WARD IDENTITIES AND VECTOR-BOSON FIELD THEORIES

Thesis

Submitted by

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ABSTRACT

The object of this thesis is to investigate, using Ward identities, two aspects of vector-boson field theories.

The first is to examine, in detail, how the renormalisation counter-terms for gauge field theories are accommodated without destroying the symmetry or corresponding Ward identities. In Chapter One the wave function and coupling constant renormalisations are studied and in Chapter Two the mass renormalisations. The conclusion is that, although there is complete freedom of choice of subtraction points for the wave function and coupling constant, the mass renormalisations are not so clear and may be restricted depending on the theory.

The second topic is the massive Yang-Mills Lagrangian. In Chapter Three, we investigate the Ward identities, and their implications, for the tree approximation. In Chapter Four, we develope the Ward identities to all orders. The massive Yang-Mills Lagrangian is shown to be identical to a Lagrangian with transverse vector-boson propagators and a compensating scalar Lagrangian with an infinite series of interactions. The Lagrangian is identical to that of Boulware which was developed in the path integral formalism. The Ward identity approach we use is shown to be equivalent to Veltman's in Chapter Five. Furthermore, it is shown that it is the S-matrices which are identical. In Chapter Six, other possible equivalent formalisms of the massive Yang-Mills Lagrangian are investigated. The formalism of Hsu & Sudarshan is shown to be for mixed spin-one spin-zero fields and not pure spin-one fields as required.

Finally a formulation is discussed which, in conjunction

with the dimensional regularisation scheme of 't. Hooft and Veltman, generates the identical S-matrix from Feynman rules which are renormalisable according to power-counting.

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INTRODUCTION

Any attempt to construct a plausible field theory of the fundamental interactions must include spin one bosons as one with zero mass, viz. the photon, is known to mediate the electromagnetic interaction. It is also suspected that massive ones are present in the weak interactions.

The presence of vector bosons gives rise to propagators with momenta terms present in the numerator. The free Lagrangian for a massive spin-one field $A^{\mu}(x)$ is

$$-\frac{1}{4}(\partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu})(\partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}) + \frac{1}{2}M^2 A_{\mu}A^{\mu} \qquad (0.1)$$

for which the propagator is $\dfrac{g^{\mu\nu} - k^{\mu}k^{\nu}/M^2}{k^2 - M^2 + i\varepsilon}$ [†]. Theories with these propagators in general are not renormalisable; this can be assessed by simple power counting of the momenta[1-3]. However, this is not a strict criterion and a Lagrangian may be renormalisable although power counting indicates otherwise. To investigate the contribution of the $k^{\mu}k^{\nu}$ terms of the propagator it has been found useful to exploit Ward-Takahashi identity relations between amplitudes with physical polarisation vectors on external lines replaced by momenta contractions. The simplest example of Ward identities occurs in Quantum Electrodynamics where they may be expressed as

$$q_{\mu}\,\Gamma^{\mu\nu\cdots}(q, k\ldots) = 0 . \qquad (0.2)$$

[†] The metric $g^{oo} = 1$, $g^{11} = g^{22} = g^{33} = -1$ is used here.

The $\Gamma^{\mu\nu\cdots}$ is any proper amplitude in which q^μ is the ingoing momentum of a particular photon.

a) The Massive Yang-Mills Lagrangian

A theory, whose renormalisability has been extensively studied, is that created by the addition of a mass term to the Yang-Mills[4] Lagrangian

$$\mathcal{L}_{Y\text{-}M} \equiv -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a \qquad (0.3)$$

where $G^{\mu\nu}_a = \partial^\mu W^\nu_a - \partial^\nu W^\mu_a - g(\underline{W}^\mu \times \underline{W}^\nu)_a$ †.

The result is the massive Yang-Mills Lagrangian

$$\mathcal{L} \equiv -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a + \frac{1}{2} M^2 W^\mu_a W^a_\mu \quad . \qquad (0.4)$$

By analogy with Q.E.D. it was suspected that the massless Yang-Mills Lagrangian, (0.3), was renormalisable and, in the belief that the massless theory can be obtained as the zero-mass limit of the massive S-matrix[5], it was thought that the massive Yang-Mills Lagrangian, (0.4), was renormalisable also. In general, the method of investigation has been to modify the Lagrangian in such a manner that the propagator becomes $\dfrac{g^{\mu\nu} - (1-\alpha^2)k^\mu k^\nu/(k^2-\alpha^2M^2 + i\varepsilon)}{k^2 - M^2 + i\varepsilon}$ for some α^2. If it were the only modification this would render the theory renormalisable. Unfortunately, in all attempts, it is accompanied by the introduction of a scalar field and the renormalisability of the Lagrangian is now governed by the interactions of this field.

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† Notation: $(\underline{A}\times\underline{B})^a \equiv f_{abc} A^b B^c$ where $i f_{abc}$ are the structure constants of a compact Lie group.

It has been shown formally, using path integral techniques[6,7], that there is an infinite series of vertices of the scalar particle, with no limit to order, which suggests their non-renormalisability. The generating functional for amplitudes of the massive Yang-Mills Lagrangian (0.4) is

$$G[J] \equiv Z^{-1} \int d[W] \exp\{i\int d^4x \left[-\frac{1}{4}\underline{G}_{\mu\nu}.\underline{G}^{\mu\nu} + \frac{1}{2}M^2\underline{W}_\mu.\underline{W}^\mu + \underline{W}_\mu.\underline{J}^\mu \right]\}. \quad (0.5)$$

For convenience we define the field matrices

$$\mathcal{W}^\mu(x) \equiv T^a W_a^\mu(x)$$

$$\mathcal{G}^{\mu\nu}(x) \equiv T^a G_a^{\mu\nu} = \partial^\mu \mathcal{W}^\nu(x) - \partial^\nu \mathcal{W}^\mu(x) + ig[\mathcal{W}^\mu(x), \mathcal{W}^\nu(x)] \quad (0.6)$$

$$\mathcal{J}^\mu(x) \equiv T^a J_a^\mu$$

where T^a are the generators of the m-dimensional representation of the group chosen such that

$$\mathrm{tr}(T^a T^b) = \lambda \delta^{ab} . \quad (0.7)$$

The Lagrangian (0.4) may then be rewritten in (0.5) as

$$\mathcal{L} \equiv -\lambda^{-1} \frac{1}{4} \mathrm{tr}\mathcal{G}^{\mu\nu}\mathcal{G}_{\mu\nu} + \lambda^{-1} \frac{1}{2}M^2 \mathrm{tr}\mathcal{W}^\mu\mathcal{W}_\mu + \lambda^{-1} \mathrm{tr}\mathcal{W}^\mu\mathcal{J}_\mu . \quad (0.8)$$

The massless Lagrangian (0.3) is invariant under the infinitesimal transformation

$$\underline{W}^\mu \to \underline{W}^\mu + g\,\underline{W}^\mu \times \underline{n}(x) - \partial^\mu \underline{n}(x) . \quad (0.9)$$

(0.9) is known as the infinitesimal gauge transform for the Yang-Mills Lagrangian which is a gauge invariant theory (or simply, a gauge theory). The finite transformation corresponding to (0.9) is

$$W_a^\mu(x) \rightarrow \Omega^A_{ab}(x) W_b^\mu - \frac{1}{\lambda g} f_{abc} \Omega^A_{cd} (\partial^\mu \Omega^{A-1}(x))_{db} \tag{0.10}$$

which can be reformulated in terms of the field matrices as

$$\mathcal{W}^\mu(x) \rightarrow \Omega(x) \mathcal{W}^\mu(x) \Omega^{-1}(x) + \Omega \partial^\mu \Omega^{-1}/ig \tag{0.11}$$

where $\Omega(x)$ is a local element of the m-dimensional representation of the group:

$$\Omega(x)_{ab} \equiv \{\exp[ig T^c \eta_c(x)]\}_{ab} \tag{0.12}$$

and $\Omega^A(x)$ is an element of the adjoint representation, i.e. when

$$T^a_{bc} \equiv -i f_{abc} .$$

Boulware[6] showed that any vector field, $\mathcal{W}^\mu(x)$, can be written as the transformation of a transverse field, $\mathcal{W}^{T^\mu}(x)$:

$$\mathcal{W}^\mu(x) \equiv \Omega \mathcal{W}^{T^\mu}(x) \Omega^{-1} + \Omega \partial^\mu \Omega^{-1}/ig . \tag{0.13}$$

The spin-one propagator was modified by making the replacement (0.13) in (0.5). The first term is unchanged as the replacement is simply the gauge transform for the massless Lagrangian. The generating functional becomes

$$\begin{aligned} G[J] \equiv Z^{-1} \int d[W^T] d[\chi] (\det M) \exp\{i \int d^4x [& -\lambda^{-1} \tfrac{1}{4} \mathrm{tr}\, \mathcal{G}^{T^{\mu\nu}} \mathcal{G}^T_{\mu\nu} \\ & + \lambda^{-1} \tfrac{1}{2} M^2 \mathrm{tr}\, \mathcal{W}^{T^\mu} \mathcal{W}^T_\mu + \lambda^{-1} M^2 \mathrm{tr}\, \ell^\mu \mathcal{W}^T_\mu + \lambda^{-1} \tfrac{1}{2} M^2 \mathrm{tr} \ell^\mu \ell_\mu \\ & + \lambda^{-1} \mathrm{tr}[\Omega \mathcal{W}^{T\mu} \Omega^{-1} + \bar{\ell}^\mu] \mathcal{J}_\mu] \} \end{aligned} \tag{0.14}$$

where

$$\begin{aligned} \bar{\ell}^\mu(x) &= \Omega \partial^\mu \Omega^{-1}/ig \\ \ell^\mu(x) &= -\Omega^{-1} \partial^\mu \Omega/ig \end{aligned} \tag{0.15}$$

and

$$\delta\chi^a T^a = \delta\Omega \Omega^{-1}/-ig . \tag{0.16}$$

The term det M is the Jacobian factor for the substitution where

$$\det M = c \det\left[g^{\mu\nu}\delta_{ab}\,\delta(x-y) + ig\mathcal{W}^{T^{\mu}}_{A}(x)_{ab}\,\partial^{\nu}D(x-y)\right] \quad (0.17)$$

with $D(x-x') = \int \frac{d^4p}{(2\pi)^4} - \frac{e^{-ip.(x-x')}}{p^2}$. $\mathcal{W}^{T^{\mu}}_{A_{ab}}$ are the field matrices in the adjoint representation. The determinant can be expanded using the identity

$$\det M \equiv \exp\left[\text{tr log } M\right] \quad (0.18)$$

such that

$$\det M = c \exp\{-\sum_{n=1}^{\infty} (-ig)^n/n \text{ tr}\left[\mathcal{W}^{T^{\mu}}_{A}\partial_{\nu} D\right]^n\} \quad (0.19)$$

These contributions are easily represented as closed loops generated by the "quasi-charged" scalar Feynman rules of Fig. 1(b) with $\alpha = 0$ and a factor (-1) associated with each loop. The Feynman rules for the vector-bosons are as Fig. 1(a) for the first two terms of the Lagrangian in (0.14).

Before the remaining terms involving $\Omega(x)$ can be interpreted it is necessary to find a convenient way of representing them. Boulware[6] restricted (0.6) to the adjoint representation and chose

$$\Omega(x) = \exp\left[ig\phi(x)\right] \quad (0.20)$$

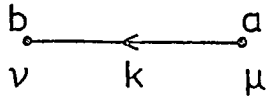
where $\phi(x)_{bc} \equiv \phi^a(x)(-i)f_{abc}$.

Then

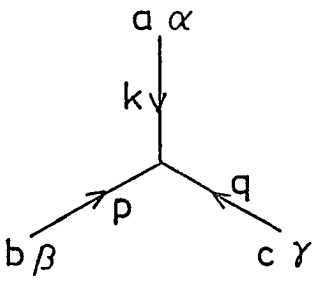
$$\partial^{\mu}\,\Omega(x) = +ig\;\partial^{\mu}\phi_a\,E_{ab}\,T^b\Omega(x) \quad (0.21)$$

where $E_{ba}(x) \equiv \left[\{\exp(-ig\phi) - I\} \Big/ -ig\phi\right]_{ba}$

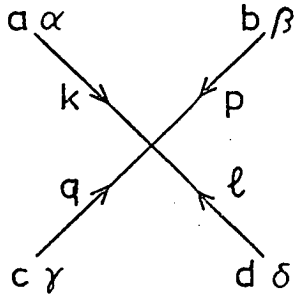
and the necessary Jacobian factor



$$\frac{g^{\mu\nu} - \frac{(1-\alpha^2)k^\mu k^\nu}{k^2 - \alpha^2 M^2 + i\varepsilon}}{k^2 - M^2 + i\varepsilon}\ \delta_{ab}$$



$$-igf_{abc}\left[g^{\beta\gamma}(q-p)^\alpha + g^{\gamma\alpha}(k-q)^\beta + g^{\alpha\beta}(p-k)^\gamma\right]$$

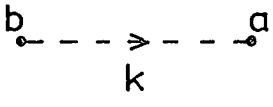


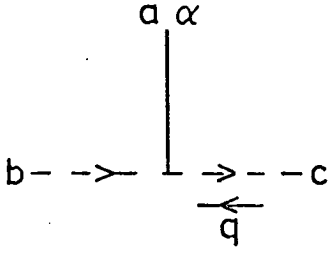
$$-g^2\ f_{abc}f_{gcd}(g^{\alpha\gamma}\ g^{\beta\delta} - g^{\alpha\delta}\ g^{\beta\gamma})$$

$$-g^2\ f_{adg}f_{gbc}(g^{\alpha\beta}\ g^{\delta\gamma} - g^{\alpha\gamma}\ g^{\delta\beta})$$

$$-g^2\ f_{acg}f_{gbd}(g^{\alpha\beta}\ g^{\gamma\delta} - g^{\alpha\delta}\ g^{\gamma\beta})$$

FIG 1(a)





(b) $\dfrac{-\delta^{ab}}{k^2-\alpha^2M^2+i\varepsilon}$ $\qquad ig\ f_{abc}\ q\alpha$

(c) $\dfrac{-\delta^{ab}}{\frac{1}{\alpha}k^2-\alpha M^2+i\varepsilon}$ $\qquad \dfrac{i}{\alpha}\ g\ f_{abc}\ q^\alpha$

(d) $\dfrac{-\delta^{ab}}{\frac{1}{\alpha^2}k^2-M^2+i\varepsilon}$ $\qquad \dfrac{i}{\alpha^2}\ g\ f_{abc}\ q^\alpha$

FIG 1(b)-(d) Alternative forms of the ghost rules.

$$\delta\chi_a(x) / \delta\phi_b(x') = -\delta(x-x')\, E_{ba}(x) \qquad (0.22)$$

for the change of variables. The functional integral over the the gauge degree of freedom in (0.14) can be written

$$G_\Omega[W^T] \equiv Z_\Omega^{-1} \int d[\phi] \{\det E_{ab}(x)\} \exp\{i \int d^4x \mathcal{L}_\Omega(\phi, W^T)\} \qquad (0.23)$$

where
$$\mathcal{L}_\Omega(\phi, W^T) = \tfrac{1}{2}M^2 \partial^\mu \phi_a \tilde{E}_{ab} E_{bc} \partial_\mu \phi_c$$
$$-M^2 \partial^\mu \phi_b \tilde{E}_{ba} W^T_{a\mu}(x) \qquad (0.24)$$
$$+\left[W^{T\mu}_a \Omega^A_{ab} - \partial^\mu \phi_a E_{ab}\right] J_{b\mu}(x) \,.$$

Hence, the gauge parameter is interpreted as a scalar field but its exact interaction depends on the representation of the group.

Salam and Strathdee[7] utilized a generalisation of the Stückelberg split[8] which incorporated the transform (0.13). One of their formulations is formally identical to that of Boulware, i.e. (0.14), but they used a different realization. Salam and Strathdee restricted the group to SU(2) and chose the fundamental representation in (0.6) with the parametrization

$$\Omega(x) = \sigma(x) + i\underline{\tau}.\underline{\Pi}(x) \,. \qquad (0.25)$$

$\Omega(x)$ is unitary with determinant (+1) if

$$\sigma^2(x) + \underline{\Pi}.\underline{\Pi} = 1. \qquad (0.26)$$

The corresponding change in the volume element including the Jacobian is

$$\int d[\chi_a] \rightarrow \int d[\pi_a]\, d[\sigma]\, \delta(\sigma^2 + \underline{\Pi}\cdot\underline{\Pi} - I) \qquad (0.27)$$

as expected. In executing the integration of $\sigma(x)$ and replacing it everywhere by $\sqrt{I - \underline{\Pi}\cdot\underline{\Pi}}$ the scalar interactions are again an infinite series but it is suggested they are of the Efimov-Fradkin normal type[9-11], i.e. all divergences can be absorbed by a finite number of renormalisation counter-terms.

To investigate whether the divergen ces caused by the scalar vertices would cancel and to avoid questions over the validity of the path integral approach, it is necessary to investigate the Lagrangian directly. Veltman[5] introduced a free scalar particle to the massive SU(2) Yang-Mills Lagrangian:

$$\mathcal{L} \equiv -\frac{1}{4}\underline{G}_{\mu\nu}\cdot\underline{G}^{\mu\nu} + \frac{1}{2}M^2\underline{W}_\mu\cdot\underline{W}^\mu - \frac{1}{2}\partial_\mu\phi^a\partial^\mu\phi_a + \frac{1}{2}M^2\phi_a\phi^a \qquad (0.28)$$

The vector boson fields were transformed first according to (0.10) with

$$f_{abc}\,\Omega^A_{cd}(\partial^\mu\Omega^{A-1})_{db} = \frac{2\lambda g}{M}\,\partial_\mu\phi^a + \frac{\lambda^2 g^2}{M^2}\,R^d_\mu \qquad (0.29)$$

and second according to

$$W^a_\mu \rightarrow W^a_\mu + \frac{\lambda}{M}\,\partial_\mu\phi^a\ . \qquad (0.30)$$

Under these transforms the Lagrangian (0.28) becomes

$$\mathcal{L}' \equiv \mathcal{L}_{YM}(W^a_\mu + \frac{\lambda}{M}\partial_\mu\phi^a) + \frac{1}{2}M^2W^a_\mu W^\mu_a - \frac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^a - \frac{1}{2}M\,\phi_a\phi^a \qquad (0.31)$$

$$- \frac{1}{2}\lambda^2 g(W^d_\mu + \frac{\lambda}{M}\partial_\mu\phi^d)R^\mu_d + \frac{\lambda^3 g}{2M}\,\partial_\mu\phi^d\,R^\mu_d + \frac{\lambda^4 g^2}{8M^2}R^d_\mu\,R^\mu_d$$

(0.29) can be realised with the choice of parametrization

$$\Omega^A \equiv \exp\left[i\frac{g\lambda}{M}\,\phi^a(x)T^{Aa}\right]\ . \qquad (0.32)$$

Choosing $\lambda = 1$ the combination of fields,

$$\psi^\mu = W^\mu + \frac{1}{M}\,\partial^\mu\phi \qquad (0.33)$$

which replaces all the legs of the vector vertices, has the Feynman rules of Fig. 1(a), with $\alpha = 0$. All other terms in the Lagrangian (0.31) generate the interactions of the scalar fields. An infinite series results but in this case as the scalar fields were originally free the vertices must be inter-related and many redundant. Veltman investigated the redundancy as far as the one loop approximation which was shown to have an equivalent explicitly renormalisable formulation, since only one three point scalar vertex is introduced. These rules are as in Fig. 1(b) in the limit $\alpha = 0$ and the rules of Figs. 1(a) and 1(b) are henceforth known as the soft rules following Mohapatra, Sakakibara and Sucher[12].

Reiff and Veltman[13] extended the analysis to the two loop approximation at which level the self energy diagrams do not satisfy unitarity for the simple soft rules employed at the one loop level. It was found necessary to introduce an additional four point vertex which was non-renormalisable. No other amplitudes were considered.

Finally Veltman[14] approached the whole problem more systematically using the "free field" technique to establish generalised Ward identities for the massive Yang-Mills Lagrangian. To (0.28) a source term $W_a^\mu F_\mu^a$ is added and the vector-boson fields transformed:

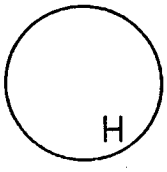
$$\underline{W}^\mu \rightarrow \underline{W}^\mu + \frac{g\lambda}{M} \underline{W}^\mu \times \underline{\phi} - \frac{\lambda}{M} \partial^\mu \underline{\phi} \, . \qquad (0.34)$$

If λ is infinitesimal the first term of the Lagrangian is invariant under (0.34) and the Lagrangian becomes

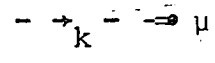
$$\mathcal{L}' \equiv -\frac{1}{4}\underline{G}_{\mu\nu}\cdot\underline{G}^{\mu\nu} + \frac{1}{2}M^2\underline{W}_\mu\cdot\underline{W}^\mu - \frac{1}{2}\partial_\mu\underline{\phi}\cdot\partial^\mu\underline{\phi} + \frac{1}{2}m^2\underline{\phi}\cdot\underline{\phi}$$
$$+ \underline{W}_\mu\cdot\underline{F}^\mu - \lambda M\,\underline{W}^\mu\cdot\partial_\mu\underline{\phi} + \frac{g\lambda}{M}\,\underline{W}_\mu\times\underline{\phi}\cdot\underline{F}^\mu - \frac{\lambda}{M}\,\partial_\mu\underline{\phi}\cdot\underline{F}^\mu \,. \tag{0.35}$$

The fields ϕ^a remain free up to first order in λ and the amplitudes to first order in λ, but any given order in g and F_μ, must be identically zero. The Feynman rules for (0.35) are those of Fig. 2.

The following conventions are required.

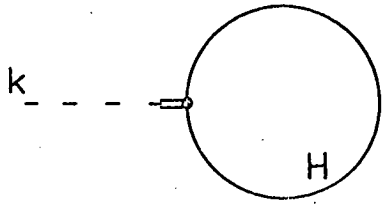


stands for the set of diagrams constructed with the vector-boson rules of Fig. 2, i.e. the manifestly unitary rules, for a given order in g with any no. of external physical W-lines. The F-sources or external ϕ-lines are to be indicated explicitly.

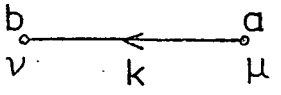


indicates that on an external W-line the polarization vector has been replaced by (ik^μ) where now the momentum may be off-mass shell.

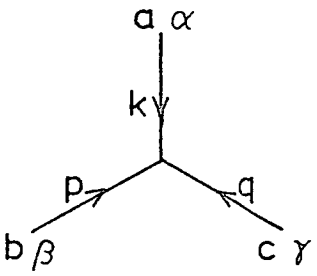
Veltman[14] first demonstrated that the Ward identities, (0.36) - (0.38) and Fig. 3, shown below, hold using the free field property.



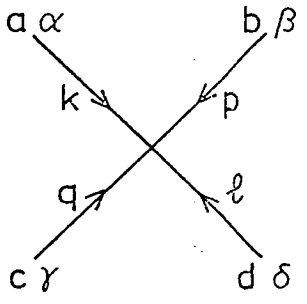
$$= 0 \tag{0.36}$$



$$\frac{g^{\mu\nu} - k^{\mu}k^{\nu}/M^2}{k^2 - M^2 + i\varepsilon}\,\delta^{ab}$$



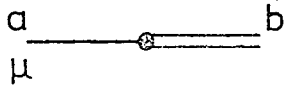
$$-ig\, f_{abc}\left[g^{\beta\gamma}(q-p)^{\alpha} + g^{\gamma\alpha}(k-q)^{\beta} + g^{\alpha\beta}(p-k)^{\gamma}\right]$$



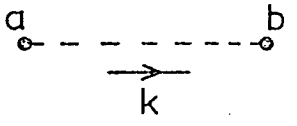
$$-g^2 f_{abg} f_{gcd}(g^{\alpha\gamma}\, g^{\beta\delta} - g^{\alpha\delta}\, g^{\beta\gamma})$$

$$-g^2 f_{adg} f_{gbc}(g^{\alpha\beta}\, g^{\delta\gamma} - g^{\alpha\gamma}\, g^{\delta\beta})$$

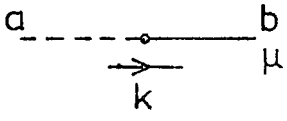
$$-g^2 f_{acq} f_{gbd}(g^{\alpha\beta}\, g^{\gamma\delta} - g^{\alpha\delta}\, g^{\gamma\beta})$$



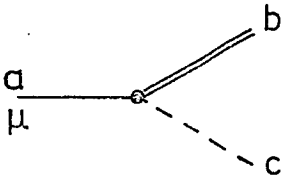
$$\delta^{ab}\, F^{b}_{\mu}$$



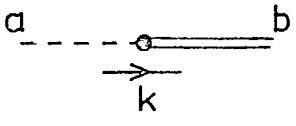
$$\frac{\delta^{ab}}{k^2 - M^2 + i\varepsilon}$$



$$i\lambda M k^{\mu}\delta^{ab}$$

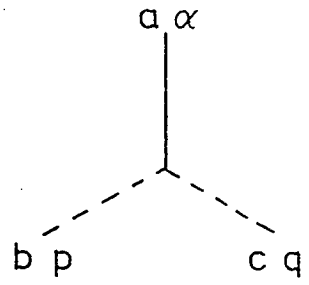


$$\frac{g\lambda}{M}\, f_{acb}\, F^{\mu}_{b}$$

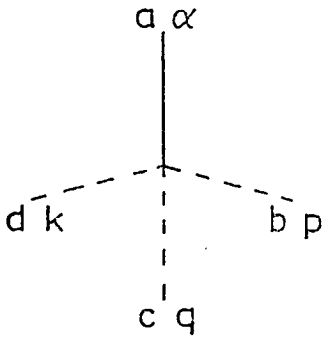


$$i\frac{\lambda}{M}\, k^{\mu}\, \delta^{ab}\, F_{b\mu}$$

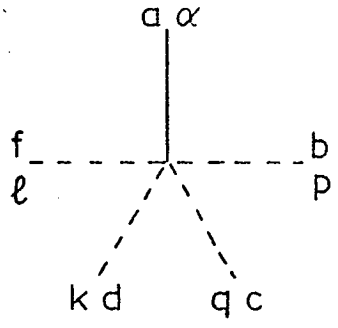
FIG 2



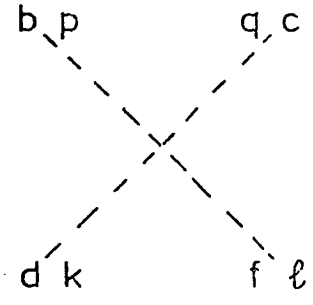
$$-\frac{i}{2} M^2 g\, f_{abc}(q-p)^{\alpha}$$



$$+ ig^2M^2(\delta_{db}\delta_{ca}q^{\alpha} + \delta_{ba}\delta_{cd}p^{\alpha} + \delta_{da}\delta_{cb}k^{\alpha})$$



$$-\frac{ig^3M^2}{4}\{f_{abf}\,\delta_{cd}\,(p-\ell)^{\alpha} + f_{abd}\,\delta_{bc}\,(p-k)^{\alpha}$$
$$+ f_{abc}\delta_{df}(p-q)^{\alpha} + f_{acf}\delta_{bd}(q-\ell)^{\alpha}$$
$$+ f_{acd}\delta_{bf}(q-k)^{\alpha} + f_{adf}\delta_{bc}(k-\ell)^{\alpha}\}$$



$$-\frac{q^2M^2}{4}\{\delta_{bc}\delta_{df}(2p.q+2k.\ell - k.q-p.k-\ell.q-\ell.p)$$
$$+\delta_{bf}\delta_{dc}(2p.\ell+2k.q-k.\ell -p.k-\ell.q-q.p)$$
$$+\delta_{bd}\delta_{fc}(2p.k+2q.\ell-k.q-p.q-\ell.k-\ell.p)\}$$

FIG 2(a) The first vertex is the symmetrized form of the scalar vertex in Fig. 1(b) with the appropriate factors. p,q,k,ℓ are the momenta of the scalar lines.

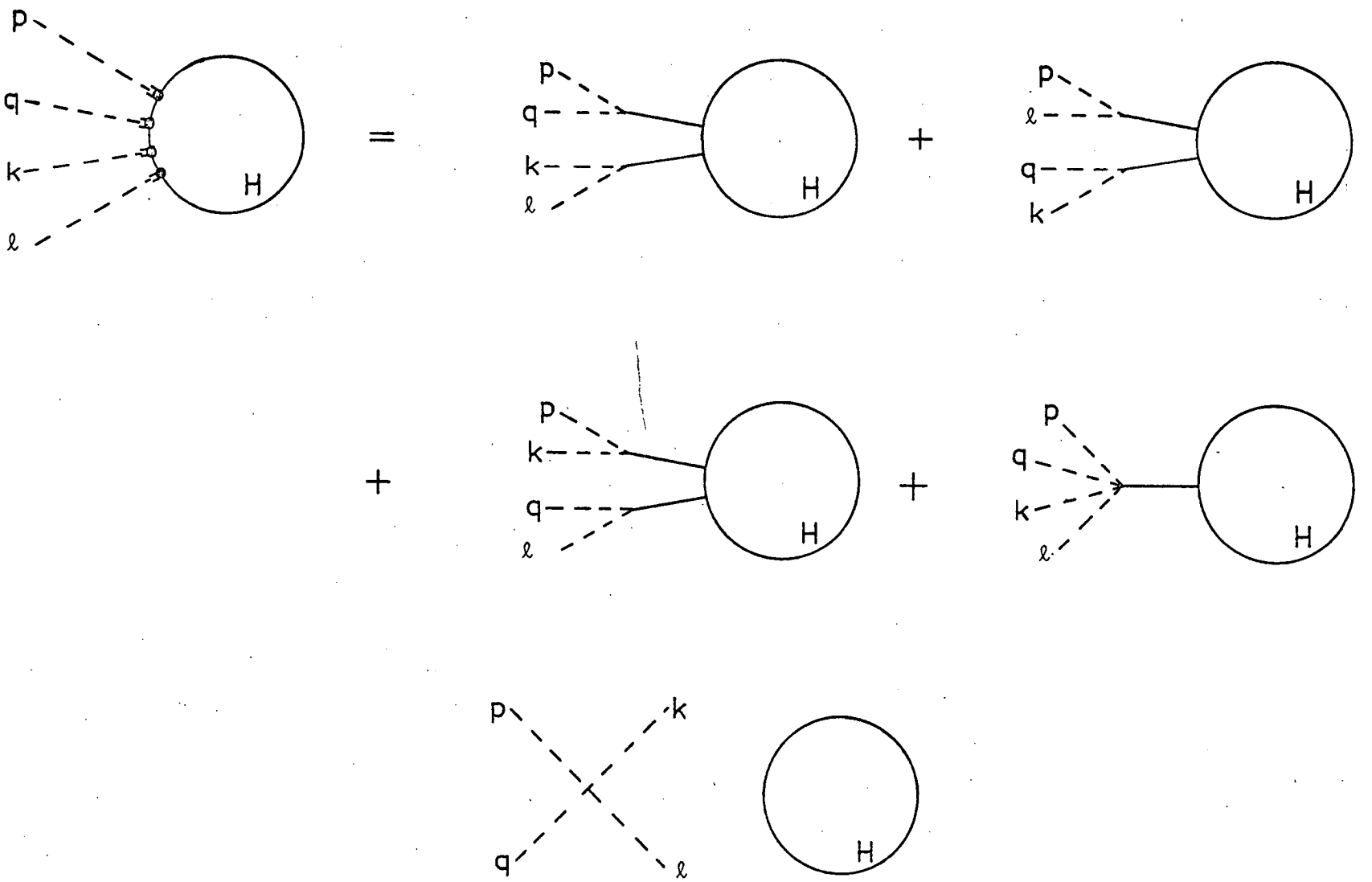
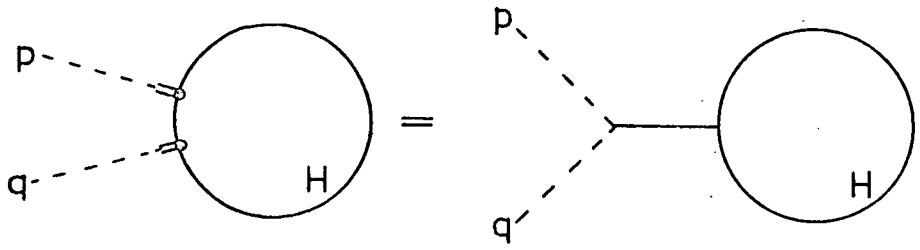
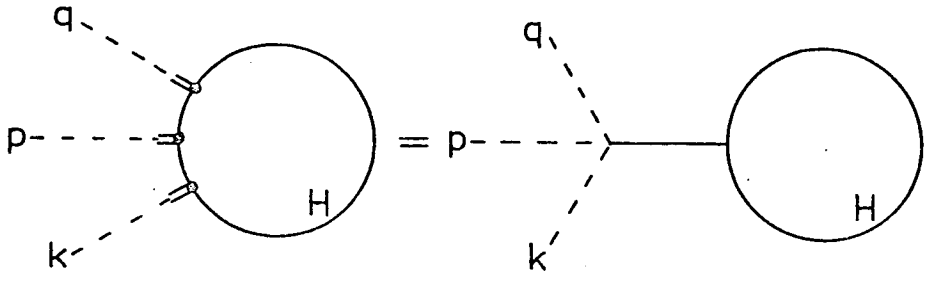


FIG 3



(0.37)



(0.38)

The Ward identities with any number of momenta contractions could have been obtained but the above were all that were required. The additional vertices are given in Fig. 2(a).

These Ward identities were used to remove the $k^{\mu}k^{\nu}$ terms from the vector-boson propagators and reduce the amplitudes to their least divergent form. It should be noted that in the derivation of the Ward identities a transformation, in the form of the gauge transform for the massless Yang-Mills Lagrangian, is exploited. Only the two loop approximation to the self energy amplitude was investigated but it was found to require the additional scalar vertices of Fig. 2(a), some of which are of a non-renormalisable nature.

Meanwhile Slavnov and Faddeev[15] and Van Dam and Veltman[16] showed that the massless theory does not result from taking the zero-mass limit of the massive case. It thus became generally accepted that the massive Yang-Mills Lagrangian

is not a renormalisable field theory.

Hsu and Sudarshan[17] re-examined the theory by introducing a Lagrange multiplier into the Lagrangian which gives rise to a soft vector-boson propagator[18-22]. The generating functional is

$$G[\underline{J}] \equiv Z^{-1}\int d[\underline{W}]\,d[\underline{\chi}]\exp\{i\int d^4x\Big[-\frac{1}{4}\underline{G}_{\mu\nu}\cdot\underline{G}^{\mu\nu} + \frac{1}{2}M^2\underline{W}_\mu\cdot\underline{W}^\mu + \alpha M\,\partial_\mu\,\underline{W}^\mu\cdot\underline{\chi} + \frac{1}{2}\beta\underline{\chi}^2 + \underline{W}_\mu\cdot\underline{J}^\mu\Big]\} \tag{0.39}$$

with $\alpha^2M^2 = \beta$. On integrating over the multiplier fields χ^a (0.39) becomes

$$G[\underline{J}] \equiv Z^{-1}\int d[\underline{W}]\exp\{i\int d^4x\Big[-\frac{1}{4}\underline{G}_{\mu\nu}\cdot\underline{G}^{\mu\nu} + \frac{1}{2}M^2\underline{W}_\mu\cdot W^\mu - \frac{1}{2}\,\frac{\alpha^2M^2}{\beta}\,(\partial_\mu\underline{W}^\mu)^2 + \underline{W}_\mu\cdot\underline{J}^\mu\Big]\} \tag{0.40}$$

i.e. the soft formulation of Fig. 1(a). The equations of motion for the χ^a are

$$(\partial^2 + M^2)\underline{\chi} = +g\,\underline{W}_\mu \times \partial^\mu\underline{\chi} \tag{0.41}$$

when $\alpha = \beta^{\frac{1}{2}}/M$. Hsu and Sudarshan defined the physical state by

$$\underline{\chi}^+(x)\,|\text{phys}\rangle = 0\ . \tag{0.42}$$

Thus the modification can also be interpreted as equivalent to a scalar field being added to the Lagrangian with the effective Lagrangian

$$\mathcal{L}(\chi) = -\frac{1}{2}\Big[\partial_\mu\,\underline{\chi}\cdot\partial^\mu\underline{\chi} - M^2\underline{\chi}^2 + g\underline{\chi}\cdot(\underline{W}_\mu\times\partial^\mu\underline{\chi})\Big] \tag{0.43}$$

and the subsidiary condition

$$\partial_\mu \underline{W}^\mu + \beta^{\frac{1}{2}} \underline{\chi} = 0 \ . \qquad (0.44)$$

Hsu and Sudarshan claim that any contribution of the scalar field to the S-matrix generated by (0.39) or equivalently (0.40) can be removed by adding the determinant $D_M^{\frac{1}{2}}$ to the generating functional (0.40) where

$$D_M^{-\frac{1}{2}} \equiv \left[\det(\delta^{ac} - (\partial^2 + M^2)^{-1} g f_{abc} W_\mu^b \partial^\mu)\right]^{-\frac{1}{2}} \qquad (0.45)$$

$$\equiv \int d[\chi^a] \exp\{i \int d^4x \, \mathcal{L}(\underline{\chi})\} \ .$$

(0.45) should be compared to (0.17) and interpreting $D_M^{\frac{1}{2}}$ similarly the complete Feynman rules for the spin-one massive Yang-Mills Lagrangian are Fig. 1(a) and 1(b) with $\alpha^2 = 1$ but a factor $(-\frac{1}{2})$ associated with each scalar loop. The scalar interactions are no longer an infinite series, and so the massive Yang-Mills theory would appear to be renormalisable. The Feynman rules are the same as those of other authors for the one-loop approximation.

Subsequently Mohapatra, Sakakibara and Sucher[12] extended the analysis of Veltman[13, 14] to the four-point interaction

$$W_a + W_b \rightarrow W_c + W_d$$

as the ingestigation of the self-energy amplitude is not wholly relevant to the S-matrix. They found the rules of Hsu and Sudarshan insufficient to obtain a theory identical to the normal canonically quantized version or even to satisfy unitarity.

The two major ways of finding the equivalent "soft" form of the massive Yang-Mills Lagrangian, viz. Boulware's and

Veltman 's generate scalar vertices which at face value seem to differ from each other. Specifically, the factors associated with the vertices and scalar loops do not seem to tally.

The analysis of Veltman was the first systematic use of the Ward identities to investigate the role of the $k^\mu k^\nu$ terms in vector-boson propagators. This approach was developed and most successfully applied to the examination of the properties of gauge field theories, an example of which has already been met in the form of the massless Yang-Mills Lagrangian, (0.3). For these theories the Ward identities are associated with the gauge invariance of the Lagrangian, e.g. (0.9) for the massless Yang-Mills theory, and are sometimes referred to as Slavnov-Taylor identities.

b. Gauge Field Theories

The study of gauge theories was initiated by Feynman[23] who, on examining the S-matrix for the massless Yang-Mills Lagrangian constructed with the propagators $\frac{g^{\mu\nu}}{k^2+i\varepsilon}$, found the theory not to be unitary. He recognised the need to introduce a fictitious scalar field to restore unitarity, the problem being that there are no covariant and unitary Feynman rules for the Lagrangian (0.3) as the fields $W_a^\mu(x)$ have both a spin-one and a spin-zero component. Faddeev and Popov[24,25] derived the correct rules for this scalar field, by path integral techniques but only for the transverse propagator formalism.

Faddeev and Popov noted that many of the problems in quantizing a gauge invariant Lagrangian are associated with

the degeneracy of the free Lagrangian. The action in the functional generator for the massless Yang-Mills Lagrangian

$$G[J] \equiv Z^{-1} \int d[W] \exp\{i\int d^4x \left[-\frac{1}{4}\underline{G}_{\mu\nu}\cdot\underline{G}^{\mu\nu} + \underline{W}_\mu\cdot\underline{J}^\mu \right] \quad (0.46)$$

is constant over the orbits of the fields, i.e. for a field $W^\mu_a(x)$, all $W'^\mu_a(x)$ such that $W'^\mu_a(x)$ is the gauge transform of $W^\mu_a(x)$ for some value of the gauge parameter:

$$W'^\mu_a(x) \equiv \Omega^A_{ab}(x)W^\mu_b(x) - \frac{1}{\lambda g} f_{abc}\, \Omega^A_{cd}(\partial^\mu \Omega^{A^{-1}}(x))_{db} \quad (0.47)$$

for some Ω^A_{ab}, i.e. some $\eta^a(x)$ in (0.12). These problems disappear if the functional integral over the vector fields in (0.46) is restricted to a surface which intersects each orbit once only. This is achieved by multiplying the generating functional by

$$\Delta_f[W] \int d[\Omega]\ \delta(f[W^\Omega]) = \text{const.} \quad (0.48)$$

W^Ω is defined by (0.47) and the δ-function picks out some surface according to the choice of f. (0.46) becomes

$$G[J] \equiv Z^{-1} \int d[W]\ d[\Omega]\ \Delta_f[W]\delta(f[W])\exp\{i\int d^4x \left[-\frac{1}{4}\underline{G}_{\mu\nu}\cdot\underline{G}^{\mu\nu} + \underline{W}^{\Omega^{-1}}_\mu\cdot\underline{J}^\mu\right]\} \quad . \quad (0.49)$$

Choosing

$$f[W] = \partial_\mu W^\mu_a(x) \quad (0.50)$$

Faddeevand Popov restricted the vector fields to the transverse formulation and all that is required is that the contribution of the functional $\Delta_f[W]$, defined by (0.48), is evaluated. It is straightforward to show

$$\Delta_f[W] = \det\left[\delta^{ac} - (\partial^2)^{-1} g f_{abc} W_\mu^b \partial^\mu\right] \tag{0.51}$$

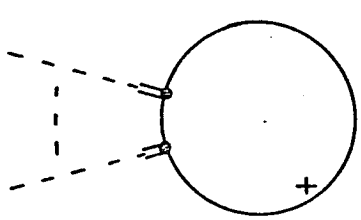
This should be compared with (0.45) and (0.17) and interpreted similarly. The choice of $f[W]$ is known as the choice of gauge and for the transverse or Landau gauge for (0.50) the Feynman rules are those of Fig. 4(a) with $\alpha = 0$ and Fig. 4(b) with a factor (-1) associated with each fictitious particle loop. Within the context of gauge theories this scalar ghost is called the Faddeev-Popov ghost.

Fradkin and Tyutin[26] developed a more flexible formalism in which the choice of gauge was made by adding a term of the form

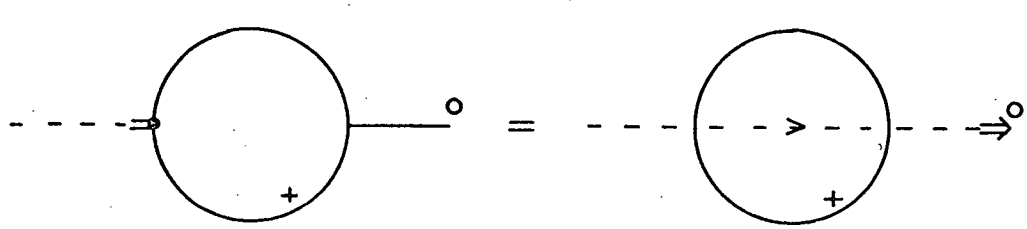
$$-\tfrac{1}{2} f_a[W] f^a[W] \tag{0.52}$$

to the Lagrangian.

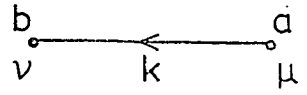
't Hooft[27], using combinatoric techniques[28], established the following Ward identities for the massless Yang-Mills theory



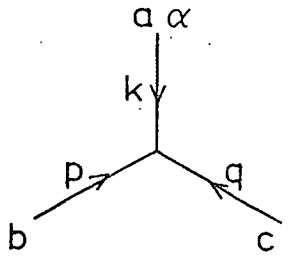
$= 0$ (0.53)



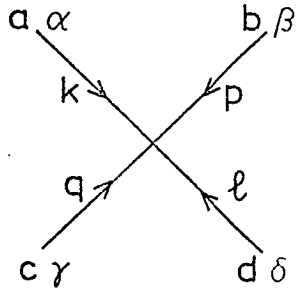
(0.54)



$$\frac{g^{\mu\nu}(1-\alpha^2)\ k^\mu k^\nu/k^2+i\varepsilon}{k^2+i\varepsilon}\ \delta^{ab}$$



$$-ig\ f_{abc}\left[g^{\beta\gamma}(q-p)^\alpha + g^{\gamma\alpha}(k-q)^\beta + g^{\alpha\beta}(p-k)^\gamma\right]$$

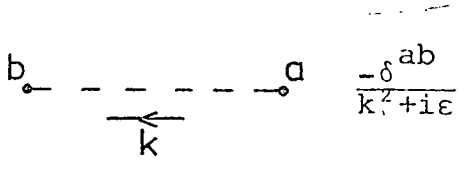


$$-g^2 f_{abg} f_{gcd}(g^{\alpha\gamma}g^{\beta\delta} - g^{\alpha\delta}g^{\beta\gamma})$$

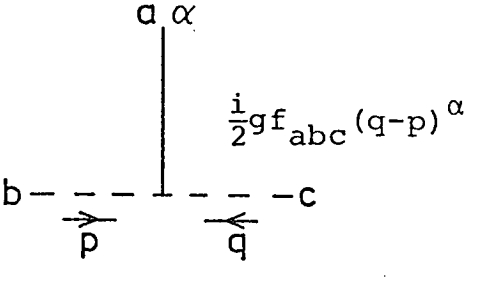
$$-g^2 f_{adg} f_{gbc}(g^{\alpha\beta}g^{\delta\gamma} - g^{\alpha\gamma}g^{\delta\beta})$$

$$-g^2 f_{acg} f_{gbd}(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\delta}g^{\gamma\beta})$$

FIG 4(a)

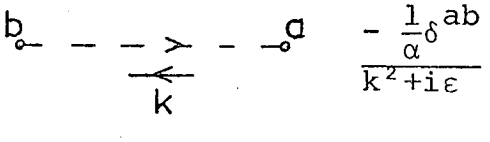


$$\frac{-\delta^{ab}}{k^2+i\varepsilon}$$

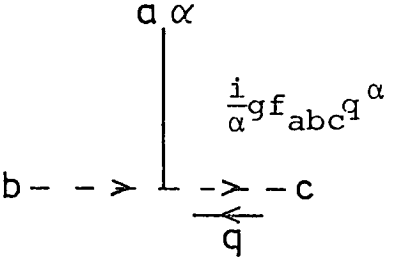


$$\frac{i}{2}gf_{abc}(q-p)^\alpha$$

FIG 4(b)



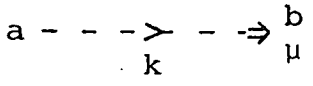
$$-\frac{1}{\alpha}\frac{\delta^{ab}}{k^2+i\varepsilon}$$



$$\frac{i}{\alpha}gf_{abc}q^\alpha$$

FIG 4(c)

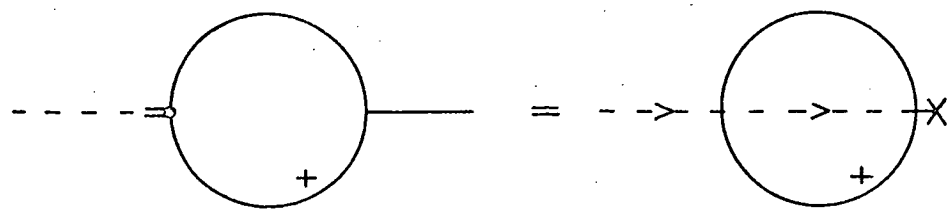
The "blobs" have any number of external physical lines and are constructed with the rules of Fig. 4(a) and 4(c) with $\alpha = 1$ and a factor (-1) with each closed ghost loop. The + indicates the presence of Faddeev-Popov ghost contributions and the 0 implies the line which has no polarization vector is on-mass shell, i.e. $k^2 = 0$. The additional vertex



$$\equiv \quad i\delta^{ab}\, k^{\mu} \quad . \tag{0.55}$$

Using (0.53) and (0.54) 't Hooft demonstrated the unitarity of the theory with the addition of an appropriate ghost Lagrangian to generate the Feynman rules of Fig. 4(c). Furthermore, he extended the formalism of Faddeev and Popov and Fradkin and Tyutin to a wide range of gauges by giving the Faddeev-Popov ghost an orientation; i.e. the ghost field becomes quasi-charged and an asymmetry between the scalar legs of the scalar vertex is introduced. The result of this for the massless Yang-Mills rules, for a general choice of α, is shown in Fig. 4(c) where the orientation is indicated by the arrow on the scalar lines. This is the arrow in (0.54).

However, 't Hooft had not established, completely, the renormalisability of the Lagrangian. It had still to be shown that the Ward identities hold after renormalisation to ensure the unitarity of the theory. The coupling constants of the vector and scalar-ghost vertices must be identical after renormalisation as well as before. Taylor[24] verified this using the further generalised Ward identities (which were also obtained by Slavnov[30]).



(0.56)

where the symbol X denotes the attachment $(q_\mu q_\nu - q^2 g_{\mu\nu})$. (0.56) trivially reduces to (0.54) and (0.53) as required.

Although neither the massive nor massless Yang-Mills theory is applicable, with much relevance, to the physical situation; the importance of the above work lay in the belief that spontaneously breaking a symmetry does not increase the degree of divergence of the theory. This was based on the work of Lee et al.[31-34] who had shown this to be the case in the σ-model. There is then the possibility of constructing theories with vector-boson fields where mass is generated by breaking the gauge symmetry of a renormalisable massless theory, such as the Salam-Weinberg model[35].

The basic concept involved in the construction of such models was first enunciated by Goldstone[36]. The Lagrangian

$$\mathcal{L}_G \equiv \partial_\mu \phi^* \, \partial^\mu \phi - \mu^2 \phi^* \phi - h(\phi^* \phi)^2 \qquad (0.57)$$

which is invariant under the global phase transformation

$$\phi \rightarrow \phi e^{i\alpha} \quad ; \quad \phi^* \rightarrow \phi^* e^{-i\alpha} \qquad (0.58)$$

can be treated as a perturbation expansion, as usual, if $\mu^2 > 0$ to ensure the expansion is about a local minimum. When $\mu^2 < 0$ the local minimum is no longer at $|\phi| = 0$, which is now a local maximum, but at $|\phi| = \lambda$ where

$\lambda = \sqrt{-\mu^2/h}$. To treat (0.57) as a normal field theory, for $\mu^2 < 0$, it is first necessary to make the replacement

$$\phi \rightarrow \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2 + \lambda) \quad . \tag{0.59}$$

The theory is said to have had its symmetry spontaneously broken as a particular choice for the vacuum expectation value has been made and the complete theory, i.e. Lagrangian plus choice of the vacuum expectation value, is no longer invariant under (0.58). The Lagrangian, (0.57), becomes

$$\begin{aligned} \mathcal{L}'_G \equiv {} & \tfrac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \tfrac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - \tfrac{1}{2}\mu^2\lambda^2 - \tfrac{1}{4}h\lambda^4 - \\ & - \phi_1\lambda(\mu^2 + h\lambda^2) - \tfrac{1}{2}\phi_1^2(\mu^2 + 3h\lambda^2) - \tfrac{1}{2}\phi_2^2(\mu^2 + h\lambda^2) \\ & - h\lambda\phi_1(\phi_1^2 + \phi_2^2) - \frac{h}{4}(\phi_1^2 + \phi_2^2)^2 \quad . \end{aligned} \tag{0.60}$$

The constant has no physical consequences and can be dropped. From the definition of λ it satisfies the equation

$$\mu^2 + h\lambda^2 = 0 \tag{0.61}$$

and on applying this relation to (0.60) the term linear in the field ϕ_1 vanish as required but so does the mass term for ϕ_2. That the field ϕ_2 becomes massless on spontaneously breaking the symmetry is an example of the Goldstone theorem which states that when a symmetry is broken by a field acquiring a non-zero vacuum expectation value, a massless scalar appears for each parameter of the symmetry which ceases to govern an exact symmetry, i.e. under which the vacuum expectation value is not invariant.

