it is positive and since $\mathcal{F}(\varphi) \notin L^1$, Lemma 12 allows us to say that $\varphi \notin L^\infty$. □

Remark 26. *One may wonder if $\varphi \in M'^2$ then under what more conditions φ will be in L^∞? The answer is given in the following proposition:*

Proposition 28. *Let $\varphi \in L^2(\mathbf{R}^2)$ such that $\frac{\partial \varphi}{\partial x}$, $\frac{\partial \varphi}{\partial y}$ and $\frac{\partial^2 \varphi}{\partial x \partial y}$ are all in $L^2(\mathbf{R}^2)$. Then $\hat{\varphi} \in L^1(\mathbf{R}^2)$ and hence $\varphi \in L^\infty(\mathbf{R}^2)$.*

Proof. Let $\varphi \in L^2(\mathbf{R}^2)$ such that $\frac{\partial \varphi}{\partial x}$, $\frac{\partial \varphi}{\partial y}$ and $\frac{\partial^2 \varphi}{\partial x \partial y}$ are all in $L^2(\mathbf{R}^2)$. Then by the Plancherel theorem we have $\hat{\varphi}, \eta\hat{\varphi}, \xi\hat{\varphi}, \eta\xi\hat{\varphi} \in L^2(\mathbf{R}^2)$. Hence

$$(1 + |\eta| + |\xi| + |\eta\xi|)\hat{\varphi} \in L^2(\mathbf{R}^2) \text{ and so}$$

$$\|\hat{\varphi}\|_{L^1(\mathbf{R}^2)} = \left\| \frac{1}{1 + |\eta| + |\xi| + |\eta\xi|}(1 + |\eta| + |\xi| + |\eta\xi|)\hat{\varphi} \right\|_{L^1(\mathbf{R}^2)}.$$

But $\frac{1}{1+|\eta|+|\xi|+|\eta\xi|} \in L^2(\mathbf{R}^2)$ since

$$\iint_{\mathbf{R}^2} \frac{d\eta d\xi}{(1 + |\eta| + |\xi| + |\eta\xi|)^2} = \iint_{\mathbf{R}^2} \frac{d\eta d\xi}{(1 + |\eta|)^2(1 + |\xi|)^2} = \int_{\mathbf{R}} \frac{d\eta}{(1 + |\eta|)^2} \int_{\mathbf{R}} \frac{d\xi}{(1 + |\xi|)^2}$$

which is finite, say equal to a positive number c, hence by the Cauchy-Schwarz's inequality

$$\|\hat{\varphi}\|_{L^1(\mathbf{R}^2)} \leq \left\| \frac{1}{1 + |\eta| + |\xi| + |\eta\xi|} \right\|_{L^2(\mathbf{R}^2)} \|(1 + |\eta| + |\xi| + |\eta\xi|)\hat{\varphi}\|_{L^2(\mathbf{R}^2)}.$$

So

$$\|\hat{\varphi}\|_{L^1(\mathbf{R}^2)} \leq c[\|\hat{\varphi}\|_{L^2(\mathbf{R}^2)} + \|\eta\hat{\varphi}\|_{L^2(\mathbf{R}^2)} + \|\xi\hat{\varphi}\|_{L^2(\mathbf{R}^2)} + \|\eta\xi\hat{\varphi}\|_{L^2(\mathbf{R}^2)}].$$

Thus

$$\|\varphi\|_\infty \leq c \left(\left\| \frac{\partial^2 \varphi}{\partial x \partial y} \right\|_{L^2(\mathbf{R}^2)} + \left\| \frac{\partial \varphi}{\partial x} \right\|_{L^2(\mathbf{R}^2)} + \left\| \frac{\partial \varphi}{\partial y} \right\|_{L^2(\mathbf{R}^2)} + \|\varphi\|_{L^2(\mathbf{R}^2)} \right).$$

□

Now we come to the counterexample that shows that $\Box + V$ can fail to be essentially self-adjoint if $V \in L^2_{loc}(\mathbf{R^2})$ and we have:

Proposition 29. *Let $\Box$ be the wave operator defined on $L^2(\mathbf{R^2})$. Then there exists a real-valued V, $V \in L^2_{loc}(\mathbf{R^2})$ such that $\Box + V$ is not essentially self-adjoint on C_0^∞.*

Proof. Basically we want to show that the following PDE has a non-zero solution in $L^2(\mathbf{R^2})$ for some $V \in L^2_{loc}(\mathbf{R^2})$:

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\varphi(x,t) + V(x,t)\varphi(x,t) = 2i\varphi(x,t).$$

By Example 16 we can say that the following ODE:

$$-\frac{d^2}{dx^2}f(x) - x^4 f(x) = if(x) \tag{4.9}$$

has a non-zero solution in $L^2(\mathbf{R})$. And we can say the same thing about

$$-\frac{d^2}{dt^2}g(t) - t^4 g(t) = -ig(t). \tag{4.10}$$

Now by multiplying (4.9) by $g(t)$ and (4.10) by $-f(x)$ we obtain:

$$-g(t)\frac{d^2}{dx^2}f(x) - x^4 f(x)g(t) = if(x)g(t) \tag{4.11}$$

and

$$f(x)\frac{d^2}{dt^2}g(t) + t^4 f(x)g(t) = if(x)g(t). \tag{4.12}$$

Now by adding up (4.11) and (4.12) we get

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)f(x)g(t) + (t^4 - x^4)f(x)g(t) = 2if(x)g(t). \tag{4.13}$$

Take $\varphi(x,t) = f(x)g(t)$. Since f, g are both in $L^2(\mathbf{R})$ then φ will be in $L^2(\mathbf{R^2})$ and (4.13) will have a non-zero solution in $L^2(\mathbf{R^2})$ with $V(x,t) = t^4 - x^4 \in L^2_{loc}(\mathbf{R^2})$.

Thus $\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + t^4 - x^4$ is not essentially self-adjoint on $\mathcal{D}(\Box)$. $\Box$

4.5 Open problems

1) Let V be a real-valued function such that $V \in L^2(\mathbf{R^2})$. The question is: is $\Box + V$ essentially self-adjoint on C_0^∞?

This is more likely to be wrong since we did not obtain (for $p = \infty$)

$$\|f\|_p \leq a\|\Box f\|_2 + b\|f\|_2,$$

which would have allowed us to conclude that $\Box + V$ is self-adjoint for V real-valued and in $L^2(\mathbf{R^2})$ or essentially self-adjoint on C_0^∞.

It is worth mentioning that $-\Delta + V$ is self-adjoint on $\mathcal{D}(-\Delta) \subset L^2(\mathbf{R^3})$ (c.f. Example 14) and the proof of that exploits

$$\|f\|_\infty \leq a\| - \Delta f\|_2 + \|f\|_2$$

(Theorem 6) and the Kato-Rellich theorem.

So one may even conjecture that if for some self-adjoint partial differential operator P one *does not have* an inequality of the type

$$\|f\|_\infty \leq a\|Pf\|_2 + b\|f\|_2,$$

then there exists a real-valued V, $V \in L^2(\mathbf{R^n})$, $n \geq 1$ for which $P + V$ is not essentially self-adjoint on C_0^∞.

Now we go back to our open problem. One way of showing that $\Box + V$ is not self-adjoint (or at least not essentially self-adjoint on $C_0^\infty(\mathbf{R^2})$) is to construct a V which is in $L^2(\mathbf{R^2})$ and not in $L^s(\mathbf{R^2})$ for $s > 2$ and show that

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) f(x,t) + V(x,t))f(x,t) = \pm i f(x,t)$$

has a non-zero solution which belongs to $L^2(\mathbf{R^2})$.

2) If $V \geq 0$ is real-valued and in L^2_{loc}, is $\Box + V$ essentially self-adjoint on C_0^∞?

If $V \geq 0$ we cannot use the same method as the proof of Proposition 29 since it is known (see

[19]) that for $V \geq 0$ and $V \in L^2_{loc}$, $-\frac{d^2}{dx^2} + V$ is essentially self-adjoint on C_0^∞.

This question too has probably a negative answer. A possible counterexample would be $V(x,t) = |t^4 - x^4|$ but one has to investigate that.

Chapter 5
Self-adjointness of the perturbed wave operator on $L^2(\mathbf{R}^n), n \geq 3$

5.1 Introduction

In this chapter we investigate the self-adjointness of $\Box + V$ for $n \geq 3$ and V real-valued and unbounded. The wave operator worsens in higher dimensions and as a result we will have smaller classes of Vs and hence more open problems.

5.2 A class of self-adjoint $\Box + V$ on $L^2(\mathbf{R}^n), n \geq 3$

Definition 36. *We set*

$$M^n = \{u \in L^2(\mathbf{R}^n) : \Box u, \text{ as a distribution, is an } L^2(\mathbf{R}^n) \text{ function } \}.$$

We first start by the case $n = 3$. Before we give the first proposition let us discuss the following Cauchy problem:

$$(I)\ldots \begin{cases} u_{tt} - u_{xx} - u_{yy} = f(x, y, t), & (x, y, t) \in \mathbf{R}^2 \times \mathbf{R}^+ \\ u(x, y, 0) = \varphi(x, y); u_t(x, y, 0) = \psi(x, y). \end{cases}$$

Now let us take the Fourier transform of (I) in the (x, y)-plane only. We get:

$$(\hat{I})\ldots \begin{cases} \hat{u}_{tt} + (\eta^2 + \xi^2)\hat{u} = \hat{f}(\eta, \xi, t) \\ \hat{u}(\eta, \xi, 0) = \hat{\varphi}(\eta, \xi); \hat{u}_t(\eta, \xi, 0) = \hat{\psi}(\eta, \xi). \end{cases}$$

$(\hat{I})$ is a second order ODE in t with constant coefficient (with respect to t) and it has the following solution in the homogeneous case:

$$\hat{u}_{\eta,\xi}(t) = \hat{\varphi}(\eta,\xi)\cos(t\sqrt{\eta^2+\xi^2}) + \frac{\hat{\psi}(\eta,\xi)}{\sqrt{\eta^2+\xi^2}}\sin(t\sqrt{\eta^2+\xi^2}).$$

Then by using the Duhamel's principle the general solution of $(\hat{I})$ will then be:

$$\begin{aligned}\hat{u}_{\eta,\xi}(t) = \hat{\varphi}(\eta,\xi)\cos(t\sqrt{\eta^2+\xi^2}) + \frac{\hat{\psi}(\eta,\xi)}{\sqrt{\eta^2+\xi^2}}\sin(t\sqrt{\eta^2+\xi^2}) \\ + \int_0^t \hat{f}(\eta,\xi,s)\frac{\sin\sqrt{\eta^2+\xi^2}(t-s)}{\sqrt{\eta^2+\xi^2}}ds.\end{aligned} \tag{5.1}$$

The previous holds for $t \geq 0$. Also for $t \geq 0$ (I) becomes, after setting $\tilde{u}(x,y,t) = u(x,y,-t)$,

$$(\tilde{I})\ldots\begin{cases} \tilde{u}_{tt} - \tilde{u}_{xx} - \tilde{u}_{yy} = f(x,y,-t), & (x,y,t) \in \mathbf{R^2}\times\mathbf{R^-} \\ \tilde{u}(x,y,0) = \varphi(x,y); \tilde{u}_t(x,y,0) = -\psi(x,y). & \end{cases}$$

Now we "fourier" everything in the (x,y)-plane to obtain

$$(\hat{\tilde{I}})\ldots\begin{cases} \hat{\tilde{u}}_{tt} + (\eta^2+\xi^2)\hat{\tilde{u}} = \hat{f}(\eta,\xi,-t) \\ \hat{\tilde{u}}(\eta,\xi,0) = \hat{\varphi}(\eta,\xi); \hat{\tilde{u}}_t(\eta,\xi,0) = -\hat{\psi}(\eta,\xi) \end{cases}$$

which has the following solution:

$$\begin{aligned}\hat{u}_{\eta,\xi}(-t) = \hat{\tilde{u}}_{\eta,\xi}(t) = \hat{\varphi}(\eta,\xi)\cos(t\sqrt{\eta^2+\xi^2}) - \frac{\hat{\psi}(\eta,\xi)}{\sqrt{\eta^2+\xi^2}}\sin(t\sqrt{\eta^2+\xi^2}) \\ + \int_0^{-t} \hat{f}(\eta,\xi,s)\frac{\sin\sqrt{\eta^2+\xi^2}(-t-s)}{\sqrt{\eta^2+\xi^2}}ds\end{aligned} \tag{5.2}$$

and this holds for $t \geq 0$. After adding up (5.1) and (5.2) one gets (still for $t \geq 0$)

$$\begin{aligned}\hat{u}_{\eta,\xi}(t) = 2\hat{\varphi}(\eta,\xi)\cos(t\sqrt{\eta^2+\xi^2}) + \int_0^t \hat{f}(\eta,\xi,s)\frac{\sin\sqrt{\eta^2+\xi^2}(t-s)}{\sqrt{\eta^2+\xi^2}}ds \\ + \int_0^{-t} \hat{f}(\eta,\xi,s)\frac{\sin\sqrt{\eta^2+\xi^2}(-t-s)}{\sqrt{\eta^2+\xi^2}}ds - \hat{u}_{\eta,\xi}(-t)\end{aligned} \tag{5.3}$$

Now Equation (5.3) is unchanged if t is replaced by $-t$. So (5.3) holds for all $t \in \mathbf{R}$.

Now we can change our Cauchy problem by introducing different initial conditions mainly for the t variable i.e. instead of working on the intervals $(0,t)$ and $(-t,0)$ we will be working on $(\frac{t+\alpha}{2}, t)$ and $(\alpha, \frac{t+\alpha}{2})$ where α is any real number that will be chosen freely (observe that if we set $\alpha = -t$ then we go back to the initial problem).

Now given $u \in C_0^\infty$ we can regard u as a solution to the Cauchy problem with $u \in C_0^\infty$ where $f = \Box u$, $u(x,y,0) = \varphi(x,y)$ and $u_t(x,y,0) = \psi(x,y)$. Then (5.3) will be:

$$\hat{u}(\eta,\xi,t) = 2\hat{u}(\eta,\xi,\frac{t+\alpha}{2}) \cos\left[\sqrt{\eta^2+\xi^2}\left(\frac{t-\alpha}{2}\right)\right] - \hat{u}(\eta,\xi,\alpha) + \int_{\frac{t+\alpha}{2}}^{t} \hat{f}(\eta,\xi,s)\frac{\sin\sqrt{\eta^2+\xi^2}(t-s)}{\sqrt{\eta^2+\xi^2}}ds + \int_{\frac{t+\alpha}{2}}^{\alpha} \hat{f}(\eta,\xi,s)\frac{\sin\sqrt{\eta^2+\xi^2}(\alpha-s)}{\sqrt{\eta^2+\xi^2}}ds. \tag{5.4}$$

Proposition 30. *For all $a > 0$, there exists $b > 0$ such that*

$$\operatorname*{ess\,sup}_{t\in\mathbf{R}} \|u(.,.,t)\|_{L^2(\mathbf{R}^2)} \leq a\|\Box u\|_{L^2(\mathbf{R}^3)} + b\|u\|_{L^2(\mathbf{R}^3)}$$

for all $u \in M^3$.

Proof. We shall prove the proposition for functions in C_0^∞ first then the result follows for functions in M^3 since C_0^∞ is dense in M^3 in the graph norm of $\Box$ (c.f. Remark 7 in Chapter1).

We choose α such that $|t-\alpha| \leq 1$ and one then obtains:

$$|\hat{u}(\eta,\xi,t)| \leq 2|\hat{u}(\eta,\xi,\frac{t+\alpha}{2})| + \tilde{c}\int_{\frac{t+\alpha}{2}}^{t} |\hat{f}(\eta,\xi,s)|ds + \tilde{c}\int_{\frac{t+\alpha}{2}}^{\alpha} |\hat{f}(\eta,\xi,s)|ds + |\hat{u}(\eta,\xi,\alpha)|. \tag{5.5}$$

where we have used the fact that $|\frac{\sin X}{X}| \leq 1$. Using Cauchy-Schwarz inequality shows that

$$|\hat{u}(\eta,\xi,t)| \leq 2|\hat{u}(\eta,\xi,\frac{t+\alpha}{2})| + \tilde{c}\left(\int_{\frac{t+\alpha}{2}}^{t} |\hat{f}(\eta,\xi,s)|^2 ds\right)^{\frac{1}{2}} + \tilde{c}\left(\int_{\frac{t+\alpha}{2}}^{\alpha} |\hat{f}(\eta,\xi,s)|^2 ds\right)^{\frac{1}{2}} + \tilde{c}|\hat{u}(\eta,\xi,\alpha)|.$$

Now square the previous inequality to get

$$|\hat{u}(\eta,\xi,t)|^2 \leq \tilde{c}|\hat{u}(\eta,\xi,\frac{t+\alpha}{2})|^2 + \tilde{c}\int_{\frac{t+\alpha}{2}}^{t} |\hat{f}(\eta,\xi,s)|^2 ds + \tilde{c}\int_{\frac{t+\alpha}{2}}^{\alpha} |\hat{f}(\eta,\xi,s)|^2 ds + \tilde{c}|\hat{u}(\eta,\xi,\alpha)|^2. \tag{5.6}$$