Extensive investigations were made to see if this rather unphysical result of the Goldstone theorem could be avoided[37-44]. Higgs[37,41,43] established that the massless scalars could be absorbed by massless gauge bosons (with two transverse polarisations) to produce massive vector-bosons (with three polarisations). A minimally coupled vector boson is added to the Goldstone Lagrangian (0.57) to construct the Higgs model

$$\mathcal{L}_H \equiv -\tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + |(\partial_\mu - ieA_\mu)\phi|^2 - \mu^2|\phi|^2 - h|\phi|^4 . \tag{0.62}$$

It is invariant under the gauge transformation

$$A^\mu \to A^\mu + \partial^\mu \Lambda(x), \qquad \phi \to e^{ie\Lambda(x)}\phi \tag{0.63}$$

and, for $\mu^2 > 0$, corresponds to a massive complex scalar and a massless vector. If $\mu^2 < 0$, the substitution (0.54) has again to be made and the Lagrangian becomes

$$\begin{aligned} \equiv & -\tfrac{1}{4}(\partial_\mu A_\nu - \partial_\mu A_\mu)^2 + \tfrac{1}{2}e^2\lambda^2 A_\mu A^\mu - \tfrac{1}{2}\mu^2\lambda^2 - \tfrac{1}{4}h\lambda^4 \\ & - \phi_1\lambda(\mu^2 + h\lambda^2) - \tfrac{1}{2}\phi_2^2(\mu^2 + h\lambda^2) \\ & + \tfrac{1}{2}(\partial_\mu\phi_1 + eA_\mu\phi_2)^2 + \tfrac{1}{2}(\partial_\mu\phi_2 - eA_\mu\phi_1)^2 - \tfrac{1}{2}\phi_1^2(\mu^2 + 3h\lambda^2) \\ & - e\lambda A_\mu(\partial^\mu\phi_2 - eA^\mu\phi_1) - h\lambda\phi_1(\phi_1^2 + \phi_2^2) - \tfrac{h}{4}(\phi_1^2 + \phi_2^2)^2 . \end{aligned} \tag{0.64}$$

Now, however, the scalar which has become massless, ϕ_2, is a ghost, i.e. not a physical field, as can be seen by making the substitution

$$\phi \to \frac{1}{\sqrt{2}}(\lambda + \chi(x))e^{i\theta(x)/\lambda} . \tag{0.65}$$

instead of (0.59). The gauge transformation (0.63) leaves $\chi(x)$ unchanged but $\theta(x)$ transforms as

$$\theta(x) \rightarrow \theta(x) + e\lambda\, \Lambda(x) \qquad (0.66)$$

Therefore, if the gauge transformation is exploited, the field $\theta(x)$ can be removed from the Lagrangian, i.e. $\Lambda(x)$ is chosen to be $-\frac{1}{e\lambda}\,\theta(x)$. The corresponding transformation of the field A is

$$A^{\mu} \rightarrow B^{\mu} = A^{\mu} - \frac{1}{e\lambda}\,\partial^{\mu}\theta(x) \quad . \qquad (0.67)$$

Expressed in terms of these fields (0.62) takes the form

$$\begin{aligned} \mathcal{L}'_{H} \equiv & -\tfrac{1}{4}(\partial_{\mu}B_{\nu}-\partial_{\nu}B_{\mu})^{2} + \tfrac{1}{2}e^{2}\lambda^{2}B_{\mu}B^{\mu} + \tfrac{1}{2}\partial_{\mu}\chi\,\partial^{\mu}\chi \\ & + \tfrac{1}{2}\,e^{2}B_{\mu}^{2}(2\lambda\chi(x) + \chi(x)^{2}) - \tfrac{1}{2}(\mu^{2}+3h\lambda^{2})\chi^{2}(x) \qquad (0.68) \\ & - \tfrac{h}{4}(4\lambda\chi^{3}(x) + \chi^{4}(x)) - (\mu^{2}+h\lambda^{2})\lambda\chi(x) - \tfrac{\mu^{2}}{2}\lambda^{2} - \tfrac{h}{4}\lambda^{4} \end{aligned}$$

and the vector field has acquired a mass $\sqrt{e^{2}\lambda^{2}}$.

With the successful completion of the investigation of the massless Yang-Mills theory, attention centred on theories with massive bosons constructed by the Higgs mechanism. 't Hooft[45] made the important step of using the fact, that a spontaneously broken gauge theory still has a gauge invariance, to reformulate massive vector-boson theories, of the spontaneously broken type, in an explicitly renormalisable manner through a judicious choice of gauge. For example, (0.64) is invariant under

$$A^{\mu} \rightarrow A^{\mu} + \partial^{\mu}\,\Lambda(x) \quad .$$

$$\phi_1(x) \to \phi_1(x) - 2\sin(\Lambda(x)/2)\{\sin(\Lambda(x)/2)\phi_1 + \cos(\Lambda(x)/2)\phi_2 + \sin(\Lambda(x)/2)\lambda\}$$
$$\phi_2(x) \to \phi_2(x) - 2\sin(\Lambda(x)/2)\{\sin(\Lambda(x)/2)\phi_2 - \cos(\Lambda(x)/2)\phi_1 + \cos(\Lambda(x)/2)\lambda\} \quad (0.69)$$

and the choice of gauge is made by adding the term

$$-\tfrac{1}{2}C(x)\,C(x) \quad (0.70)$$

to the Lagrangian, where

$$C(x) = \left(\frac{1}{\alpha}\,\partial^\mu A_\mu + \alpha\, e\, \lambda\, \phi_2\right) . \quad (0.71)$$

The form of the gauge function (0.71) is chosen so that the cross-terms cancel the term $-e\lambda A_\mu \partial^\mu \phi_2$ in the Lagrangian, (0.64), which is now

$$\begin{aligned} \equiv & -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \frac{1}{2}\frac{1}{\alpha^2}(\partial^\mu A_\mu)^2 + \frac{1}{2}e^2\lambda^2 A_\mu A^\mu \\ & + \frac{1}{2}\partial_\mu\phi_1\,\partial^\mu\phi_1 - \frac{1}{2}(\mu^2 + 3h\lambda^2)\phi_1^2 + \frac{1}{2}\partial_\mu\phi_2\,\partial^\mu\phi_2 - \frac{1}{2}\alpha^2 e^2\lambda^2\phi_2^2 \\ & + e\partial_\mu\phi_1 A^\mu\phi_2 + \frac{1}{2}e^2 A_\mu^2\phi_2^2 - e\partial_\mu\phi_2\, A^\mu\phi_1 + \frac{1}{2}e^2 A_\mu^2\phi_1^2 \\ & + e^2\lambda A^2\phi_1 - h\lambda\phi_1(\phi_1^2 + \phi_2^2) - \frac{h}{4}(\phi_1^2 + \phi_2^2)^2 \end{aligned} \quad (0.72)$$

The vector propagator is $\dfrac{g^{\mu\nu} - (1-\alpha^2)k^\mu k^\nu/(k^2 - \alpha^2 e^2\lambda^2 + i\varepsilon)}{k^2 - e^2\lambda^2 + i\varepsilon}$ and to regain the explicitly unitary formalism the limit as $\alpha \to \infty$ must be taken. In this limit the mass of the scalar field ϕ_2 also tends to infinity and it, obviously, must be a ghost, i.e. unphysical particle. To construct the correct S-matrix the contributions of the Faddeev-Popov ghost, appropriate for the gauge transform (0.69) and the gauge function, (0.71), must be included. 't Hooft, also, demonstrated for

a certain model, that the electromagnetic mass differences are finite as suggested by Weinberg[46].

Spontaneously broken gauge theories had only been shown to be renormalisable according to power counting. Within the framework of the path integral formulation, Lee and Zinn-Justin[47-49] showed the renormalisability and unitarity of various examples of these theories. Also, the invariance of the S-matrix under different choice of the gauge function was demonstrated. This had been assumed in much of the above work. Again Ward identities were heavily relied upon.

't Hooft and Veltman[50] dispelled any doubts, over the use of path integral methods, by demonstrating that the functional manipulations have diagrammatic equivalents which can be fully justified by combinatoric manipulations, i.e. by direct manipulation of the vertices and propagators in an amplitude. Consider a general Lagrangian, $\mathcal{L}_{INV}(A_i)$, of the fields $A_i(x)$ invariant under the infiniestimal gauge transform

$$A_i \rightarrow A_i' = A_i + g\,\hat{s}_{ia}(A)\,\Lambda^a(x) + \hat{t}_{ia}\Lambda^a(x) \qquad (0.73)$$

where the $\Lambda^a(x)$ are the parametrisation of the transformation. The circumflex on the $\hat{s}$ and $\hat{t}$ denotes there may be derivatives present which act, also, on the $\Lambda^a(x)$. The $\hat{t}$ is independent of the fields. For example, if $\mathcal{L}_{INV}(A_i)$ is the massless Yang-Mills Lagrangian, (0.3),

$$\begin{aligned} g\,\hat{s}_{ia}(A)\,\Lambda^a(x) &\equiv \left[g\,\underline{W}^\mu \times \underline{\eta}(x)\right]^b \\ \hat{t}_{ia}\,\Lambda^a(x) &\equiv \left[-\partial^\mu\,\underline{\eta}(x)\right]^b \end{aligned} \qquad (0.74)$$

and (0.73) is equivalent to the infinitesimal transformation

(0.9). For the spontaneously broken gauge theory (0.64) the transformation (0.73) is (0.69). 't Hooft and Veltman[50] showed how to generate the correct Feynman rules by choosing a gauge function $C_a(x)$ of the fields, $A_i(x)$, with $C_a(x)$ transforming as

$$C_a(x) \rightarrow C_a(x) + g\,\hat{\ell}_{ab}(A)\Lambda^b(x) + \hat{m}_{ab}\Lambda^b(x) \qquad (0.75)$$

under (0.73). $\frac{1}{2}C_a^{\,2}$ is subtracted from, and a Faddeev-Popov ghost Lagrangian

$$L_\phi \equiv \phi_a^*\,(\hat{m}_{ab} + g\,\hat{\ell}_{ab}(A))\phi_b \qquad (0.76)$$

added to, the Lagrangian:

$$\mathcal{L} \equiv \mathcal{L}_{INV} - \tfrac{1}{2}C_a^{\,2} + L_\phi \ . \qquad (0.77)$$

The Faddeev-Popov ghost loops generated by (0.76) have an associated factor of (-1) as usual. The scalar ϕ^a is treated as if complex to create asymmetric vertices when required, e.g. for the massless Yang-Mills Lagrangian (0.76) gives rise to the rules of Fig. 4(c) when the gauge function is

$$C^a(x) \equiv \frac{1}{\alpha}\,\partial_\mu W_\mu^a(x) \ . \qquad (0.78)$$

If a source term

$$R_i(A)J^i \qquad (0.79)$$

is added to the Lagrangian such that it transforms as

$$R_i(A) \rightarrow R_i(A) + g\,\hat{\rho}_{ia}(A)\Lambda^a(x) + \hat{r}_{ia}\Lambda^a(x) \qquad (0.80)$$

under (0.73), 't Hooft and Veltman[50] demonstrated that the amplitudes obey the generalised Ward identities of Fig. 5. The notation used is shown in Fig. 6. The source functions $R_i(A)$

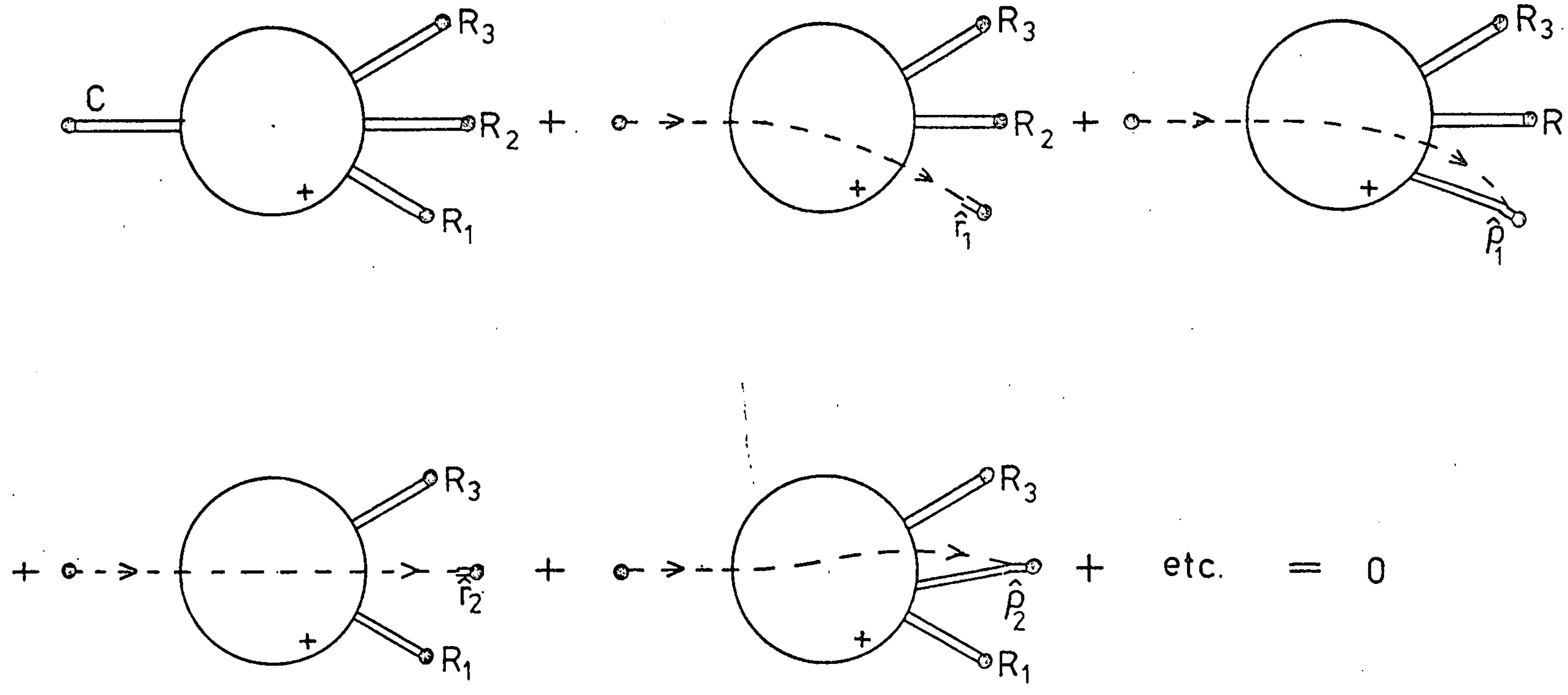
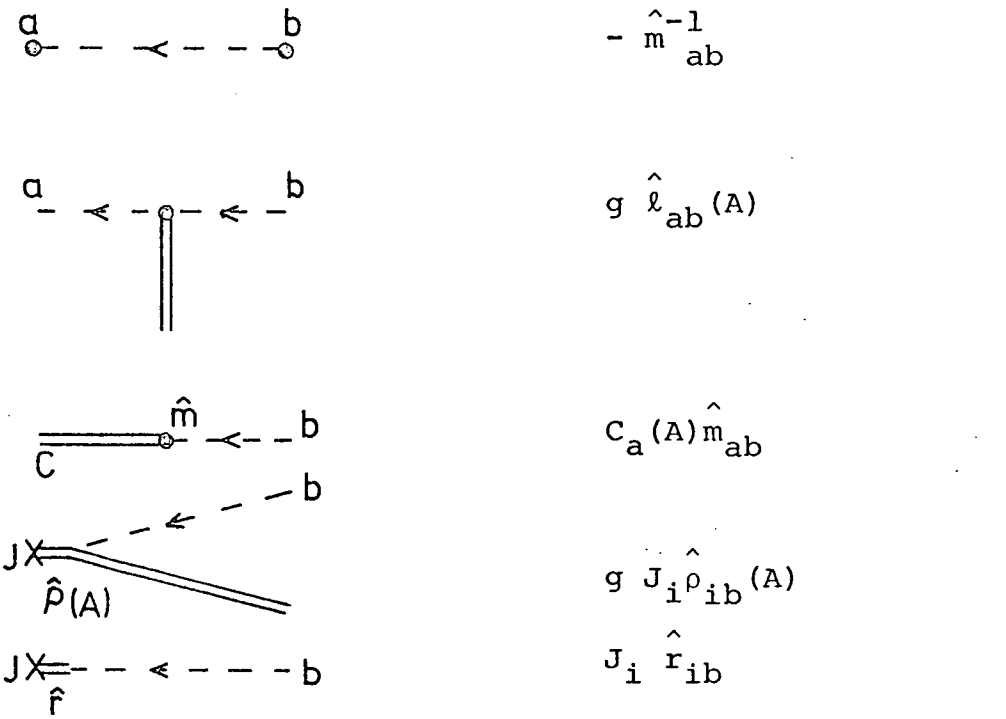
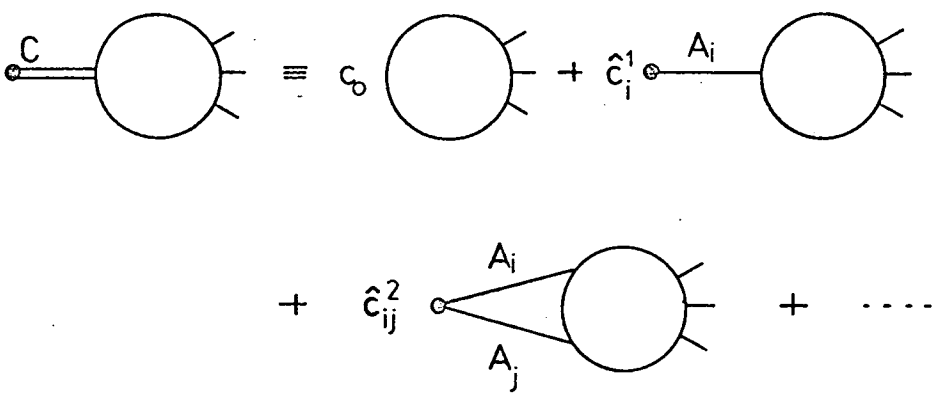


FIG 5 The + designates the inclusion of Faddeev-Popov ghost loops.





where $C(A) = c_o + \hat{c}^1_i . A_i + \hat{c}^2_{ij} A_i A_j + \ldots$

The same notation is used for

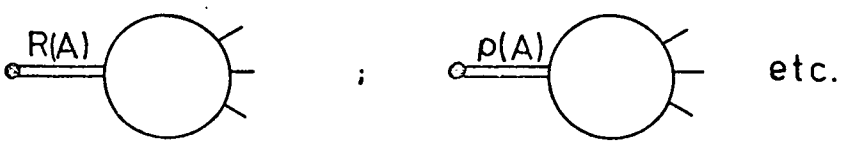


FIG 6 The "blobs" contain all diagrams of a set order in g constructed with the Feynman rules for the appropriate Lagrangian.

can be chosen to be anything and the J^i's may be, subsequently, dropped from the identities. Using the Ward identities the invariance of the S-matrix of a gauge theory (spontaneously broken gauge theories are covered by the term gauge theory in this context) under a change of gauge was proved combinatorically and hence unitarity established for all gauges. The renormalisability of these theories was also shown by demonstrating that the original theory plus divergent subtractions was itself a gauge theory. Therefore, doing the substractions in a manifestly renormalisable gauge results in a well-behaved theory. Instrumental in this is the Tree-Loop theorem[50]:

If there exists functions $C_a(A)$, of the fields A_i, and matrices $\hat{s}_{ia}(A)$, $\hat{t}_{ia}$, $\hat{\ell}_{ab}(A)$ and $\hat{m}_{ab}$ such that

$$C_a(A_i + g\hat{s}_{ia}(A)\Lambda^a + \hat{t}_{ia}\Lambda^a) \equiv C_a(A) + g\hat{\ell}_{ab}(A)\Lambda^b + \hat{m}_{ab}\Lambda^b \qquad (0.81)$$

and the Feynman rules for the Lagrangian, $\mathcal{L}$, obey the Ward identities for tree diagrams, Fig. 7, constructed with Faddeev-Popov ghost rules $\hat{\ell}_{ab}(A)$ and $\hat{m}_{ab}$ and source vertices $R^i(A)$, $\hat{\rho}_{ia}(A)$ and $\hat{r}_{ia}$ defined by

$$R_i(A_i + g\,\hat{s}_{ia}(A)\Lambda^a + \hat{t}_{ia}\Lambda^a) = R_i(A) + g\hat{\rho}_{ia}(A)\Lambda^a + \hat{r}_{ia}\Lambda^a \qquad (0.82)$$

for some function R_i; then the Lagrangian $\mathcal{L}$ can be rewritten

$$\mathcal{L} \equiv L_1 - \tfrac{1}{2}C_a{}^2 \qquad (0.83)$$

where L_1 is invariant under the infinitesimal transformation

$$A_i \to A_i' = A_i + g\,\hat{s}_{ia}(A)\Lambda^a + \hat{t}_{ia}\Lambda^a \quad . \qquad (0.84)$$

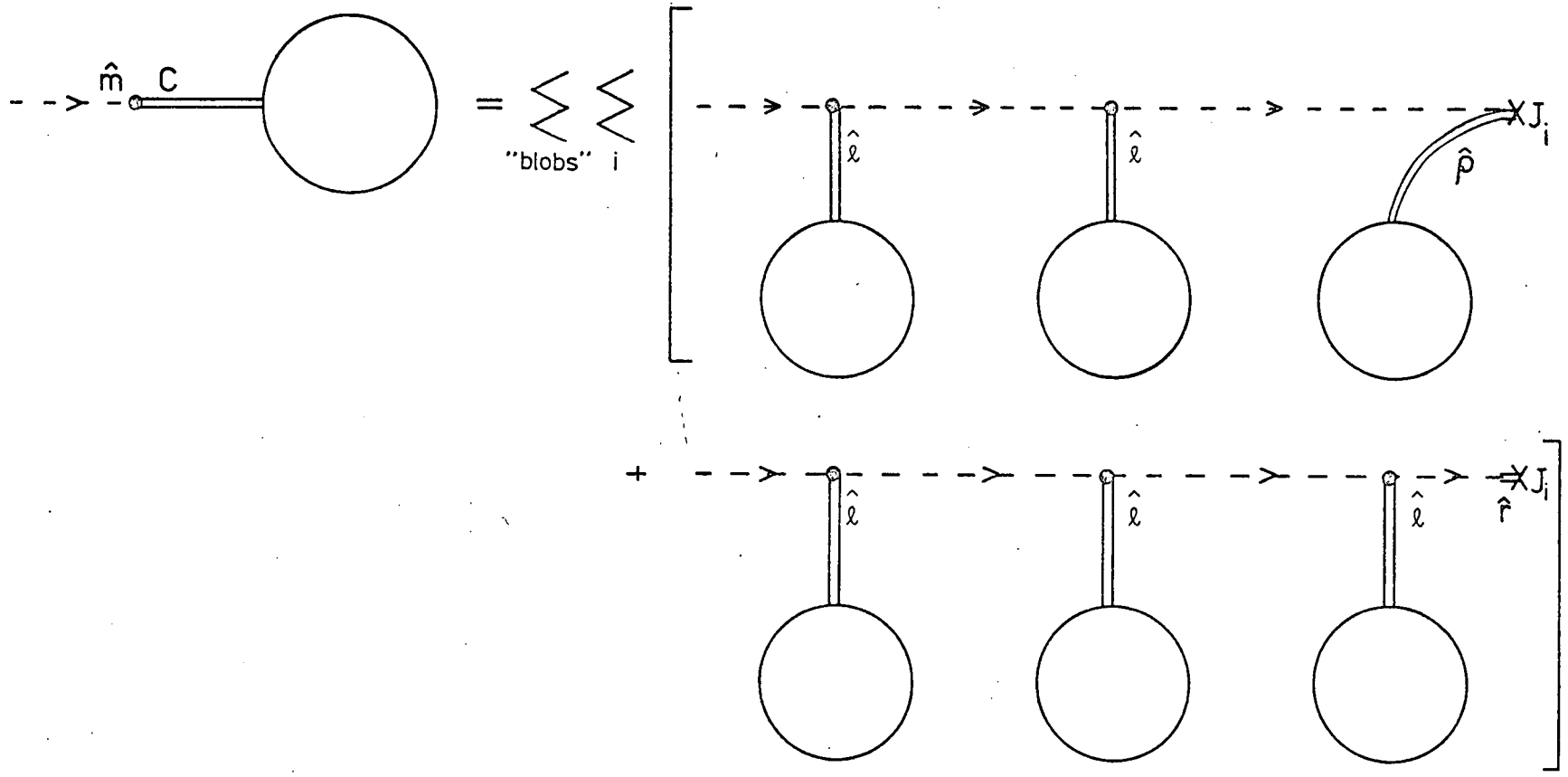


FIG 7 Generalised Ward identities for tree diagrams.

Furthermore, the generalised Ward identities will hold for diagrams with loops.

The theorem has the important implication that the Ward identiies contain the full symmetry of the theory in that the invariance can be deduced from them.

However, it is still necessary to present and execute a programme of renormalisation for a theory. Such a programme for the Salam-Weinberg model[35] was demonstrated by Ross and Taylor[51]. 't. Hooft and Veltman[50] had only considered the purely divergent contributions to the subtraction constants and not the finite contributions to them. Ross and Taylor found that the renormalisation counter-terms could not be freely chosen[51,52] on-mass shell in the conventional manner, as they are interrelated through the Ward identities and each constant is involved with more than one process.

To execute many of the concepts discussed above, it is necessary to have a regularisation procedure strong enough not to disturb the symmetry or the Ward identities of the field theory. The dimensional regularisation scheme of 't. Hooft and Veltman[53], which continues the dimensions of the integration variables analytically from a region in which the integration is finite, is just such a scheme.

To obtain the manifestly unitary formulation of a spontaneously broken gauge theory the limit $\alpha \to \infty$, in for example the Higgs' model (0.72), must be taken. To be sure this does not affect the renormalisability it must be checked that the theory may be renormalised in the unitary gauge. The study of

theories in the U-gauge[54-56] culminated in the demonstration by Mainland, O'Raifeartaigh and Sherry[57] that the renormalisation and unitary gauges are connected by a point transformation. Thus the renormalisation in the unitary gauge can be realised.

c) Synopsis

The object of this thesis is to investigate the following two aspects of vector-boson theories.

The first is to examine the renormalisation of gauge theories. We have mentioned that the renormalisability[47,48,50] of gauge field theories has been shown in that the original Lagrangian plus the counter-terms, necessary to remove the purely divergent contributions to the amplitudes, form a gauge theory themselves. Explicit renormalisation programmes for various models have been investigated[51,52] and it has been found that the choice of the complete renormalisation counter-terms, i.e. finite plus divergent parts, is restricted by the symmetries present and the need to absorb the counter-terms in scaling constants. We explore the renormalisation programme for any gauge theory, in a model independent manner, to see what types of counter-terms may be accommodated. In particular, we examine the extent of the restrictions on the points about which the renormalisation subtractions may be made, i.e. the choice of the finite parts of the counter-terms. In Chapter One we consider the wave function and coupling constant renormalisations and find we have complete freedom of choice, including the ability to renormalise on-mass shell and absorb

any infra-red terms then necessary. The infra-red terms are those associated with an abelian field as in Quantum Electrodynamics and not with a set of non-abelian fields as in the massless Yang-Mills Lagrangian. The effect of the choice of renormalisation counter-terms is simply to change the representation of the gauge invariance of the renormalised Lagrangian. In Chapter Two we consider the mass renormalisations. The situation is not quite so simple in this case. Some, but not necessarily all, of the masses may be renormalised independently with complete freedom of choice of the finite contributions. The number is dependent on the structure of the Lagrangian but usually onlythe Higgs scalars are restricted. The choice of counter-terms for the masses does not affect the representation of the invariance only the form of the Lagrangian. One interesting subsidiary result is that the Tree-Loop theorem of 't. Hooft and Veltman[50] is not as strong as it seems. We find that for a Lagrangian and a transform which fulfil all the required conditions L_1, of (0.83), is not necessarily invariant under that transform, although the Lagrangian must be a gauge theory. The method of investigation in Chapters One and Two is by means of the Ward identities.

The other topic investigated here is the massive Yang-Mills Lagrangian. In Chapter Three, we investigate the Ward identities of the theory for the tree approximation. They are found to be very similar to those of the massless Yang-Mills Lagrangian and a comparison of the implications is made. One result is that the abelian theory is renormalisable as is well known[16,18]. The analysis of the massive Yang-Mills Lagrangian in terms of

transverse vector-boson propagators and compensating scalar fields, which was executed for the self-energy amplitude by Veltman[14] and for the four-point interaction by Mohapatra, Sakakibara and Sucher[12] to the two loop approximation only, is extended to all orders for all interactions by means of generalised Ward identities in Chapter Four. The effective scalar Lagrangian is identical to that obtained by Boulware[6]. In Chapter Five we demonstrate that our approach is equivalent to that of Veltman's[14]. Hence we have demonstrated the equivalence of Boulware's [6] and Veltman's[14] reformulations of the massive Yang-Mills Lagrangian. We also explore whether the massive theory can be reformulated in terms of the soft rules of Fig. 1, i.e. if we choose the factor associated with each scalar loop to be $(-\frac{1}{2})$ we have the formulation proposed by Hsu and Sudarshan[17]. It appears we cannot. However, we further show that it is the S-matrix which is equivalent in the reformulations of the theory and not the amplitudes. Hence, these investigations do not entirely rule out the possibility that the massive Yang-Mills Lagrangian is renormalisable or that the reformulation of Hsu and Sudarshan is equivalent although they make it highly improbable. In Chapter Six we investigate the renormalisability of the massive Yang-Mills Lagrangian. First we show the reformulation of Hsu and Sudarshan[17] is not of a purely spin-one field but incorporates a component with spin-zero. By means of path-integral techniques two possible altern ative formulations are derived for the SU(n) Lagrangian. The second is substantiated by direct combinatorical analysis. It is quasi-renormalisable in the sense that in conjunction with the dimensional regularisation

scheme of 't. Hooft and Veltman[53] the theory can be expressed in terms of the soft rules of Fig. 1 with $\alpha = 0$ and a factor $(-\frac{1}{2})$ associated with each scalar loop.

CHAPTER 1

WAVE FUNCTION AND COUPLING CONSTANT RENORMALISATION

The points, about which subtractions are made, i.e. the choice of finite contributions, in the renormalisation of a gauge theory, appear to be restricted by the very symmetry that enables the Lagrangian to be renormalisable[51,52]. Here we shall examine the extent to which the wave function and coupling constant renormalisation is constrained. An important consideration is that the counter-terms must be consistent with the Ward identities, i.e. if initially the theory obeys Ward identities, the theory plus counter-terms must also obey Ward identities. This is necessary to ensure that the s-matrix is invariant under the choice of gauge and that unitarity continues to hold.

't. Hooft and Veltman[50] have shown gauge theories are renormalisable to the extent of adding purely divergent counter-terms to the Lagrangian. The approach in Chapters One and Two is to add finite counter-terms to the original gauge invariant Lagrangian:

$$\mathcal{L} \rightarrow \mathcal{L}' = \mathcal{L} + \text{C.T. (finite)} \quad .$$

Obviously $\mathcal{L}'$ is renormalisable in the sense that the renormalisation counter-terms may be chosen as those for $\mathcal{L}$ plus the finite counter-terms previously added. The renormalisation becomes valid if the Lagrangian $\mathcal{L}'$ is itself a gauge theory as then $\mathcal{L}'$ may be treated in the manner of (50) and the purely polar terms removed. To realise this, it is only necessary to show $\mathcal{L}'$ obeys Ward identities. These are established using

the identities for the original Lagrangian $\mathcal{L}$.

The Ward identity restriction on the counter-terms is automatically catered for by the method of investigation. The finite additions to the Lagrangian are divided into two types. Those which modify the Ward identities through a multiplicative factor (which may be dropped) treated in this chapter; and those, for which the Ward identities for $\mathcal{L}'$ are set up by iterating the identities for $\mathcal{L}$, which are treated in Chapter Two.

a) Treatment of Self-Energy Terms

The notation and treatment follow 't. Hooft and Veltman[28,50]. Consider a Lagrangian $\mathcal{L}_{INV}(A_i)$ invariant under the infinitesimal gauge transformation

$$A_i \to A_i' = A_i + g\,\hat{s}_{ia}(A)\Lambda^a(x) + \hat{t}_{ia}\Lambda^a(x) \qquad (1.1)$$

and a gauge function $C_a(x)$ of the fields $A_i(x)$ with $C_a(x)$ transforming as

$$C_a(x) \to C_a(x) + g\,\hat{\ell}_{ab}(A)\Lambda^b(x) + \hat{m}_{ab}\Lambda^b(x) \qquad (1.2)$$

As discussed in the introduction the complete Lagrangian is

$$\mathcal{L} = \mathcal{L}_{INV} - \tfrac{1}{2}C_a^2 + L_\phi \qquad (1.3)$$

with the Faddeev-Popov ghost Lagrangian

$L_\phi = \phi_a^*(\hat{m}_{ab} + g\,\hat{\ell}_{ab}(A))\phi_b$.

The s-matrix constructed with Feynman rules obtained directly from this Lagrangian, but with a factor (-1) associated with each ghost loop, is invariant under the choice of the gauge function C_a and unitary[50]. Proof of these properties follows from the manipulation of Ward identities.

If a source term $J_i R^i(A)$, where $R^i(A)$ is arbitrary and transforms

$$R_i(A) \rightarrow R_i(A) + g\,\hat{\rho}_{ia}(A)\Lambda^a(x) + \hat{r}_{ia}\Lambda^a(x), \qquad (1.4)$$

is added to the Lagrangian (1.3), the theory obeys the generalised Ward identities shown in Fig. 5. A particular subset is the Ward identities for tree diagrams in Fig. 7.

For the rest of this chapter in discussing self-energy terms the mass-like contributions are absorbed into the denominators of the propagators but not renormalised yet. The self-energy terms are then of the form of wave function renormalisations.

The facility to add counter-terms (not necessarily divergent) to the Lagrangian to renormalise the wave functions is best shown by the following construction:

$$\text{If } \hat{S}_{ia}(A) \equiv \hat{\mathcal{S}}^1_{iaj}A_j + \hat{\mathcal{S}}^2_{iak}A_j A_k + \cdots\cdots$$

$$\text{define } \hat{S}'_{ia}(A) \equiv \hat{\mathcal{S}}^1_{iam}N_{mj}A_j + \hat{\mathcal{S}}^2_{iamn}N_{mj}N_{nk}A_jA_k + \cdots\cdots$$

where N is any non-singular matrix.

Set $\hat{T}^S_{ia}(A) \equiv N^{-1}_{ij}\,\hat{S}'(A)_{jb}\,N'_{ba}$ where N' is any non-singular matrix and $\hat{T}^t_{ia} \equiv N^{-1}_{ij}\,\hat{t}_{jb}\,N'_{ba}$.

A new field theory invariant under the transform

$$A_i \rightarrow A'_i = A_i + g\,\hat{T}^S_{ia}(A)\Lambda^a + \hat{T}^t_{ia}\Lambda^a \qquad (1.5)$$

is constructed as below.

In the original theory, $\mathcal{L}_{INV}$ with gauge function C_a, C_a is restricted to be linear in the fields A_i, let

$C_a \equiv \hat{M}_{ai} A_i$, for example. Now we define the new gauge function

$$C'_a \equiv \hat{G}_{ai} A_i = N'_{ac} \hat{M}_{cj} N_{ji} A_i \quad . \tag{1.6}$$

Under (1.5) the function C_a' transforms as

$$C'_a + g \hat{G}_{ai} \hat{T}^s_{ib}(A)\Lambda^b + \hat{G}_{ai} \hat{T}^t_{ib} \Lambda^b \quad . \tag{1.7}$$

Hence the ghost vertices and propagator functions become

$\hat{V}_{ab}(A) \equiv N'_{ad} \hat{\ell}'(A)_{de} N'_{ef}$ where $\hat{\ell}'_{de}(A) = \hat{M}_{di} \hat{S}'(A)_{ie}$ and $\hat{B}_{ab} \equiv N'_{ad} \hat{m}_{de} N'_{eb}$.

As N' is non-singular and the inverse of $\hat{m}$ exists, $\hat{B}$ is a proper ghost propagator function which has an inverse.

We construct from $\mathcal{L}_{INV} - \frac{1}{2}C_a^2$ a new Lagrangian such that if $A_i \hat{V}_{ij} A_j$ is any bilinear it is replaced by $A_i \tilde{N}_{ik} \hat{V}_{k\ell} N_{\ell j} A_j$ and the vertex terms $\hat{\alpha}_{ijk} \ldots A_i A_j A_k \ldots$ are replaced by $\hat{\alpha}_{ijk} \ldots N_{i\ell} N_{jm} N_{kn} \ldots A_\ell A_m A_n \ldots$. Denote this Lagrangian by $\mathcal{L}'$.

We have here omitted the possibility of non-Hermitean fields but they can be easily accommodated within the construction.

The source terms for the original Lagrangian $\mathcal{L}_{INV}$ are also chosen to be linear combinations of the fields, e.g. $R_i = \hat{R}_{ij} A_j$. For the reconstructed theory we choose for the sources $R_i' = \hat{R}_{ik} N_{kj} A_j$. They transform under (1.5) as

$$\hat{R}_{ik} N_{kj} A_j + g \hat{R}_{ik} N_{kj} \hat{T}^s_{ja}(A)\Lambda^a + \hat{R}_{ik} N_{kj} \hat{T}^t_{ja} \Lambda^a \tag{1.8}$$

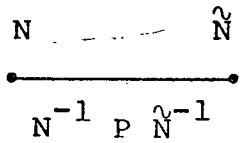
giving the new source-ghost vertices.

$$\hat{\rho}'_{ia}(A) \equiv \hat{R}_{ik}N_{kj}\hat{T}^{s}_{ja}(A) \equiv \hat{R}_{ik}\hat{S}'(A)_{kb}N'_{ba} \equiv \hat{\rho}''_{ib}N'_{ba}$$

and $$\hat{r}'_{ia} \equiv \hat{R}_{ik}N_{kj}\hat{T}^{t}_{ja} \equiv \hat{R}_{ik}\hat{t}_{kb}N'_{ba} \equiv \hat{r}_{ib}N'_{ba} .$$

It can now be shown that the theory $\mathcal{L}' + R_i'J_i$, constructed above with the ghost Lagrangian generated from C_a' by the transformation (1.5), satisfies the Ward identities for tree diagrams (Fig. 7) with a particular choice of N'.

The manipulation of the terms for $\mathcal{L}'$ equivalent to the first set of diagrams on the right hand side of Fig. 7 is as in Fig. 8. The direction in which the diagrams are constructed is reversed, so that the vertices etc. can be written in naturally with the same expressions as in the above derivations and definitions. The final line is obtained by noting that each "physical" propagator appears in the combination

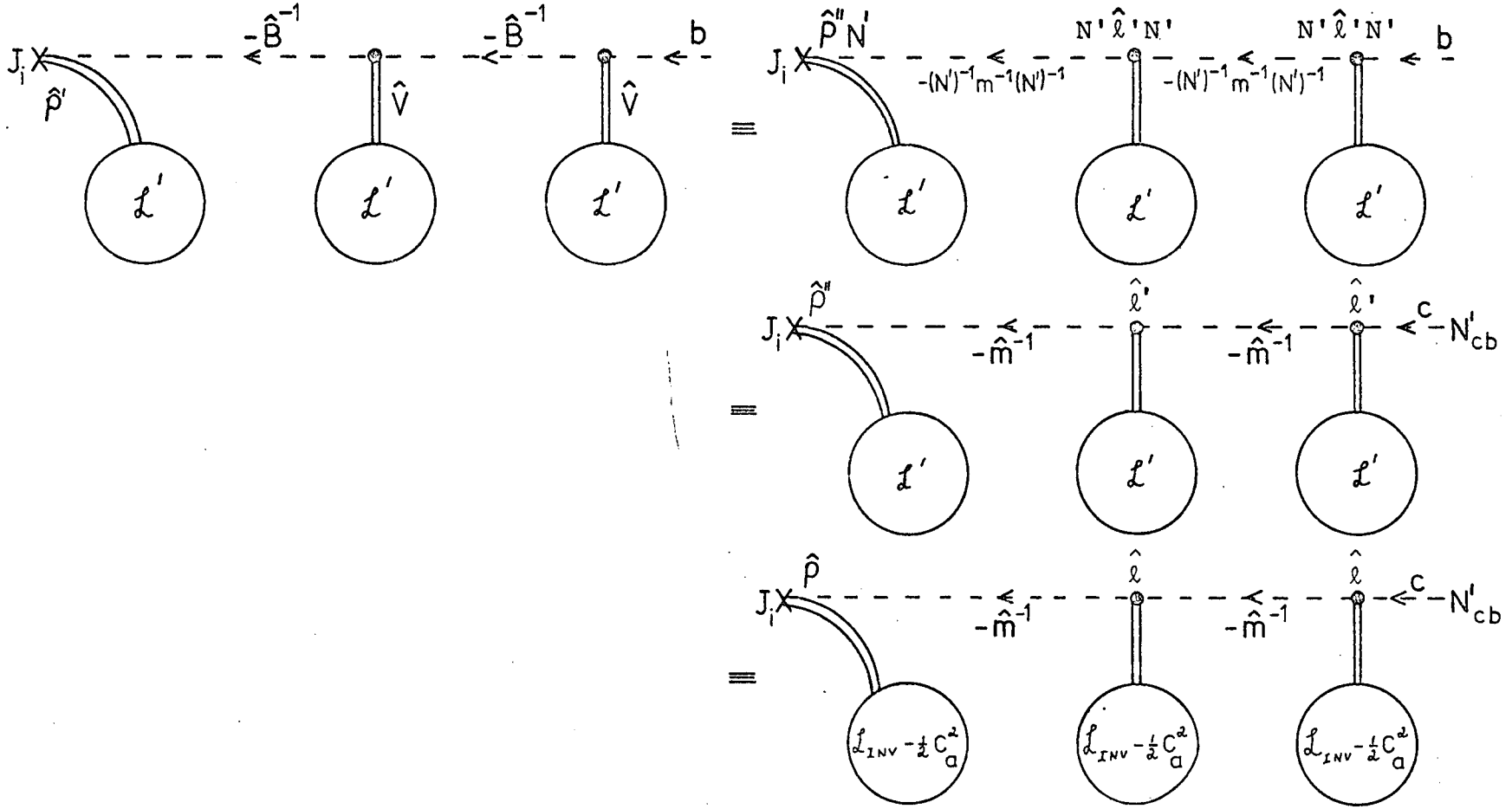


The equivalent identity for the second subset of diagrams on the right hand side of Fig. 7 is shown in Fig. 9 and the identity for the left hand side in Fig. 10.

If N' is chosen such that

$$\tilde{N}' N' = I$$

e.g. $N' = I$, the Ward identities for tree diagrams, of Fig. 7, for the original field theory $\mathcal{L}$ may be used to prove that the new theory $\mathcal{L}'$, constructed as above, obeys the Ward identities of Fig. 11. By the Tree-Loop theorem of (50), $\mathcal{L}'$ may be written as $\mathcal{L}'_{INV} - \frac{1}{2} C_a'^2$ where $\mathcal{L}'_{INV}$ is invariant under (1.5). $\mathcal{L}'_{INV}$ obeys the full Ward identities, which


J_i
$\hat{\rho}'$
$-\hat{B}^{-1}$
$-\hat{B}^{-1}$
b
$\hat{V}$
$\hat{V}$
$\mathcal{L}'$
$\mathcal{L}'$
$\mathcal{L}'$
≡
$\hat{\rho}''N'$
$N'\hat{\ell}'N'$
$N'\hat{\ell}'N'$
b
$-(N')^{-1}m^{-1}(N')^{-1}$
$-(N')^{-1}m^{-1}(N')^{-1}$
$\mathcal{L}'$
$\mathcal{L}'$
$\mathcal{L}'$
$\hat{\rho}''$
$\hat{\ell}'$
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c
N'_{cb}
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c
N'_{cb}
$-\hat{m}^{-1}$
$-\hat{m}^{-1}$
$\mathcal{L}_{INV}-\frac{1}{2}C_a^2$
$\mathcal{L}_{INV}-\frac{1}{2}C_a^2$
$\mathcal{L}_{INV}-\frac{1}{2}C_a^2$

FIG 8

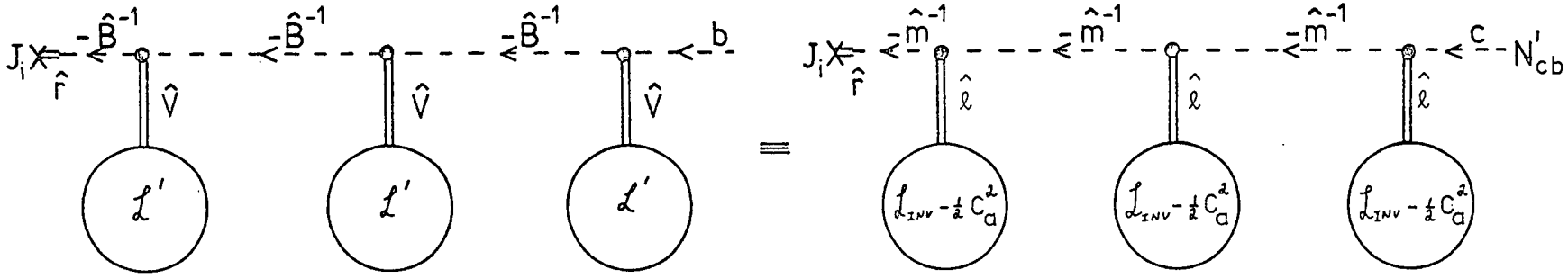

J_i
$\hat{r}$
$-\hat{B}^{-1}$
$-\hat{B}^{-1}$
$-\hat{B}^{-1}$
b
$\hat{V}$
$\hat{V}$
$\hat{V}$
$\mathcal{L}'$
$\mathcal{L}'$
$\mathcal{L}'$
=
J_i
$\hat{r}$
$-\hat{m}^{-1}$
$-\hat{m}^{-1}$
$-\hat{m}^{-1}$
c
N'_{cb}
$\hat{\ell}$
$\hat{\ell}$
$\hat{\ell}$
$\mathcal{L}_{INV} - \frac{1}{2} C_a^2$
$\mathcal{L}_{INV} - \frac{1}{2} C_a^2$
$\mathcal{L}_{INV} - \frac{1}{2} C_a^2$

FIG 9

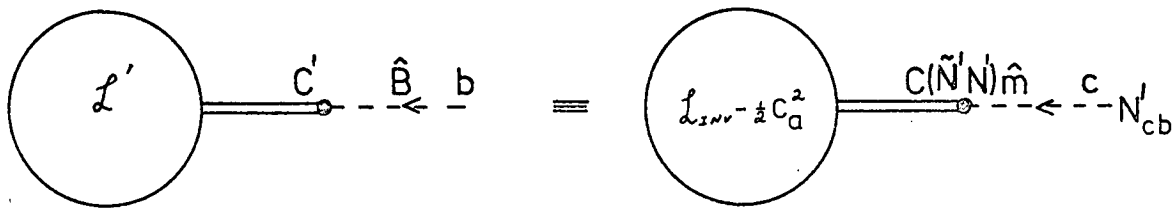

$\mathcal{L}'$
C'
$\hat{B}$
b
≡
$\mathcal{L}_{INV} - \frac{1}{2} C_a^2$
$C(\tilde{N}'N')\hat{m}$
c
N'_{cb}

FIG 10

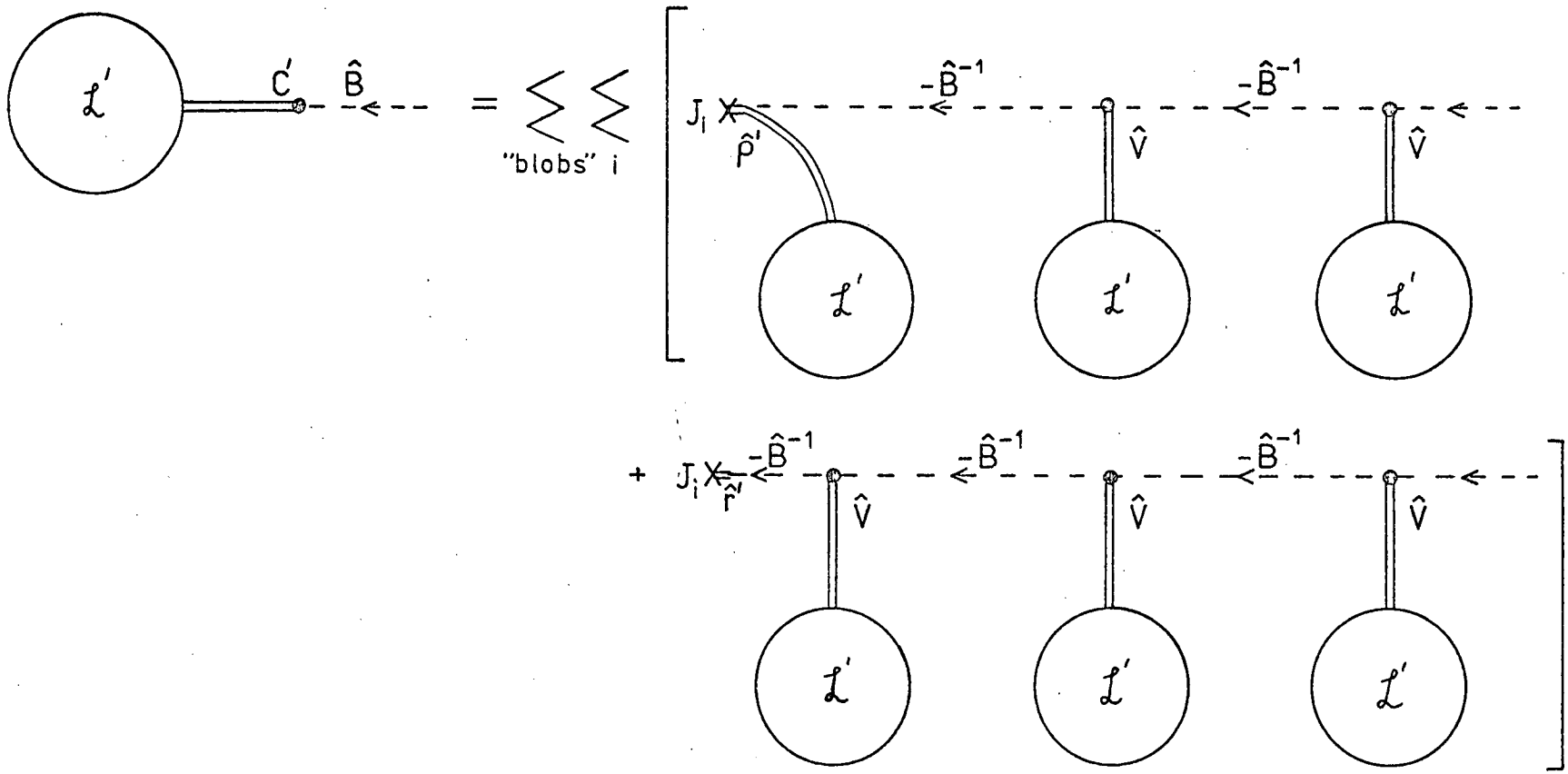
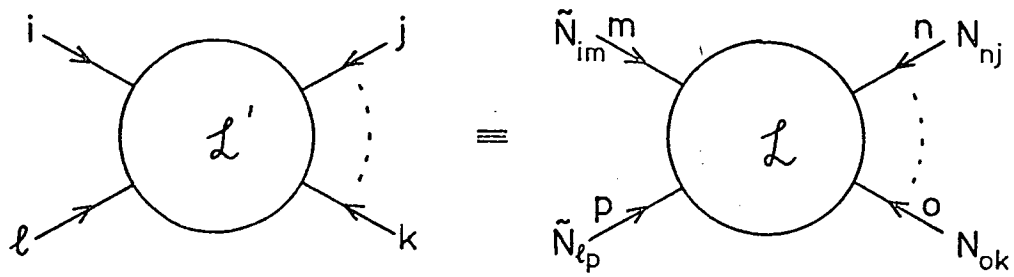


FIG 11

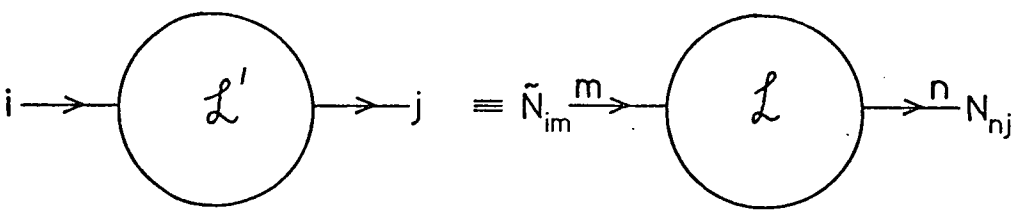
may be shown directly or inferred from the Tree-Loop theorem. It has thus been demonstrated that the finite counter-terms $(\mathcal{L} - \mathcal{L}')$ may be added to the original Lagrangian. In this context the N has terms dependent on the coupling constant, g, but the bilinear terms may be separated into a propagator plus vertices in the perturbation expansion as required.