Then integrate (5.6) with respect to η and ξ in $\mathbf{R^2}$ to obtain

$$\iint\limits_{\mathbf{R^2}} |\hat{u}(\eta,\xi,t)|^2 d\eta d\xi \leq \tilde{c} \iint\limits_{\mathbf{R^2}} |\hat{u}(\eta,\xi,\frac{t+\alpha}{2})|^2 d\eta d\xi + \tilde{c} \iint\limits_{\mathbf{R^2}} \int_{\frac{t+\alpha}{2}}^{t} |\hat{f}(\eta,\xi,s)|^2 d\eta d\xi ds +$$
$$\tilde{c} \iint\limits_{\mathbf{R^2}} \int_{\frac{t+\alpha}{2}}^{\alpha} |\hat{f}(\eta,\xi,s)|^2 d\eta d\xi ds + \tilde{c} \iint\limits_{\mathbf{R^2}} |\hat{u}(\eta,\xi,\alpha)|^2 d\eta d\xi. \tag{5.7}$$

And hence

$$\iint\limits_{\mathbf{R^2}} |\hat{u}(\eta,\xi,t)|^2 d\eta d\xi \leq \tilde{c} \iint\limits_{\mathbf{R^2}} |\hat{u}(\eta,\xi,\frac{t+\alpha}{2})|^2 d\eta d\xi + \tilde{c} \iiint\limits_{\mathbf{R^3}} |\hat{f}(\eta,\xi,s)|^2 d\eta d\xi ds$$
$$+\tilde{c} \iint\limits_{\mathbf{R^2}} |\hat{u}(\eta,\xi,\alpha)|^2 d\eta d\xi.$$

Now integrate everything with respect to α in the segment $|t-\alpha| \leq 1$ i.e. $t-1 \leq \alpha \leq t+1$ to get:

$$\int_{t-1}^{t+1} \iint\limits_{\mathbf{R^2}} |\hat{u}(\eta,\xi,t)|^2 d\eta d\xi d\alpha \leq \tilde{c} \int_{t-1}^{t+1} \iint\limits_{\mathbf{R^2}} |\hat{u}(\eta,\xi,\frac{t+\alpha}{2})|^2 d\eta d\xi d\alpha$$
$$+\tilde{c} \int_{t-1}^{t+1} \iiint\limits_{\mathbf{R^3}} |\hat{f}(\eta,\xi,s)|^2 d\eta d\xi ds d\alpha + \tilde{c} \int_{t-1}^{t+1} \iint\limits_{\mathbf{R^2}} |\hat{u}(\eta,\xi,\alpha)|^2 d\eta d\xi d\alpha.$$

Hence

$$\iint\limits_{\mathbf{R^2}} |\hat{u}(\eta,\xi,t)|^2 d\eta d\xi \leq \tilde{c} \iiint\limits_{\mathbf{R^3}} |\hat{u}(\eta,\xi,\frac{t+\alpha}{2})|^2 d\eta d\xi d\alpha$$
$$+\tilde{c} \iiint\limits_{\mathbf{R^3}} |\hat{f}(\eta,\xi,s)|^2 d\eta d\xi ds + \tilde{c} \iiint\limits_{\mathbf{R^3}} |\hat{u}(\eta,\xi,\alpha)|^2 d\eta d\xi d\alpha.$$

Thus by the Plancherel theorem one has:

$$\|u(.,.,t)\|^2_{L^2(\mathbf{R^2})} \leq a\|\Box u\|^2_{L^2(\mathbf{R^3})} + b\|u\|^2_{L^2(\mathbf{R^3})}. \tag{5.8}$$

Taking square roots of both sides then the essential supremum in t over $\mathbf{R}$:

$$\operatorname*{ess\,sup}_{t\in\mathbf{R}} \|u(.,.,t)\|_{L^2(\mathbf{R^2})} \leq a\|\Box u\|_{L^2(\mathbf{R^3})} + b\|u\|_{L^2(\mathbf{R^3})}. \tag{5.9}$$

The result for functions in M^3 follows by the density of C_0^∞ in M^3. □

Before giving the main theorem here, we first have the following proposition:

Proposition 31. *The constant a in (5.9) can be made as small as we want.*

Proof. Take $u_r(x,y,t) = u(rx,ry,rt), r > 0$. Then

$$\|u_r(.,.,t)\|^2_{L^2(\mathbf{R^2})} = \iint_{\mathbf{R^2}} |u_r(x,y,t)|^2 dxdy = \iint_{\mathbf{R^2}} |u(rx,ry,rt)|^2 dxdy$$
$$= \iint_{\mathbf{R^2}} \frac{1}{r^2}|u(rx,ry,rt)|^2 d(rx)d(ry).$$

which implies that:

$$\|u_r(.,.,t)\|_{L^2(\mathbf{R^2})} = \frac{1}{r}\|u_r(.,.,rt)\|_{L^2(\mathbf{R^2})}.$$

We also have:

$$\|u_r\|^2_{L^2(\mathbf{R^3})} = \iiint_{\mathbf{R^3}} |u_r(x,y,t)|^2 dxdydt = \iiint_{\mathbf{R^3}} |u(rx,ry,rt)|^2 dxdydt$$
$$= \frac{1}{r^3}\iiint_{\mathbf{R^3}} |u(rx,ry,rt)|^2 d(rx)d(ry)d(rt).$$

that is

$$\|u_r\|_{L^2(\mathbf{R^3})} = \frac{1}{r^{\frac{3}{2}}}\|u\|_{L^2(\mathbf{R^3})}.$$

Finally

$$\|\Box u_r\|^2_{L^2(\mathbf{R^3})} = \iiint_{\mathbf{R^3}} |\Box u_r(x,y,t)|^2 dxdydt = \iiint_{\mathbf{R^3}} r^4|\Box u(rx,ry,rt)|^2 dxdydt$$
$$= \iiint_{\mathbf{R^3}} r|\Box u(rx,ry,rt)|^2 d(rx)d(ry)d(rt).$$

Hence

$$\|\Box u_r\|_{L^2(\mathbf{R^3})} = \sqrt{r}\|\Box u\|_{L^2(\mathbf{R^3})}.$$

Thus (5.9) [1] becomes:

$$\frac{1}{r}\operatorname*{ess\,sup}_{t\in\mathbf{R}} \|u(.,.,rt)\|_{L^2(\mathbf{R}^2)} \leq a\sqrt{r}\|\Box u\|_{L^2(\mathbf{R}^3)} + \frac{1}{r^{\frac{3}{2}}}b\|u\|_{L^2(\mathbf{R}^3)}.$$

Or

$$\operatorname*{ess\,sup}_{t\in\mathbf{R}} \|u(.,.,rt)\|_{L^2(\mathbf{R}^2)} = \operatorname*{ess\,sup}_{t\in\mathbf{R}} \|u(.,.,t)\|_{L^2(\mathbf{R}^2)} \leq ar^{\frac{3}{2}}\|\Box u\|_{L^2(\mathbf{R}^3)} + \frac{b}{\sqrt{r}}\|u\|_{L^2(\mathbf{R}^3)}.$$

Choosing r small enough makes the constant in front of $\|\Box u\|_{L^2(\mathbf{R}^3)}$ arbitrarily small. □

Now we have the following theorem:

Theorem 30. *Let □ be the wave operator defined on $L^2(\mathbf{R}^3)$. Let V be a real-valued function such that $\int_{\mathbf{R}} \|V(.,.,t)\|^2_{L^\infty(\mathbf{R}^2)}dt < \infty$. Then $\Box + V$ is self-adjoint on $\mathcal{D}(\Box)$.*

Proof. We have by the generalized Hölder's inequality:

$$\iint_{\mathbf{R}^2} |V(x,y,t)u(x,y,t)|^2 dxdy \leq \|V(.,.,t)\|^2_{L^\infty(\mathbf{R}^2)}\|u(.,.,t)\|^2_{L^2(\mathbf{R}^2)}$$
$$\leq \operatorname*{ess\,sup}_{t\in\mathbf{R}}\|u(.,.,t)\|^2_{L^2(\mathbf{R}^2)}(\|V(.,.,t)\|^2_{L^\infty(\mathbf{R}^2)}).$$

Then

$$\|Vu\|^2_{L^2(\mathbf{R}^3)} \leq \operatorname*{ess\,sup}_{t\in\mathbf{R}} \|u(.,.,t)\|^2_{L^2(\mathbf{R}^2)} \int_{\mathbf{R}} \|V(.,.,t)\|^2_{L^\infty(\mathbf{R}^2)}dt.$$

Therefore,

$$\|Vu\|^2_{L^2(\mathbf{R}^3)} \leq \left(\int_{\mathbf{R}} \|V(.,.,t)\|^2_{L^\infty(\mathbf{R}^2)}dt\right)\left(a\|\Box u\|^2_{L^2(\mathbf{R}^3)} + b\|u\|^2_{L^2(\mathbf{R}^3)}\right).$$

Since we can choose a small enough to have $a\int_{\mathbf{R}} \|V(.,.,t)\|^2_{L^\infty(\mathbf{R}^2)}dt < 1$ we conclude by the Kato-Rellich perturbation theorem that $\Box + V$ is self-adjoint on $\mathcal{D}(\Box)$. □

Remark 27. *It has been proved previously that $\Box + V$ is self-adjoint on $\mathcal{D}(\Box)$ for V real-valued and $\int_{-\infty}^{\infty} \|V(.,y)\|^2_{L^\infty(\mathbf{R})}dy < \infty$ (Theorem 27). So Theorem 30 is an analogue of that result.*

[1] We use u_r instead of u in (5.9).

Besides one has another method to find that norm of V as the method in this work is applicable to the two-dimensional case and even to any dimension and we have:

Proposition 32. *For all $a > 0$, there exists $b > 0$ such that*

$$\operatorname*{ess\,sup}_{t\in\mathbf{R}} \|u(\ldots,t)\|_{L^2(\mathbf{R}^n)} \leq a\|\Box u\|_{L^2(\mathbf{R}^{n+1})} + b\|u\|_{L^2(\mathbf{R}^{n+1})}$$

for all $u \in M^{n+1}$, $n \geq 1$.

Proof. The same as for Proposition 30 with the obvious changes. □

We also have the following theorem whose proof is a word for word translation of that of Theorem 30 [2]:

Theorem 31. *Let $\Box$ be the wave operator defined on $L^2(\mathbf{R}^{n+1})$. Let V be a real-valued function such that $\int_{-\infty}^{\infty} \|V(\ldots,t)\|^2_{L^\infty(\mathbf{R}^n)}dt < \infty$. Then $\Box + V$ is self-adjoint on $\mathcal{D}(\Box)$.*

We also have

Proposition 33. *Let V_1 be as in Theorem 31. Let $V_2 \in L^\infty(\mathbf{R}^{n+1})$ and real-valued. Let $V = V_1 + V_2$. Then $\Box + V$ is self-adjoint on $\mathcal{D}(\Box)$.*

Example 18. *Take $V(x,t) = \frac{1}{|t|^{\frac{1}{4}}}$ where $x \in \mathbf{R}^n$ and $t \in \mathbf{R}$. Then $\Box + V$ is self-adjoint on $\mathcal{D}(\Box)$ since*

$$V(x,t) = \frac{1}{|t|^{\frac{1}{4}}} = V_1(x,t) + V_2(x,t) = \frac{\mathbf{1}_{\{x\in\mathbf{R}^n,|t|<1\}}(x,t)}{|t|^{\frac{1}{4}}} + \frac{\mathbf{1}_{\{x\in\mathbf{R}^n,|t|\geq1\}}(x,t)}{|t|^{\frac{1}{4}}}$$

The result then follows since $\int_{-\infty}^{\infty} \|V_1(\ldots,t)\|^2_{L^\infty(\mathbf{R}^n)}dt < \infty$ and $V_2 \in L^\infty(\mathbf{R}^{n+1})$.

We can also improve the norm on the left hand-side of the inequality in Proposition 30 in order to get a better norm of V for which the operator $\Box + V$ will be self-adjoint. We have:

Proposition 34. *For all $a > 0$, there exists $b > 0$ such that*

$$\sum_{k=-\infty}^{\infty} \operatorname*{ess\,sup}_{k\leq t\leq k+1} \|u(.,.,t)\|^2_{L^2(\mathbf{R}^2)} \leq a\|\Box u\|^2_{L^2(\mathbf{R}^3)} + b\|u\|^2_{L^2(\mathbf{R}^3)}$$

for all $u \in M^3$.

[2] The only difference lies in considering the Cauchy problem in $\mathbf{R}^n \times \mathbf{R}^+$.

Proof. Let $u \in C_0^\infty$. We have by (5.7)

$$\|u(.,.,t)\|^2_{L^2(\mathbf{R^2})} \leq a \iint\limits_{\mathbf{R^2}} |\hat{u}(\eta, \xi, \frac{t+\alpha}{2})|^2 d\eta d\xi + b \int_{\frac{t+\alpha}{2}}^{t} \iint\limits_{\mathbf{R^2}} |\hat{f}(\eta, \xi, s)|^2 d\eta d\xi ds + c \int_{\frac{t+\alpha}{2}}^{\alpha} \iint\limits_{\mathbf{R^2}} |\hat{f}(\eta, \xi, s)|^2 d\eta d\xi ds + d \iint\limits_{\mathbf{R^2}} |\hat{u}(\eta, \xi, \alpha)|^2 d\eta d\xi.$$

Let $k \in \mathbf{Z}$ and let t and α be such that: $k \leq t \leq k+1$ and $k \leq \alpha \leq k+1$ (then $|t-\alpha| \leq 1$ which does not contradict our choice). Then $k \leq \frac{t+\alpha}{2} \leq k+1$ and so:

$$\int_{\frac{t+\alpha}{2}}^{t} \iint\limits_{\mathbf{R^2}} |\hat{f}(\eta, \xi, s)|^2 d\eta d\xi ds \leq \int_k^{k+1} \iint\limits_{\mathbf{R^2}} |\hat{f}(\eta, \xi, s)|^2 d\eta d\xi ds \text{ since } \left(\frac{t+\alpha}{2}, t\right) \subset (k, k+1)$$

and

$$\int_{\frac{t+\alpha}{2}}^{\alpha} \iint\limits_{\mathbf{R^2}} |\hat{f}(\eta, \xi, s)|^2 d\eta d\xi ds \leq \int_k^{k+1} \iint\limits_{\mathbf{R^2}} |\hat{f}(\eta, \xi, s)|^2 d\eta d\xi ds \text{ since } \left(\frac{t+\alpha}{2}, \alpha\right) \subset (k, k+1).$$

Also

$$\int_k^{k+1} \iint\limits_{\mathbf{R^2}} |\hat{u}(\eta, \xi, \frac{t+\alpha}{2})|^2 d\eta d\xi d\alpha = 2 \int_{\frac{t+k}{2}}^{\frac{t+k+1}{2}} \iint\limits_{\mathbf{R^2}} |\hat{u}(\eta, \xi, r)|^2 d\eta d\xi dr$$
$$\leq 2 \int_k^{k+1} \iint\limits_{\mathbf{R^2}} |\hat{u}(\eta, \xi, r)|^2 d\eta d\xi dr \text{ since } \left(\frac{t+k}{2}, \frac{t+k+1}{2}\right) \subset (k, k+1).$$

Then we get

$$\|u(.,.,t)\|_{L^2(\mathbf{R^2})} \leq \tilde{c} \int_k^{k+1} \iint\limits_{\mathbf{R^2}} |\hat{u}(\eta, \xi, r)|^2 d\eta d\xi dr + \tilde{c} \int_k^{k+1} \iint\limits_{\mathbf{R^2}} |\hat{f}(\eta, \xi, s)|^2 d\eta d\xi ds.$$

Hence

$$\operatorname*{ess\,sup}_{k \leq t \leq k+1} \|u(.,.,t)\|_{L^2(\mathbf{R^2})} \leq \tilde{c} \int_k^{k+1} \iint\limits_{\mathbf{R^2}} |\hat{u}(\eta, \xi, r)|^2 d\eta d\xi dr + \tilde{c} \int_k^{k+1} \iint\limits_{\mathbf{R^2}} |\hat{f}(\eta, \xi, s)|^2 d\eta d\xi ds.$$

Thus

$$\sum_{k=-\infty}^{\infty} \operatorname*{ess\,sup}_{k \leq t \leq k+1} \|u(.,.,t)\|^2_{L^2(\mathbf{R^2})} \leq a\|\Box u\|^2_{L^2(\mathbf{R^3})} + b\|u\|^2_{L^2(\mathbf{R^3})}$$

where we have used the Plancherel theorem and the fact that $u_{tt} - u_{xx} - u_{yy} = f(x, y, t)$.

Using the usual density argument allows us to obtain the desired result for M^3 functions (see the proof of Proposition 24). $\square$

Theorem 32. *Let $\square$ be the wave operator defined on $L^2(\mathbf{R}^3)$. Let V be a real-valued function such that* $\sup_{k\in\mathbf{Z}} \int_k^{k+1} \|V(.,.,t)\|^2_{L^\infty(\mathbf{R}^2)}dt < \infty$. *Then $\square + V$ is self-adjoint on $\mathcal{D}(\square)$.*

Proof. We have

$$\begin{aligned}\|Vu\|^2_{L^2(\mathbf{R}^3)} &= \sum_{k=-\infty}^{\infty} \int_k^{k+1} \iint_{\mathbf{R}^2} |V(x,y,t)u(x,y,t)|^2 dxdydt \\ &\leq \sum_{k=-\infty}^{\infty} \int_k^{k+1} \|u(.,.,t)\|^2_{L^2(\mathbf{R}^2)} \|V(.,.,t)\|^2_{L^\infty(\mathbf{R}^2)}dt.\end{aligned}$$

Hence

$$\|Vu\|^2_{L^2(\mathbf{R}^3)} \leq \sum_{k=-\infty}^{\infty} \operatorname{ess} \sup_{k\leq t\leq k+1} \|u(.,.,t)\|^2_{L^2(\mathbf{R}^2)} \int_k^{k+1} \|V(.,.,t)\|^2_{L^\infty(\mathbf{R}^2)}dt.$$