The counter-terms, added to $\mathcal{L}$ by the above construction, can be used to renormalise the self-energy terms of the physical amplitude since



(1.9)

In particular



(1.10)

The N may be chosen such that, with the subsequent removal of the divergences, the self-energy terms for the "physical" fields are renormalised on mass shell. This varies with the gauge but the most logical choice of counter-terms would be those that set the U-gauge representation renormalisation on-mass shell.

b) Infra-red Considerations

If the theory contained the electromagnetic field, the on-mass shell counter-terms would carry infra-red divergent quantities[58] for the charged fields. The requirement that the infra-red terms sum in the usual manner[59], imposes relations between the infra-red contributions to the counter-terms which must be shown to be consistent with the relations imposed by the symmetry of the Lagrangian, i.e. as the infra-red contributions to an amplitude factorize the infra-red contributions to the counter-terms must allow a similar factorisation. Here we shall show the addition of infra-red divergent counter-terms, which do not affect the infra-red summation but leave the on mass shell self-energy terms infra-red finite, may be made consistently.

It is interesting to note that the solution of the infra-red catastrophe contains an early use of the Ward identity in quantum electrodynamics. There because of the equality of Z_1 and Z_2[58] (also proved by a Ward identity) the charge and electron self-energy renormalisations are spurious. Yennie, Frautschi and Suura[59], for simplicity, assume initially there are no self-energy parts on external lines. The spurious charge renormalisation, connected to the usual wave function renormalisation diagram by the Ward identity of Fig. 12[60], is then removed after summation of all possible insertions of an additional virtual photon to give

$$B(p,p')\ \bar{u}(p')\Gamma(p',p)u(p) + K(p',p;k) \qquad (1.11)$$

where

$$B(p,p') = \tfrac{1}{2}\frac{ie^2}{(2\pi)^4}\int_{\lambda}^{\infty}\frac{d^4k}{k^2}\left(\frac{2p'^{\mu}-k^{\mu}}{k^2-2k.p'}-\frac{2p^{\mu}-k^{\mu}}{k^2-2k.p}\right)^2$$

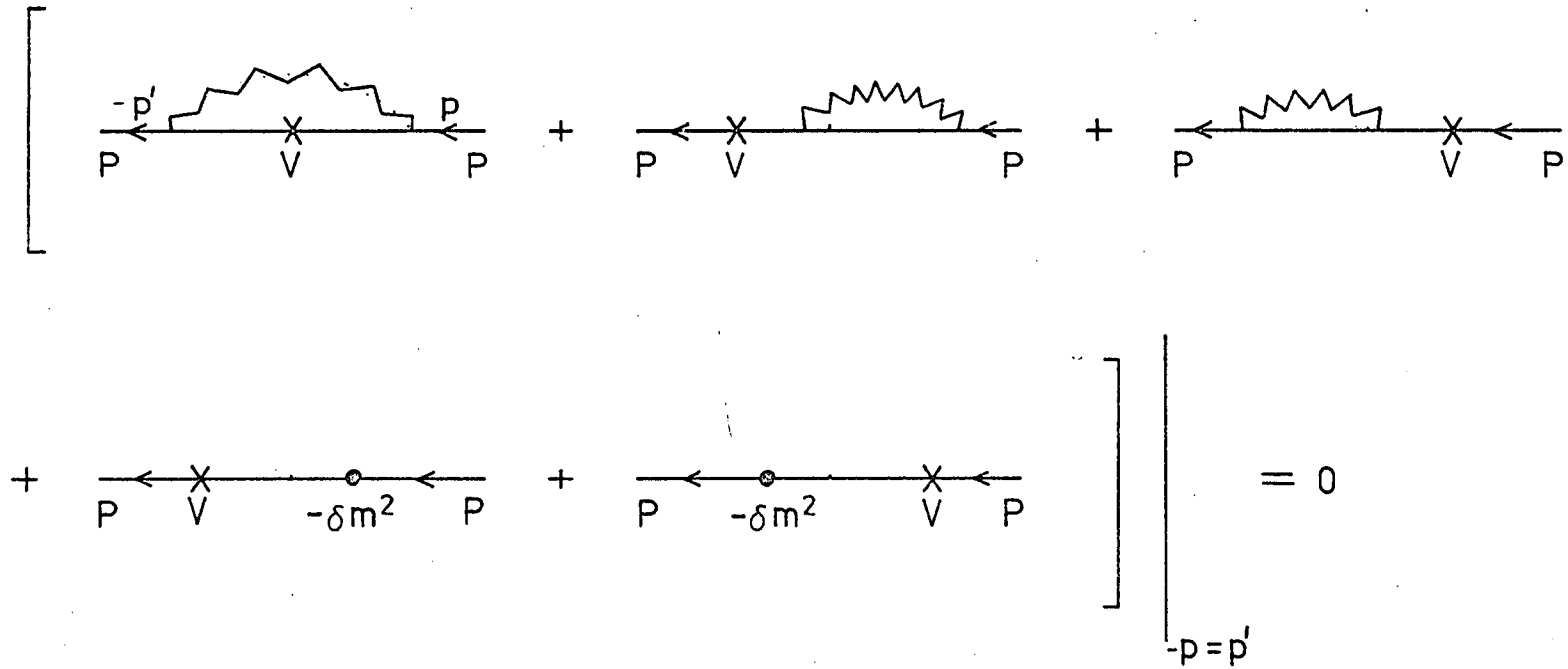


FIG 12 P indicates that the line is external and physical.

~~~~ is the photon; ——— the charged fermion.
~~~~

$\bar{u}(p')\Gamma(p',p)u(p)$ is the amplitude before the addition of the virtual photon; $K(p'p;\ k)$ are the contributions to the amplitude infra-red finite in k; λ is the infra-red cut-off. As expected the k-integral is ultra-violet convergent and no cut-off is necessary.

As it cannot always be guaranteed $Z_1 = Z_2$ we shall include self-energy terms explicitly and evaluate their infra-red component before renormalisation. The discussion is held within the context of quantum electrodynamics but the value of the infra-red components will not be different in another field theory and the conclusions are quite general.

In Appendix A it is shown that if an additional photon is added, to an amplitude including self-energy terms, in all possible ways the infra-red contribution factorizes as

$$B_o\ \bar{u}(p')\Gamma(p',p)u(p) + K(p',p;\ k) \tag{1.12}$$

where

$$B_o = \frac{ie^2}{(2\pi)^2}\int_\lambda^\Lambda \{\frac{(2p'-k)^2}{(k^2-2p'.k)^2} - \frac{(2p'-k).(2p-k)}{(k^2-2p'.k)(k^2-2p.k)} + \frac{(2p-k)^2}{(k^2-2p.k)^2}\}\ \frac{d^4k}{k^2}\ .$$

Λ is an ultra-violet cut-off. The virtual infra-red divergences can now be summed, as in (59), to

$$M_{TOT} \equiv \exp(B_o)\ \sum_{n=0}^{\infty}\ m^o{}_n \tag{1.13}$$

where m_n^o are the infra-red finite contributions from the amplitudes with n virtual photons. From (1.13) (cf. Appendix A) we obtain the total contribution to the self-energy terms

$$M_{S-E} \equiv \exp(B_{S-E})\ \sum_{n=0}^{\infty}\ m_n^{S-E} \tag{1.14}$$

where $B^{S-E} = \frac{ie^2}{(2\pi)^4} \int_{\lambda}^{\Lambda} \frac{d^4k}{k^2(k^2-2p.k)^2} \frac{(2p-k)^2}{}$.

Hence the total contribution to the S-matrix is,

$$M^{S-M} \equiv \exp(B) \sum_{n=0}^{\infty} m_n \qquad (1.15)$$

which agrees with Yennie, Frautschi and Suura[59].

Using this factorisation the infra-red divergent terms from the virtual photons can be shown to cancel with the infra-red divergent terms from the real photons[59]. The summation, obviously the same for any theory, as above is valid for the unrenormalised S-matrix or for the renormalised S-matrix provided the counter-terms are infra-red finite. Q.E.D. may be considered as an example of either as the relevant renormalisations are spurious.

Suppose counter-terms are added, as in section a), to the Lagrangian with the choice for N of

$$N_{ij} = \delta_{ij} n_j \quad \text{(no summation over j)}$$

where $n_j = 1$ if j refers to a neutral particle

or $n_j = \exp(\alpha)$ if j refers to a charged particle.

Using the relation (1.9), between the amplitudes for the original Lagrangian and the amplitudes for the Lagrangian plus counter-terms, it is easily seen that the only difference this makes to the summation to all orders of the virtual photon contributions to an amplitude is to modify (1.13) to

$$M'_{TOT} \equiv \exp(B_o + 2\alpha) \sum_{n=0}^{\infty} m^o{}_n \qquad (1.16)$$

(1.16) could now be reduced to the correct form for the infra-red summation, as in (1.15), by a judicious choice of α. The most obvious

$$\alpha = -\tfrac{1}{2}\frac{ie^2}{(2\pi)^4}\int_\lambda^\infty \frac{d^4k}{k^2}\,\frac{(2p-k)^2}{(k^2-2p.k)^2} \qquad (1.17)$$

is inappropriate as it is both ultra-violet divergent and dependent on p, the momentum of the charged particle. A choice which avoids these difficulties is

$$\alpha = -\tfrac{1}{2}\frac{ie^2}{(2\pi)^4}\int_\lambda^\infty \frac{d^4k}{k^2}\,\frac{(2p)^2}{(k^2-2p.k)^2}$$

$$= -\tfrac{1}{2}\frac{e^2}{(2\pi)^2}\int_\lambda^1 \frac{dx}{x}\,. \qquad (1.18)$$

The differences between (1.17) and (1.18) can be assigned to the infra-red finite terms and (1.16) becomes

$$M'_{TOT} \equiv \exp(B)\sum_{n=0}^{\infty} m^{o\,\prime}_n\,. \qquad (1.19)$$

Immediately we see from (1.10) and (1.14) that the above choice for α makes the self-energy terms infra-red finite as required (and as necessary, as on generating the S-matrix no more infra-red divergences will be introduced and the summation as in (59) remains valid.)

Counter-terms can hence be added to the Lagrangian in such a manner that the on-mass shell self-energy terms are infra-red finite, while keeping the usual form for the summation of all contributions of the virtual photons to the infra-red divergences. Further infra-red finite counter-terms

could then be added to complete the on-mass shell renormalisation of the self-energy terms. Thus the normal on-mass shell renormalisation of self-energy terms may be realised for a gauge theory.

c) Coupling Constant Renormalisation

The coupling constants of a gauge theory are readily renormalised similarly to the self-energy terms in section a). If the coupling constant g is replaced by βg whenever it occurs in the Lagrangian and the ghost vertices and source terms are modified likewise, the only change in the diagrams of the Ward identities for tree diagrams, Fig. 7, is that each side gains a factor $(\beta)^n$, where n is the order in coupling constant of the identity. For simplicity let the gauge and source functions be linear as before. It is then obvious that to obtain the modified ghost vertices and source terms the only change necessary is to replace g by βg whenever it appears in the gauge transform. The new Lagrangian is therefore invariant under the modified transform and the Ward identities hold to all orders by the Tree-Loop theorem. Hence the coupling constant can be suitably renormalised with the counter-terms $(\mathcal{L}' - \mathcal{L})$.

If the theory under consideration contains the electromagnetic field the electric charge may be connected to other coupling constants. However, the normal finite renormalisation[58] for the electromagnetic charge may still be used as the appropriate amplitude, $\Lambda^{\mu}(p',p)$,[68] defining the charge renormalisation, is infra-red finite in the limit $p = p'$ and

with p set on-mass shell, if the self-energy terms have already been renormalised as in section b). No additional infra-red divergences arise.

The analysis is easily extended to Lagrangians with more than one independent coupling constant whereby each is renormalised separately.

d) Discussion

The three sets of additions to the Lagrangian discussed above start from a gauge invariant Lagrangian which obeys the Ward identities and end with a gauge invariant Lagrangian which obeys the Ward identities. As the same is true of the purely divergent counter-terms discussed by 't Hooft and Veltman[50] the different sets of counter-terms may be added in any order, i.e. they commute. The most convenient order of application would be

(i) add counter-terms necessary to render on-mass shell self-energy terms infra-red finite if necessary as in section b).

(ii) add finite counter-terms to complete the finite renormalisation of the self-energy terms of "physical" fields as in section a), i.e. render self-energy terms purely divergent.

(iii) add finite counter-terms to renormalise the coupling constant(s) as in section c).

(iv) add purely divergent counter-terms to complete renormalisation as in (50).

The renormalisations should be done order by order in the loop expansion approximation.

We thus see that the self-energy and coupling constant renormalisation can be done, in the conventional manner, on-mass shell. As the coupling constant appears in several vertices one may be chosen to define it and be renormalised on-mass shell, e.g. if the Lagrangian includes photons and electrons the interaction $i\, e\, \bar{\psi}\, \not{A}\, \psi$ could be renormalised as in Q.E.D. The charge renormalisation of all other interactions then follow accordingly.

Since no other way of adding suitable counter-terms consistent with the Ward identities could be found, it seems likely that the divergent counter-terms of (50) are of the form of sections a) and c). It is the invariance of the Lagrangian which makes it possible to render all the vertices finite with only one counter-term for each coupling constant[31,32,48,51]. Similarly the symmetry could be utilized to remove the divergences from the self-energy terms by introducing far fewer independent counter-terms than one for each physical field as proposed[48,50].

The reformalised theory is indeed invariant under the proposed transform as in essence the modifications are only changes in the representation of the original invariance. However, it has been demonstrated that the representation depends on the choice of subtraction points in the renormalisation and is therefore independent of gauge as required[50,51]. The counter-terms can be represented as scaling constants quite readily as only one is needed for each "physical" field and one for each independent coupling constant. This is particularly acceptable in the U-gauge representation although applicable to all gauges. Relying on the equality of divergences (but not finite terms) many of the scaling constants may be made equal, e.g. one for each multiplet of

fields as in Ross and Taylor[51]. The latter is much restricted and does not allow on-mass shell renormalisation.

None of the renormalisations examined in this chapter have any relevance to the mass renormalisation. In Chapter Two the possible additions to the Lagrangian which may be interpreted as mass renormalisation counter-terms will be examined.

CHAPTER 2

MASS RENORMALISATION

So far only additions to the Lagrangian, which may be interpreted as charge or wave-function renormalisation counter-terms, have been investigated. In Chapter Two we consider additions which could be used for mass renormalisation whether finite or infinite. The technique is similar to Chapter One except that the additions preserve the Ward identities by iteration rather than factorisation.

a) General Case

Again we start from the Lagrangian $\mathcal{L}_{INV} - \frac{1}{2}C_a^2$ where $\mathcal{L}_{INV}$ is invariant under

$$A_i \to A_i' = A_i + g\,\vec{S}_{ia}(A)\Lambda^a + \vec{t}_{ia}\,\Lambda^a \quad . \tag{2.1}$$

The notation has been changed by replacing the circumflex, which denotes the presence of derivatives, by an arrow which indicates the presence of derivatives and points in their direction of application. C_a correspon dingly transforms as

$$C_a + g\,\vec{\ell}_{ab}(A)\Lambda^b + \vec{m}_{ab}\,\Lambda^b \tag{2.2}$$

and supplies the ghost vertices $\vec{\ell}_{ab}(A)$ and propagators $-\vec{m}_{ab}^{-1}$. If sources J_{1i}, J_{2i} etc. couple to the field combinations R_{1i}, R_{2i} ... where R_{ij} transform as

$$R_{ij} + g\,\vec{\rho}_{ija}(A)\Lambda^a + \vec{r}_{ija}\Lambda^a$$

then the $\vec{\rho}_{ija}(A)$ and $\vec{r}_{ija}$ are the required ghost-source

terms for the Ward identity in Fig. 5 .

The source functions R_2, R_3, etc. are chosen to be equal, R say, only R_1 being different and henceforth denoted by $\mathcal{R}$. $\mathcal{R}$ is restricted to be linear in the fields, for example $\mathcal{R}_i \equiv \vec{\mathcal{R}}_{ij} A_j$. The corresponding ghost-source functions for R and $\mathcal{R}$ are obtained from

$$\begin{aligned} R_i &\to R_i + g\,\vec{\rho}_{ia}(A)\Lambda^a + \vec{r}_{ia}\Lambda^a \\ \mathcal{R}_i &\to \mathcal{R}_i + g\,\vec{\mathcal{R}}_{ij}\,\vec{s}_{ja}(A)\Lambda^a + \vec{\mathcal{R}}_{ij}\,\vec{t}_{ja}\Lambda^a \ . \end{aligned} \tag{2.3}$$

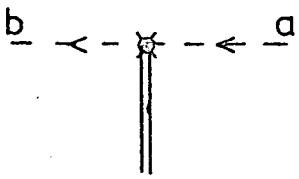
The double-headed arrow indicates the presence of some derivatives acting to the right, some to the left. In practice, this only affects diagrams through a change in sign of some momenta terms and the distinction is really superfluous. However, it is retained for ease of interpretation. The gauge function C_a is again restricted to be linear such that $C_a \equiv \vec{M}_{ai} A_i$ and $\vec{\mathcal{R}}_{ij}$ chosen to obey the relations

$$\begin{aligned} \vec{\mathcal{R}}_{ij}\ \vec{s}_{ja}(A) &\equiv \overleftarrow{\tilde{M}}_{ib}\ \vec{L}_{1bj}\ \vec{s}_{ja}(A) \\ \vec{\mathcal{R}}_{ij}\ \vec{t}_{ja} &\equiv \overleftarrow{\tilde{M}}_{ib}\ \vec{L}_{2bj}\ \vec{t}_{ja} \end{aligned} \tag{2.4}$$

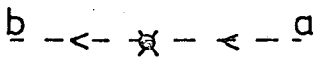
which introduces the additional vertices of Fig. 13. The original Lagrangian $\mathcal{L}_{INV} - \tfrac{1}{2}C_a^2$, with the above choice of source functions, obeys the Ward identities for tree diagrams of Fig. 14. It also obeys the Ward identities with no $\mathcal{R}$ sources of Fig. 15.

If a Lagrangian $\mathcal{L}'$ is constructed such that

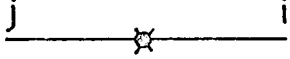
$$\begin{aligned} \mathcal{L}' \equiv \mathcal{L}_{INV} &- \tfrac{1}{2}C_a^2 + \phi_a^*(\vec{m}_{ab} + g\vec{\ell}_{ab}(A))\phi_b + J_i R_i + J_i'\mathcal{R}_i \\ &+ \tfrac{1}{2}A_i \cdot \vec{\mathcal{R}}_{ij}\,A_j + \phi_a^*(\vec{m}'_{ab} + g\,\vec{\ell}'_{ab}(A))\phi_b . \end{aligned} \tag{2.5}$$



$$\overleftrightarrow{\ell'}_{ba}(A) \equiv \overleftrightarrow{L}_{1_{bj}} \overleftrightarrow{S}_{ja}(A)$$



$$\overleftrightarrow{m'}_{ba} \equiv \overleftrightarrow{L}_{2_{bj}} \overleftrightarrow{t}_{ja}$$



$$\overleftrightarrow{\mathcal{R}}_{ij}$$

FIG 13 Additional vertices.

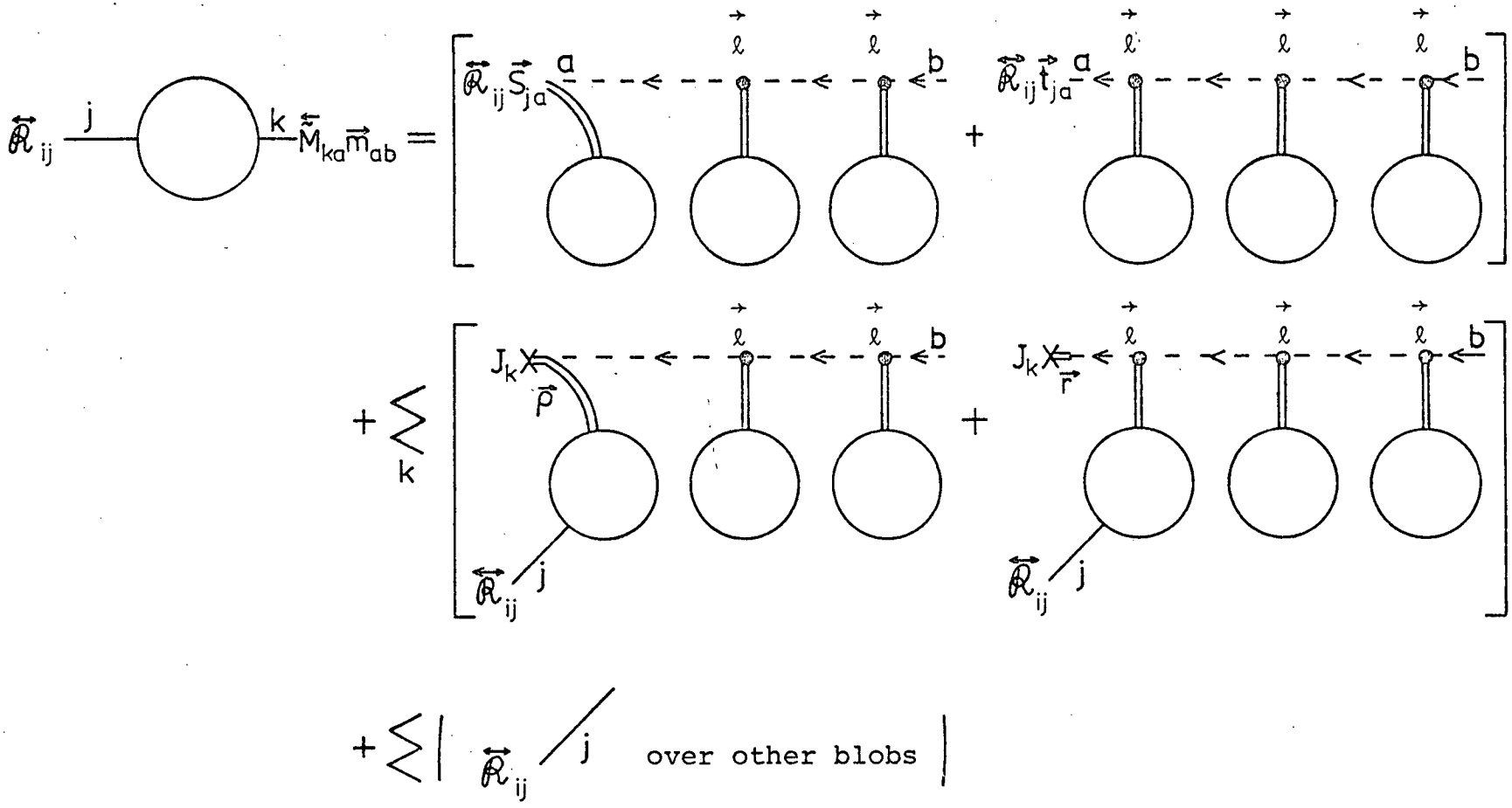


FIG 14 Only the sources $\mathcal{R}$ are shown specifically attached to blobs.

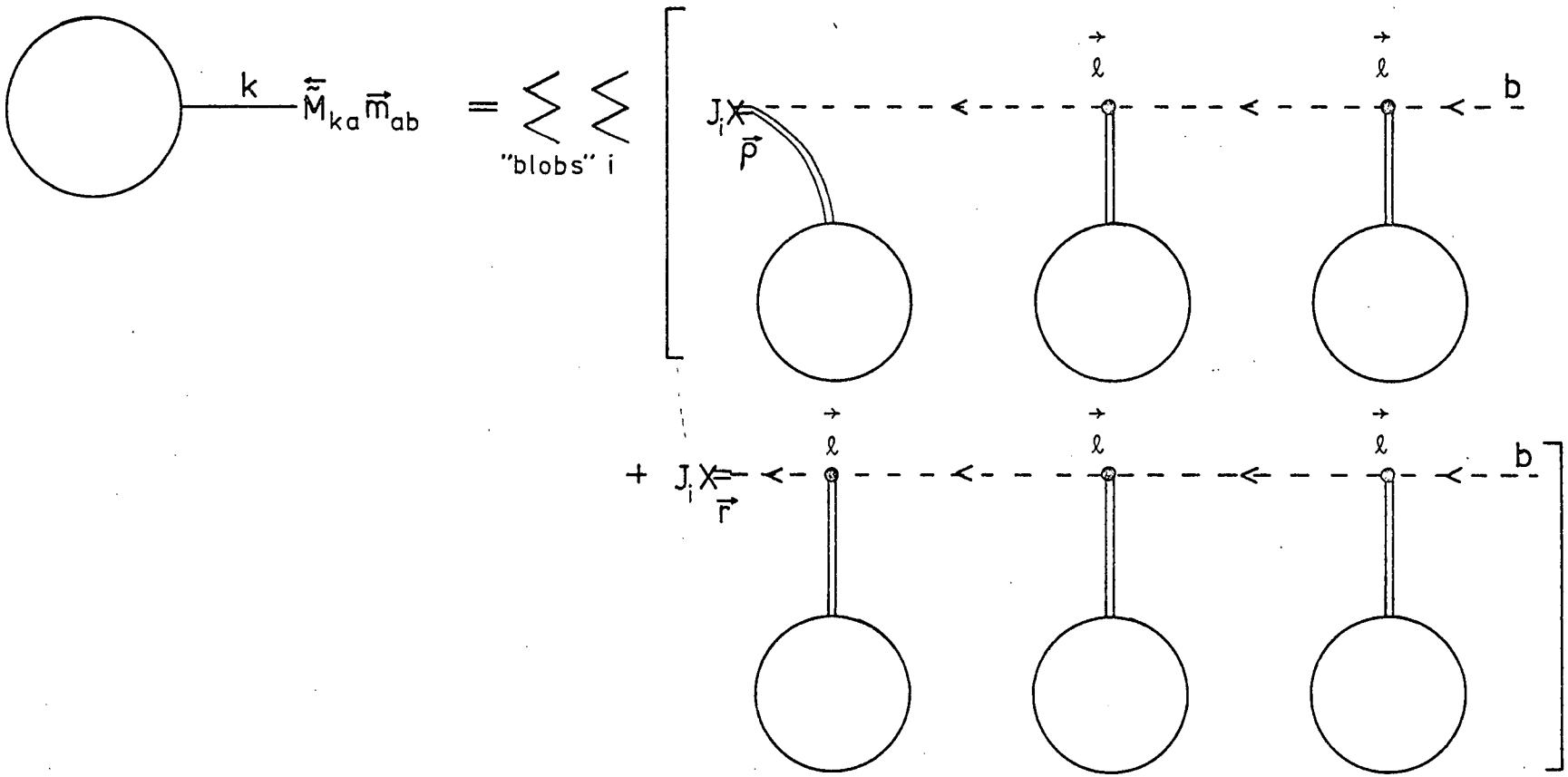
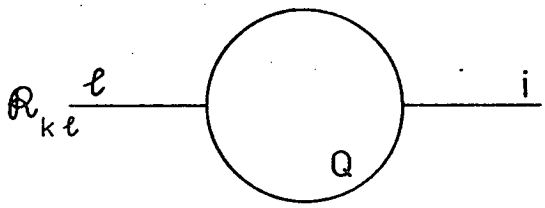


FIG 15

$\mathcal{L}'$ will also obey the Ward identities in Fig. 14 and 15. The proof is by induction.

The Ward identity, Fig. 14, is assumed to hold for diagrams containing all possible choices of n vertices of the types of Fig. 13. (The choices have to remain compatible with the order in coupling constant of the diagrams.) Both sides of the identify Fig. 14 are multiplied on the left by the set of diagrams



(2.6)

where the "blob" is constructed from the original vertices only and contains any number of the sources R as required. The i's are summed over to give the identity in Fig. 16, which is obtained using the relations (2.4) to introduce the gauge function to the set of diagrams (2.6). The Ward identity of Fig. 14 is now applied to the diagrams of (2.6) and we get the identity of Fig. 17. The identity has been proved for one value of x and y only. If the identities for all values of x and y, such that x plus y is constant, are summed then the Ward identity Fig. 14 has been established for (n+1). The identity is known to hold for n=0 when it reduces to the Ward identity of Fig. 14 constructed from the original Lagrangian. Hence the identity for the Lagrangian (2.5) has been established for all n.

If, at any level of the iteration, the identity of Fig. 14 had been multiplied by the set of diagrams

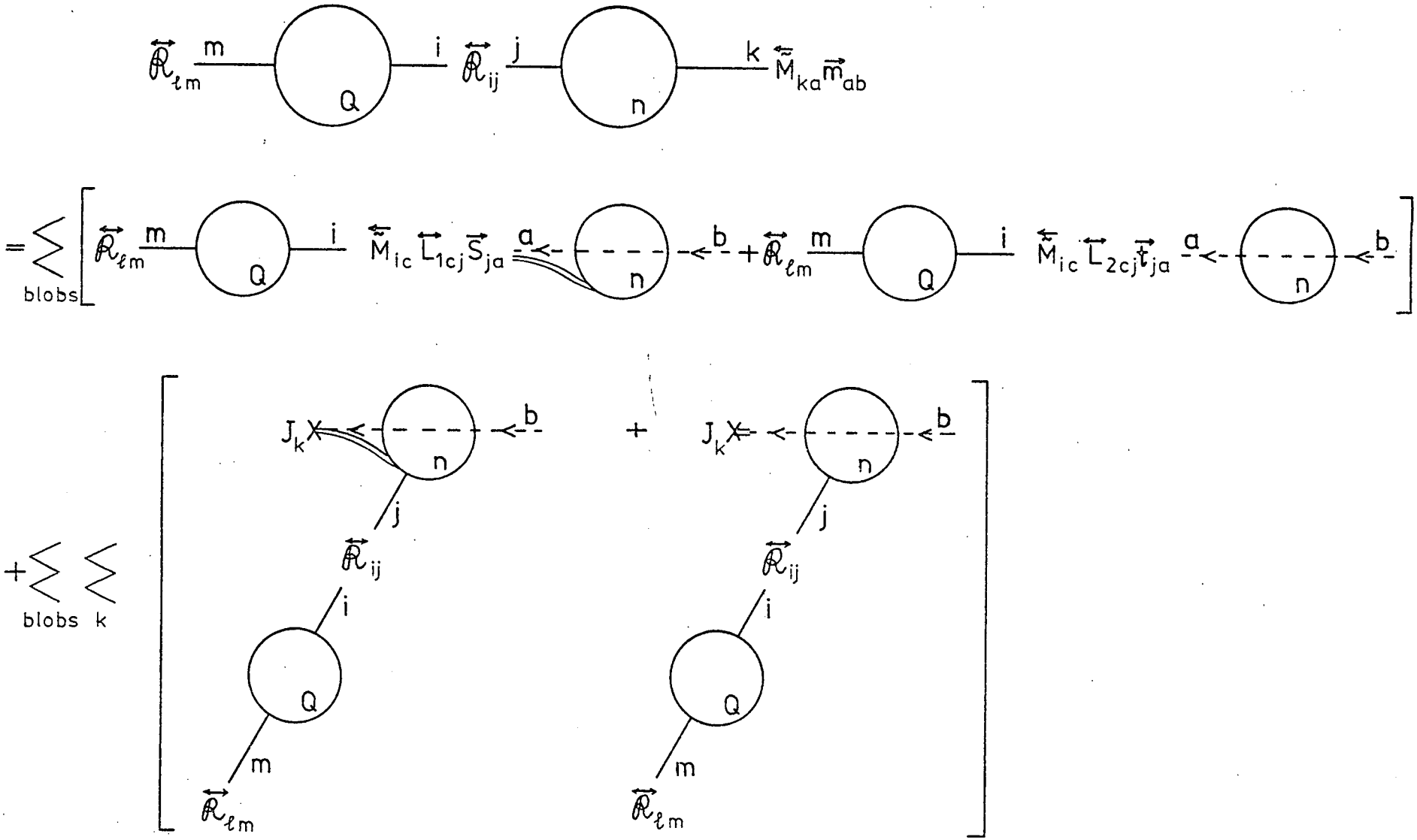


FIG 16 The diagrams are all of the tree variety. The notation, more usually used when loops are present, has been adopted for brevity.

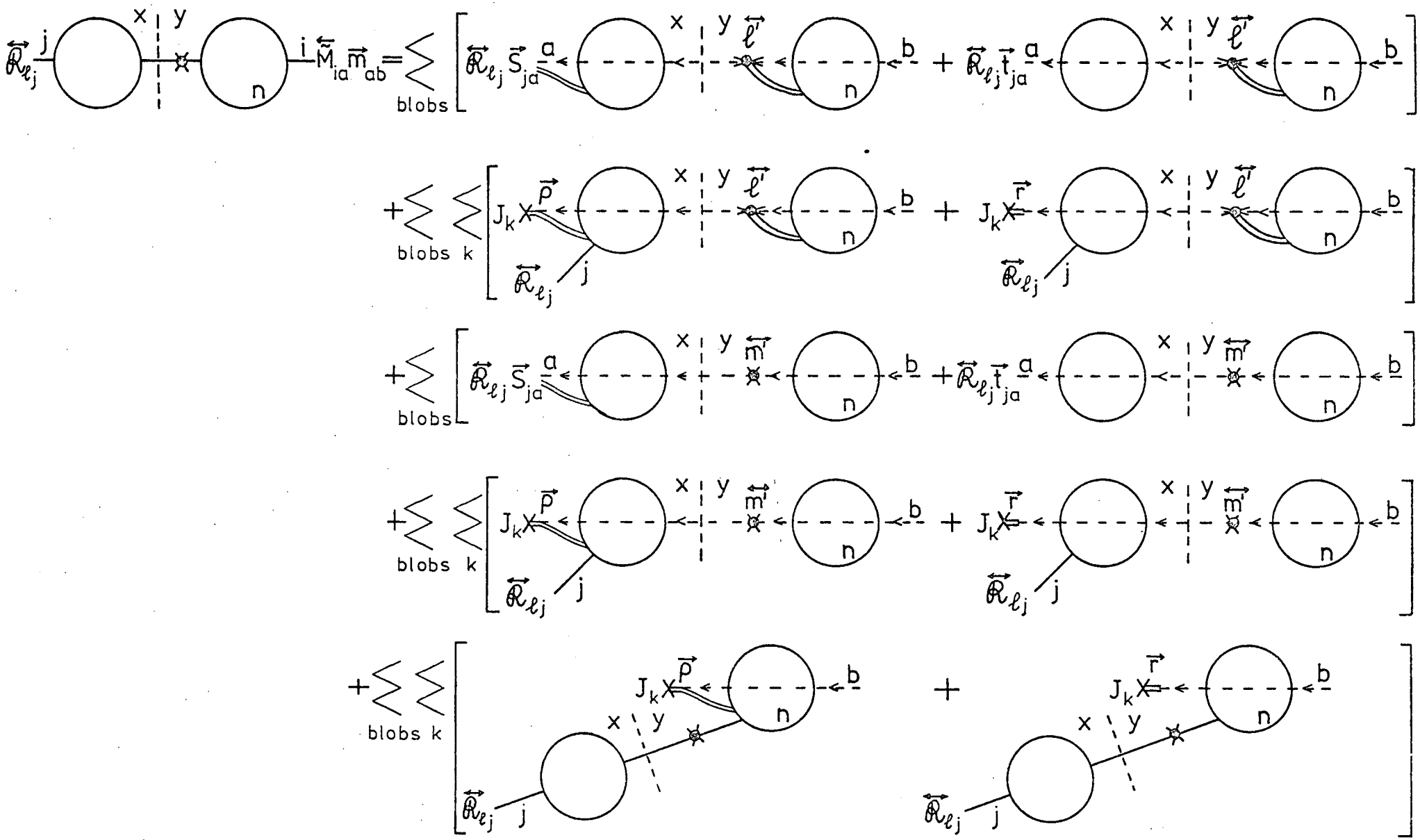
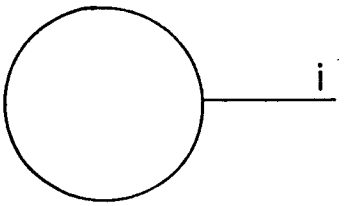


FIG 17

y is the power in coupling constant of the part of the diagrams to the right of the dashed line
x " " " " " " " " " " " " left " " "



rather than (2.6), i.e. the "blob" now has no $\mathcal{R}$-sources, the identity would have been established at that value of n with only the sources R present. The complete set of diagrams constructed from (2.5) for any fixed order in the coupling constant may be divided into subsets, each of which is distinguished by having the same number of vertices of the types in Fig. 13, n. Each subset obviously contains all possible diagrams for that value of n and the previously established identities for each value of n may be applied. Therefore, it has been proved that the Lagrangian (2.5) obeys Ward identities for tree diagrams with sources (2.3) and constraints (2.4).

By the Tree-Loop theorem[50] all that is necessary now is to find a transformation, a gauge function and a source function which generate the ghost and source terms appropriate for the Ward identity. If the gauge function is kept the same and we define the transform

$$A_i \to A_i' = A_i + g \overleftrightarrow{T}^{s}_{ia}(A)\Lambda^a + \overleftrightarrow{T}^{t}_{ia}\Lambda^a \qquad (2.7)$$

where

$$\overleftrightarrow{T}^{s}_{ia}(A) = \vec{S}_{ia}(A) + \overleftrightarrow{S}'_{ia}(A) \text{ and } \overleftrightarrow{T}^{t}_{ia} = \vec{t}_{ia} + \overleftrightarrow{t}'_{ia}$$

then we require $\vec{M}_{ai}\,\overleftrightarrow{S}'_{ib}(A)$ and $\vec{M}_{ai}\,\overleftrightarrow{t}'_{ib}$ to be the additional ghost vertices $\overleftrightarrow{\ell}'_{ab}(A)$ and $\overleftrightarrow{m}'_{ab}$. A solution of this is

$$\overleftrightarrow{S}'_{ia}(A) \equiv \vec{t}_{ic}\,\vec{m}^{-1}_{cd}\,\overleftrightarrow{L}_{1dj}\,\vec{S}_{ja}(A)$$
$$\overleftrightarrow{t}'_{ia} \equiv \vec{t}_{ic}\,\vec{m}^{-1}_{cd}\,\overleftrightarrow{L}_{2dj}\,\vec{t}_{ja} \quad . \tag{2.8}$$

From Appendix B the solution of (2.4) for L_1 and L_2 in terms of $\mathcal{R}$ is

$$\overleftrightarrow{L}_{1dj} = \overleftrightarrow{L}_{2dj} = \overleftarrow{\tilde{m}}^{-1}_{dc}\,\overleftarrow{\tilde{t}}_{ci}\,\overleftrightarrow{\mathcal{R}}_{ij} \equiv \overleftrightarrow{L}_{dj} \tag{2.9}$$

where $\mathcal{R}$ satisfies the conditions (B.13). If the source term is linear in the fields, i.e. $R_i \equiv \vec{R}_{ij}A_j$, it would transform under (2.7) as

$$\vec{R}_{ij}A_j \to \vec{R}_{ij}A_j + g\,\vec{R}_{ij}(\vec{S}_{ja}(A) + \vec{t}_{jc}\vec{m}^{-1}_{cd}\overleftrightarrow{L}_{dk}\vec{S}_{ka}(A))\Lambda^a$$
$$+ \vec{R}_{ij}(\vec{t}_{ja} + \vec{t}_{jc}\vec{m}^{-1}_{cd}\overleftrightarrow{L}_{dk}\vec{t}_{ka})\Lambda^a \quad . \tag{2.10}$$

Thus if for the original gauge invariant Lagrangian the source functions R_iJ_i had been chosen linear with

$$\overleftrightarrow{R}_{ik} \equiv \overleftrightarrow{R}'_{ij}(\delta_{jk} + \vec{t}_{ja}\,\vec{m}^{-1}_{ab}\,\overleftrightarrow{L}_{bk}) \tag{2.11}$$

the Ward identities for (2.5) with source term $\vec{R}'_{ik}A_k$ would have been demonstrated to hold. We thus expect the Lagrangian $\mathcal{L}_{INV} + A_i\overleftrightarrow{\mathcal{R}}_{ij}A_j$ to be invariant under (2.7).

The generalised Ward identies can be shown to be satisfied directly or inferred by the Tree-Loop theorem when $\overleftrightarrow{\mathcal{R}}$ is well-behaved. The final theory is a gauge theory and the divergences may be removed as in (50). From the form of the

additional vertices of Fig. 13 it is hoped the added term $A_i \overleftrightarrow{\mathcal{R}}_i A_j$ may be used as finite counter-terms for the mass renormalisations.

One further restriction on $\overleftrightarrow{\mathcal{R}}_{ij}$ is that it must be hermitean. The construction for tree diagrams could accommodate an asymmetric $\mathcal{R}$ through the asymmetry of the source-terms $A_i \overleftrightarrow{\mathcal{R}}_{ij} J_j$ but it would not be possible to identify the modified Feynman rules with a Lagrangian and the extension to diagrams including loops would fail. It should be noted that the construction may involve a change in the gauge parameter in the transform as well, i.e. $\Lambda_a \to Z_{ab}\Lambda_b$.

b) Indeterminacy of the Tree-Loop Theorem

In section a) we found a transformation (2.7) and source term (2.11) which, with the original gauge function, gave rise to the necessary terms for the Ward identities, proved to hold for the Lagrangian (2.5). This, however, may not be the only solution.

As a preliminary step gather all the bilinear terms together to obtain the complete propagator function

$$\overleftrightarrow{m''_{ab}} \equiv \overrightarrow{m_{ab}} + \overleftrightarrow{m'_{ab}} \,. \tag{2.12}$$

Similarly define

$$\overleftrightarrow{\ell''_{ab}}(A) \equiv \overrightarrow{\ell_{ab}}(A) + \overleftrightarrow{\ell'_{ab}}(A) \,.$$

The tree-diagram Ward identities for the Lagrangian (2.5) can now be expressed as in Fig. 18. If we further assume (2.12) can be rewritten as

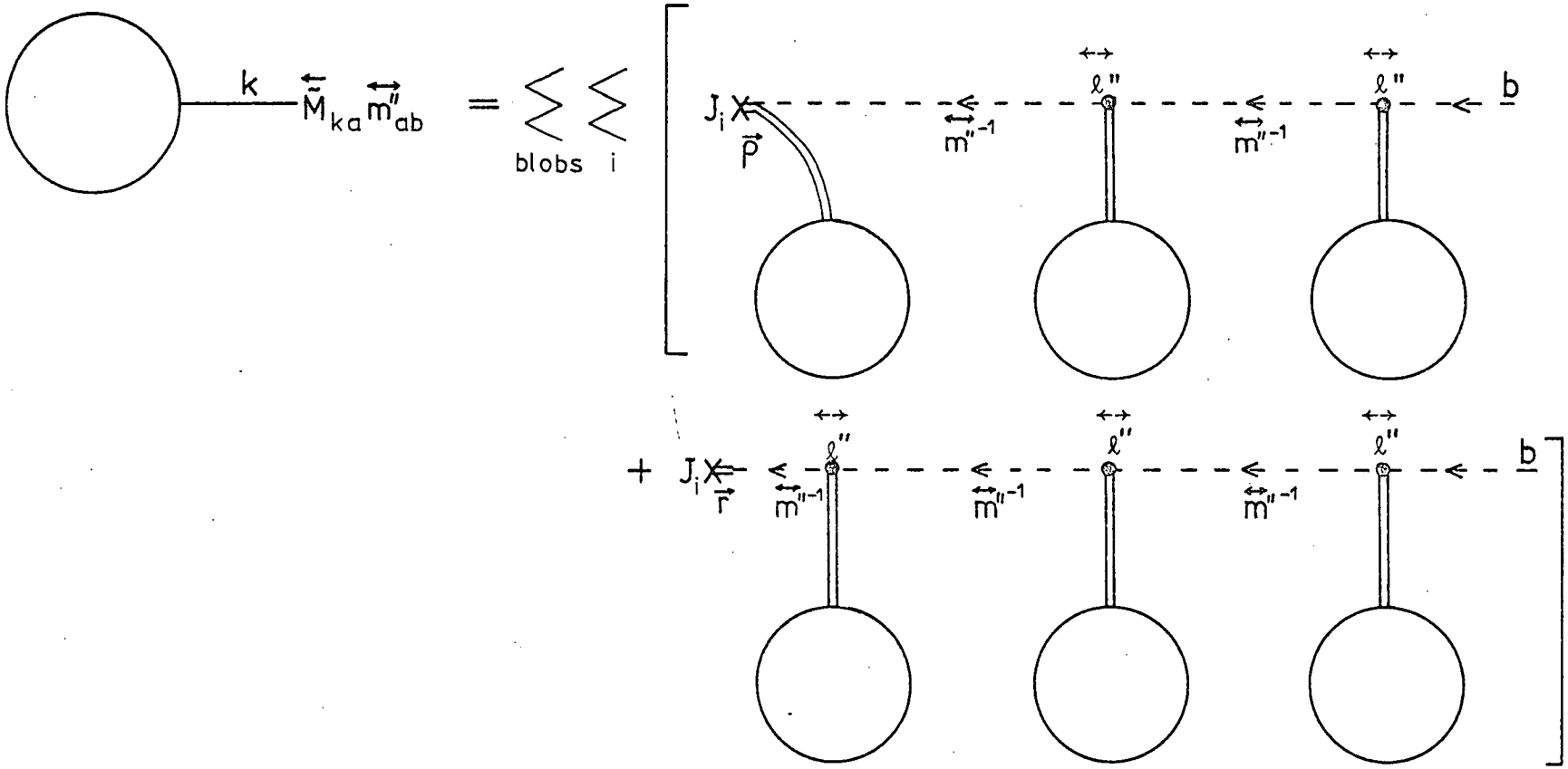

k
$\overleftarrow{\overleftrightarrow{M}}_{ka}\overleftrightarrow{m''}_{ab}$
=
blobs i
J_i
$\vec{p}$
$\overleftrightarrow{m''}^{-1}$
$\overleftrightarrow{\ell''}$
b
+
J_i
$\vec{r}$

FIG 18

$$\overleftrightarrow{m''}_{ab} \equiv \overleftrightarrow{\tilde{X}}_{ac} \overleftrightarrow{X}_{cd} \overrightarrow{m}_{db}$$

and (2.13)

$$\overleftrightarrow{\ell''}_{ab} \equiv \overleftrightarrow{\tilde{X}}_{ac} \overleftrightarrow{X}_{cd} \overrightarrow{\ell}_{db}(A)$$

where $\overleftrightarrow{X}_{cd}$ must be non-singular, Fig. 18 immediately gives the Ward identity of Fig. 19. Since the transform (2.1) acts on the gauge function $\overleftrightarrow{X}_{ab} \overrightarrow{M}_{bi} A_j$, indicated by Fig. 19, to generate the appropriate ghost propagator $- (\overleftrightarrow{X} \overrightarrow{m})^{-1}_{ab}$ and vertices $(X\ell)_{ab}$ we have shown the additions may be considered to have changed the gauge of the original Lagrangian only.

Hence the subset of additions, $A_i \overleftrightarrow{\mathcal{R}}_{ij} A_j$, for which the conditions (2.13) hold are in most cases a change of gauge. The most general addition can be any mixture of the two solutions, i.e. those represented by a change of the invariant Lagrangian and corresponding transformation and those by a change of gauge, the precise nature of which is model dependent and can only be found by direct investigation. However, the demonstration of section a) of the existence of a gauge function, a source function and a transform which generates the correct ghost and source terms for the Ward identities is still valid to the extent of proving that the modified Lagrangian is a gauge theory although the gauge function etc. found may not be the correct ones.