So that

$$\begin{aligned}\|Vu\|^2_{L^2(\mathbf{R}^3)} &\leq \sup_{k\in\mathbf{Z}} \int_k^{k+1} \|V(.,.,t)\|^2_{L^\infty(\mathbf{R}^2)}dt \sum_{k=-\infty}^{\infty} \operatorname{ess} \sup_{k\leq t\leq k+1} \|u(.,.,t)\|^2_{L^2(\mathbf{R}^2)} \\ &\leq \sup_{k\in\mathbf{Z}} \int_k^{k+1} \|V(.,.,t)\|^2_{L^\infty(\mathbf{R}^2)}dt \left(a\|\square u\|^2_{L^2(\mathbf{R}^3)} + b\|u\|^2_{L^2(\mathbf{R}^3)}\right)\end{aligned}$$

where a can arbitrarily small (by the same argument). Hence $\square+V$ is self-adjoint on $\mathcal{D}(\square)$. $\square$

Example 19. *We show that Theorem 32 is stronger than Theorem 30 by giving an example. We want a $\varphi : \mathbf{R} \mapsto \mathbf{R}$ such that $\varphi \notin L^\infty(\mathbf{R})$,*

$$\sup_{k\in\mathbf{Z}} \int_k^{k+1} |\varphi(t)|^2 dt < \infty \text{ and } \sum_{k=-\infty}^{\infty} \int_k^{k+1} |\varphi(t)|^2 dt = \infty.$$

The condition on the right hand side of the last equation means that $\varphi \notin L^2(\mathbf{R})$. We do not want $\varphi \in L^2(\mathbf{R})$ only because this case is already included in the class of Vs that was found in Theorem 30.

Fix $-\frac{1}{2} < \alpha < 0$ *and define* φ *in each interval* $k < t \leq k+1$, $k \in \mathbf{Z}$, *by*

$$\varphi(t) = (t-k)^{\alpha}.$$

Then φ *is certainly not essentially bounded on* $\mathbf{R}$. *Moreover one has*

$$\int_k^{k+1} |\varphi(t)|^2 dt = \int_k^{k+1} (t-k)^{2\alpha} dt = \frac{1}{2\alpha+1}.$$

Hence

$$\sup_{k\in\mathbf{Z}} \int_k^{k+1} |\varphi(t)|^2 dt = \sup_{k\in\mathbf{Z}} \frac{1}{2\alpha+1} = \frac{1}{2\alpha+1} < \infty.$$

And

$$\sum_{k=-\infty}^{\infty} \frac{1}{2\alpha+1} = \infty.$$

Now take $V(x,y,t) = \varphi(t)$. *Thus* $\Box + V$ *is self-adjoint on* $\mathcal{D}(\Box)$.

Remark 28. *The* V *constructed in Example 19 does not satisfy the conditions of Proposition 33.*

Again we have the same results in n-dimensions.

Proposition 35. *For all* $a > 0$, *there exists* $b > 0$ *such that*

$$\sum_{k=-\infty}^{\infty} \operatorname{ess} \sup_{k\leq t\leq k+1} \|u(\ldots,t)\|^2_{L^2(\mathbf{R}^n)} \leq a\|\Box u\|^2_{L^2(\mathbf{R}^{n+1})} + b\|u\|^2_{L^2(\mathbf{R}^{n+1})}$$

for all $u \in M^{n+1}$.

Theorem 33. *Let* $\Box$ *be the wave operator defined on* $L^2(\mathbf{R}^{n+1})$. *Let* V *be a real-valued function such that* $\sup_{k\in\mathbf{Z}} \int_k^{k+1} \|V(\ldots,t)\|^2_{L^\infty(\mathbf{R}^n)} dt < \infty$. *Then* $\Box + V$ *is self-adjoint on* $\mathcal{D}(\Box)$.

By going back to equation (5.4) we can do more, i.e., we have a better estimate than the one in Proposition 30.

Proposition 36. *For all* $a > 0$, *there exists* $b > 0$ *such that*

$$\operatorname{ess}\sup_{t\in\mathbf{R}} \|u(.,.,t)\|_{L^r(\mathbf{R}^2)} \leq a\|\Box u\|_{L^2(\mathbf{R}^3)} + b\|u\|_{L^2(\mathbf{R}^3)}, \text{ where } 2 \leq r < 4.$$

for all $u \in M^3$.

We first need several lemmas.

Lemma 13 (Sobolev's inequality). *For $f \in H^1(\mathbf{R^2})$ the inequality*

$$\|f\|_q^2 \leq a\|\nabla f\|^2_{L^2(\mathbf{R^2})^2} + b\|f\|^2_{L^2(\mathbf{R^2})} \text{ holds for all } 2 \leq q < \infty.$$

For a proof see [7] (Theorem 8.5). We prefer here to use a proof which uses Fourier transforms where we do not care about the constant a because we can always make it as small as we want.

Proof. Since $f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \in L^2(\mathbf{R^2})$, then by the Plancherel theorem we get: $\hat{f}, \eta\hat{f}, \xi\hat{f} \in L^2(\mathbf{R^2})$ and hence $\hat{f}, |\eta|\hat{f}, |\xi|\hat{f} \in L^2(\mathbf{R^2})$ which gives us:
$(1 + |\eta| + |\xi|)\hat{f} \in L^2(\mathbf{R^2})$. Now let $2 \leq q < \infty$ and let p be the conjugate of q (this implies that $1 < p \leq 2$). So:

$$\|\hat{f}\|^p_{L^p(\mathbf{R^2})} = \|(1 + |\eta| + |\xi|)^{-p}.(1 + |\eta| + |\xi|)^p|\hat{f}|^p\|_{L^1(\mathbf{R^2})}.$$

Then by Hölder's inequality:

$$\|\hat{f}\|^p_{L^p(\mathbf{R^2})} \leq \|(1 + |\eta| + |\xi|)^{-p}\|_{L^r(\mathbf{R^2})}.\|(1 + |\eta| + |\xi|)^p|\hat{f}|^p\|_{L^s(\mathbf{R^2})}$$

where $\frac{1}{r} + \frac{1}{s} = 1$. Let

$$c = \|(1 + |\eta| + |\xi|)^{-p}\|_{L^r(\mathbf{R^2})}$$

which is finite if $pr > 2$.
Now define $r = \frac{2}{2-p}$ (this gives us $pr > 2$ so c is finite) and define $s = \frac{2}{p}$ (the exponents s and r are conjugate).
We now use the fact that $f \in H^1(\mathbf{R^2})$. We have

$$\|(1 + |\eta| + |\xi|)^p|\hat{f}|^p\|_{L^s(\mathbf{R^2})} = \left(\iint_{\mathbf{R^2}} (1 + |\eta| + |\xi|)^{ps}|\hat{f}|^{ps}d\eta d\xi\right)^{\frac{p}{2}}$$

$$= \left(\sqrt{\iint_{\mathbf{R^2}} (1 + |\eta| + |\xi|)^2|\hat{f}|^2 d\eta d\xi}\right)^p = \|(1 + |\eta| + |\xi|)|\hat{f}|\|^p_{L^2(\mathbf{R^2})}.$$

Thus for $1 < p \leq 2$ we have:

$$\|\hat{f}\|_{L^p(\mathbf{R}^2)} \leq c^{\frac{1}{p}}\|(1+|\eta|+|\xi|)|\hat{f}|\|_{L^2(\mathbf{R}^2)}.$$

Hence

$$\|\hat{f}\|_{L^p(\mathbf{R}^2)} \leq c^{\frac{1}{p}}\|\hat{f}\|_{L^2(\mathbf{R}^2)} + c^{\frac{1}{p}}\|(|\eta|+|\xi|)|\hat{f}|\|_{L^2(\mathbf{R}^2)}.$$

Thus:

$$\|\hat{f}\|_{L^p(\mathbf{R}^2)} \leq \tilde{c}\|\hat{f}\|_{L^2(\mathbf{R}^2)} + \tilde{c}\|\widehat{\nabla f}\|_{L^2(\mathbf{R}^2)^2}.$$

Thus by the Hausdorff-Young inequality (Theorem 5), we deduce that for $2 \leq q < \infty$:

$$\|f\|_{L^q(\mathbf{R}^2)} \leq a\|\nabla f\|_{L^2(\mathbf{R}^2)^2} + b\|f\|_{L^2(\mathbf{R}^2)}.$$

□

Lemma 14. *Let $w \in L^1_{loc}(\mathbf{R}^2)$. Assume for all $\epsilon > 0$, there exist V, g such that $w = V + g$, $\|V\|_2 \leq d\epsilon$ and $\|g\|_q \leq \frac{c}{\epsilon}$; c, d being two constants. Then $w \in L^p_w(\mathbf{R}^2)$ and*

$$\|w\|_{p,w} \leq \tilde{c}c + \tilde{c}d \tag{5.10}$$

where $p = \frac{4q}{2+q}$ and $2 \leq q < \infty$.

Proof. Let $E_\lambda = \{x : |u(x)| \geq \lambda\}$. Then, since $u = V + g$

$$E_\lambda \subset \{x : |V(x)| \geq \frac{\lambda}{2}\} \cup \{x : |g(x)| \geq \frac{\lambda}{2}\}.$$

So that

$$|E_\lambda| \leq |\{x : |V(x)| \geq \frac{\lambda}{2}\}| + |\{x : |g(x)| \geq \frac{\lambda}{2}\}|.$$

Then using Chebyshev inequality

$$|E_\lambda| \leq 4\lambda^{-2}\|V\|_2^2 + 2^q\lambda^{-q}\|g\|_q^q.$$

But we have the freedom to choose any $\epsilon(\lambda) > 0$. So take $\epsilon(\lambda) = \lambda^b$ (b to be determined). With this choice we obtain:

$$|E_\lambda| \leq (4d^2\lambda^{2b-2} + 2^q c^q \lambda^{-bq-q}) \text{ or}$$

$$\lambda^p |E_\lambda| \leq 4d^2 \lambda^{p+2b-2} + 2^q c^q \lambda^{p-bq-q}.$$

So in order that u belongs to $L^p_w(\mathbf{R}^2)$ it suffices to have:

$$\sup_{\lambda>0}[4d^2\lambda^{p+2b-2} + 2^q c^q \lambda^{p-bq-q}] < \infty.$$

which occurs if $b = \frac{2-p}{2}$ and $p = \frac{4q}{2+q}$.
Therefore,

$$\|w\|_{p,w} \leq \tilde{c}d^{\frac{2}{p}} + \tilde{c}c^{\frac{q}{p}}.$$

Finally, apply the same method as in the end of the proof of Theorem 3 to establish (5.10). □

We shall now prove Proposition 36.

Proof. Let $u \in C_0^\infty(\mathbf{R}^3)$. For a fixed t, let $w(x,y) = u(x,y,t)$. We let

$$\hat{V}(\eta,\xi) = 2\hat{u}\left(\eta,\xi,\frac{t+\alpha}{2}\right)\cos\left[\sqrt{\eta^2+\xi^2}\left(\frac{t-\alpha}{2}\right)\right] - \hat{u}(\eta,\xi,\alpha) \tag{5.11}$$

and

$$\hat{g}(\eta,\xi) = \int_{\frac{t+\alpha}{2}}^{t} \hat{f}(\eta,\xi,s)\frac{\sin\sqrt{\eta^2+\xi^2}(t-s)}{\sqrt{\eta^2+\xi^2}}ds + \int_{\frac{t+\alpha}{2}}^{t} \hat{f}(\eta,\xi,s)\frac{\sin\sqrt{\eta^2+\xi^2}(\alpha-s)}{\sqrt{\eta^2+\xi^2}}ds \tag{5.12}$$

where $f = \Box u$ and α is yet to be chosen. We observe that both $\hat{V}$ and $\hat{g}$ depend on α.
Note that $\hat{w} = \hat{V} + \hat{g}$, by (5.4). It is clear that $\hat{V} \in L^2(\mathbf{R}^2)$ and hence $\hat{g} \in L^2(\mathbf{R}^2)$. We let V and g be their inverse Fourier transforms, so $w = V + g$. We aim to apply Lemma 14 by showing that, given $\epsilon > 0$, we can choose α in such a way that $\|V\|_2 \leq d\epsilon$ and $\|g\|_q \leq \frac{c}{\epsilon}$. Here, q will satisfy $2 \leq q < \infty$, and c and d will be constants depending on $\|u\|_2$ and $\|\Box u\|_2$. The estimate for $\|g\|_q$ will follow from Lemma 13 once we have shown that $\nabla f \in L^2(\mathbf{R}^2)^2$ with suitable estimates.

We shall only consider the term $\hat{u}\left(\eta,\xi,\frac{t+\alpha}{2}\right)\cos\left[\sqrt{\eta^2+\xi^2}\left(\frac{t-\alpha}{2}\right)\right]$ in (5.11) and the term $\int_{\frac{t+\alpha}{2}}^{t} \hat{f}(\eta,\xi,s)\frac{\sin\sqrt{\eta^2+\xi^2}(t-s)}{\sqrt{\eta^2+\xi^2}}ds$ in (5.12) and we denote them by $\hat{V}(\eta,\xi)$ and $\hat{g}(\eta,\xi)$. The proofs for the other two terms are similar.

Let us start by showing that $\nabla f \in L^2(\mathbf{R}^2)^2$. Since $\hat{g} \in L^2(\mathbf{R}^2)$ and since

$$-i\eta\frac{\sin\sqrt{\eta^2+\xi^2}(t-s)}{\sqrt{\eta^2+\xi^2}}, -i\xi\frac{\sin\sqrt{\eta^2+\xi^2}(t-s)}{\sqrt{\eta^2+\xi^2}} \in L^\infty(\mathbf{R}^2),$$

then $-i\eta\hat{g}, -i\xi\hat{g} \in L^2(\mathbf{R}^2)$ which implies that $\widehat{\nabla g} \in L^2(\mathbf{R}^2)^2$ since

$$\begin{aligned}
\|\nabla g\|^2_{L^2(\mathbf{R}^2)^2} &= \iint\limits_{\mathbf{R}^2} |\nabla g(x,y)|^2 dxdy = \|\widehat{\nabla g}\|^2_{L^2(\mathbf{R}^2)^2} \\
&= \iint\limits_{\mathbf{R}^2} |\eta\hat{g}(\eta,\xi)|^2 d\eta d\xi + \iint\limits_{\mathbf{R}^2} |\xi\hat{g}(\eta,\xi)|^2 d\eta d\xi, \text{ but}
\end{aligned}$$

$$|\eta\hat{g}(\eta,\xi)| \leq \int_{\frac{t+\alpha}{2}}^t |\hat{f}(\eta,\xi,s)|\left|\frac{\eta\sin\sqrt{\eta^2+\xi^2}(t-s)}{\sqrt{\eta^2+\xi^2}}\right|ds \leq \int_{\frac{t+\alpha}{2}}^t |\hat{f}(\eta,\xi,s)|ds.$$

Then by using Cauchy-Schwarz

$$|\eta\hat{g}(\eta,\xi)| \leq \sqrt{\int_{\frac{t+\alpha}{2}}^t |\hat{f}(\eta,\xi,s)|^2 ds}\sqrt{\int_{\frac{t+\alpha}{2}}^t 1^2 ds} \leq a\sqrt{\int_{\frac{t+\alpha}{2}}^t |\hat{f}(\eta,\xi,s)|^2 ds}|t-\alpha|^{\frac{1}{2}}.$$

And so

$$\eta^2|\hat{g}(\eta,\xi)|^2 \leq a^2|t-\alpha|\int_{\frac{t+\alpha}{2}}^t |\hat{f}(\eta,\xi,s)|^2 ds.$$

By applying the same method one will also get

$$\xi^2|\hat{g}(\eta,\xi)|^2 \leq a^2|t-\alpha|\int_{\frac{t+\alpha}{2}}^t |\hat{f}(\eta,\xi,s)|^2 ds.$$

Now we choose α such that $|t-\alpha| \leq \frac{1}{\epsilon^2}$. Hence

$$\begin{aligned}
\|\nabla g\|^2_{L^2(\mathbf{R}^2)^2} &= \iint\limits_{\mathbf{R}^2} |\eta\hat{g}(\eta,\xi)|^2 d\eta d\xi + \iint\limits_{\mathbf{R}^2} |\xi\hat{g}(\eta,\xi)|^2 d\eta d\xi \\
&\leq \frac{2a^2}{\epsilon^2}\iint\limits_{\mathbf{R}^2}\int_{\frac{t+\alpha}{2}}^t |\hat{f}(\eta,\xi,s)|^2 d\eta d\xi ds.
\end{aligned}$$

Thus

$$\|\nabla g\|_{L^2(\mathbf{R}^2)^2} \leq \frac{c}{\epsilon}\|\Box u\|_{L^2(\mathbf{R}^3)}.$$

In a similar way one gets $\|g\|_{L^2(\mathbf{R}^2)} \leq \frac{c}{\epsilon^{\frac{3}{2}}}\|\Box u\|_{L^2(\mathbf{R}^3)}$. So for $\epsilon \geq 1$ we have $\|g\|_{L^2(\mathbf{R}^2)} \leq$

$\frac{c}{\epsilon}\|\Box u\|_{L^2(\mathbf{R}^3)}$. In such case we get

$$\|g\|_{L^2(\mathbf{R}^2)} \leq \frac{c}{\epsilon}\|\Box u\|_{L^2(\mathbf{R}^3)} \text{ so } \|g\|_{L^q(\mathbf{R}^2)} \leq \frac{c}{\epsilon}\|\Box u\|_{L^2(\mathbf{R}^3)}$$

by Lemma 13 where $2 \leq q < \infty$.