Thus the equivalence, between the Ward identities and the gauge theory nature of the Lagrangian, implied by the Tree-Loop theorem of (50) is weaker than the statement of the theorem would give reason to believe. The theorem should be modified to state that if all the appropriate conditions are fulfilled the Lagrangian under consideration is a gauge theory

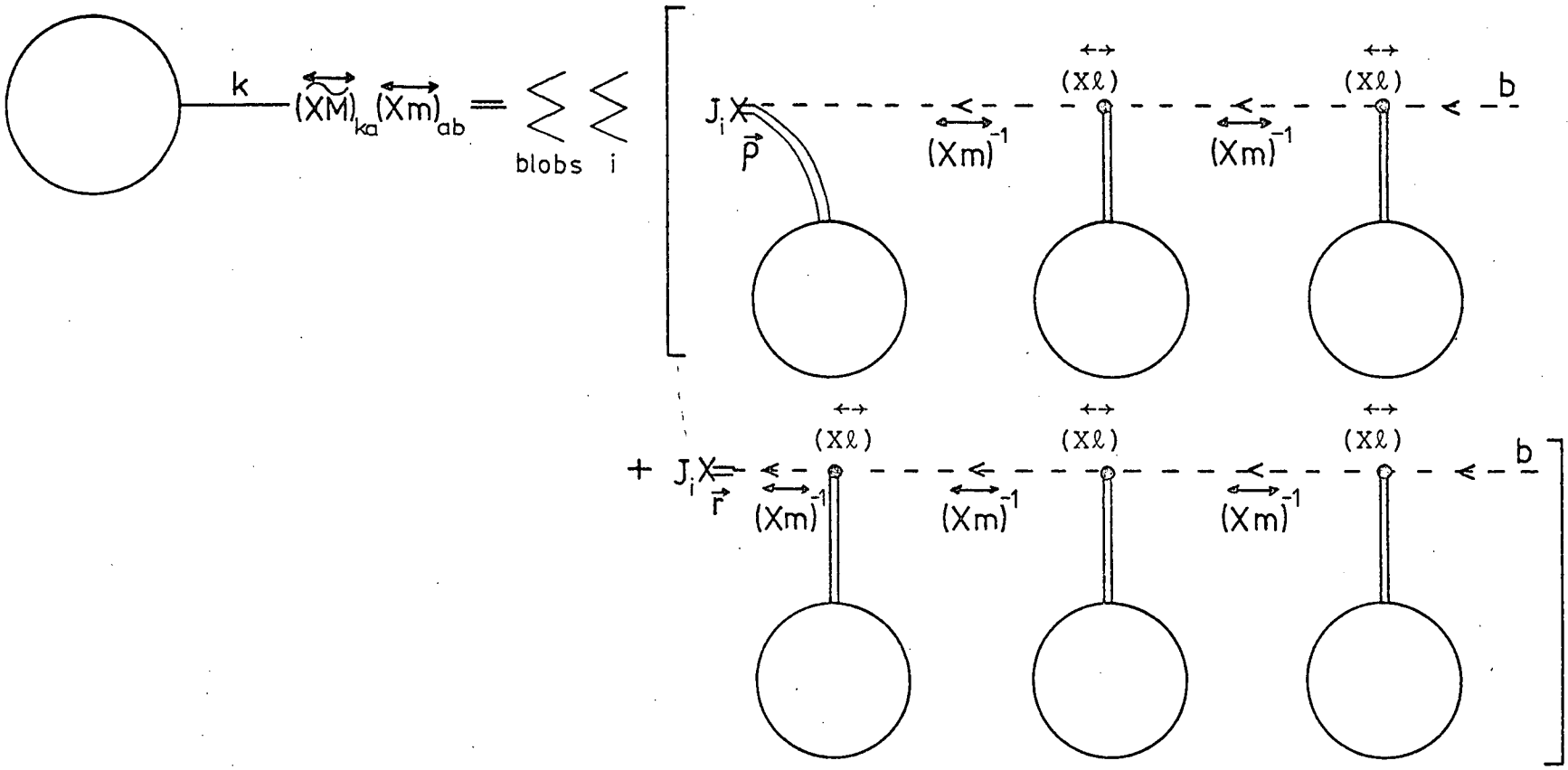


FIG 19

but the gauge transform may not be that used in fulfilling the conditions.

A simple example of the additions to a Lagrangian, discussed in this section, is the possible additions to the Yang-Mills[4] Lagrangian invariant under local SU(2) transforms.

$$\mathcal{L}_{INV} \equiv -\frac{1}{4}\underline{G}_{\mu\nu} \cdot \underline{G}^{\mu\nu} \tag{2.14}$$

where $$\underline{G}^{\mu\nu} = \partial^{\mu}\underline{W}^{\nu} - \partial^{\nu}\underline{W}^{\mu} - g\,\underline{W}^{\mu} \times \underline{W}^{\nu} \tag{2.15}$$

is invariant under the infinitesimal gauge transform

$$\underline{W}^{\mu} \to \underline{W}^{\mu} + g\,\underline{W}^{\mu} \times \underline{\eta} - \partial^{\mu}\underline{\eta} \tag{2.16}$$

i.e. $$\vec{S}_{ia} \equiv \begin{pmatrix} 0 & -W_3^{\mu} & W_2^{\mu} \\ W_3^{\mu} & 0 & -W_1^{\mu} \\ -W_2^{\mu} & W_1^{\mu} & 0 \end{pmatrix} \quad ; \quad \vec{t}_{ia} \equiv \begin{pmatrix} -\partial^{\mu} & 0 & 0 \\ 0 & -\partial^{\mu} & 0 \\ 0 & 0 & -\partial^{\mu} \end{pmatrix}$$

Here all derivatives will be taken to act to the right with signs adjusted accordingly. The gauge functions are selected to be

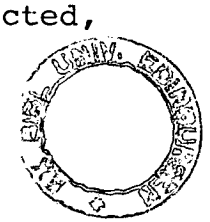
$$C_a \equiv \partial_{\nu} W_a^{\nu} \tag{2.17}$$

i.e. $$\vec{M}_{ai} \equiv \begin{pmatrix} \partial_{\nu} & 0 & 0 \\ 0 & \partial_{\nu} & 0 \\ 0 & 0 & \partial_{\nu} \end{pmatrix}$$

which gives the characteristic operator of Appendix B

$$(I - \tilde{M}\,\tilde{m}^{-1}\,\tilde{t}) \equiv (\delta^{\mu}{}_{\nu} - \frac{\partial^{\mu}\partial_{\nu}}{\partial^2}) I^{(3)} \,. \tag{2.18}$$

(2.18) has three eigenvectors of eigenvalue 0 as expected, viz.



$$\begin{pmatrix} \partial^\mu \\ 0 \\ 0 \end{pmatrix} ; \quad \begin{pmatrix} 0 \\ \partial^\mu \\ 0 \end{pmatrix} ; \quad \begin{pmatrix} 0 \\ 0 \\ \partial^\mu \end{pmatrix} \tag{2.19}$$

Keeping to an explicitly Lorentz covariant expression each element of $\mathcal{R}$ is of the form $xg^{\nu\mu} + y\,\partial^\mu\partial^\nu$ where x and y may involve derivatives. To satisfy the necessary conditions (B.13) all the x must be chosen to be zero when

$$\mathcal{R} \equiv \partial^\mu Y \partial_\nu \quad . \tag{2.20}$$

Of course a solution of the form of section a) could be constructed but it would be inappropriate since

$$Y \equiv (-I + \tilde{X}X) \tag{2.21}$$

for any Y, as $\mathcal{R}$ is symmetric. The construction is thus equivalent to changing the gauge function from (2.17) to $C_a' \equiv X_{ab}C_b$ for any X_{ab} without changing $\mathcal{L}_{INV}$, i.e. the construction allows us to go to any other Lorentz covariant gauge. This is as expected as the massive Yang-Mills Lagrangian does not exhibit gauge invariance.

Another example of interest is the Abelian gauge invariant Lagrangian of Higgs[43,47]

$$\mathcal{L}_{INV} \equiv -\tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + (\partial_\mu + ieA_\mu)\phi^*(\partial^\mu - ieA^\mu)\phi - \mu^2(\phi^*\phi) - \lambda(\phi^*\phi)^2 \tag{2.22}$$

unvariant under

$$\begin{aligned} \phi &\to \phi - ie\phi\Lambda \\ \phi^* &\to \phi^* + ie\phi^*\Lambda \\ A^\mu &\to A^\mu - \partial^\mu\Lambda \end{aligned} \tag{2.23}$$

i.e. $\vec{S}_{ia} \equiv \begin{pmatrix} -i\phi \\ i\phi^* \\ 0 \end{pmatrix}$; $\vec{t}_{ia} \equiv \begin{pmatrix} 0 \\ 0 \\ -\partial^\mu \end{pmatrix}$

The gauge function is chosen to be

$$C \equiv \partial_\nu A^\nu \quad \text{i.e.} \quad \vec{M}_{ai} \equiv (0,0,\partial_\nu) \tag{2.24}$$

when the characteristic operator is identical to (2.18) with the single eigenvector for eigenvalue zero, simply ∂^μ. This time, however, the conditions (B.13) are satisfied by the covariant choice of

$$\overleftrightarrow{R}_{ij} \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a\delta^\mu_\nu + b\partial^\mu\partial_\nu \end{pmatrix} \tag{2.25}$$

The additional terms arising from (2.25) must be broken into the two terms

$$\tfrac{1}{2}A_\mu\, a(\delta^\mu_\nu - \frac{\partial^\mu\partial_\nu}{\partial^2})A^\nu + \tfrac{1}{2}A_\mu\partial^\mu(b + \frac{a}{\partial^2})\partial_\nu A^\nu \tag{2.26}$$

whereby the first is an addition which is interpreted as of the type of section a) with the basic invariant Lagrangian being modified but in this particular case the transformation is not. The second term is incorporated in a change in gauge from $\partial_\mu A^\mu$ to $(1 + b + a/\partial^2)^{\frac{1}{2}}\,\partial_\mu A^\mu$. Thus this example is in the form of a mixture of the two types of modification

In particular choose $a = M^2$ and $b = \frac{1}{\alpha^2} - 1$ when the complete Lagrangian becomes

$$- \tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + (\partial_\mu + ieA_\mu)\phi^*(\partial^\mu - ieA^\mu)\phi - \mu^2(\phi^*\phi)$$

$$- \lambda(\phi^*\phi)^2 + \tfrac{1}{2}M^2 A_\mu A^\mu - \tfrac{1}{2}(\frac{1}{\alpha}\,\partial_\mu A^\mu)^2 \ . \qquad (2.27)$$

In (2.27) the vector boson has become massive with propagator $\dfrac{g^{\mu\nu} - \frac{(1-\alpha^2)q^\mu q^\nu}{q^2 - \alpha^2 M^2}}{q^2 - M^2}$. It can be interpreted, as discussed, as a gauge theory which obeys Ward identities with source terms generated by the transformation (2.23). The appropriate ghost Lagrangian is generated from the gauge function $(\frac{1}{\alpha^2} + M^2/\partial^2)^{\frac{1}{2}}\,\partial_\mu A^\mu$, i.e. there are no ghost vertices and the ghost propagator function is $-\frac{1}{\alpha}(q^2 - \alpha^2 M^2)^{\frac{1}{2}}(q^2)^{\frac{1}{2}}$. A consequence of this is that the S-matrix is invariant under variation of α and the Lagrangian (2.27) is renormalisable as is well known[6]. When the mass of the vector-boson is renormalised the counter-term is equivalent to an addition of the form (2.25) or (2.26), i.e. the Lagrangian and gauge function is modified by a construction of the form discussed in this chapter. The massive abelian theory is further discussed in Chapter 3.

c) Other Modifications

So far we have considered only source terms linear in the fields being absorbed into the Lagrangian. These led to the introduction of bilinear terms with accompanying two-point and three-point ghost vertices. The technique may, obviously, be extended to bilinear and higher order source functions with their corresponding generation of three-point and higher ghost vertices. Combinations of different order source terms

introduce more flexibility into the choice of functions but as these would tend to introduce non-linear terms to the gauge functions, and we are chiefly interested in two-point vertex additions which may be interpreted as mass counter-terms, we restrict ourselves to the linear source situation examined in sections a) and b).

The one exception to the above restriction is within additions being added to the Lagrangian which have no corresponding ghost vertices. If $J_i \overleftrightarrow{\mathcal{R}}_{ij} A_j$ is the additional source term for physical fields the ghost source terms for the above construction are $J_i \overleftrightarrow{\mathcal{R}}_{ij} \overrightarrow{S}_{ja}(A)\Lambda^a$ and $J_i \overleftrightarrow{\mathcal{R}}_{ij} \overrightarrow{t}_{ja}\Lambda^a$. On multiplying by (2.6) these become equivalent to vertices $A_i \overleftrightarrow{\mathcal{R}}_{ij} \overrightarrow{t}_{ja}$ and $A_i \overleftrightarrow{\mathcal{R}}_{ij} \overrightarrow{S}_{ja}(A)$. If these are identically zero the additional term $A_i \overleftrightarrow{\mathcal{R}}_{ij} A_j$ may be added to the Lagrangian without changing the Faddeev-Popov ghost Lagran gian but preserving the Ward identities. This genre of additions can be extended to higher orders in the fields as usual and includes the set of terms invariant under the gauge transform. For example, for the Lagrangian (2.22)

$\overleftrightarrow{\mathcal{R}}_{ij}$ could have been chosen $\begin{pmatrix} 0 & a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

when $J_i \overleftrightarrow{\mathcal{R}}_{ij} \overrightarrow{t}_{ja}\Lambda^a \equiv 0$ and $J_i \overleftrightarrow{\mathcal{R}}_{ij} \overrightarrow{S}_{ja}(A)\Lambda^a$

$$\equiv (J_i) \begin{pmatrix} 0 & -iea & 0 \\ iea & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \phi \\ \phi^* \\ A^\mu \end{pmatrix} \Lambda$$

$A_i \overleftrightarrow{\mathcal{R}}_{ij} \overrightarrow{S}_{ja}(A)\Lambda^a$ is identically zero and the term $a\phi^*\phi$ may be added to the Lagrangian.

These additions may be disguised by being combined with additions to the Lagrangian of the form of Chapter 1. An example is the Salam-Weinberg[35,51] model in which the mass renormalisation counter-terms are made explicit by the reformulation:

$$\begin{aligned}\mathcal{L} \equiv & -\tfrac{1}{4}(M_o)^2(\partial_\mu \underline{W}_\nu - \partial_\nu \underline{W}_\mu - g\underline{W}_\mu \times \underline{W}_\nu)^2 - \tfrac{1}{4}(\tfrac{M}{M'})^2(\partial_\mu B_\nu - \partial_\nu B_\mu)^2 \\ & + (M^2)\left|(\partial_\mu + \tfrac{1}{2}ig\underline{\tau}.\underline{W}_\mu - \tfrac{1}{2}igIB_\mu)\Phi + \frac{i}{\sqrt{2}}(\underline{\tau}.\underline{W}_\mu - IB_\mu)\begin{pmatrix}0\\1\end{pmatrix}\right|^2 \\ & -\tfrac{1}{2}\mu^2M^2(g\,\Phi^+\Phi + \sqrt{2}\,\Phi^+\begin{pmatrix}0\\1\end{pmatrix} + \sqrt{2}\,(0,1)\Phi)^2 \\ & -\frac{\mu^2M^2K^2}{g}(g\,\Phi^+\Phi + \sqrt{2}\,\Phi^+\begin{pmatrix}0\\1\end{pmatrix} + \sqrt{2}\,(0,1)\Phi) \\ & + \alpha\,\bar{L}(i\partial_\mu - \tfrac{1}{2}g\underline{\tau}.\underline{W}_\mu - \tfrac{1}{2}gIB_\mu)\gamma^\mu L + \beta\bar{R}(i\partial_\mu - gB_\mu)\gamma^\mu R \\ & - \frac{m}{2}(\frac{g}{\sqrt{2}}\,\bar{L}\,\Phi\,R + \bar{L}\begin{pmatrix}0\\1\end{pmatrix}R + \frac{g}{\sqrt{2}}\,\bar{R}\,\Phi^+L + R(0,1)L)\end{aligned} \qquad (2.27)$$

invariant under the transformation

$$\begin{aligned}\underline{W} &\to \underline{W}_\mu + g(\underline{W}_\mu \times \underline{\eta}) - \partial_\mu \underline{\eta} \\ B_\mu &\to B_\mu - \partial_\mu \eta^4 \\ \Phi &\to \Phi + \tfrac{1}{2}ig(\underline{\tau}.\underline{\eta} - I\eta^4)\Phi + \frac{i}{\sqrt{2}}(\underline{\tau}.\underline{\eta} - I\eta^4)\begin{pmatrix}0\\1\end{pmatrix} \\ L &\to L + \tfrac{1}{2}ig(\underline{\tau}.\underline{\eta} + I\eta^4)L \\ R &\to R + ig\,\eta^4\,R \quad .\end{aligned}$$

As each separate term is invariant the constants M_o^2, $(\frac{M}{M'})^2$, M^2, μ^2M^2, α, β and m may be scaled to facilitate mass renormalisation without affecting the Ward identities. Similarly K can be used to remove the tadpole terms as necessary[47,48,51]. To obtain the more usual form[51] the following replacements should be made

$$\Phi \to \frac{1}{M}\Phi, \qquad B_\mu \to \frac{M'}{M} B_\mu, \qquad \eta^4 \to \frac{M'}{M}\eta^4$$

$$g\frac{M'}{M} \to g' \ , \qquad M \to \tfrac{1}{2}gF \ , \qquad \mu^2 \to \frac{F^2\lambda_o}{4}, \quad K \to 2(F^2-E^2)/F^2$$

$$m \to \sqrt{2}\,GF, \qquad M_o \to 1, \qquad \alpha \to 1, \ \beta \to 1 \ .$$

We have not included wave function renormalisation as it can always be treated separately as in the previous chapter. It can also be seen from the above form that the coupling constant renormalisation can be facilitated as suggested in Chapter 1.

d) Conclusions

From the preceding analysis it is evident that for any gauge theory there is complete freedom of choice of the finite counter-terms for the self-energy terms and the coupling constant. It is only in the mass renormalisation that the choice may be restricted as in the Salam-Weinberg model (2.27) where the counter-terms for the Higgs scalars are related. In the likelihood that there are no further consistent additions to the Lagrangian possible the same counter-terms as discussed must also be responsible for removing the divergences. By considering the counter-terms broken down to the forms discussed we see, for example in (2.27), the relations between couplings and masses must hold for renormalised we well as the bare theory.

One further point to note is that the connection between the Ward identities and a specific invariance is not as strong as implied by the Tree-Loop theorem[50]. However, the conclusion that there exists an invariance is not invalidated.

CHAPTER 3

MASSIVE AND MASSLESS YANG-MILLS LAGRANGIANS

In section b) of the previous chapter a certain overlap between a massive and massless abelian Lagrangian was shown to exist to the extent that the existence of Ward identities for the massless theory could be used to establish them in the massive situation. A similar extension is also possible for the non-abelian Yang-Mills Lagrangian.

a) Existence of Ward Identities

Following the prescription of Chapter 2 we consider the non-abelian Lagrangian

$$\mathcal{L}_{INV} = -\frac{1}{4}\underline{G}_{\mu\nu}\cdot\underline{G}^{\mu\nu} \tag{3.1}$$

where $\underline{G}^{\mu\nu} = \partial^{\mu}\underline{W}^{\nu} - \partial^{\nu}\underline{W}^{\mu} - g\,\underline{W}^{\mu}\times W^{\nu}$

which is invariant under the infinitesimal gauge transform

$$\underline{W}^{\mu} \rightarrow \underline{W}^{\mu} + g\,\underline{W}^{\mu}\times\underline{\eta} - \partial^{\mu}\underline{\eta} \quad . \tag{3.2}$$

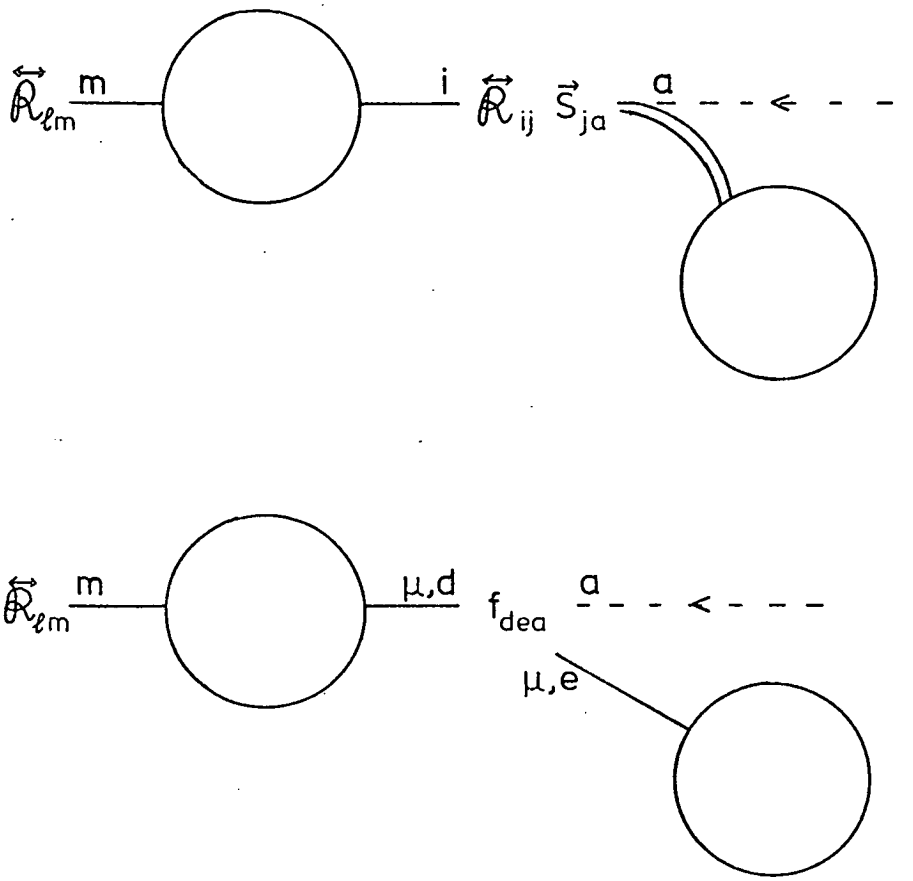
The gauge function is chosen to be

$$C_a \equiv \frac{1}{\alpha}\,\partial_{\mu}W^{\mu}_{a} \tag{3.3}$$

and the source function

$$\overleftrightarrow{\mathcal{R}}_{ij} \equiv M^2\delta^{a}{}_{b}\,\delta^{\mu}{}_{\nu} \quad . \tag{3.4}$$

When the Ward identity of Fig. 16 is constructed for this Lagrangian the contribution of the form



vanishes identically as f_{dea} is anti-symmetric in d and e. The other ghost-source additions are of the necessary form

$$\overleftrightarrow{R}_{ij}\ \vec{t}_{ja} \equiv \frac{1}{\alpha}\overleftarrow{\partial}_{\mu}\ (\alpha M^2)\,\delta^{a}_{c} \quad . \tag{3.5}$$

Hence it is possible to interpret the additions as (2.4) but with $\overleftrightarrow{L}_{lbj} \equiv 0$. On absorbing the mass terms into the propagators we obtain the Feynman rules of Figs. 1(a) and 1(c) which obey the Ward identities for tree diagrams of Fig. 7 with the appropriate source terms generated by the transform (3.2).

The gauge function (3.3) is linear in the fields and we shall also restrict the source functions to be linear, i.e. $C_a \equiv \vec{M}_{ai}\ A_i$ and $R_i \equiv \overleftrightarrow{R}_{ij}\ A_j$. The conditions which need to be satisfied in order that the Tree-Loop theorem may be applied are

$$\vec{R}_{ij}\,\vec{S}_{ja}(A) \equiv \overleftrightarrow{R'}_{ij}\,\overleftrightarrow{S'}_{ja}(A); \quad \vec{R}_{ij}\,\vec{t}_{ja} \equiv \overleftrightarrow{R'}_{ij}\,\overleftrightarrow{t'}_{ja}$$
$$\text{and } \vec{M}_{aj}\,\vec{S}_{jb}(A) \equiv \overleftrightarrow{M'}_{aj}\,\overleftrightarrow{S'}_{jb}(A); \quad \vec{m}'_{ab} \equiv \overleftrightarrow{M'}_{aj}\,\overleftrightarrow{t'}_{jb} \tag{3.6}$$

for some R'_{ij}, $S'_{ja}(A)$ and t'_{ja}, where m'_{ab} is the complete ghost propagator function for the massive theory. On assuming $\overleftrightarrow{R'}_{ij}$ is non-singular

$$\overleftrightarrow{S'}_{ja}(A) \equiv \overleftrightarrow{R'}^{-1}_{jk}\,\vec{R}_{ki}\,\vec{S}_{ia}(A); \quad \overleftrightarrow{t'}_{ja} \equiv \overleftrightarrow{R'}^{-1}_{jk}\,\vec{R}_{ki}\,\vec{t}_{ia} \tag{3.7}$$

and multiplying on left by M'_{bj}

$$\vec{M}_{bj}\,\vec{S}_{ja}(A) = \overleftrightarrow{M'}_{bj}\,\overleftrightarrow{S'}_{jb}(A) \equiv \overleftrightarrow{M'}_{bj}\,\overleftrightarrow{R'}^{-1}_{jk}\,\vec{R}_{ki}\,\vec{S}_{ia}(A)$$
$$\text{i.e. } (\vec{M}_{bi} - \overleftrightarrow{M'}_{bj}\,\overleftrightarrow{R'}^{-1}_{jk}\,\vec{R}_{ki})\vec{S}_{ia}(A) = 0. \tag{3.8}$$

However, all eigenvectors, with eigenvalue zero, for $\vec{S}_{ia}(A)$ as defined by (3.2) involve the fields W^{μ}_{a} whereas $M - M'R'^{-1}R$ does not and we are forced to conclude

$$\vec{M}_{bi} = \overleftrightarrow{M'}_{bj}\,\overleftrightarrow{R'}^{-1}_{jk}\,\vec{R}_{ki} \tag{3.9}$$

(3.9) implies

$$\vec{M}_{bi}\,\vec{t}_{ia} = \overleftrightarrow{M'}_{bj}\,\overleftrightarrow{R'}^{-1}_{jk}\vec{R}_{ki}\,\vec{t}_{ia} = \overleftrightarrow{M'}_{bj}\,\overleftrightarrow{t'}_{ja} = \vec{m}'_{ba} \tag{3.10}$$

which precludes any additions of the type (3.4) from satisfying the conditions (3.6). The Tree-Loop theorem, therefore, cannot be invoked on this occasion to imply the invariance of the Lagrangian corresponding to the Feynman rules of Figs. 1(a) and 1(c). If $\overleftrightarrow{R'}_{ij}$ had been chosen to be singular a solution to the conditions (3.6) could have been found but the Tree-Loop theorem would again no longer have been applicable.

To see whether the Ward identities hold for diagrams

involving loops must now be examined directly. This will be done later.

b) Direct Derivation of Ward Identities

The Ward identities for the massive Yang-Mills Lagrangian can be deduced more directly than above.

First we consider a Lagrangian $\mathcal{L}_{INV}(A_i)$ invariant under (1.1). Instead of following the usual prescription for generating a gauge invariant field theory as in section a) of Chapter 1, we use a more general "gauge" fun ction $\frac{1}{2}G(A)$ (which may not be a perfect square) and define a Lagrangian

$$\mathcal{L} \equiv \mathcal{L}_{INV} - \tfrac{1}{2}G(A) + J_i R_i(A) \qquad (3.11)$$

with source term $J_i R_i(A)$. The Lagrangian is not a gauge theory and this method is only a device to enable the Ward identities to be found.

Under the infinitesimal transform (1.1) the source term transforms as (1.4) and

$$G(A) \to G'(A) = G(A) + g\hat{P}_a(A)\Lambda^a + \hat{Q}_a(A)\Lambda^a . \qquad (3.12)$$

The "gauge" is restricted to the set of functions such that

$$\hat{P}_a(A) \equiv 2\hat{C}_b(A)\hat{L}_{ba}(A); \quad \hat{Q}_a(A) \equiv 2\hat{C}_b(A)\hat{M}_{ba} \qquad (3.13)$$

and $\hat{M}_{ba}$ is non-singular. Following the technique of (50) free particle fields of mass m are added to the Lagrangian:

$$\mathcal{L} \equiv \mathcal{L}_{INV} - \tfrac{1}{2}G(A) + J_i R_i(A) + \tfrac{1}{2}(\partial_\mu B^a)^2 - \tfrac{1}{2}m^2 B_a^2 \qquad (3.14)$$

which under the transform

$$A_i \to A_i' = A_i + \varepsilon g \hat{S}_{ia}(A) B^a + \varepsilon \hat{t}_{ia} B^a \tag{3.15}$$

becomes

$$\mathcal{L}' \equiv \mathcal{L} - \varepsilon \hat{C}_b(A) (\hat{M}_{ba} + g\hat{L}_{ba}(A)) B^a + \varepsilon J_i (g\hat{\rho}_{ia}(A) + \hat{r}_{ia}) B^a \tag{3.16}$$

where ε is infinitesimal. It is immediately evident as in (50) that the field theory constructed by adding a ghost Lagrangian

$$L_\phi \equiv \phi_b^* [\hat{M}_{ba} + g\hat{L}_{ba}(A)] \phi_a \tag{3.17}$$

to (3.11):

$$\mathcal{L} \equiv \mathcal{L}_{INV} - \tfrac{1}{2} G(A) + J_i R_i(A) + \phi_b^* [\hat{M}_{ba} + g\hat{L}_{ba}(A)] \phi_a \tag{3.18}$$

obeys the Ward identities for tree diagrams, cf. Fig. 7.

Let $\hat{L}_{ab}(A)$ transform according to

$$\hat{L}_{ab}(A) \to \hat{L}'_{ab}(A) = \hat{L}_{ab}(A) + g\hat{D}_{abc}(A)\Lambda^c + \hat{E}_{abc}\Lambda^c \quad . \tag{3.19}$$

To extend the Ward identities to diagrams including loops, Fig. 5, it is sufficient to demonstrate that the auxiliary vertices $\hat{D}_{abc}(A)$ and $\hat{E}_{abc}$ satisfy the "group property"[50] of Fig. 20(a). Let $R_i(A) = A_i$ in the Lagrangian (3.11) and vary the fields according to

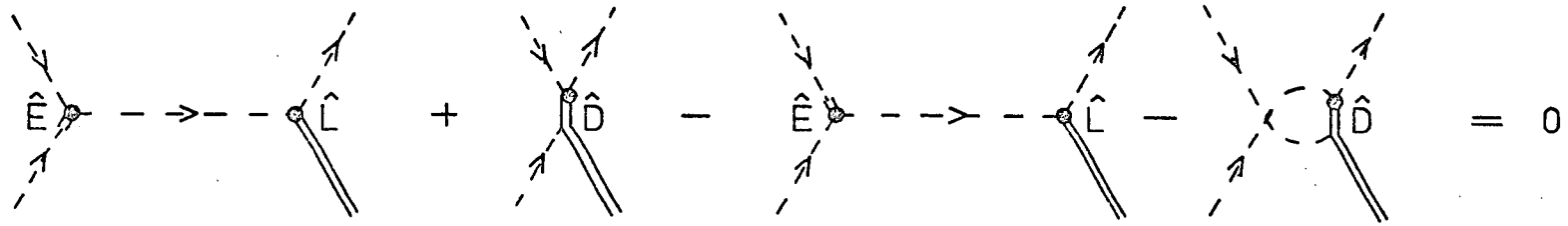
$$A_i \to A_i + g\hat{S}_{ia}(A) B^a + \hat{t}_{ia} B^a$$

and $$A_i \to A_i + g\hat{S}_{ia}(A) B'^a + \hat{t}_{ia} B'^a \ ;$$

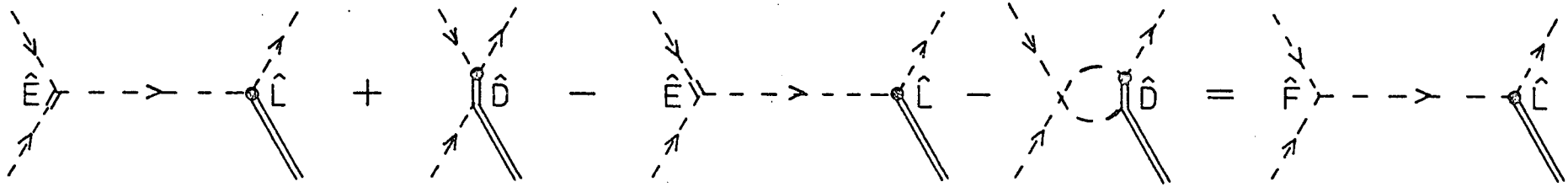
then their inverses

$$A_i \to A_i - g\hat{S}_{ia}(A) B'^a - \hat{t}_{ia} B'^a$$

and $$A_i \to A_i - g\hat{S}_{ia}(A) B^a - \hat{t}_{ia} B^a \ .$$



(a)



(b)

FIG 20 For vertex notation cf. Fig. 16(i), (ii) and (iii).

Finally on redefining the fields:

$$A_i = A_i - g\hat{S}_{ia}g\hat{M}^{-1}_{ad}\hat{E}_{dbc}(B^bB'^c - B'^bB^c) - \hat{t}_{ia}g\hat{M}^{-1}_{ad}\hat{E}_{dbc} \times (B^bB'^c - B'^bB^c)$$

(3.11) becomes

$$\begin{aligned}
&\mathcal{L}_{INV} + J_iA_i - C_b(A)g(g\hat{D}_{bac}(A) - g\hat{L}_{bd}(A)\hat{M}^{-1}_{de}\hat{E}_{eac})(B^aB'^c - B'^aB^c) \\
&+ J_ig(g\hat{v}_{iab}(A) + \hat{u}_{iab} - g\hat{S}_{ic}(A)\hat{M}^{-1}_{cd}\hat{E}_{dab} - \hat{t}_{ic}\hat{M}^{-1}_{cd}\hat{E}_{dab})(B^aB'^b - B'^aB^b) \\
&+ (g\hat{\ell}_{bc}(A) + \hat{m}_{bc})B^c(g\hat{L}_{ba}(A) + M_{ba})B'^a - (g\hat{\ell}_{bc}(A) + \hat{m}_{bc})B'^c(g\hat{L}_{ba}(A) + \hat{M}_{ba}) \\
&- J_ig(g\hat{v}_{iab}(A) + \hat{u}_{iab})(B^aB^b + B'^aB'^b) + C_b(A)g(g\hat{D}_{bac}(A) \\
&\qquad + \hat{E}_{bac})(B^aB^c + B'^aB'^c) \qquad (3.20) \\
&+ (g\hat{\ell}_{bc} + \hat{m}_{bc})B^c(g\hat{L}_{ba}(A) + \hat{M}_{ba})B^a + (g\hat{\ell}_{bc}(A) + \hat{m}_{bc}) \\
&\qquad B'^c(g\hat{L}_{ba}(A) + \hat{M}_{ba})B^a
\end{aligned}$$

where $\hat{\ell}_{ba}(A)$, $\hat{m}_{ba}$, $\hat{v}_{iab}$ and $\hat{u}_{iab}$ are defined according to

$$C_b(A_i + g\hat{S}_{ia}(A)\Lambda^a + \hat{t}_{ia}\Lambda^a) \equiv C_b(A) + g\hat{\ell}_{ba}(A)\Lambda^a + \hat{m}_{ba}\Lambda^a \qquad (3.21)$$

and

$$\hat{S}_{ia}(A_j + g\hat{S}_{jb}(A)\Lambda^b + \hat{t}_{jb}\Lambda^b) \equiv \hat{S}_{ia}(A) + g\hat{v}_{iab}(A)\Lambda^b + \hat{u}_{iab}\Lambda^b \qquad (3.22)$$

For a well behaved theory it can be established[50], by considering diagrams involving one BB' pair only, that the group property holds if

$$(g\hat{\ell}_{bc}(A) + \hat{m}_{bc})B^c(g\hat{L}_{ba}(A) + \hat{M}_{ba})B'^a - (g\hat{\ell}_{bc}(A) + \hat{m}_{bc})B'^c(g\hat{L}_{ba}(A) + \hat{M}_{ba})B^a = 0 \qquad (3.23)$$

Therefore the theory (3.18) with sources generated by (1.4), i.e. under the gauge transform for the corresponding gauge invariant Lagrancian, obeys the Ward identities of Fig. 5 to all orders if the condition (3.21) is satisfied. There are two trivial examples. The normal gauge theory constructed with $G(A) = C_a^2(A)$ and $\hat{\ell}_{ab}(A) = \hat{L}_{ab}(A)$ and $\hat{m}_{ab} = \hat{M}_{ab}$. Or the abelian theory with bilinear $G(A)$ when $\hat{\ell}_{ab}(A) = \hat{L}_{ab}(A) = 0$.

Let $\mathcal{L}_{INV}$ be the normal massless Yang-Mills Lagrangian (3.1) and choose the "gauge" function

$$G(A) \equiv - M^2 \underline{W}_\mu . \underline{W}^\mu + (\tfrac{1}{\alpha}\partial^\mu \underline{W}_\mu)^2 . \tag{3.24}$$

(3.24) transforms under (3.2) to

$$2M^2 \underline{W}^\mu . \partial_\mu \underline{\eta} + 2 \frac{1}{\alpha^2} \partial^\mu \underline{W}_\mu . \partial_\nu (g \underline{W}^\nu \times \underline{\eta} - \partial^\nu \underline{\eta})$$

$$\equiv 2(\tfrac{1}{\alpha}\partial_\mu W^\mu) . (- \tfrac{1}{\alpha}\partial^2 - \alpha M^2)\underline{\eta} + 2(\tfrac{1}{\alpha}\partial_\mu \underline{W}^\mu) . g\partial_\nu (\underline{W}^\nu \times \underline{\eta}) \tag{3.25}$$

which is open to interpretation (3.13) with

$$C_a \equiv \tfrac{1}{\alpha}\partial_\mu W_a^\mu ; \quad \hat{L}_{ab} \equiv \tfrac{1}{\alpha} g \dot{\partial}_\nu f_{acb} W_c^\nu ; \quad \hat{M}_{ab} \equiv (- \tfrac{1}{\alpha}\dot{\partial}^2 - \alpha M^2)\delta_{ab} \tag{3.26}$$

where the dotted derivatives act on everything on the right. The massive Yang-Mills Lagrangian

$$\mathcal{L} \equiv - \tfrac{1}{4} G_{\mu\nu}^a \, G_a^{\mu\nu} + \tfrac{1}{2} M^2 W_\mu^a W_a^\mu - \frac{1}{2} \frac{1}{\alpha^2} (\partial_\mu W_a^\mu)(\partial_\nu W_a^\nu) \tag{3.27}$$

has been constructed, with Feynman rules as in Figs. 1(a) and 1(c), i.e. the identical theory to that of section a) of this chapter. We have thus verified that the massive Yang-Mills Lagrangian satisfies the Ward identities constructed

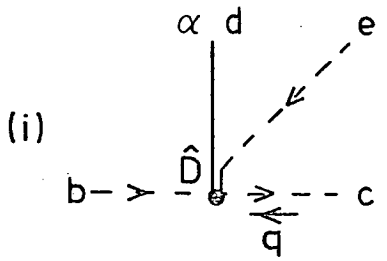
with rules of Figs. 1(a) amd 1(c) and the same source terms as the massless case.

Variation of $\hat{L}'_{ab}$ (3.26) with respect to (3.2) generates the auxiliary vertices $\hat{D}$ and $\hat{E}$, (i) and (ii) of Fig. 21. For these vertices the group property is not satisfied; instead the equation expressed as Fig. 20(b) holds. The additional vertex $\hat{F}$ is defined as (iii) of Fig. 21, i.e. the left hand side of identity Fig. 13(b) is

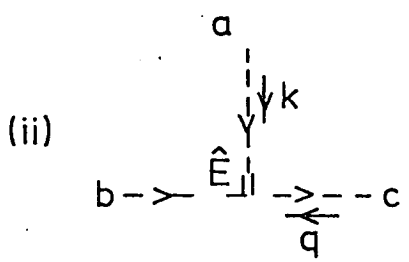
$$-\frac{1}{\alpha} i g^2 f_{eab} f_{ecd} \frac{\alpha M^2}{\frac{1}{\alpha}(p+q)^2 - \alpha M^2} k^\mu \quad . \tag{3.28}$$

This is as expected as the sufficiency condition (3.23) obviously does not hold with the vertices $\hat{\ell}_{ab} = \frac{1}{\alpha} g \partial_\nu f_{acb} W^\nu_c$ and $\hat{m}_{ab} = -\frac{1}{\alpha}\partial^2 \delta_{ab}$, and $\hat{L}_{ab}$ and $\hat{M}_{ab}$ as (3.26). Therefore, although the Ward identities hold for tree diagrams for this massive Yang-Mills Lagrangian for all α, they do not hold for diagrams involving loops generally. However, for $\alpha = 0$ the contributions from (3.28) vanish when the ghost vertices reduce to those for the massless Yang-Mills Lagrangian[27] with the same transform (3.2). The condition (3.23) is also satisfied. The Ward identities, therefore, hold to all orders for this "gauge".

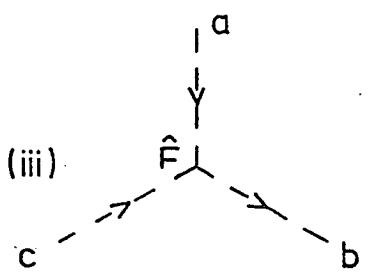
So far no physical lines have been included in the diagrams contributing to the Ward identities. Let the source terms include $J^a_\mu W^\mu_a$ (no summation implied over a), for each vector field, with vertices $\hat{r}$ and $\hat{\rho}$ as Fig. 21(iv) and (v). For an example consider the Ward identity with one source only shown in Fig. 22. To make the external vector boson line



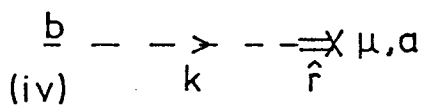
$$\hat{D}_{cbe}(A) \equiv \frac{1}{\alpha} i g^2 f_{abc} f_{ade} q^{\alpha}$$



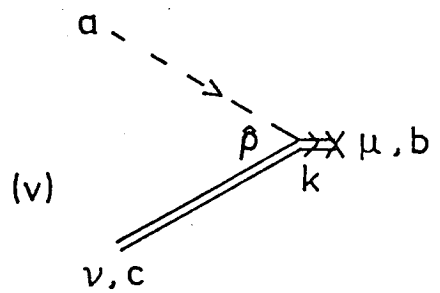
$$E_{cba} \equiv -\frac{1}{\alpha} g f_{abc} k.q$$



$$\hat{F}_{bca} \equiv \alpha g f_{abc} M^2$$



$$i k^{\mu} \delta_{ab}$$



$$g f_{abc} g^{\mu\nu}$$

FIG 21

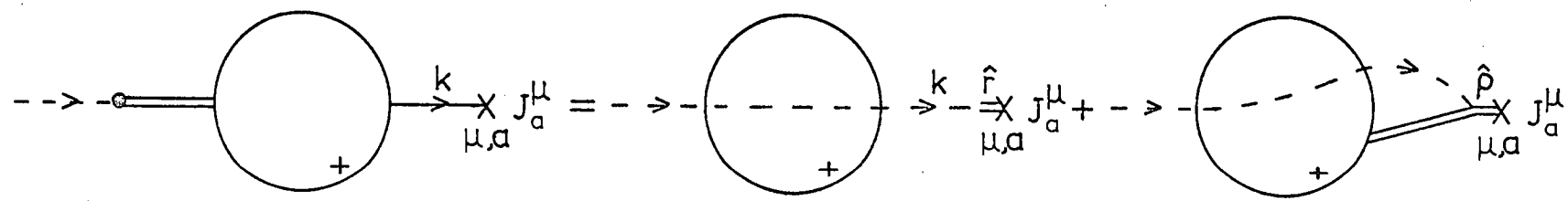


FIG 22 No summation is implied over a.

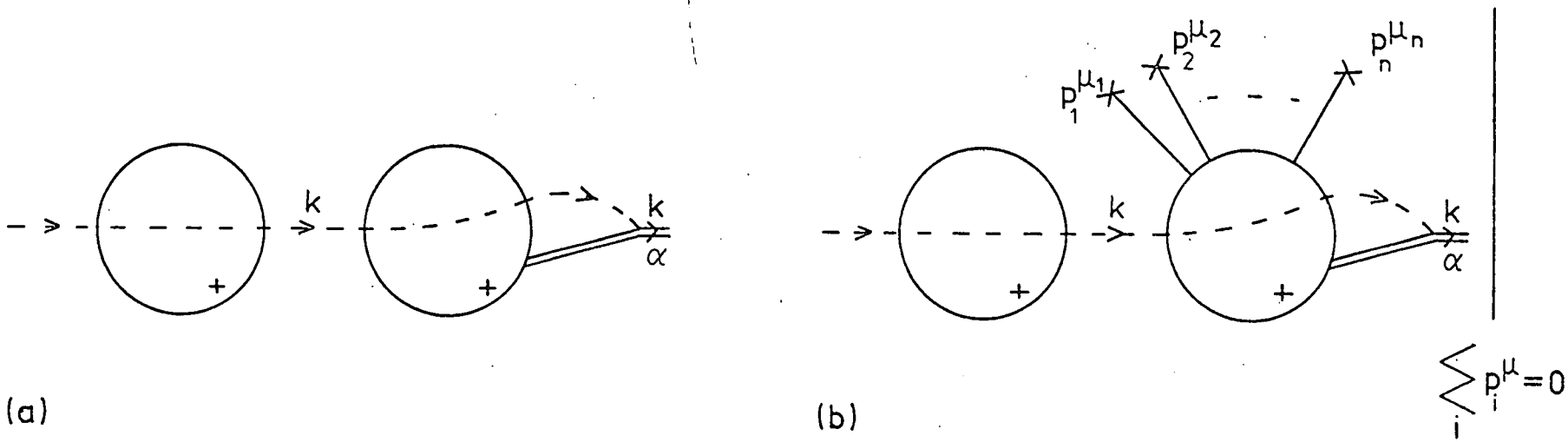


FIG 23 Diagrams with poles at $k^2 = M^2$.

on the left of Fig. 22 physical, drop the J^{μ}_{a}, multiply by the inverse of the propagator, viz. $(k^2- M^2)g^{\mu\nu}-(1- \frac{1}{\alpha_2})k^{\mu}k^{\nu}$, then by a physical polarization vector $e_{\nu}(k)$ and finally set k^2 on-mass shell, i.e. $k^2= M^2$. Nothing is assumed about the form of the polarization vector other than it is perpendicular to the 4-momentum

$$\text{i.e.} \qquad e(k).k \;=\; 0 \quad . \tag{3.29}$$

Hence the above procedure is equivalent to multiplying by $e_{\mu}(k)$ followed by $(k^2 - M^2)$ and only then is k^2 set on-mass shell.

The diagrams involving $\hat{r}$ vanish automatically. If the ghost mass is different from the vector mass then the diagrams with $\hat{\rho}$ vanish when the factor $(k^2 - M^2)$ is set on-mass shell. When the masses are the same diagrams of the form of Fig. 18(a) or (b) would not vanish in the on-mass shell limit. Through considerations of Lorentz invariance these diagrams must be proportional to k_{α}. Therefore, they also vanish under condition (3.29).

The diagrams in the Ward identities can now be extended to include physical external W-lines with no corresponding ghost-source terms. The Ward identities are as for the massless Yang-Mills Lagrangian and the extra physical polarization in the massive theory does not affect them as the condition (3.29) is automatically satisfied.

c) Invariance of the S-matrix

The Lagrangian of sections a) and b) may be written

$$\mathcal{L} \equiv \mathcal{L}_{INV} - \tfrac{1}{2}(-M^2 \underline{W}_\mu \cdot \underline{W}^\mu + \frac{1}{\alpha^2}(\partial_\mu \underline{W}^\mu)^2) + J_i R^i(A)$$

$$+\phi_a^* \left[(-\frac{1}{\alpha^2}\partial^2 - M^2)\delta^{ab} + \frac{1}{\alpha^2} g \partial_\nu f^{acb} W_c^\nu \right] \phi_b \qquad (3.30)$$

which gives Feynman rules equivalent to Figs. 1(a) and 1(d).

If the parameter α is varied infinitesimally such that

$$\frac{1}{\alpha} \rightarrow \frac{1}{\alpha} + \varepsilon \qquad (3.31)$$

then the Lagrangian (3.30) changes by the amount

$$\Delta \mathcal{L} = -\tfrac{1}{2}(\frac{2\varepsilon}{\alpha^2} + \varepsilon^2)(\partial_\mu \underline{W}^\mu)^2 + (\frac{2\varepsilon}{\alpha^2} + \varepsilon^2)\phi_a^* \left[-\partial^2 \delta^{ab} + g\partial_\nu f^{acb} W_c^\nu \right] \phi_b . \qquad (3.32)$$

Invariance of the S-matrix under (3.31) follows if a change in the Lagrangian proportional to (3.32) does not change the S-matrix elements between physical states constructed with (3.30)[50]. In addition to (3.32), for the S-matrix, the changes in the self-energy factors Z_e, multiplying each external physical line, must be taken into account. This is expressed graphically in Fig. 24 where the negative sign associated with the ghost loop is shown explicitly. Fig. 24 differs from the usual relation[50] for invariance in that an additional factor of $\frac{1}{2}$ appears in association with the first set of diagrams. It arises from combinatorical considerations as explained later.

Consider the Ward identity with one source term

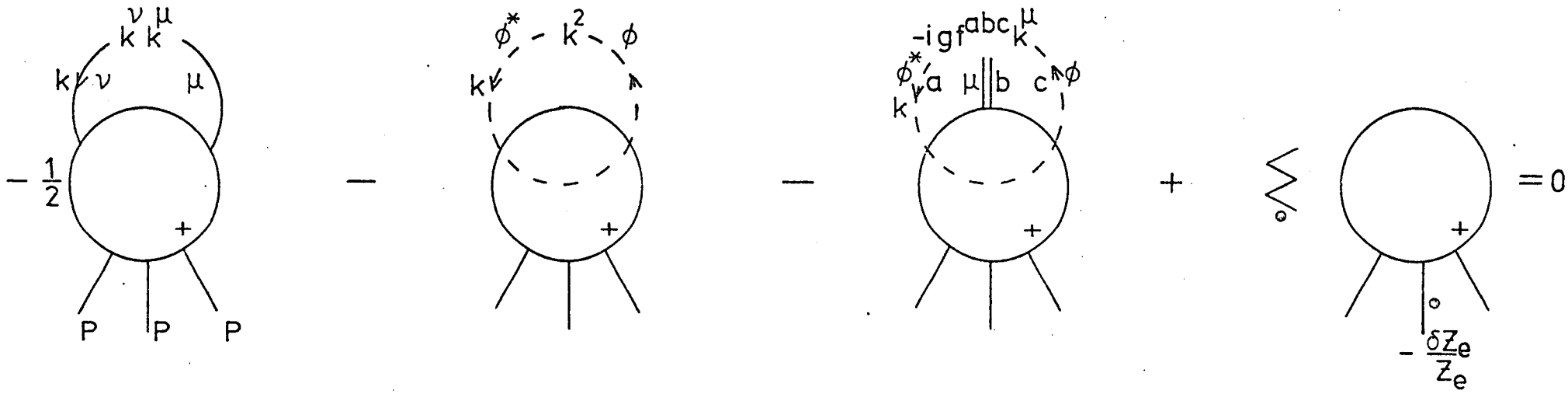


FIG 24 The lines with a P are physical lines including external line factors Z_e. The quantities δZ_e are the changes in Z_e due to the change of "gauge".