We also obtain the estimates when using $\int_{\frac{t+\alpha}{2}}^{\alpha} \hat{f}(\eta, \xi, s)\frac{\sin\sqrt{\eta^2+\xi^2}(\alpha-s)}{\sqrt{\eta^2+\xi^2}}ds$.

Now we come back to the case $\epsilon < 1$ after finding the bound for V which will be needed for the case $\epsilon < 1$.

For V we have

$$\hat{V}(\eta,\xi) = 2\hat{u}\left(\eta, \xi, \frac{t+\alpha}{2}\right)\cos\left[\sqrt{\eta^2+\xi^2}\left(\frac{t-\alpha}{2}\right)\right], \text{ then}$$

$$\|\hat{V}\|^2_{L^2(\mathbf{R}^2)} \leq 4\iint_{\mathbf{R}^2} |\hat{u}(\eta, \xi, \frac{t+\alpha}{2})|^2 d\eta d\xi$$

and by integrating with respect to $\alpha \in \{\alpha : |t-\alpha| \leq \frac{1}{\epsilon^2}\}$ one obtains:

$$\int_{t-\frac{1}{\epsilon^2}}^{t+\frac{1}{\epsilon^2}} \|\hat{V}\|^2_{L^2(\mathbf{R}^2)}d\alpha \leq 4\iint_{\mathbf{R}^2}\int_{t-\frac{1}{\epsilon^2}}^{t+\frac{1}{\epsilon^2}} |\hat{u}(\eta, \xi, \frac{t+\alpha}{2})|^2 d\eta d\xi d\alpha \leq 4\|\hat{u}\|^2_{L^2(\mathbf{R}^3)}.$$

Then there exists $\alpha \in (t-\frac{1}{\epsilon^2}, t+\frac{1}{\epsilon^2})$ such that

$$\|\hat{V}\|^2_{L^2(\mathbf{R}^2)} \leq \frac{4\|\hat{u}\|^2_{L^2(\mathbf{R}^3)}}{\frac{2}{\epsilon^2}} \text{and so } \|V\|_{L^2(\mathbf{R}^2)} \leq d\epsilon\|u\|_{L^2(\mathbf{R}^3)}.$$

The same applies to $\hat{u}(\eta, \xi, \alpha)$ and we will get the same estimate.

Now for the case $\epsilon \leq 1$ one has to use $\hat{g} = \hat{w} - \hat{V}$ to obtain

$$\|g\|_{L^2(\mathbf{R}^2)} \leq \tilde{c}(\|w\|_{L^2(\mathbf{R}^2)} + \|V\|_{L^2(\mathbf{R}^2)}).$$

Then using the estimate for V, that is, $\|V\|_{L^2(\mathbf{R}^2)} \leq d\epsilon\|u\|_{L^2(\mathbf{R}^3)}$ and Equation (5.8) give us

$$\|g\|_{L^2(\mathbf{R}^2)} \leq a\|\Box u\|_{L^2(\mathbf{R}^3)} + b\|u\|_{L^2(\mathbf{R}^3)} + \epsilon d\|u\|_{L^2(\mathbf{R}^3)} \leq \tilde{c}(\|\Box u\|_{L^2(\mathbf{R}^3)} + b\|u\|_{L^2(\mathbf{R}^3)})$$

since $\epsilon \leq 1$. Thus

$$\|g\|_{L^q(\mathbf{R}^2)} \leq \frac{\tilde{c}}{\epsilon}\left[\|\Box u\|_{L^2(\mathbf{R}^3)} + \|u\|_{L^2(\mathbf{R}^3)}\right].$$

So one has by Lemma 13

$$\|w\|_{p,w} \leq \tilde{c}\|u\|_{L^2(\mathbf{R}^3)} + \tilde{c}\|\Box u\|_{L^2(\mathbf{R}^3)}. \tag{5.13}$$

Equation (5.13) shows that $w \in L^p_w$ and since $w \in L^2_w$ (in fact, it belongs to L^2) then by (1.1), $w \in L^r$ for $2 < r < p$ and

$$\|w\|_r \leq \tilde{c}\|w\|_{2,w} + \tilde{c}\|w\|_{p,w}. \tag{5.14}$$

We then get

$$\|w\|_{L^r(\mathbf{R}^2)} \leq \tilde{c}\|w\|_{2,w} + \tilde{c}\|w\|_{p,w} \leq \tilde{c}(\|w\|_2 + \|\Box u\|_{L^2(\mathbf{R}^3)} + \|u\|_{L^2(\mathbf{R}^3)}).$$

Now taking the essential supremum in t over $\mathbf{R}$ and using Proposition 30 shows that

$$\operatorname*{ess\,sup}_{t\in\mathbf{R}}\|u(.,.,t)\|_{L^r(\mathbf{R}^2)} \leq \tilde{c}\|\Box u\|_{L^2(\mathbf{R}^3)} + \tilde{c}\|u\|_{L^2(\mathbf{R}^3)}. \tag{5.15}$$

Equation (5.15) holds whenever $2 \leq r < 4$ since we have the constraints: $p = \frac{4q}{2+q}$, $2 \leq q < \infty$ and $2 \leq r < p$.
Finally, to obtain the result for arbitrary $\varphi \in M^3$ we use the usual approximation argument. □

We also have the result in higher dimensions.

Proposition 37. *For all $a > 0$, there exists $b > 0$ such that*

$$\operatorname*{ess\,sup}_{t\in\mathbf{R}}\|u(.,.,t)\|_{L^r(\mathbf{R}^n)} \leq a\|\Box u\|_{L^2(\mathbf{R}^{n+1})} + b\|u\|_{L^2(\mathbf{R}^{n+1})}$$

for all $u \in M^{n+1}$ where $n \geq 2$ and where $2 \leq r < \frac{2n}{n-1}$.

We are going to need another Sobolev's inequality in higher dimensions.

Proposition 38. *Let $n \geq 2$. Let $f \in H^1(\mathbf{R}^n)$ then $f \in L^q(\mathbf{R}^n)$ for $2 \leq q < \frac{2n}{n-2}$ and we have:*

$$\|f\|_q \leq a\|\nabla f\|_{L^2(\mathbf{R}^n)^n} + b\|f\|_{L^2(\mathbf{R}^n)}.$$

Remark 29. *Observe that the case $n = 2$ gives us $2 \leq q < \infty$ which was Lemma 13.*

Remark 30. *Observe that the proof in [7] (Theorem 8.3), is true even for the case $q = \frac{2n}{n-2}$. But for our problem we do not mind whether this q is sharp or not.*

We also have the following theorem:

Theorem 34. *Let $\Box$ be the wave operator defined on $L^2(\mathbf{R^3})$. Let V be a real-valued function such that $\int_{-\infty}^{\infty} \|V(.,.,t)\|^2_{L^s(\mathbf{R^2})} dt < \infty$ where $\frac{1}{s} = \frac{1}{2} - \frac{1}{r}$ and $2 \leq r < 4$. Then $\Box + V$ is self-adjoint on $\mathcal{D}(\Box)$.*

Proof. We have by the generalized Hölder's inequality, for $\frac{1}{2} = \frac{1}{r} + \frac{1}{s}$, $2 \leq r < 4$:

$$\iint_{\mathbf{R^2}} |V(x,y,t)u(x,y,t)|^2 dxdy \leq \|V(.,.,t)\|^2_{L^s(\mathbf{R^2})} \|u(.,.,t)\|^2_{L^r(\mathbf{R^2})}$$

$$\leq \operatorname*{ess\,sup}_{t \in \mathbf{R}} \|u(.,.,t)\|^2_{L^r(\mathbf{R^2})} (\|V(.,.,t)\|^2_{L^s(\mathbf{R^2})}).$$

Then by using Proposition 36 and by integrating with respect to t over $\mathbf{R}$ one gets

$$\|Vu\|^2_{L^2(\mathbf{R^3})} \leq a \left(\int_{\mathbf{R}} \|V(.,.,t)\|^2_{L^s(\mathbf{R^2})} dt \right) \|\Box u\|^2_2 + b \left(\int_{\mathbf{R}} \|V(.,.,t)\|^2_{L^s(\mathbf{R^2})} dt \right) \|u\|^2_2.$$

Since we can choose a small enough to have $a \int_{\mathbf{R}} \|V(.,.,t)\|^2_{L^\infty(\mathbf{R^2})} dt < 1$ we conclude by the Kato-Rellich perturbation theorem that $\Box + V$ is self-adjoint on $\mathcal{D}(\Box)$. $\Box$

Also

Theorem 35. *Let $n \geq 3$. Let $\Box$ be the wave operator defined on $L^2(\mathbf{R^{n+1}})$. Let V be a real-valued function such that $\int_{-\infty}^{\infty} \|V(...,t)\|^2_{L^s(\mathbf{R^n})} dt < \infty$ for $\frac{1}{s} = \frac{1}{2} - \frac{1}{r}$ and $2 \leq r < \frac{2n}{n-1}$. Then $\Box + V$ is self-adjoint on $\mathcal{D}(\Box)$.*