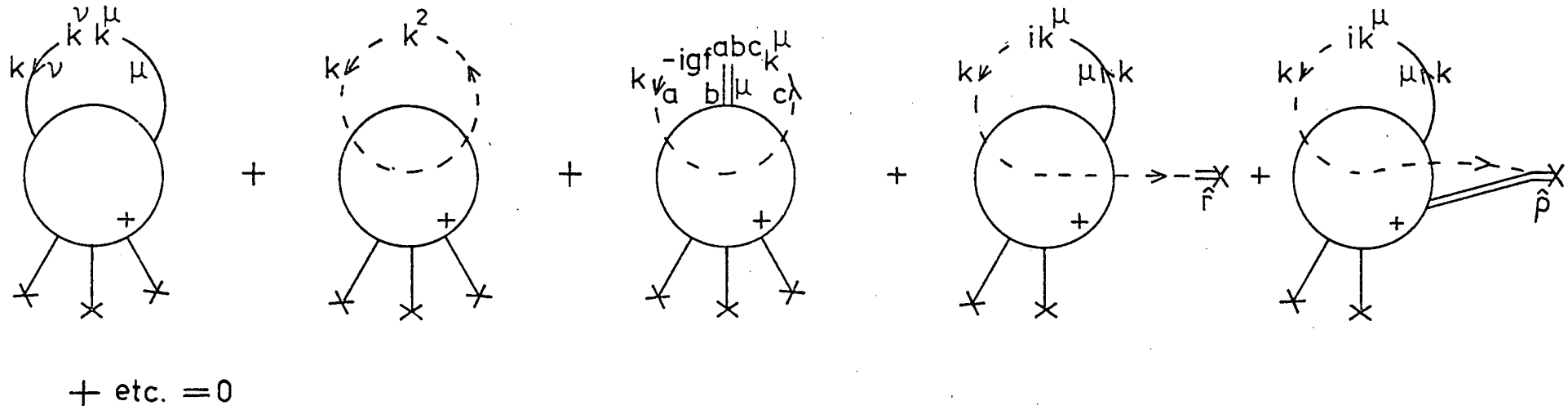
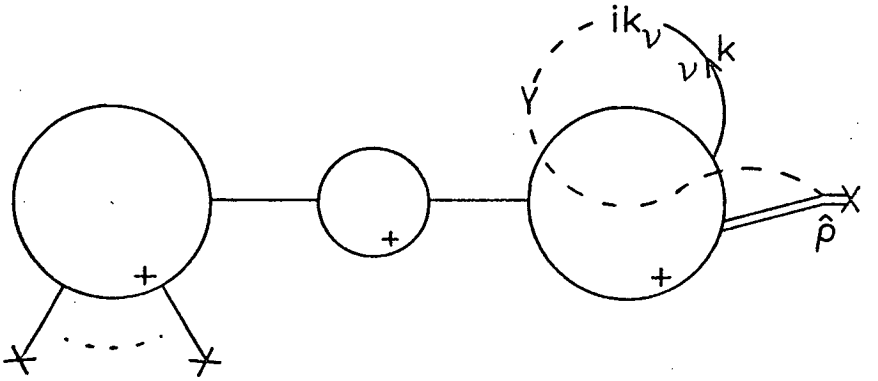


FIG 25

$J^a \partial_\mu W_a^\mu$. All other sources are $J_\mu^a W_a^\mu$ as in the discussion of external physical lines. The "C-source" and the source $J^a \partial_\mu W_a^\mu$ can then be folded together to obtain Fig. 25 with $\hat{r}$ and $\hat{\rho}$ as Fig. 21(iv) and (vi). All other sources are now made physical as in section b). The same arguments apply to the last two diagrams of Fig. 25 except for those such as



(3.33)

As in (50) these terms are the change in the external line factors, Z_e, due to (3.31). It can be seen that in this case Fig. 25 does not reduce to Fig. 24. If, however, each ghost loop had associated with it a factor of $(-\frac{1}{2})$ instead of (-1) then the middle two sets of diagrams of Fig. 24 would also have a factor $\frac{1}{2}$ associated with them and Fig. 25 would reduce to the appropriate relation corresponding to Fig. 24. The S-matrix is therefore invariant under variation of α if each ghost loop has a factor $(-\frac{1}{2})$ and the Ward identities are valid. In this proof the Ward identities for the (n-1) loop approximation are used to prove the invariance of the n loop approximation of the S-matrix.

The invariance immediately shows that the massive non-abelian theory, with Feynman rules of Figs. 1(a) and 1(c) and a factor $(-\frac{1}{2})$ associated with the ghost loops, is both

renormalizable and unitary as is well known[5,6,13,17]. When $\alpha = 1$ the Feynman rules become renormalizable according to power counting and in the limit $\alpha \to \infty$ the rules are the normal ones associated with the canonical quantization of the massive Yang-Mills field when no ghosts are present. The theory is therefore unitary, and identical to the usual massive theory, up to the one loop approximation.

The massive abelian theory is similarly renormalisable and unitary to all orders and the S-matrix for all α is identical to the canonically quantized formulation as anticipated in section b) of Chapter 2. By considering the formalism for $\alpha = 1$ or 0 it would appear not to be necessary to associate a multiplicative factor $\exp[i\frac{1}{2}(g/m)^2 D(0)]$ with each "charged" field as suggested by the investigation of Boulware[6]. Similarly Nakanishi's quantization[18] is that for $\alpha = 0$ and generates an S-matrix identical to the normal canonical quantization.

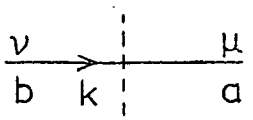
d) Unitarity

In section c) it was shown that the massive non-abelian Yang-Mills Lagrangian formulated as in section b) must be unitary up to the one loop approximation when a factor $(-\frac{1}{2})$ is associated with each loop. Thus at the one loop level it must be possible to prove the unitarity directly using the Ward identities of sections a) and b) in any "gauge". The notation is as (27).

As the vector boson propagator

$$\frac{g^{\mu\nu} - \frac{(1-\alpha^2)k^\mu k^\nu}{k^2 - M^2 + i\varepsilon}\delta^{ab}}{k^2 - M^2 + i\varepsilon} = \frac{g^{\mu\nu} - \frac{k^\mu k^\nu}{M^2}\delta^{ab}}{k^2 - M^2 + i\varepsilon} + \frac{k^\mu k^\nu / M^2}{k^2 - \alpha^2 M^2 + i\varepsilon}\delta^{ab}$$

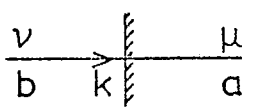
the appropriate cutting rule[28] is



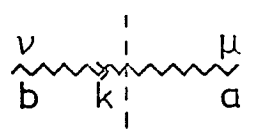
$$\equiv -\frac{1}{(2\pi)^3}\delta(k^2-M^2)\theta(k_o)(g^{\mu\nu} - \frac{k^\mu k^\nu}{M^2})\delta_{ab} -$$

$$- \frac{1}{(2\pi)^3}\delta(k^2-\alpha^2M^2)\theta(k_o)\frac{k^\mu k^\nu}{M^2}\delta_{ab}$$

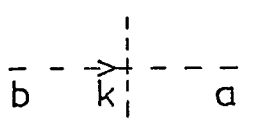
If we define



$$\equiv -\frac{1}{(2\pi)^3}\delta(k^2-M^2)\theta(k_o)(g^{\mu\nu} - k^\mu k^\nu/M^2)\delta_{ab}$$



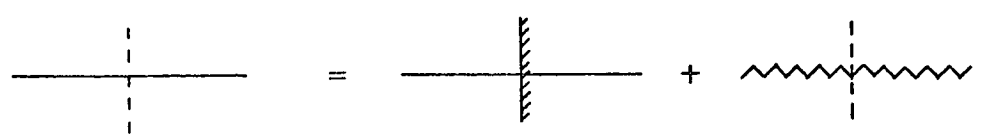
$$\equiv -\frac{1}{(2\pi)^3}\delta(k^2-\alpha^2M^2)\theta(k_o)k^\mu k^\nu/M^2\ \delta_{ab}$$



$$\equiv \frac{1}{(2\pi)^3}\delta(\frac{k^2}{\alpha} - \alpha M^2)\theta(k_o)\delta_{ab}$$

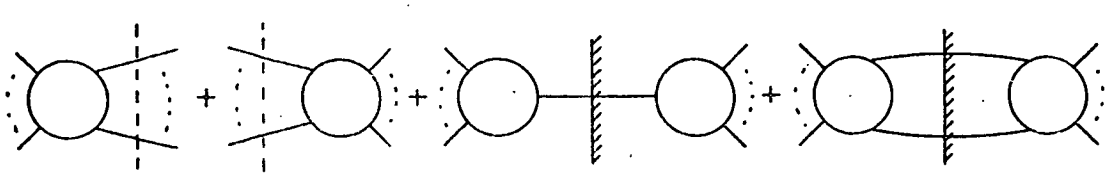
$$= \frac{1}{(2\pi)^3}\alpha\delta(k^2 - \alpha^2M^2)\theta(k_o)\delta_{ab}$$

then



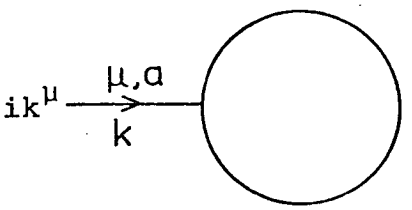
(3.34)

If the equation



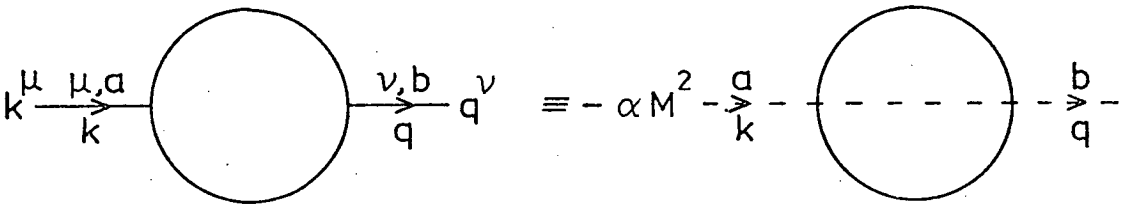
(3.35)

can be proved, unitarity is verified. Two Ward identities are needed. The first is the identity for diagrams with no unphysical sources which is equivalent to



$\equiv 0$. (3.36)

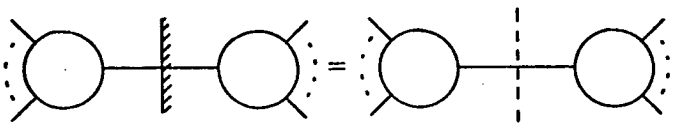
The second has a single source $J_a(-\partial^2 - \alpha^2 M^2)\partial^\nu W^a_\nu$ such that



(3.37)

For this the diagrams must be at least first order in g. These identities hold for tree diagrams only for all α.

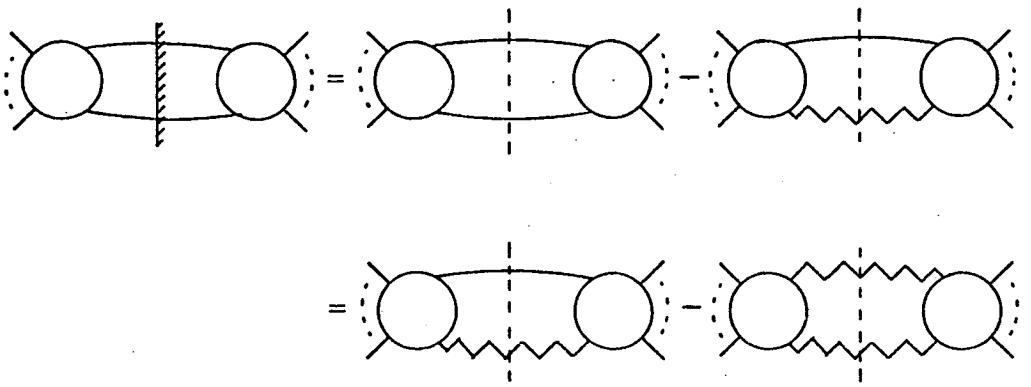
For the one loop approximation, using (3.34),



(3.38)

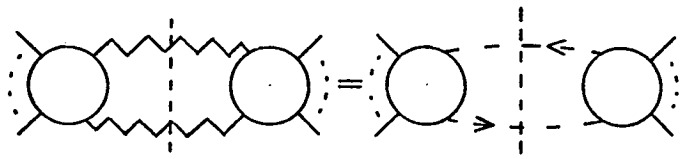
since at least one of the sub-diagrams is a tree diagram and (3.36) can be applied to it.

Similarly



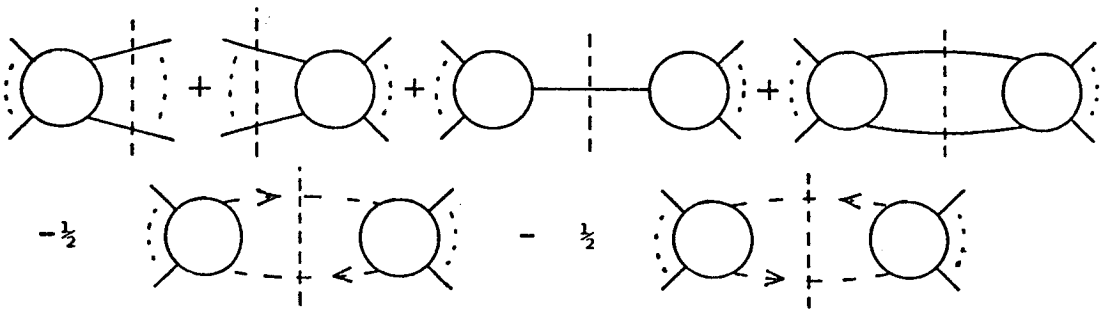
(3.39)

Applying (3.37) to both sub-diagrams which must be tree diagrams in the one loop approximation



(3.40)

With (3.38), (3.39) and (3.40) the left hand side of (3.35) becomes



(3.41)

(3.41) is zero and unitarity is proved as this is the correct form for the Cutkosky rules[28,61,62] to be applied if the ghost loops have an associated factor $(-\frac{1}{2})$.

As the Ward identities hold for the rules of Fig. 1 to all orders when $\alpha = 0$ the question now is whether unitarity can be proved directly to all orders for that gauge even though invariance of the S-matrix may no longer be invoked. The answer must be no, since to generalize (3.41) would require that the Ward identities hold when the factor associated with the ghost loops is $(-\frac{1}{2})$ and not (-1) as is the case. It would thus seem that the "soft" rules of Hsu and Sudarshan[17] are inappropriate to describe the massive Yang-Mills Lagrangian to all orders. We shall, however, examine this in more detail in Chapters Five and Six.

This chapter has been confined to the massive Yang-Mills

Lagrangian but the implications of the above for the massless theory is discussed in Appendix C. It may be noted that, as the massive rules obey the same Ward identities to all orders in the Landau gauge, i.e. $\alpha = 0$, infra-red divergent terms in the massless theory may be regularized by simply adding a mass term in the Landau gauge.

CHAPTER 4

EQUIVALENT FORMULATIONS OF THE MASSIVE YANG MILLS LAGRANGIAN - I

In Chapter 3 it was shown that the Ward identity techniques developed in the context of gauge theories[27,45,50,53] are applicable to the Massive Yang-Mills Lagrangian in the tree and one loop approximation.

In investigating whether the massive Yang-Mills theory is renormalisable it is advantageous to reformulate the theory such that the vector boson propagator becomes

$$\frac{g^{\mu\nu} - (1-K)k^{\mu}k^{\nu}/(k^2 - KM^2)}{k^2 + M^2 + i\varepsilon}$$

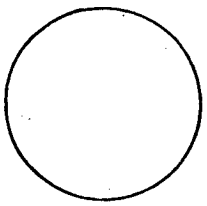
for some K. The reformulation has been achieved by Veltman et al.[5,13,14], using Ward identities, for the self energy terms and by Mohapatra, Sakakibara and Sucher[12] for the four-point interaction, but only to the two loop approximation. In Chapter 4 we extend the Ward identities of Chapter 3 to all orders in loops to reformulate all possible interactions to all orders in the loops. Our approach is shown to be equivalent to that of Veltman et al. in Chapter 5, and the resultant Feynman rules are identical to those derived, in the path integral formulation, by Boulware[61].

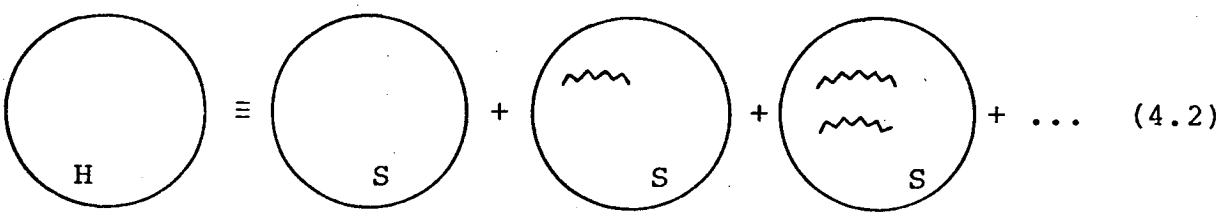
a) Combinatorial Factor Considerations

The normal vector-boson propagator can be factorized:

$$\frac{g^{\mu\nu} - k^\mu k^\nu/M^2}{k^2 - M^2 + i\varepsilon}\,\delta^{ab} \equiv \frac{g^{\mu\nu} - \frac{(1-\alpha^2)k^\mu k^\nu}{k^2-\alpha^2M^2+i\varepsilon}\,\delta^{ab}}{k^2 - M^2 + i\varepsilon} + \frac{-k^\mu k^\nu/M^2}{k^2 - \alpha^2M^2 + i\varepsilon}\,\delta^{ab} \qquad (4.1)$$

where the terms spoiling the renormalizability according to power counting have been separated. The left hand side is the hard vector-boson propagator and the first term on the right hand side of (4.1) the soft propagator, as in (12).

If  is equivalent to the set of all diagrams contributing to an amplitude and the replacement (4.1) is made for all propagators, then

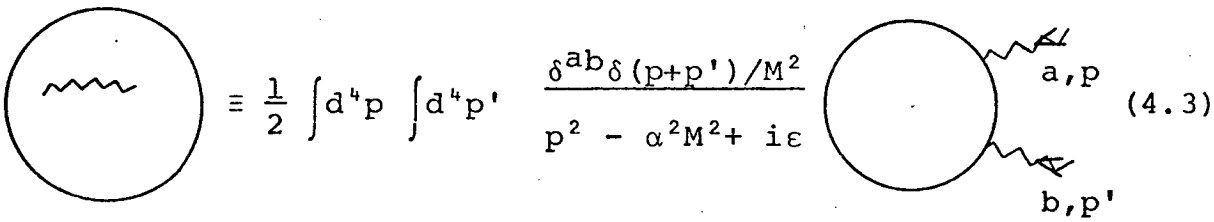


(4.2)

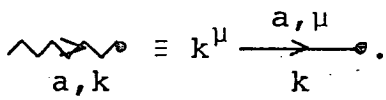
The suffix H indicates that the hard propagator is used for all vector-boson propagators; similarly S for the soft propagator. The ∿∿∿ in the "blob" denote the replacement, in all possible ways, of a vector-boson propagator by the second term on the right hand side of (4.1). The series continues up to the set of diagrams in which all propagators are replaced by ∿∿∿ . We wish to separate the terms

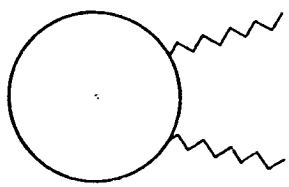
~~~~ into two independent momenta contractions which may then be treated as sources for the Ward identities. However, the amplitudes from which we start and those to which the Ward identities are applied have different combinatorial factors[14], the difference between which must be taken into account on splitting an internal line. An example of this has already been met in the factor $\frac{1}{2}$ present in Fig. 24.

Firstly, we consider the second term on the right hand side of (4.2) which can be represented as



(4.3)

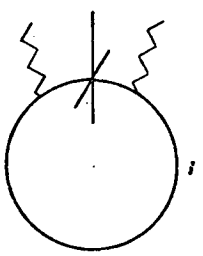
where 

. It should be noted that with this construction there is at least one vertex connected to each source ~~~~ . The set of diagrams  is symmetrized in the external legs, i.e. the notation is the same as 't. Hooft[27] except for the symbol for contraction with the momentum vector. That $\frac{1}{2}$ is the correct combinatorial factor is demonstrated as follows.

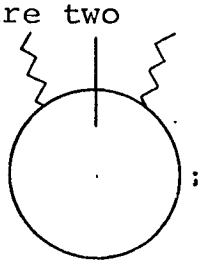
The same diagram with two external sources cannot be generated by splitting a propagator in different diagrams as reversing the process would imply the original diagrams were identical. The only way a diagram with external legs can be generated repeatedly is by replacing indistinguishable propagators
~~~~

in the same diagram by ~~~~~ . The number of ways of constructing the original diagram from the individual vertices is the number of ways of constructing the corresponding diagram with the vertex legs which would form the n indistinguishable propagators unconnected and undesignated, N say, times the number of ways of connecting these legs to form the required diagram, i.e. $n!\times N$. The number of ways of constructing the associated diagram, leaving two unconnected legs to form the pair of external sources, is N times the number of ways of choosing which two legs are not to be connected, times the number of ways of constructing the remaining $(n-1)$ indistinguishable propagators, i.e. $n^2(n-1)!\times N = n(n!\times N)$. But n is the number of repetitions of the diagram constructed with a pair of external sources and the repeated generation of the same diagram supplies the factor necessary to modify the original combinatorial factor associated with each diagram to its correct value, cf. Appendix of Veltman[14].

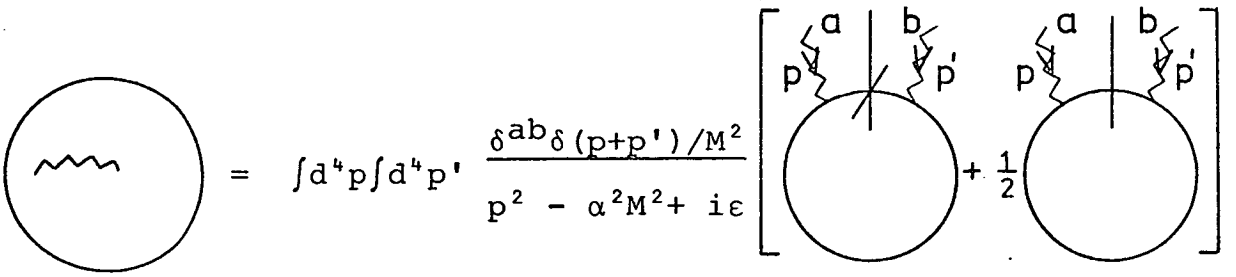
It remains for the unconnected pair of sources to be labelled. For the set of diagrams non-symmetric in the unconnected sources there is only one way of doing so as they are already distinguishable. This set is denoted by



their combinatorial factors are automatically correct. For the set of diagrams symmetric in the sources there are two ways of labelling the legs. This set is denoted by

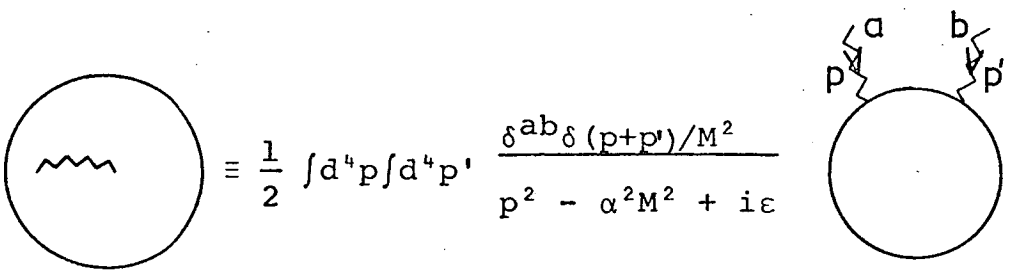


their factorials are thus half the required value. Hence,



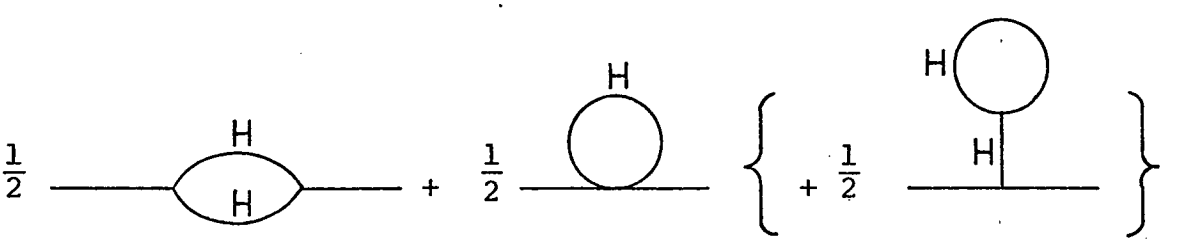
$$= \int d^4p \int d^4p' \, \frac{\delta^{ab}\delta(p+p')/M^2}{p^2 - \alpha^2M^2 + i\varepsilon} \left[\; + \frac{1}{2} \; \right]$$

To regain the usual notation the external sources must be symmetrized. This is already so for the second set of diagrams on the right hand side but for the first set it will entail a doubling of the number of diagrams. Thus

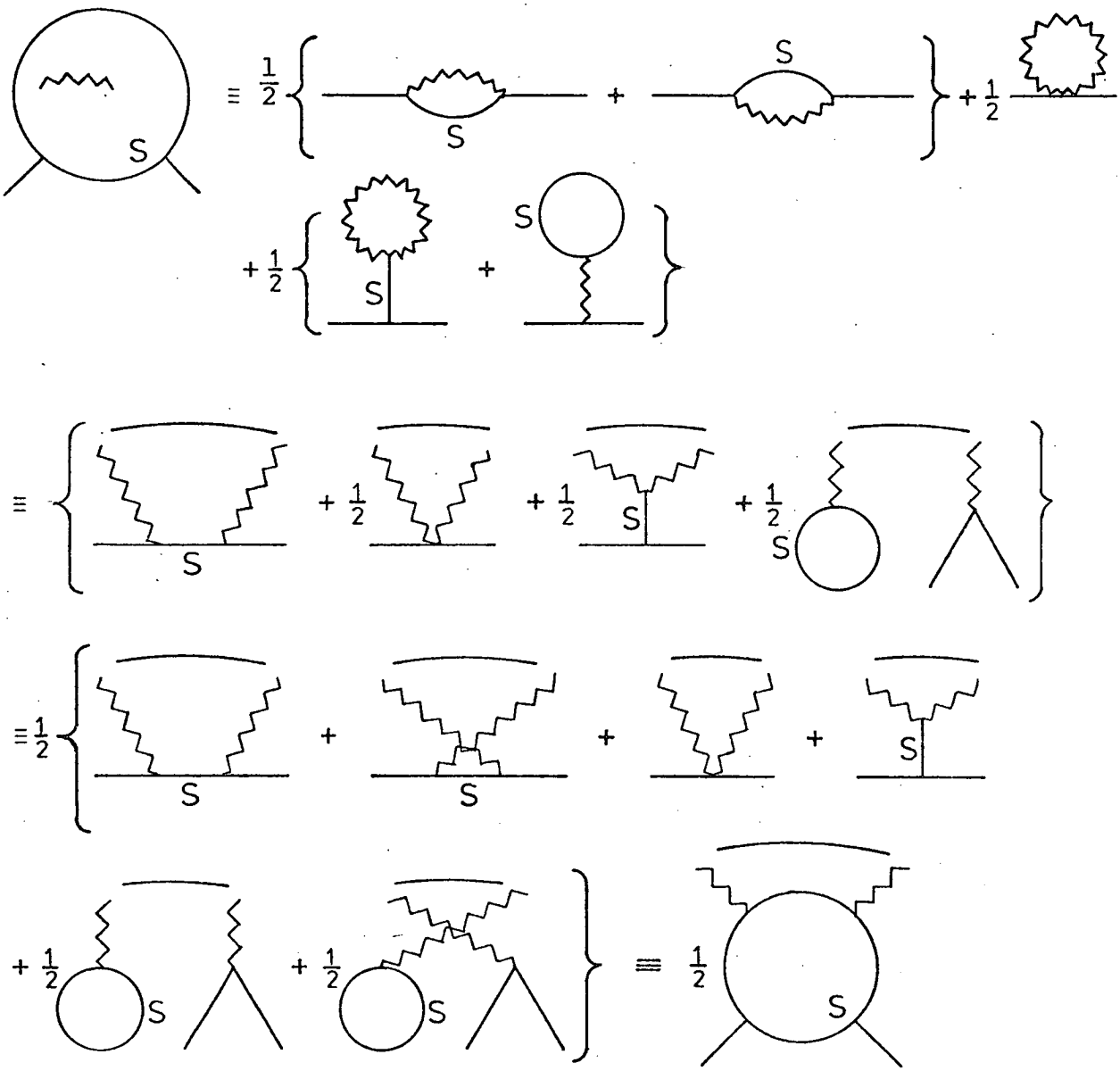


$$\equiv \frac{1}{2} \int d^4p \int d^4p' \, \frac{\delta^{ab}\delta(p+p')/M^2}{p^2 - \alpha^2M^2 + i\varepsilon} \qquad (4.4)$$

The set of diagrams represented by the "blob" on the right hand side contains all possible diagrams contributing to the new amplitude; the diagram from which any diagram with a pair of sources was constructed can be obtained by reconnecting the sources. Included are diagrams which correspond to those on the left, with tadpole terms, which vanish anyway on contracting. For example, consider the self energy term of order g^2 (the factorials are shown explicitly)

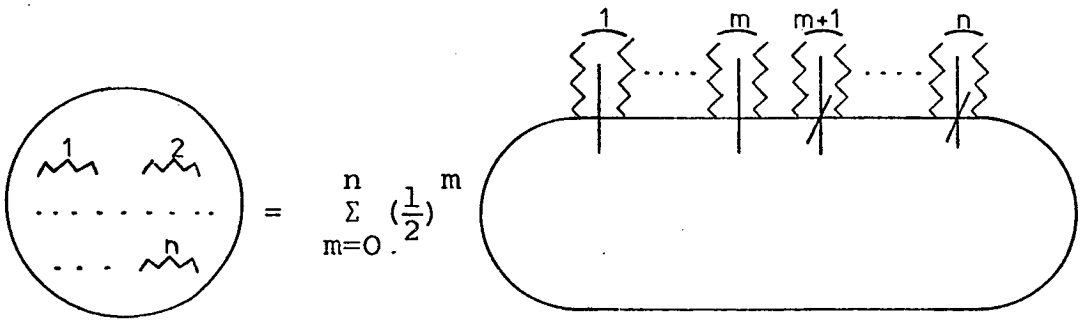


gives rise to



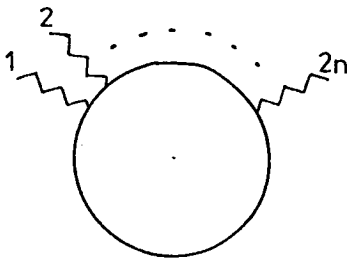
where ——— stands for the integral contraction.

The analysis as above can be repeated indefinitely to give before symmetrization of the external sources

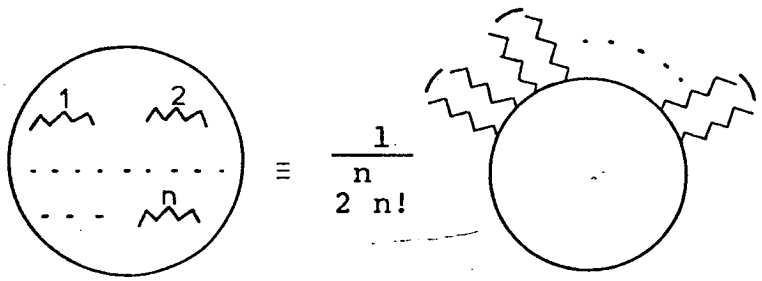


$$= \sum_{m=0}^{n} \left(\frac{1}{2}\right)^{m} \qquad (4.5)$$

where the contractions have been explicitly labelled 1 to n. Let us consider the subset of diagrams



which only differ through the labelling of the external sources. Any diagram produced by connecting two sources can contribute to no other subset. Hence, as this subset contains all the permutations of labelling the sources, any diagram and an associated one, which differs only through an interchange of the labels on a contracted pair of non-symmetric sources, produce the same result on evaluation. Also any permutation of the labels on the contractions have the same outcome. Therefore, on symmetrizing the "blob" as usual we find



(4.6)

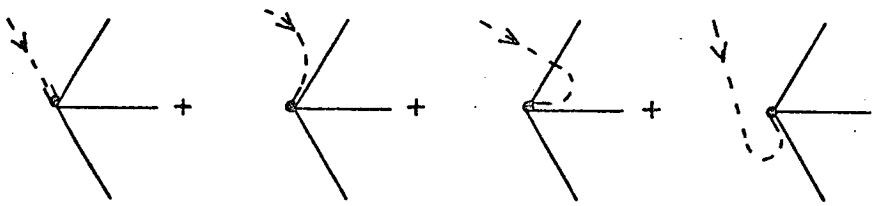
The set of diagrams on the right of (4.6) are now of the form to which the Ward identities may be applied.

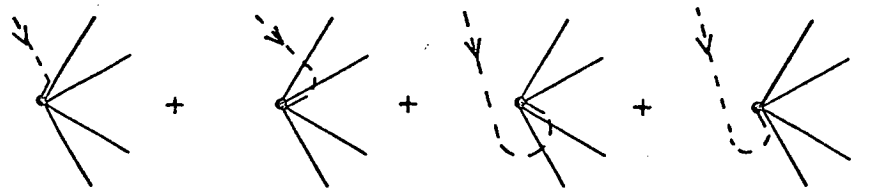
b) Generalised Ward Identities

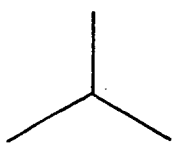
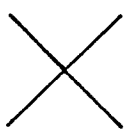
To evaluate the contribution of the momenta contractions we wish to treat them as the sources (either "C-sources" or "R-sources", as required, in the language of (50)) in Ward identities and so generate the equivalent Feynman rules for a scalar particle. If there were only two sources present in an amplitude which was of no higher than second order in the loops the Ward identities of Chapter 3 could be used to generate

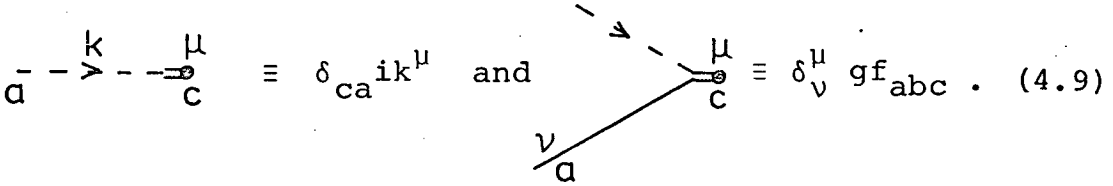
the soft rules of Fig. 1 which were shown to be equivalent to that order of approximation. To generalize the Ward identities it is easier to follow the original method of 't. Hooft[27] (all references in this section are to this paper). Initially we shall not include any ghost terms in the amplitudes unlike (27) or Chapter 3 where there were ghost loops present from the beginning.

As in §4 of (27) the identities 4.4b) and 4.4c) hold as they involve pure vector-boson vertex identities which are unchanged. The identity 4.4a) is not necessary here and has in fact no content in this context.

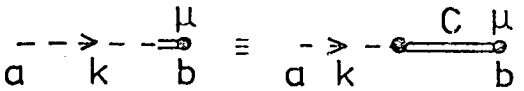
i.e.  (4.7)

and  (4.8)

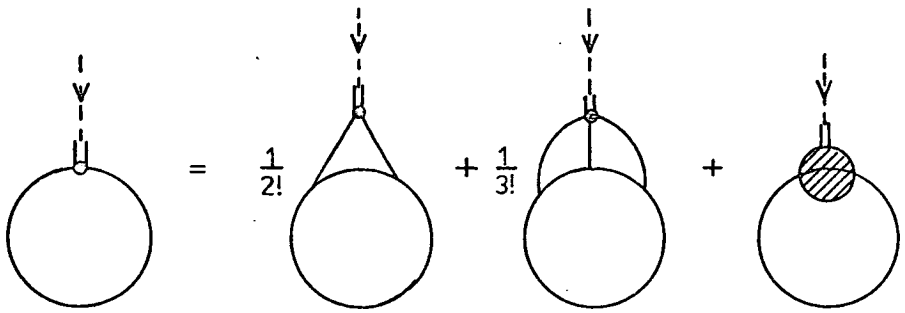
where  and  are the vector boson vertices of Fig. 1(a) and

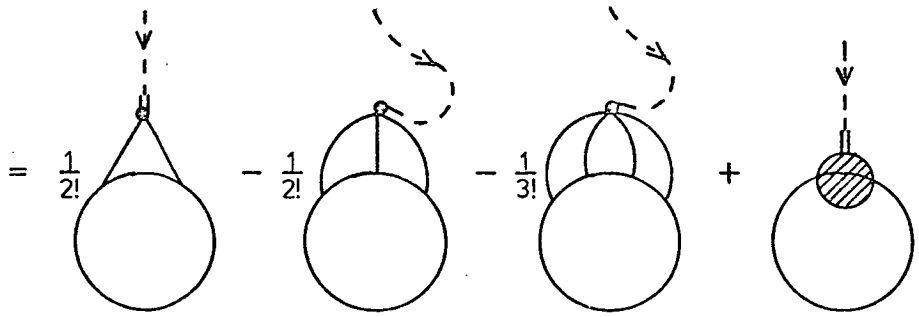
 (4.9)

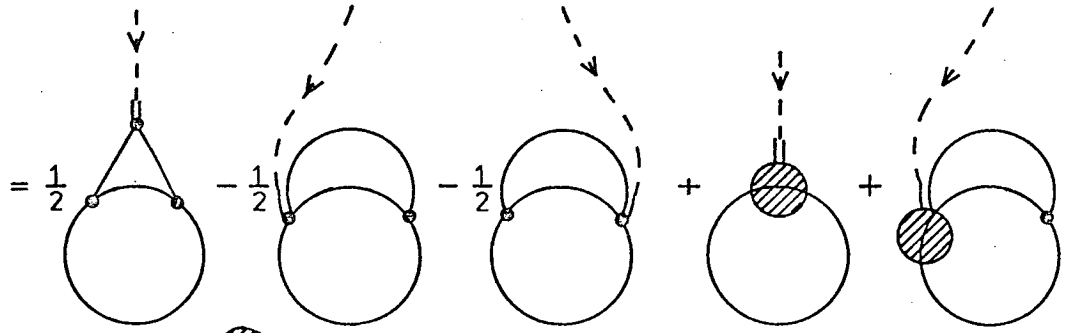
In terms of the notation used previously

 (4.10)

Similar manipulations to (4.5), (4.6) and (4.7) of (27) can be made :

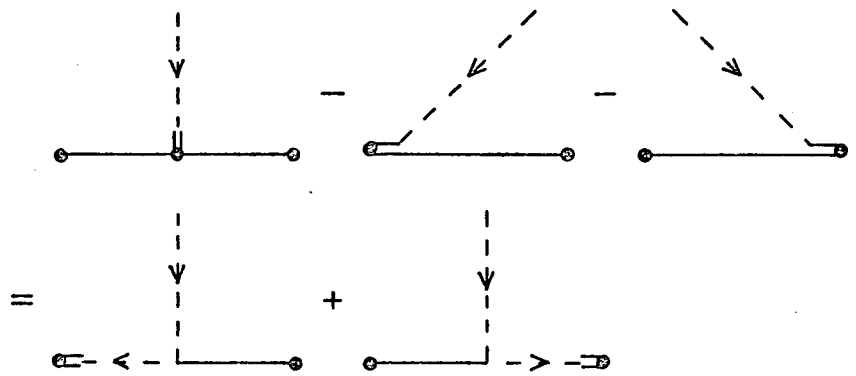






(4.11)

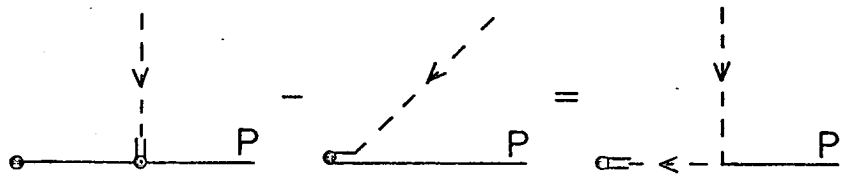
where  indicates any other termination of the ghost line other than at a pure vector-boson vertex. The identity equivalent to 4.8a) of (27) is



(4.12)

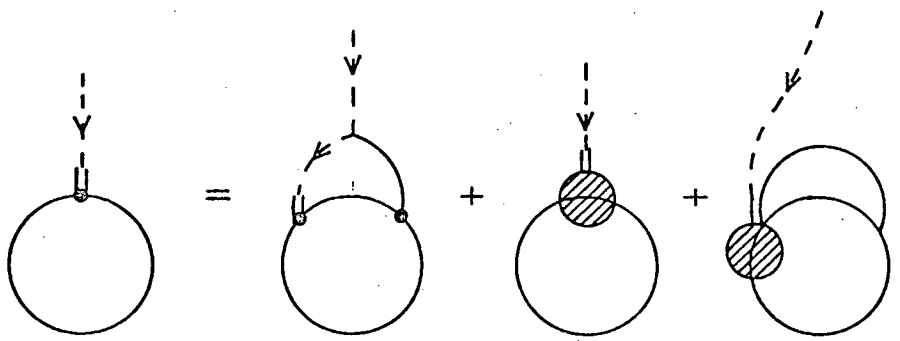
in which the ghost propagators and vertices are as in Fig. 1(b). (4.12) can be verified either directly or by noting that it is merely the Ward identity of Chapter 3 with two sources $J^{\mu}_{a} W^{a}_{\mu}$ and of order one in g. Also the Ward

identity for one source $J^{\mu}_{a}\, W^{a}_{\mu}$ and a physical particle of order one in g is the identity corresponding to 4.8b) of (27), viz.



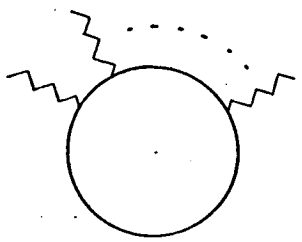
(4.13)

On applying (4.12) and (4.13) to (4.11) we obtain corresponding to (4.9) of (27)



(4.14)

By iterating (4.14) the scalar ghost line can be traced through the diagrams until it is terminated by either the second or third term on the right hand side.

For the general amplitude  , starting from the first external source on the left, there are two sets of possible terminations:-

(i) the scalar line turning and terminating on itself

(ii) terminating at another external source.

These are illustrated in Fig. 26. For the subset (i) the additional vertices required are

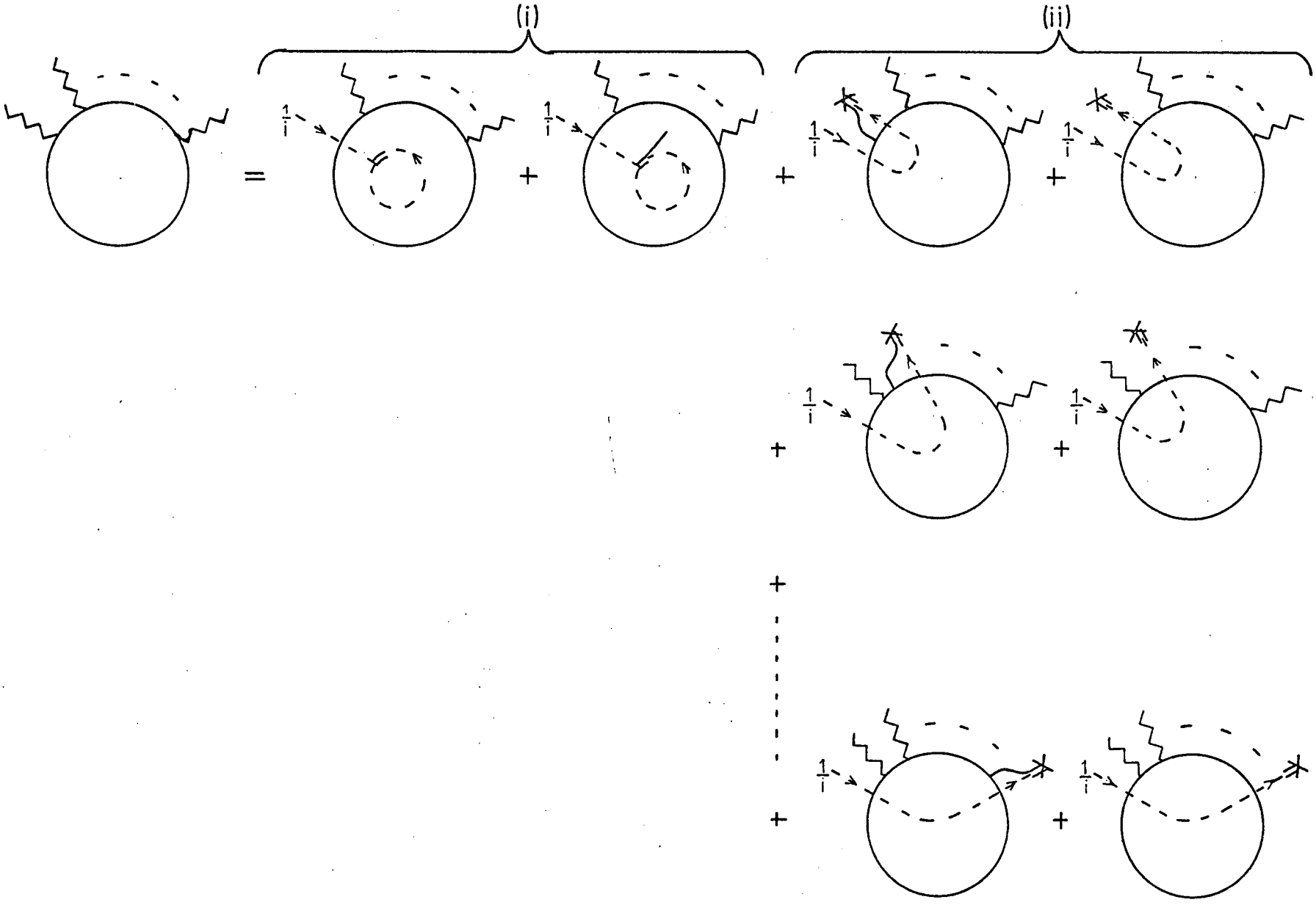
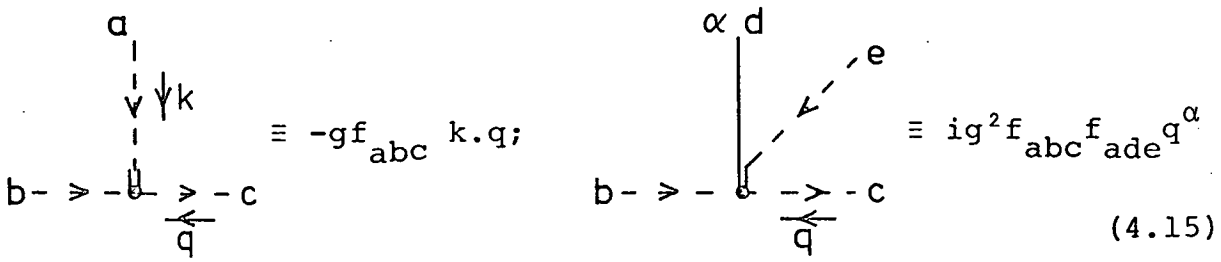
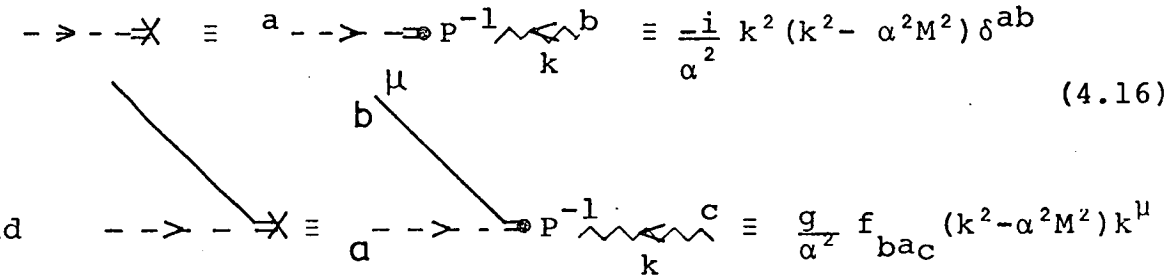


FIG 26 The $(\frac{1}{i})$ is present since ≡ $\frac{1}{i}$



$$\equiv -gf_{abc}\, k.q; \qquad \equiv ig^2 f_{abc} f_{ade} q^{\alpha} \tag{4.15}$$

cf. Fig. 16 (i) and (ii). For (ii) the additional vertices are

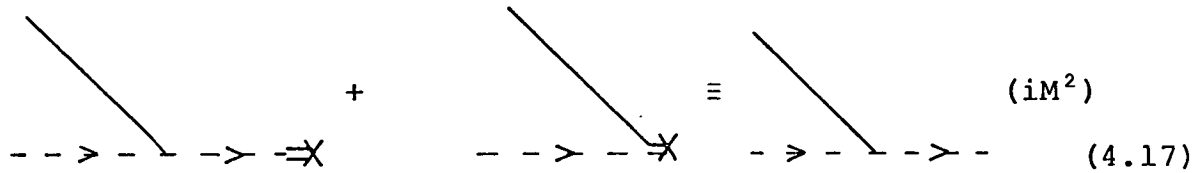


$$\equiv \frac{-i}{\alpha^2} k^2 (k^2 - \alpha^2 M^2) \delta^{ab} \tag{4.16}$$

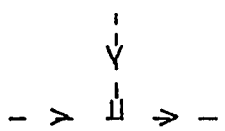
and

$$\equiv \frac{g}{\alpha^2} f_{bac} (k^2 - \alpha^2 M^2) k^{\mu}$$

where P is the vector boson propagator. As each source is attached to at least one vertex the two terms in (4.16) can be combined such that



(4.17)

cf. (3.37). As a shorthand let both amendments to vertices, (4.9), be represented in diagrams by the first, e.g. both vertices in (4.15) are denoted by  except where explicitly indicated. Thus Fig. 26 can be rewritten as Fig. 27.

The process may be repeated starting from the next unexploited external source in any subset of diagrams on the

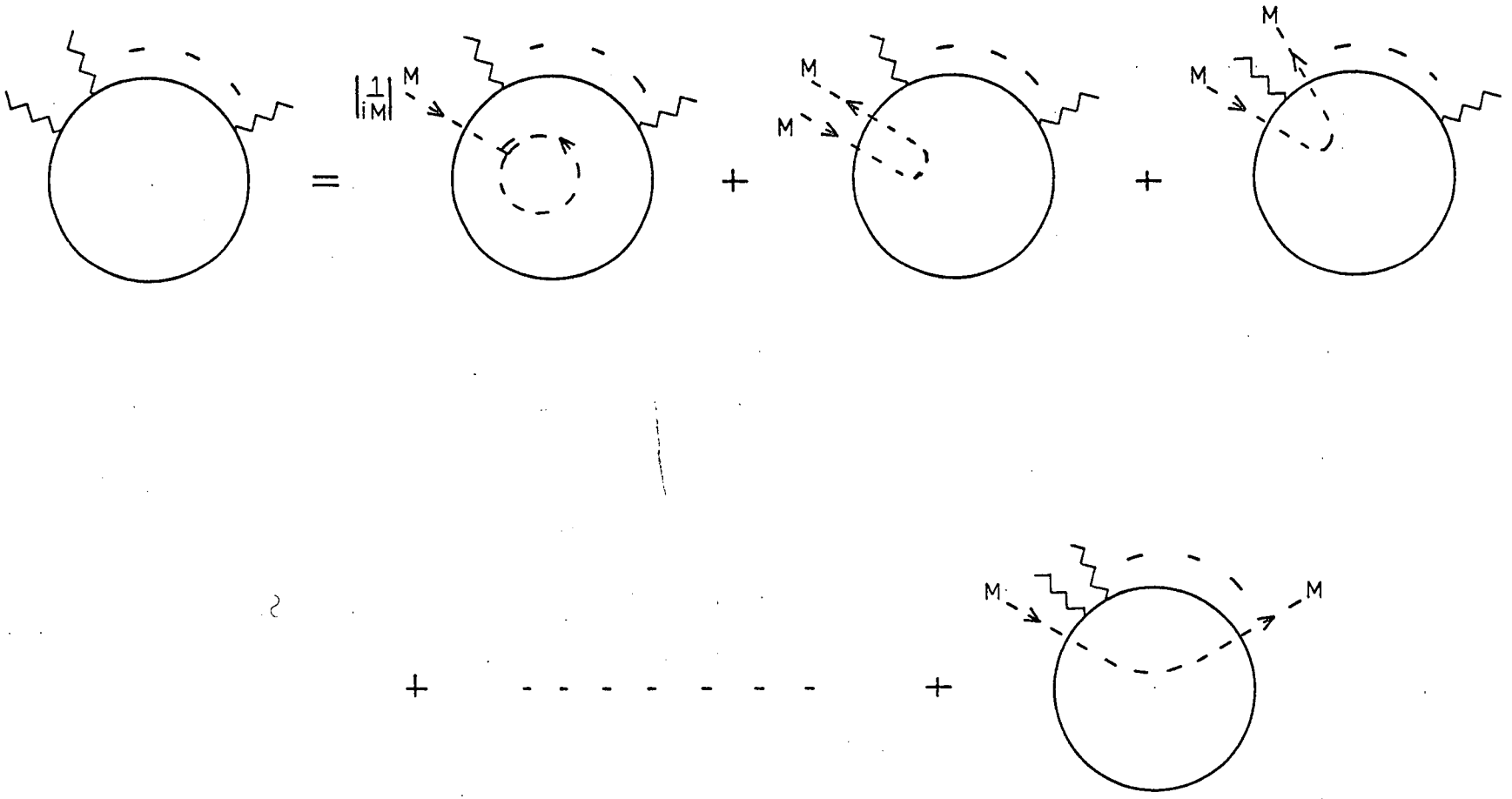


FIG 27 The M^2 associated with a scalar line which does not terminate internally has been split between both sources and a factor M has been given to all other scalar lines.

right hand side of Fig. 27, except that there are now additional points at which the scalar line may terminate, i.e. at the additional vertices introduced above or on a previously created scalar line. The result of tracing an additional line for the first subset on the right hand side of Fig. 27 is shown in Fig. 28. By repeating until all external sources have been utilized in all diagrams we obtain the general Ward identities which are constructed with a family of scalar vertices, with no limit on the order, in addition to those of Fig. 1(b). When the factor $(\frac{1}{iM})$, associated with each ingoing scalar line for which there is no corresponding outgoing line, is incorporated in the vertices, the family is as in Fig. 29.

c) The Duplication Factors

Having absorbed the factors $(\frac{1}{iM})$ into the vertices we are left with a factor M on each external scalar line. However, adjacent pairs of external sources were originally contracted, as in (4.6), by $\int d^4p \int d^4p' \frac{\delta^{ab}\delta(p+p')/M^2}{p^2-\alpha^2M^2+i\varepsilon}$ where p and p' are the momenta of the sources. Thus the factors M on the external lines and the $\frac{1}{M^2}$ in the contraction completely cancel and the remainder of the contraction may be identified as (-1) times the scalar propagator. If these externally created propagators could be reabsorbed into the diagrams by the reverse process to that of section a), the contributions of the various scalar configurations (by configuration here we mean the overall topology of the scalar lines including external propagators without considering the

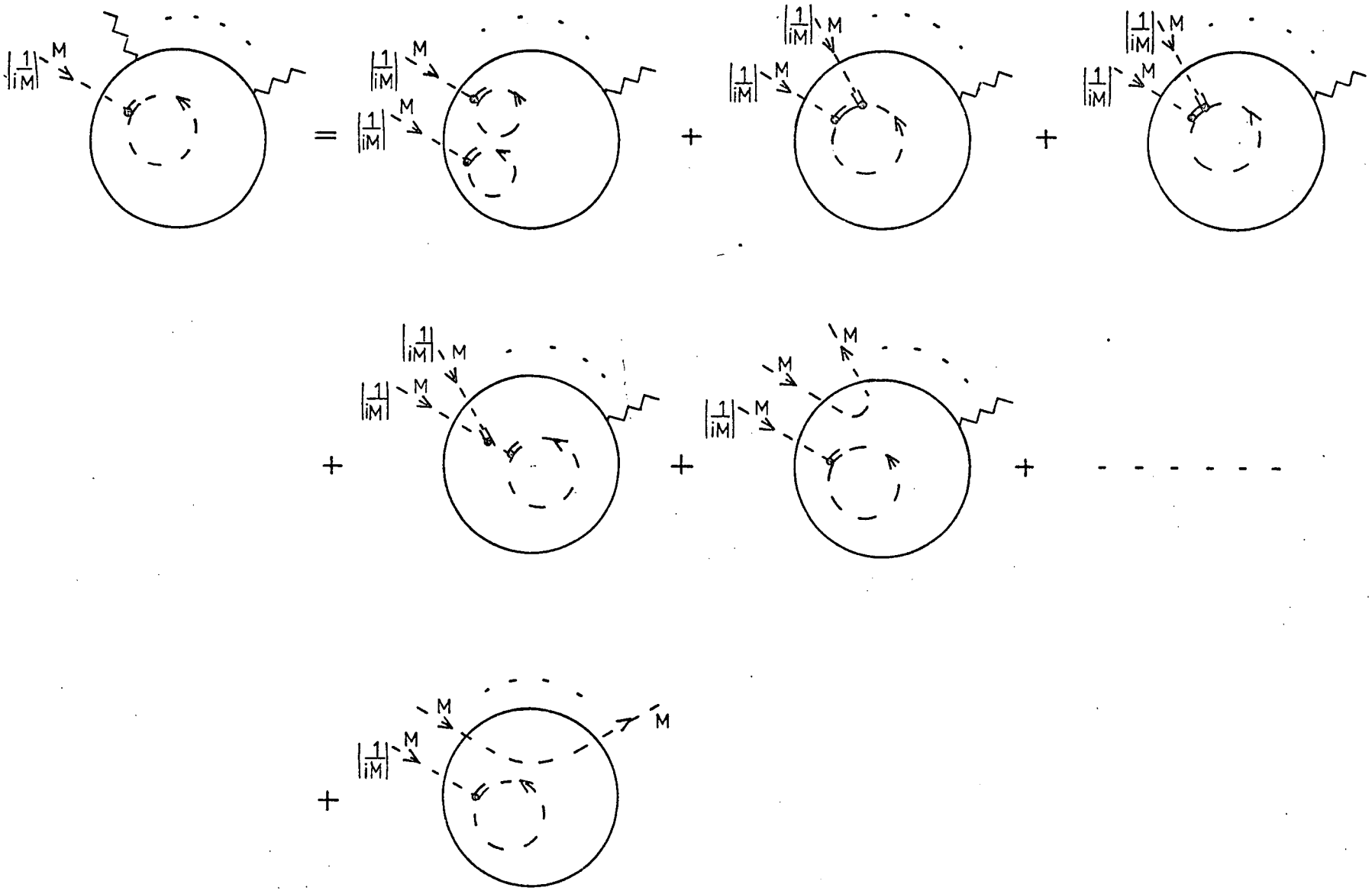


FIG 28

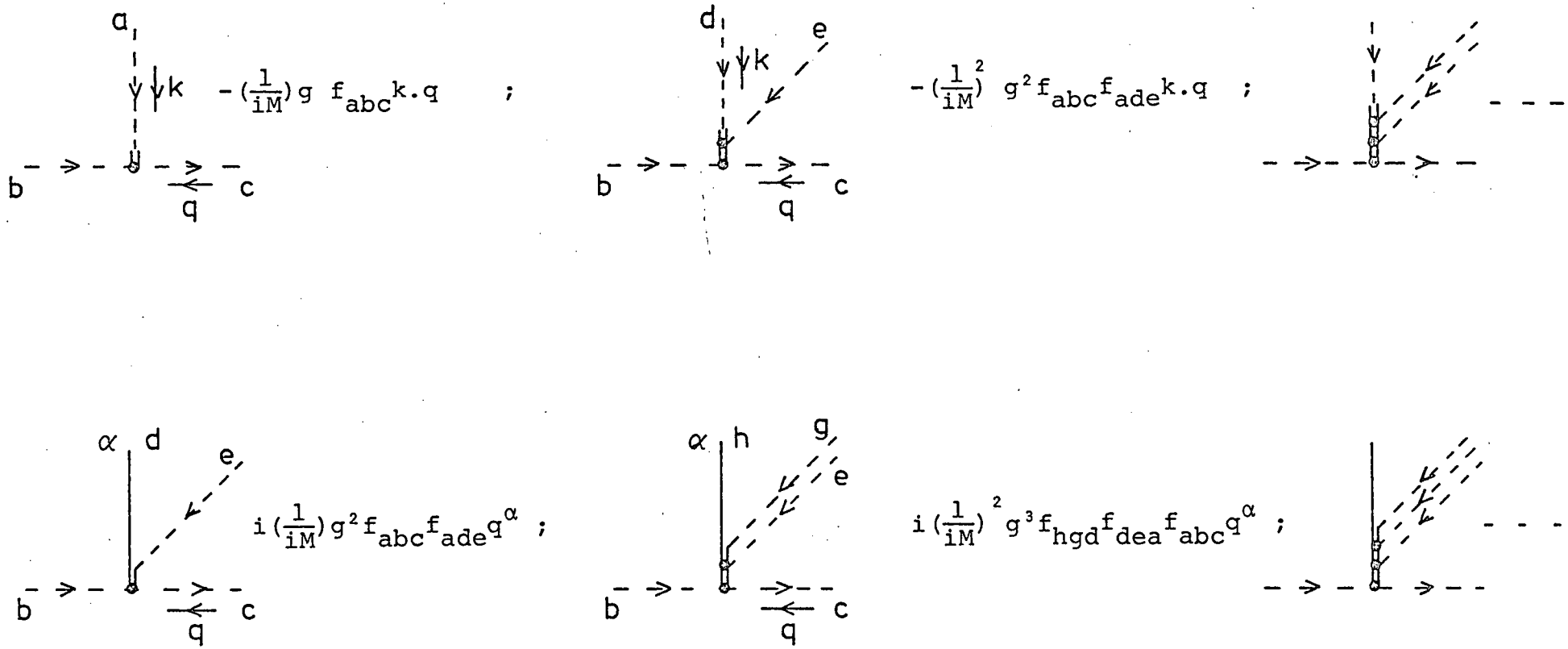


FIG 29 The next pair of vertices is constructed by making the additions (4.9) to the lower one of the previous pair.

vector-boson vertices) might be described by a conventional Lagrangian. However, the same final scalar configuration can be generated repeatedly with different numbers of external propagators from different subsets on the right hand side of (4.2). To obtain the Feynman rules for the scalar ghosts the number of duplications of each different scalar configuration must be evaluated. To ease the problem of counting we use the rule that when a scalar line terminates at an external source or internally, the first available external source from the left is used to originate the next scalar line, cf. for example Figs. 30 and 31.

First we consider the scalar configurations before absorption into the diagrams. They consist of sections of the following forms:-

(i) A simple line starting at an external source and terminating at another.

(ii) Trees involving any number of vertices of any order with only one outgoing scalar line. In the diagrams with one particular set of external sources used in the construction of a tree, the outgoing line may terminate at any one of the sources involved except the one furthest left. Note that in the trees there are lines from vertex to vertex which include no externally constructed propagators.

(iii) Trees with only one closed scalar loop incorporated. There is no way in which two closed scalar loops could be constructed and joined together without involving externally created propagators. This set includes the simplest possible configurations  , the ρ-loop. There is no

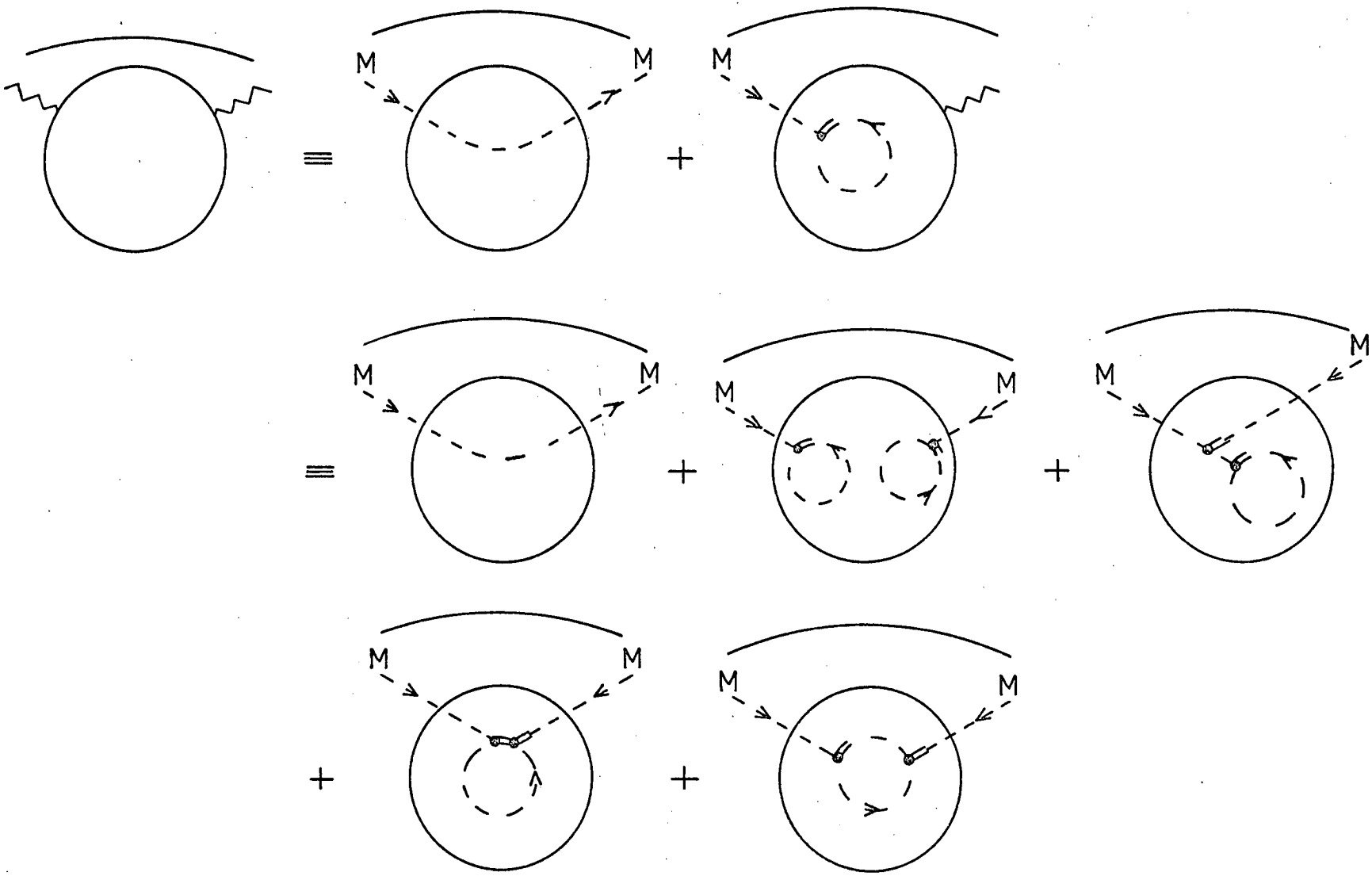


FIG 30 The Ward identity for the second set of diagrams on the right of (4.2) constructed using the rules suggested in the text.

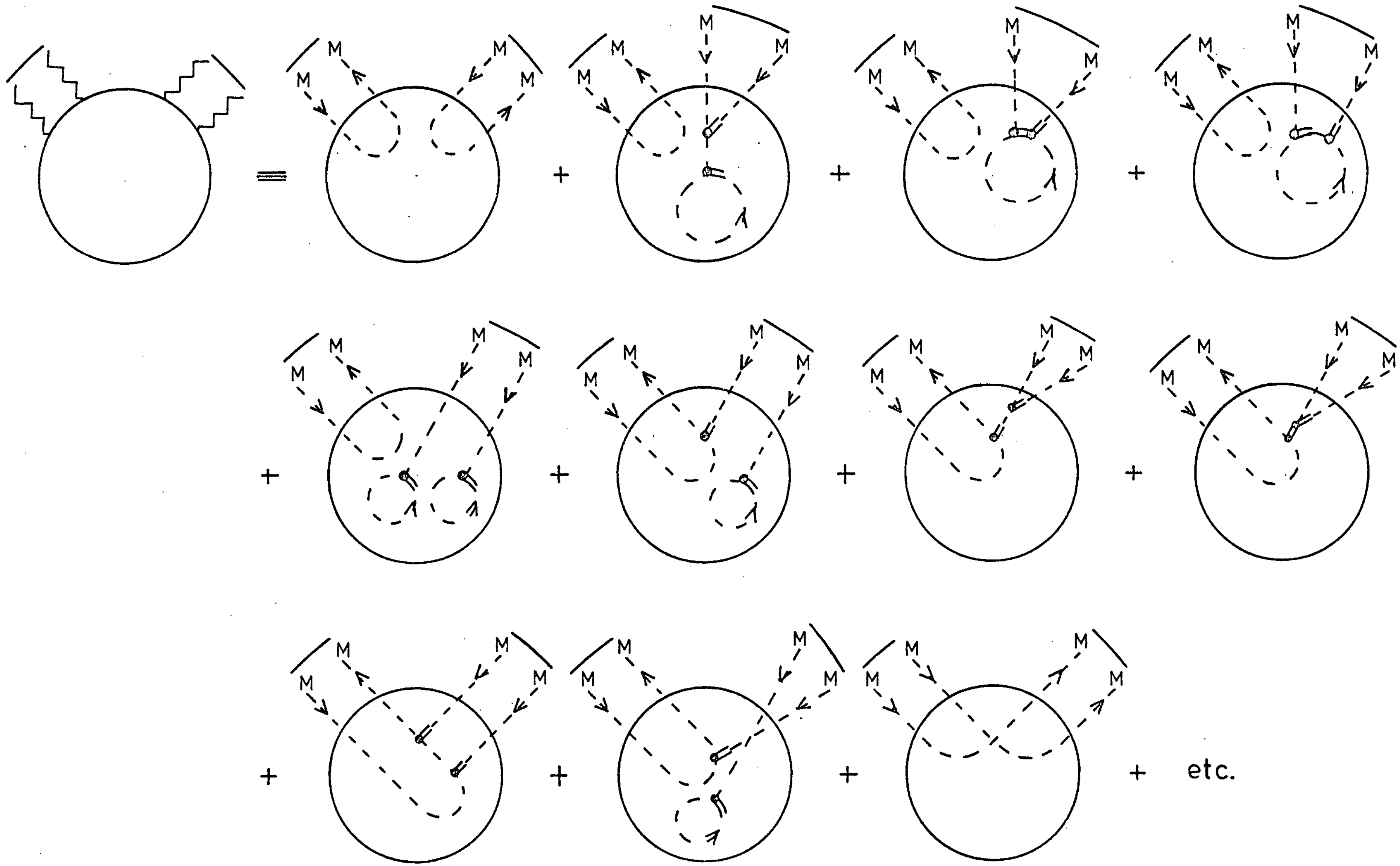


FIG 31 The Ward identity for the third set of diagrams on the right of (4.2).

outgoing line for the trees involved in this set.

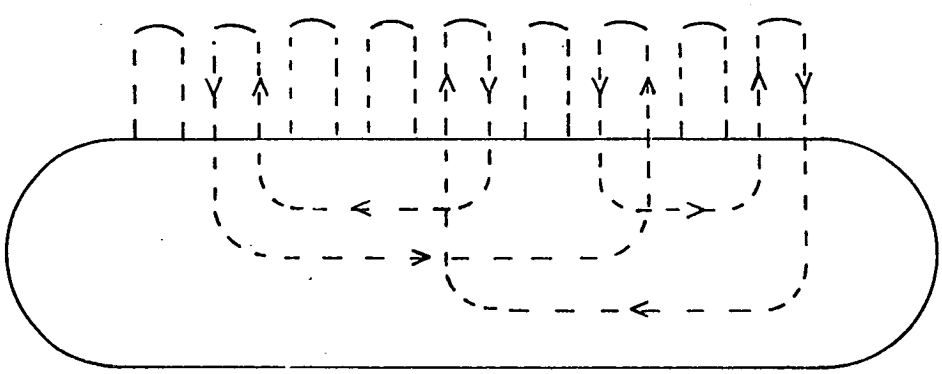
The branches of the trees in (ii) and (iii) can be distinguished by the vector-boson attachments but, having labelled them, the branches can originate from any selection of sources (this is without reference to the directional arrows). If two branches originate from a contracted pair of external sources, they do so symmetrically. Members of (i) may be thought of as forming chains whereby a line may end at an external source which is contracted to the beginning of another etc. The chains may connect members of (ii) or (iii) through external propagators or two legs of the same tree. Within the chains the choice of source-pairs are again free and symmetric within each pair. The only other possibility is that the head and tail of a chain are themselves connected by an external propagator to form a closed scalar loop when absorbed into the diagrams. However, for loops although any selection of source-pairs may be involved, the total symmetricity of choice within a pair is no longer available and these configurations will be treated separately.

We, thus, see that any n-point interaction of the original explicitly unitary formalism of the massive Yang-Mills field can be expressed in terms of the soft rules but with the additional vertices of Fig. 29. Using the Ward identities it is straightforward though tedious to do the conversion for any specific example although it may not generally be reduced to the simple algebra of a normal field theory. As the externally created propagators carry a factor (-1) many

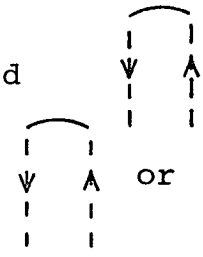
cancellations can be expected but not enough to render the formalism explicitly renormalizable. To find a lower bound to the cancellations for a general choice of α the scalar vertices shall be treated as if totally symmetric and hence the directional arrows cease to have any meaning. Obviously if the contributions of the vertices of Fig. 29 do not cancel exactly with this simplification, they cannot do so in the original form. With this assumption of symmetry which will be discussed in section e), the scalar contributions can be reduced to a standard field theory.

In the formation of scalar loops the arrows on the separate lines may now be reversed. Thus starting from the source in the loop furthest to the left, the directional arrow is followed to the source at which the line terminates. The arrow must automatically point away from the original source by virtue of the construction procedure. The loop is followed through the externally created propagator and the line attached to its other end, reversing the arrow if necessary to point in the required direction. Continue until the chain reaches the external propagator connected to the original source. The arrows now point round the loop in the same direction

e.g.



The asymmetry within the originating source-pair can now be

seen explicitly as it may only be constructed 

whereas any other pair might be constructed

If there was a tree involved this asymmetry would be removed by the method of construction. Let a loop involve n_L source-pairs in a configuration with a total of n source-pairs. The loop can be constructed by any selection of n_L external propagators, i.e. a duplication of $\frac{n!}{(n-n_L)!}$. Also, as all but the originating source-pair, furthest to the left, may be connected to the rest of the loop in either of the two ways illustrated, there is further duplication by the amount 2^{n_L-1}. Keeping the directional arrow on the vertices the loops may be treated as those for a pseudo-charged field with Feynman rules of Fig. 1b), when the appropriate combinatorial factor is one. The loop can therefore be absorbed into the diagrams to give Fig. 32. There the extra factor $\frac{1}{2}$ is associated directly with the scalar loop.

In Fig. 32 the externally created propagators have been indicated by an asterisk and each carries a factor (-1). The example shown has four such external propagators which may be any selection of four from all the scalar propagators in the loop. There is thus an additional duplication factor $\binom{n_L}{4}(-1)^4$ where the (-1) of the external propagators has been included in the duplication factor. However, with the rest of the scalar configuration remaining unchanged, the same loop can be created by the different subsets on the right of (4.2) with any number of external propagators from 1 to n_L and the total duplication is

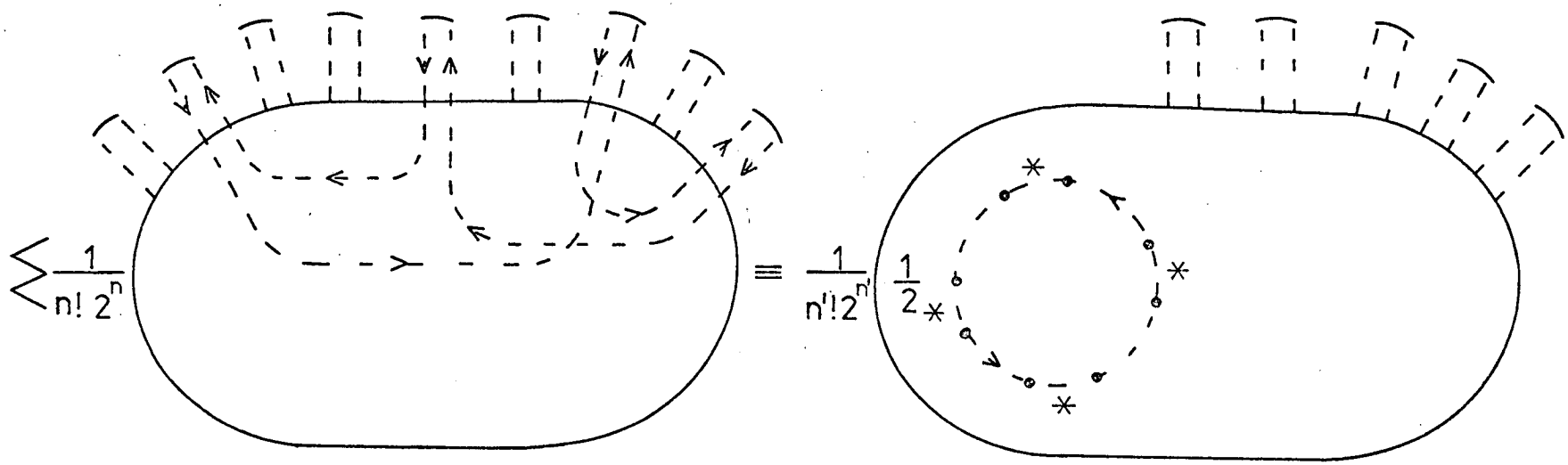


FIG 32 The summation is over all equivalent configurations in the loop. n and n' are the number of source-pairs in the diagrams and * denotes the externally created propagators absorbed into the diagrams.

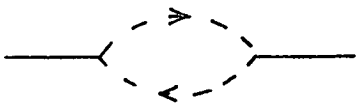
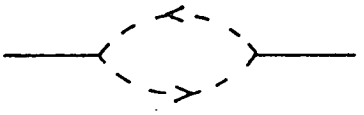
$$\{(-1)\binom{n_L}{1} + (-1)^2\binom{n_L}{2} + \ldots\ldots (-1)^{n_L}\binom{n_L}{n_L}\}$$

$$\equiv (1-1)^{n_L} - 1 = -1 .$$

Hence any loop created can be incorporated in the diagrams if it has an associated factor $(-\frac{1}{2})$ and a factor $\frac{1}{n'!2^{n'}}$ remains with the rest of the scalar configuration where n' is the number of source-pairs left in the rest of the configuration. The loops continue to be treated as pseudo-charged fields.

Having absorbed all the loops into the diagrams we are left with all other scalar configurations. To treat them the assumed symmetry of the scalar vertices is exploited to drop the arrows on the scalar chains and trees involved. If for any purely tree section of the configuration only one exit source for the outgoing arrow were allowed the diagrams could be considered as those appropriate for a standard field theory with vertices as Figs. 1(a), 1(b) and 29 and the appropriate combinatorical factor. But all but the source furthest to the left may be the exit point. For the chains of simple lines there is no problem but for the sections of configuration which are trees with a closed loop attached, the combinatorical factor is not appropriate. The arrow in the construction of the scalar loop discriminates between the same loop created with a clockwise or anti-clockwise ordering of the vertices in the loop, when these are not identical. Here the combinatorical factor should be one but each diagram appears twice on dropping the arrow. When the clockwise and

anti-clockwise orderings are identical

e.g.  and 

they are created once only but the associated combinatorical factor should be $\frac{1}{2}$ for a normal scalar field theory cf. Section a). Thus, on dropping the arrows and associating with these trees, the appropriate combinatorical factor, each diagram carries a duplication factor of two. Hence, when any particular diagram is given the correct combinatorical factor consistent with the ghosts being described by a normal scalar field theory, it has a duplication factor associated only with the ghost configuration. The factor has a multiple of two for each tree with an attached closed loop and a multiple $(n-1)$ for each tree when n is the number of branches terminating at a source. When a tree has an attached loop there is no factor of $(n-1)$ since there is no outgoing scalar line to produce the repetition of construction.

If we consider subsets of diagrams with the same scalar tree and loop structure but with different numbers of lines and eternally created propagators the duplication factor for each diagram in the subset is the same. We now redefine a line to start from a member of Fig. 29 and to end at a member of Fig. 29 without reference as to whether external propagators are involved. For each subset the final overall scalar configuration is identical for each diagram with a fixed selection of lines containing no external propagators.

For those lines with external propagators any number can be present, up to and including the case when all the propagators in a line are externally created, but at least one must be. From the method of constructing the scalar lines it is obvious that each must start and finish at a vertex of Fig. 29. The same final configuration can be generated from different subsets of diagrams but with a different selection of lines having no external propagators.

Because of the complete symmetry in external sources of all sections of a configuration except those that form isolated closed loops, as noted earlier, the configuration under consideration here can have its external propagators reabsorbed into the diagrams by the reverse process of section a). As before the combinatorical factors have to be reconciled with the new diagrams but it only entails the absorption of the factors $\frac{1}{n'!}\frac{1}{2^{n'}}$. It does not matter that all possible diagrams do not have the same duplication factors as the insertion of the external propagators can be done diagram by diagram. For any subset we obtain a fixed scalar configuration with some of the lines containing all selections from one to n_L propagators carrying a factor (-1) as they were originally generated eternally. n_L is the total number of scalar propagators in the line. As the only weighting factors associated with the diagrams are now the duplication factors, the combinatorial factors having been completely removed, they are not dependent on the number of external propagators involved, only on the topology. Hence, the duplications due to having all selections of propagators in any line, with at least one external propagator, each carrying a factor (-1) can be summed to give a total factor of (-1) as for the isolated loop.

From here we only consider connected configurations. There is a minimum number of lines which can be chosen to contain external propagators for any set scalar configuration. This is easily seen as there must be at least one propagator externally created for each attachment, (4.9), to a scalar vertex, i.e. if V_i is the number of scalar lines attached to the i^{th} scalar vertex the minimum number is $N \equiv \Sigma(V_i-2)/2$. Within this restriction any selection of lines in a configuration may be chosen not to contain any external propagators. These correspond to the various subsets above which would generate the same configurations. The total duplication factor associated with any scalar configuration may be calculated from the following rules:-

(i) Draw a diagram of the purely scalar configuration for each possible selection of from O to L-N lines not containing any external propagators where L is the total number of lines in the configuration. Here lines with no external propagators will be indicated by a wavy line with all other lines indicated by a solid line.

(ii) A factor zero is given to all diagrams with two loops made up of and connected solely by wavy lines. By the construction none can appear. This rule is to some extent precluded by (i).

(iii) For a vertex composed solely of hard lines multiply by a factor (V-1) where V is the number of lines in the vertex.

(iv) For a tree structure composed solely of wavy lines multiply by a factor (T-1) where T is the total number of hard lines emanating from the tree of wavy lines. (iii)

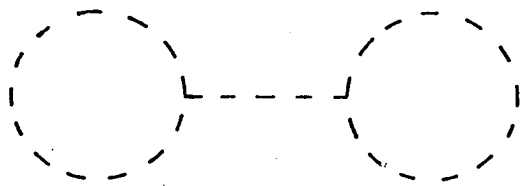
and (iv) are the factors (n-1) associated with the different exits for the outgoing scalar line in a tree with n branches.

(v) For a tree of wavy lines with an attached loop made up solely of wavy lines, a factor 2.

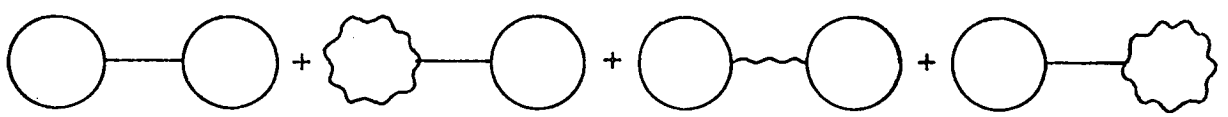
(vi) For each hard line a factor (-1).

(vii) The total duplication factor is the sum of the factors for each diagram of (i). The factors are calculated by the rules (ii) to (vi).

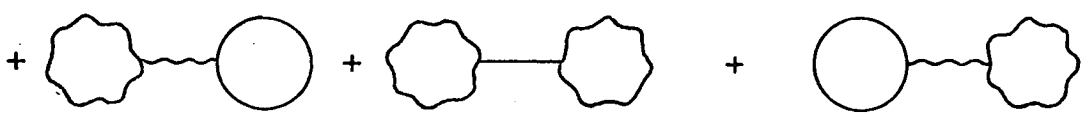
For example consider the dumb-bell shaped scalar configuration



Its duplication factor is calculated as follows. As N = 1 the following possibilities arise



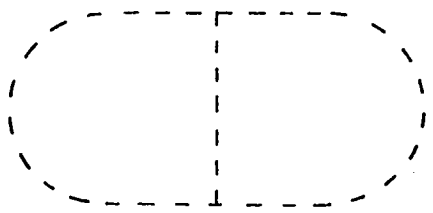
$(-1)^3 2 \times 2$ $(-1)^2 2 \times 2$ $(-1)^2 3$ $(-1)^2 2 \times 2$



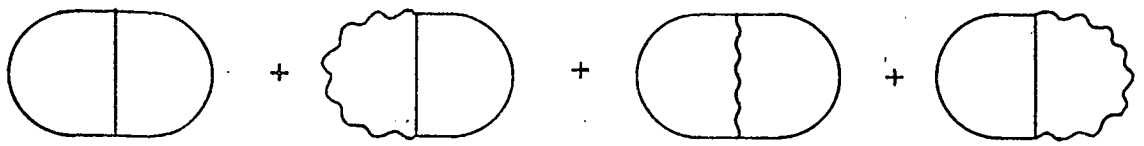
$(-1)\ 2$ $(-1)\ 2 \times 2$ $(-1)2$

$$\equiv -4 + 4 + 3 + 4 - 2 - 4 - 2 = -1.$$

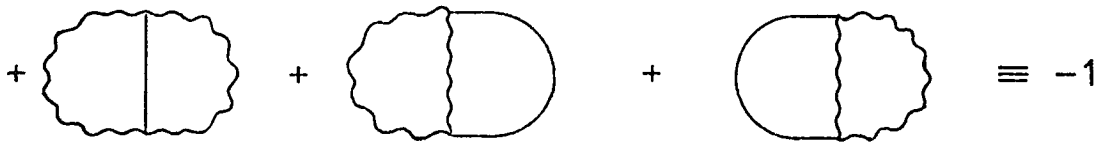
Similarly



; N = 1



$(-1)^3 2 \times 2$ $(-1)^2 3$ $(-1)^2 3$ $(-1)^2 3$

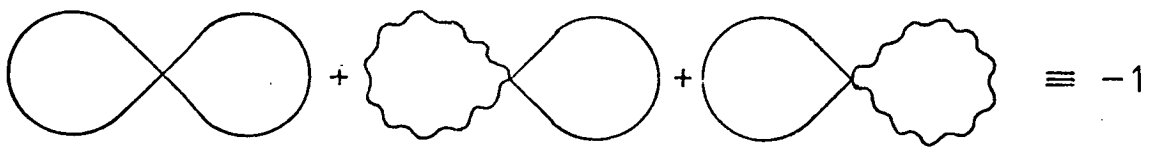


$\equiv -1$

$(-1)2$ $(-1)2$ $(-1)2$

And

; N = 1



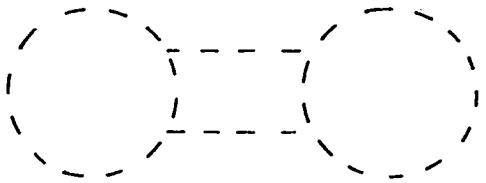
$\equiv -1$

$(-1)^2 3$ $(-1)2$ $(-1)2$

The above examples are the only ways in which a scalar configuration can be created with N = 1 and in each case the total duplication factor is (-1).

Any scalar configuration with a certain value for N can be considered to have been constructed by the addition of a line or a ρ-loop to a scalar configuration with "minimum number" (N-1). It is shown in Appendix D that the duplication factor, F', for a configuration created by the

addition of a line or loop in any manner to a configuration with duplication factor F is just (-1)F. Hence by induction any scalar configuration has a duplication factor $(-1)^N$. This predicts for



a duplication factor

$(-1)^2 = 1$ which can be verified by direct calculation.

d) The Scalar Lagrangian

In section c) we saw that the connected configurations of scalar lines constructed from the Feynman rules of Figs. 1(b) and 29 have an associated duplication factor of $(-1)^N$ which can be absorbed into the vertices of Fig. 29 by attaching a factor $(i)^{V-1}$ to each vertex where V is the number of scalar lines in the vertex. As we have assumed that the vertices are totally symmetric the scalar Lagrangian, which gives rise to these Feynman rules, is

$$\begin{aligned}\mathcal{L}_\phi \equiv\ & \frac{1}{2}\,\partial^\mu\phi_a\partial_\mu\phi^a - \frac{1}{2}\,\alpha^2M^2\phi^a\phi_a \\ & -\partial^\mu\phi_a\left\{\frac{1}{3!}\left(\frac{ig}{M}\,\underline{\underline{\phi}}\right)+\frac{1}{4!}\left(\frac{ig}{M}\,\underline{\underline{\phi}}\right)^2+\frac{1}{5!}\left(\frac{ig}{M}\,\underline{\underline{\phi}}\right)^3+\ldots\right\}^{ab}\partial_\mu\phi_b \\ & -M\partial^\mu\phi_a\left\{\frac{1}{2!}\left(\frac{ig}{M}\,\underline{\underline{\phi}}\right)+\frac{1}{3!}\left(\frac{ig}{M}\,\underline{\underline{\phi}}\right)^2+\frac{1}{4!}\left(\frac{ig}{M}\,\underline{\underline{\phi}}\right)^3+\ldots\right\}^{ab}W_{\mu b}\end{aligned} \tag{4.18}$$

where $\underline{\underline{\phi}}_{ab} \equiv -i\,f_{abc}\,\phi^c$. In the second term of (4.18) every interaction term odd in ϕ vanishes and (4.18) may be rewritten

$$\mathcal{L}_{\phi} \equiv \frac{1}{2}\partial^{\mu}\phi_a \left[E(\phi)\, E^{T}(\phi)\right]^{ab} \partial_{\mu}\phi_b - \frac{1}{2}\alpha^2 M^2\, \phi^a\phi a$$
$$- M\, \partial^{\mu}\phi^a\, E(\phi)_{ab}\, W^b_{\mu} + M\, \partial^{\mu}\phi_a\, W^a_{\mu} \qquad (4.19)$$

when $E_{ab}(\phi) \equiv \left[(\exp(\frac{ig}{M}\underline{\underline{\phi}}) - 1)/\frac{ig}{M}\underline{\underline{\phi}}\right]_{ab}$. On setting $\alpha = 0$ (4.19) is identical to the scalar Lagrangian, found to be necessary by Boulware[6], within the transform $\phi^a \to M\,\phi^a$, cf. (0.24).

These rules, however, give rise to closed scalar loops with the normal combinatorical factor compared with the pseudo-charged loops of section c) and their factors of $(-\frac{1}{2})$. When the arrow on pseudo-charged loops is dropped, clockwise and anti-clockwise loops which would otherwise be distinguishable become identical and double counting occurs. When the clockwise and anti-clockwise loops are indistinguishable the appropriate combinatoric factor for a scalar Lagrangian is $(\frac{1}{2})$ and again double counting occurs. Hence the loops generated by (4.19) can be reformulated as pseudo-charged loops but with an associated factor $(+\frac{1}{2})$. To get the correct amplitudes, in addition to (4.19), a Lagrangian

$$\mathcal{L}_{\psi} \equiv \partial^{\mu}\psi^*\partial_{\mu}\psi - \alpha^2 M^2\psi^*\psi - g(\partial^{\mu}\psi^*_a) f_{abc}\, W^b_{\mu}\, \psi_c \qquad (4.20)$$

with a factor (-1) associated with each loop is required.

We have thus demonstrated directly that the explicitly unitary Lagrangian for the massive Yang-Mills fields

$$-\frac{1}{4}\,\underline{G}_{\mu\nu}\cdot\underline{G}^{\mu\nu} + \frac{1}{2}\,M^2\underline{W}_{\mu}\cdot\underline{W}^{\mu} \qquad (4.21)$$

is equivalent to

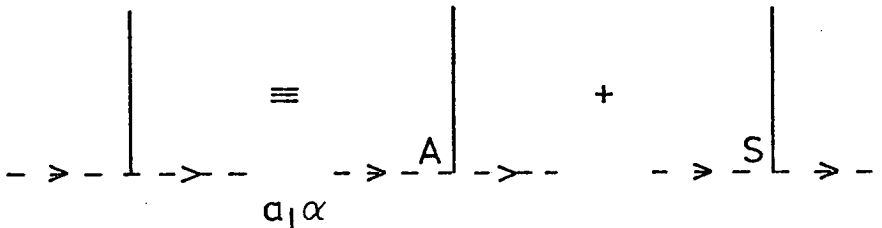
$$-\frac{1}{4}\,\underline{G}_{\mu\nu}\cdot\underline{G}^{\mu\nu} + \frac{1}{2}\,M^2\underline{W}_{\mu}\cdot\underline{W}^{\mu} - \frac{1}{2}\left(\frac{1}{\alpha}\,\partial^{\mu}\underline{W}_{\mu}\right)^2 + \mathcal{L}_{\phi} + \mathcal{L}_{\psi} \qquad (4.22)$$

where $\underline{G}_{\mu\nu}$, $\mathcal{L}_{\phi}$, and $\mathcal{L}_{\psi}$ are defined by (3.1), (4.19) and (4.20).

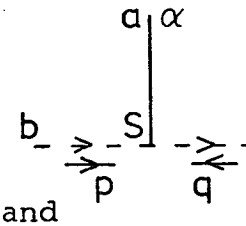
e) Symmetrization of the Scalar Vertices

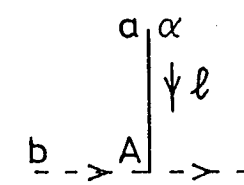
In the derivation of the Lagrangian (4.22) it was assumed that the scalar vertices of $\mathcal{L}_{\phi}$ were totally symmetric. Although it is not generally true the assumption can be justified for the "gauge" $\alpha = 0$.

First it is necessary to remove the directional dependence of the basic scalar vertex of Fig. 1(b) and so render the arrow on the scalar lines redundant. For $\alpha = 0$ it is self-evident as



(4.23)

where  $\equiv \frac{1}{2}\,ig\,f_{abc}(q-p)^{\alpha}$ the symmetrized vertex, and

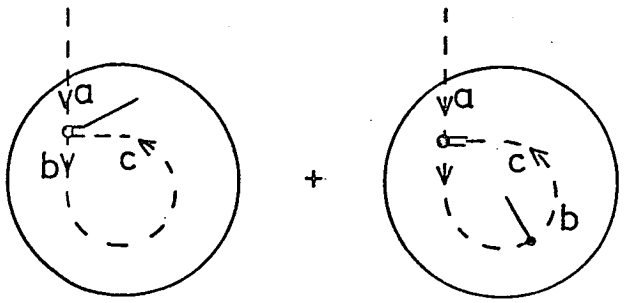


$\equiv \frac{1}{2}\,ig\,f_{abc}\,\ell^{\alpha}$ the anti symmetric vertex.

Thus in any diagram the basic scalar vertex can be symmetrized in the transverse "gauge".

Let us now consider the simple tree sections of a configuration and in particular any component scalar vertex of Fig. 29. Because the scalar contributions were generated from diagrams, which had been totally symmetrized in their sources in section a), the trees must be symmetric in the labelling of the branches. However, the detailed structure of a vertex is only dependent on the order of creation of its legs, i.e. on their left to right ordering in the construction, and the lines themselves have no directional bias when $\alpha = 0$. Hence, in this case, each possible ordering of the attachment of the scalar lines, labelled by their vector connections, occurs with equal weighting and the vertices themselves can be symmetrized over those legs.

For the trees with an associated loop the symmetrization is not quite so straightforward. There, the legs of the scalar vertices which terminate at external sources may be symmetrized as above. Also the two legs of the vertex which are connected to form the closed loop may be symmetrized as the connecting line is totally symmetric when $\alpha = 0$. In the transverse gauge the scalar propagators and vertices are identical to those for the gauge invariant massless Yang-Mills Lagrangian and any identities valid for the latter are also valid for the former. The simplest identity is the "group property" of Fig. 20(a). In the set of diagrams



(4.24)

the "group property" can be used to symmetrize the legs a, b of the scalar vertex. Hence the first pair of vertices of Fig. 29 can be symmetrized when they are generated by a closed loop.

For the second pair of vertices of Fig. 29 an equivalent identity to Fig. 20(a) is required. For the massless Yang-Mills Lagrangian it may be obtained by generalizing the technique of Appendix B of (50). The vertices in question are denoted by $\hat{d}^1_{abcd}(A)$ and $\hat{e}^1_{abcd}$ where

$$\hat{d}_{abc}(A) \rightarrow \hat{d}_{abc}(A) + g\,\hat{d}^1_{abcd}\Lambda^d + \hat{e}^1_{abcd}\Lambda^d$$

under the transform (3.2). With the usual notation and $\hat{v}_{iab}(A)$ transforming as

$$\hat{v}_{iab}(A) \rightarrow \hat{v}_{iab}(A) + g\,\hat{v}^1_{iabc}(A)\Lambda^c + \hat{u}^1_{iabc}\Lambda^c$$

transform $\mathcal{L}_{INV} - \frac{1}{2}C_a^2 + J_i A^i$ through B, B', B" and - B, -B', -B" .

$$\mathcal{L}_{INV} - \tfrac{1}{2}C^2 + J_i A_i \rightarrow \mathcal{L}_{INV} - \tfrac{1}{2}C^2 + J_i A_i$$

$$- J_i(g)(g\hat{v}_{iab} + \hat{u}_{iab})\{(B''^{a}B'^{,b} - B'^{,a}B''^{b}) + (B''^{a}B^{b} - B^{a}B''^{b}) + (B'^{,a}B^{b} - B^{a}B'^{,b})\}$$

$$+ C_a(g)(g\,\hat{d}_{abc} + \hat{e}_{abc})\{(B'^{,b}B^{c} - B^{b}B'^{,c}) + (B''^{b}B^{c} - B^{b}B''^{c}) + (B'^{,b}B^{c} - B^{b}B'^{,c})\}$$

$$- C_a(g^2)(g\,\hat{d}^{1}_{abcd} + \hat{e}^{1}_{abcd})\{B'^{,b}B^{c}B''^{d} - B^{b}B''^{c}B'^{,d} + B''^{b}B^{c}B'^{,d} - B'^{,b}B''^{c}B^{d}\}$$

$$+ J_i(g^2)(g\,\hat{v}^{1}_{iabc} + \hat{u}^{1}_{iabc})\{B'^{,a}B^{b}B''^{c} - B^{a}B''^{b}B'^{,c} + B''^{a}B^{b}B'^{,c} - B'^{,a}B''^{b}B^{c}\}$$

The second and third terms of (4.25) simply reflect the "group property" of Fig. 20(a) and may be dropped. Finally on transforming by

$$-g^2\,\hat{m}^{-1}_{ef}\,\hat{e}^{1}_{fbcd}\{B'^{,b}B^{c}B''^{d} - B^{b}B''^{c}B'^{,d} + B''^{b}B^{c}B'^{,d} - B'^{,b}B''^{c}B^{d}\}$$

we obtain

$$\mathcal{L}_{INV} - \tfrac{1}{2}C_a^{\,2} + J_iA_i \;\rightarrow\; \mathcal{L}_{INV} - \tfrac{1}{2}C_a^{\,2} + J_iA_i \qquad (4.26)$$

$$-g^2C_a(g\,\hat{d}^{1}(A)_{abcd} - g\,\hat{\ell}(A)_{ae}\hat{m}^{-1}_{ef}\,\hat{e}^{1}_{fbcd})\{B'^{,b}B^{c}B''^{d} - B^{b}B''^{c}B'^{,d} + B''^{b}B^{c}B'^{,d} - B'^{,b}B''^{c}B^{d}\}$$

$$+g^2J_i(g\hat{v}^{1}_{iabc}(A) + \hat{u}^{1}_{iabc} - g\hat{S}_{ie}(A)\,\hat{m}^{-1}_{ef}\,\hat{e}^{1}_{fbcd} - \hat{t}_{ie}\hat{m}^{-1}_{ef}\hat{e}^{1}_{fbcd})$$

$$\{B'^{,b}B^{c}B''^{d} - B^{b}B''^{c}B'^{,d} + B''^{b}B^{c}B'^{,d} - B'^{,b}B''^{c}B^{d}\}.$$

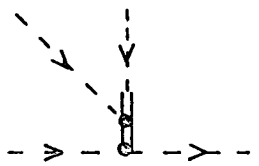
From (4.26) we get the identity

$$\hat{u}^{1}_{iabc} - \hat{t}_{ie}\,\hat{m}^{-1}_{ef}\,\hat{e}^{1}_{fabc} = 0 \qquad (4.27)$$

and the required vertex identity, Fig. 33. The legs a and c are joined to form the closed loop and the other two legs b and d are symmetrized to give the identity of Fig. 34(a). Using this identity the second pair of vertices in Fig. 29 may be symmetrized also when they occur in a closed loop.

The technique can be systematically extended to vertices of all orders. However, it is not really necessary as the required identities can be obtained from Fig. 20(a) by building up the vertices with the additions (4.9). For example instead of Fig. 33 the identities of Figs. 35(a) and (b) together with Fig. 20(a) could be used. Thus, all vertices involved with scalar loops can be symmetrized. In each case above, in symmetrizing the purely scalar vertices of Fig. 29, it was necessary to have an accompanying vertex in the loop. Hence we still need to examine the case of a loop with no other vertices.