5.3 Counterexamples:

We show that there exists a $\varphi \in L^2(\mathbf{R^n})$ such that $\Box\varphi \in L^2(\mathbf{R^n})$ and $\varphi \notin L^\infty(\mathbf{R^n})$. We do the same as for the second proof of Proposition 27. So one only need show that $f = \frac{1}{1+|t^2 - x_2^2 - ... - x_n^2|} \notin L^2(\mathbf{R^n})$ (in fact f is not in $L^p(\mathbf{R^n})$ for any $1 \leq p < \infty$). We have

$$\|f\|^2_2 = \int_{\mathbf{R^n}} \frac{dtdx_2...dx_n}{(1 + |t^2 - x_2^2 - ... - x_n^2|)^2} \geq \int_D \frac{dtdx_2...dx_n}{(1 + x_2^2 + ... + x_n^2 - t^2)^2}$$

where $D = \{(x_2, ...x_n, t) \in \mathbf{R}^n | x_2^2 + ... + x_n^2 \geq t^2, t > 0\}$. Now using "generalized polar coordinates" we get

$$\|f\|_2^2 \geq c \int_0^\infty \int_t^\infty \frac{r^{n-2} dr dt}{(1 + r^2 - t^2)^2} = c \int_0^\infty \int_t^\infty \frac{r.r^{n-3} dr dt}{(1 + r^2 - t^2)^2}.$$

So

$$\|f\|_2^2 \geq c \int_0^\infty t^{n-3} dt \int_t^\infty \frac{r dr}{(1 + r^2 - t^2)^2} = c \int_0^\infty \frac{1}{2} t^{n-3} dt = \infty.$$

Remark 31. *As it is known for the Laplacian that if φ is in $L^2(\mathbf{R}^n)$ such that $-\Delta\varphi \in L^2(\mathbf{R}^n)$ then $\varphi \in L^\infty(\mathbf{R}^n)$ for $n \leq 3$ and $\varphi \in L^q(\mathbf{R}^n)$ for $n \geq 4$ and where $2 \leq q < \frac{2n}{n-4}$ (c.f. Theorem 6). Here we show the existence of a $\varphi \in L^2(\mathbf{R}^4)$ such that $-\Delta\varphi \in L^2(\mathbf{R}^4)$ and while $\varphi \notin L^\infty(\mathbf{R}^4)$.*

Proposition 39. *Let $\varphi \in L^2(\mathbf{R}^4)$ such that $-\Delta\varphi \in L^2(\mathbf{R}^4)$. Then φ need not be essentially bounded on $\mathbf{R}^4$.*

Proof. One only need check that

$$(x, y, z, t) \mapsto f(x, y, z, t) = \frac{1}{1 + x^2 + y^2 + z^2 + t^2} \notin L^2(\mathbf{R}^4)$$

(which is an easy integration exercise). Then one has only to apply the same method as the second proof of Proposition 27. □

We also show that there exists a real-valued $V \in L^2_{loc}(\mathbf{R}^{n+1})$ such that $\Box + V$ is not essentially self-adjoint on C_0^∞. We have

Proposition 40. *Let $\Box$ be the wave operator in n-dimensions defined on $L^2(\mathbf{R}^{n+1})$. Then there exists a real-valued $V \in L^2_{loc}(\mathbf{R}^{n+1})$ such that $\Box + V$ is not essentially self-adjoint on C_0^∞.*

Proof. We know by Example 16 that $-\frac{d^2}{dx_1^2} - x_1^4$ is not essentially self-adjoint on $C_0^\infty(\mathbf{R})$ hence the ODE:

$$-\frac{d^2}{dx_1^2} f_1(x_1) - x_1^4 f_1(x_1) = i f_1(x_1)$$

has a non-zero solution in $L^2(\mathbf{R})$. We can say the same thing about:

$$-\frac{d^2}{dx_2^2} f_2(x_2) - x_2^4 f_2(x_2) = i f_2(x_2)$$

and

$$-\frac{d^2}{dx_n^2} f_n(x_n) - x_n^4 f_n(x_n) = i f_n(x_n)$$

Also the following ODE has a non-zero solution in $L^2(\mathbf{R})$:

$$-\frac{d^2}{dt^2} g(t) - t^4 g(t) = -ig(t)$$

or

$$\frac{d^2}{dt^2} g(t) + t^4 g(t) = +ig(t).$$

Multiplying each of the previous equations by the functions that are solutions to other equations. For instance we multiply the first equation by $f_2(x_2)...f_n(x_n).g(t)$ and so on. Then by adding them up together, we have

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} - \dots - \frac{\partial^2}{\partial x_n^2}\right) f_1(x_1)...f_n(x_n)g(t) + \left(t^4 - \sum_{k=1}^{n} x_k^4\right) f_1(x_1)...f_n(x_n)g(t)$$
$$= (n+1)if_1(x_1)...f_n(x_n)g(t). \tag{5.16}$$

Take $\varphi(x_1, ..., x_n, t) = f_1(x_1)...f_n(x_n)g(t)$. Since $f_1, ..., f_n, g$ are all in $L^2(\mathbf{R})$ then φ is in $L^2(\mathbf{R^{n+1}})$ and is a solution of (5.16).

So $V(x_1, ..., x_n, t) = t^4 - \sum_{k=1}^{n} x_k^4 \in L^2_{loc}(\mathbf{R^{n+1}})$ and

$$\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} - \dots - \frac{\partial^2}{\partial x_n} + t^4 - \sum_{k=1}^{n} x_k^4$$

is not essentially self-adjoint on $C_0^\infty(\mathbf{R^{n+1}})$. □

5.4 Open problems

In this chapter there are more open problems than the previous one. They are:

1) Is $\Box + V$ self-adjoint for V real-valued and in $L^2(\mathbf{R^n})$? or essentially self-adjoint on C_0^∞? (see comments on Section 4.5, Question 1).

2) Assume $V \geq 0$ and such that $V \in L^2_{loc}(\mathbf{R^n})$. Is $\Box + V$ essentially self-adjoint on C_0^∞?

(also, see comments on Section 4.5).

3) Do we have $M^n \subset BMO(\mathbf{R^n})$? Observe that for $n = 2$ we do have $M^2 \subset BMO(\mathbf{R^2})$ (see Remark 23).

4) If Question 3) has a negative answer then do we have $M^n \subset L^p(\mathbf{R^n})$ for some $p > 2$? This question may have a negative answer simply because we have not obtained any global estimate of the type

$$\|f\|_p \leq a\|\Box f\|_2 + b\|f\|_2,$$

for any $p > 2$ (p is far from infinity).

We say a few words concerning this question. There are known estimates for the wave equation of the type

$$\|f\|_q \leq C\|\Box f\|_p$$

for some p and q (see [20] and [21]) but none of these is helpful for our purpose. For instance, The estimate, J. Harmse gets in [20], is

Theorem 36. *Assume $n \geq 2$. Suppose $\frac{1}{p} - \frac{1}{q} = \frac{2}{n+1}$ and*

$$\frac{n+1}{2n} - \frac{2}{n+1} < \frac{1}{q} < \frac{n-1}{2n}. \tag{5.17}$$

Then there is a constant C such that for every $f \in C_0^\infty(\mathbf{R^{n+1}})$,

$$\|f\|_q \leq C\|\Box f\|_p. \tag{5.18}$$

If we want to apply this theorem to our problem one has to start with $p = 2$. But, with this choice, $q = \frac{2(n+1)}{n-3}$ does not satisfy the condition (5.17) for any $n \geq 2$.

Also, in [21], it is proved that

$$\|f\|_q \leq M\|\Box u\|_p$$

for $q = \frac{2(n+1)}{n-1}$ and $p = \frac{2(n+1)}{n+3}$. Observe that the case $p = 2$ cannot occur in this case.

Finally, more 'Strichartz estimates' have been proved since Strichartz's paper [21] and one of the important papers is [22].

5) We have by Proposition 37 an estimate which is true for $2 \leq r < \frac{2n}{n-1}$. So the natural

question is: can we push the result beyond $\frac{2n}{n-1}$, i.e., can we have the following estimate for $r \geq \frac{2n}{n-1}$ and $n \geq 3$:

$$\operatorname*{ess\,sup}_{t \in \mathbf{R}} \|u(\ldots, t)\|_{L^r(\mathbf{R}^n)} \leq a\|\Box u\|_{L^2(\mathbf{R}^{n+1})} + b\|u\|_{L^2(\mathbf{R}^{n+1})}?$$

5.5 Conclusion

Finally, most of the results obtained in Chapters 4 and 5 (mainly Theorem 28, Proposition 25, Proposition 27 (the first proof), Proposition 30, Proposition 37 and Theorem 35) form a paper by myself [23] which has been accepted for publication.

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