The simplest such case with the vertex of Fig. 21(ii) can be ignored as $f_{abc}\,\delta_{bc} \equiv 0$. For the 4-point vertex



the selections of the two legs to form the loop may carry zero, one or two momentum vectors. The net effect of the zero momentum ones are equivalent to those carrying one momentum vector as on connecting b to c, instead of a to c in Fig. 33, we obtain Fig. 34(b). The outgoing line a of Fig. 34(b) must involve another vertex but this may be accommodated

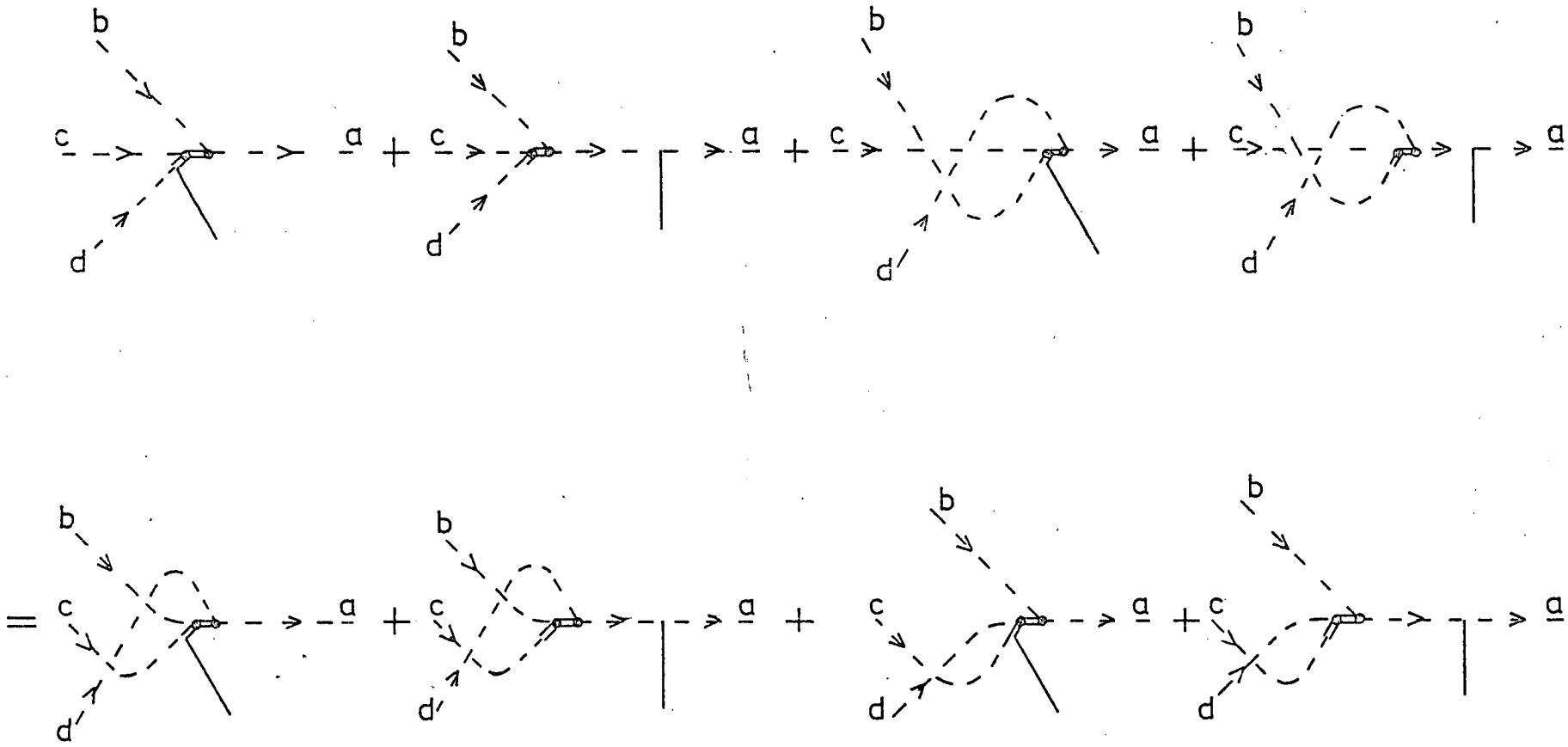
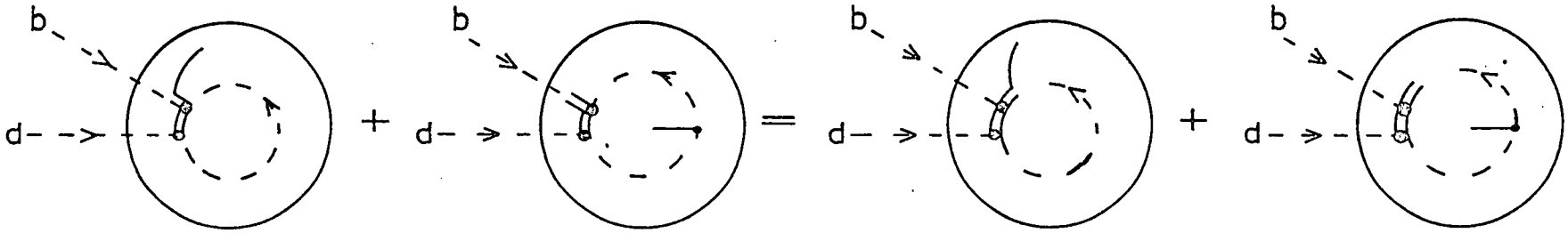
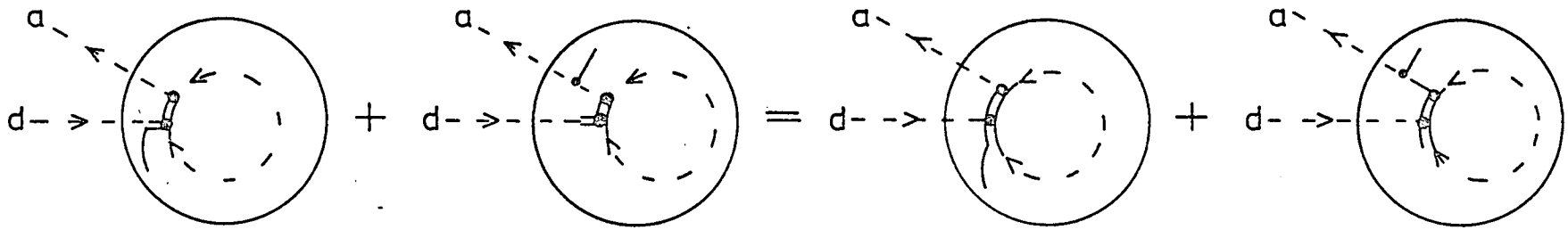


FIG 33



(a) Notice the similarity to the identity Fig. 15(a).



(b)

FIG 34

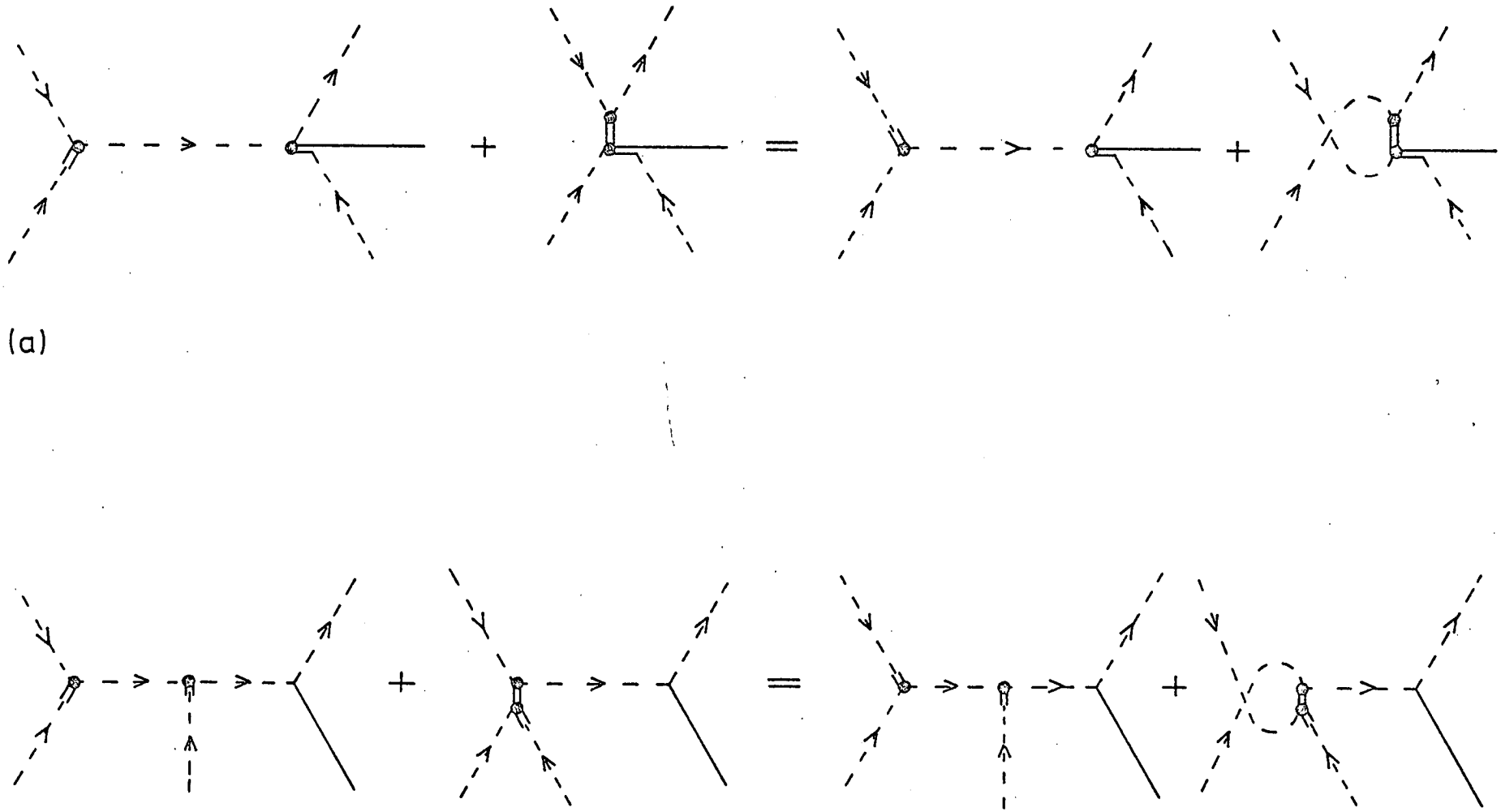


FIG 35 The basic 3-point vertex must be included in first diagram on both sides of (b) in order that the preceding scalar vertex can be symmetrized using Fig. 20 (a).

by going through an externally created propagator if necessary. For higher ordered vertices we can again attach bits of the form (4.9) until it is of the desired form. The only exception is if the lines a and d of Fig. 34(b) connect to each other immediately but in that situation the loop formed by the external propagator can be treated as the original when it no longer has zero momentum dependence. However, with a suitable regularisation[53] the loops, dependent on the momentum vector once, vanish on integrating. We are left only with those loops dependent on the momentum squared.

Therefore, on symmetrizing the vertices involved with scalar loops it is necessary to include an additional purely scalar term in the Lagrangian to cancel these momentum squared integrals. This is just the $\delta^4(0)$ term of Boulware[6], cf. (0.22), and we have exactly reproduced his scalar terms in the equivalent massive Yang-Mills Lagrangian in the transverse "gauge".

CHAPTER 5

EQUIVALENT FORMULATIONS OF THE MASSIVE YANG-MILLS LAGRANGIAN - II

In Chapter 4 Boulware's[6] equivalent formulation of the massive Yang-Mills Lagrangian in the transverse gauge was verified directly using Ward identities. One of the aims of this chapter is to show that our approach is exactly equivalent to that of Veltman et al.[5,12,13,14]. Hence, the two approaches to obtaining equivalent formulations, viz. Boulware and Veltman et al., are equivalent and the latter does not give rise to a form any less divergent[63].

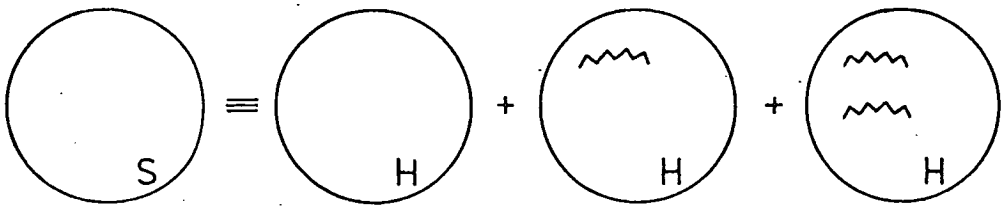
The second topic discussed is the role of the self-energy terms. These were ignored in Chapter 4 for clarity but the necessary amendments are discussed here.

a) The Veltman Ward identities

Instead of expanding the amplitudes, with hard propagators, in terms of amplitudes with soft propagators, as in section a) of Chapter 4, the very opposite could be done, i.e. expand the amplitudes with soft propagators in terms of the hard propagators. The factorisation corresponding to (4.1) is

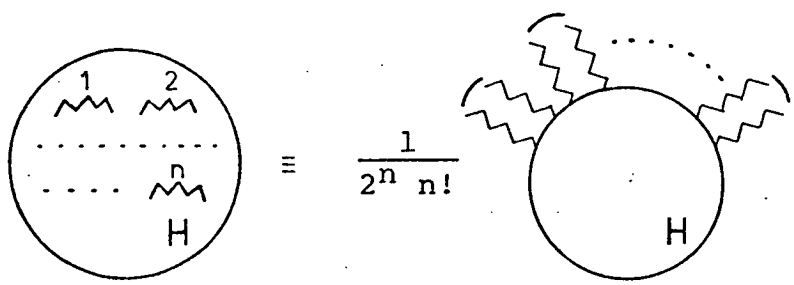
$$\frac{g^{\mu\nu} - \frac{(1-\alpha^2)k^\mu k^\nu}{k^2 - \alpha^2 M^2 + i\varepsilon}}{k^2 - M^2 + i\varepsilon}\,\delta^{ab} \equiv \frac{g^{\mu\nu} - \frac{k^\mu k^\nu}{M^2}}{k^2 - M^2 + i\varepsilon}\,\delta^{ab} + \frac{k^\mu k^\nu / M^2}{k^2 - \alpha^2 M^2 + i\varepsilon}\,\delta^{ab} \qquad (5.1)$$

If (5.1) is substituted for all propagators we obtain, cf. (4.2),



(5.2)

where ~~~~ is now $\frac{+k^\mu k^\nu/M^2}{k^2-\alpha^2M^2+i\epsilon}\,\delta^{ab}$. On treating the momentum contractions as sources



(5.3)

where ——— $\equiv \int d^4p \int d^4p' \frac{(-1)\,\delta^{ab}\,\delta(p+p')/M^2}{p^2-\alpha^2M^2+i\epsilon}$.

We now require the Ward identities for the right hand side of (5.3) to re-express the momentum contractions as a scalar particle and its interactions. These are easily obtained by considering the Ward identities for the soft amplitudes of section b), Chapter 4, for any α and taking the limit $\alpha \to \infty$. Obviously any diagram with scalar propagators vanishes to leave Fig. 36, i.e. each scalar leg of a vertex immediately terminates at a source and there are no scalar propagators. The scalar vertices are the same as Fig. 29. The identities are the same as those of Veltman[14].

On cancelling the $\frac{1}{M^2}$ of the external contractions,

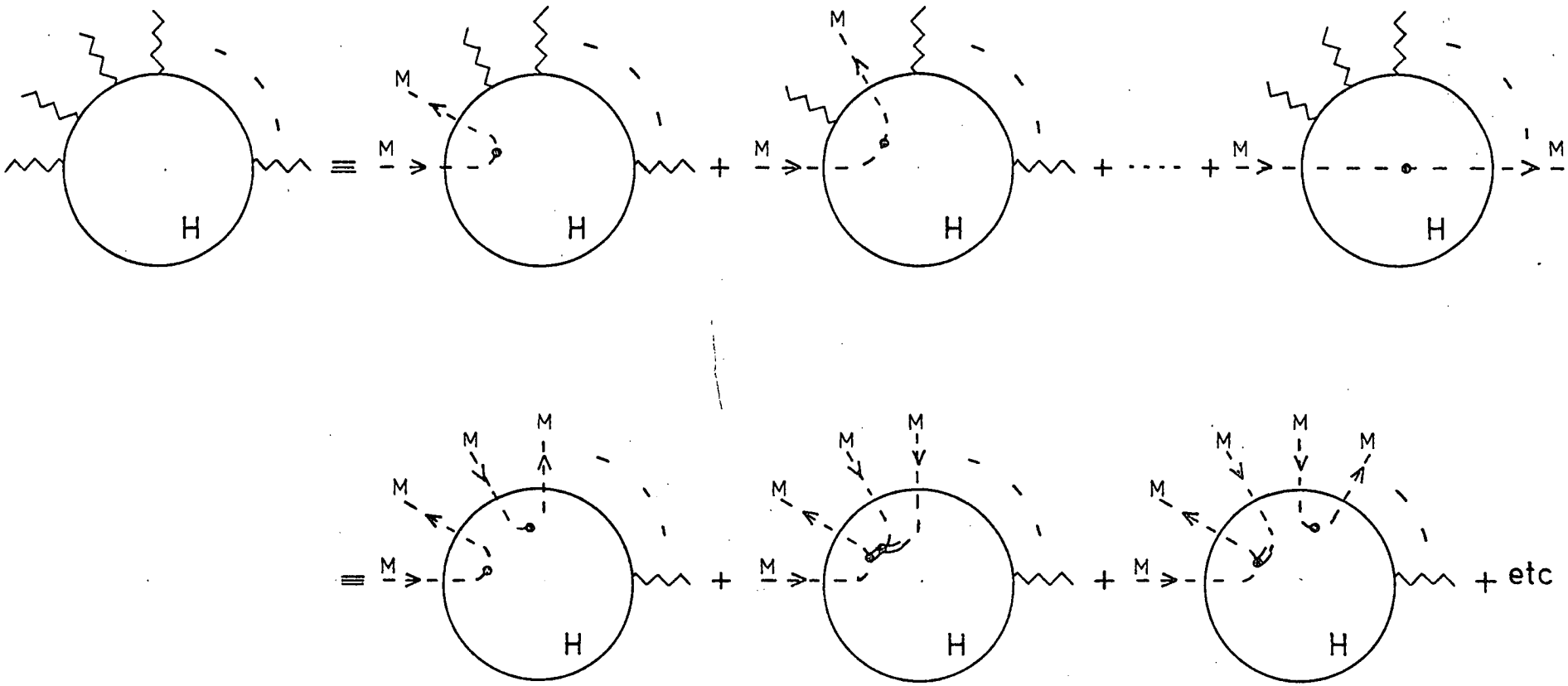


FIG 36 All scalar vertices are explicitly indicated.

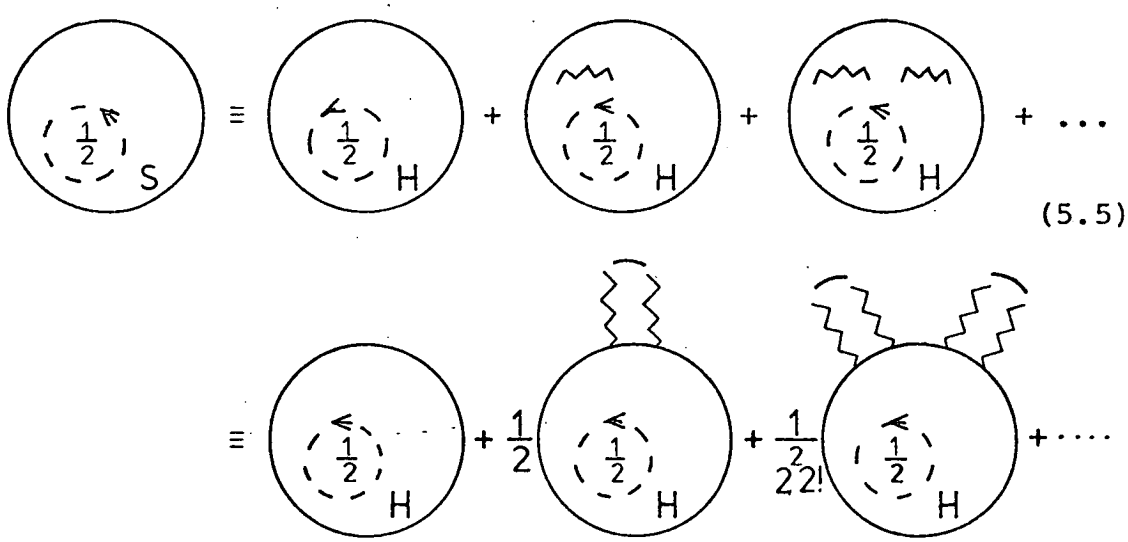
with the factors of M with each scalar line, they can be interpreted as propagators for a scalar field of mass α^2M^2. There is no additional factor of (-1) this time and all propagators in a scalar configuration are created externally. As all sections making up a configuration are simple trees, with only one vertex, the vertices including the basic one of Fig. 1(b) may be symmetrized in the scalar lines immediately, for all α, and the external propagators re-absorbed into the diagrams as in Chapter 4. The only duplication factors are due to the multiplicity of choice of branch for the outgoing scalar line of each tree. This gives a factor $(v-1)$ for every vertex, (where v is the number of scalar legs in the vertex), which may be absorbed into the vertices . The simple closed scalar loops, also, have the correct combinatoric factor for a normal Lagrangian. Hence, there is no need for an additional scalar Lagrangian equivalent to (4.20) and no need for a term to cancel the contributions to the diagrams, when both legs of the purely scalar vertices carrying the momenta vectors are directly connected by a propagator, unlike section e) of the previous chapter. Thus, the diagrams with the soft vector boson propagators are identical, for all α, to the diagrams constructed with the hard propagator plus all scalar contributions constructed with the Lagrangian

$$\begin{aligned}\mathcal{L}_\phi \equiv{}& \tfrac{1}{2}\,\partial^\mu\phi_a\partial_\mu\phi^a - \tfrac{1}{2}\,\alpha^2M^2\phi_a\phi^a \\ &- \partial^\mu\phi_a\{\tfrac{2}{3!}(\tfrac{g}{M}\,\underline{\underline{\phi}}) + \tfrac{3}{4!}(\tfrac{g}{M}\,\underline{\underline{\phi}})^2 + \tfrac{4}{5!}(\tfrac{g}{M}\,\underline{\underline{\phi}})^3 + \ldots\}^{ab}\partial_\mu\phi^b \\ &- iM\partial^\mu\phi_a\{\tfrac{1}{2!}(\tfrac{g}{M}\,\phi) + \tfrac{2}{3!}(\tfrac{g}{M}\,\phi)^2 + \tfrac{3}{4!}(\tfrac{g}{M}\,\phi)^3 + \ldots\}^{ab}\,W_{\mu b}\end{aligned} \tag{5.4}$$

i.e. we have found a closed form for all α in this situation.

b) Elimination of the Scalar Contributions to the Soft Diagrams

Instead of (5.2) we could expand the set of diagrams with a scalar loop constructed by Fig. 1(b) already present i.e.

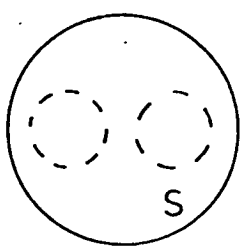


The external sources can be converted into scalar vertices, as in section a), by the usual Ward identity technique. Here, however, we have additional terminations of the scalar lines on the loop introduced to the soft diagrams. To symmetrize these vertices it is necessary to restrict α to zero and treat as in section e) of Chapter 5. If the pseudo-charged loops put into (5.5) by hand had an associated factor $(\frac{1}{2})$ as indicated, the arrow could be dropped on both sides to give the normal scalar loops as we are in the transverse "gauge". The external propagators are now re-absorbed as

usual to give the contributions of the Lagrangian (5.4) but with an additional duplication factor for each scalar configuration, viz. the number of ways in which a loop or set of scalar lines which form a closed loop can be chosen to have been the original scalar loop introduced to the left hand side of (5.5). Furthermore, the scalar vertices constructed on the loop have no outgoing line with its multiplicity of sources and if the rules of (5.4) are to be retained, a duplication factor of $\frac{1}{v-1}$ must be supplied for each vertex in the loop chosen to be the original one. This must be done for every selection of the loop.

In symmetrizing the purely scalar vertices there is no need to introduce additional terms to the Lagrangian to remove the momentum squared terms, as before, since there must be at least two vertices in the loop added to the soft diagrams. However, the purely scalar vertices are still anomalous in that when they occur in a scalar configuration in a loop with no other vertices, the loop cannot have been the original loop of the left hand side of (5.5) and the duplication factor is different from normal.

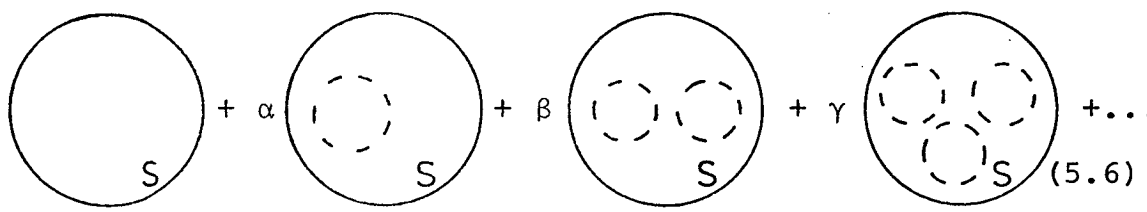
The equivalent construction can be done for the soft diagrams with two scalar loops, i.e. on dropping the unnecessary arrow for $\alpha = 0$



Again we get the normal set of diagrams with scalar configurations in the hard formulation but the duplication factor

becomes the number of ways of choosing the two original loops in that particular configuration. The construction can be done for any number of loops.

The soft rules suggested by Hsu and Sudarshan[17] as being equivalent to the massive Yang-Mills Lagrangian can now be tested directly to all orders. Indeed the investigation can be widened to considering the soft rules but with any multiplicative factor associated with the scalar loops and not necessarily the same for all loops. We consider the set of diagrams

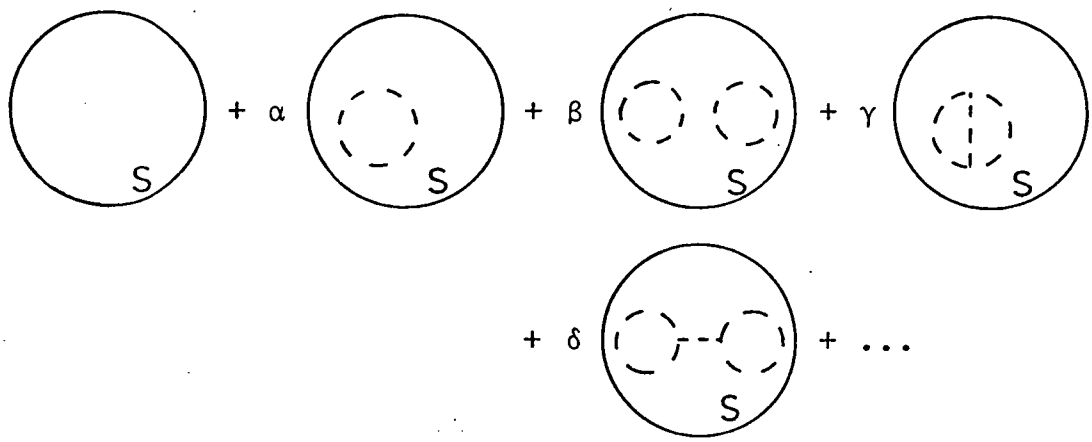


(5.6)

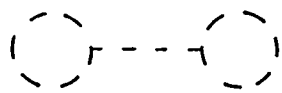
where α β γ etc. are arbitrary. We require the sum of the duplication factors for each scalar configuration to be zero so that the total set of diagrams is equivalent to the set of diagrams with hard vector-boson propagators and no scalar contributions, i.e. the explicitly unitary set. This is obviously impossible as the duplication factor for each configuration depends on the shape of that configuration and the number of different configurations far outnumbers the arbitrary constants introduced above. Hence, there is no way that the purely soft rules can be used to emulate the normal hard rules whether with the factor of $(-\frac{1}{2})$ of Hsu and Sudarshan for each loop or any arbitrary factor as above.

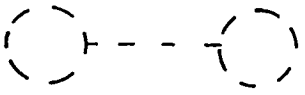
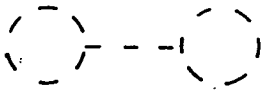
To regain the diagrams with hard Feynman rules it is

necessary to include, in the set of soft diagrams, all configurations of the scalar lines each with an associated factor, i.e. the set



(5.7)

As there is an arbitrary factor associated with every possible scalar configuration it is now possible to choose α β, γ ... etc. such that in the equivalent hard formalism the duplication factor for each scalar configuration is zero. This is the programme executed by Veltman[14] for the two loop approximationto the self-energy terms and by Mohapatra, Sakakibara and Sucher[12] for the two loop approximation to the four-point interaction. On doing the summation (5.7) to all orders one undoubtedly obtains the values for α, β, γ etc. which correspond to the duplication factors obtained by the direct construction of Chapter 4. For example, all the factors for configurations of isolated loops only, α, β, etc. are $(-1)^{\ell}$ where ℓ is the number of loops in the configuration (if the arrows are dropped); $\gamma = -1$ on reducing the vertices to Fig. 29 as the first set in (5.7) contributes 4  (where a factor 2 has been removed from each vertex) and the second set -3 as the original loop can be chosen in three ways and $\alpha = -1$. Similarly $\delta = -1$ as the first set in (5.7) contributes 4  the second

-4  and the third +1 

The anomalous duplication factors for the loops with only one purely scalar vertex leads to the necessity of introducing further terms to the Lagrangian associated with $\delta^4(0)$ as in Chapter 4.

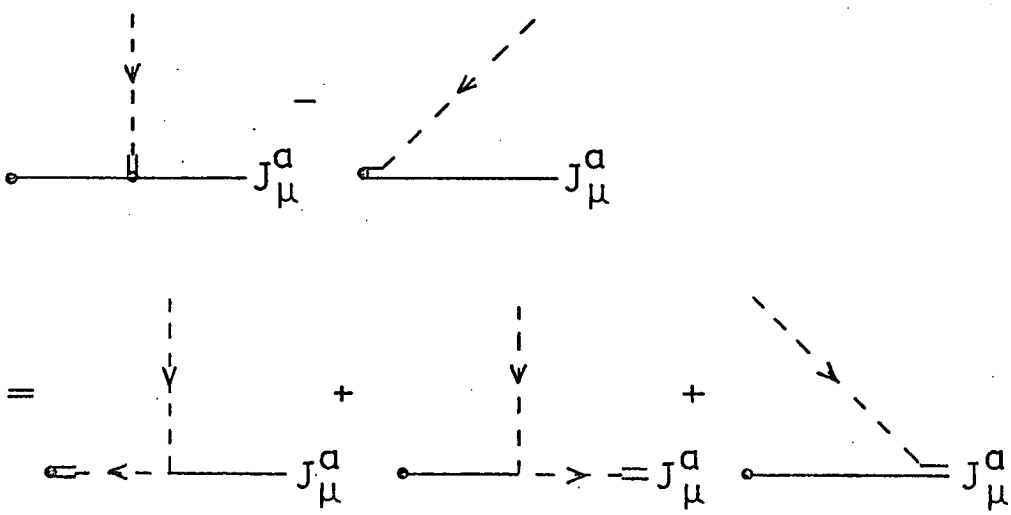
Therefore, we see the approach of Veltman et al.[12-14] is equivalent to that of Boulware[6] in that they both give rise to the same scalar ghost Lagrangian. We have also demonstrated that the soft rules of Hsu and Sudarshan[17] are insufficient for any order of loops other than the first. One fact which should be noted is the similarity of the scalar vertices in generating the equivalent soft formulation having started from the hard formulation or vice-versa. We have thus shown directly the equivalence of the various approaches to generating equivalent formulations of the massive Yang-Mills Lagrangian and the implication would seem to be that the theory is non-renormalizable.

c) The Self-Energy Terms

In deriving the Ward identities of Chapters 4 and 5 use was made of the identity (4.13). The derivation of it involves terms proportional to $(k^2 - M^2)$ which are taken to vanish as the physical particle is on-mass shell. This manifestly cannot always be true when (4.13) is applied to self-energy terms with an accompanying pole. Instead of

formulating the equivalence in terms of amplitudes we should have considered the S-matrix as pointed out by Biabynicki-Birula[64,65]. Both Veltman[14] and Mohapatra, Sakakibara and Sucher[12] fail to take these self-energy terms into account. Boulware[6] glosses over the problem for the non-abelian theory but considers them fully for the abelian Lagrangian. Unfortunately, his treatment leads to the conclusion that it is necessary to associate a factor $\exp\left[i\,\frac{1}{2}\,\left(\frac{g}{M}\right)^2 D(0)\right]$ with each "charged" field which we found unnecessary in Chapter 3. We replace any "physical" lines in the hard diagrams by an external source which is later made "physical" as in section b) of Chapter 3. The sources are $J^a_\mu\, W^\mu_a$ (no summation implied over a) for each vector field which are made physical only after all manipulations are completed.

If the previous constructions of Chapters 4 and 5 are now repeated with all the physical lines replaced by the above source we have, instead of (4.13)



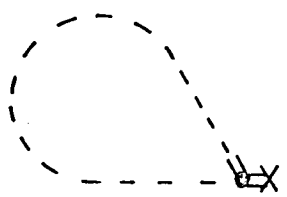
(5.8)

i.e. a scalar line may terminate at one of these sources

through (4.9) to form the vertices Fig. 21(iv) and (v). From these a hierarchy of vertices is generated as in Fig. 29, since scalar lines may terminate at previously created source-vertices to form the vertices of Fig. 37. The additional vertices as they only contain ingoing scalar lines are trivially symmetrized and straight-forward to absorb into the diagrams as usual.

In the context of Chapter 4 it remains for the duplication factor for the configurations, including the vertices of Fig. 37, to be obtained. In the configurations before absorption of the external propagators there are now trees with an external line source. These trees cannot have an associated closed loop and as all scalar lines are ingoing their multiplicity is one. The total duplication factor becomes $(-1)^N$, where N now includes the terms $\frac{V_{si}}{2}$ when V_{si} is the number of scalar legs of the i-th source vertex, as can be proved by induction following section c) of Chapter 4.

The source has to be treated like a vertex with an attached closed wavy loop with an associated factor of (+1) in Appendix D and it is found again that if a line or ρ-loop is added to a configuration, the only change to the duplication factor is to multiply by (-1). For configurations with one external source a basic shape is



(5.9)

which has a duplication factor (-1) as required since any number but at least one of the propagators in the loop may have been generated externally. The other basic configuration with one source can be deduced from (5.9). For example

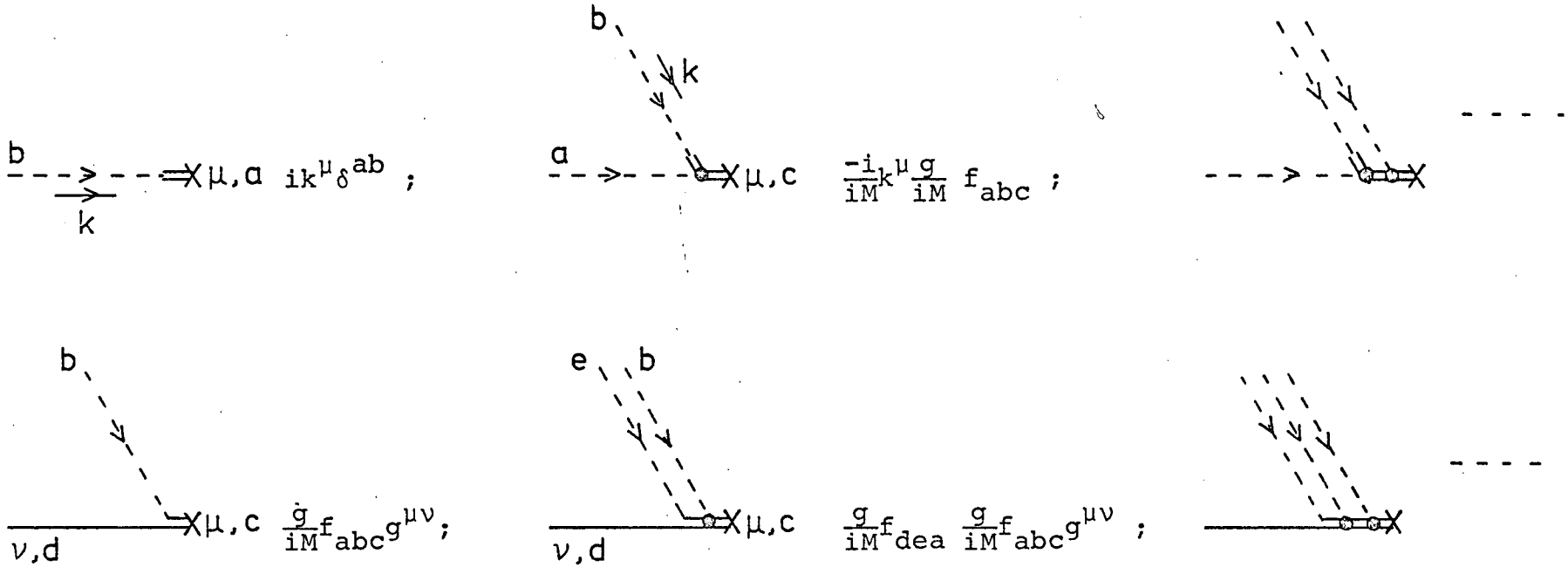
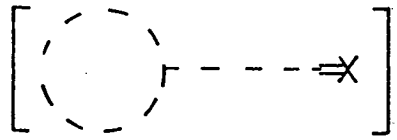
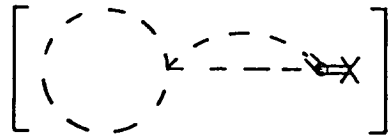
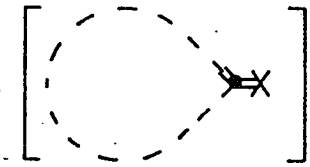
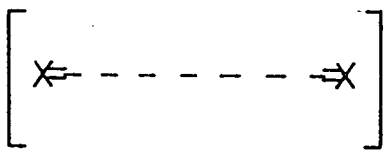
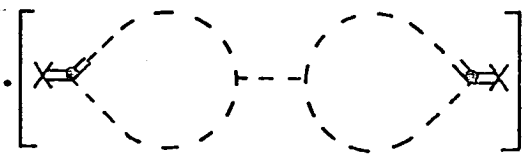


FIG 37 The factors $(\frac{1}{iM})$ from the momentum contraction sources are shown explicitly in connection with the sources, cf. Fig. 29.

D.F.  = (-1) D.F. 

= D.F.  = (-1)

As well as the duplication factor for the lowest order configurations with one source we require to show the lowest order configurations with any number of sources is consistent with $(-1)^N$. This can be deduced by starting from the configuration made up solely of the appropriate number of sections like (5.9) and connecting up the separate parts. For two sources

D.F.  = $(-1)^2$D.F. 

= (-1) D.F.  = (-1)

The duplication factors are absorbed into the scalar vertices as in section d) of Chapter 4 with a factor (i) for every scalar leg of the source-vertices of Fig. 37. Thus for the soft diagrams the Lagrangian (4.22) is modified by the addition of a source term

$$- \frac{1}{M} \partial_\mu \phi_a \{I + \frac{1}{2!}(\frac{-ig}{M} \underline{\underline{\phi}}) + \frac{1}{3!}(\frac{-ig}{M} \underline{\underline{\phi}})^2 + \ldots\}^{ab} J_b^\mu$$

$$+ W_{\mu a} \{I + (\frac{+ig}{M} \underline{\underline{\phi}}) + \frac{1}{2!}(\frac{+ig}{M} \underline{\underline{\phi}})^2 + \ldots.\}^{ab} J_b^\mu \qquad (5.10)$$

which can be re-expressed as

$$- \frac{1}{M} \partial_\mu \phi_a E^{ab}(-\phi) J_b^\mu + W_{\mu a} \{\exp(+\frac{ig}{M} \underline{\underline{\phi}})\}^{ab} J_b^\mu \qquad (5.11)$$

(5.11) is exactly the source terms found by Boulware[6] and Salam and Strathdee[7] cf. (0.24).

In the corresponding construction from soft to hard diagrams of section a) of this chapter the additional source terms for the Lagrangian (5.4) are

$$\frac{i}{M} \partial_\mu \phi_a \{I + \frac{1}{2!}(\frac{-g}{M} \underline{\underline{\phi}}) + \frac{1}{3!}(\frac{-g}{M} \underline{\underline{\phi}})^2 + \ldots\}^{ab} J_b^\mu$$

$$+ W_{a\mu} \{I + (\frac{+g}{M} \underline{\underline{\phi}}) + \frac{1}{2!}(\frac{+g}{M} \underline{\underline{\phi}})^2 + \ldots\}^{ab} J_b^\mu \quad . \qquad (5.12)$$

d) Invariance of the S-matrix

The consequence of the sources (5.11) is that the S-matrix and not the amplitudes are identical, for the soft and hard formulations, on making the external sources physical. When the sources of section c) are included, the identity proved is represented graphically by Fig. 38. The external lines are made physical in the usual manner. Of the contributions which may be detached from the rest of the

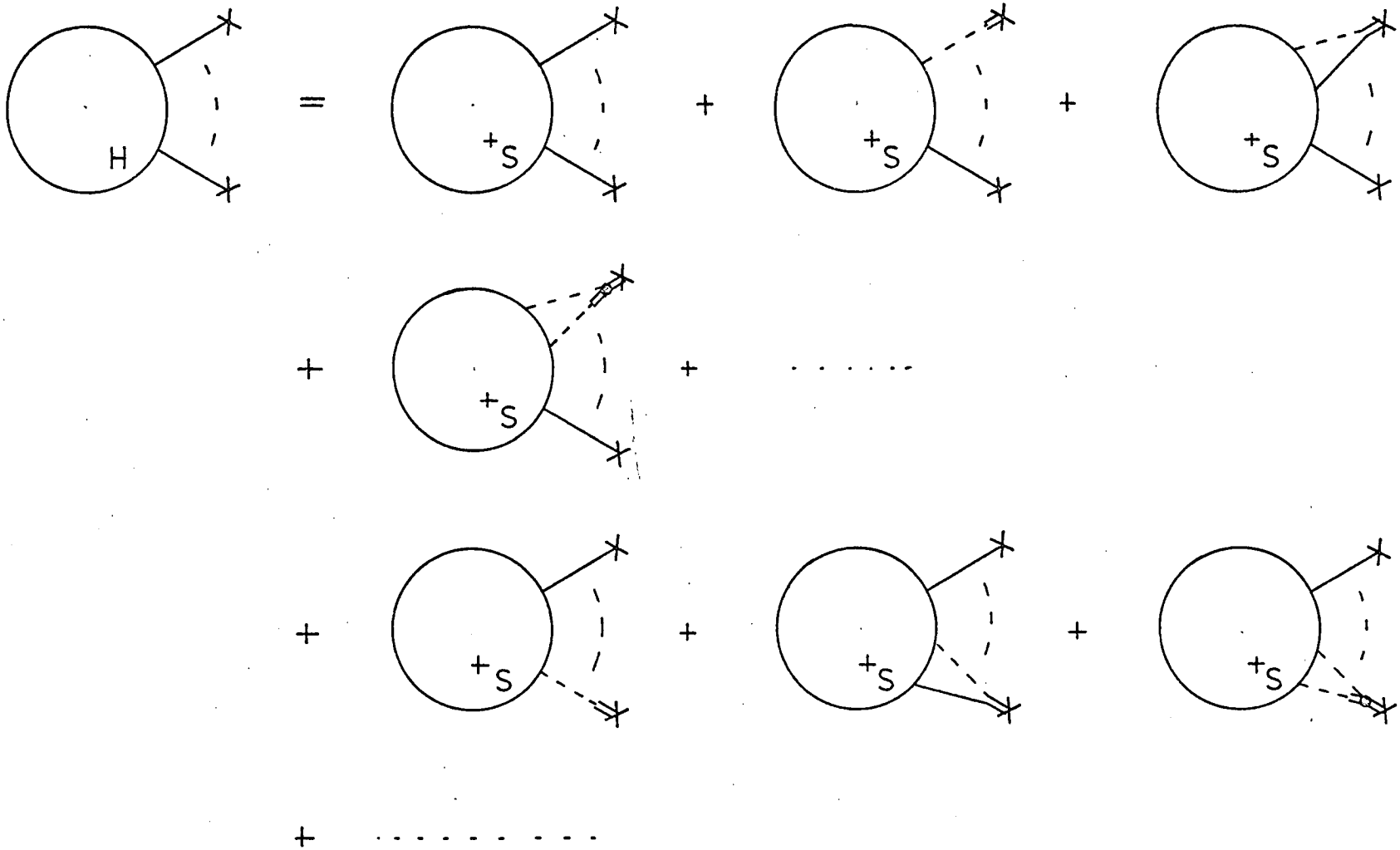


FIG 38 The + indicates the presence of terms constructed from the scalar Lagrangian (4.19). The source-vertices are symmetrized.

diagram by cutting a dressed propagator, Fig. 39, only those shown in Fig. 40 do not vanish. In particular, when there are only two external sources, the identity is as in Fig. 41 if all unnecessary terms are omitted. In order that the external lines are set on-mass shell for the physical rather than the bare particle all mass-like factors in the dressed propagators are quasi-renormalised by absorption into the bare vector boson propagator. With the definitions

$$Z_H\, e^{\nu}(\text{phys}) \equiv \lim_{k^2 \to M^2_{\text{phys}}} \Big[\nu \xrightarrow[k]{} \boxed{H} \xrightarrow[k]{} \mu \quad (k^2 - M^2_{\text{phys}})\, e^{\mu} \Big]$$

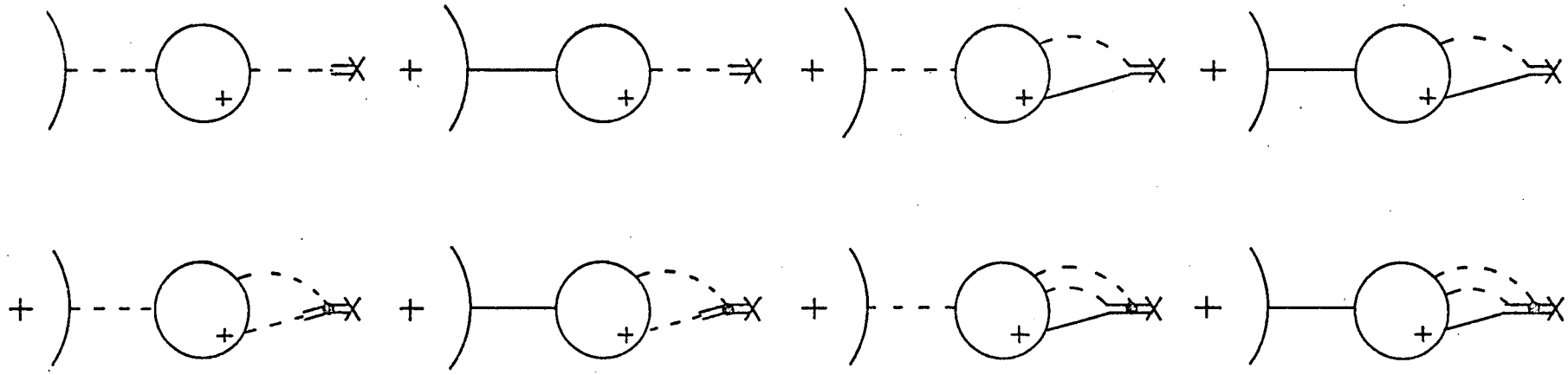
$$Z_S\, e^{\nu}(\text{phys}) \equiv \lim_{k^2 \to M^2_{\text{phys}}} \Big[\nu \xrightarrow[k]{} \boxed{+_S} \xrightarrow[k]{} \mu \quad (k^2 - M^2_{\text{phys}})\, e^{\mu} \Big] \qquad (5.13)$$

$$F\, e^{\nu}(\text{phys}) \equiv \lim_{k^2 \to M^2_{\text{phys}}} \Big[g^{\mu\nu} + \nu \xrightarrow[k]{} \boxed{+_S} \mu + \nu \xrightarrow[k]{} \boxed{+_S} \mu + \cdots \Big] e_{\mu}$$

in which the source function J^a_μ has been dropped and $e^{\nu}(\text{phys})$ obeys the condition (3.29), the identity Fig 41 is

$$Z_H = Z_S\, F^2 \qquad F = Z_H^{\frac{1}{2}} / Z_S^{\frac{1}{2}} \quad . \qquad (5.14)$$

On applying (5.14) to the terms of Fig. 40 in Fig. 38 we obtain



+ etc.

FIG 39

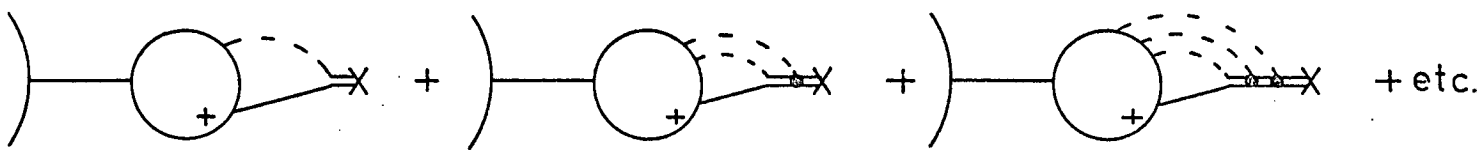


FIG 40 Only proper diagrams are included in the "blobs" shown.

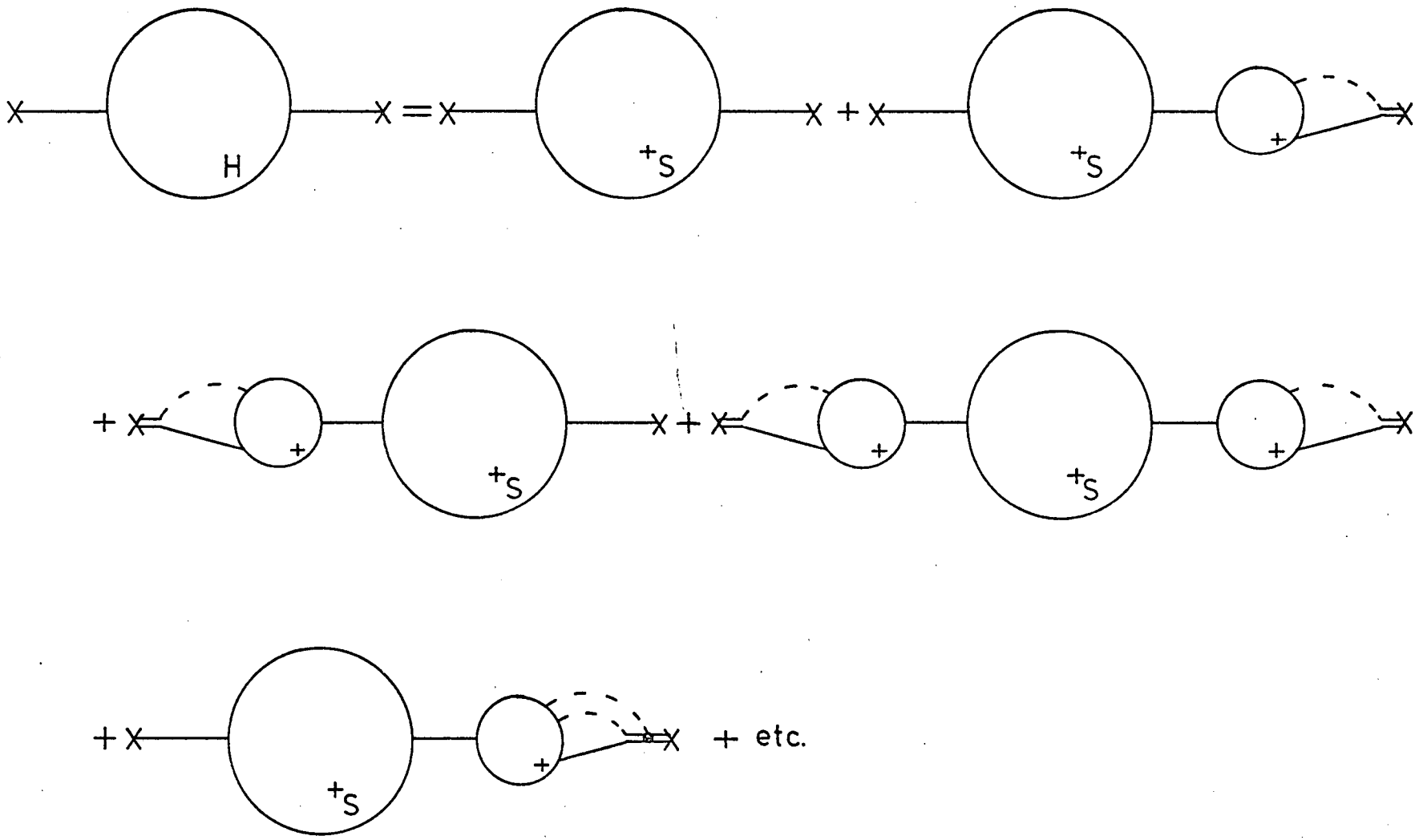
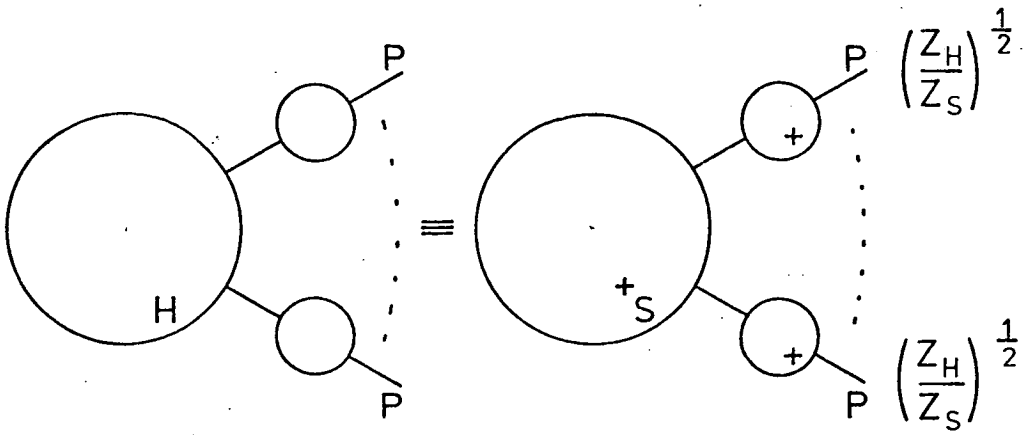
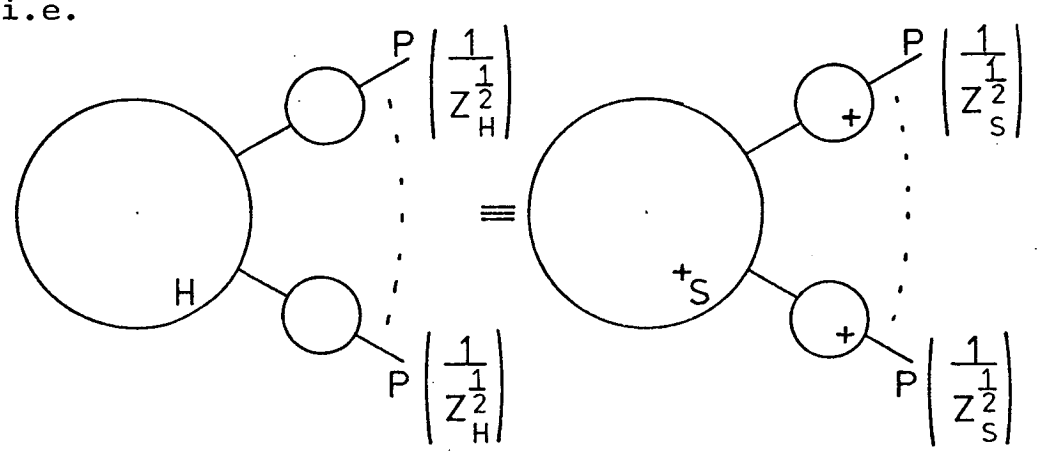


FIG 41



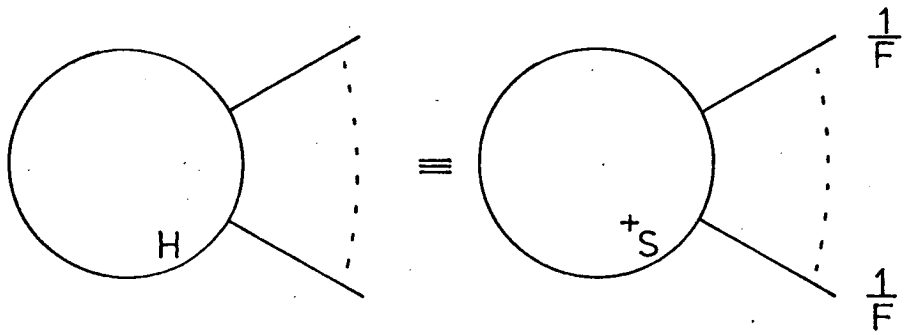
i.e.



(5.15)

Therefore, it is the S-matrices which are equal. The equivalent can be done in going from the soft to hard formalisms of section a) with the corresponding function to F, $F' = {}^{1}/F = Z_S^{\frac{1}{2}}/Z_H^{\frac{1}{2}}$.

However, in considering the renormalizability of the massive Yang-Mills Lagrangian we are more interested in the proper amplitudes



(5.16)

rather than the S-matrix. The presence of the factors ${}^{1}/F$ in (5.16) make the evaluation of the degree of divergence of the hard vertex functions much less clear. On treating

the right hand side of (5.16) as a perturbation approximation, by expanding $^1/F$ as a polynomial in g, it cannot be expected that all non-renormalisable divergences mutually cancel. However, if we consider the contributions of the vertices of Fig. 29 and Fig. 37 only, to the "blob" and the function F we see they are similar. To be more precise on using the dimensional regularisation[53] each produces a polynomial in the pole factor $\Gamma(2 - \frac{n}{2})$. The exact solution requires the diagrams to all orders to be taken into consideration, in which case, these polynomials become infinite and both the expressions are essential singularities. How the complete expression behaves cannot be determined.

The root cause of the presence of the vertices of Fig. 37 in all the formal derivations of the equivalent formalism, with soft vector-boson propagators, is that at some stage there is a rotation of the physical subspace of the Fock space of the fields[7]. The rotation is a finite gauge transform for the massless theory. In Boulware[6] the transform is between the vector-boson field and its transverse equivalent; in Salam and Strathdee[7] the transform is used to generalize the Stuckelberg split; in Veltman et al.[12-14] it is used to set up the Ward identities. Because the transform is of infinite order in g (5.16) must be considered to all orders in above.

CHAPTER 6

RENORMALISABILITY OF THE MASSIVE YANG-MILLS LAGRANGIAN

It would be advantageous to find equivalent formalisms for the massive Yang-Mills Lagrangian for which the factor F was more simply behaved than in Chapter 5. We could realise this by having a derivation which is not dependent on the finite gauge transform. The derivation of Hsu and Sudarshan[17] in which the scalar ghosts are introduced by a Lagrange multiplier is an example. One implication of (5.16) is that the investigation of Mohapatra, Sakakibara and Sucher[12] and Chapter 5, section b), as to the validity of the formalism of Hsu and Sudarshan, is not wholly valid. We discuss it in section a). In sections b) and c) we revert to the formalism of Boulware[6] and examine a parametrization of the transform which considerably simplifies F and the scalar Feynman rules.

a) The Lagrange Multiplier Scheme

Hsu and Sudarshan[17] considered the Lagrangian

$$\mathcal{L} \equiv -\frac{1}{4}\underline{G}_{\mu\nu}.\underline{G}^{\mu\nu} + \tfrac{1}{2}M^2\underline{W}_{\mu}.\underline{W}^{\mu} - \alpha M\,\underline{W}^{\mu}.\partial_{\mu}\underline{\chi} + \tfrac{1}{2}\beta\underline{\chi}^{\alpha} \qquad (6.1)$$

but with $\alpha^2M^2 = \beta$. They showed that the equations of motion for the Lagrange multiplier, χ_a, were the normal ones for a scalar particle with a renormalizable interaction. We shall examine (6.1) in the Hamiltonian formalism. The fields are W_a^k and their canonical variables

$$\pi^k_{W_a} \equiv -\partial^k W_{oa} + \partial_o W_a^k - g(\underline{W}_o \times \underline{W}^k)_a \qquad (6.2)$$

and χ_a and their canonical variables

$$\Pi_{\chi_a} \equiv -\alpha M W_a^o \tag{6.3}$$

The Hamiltonian is

$$\begin{aligned}\mathcal{H} \equiv\ & \tfrac{1}{2}\underline{\Pi}_W^k.\underline{\Pi}_W^k - \frac{1}{\alpha M}\,\underline{\Pi}_W^k.\partial^k\underline{\Pi}_\chi + \tfrac{1}{2}\partial^j\underline{W}^k.(\partial^j\underline{W}^k - \partial^k W^j) + \tfrac{1}{2}M^2\underline{W}^k.\underline{W}^k \\ & - \tfrac{1}{2}\frac{1}{\alpha^2}\,\underline{\Pi}_\chi^2 - \tfrac{1}{2}\beta\underline{\chi}.\underline{\chi} - \alpha M\underline{W}^k.\partial^k\underline{\chi} \qquad (6.4) \\ & - g(\partial^j\underline{W}^k).(\underline{W}^j\times\underline{W}^k) + \frac{g^2}{4}(\underline{W}^j\times\underline{W}^k).(\underline{W}^j\times\underline{W}^k) - \frac{g}{\alpha M}\,\underline{\Pi}_W^k.(\underline{\Pi}_\chi\times\underline{W}^k)\end{aligned}$$

with equations of motion

$$\dot{\underline{W}}^k = \underline{\Pi}_W^k - \frac{1}{\alpha M}\,\partial^k\underline{\Pi}_\chi - \frac{g}{\alpha M}\underline{\Pi}_\chi\times\underline{W}^k \tag{6.5}$$

$$\dot{\underline{\chi}} = -\frac{1}{\alpha^2}\,\underline{\Pi}_\chi + \frac{1}{\alpha M}\,\partial^k\underline{\Pi}_W^k + \frac{g}{\alpha M}\,\underline{\Pi}_W^k\times\underline{W}^k \tag{6.6}$$

$$\dot{\underline{\Pi}}_\chi = \beta\underline{\chi} - \alpha M\,\partial^k\underline{W}^k \tag{6.7}$$

$$\begin{aligned}\dot{\underline{\Pi}}_W^k =\ & -M^2\underline{W}^k + \alpha M\partial^k\underline{\chi} + \frac{g}{\alpha M}\,\underline{\Pi}_W^k\times\underline{\Pi}_\chi + g(\partial^j\underline{W}^k)\times\underline{W}^j - g(\partial^k\underline{W}^j)\times\underline{W}^j \qquad (6.8) \\ & - g^2(\underline{W}^j\times\underline{W}^k)\times\underline{W}^j - g\partial^j(\underline{W}^j\times\underline{W}^k) + (\partial^2\delta^{k\ell} - \partial^k\partial^\ell)\underline{W}^\ell .\end{aligned}$$

It is straightforward to show that (6.5) - (6.8) are identical to the equations of motion obtained by Hsu and Sudarshan with the choice $\alpha = \beta^{\frac{1}{2}}/M$, when the scalar equation is

$$(\partial^2 + M^2)\underline{\chi} = +g\underline{W}_\mu\times\partial^\mu\underline{\chi} \ . \tag{6.9}$$

In the context of the functional integral formalism the effect of the presence of the Lagrange multiplier in (6.1) can be

evaluated by integrating over the scalar field to get[17]

$$\mathcal{L}^{T} \equiv -\frac{1}{4}\,\underline{G}_{\mu\nu}\cdot\underline{G}^{\mu\nu} + \tfrac{1}{2}\,M^2\underline{W}_{\mu}\cdot\underline{W}^{\mu} - \tfrac{1}{2}\,\frac{\alpha^2M^2}{\beta^2}\,(\partial_{\mu}\underline{W}^{\mu})^2 \qquad (6.10)$$

Hsu and Sudarshan defined the physical state by

$$\chi^{(+)}(x)\;|\text{Phys.}\rangle = 0 \qquad (6.11)$$

and noted that the equation of motion (6.9) could have been obtained from

$$\mathcal{L}(\underline{\chi}) = -\tfrac{1}{2}\left[\partial_{\mu}\underline{\chi}\cdot\partial^{\mu}\underline{\chi} - M^2\underline{\chi}^2 + g\underline{\chi}\cdot(\underline{W}_{\mu}\times\partial^{\mu}\underline{\chi})\right] \qquad (6.12)$$

with the subsidiary condition

$$\partial_{\mu}\underline{W}^{\mu} + \beta^{\frac{1}{2}}\underline{\chi} = 0\ . \qquad (6.13)$$

They then removed, from the S-matrix generated by (6.10), any contributions of the scalar field by adding the determinant $D_M^{\frac{1}{2}}$ to the functional integral where

$$\begin{aligned} D_M^{-\frac{1}{2}} &\equiv \left[\det(\delta^{ac} - (\partial^2+M^2)^{-1}\,gf_{abc}\,W_{\mu}^{b}\partial^{\mu})\right]^{-\frac{1}{2}} \\ &\equiv \int d[\chi^a]\,\exp\{i\int d^4x\,\mathcal{L}(\underline{\chi})\} \end{aligned} \qquad (6.14)$$

i.e. $D_M^{-\frac{1}{2}}$ is the total contribution of the scalar fields to the original Lagrangian (6.1). The Feynman rules are, thus, those of Fig. 1 with $\alpha = 1$ and a factor $(-\tfrac{1}{2})$ associated with each scalar loop.

The above manipulation would appear not to be wholly valid as no account was taken of how the determinant would affect the equations of motion of the vector-boson. However,

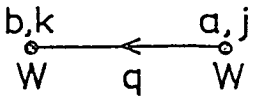
if a suitable source term were included in the Lagrangian (6.1) to generate "Ward identities" between diagrams generated by (6.10) and (6.4), the programme of section b), Chapter 5, could be executed to remove the contributions of χ^a to the diagrams for (6.4). The Feynman rules for (6.10) are as in Fig. 1(a), except for the scalar propagator and vertex. Quantizing (6.4) canonically as in Appendix E, without worrying about the indefinite metric or formulating in an explicitly Lorentz invariant manner[13,66], we obtain the Feynman rules of Fig. 42. The degree of divergence of the resultant rules would be no worse than those of (6.4) and so Hsu and Sudarshan's conclusion, that the massive Lagrangian is renormalizable, would appear to be qualitatively correct.

The free Hamiltonian is obtained by setting $g = 0$ in (6.4):

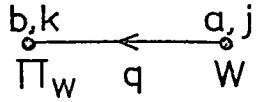
$$\mathcal{H}_{\text{FREE}} \equiv \tfrac{1}{2}\underline{\Pi}^k_W.\underline{\Pi}^k_W - \frac{1}{\alpha M}\,\underline{\Pi}^k_W.\partial^k\underline{\Pi}_\chi + \tfrac{1}{2}\partial^j\underline{W}^k.(\partial^j\underline{W}^k - \partial^k\underline{W}^j) + \tfrac{1}{2}M^2\underline{W}^k.\underline{W}^k$$
$$- \tfrac{1}{2}\frac{1}{\alpha^2}\underline{\Pi}_\chi.\underline{\Pi}_\chi - \tfrac{1}{2}\beta\underline{\chi}.\underline{\chi} - \alpha M\,\underline{W}^k.\partial^k\underline{\chi} \qquad (6.14)$$

which may be re-written as

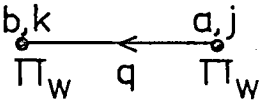
$$\mathcal{H}_{\text{FREE}} \equiv \tfrac{1}{2}\underline{\Pi}^k_W.\underline{\Pi}^k_W + \tfrac{1}{2}\frac{1}{M^2}(\partial^k\underline{\Pi}^k_W).(\partial^j\underline{\Pi}^j_W) + \tfrac{1}{2}\Big[\sum_{\ell jk}.\partial^k(\underline{W}^j - \frac{\alpha}{M}\partial^j\underline{\chi})\Big]$$
$$.\Big[\sum_{\ell mn}\partial^m(\underline{W}^n - \frac{\alpha}{M}\,\partial^n\underline{\chi})\Big] \qquad (6.15)$$
$$+ \tfrac{1}{2}\,M^2(\underline{W}^k - \frac{\alpha}{M}\,\partial^k\underline{\chi}).(\underline{W}^k - \frac{\alpha}{M}\,\partial^k\underline{\chi})$$
$$- \tfrac{1}{2}\frac{1}{\alpha^2}\,(\underline{\Pi}_\chi - \frac{\alpha}{M}\,\partial^k\underline{\Pi}^k_W).(\underline{\Pi}_\chi - \frac{\alpha}{M}\,\partial^j\underline{\Pi}^j_W)$$
$$- \tfrac{1}{2}\alpha^2\partial^k\underline{\chi}.\partial^k\underline{\chi} - \tfrac{1}{2}\beta\underline{\chi}.\underline{\chi} - \frac{1}{\alpha M}\,\partial^k(\underline{\Pi}^k_W.\underline{\Pi}_\chi)\ .$$



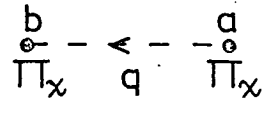
$$-\left\{\frac{\delta^{jk}}{q^2-M^2+i\varepsilon}+\frac{q^j q^k (M^2-\beta/\alpha^2)/M^2}{(q^2-M^2+i\varepsilon)(q-\beta/\alpha^2+i\varepsilon}\right\}\ \delta^{ab}$$



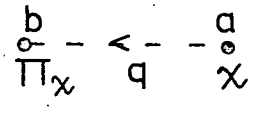
$$\frac{iq_o\delta^{k\ell}}{q^2-M^2+i\varepsilon}\ \delta^{ab}\ ;$$



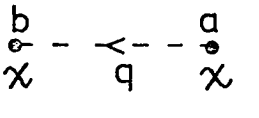
$$\frac{-q_o^2\ jk+q^j q^k}{q^2-M^2+i\varepsilon}\ \delta^{ab}$$



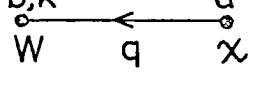
$$\left\{\frac{\alpha^2M^2}{q^2-M^2+i\varepsilon}+\frac{q_o^{\,2}(M^2\alpha^2-\beta)}{(q^2-M^2+i\varepsilon)(q^2-\beta/\alpha^2+i\varepsilon}\right\}\ \delta^{ab}$$



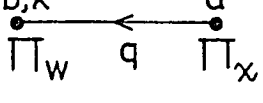
$$\frac{iq_o\delta^{ab}}{q^2-\beta/\alpha^2+i\varepsilon}\ ;$$



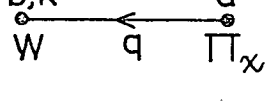
$$\frac{-1/\alpha^2\delta^{ab}}{q^2-\beta/\alpha^2+i\varepsilon}$$



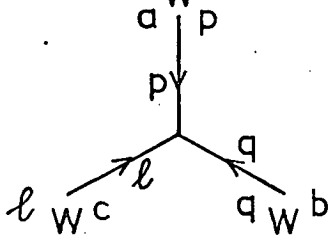
$$\frac{iq^k/\alpha M\delta^{ab}}{q^2-\beta/\alpha^2+i\varepsilon}\ ;$$



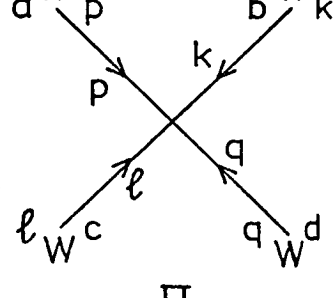
$$\frac{i\alpha Mq^k\delta^{ab}}{q^2-M^2+i\varepsilon}$$



$$\frac{iq^{k\alpha}/M(M^2-\beta/\alpha^2)(-iq_o)}{(q^2-M^2+i\varepsilon)(q^2-\beta/\alpha^2+i\varepsilon)}$$



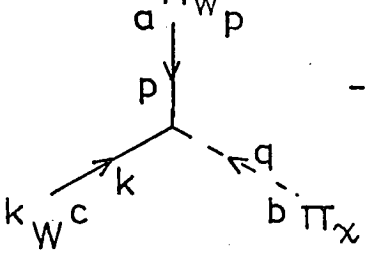
$$i\ g\ f_{abc}\left[\delta^{q\ell}(q-\ell)^p+\delta^{p\ell}(\ell-p)^q+\delta^{qp}(p-q)^\ell\right]$$



$$g^2 f_{eab}\ f_{ecd}\left[\delta^{pq}\delta^{k\ell}-\delta^{p\ell}\ \delta^{kq}\right]$$

$$+\ g^2 f_{eac}\ f_{ebd}\left[\delta^{pk}\delta^{a\ell}-\delta^{p\ell}\ qk\right]$$

$$+\ g^2 f_{ead}\ f_{ebc}\left[\delta^{pk}\delta^{\ell q}-\delta^{pa}\ \delta^{\ell k}\right]$$



$$-\ g/\alpha M\ f_{abc}\ \delta^{p\ell}$$

FIG 42

If the transformation

$$\underline{W}^k \rightarrow \underline{W}'^k = \underline{W}^k - \frac{\alpha}{M}\partial^k\underline{\chi}; \quad \underline{\Pi}_W^k \rightarrow \underline{\Pi}'^k_W = \underline{\Pi}_W^k$$

$$\underline{\chi} \rightarrow \underline{\chi}' = \alpha\underline{\chi}; \quad \underline{\Pi}_\chi \rightarrow \underline{\Pi}'_\chi = \frac{1}{\alpha}(\underline{\Pi}_\chi - \frac{\alpha}{M}\partial^k\underline{\Pi}^k_W) \tag{6.16}$$

can be made without spoiling the Hamiltonian formalism we would obtain

$$\mathcal{H}'_{FREE} \equiv \tfrac{1}{2}\underline{\Pi}'^k_W.\underline{\Pi}'^k_W + \tfrac{1}{2}\frac{1}{M^2}(\partial^k\underline{\Pi}'^k_W).(\partial^j\underline{\Pi}'^k_W) + \tfrac{1}{2}(\sum_{\ell kj}\partial^k\underline{W}'^j).(\sum_{\ell mn}\partial^m\underline{W}'^n)$$

$$+ \tfrac{1}{2}M\,\underline{W}'^k.\underline{W}'^k \tag{6.17}$$

$$-\tfrac{1}{2}\,\underline{\Pi}'_\chi.\underline{\Pi}'_\chi - \tfrac{1}{2}\partial^k\underline{\chi}'\cdot\partial^k\underline{\chi}' - \tfrac{1}{2}\beta/\alpha^2\,\underline{\chi}'\cdot\underline{\chi}'$$

on dropping the divergence term. What is required is that the transform (6.16) is canonical, i.e.

$$\int d^3\underline{x}\left[\Pi_\alpha(x)\,d\Phi_\alpha(x) - \pi_\alpha(x)\,d\phi_\alpha(x)\right] = dW(t) \tag{6.18}$$

where $\phi_\alpha(x)$ are the original fields with canonical variables $\pi_\alpha(x)$ and $\Phi_\alpha(x)$ the transformed fields with corresponding canonical variables $\Pi_\alpha(x)$. For (6.16) $dW \equiv 0$ and the transformation is canonical.

The first part of (6.17) is the free Hamiltonian for spin-one fields of mass M[67]; the second part is the free Hamiltonian for spin-zero fields of mass $\beta^{\frac{1}{2}}/\alpha$ [67]. Hence we see that just subtracting the contributions of the χ^a fields as in Hsu and Sudarshan[17] does not leave a pure

spin one Lagrangian, i.e. the unitary formalism for the massive Yang-Mills Lagrangian, which can be confirmed by considering the W-W propagator in Fig. 42. To regain the pure spin-one Lagrangian the transformation (6.16) must be made first, but that introduces scalar vertices with multiple derivatives and the rules are non-renormalizable again.

We may compare the massive formalism, above, to the massless with the Lagrangian

$$\mathcal{L} \equiv -\frac{1}{4}\underline{G}_{\mu\nu}\cdot\underline{G}^{\mu\nu} + \alpha\partial_\mu\underline{W}^\mu\cdot\underline{\chi} + \frac{1}{2}\beta\,\underline{\chi}\cdot\underline{\chi} \tag{6.19}$$

and canonical variables to the fields W_a^k and W_a^o

$$\begin{aligned} \Pi_{Wa}^k &= -\partial^k W_{oa} + \partial_o W_a^k - g(\underline{W}_o\times\underline{W}^k)_a \\ \Pi_{oa} &= \alpha\chi_a \ . \end{aligned} \tag{6.20}$$

The Hamiltonian is

$$\begin{aligned} \mathcal{H} \equiv{}& \tfrac{1}{2}\,\underline{\Pi}_W^k\cdot\underline{\Pi}_W^k + \underline{\Pi}_W^k\cdot\partial^k\underline{W}_o + \tfrac{1}{2}\partial^j\underline{W}^k\cdot(\partial^j\underline{W}^k - \partial^k\underline{W}^j) \\ &- \tfrac{1}{2}\,\beta/\alpha^2\underline{\Pi}_o\cdot\underline{\Pi}_o + \partial^k\underline{W}^k\cdot\underline{\Pi}_o \\ &- g(\partial^j\underline{W}^k)\cdot(\underline{W}^j\times\underline{W}^k) + g^2/4\,(\underline{W}^j\times\underline{W}^k)\cdot(\underline{W}^j\times\underline{W}^k) + g\,\underline{\Pi}_W^k\cdot(\underline{W}^o\times\underline{W}^k) \end{aligned} \tag{6.21}$$

If we apply the transform

$$\begin{aligned} &\underline{W}^k \to \underline{W}'^k = \underline{W}^k \ ; \quad \underline{\Pi}_W^k \to \underline{\Pi}_W'^k = \underline{\Pi}_W^k + \partial^k\underline{W}^o \\ &\underline{W}^o \to \underline{W}'^o = \underline{W}^o \ ; \quad \underline{\Pi}^o \to \underline{\Pi}'^o = \underline{\Pi}^o - \partial^k\underline{W}^k\,\beta/\alpha^2 \end{aligned} \tag{6.22}$$

to (6.21) the free part may be rewritten as

$$\mathcal{H}'_{\text{FREE}} \equiv \tfrac{1}{2}\,\underline{\Pi}_W'^{Lk}.\underline{\Pi}_W'^{Lk} - \tfrac{1}{2}\underline{W}'^{Lk}.\underline{\partial}^2\,\underline{W}'^{Lk} + \tfrac{1}{2}\left(\frac{\alpha^2}{\beta} - 1\right)\partial^k\underline{W}^{Lk}.\partial^j\underline{W}'^{Lj}$$
$$+ \tfrac{1}{2}\underline{\Pi}_W'^{Tk}.\underline{\Pi}_W'^{Tk} - \tfrac{1}{2}\underline{W}'^{Tk}.\,\underline{\partial}^2\underline{W}'^{Tk} \qquad (6.23)$$
$$+ \tfrac{1}{2}\partial^k\underline{W}'^{o}.\partial^k\underline{W}'^{o} - \tfrac{1}{2}\,\underline{\Pi}'^{o}.\underline{\Pi}'^{o}\,\beta/\alpha^2$$

where the fields $\underline{W}'^{k}$ have been split into transverse (T) and longitudinal (L) parts. The transform (6.22) is canonical as

$$dW(t) \equiv d\int d^3\underline{x}(\partial^k\underline{W}^{o}.\underline{W}^{k}) \quad . \qquad (6.24)$$

If we choose $\beta/\alpha^2 = 1$ the Hamiltonian (6.23) can be interpreted to consist of three different sets of fields. The first are spin-zero massless fields[67]; the second are massless spin-one fields in the radiation gauge[67]; the third are massless spin-one fields with a negative metric. The influence of the spin-zero fields from the amplitudes may be removed as before. In this case we are left with a renormalizable theory as the transform (6.22) does not generate any non-renormalizable vertices. In this construction of the unitary rules for the massless Yang-Mills Lagrangian gauge invariance has not been used, only the dynamics of the fields have been exploited and we obtain the Feynman rules for one gauge only.

b) A Parametrization of the SU(2) Lagrangian

The only option left in investigating whether the massive Yang-Mills Lagrangian is renormalizable is to choose a more convenient parametrization of the gauge transform than Boulware[6]. We follow Boulware as far as establishing that the explicitly unitary formalism can be rewritten in terms of the transverse fields.

For the Lagrangian with only the Yang-Mills fields and a source-function, the generating functional is

$$G[J] \equiv Z^{-1}\int d[W]\ \exp\{i\int d^4x\left[-\tfrac{1}{4}\underline{G}_{\mu\nu}\cdot\underline{G}^{\mu\nu} + \tfrac{1}{2}M^2\underline{W}_\mu\cdot\underline{W}^\mu + \underline{W}_\mu\cdot\underline{J}^\mu\right]\} \tag{6.25}$$

For convenience define the field matrices

$$\begin{aligned} \mathcal{W}^\mu(x) &\equiv T^a W_a^\mu \\ \mathcal{G}^{\mu\nu}(x) &\equiv \partial^\mu\mathcal{W}^\nu(x) - \partial^\nu\mathcal{W}^\mu(x) + ig\left[\mathcal{W}^\mu(x),\mathcal{W}^\nu(x)\right] \\ &= T^a G_a^{\mu\nu} \\ \mathcal{J}^\mu(x) &\equiv T^a J_a^\mu \end{aligned} \tag{6.26}$$

where T^a are the generators of the n-dimensional representation of SU(2) chosen such that

$$\mathrm{tr}(T^a T^b) = \lambda\,\delta^{ab}\,. \tag{6.27}$$

The Lagrangian may then be rewritten

$$\mathcal{L} \equiv -\lambda^{-1}\tfrac{1}{4}\mathrm{tr}\,\mathcal{G}^{\mu\nu}\mathcal{G}_{\mu\nu} + \lambda^{-1}\tfrac{1}{2}\mathrm{tr}\mathcal{W}^\mu\mathcal{W}_\mu + \lambda^{-1}\,\mathrm{tr}\mathcal{W}^\mu\mathcal{J}_\mu. \tag{6.28}$$

Under the finite gauge transformation

$$\mathcal{W}^{\mu}(x) \to \mathcal{W}'^{\mu}(x) \quad = \quad \Omega(x)\mathcal{W}^{\mu}(x)\Omega^{-1}(x) + \Omega\partial^{\mu}\Omega^{-1}/ig \tag{6.29}$$

the Lagrangian (6.28) transforms:

$$\mathcal{L} \to \mathcal{L}' \equiv -\lambda^{-1}\tfrac{1}{4}\mathrm{tr}\ \mathcal{G}^{\mu\nu}\mathcal{G}_{\mu\nu} + \lambda^{-1}\tfrac{1}{2}M^2\mathrm{tr}[\mathcal{W}^{\mu} + \ell^{\mu}][\mathcal{W}_{\mu} + \ell_{\mu}]$$
$$+ \lambda^{-1}\mathrm{tr}[\Omega\mathcal{W}^{\mu}\Omega^{-1} + \bar{\ell}^{\mu}]\mathcal{J}_{\mu} \tag{6.30}$$

where
$$\bar{\ell}^{\mu}(x) = \Omega\,\partial^{\mu}\Omega^{-1}/ig$$
$$\ell^{\mu}(x) = -\Omega^{-1}\partial^{\mu}\Omega/ig \tag{6.31}$$

and $\Omega(x)$ is a local element of the n-dimensional representation of SU(2).

Any vector field $\mathcal{W}^{\mu}(x)$ can be written as the gauge transform of a transverse field[6], $\mathcal{W}^{\mu}_{T}(x)$. However, in the generating functional (6.25) any transformation of the fields must be accompanied by a Jacobian factor. Hence, (6.25) is equivalent to

$$G[J] = Z^{-1}\int d[W^{T}]\,d[\chi]\,(\det M)\exp\{i\int d^4x\Big[-\lambda^{-1}\tfrac{1}{4}\mathrm{tr}\,\mathcal{G}^{T\mu\nu}\,\mathcal{G}^{T}_{\mu\nu}$$
$$+\lambda^{-1}\tfrac{1}{2}M^2\mathrm{tr}\,\mathcal{W}^{T\mu}\mathcal{W}^{T}_{\mu} + \lambda^{-1}M^2\mathrm{tr}\ell^{\mu}\mathcal{W}^{T}_{\mu} + \tfrac{1}{2}\lambda^{-1}M^2\mathrm{tr}\ell^{\mu}\ell_{\mu}$$
$$+ \lambda^{-1}\mathrm{tr}\Big[\Omega\mathcal{W}^{T\mu}\Omega^{-1} + \bar{\ell}^{\mu}\Big]\mathcal{J}_{\mu}\Big] \tag{6.32}$$

where $\det M$ is the appropriate Jacobian factor

$$\det M = C\ \det\Big[g^{\mu\nu}\delta_{ab}\delta(x-y) + ig\mathcal{W}^{T\mu}_{A\ ab}(x)\partial^{\nu}D(x-y)\Big] \tag{6.33}$$

with $D(x-x') = \int \frac{d^4p}{(2\pi)^4}\,\frac{-e^{-ip.(x-x')}}{p^2}$ and the parametrization in (6.32) is $\delta\chi = \delta\Omega\ \Omega^{-1}/-ig$. $\mathcal{W}^{T^{\mu}}_{A_{ab}}$ are the field matrices in the adjoint representation of the group i.e. $T^{a}_{bc} = -t_{abc}$

where t_{abc} are the structure constants of the group. The net effect of (6.33) is just to introduce scalar loops to the amplitudes as in section a).

An explicit parametrization of $\Omega(x)$ must now be chosen. Boulware[16] made the choice

$$\Omega(x) = \exp(-igT^a\phi_a(x)) \tag{6.34}$$

for the adjoint representation. Then

$$\partial^\mu\Omega(x) = -ig\partial^\mu\phi_a\, E_{ba}\, T_b\Omega \tag{6.35}$$

where $E_{ba}(x) \equiv \left[(\Omega^A(x) - 1)/-ig\phi_c(x)\, T^{Ac}\right]_{ba}$

and the necessary Jacobian factor

$$\delta\chi_a(x)/\delta\phi_b(x') = \delta(x-x')\, E_{ab}(x) \quad . \tag{6.36}$$

We restrict ourselves, now, to the fundamental representation of SU(2) with the parametrisation[7]

$$\Omega(x) = (1 + \sigma(x))I + i\, \underline{\tau}.\underline{\Pi} \quad . \tag{6.37}$$

$\Omega(x)$ is unitary with determinant (+1) if

$$2\sigma(x) + \sigma^2(x) + \underline{\Pi}.\underline{\Pi} = 0 \tag{6.38}$$

The corresponding change in the volume element including the Jacobian is

$$\int d[\chi_a] \rightarrow \int d[\pi_a]\, d[\sigma]\, \delta(2\sigma + \sigma^2 + \underline{\Pi}.\underline{\Pi}) \tag{6.39}$$

as expected[7]. Further, it should be noted that

$$\ell^\mu\ell_\mu \equiv \frac{1}{g^2}\, \partial^\mu\, \Omega^{-1}\, \partial_\mu\, \Omega \quad . \tag{6.40}$$

Hence the generating functional (6.32) can be rewritten

$$G[J] = Z^{-1}\int d[W^T]\,d[\pi]\,d[\sigma]\,(\det M)\,\delta(4Mg\sigma + g^2\sigma^2 + g^2\underline{\Pi}.\underline{\Pi})$$

$$\exp\{i\int d^4x\,[-\tfrac{1}{2}\underline{G}^T_{\mu\nu}.\underline{G}^{T\mu\nu} + \tfrac{1}{2}M^2\underline{W}_\mu.\underline{W}^\mu + \tfrac{1}{2}\partial^\mu\sigma\partial_\mu\sigma + \tfrac{1}{2}\partial^\mu\underline{\Pi}.\partial_\mu\underline{\Pi}$$

$$+ g\,\partial_\mu\sigma\,\underline{\Pi}.\underline{W}^{T\mu} - g\underline{\Pi}.\partial_\mu\underline{\Pi}\times\underline{W}^{T^\mu} + \underline{W}^\mu.\underline{J}_\mu + \frac{g}{M}(1 + \frac{g}{4M}\sigma)\sigma\underline{W}^{T^\mu}.\underline{J}_\mu \quad (6.41)$$

$$- \frac{2g}{M}(1 + \frac{g}{M}\,\sigma)\underline{\Pi}.\underline{W}^{T\mu}\times\underline{J}_\mu - \frac{g^2}{M^2}(\underline{W}^{T^\mu}\times\underline{\Pi}).(\underline{\Pi}\times\underline{J}_\mu)$$

$$+ \frac{g}{2M^2}\,\partial^\mu\sigma\,\underline{\Pi}.\underline{J}_\mu - \frac{1}{M}(1 + \frac{g}{2M}\sigma)\partial^\mu\underline{\Pi}.\underline{J}_\mu + \frac{g}{M^2}\underline{\Pi}.\partial^\mu\underline{\Pi}\times\underline{J}_\mu\,]\}$$

where the $\sigma(x)$ and $\pi^a(x)$ fields have been rescaled by $\frac{g}{M^2}$. The source terms in (6.41) are of no real account as they only ensure the S-matrix is identical to the unitary formalism. Hence the Feynman rules on treating the formalism as an equivalent Lagrangian are as in Figs. 1(a) and (c) with $\alpha = 0$, plus the additional vertices of Fig. 43 if the δ-function is ignored. To interpret the δ-function we first reformulate in the generating functional as

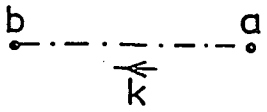
$$\delta(4Mg\sigma+g^2\sigma^2+g^2\underline{\Pi}.\underline{\Pi}) \equiv \int d[C]\exp\{-i\int d^4x C(x)(4Mg\sigma+g^2\sigma^2+g^2\underline{\Pi}.\underline{\Pi})\}$$

$$\equiv \int d[C']\det{}^{-1}(4Mg\sigma+g^2\sigma^2+g^2\underline{\Pi}.\underline{\Pi})\exp\{-i\int d^4x C'(x)\} \quad (6.42)$$

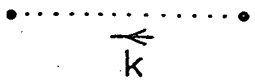
on substituting

$$C(x) \rightarrow C'(x) \equiv C(x)(4Mg\sigma + g^2\sigma^2 + g^2\underline{\Pi}.\underline{\Pi}).$$

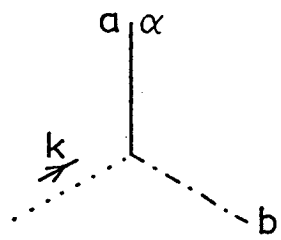
The integral over C' can be incorporated in the normalising factor and the δ-function is replaced by $\det^{-1}(4Mg\sigma + g^2\sigma^2 + g^2\underline{\Pi}.\underline{\Pi})$.



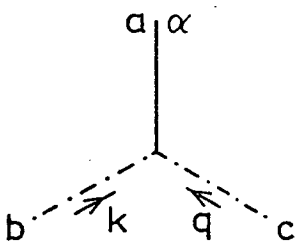
$$\frac{-\delta^{ab}}{k^2}$$



$$\frac{-\delta^{ab}}{k^2}$$



$$-ig\ k^{\alpha}\delta^{ab}$$



$$ig\ \varepsilon_{abc}(q-k)^{\alpha}$$

FIG 43 The σ field is represented by ..... and the π fields by -·-·-·-·- .

c) A Parametrization of the SU(n) Lagrangian

The parametrization of section b) is specifically for SU(2). However, in general any SU(n) Lagrangian can be reformulated similarly. To the SU(n) Lagrangian we add a free field, designated W_μ^9, and redefined

$$\mathcal{W}^\mu(x) = T^a W_a^\mu + \alpha I W_9^\mu \quad \text{etc.} \tag{6.43}$$

i.e. the Lagrangian (6.28) is rewritten for the group U(n) rather than SU(n) with generators T^a of the SU(n) algebra plus αI. The α is chosen such that condition (6.27) still holds. On transforming the generating functional with the appropriate U(n) transformation (6.29) we obtain the form equivalent to (6.32).

The choice of parametrization of the group elements is made considerably easier since there are now the same number of group parameters as matrix elements of the fundamental representation. If we again restrict ourselves to the fundamental representation we may parametrize by the n^2 functions

$$\Psi^{ab}(x) \equiv \Omega^{ab}(x) - \delta^{ab} \quad . \tag{6.44}$$

The Jacobian factor is $\det|I + \Psi|^{-n}$ and the integrations are over the n^2 dimensional surface, Γ, in the function space mapped out by the requirement that Ω is unitary. The generating functional is now

$$\begin{aligned} G[J] \equiv Z^{-1}\int_\Gamma d[W^T]\, d[\Psi]\, (\det M)\det|I+\Psi|^{-n}\exp\{i\int d^4x\Big[-\lambda^{-1}\tfrac{1}{4}\mathrm{tr}\,\mathcal{G}^{T\mu\nu}\mathcal{G}^T_{\mu\nu} \\ + \lambda^{-1}\tfrac{1}{2}M^2\,\mathrm{tr}\,\mathcal{W}^{T\mu}\mathcal{W}^T_\mu + \lambda^{-1}\tfrac{1}{2}\frac{M^2}{g^2}\,\mathrm{tr}\,\partial^\mu[I+\Psi]^{-1}\partial_\mu[I+\Psi] \\ - \frac{\lambda^{-1}M^2}{ig}\mathrm{tr}\,[I+\Psi]^{-1}\partial_\mu[I+\Psi]\mathcal{W}^{T\mu} + \lambda^{-1}\mathrm{tr}\{[I+\Psi]\mathcal{W}^{T\mu}[I+\Psi]^{-1} \\ + [I+\Psi]\partial_\mu[I+\Psi]^{-1}/ig\}\mathcal{J}_\mu\Big]\} \end{aligned} \tag{6.45}$$

We now utilize a similar trick to section b) by introducing the field $\bar{\Psi}$ such that

$$\left[I + \Psi\right]^{-1} \equiv I + \bar{\Psi} \tag{6.46}$$

when (6.45) becomes

$$\begin{aligned} G[J] \equiv Z^{-1}\int_{\Gamma} d[W^T]\, d[\Psi]\, d[\bar{\Psi}]\, (\det M)\,(\det |I+\Psi|^{-n})\,\delta([I+\Psi]^{-1} - [I+\bar{\Psi}]) \\ .\exp\{i\int d^4x\,[-\lambda^{-1}\,\tfrac{1}{4}\mathrm{tr}\, \mathcal{G}^{T\mu\nu}\mathcal{G}^{T}_{\mu\nu} + \lambda^{-1}\tfrac{1}{2}M^2\,\mathrm{tr}\, W^{T\mu} W^{T}_{\mu} \\ + \lambda^{-1}\tfrac{1}{2}\,\frac{M^2}{g^2}\mathrm{tr}\,\partial^{\mu}\bar{\Psi}\,\partial_{\mu}\Psi - \frac{\lambda^{-1}M^2}{ig}\,\mathrm{tr}\,\bar{\Psi}\,\partial_{\mu}\Psi\, W^{T\mu} \\ + \lambda^{-1}\mathrm{tr}\{[I+\Psi]\, W^{T\mu}[I+\bar{\Psi}] + [I+\Psi]\partial^{\mu}\bar{\Psi}/ig\}\mathcal{G}_{\mu}]\} \end{aligned} \tag{6.47}$$

which is equivalent to

$$\begin{aligned} G[J] \equiv Z^{-1}\int_{\Gamma} d[W^T]\, d[\Psi]\, d[\bar{\Psi}]\, (\det M)\,(\det|I+\Psi|^{-(n-1)})\,. \\ .\,(\det^{-1}|\Psi+\bar{\Psi}+\Psi\bar{\Psi}|)\exp\{i\int d^4x\,\mathcal{L}^T\}\,. \end{aligned} \tag{6.48}$$

In (6,48) the blinear in the "scalar" fields is of the correct form for a Lagrangian as

$$\mathrm{tr}\,\partial^{\mu}\bar{\Psi}\,\partial_{\mu}\Psi \equiv \partial^{\mu}\bar{\Psi}_{ab}\,\partial_{\mu}\Psi_{ab}\,.$$

Although the formalism (6.48) contains n^2 tranvsverse vector-boson fields the SU(n) Lagrangian can be abstracted from it as the additional field remains free.

d) Discussion

The two alternative parametrizations of sections b) and c) must be interpreted with care. Since there are no sources for the ghosts fields present in the formulations we cannot be certain that they can be interpreted as perturbation expansions. If the fields $\pi^a(x)$ and $\sigma(x)$ of section b) are considered as polynomials of the scalar fields of Boulware[6], i.e. the scalar fields of Chapters 4 and 5, the lowest order term of $\sigma(x)$ is $O(\phi^2)$. It thus seems improbable that the formalism of section b) can be expanded as a perturbation series and the Feynman rules of Fig. 4 3 are not justifiable.

To investigate the formalism of section c) we start from the formalism of Boulware. It should be noted that the latter formalism is open to the same query over interpretation but it has been completely verified in Chapters 4 and 5. (6.45) is then obtained by making the transformation

$$\phi^a \rightarrow \frac{1}{ig} \mathrm{tr}\{T^a \log(I+\Psi)\} \tag{6.49}$$

which is easily justified by diagrammatical means as in (28). Similar methods can be used to support (6.47) and to find the correct interpretation of the field $\bar{\Psi}$. First we note that any term of the expansion of

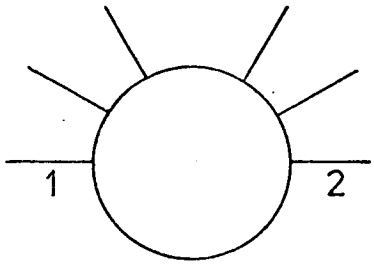
$$\mathrm{tr}\{(\partial^\mu\Psi)\partial_\mu(I+\Psi)^{-1}\} \tag{6.50}$$

e.g.
$$(-1)^n \mathrm{tr}\{(\partial^\mu\Psi)\partial_\mu(\Psi\Psi \ldots \Psi)\}$$

$$\equiv (-1)^n \mathrm{tr}\{\partial^\mu(\Psi\Psi \ldots \Psi)\partial_\mu\Psi\} \tag{6.51}$$

by the rotation property of the trace. Thus in the vertices

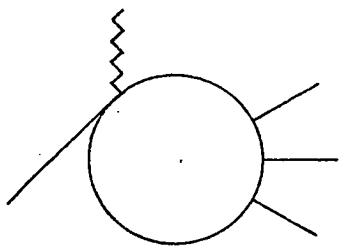
created by (6.50) there are two legs, labelled 1 and 2, either of which may be considered to carry the inverse propagator term (∂^2).



(6.52)

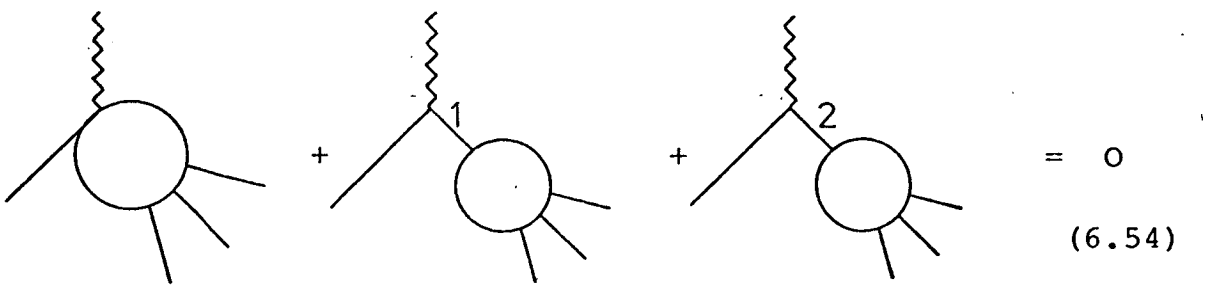
The basic reason for the property (6.51) is the symmetry of the purely scalar contribution to (4.19).

If the higher order vertices including W_μ^T are denoted by



(6.53)

the scalar vertices obey the identity



(6.54)

In the combinatorics of the usual transformation of a field Φ, with propagator term $\frac{1}{2}\Phi V\Phi$, two vertex functions of the form $\frac{1}{2}\Phi VF(\Phi)$ are obtained to give the cancellation (6.54)[28]. In the present case the two terms from the symmetry of the vertices in legs 1 and 2 are required.

Hence, in a general scalar configuration only the lowest

order scalar-vector vertices remain and the legs of the higher order, purely scalar vertices are connected such that 1 and 2 do not attach to the scalar-vector boson vertex. If we start from one of these purely scalar vertices, a path can be traced out by leaving it on leg 1. At the vertex this leads us to we may again leave on leg 1, unless that was the leg at which we arrived. In the latter case we leave from leg 2. As the path is restricted to the purely scalar vertices, it must eventually form a closed loop. The closed loops can be considered to be equivalent to a determinant term in the Lagrangian, as in (28), but here the combinatoric factors are unusual since two possible ways of constructing the configuration are deleted by (6.54) and two corresponding ways of connecting each vertex to the loop included. It is not necessary to check if these factors account for the determinants in (6.48) as the exact form is superfluous.

The way to interpret the field $\overline{\Psi}$ is obviously to keep only the lowest order scalar-vector vertex, i.e. to replace $\overline{\Psi}$ by Ψ and calculate the combinatoric factors as for a normal scalar Lagrangian. Hence, the formalism of section c) demonstrates how to group the vertices of the formalism of Boulware, (4.22), such that all higher order vertices vanish except for the determinant contributions. The resultant rules are those of Fig. 1 with $\alpha = 0$ and a factor $(-\frac{1}{2})$ associated with each scalar loop plus the determinant terms.

If the amplitudes are calculated using the dimensional regularisation scheme[53], all contributions from the determinants and loops of the previous paragraph vanish and the Feynman rules which are left are renormalisable according to

power counting. With the rules of Fig. 1 the ghost for the field W_9^μ and so W_9^μ completely decouple and the free field W_9^μ can be dropped to leave the SU(n) Lagrangian.

Finally we reflect on the interpretation of this regularisation dependent formalism. If we assume that the problems raised by the identity (5.16) reflect the inadequacy of the perturbation formalism and that in an exact solution all non-renormalizable terms converge in the limit as $n \to 4$, we could calculate the renormalizable contributions, with any regularization scheme, and add by hand those finite terms necessary to render the S-matrix unitary. The additional finite terms being assumed to be the limit of the non-renormalizable terms. As the complete theory is unitary regardless of the regularisation procedure the S-matrix obtained by the formalism of the previous paragraph must be unitary and the limit of the non-renormalizable contributions taken to be zero. If the dimensional regularization scheme is viewed as only being a convenient mathematical trick and the massive Yang-Mills theory is still taken to be inherently unrenormalizable, we could take the "renormalizable" formalism to be an alternative, unitary, spin-one Lagrangian whose S-matrix differs from the normal Lagrangians by divergent terms which vanish under the dimensional regularization. Alternatively a pragmatic attitude could be adopted in that the formalism be considered a method of calculating renormalizable amplitudes for the massive Yang-Mills Lagrangian and that the use of the dimensional regularisation scheme contains no inherent difficulties.

We have only considered here Lagrangians with a mass term

of the form $\frac{1}{2}M^2 \underline{W}_\mu . \underline{W}^\mu$ but the procedure could be generalised to addition of a mass term $\frac{1}{2}M^2 W^a_\mu \, M^{ab} \, W^{b\mu}$ as long as it could be expressed as

$$\tfrac{1}{2}\lambda^{-1} M^2 \operatorname{tr} w^\mu m \, w_\mu \tag{6.55}$$

in (6.28). It is still necessary to demonstrate that the renormalization counter-terms can be absorbed in scaling factors. A similar argument to that used by't. Hooft and Veltman[50] for gauge theories can be used. For the S-matrix obtained by the "renormalizable" rules to be unitary it is required that amplitudes in the explicitly unitary rules are related to those for the "renormalizable" rules by Ward-type identities obtainable by including a suitable source term in the Lagrangian. These identities must survive renormalization to preserve unitarity and so may restrict the form of the renormalized theory. This argument is of course not so strong as in the gauge theories, as there is no equivalent of the Tree-Loop theorem.

Finally we note the reformalism of the SU(n) Yang-Mills Lagrangian of section c) could equally well be applied to the U(n) Lagrangian.

Appendix A: Self-Energy Terms in the Infrared Summation

In Quantum Electrodynamics the proper self-energy terms for the electron may be written[58] as

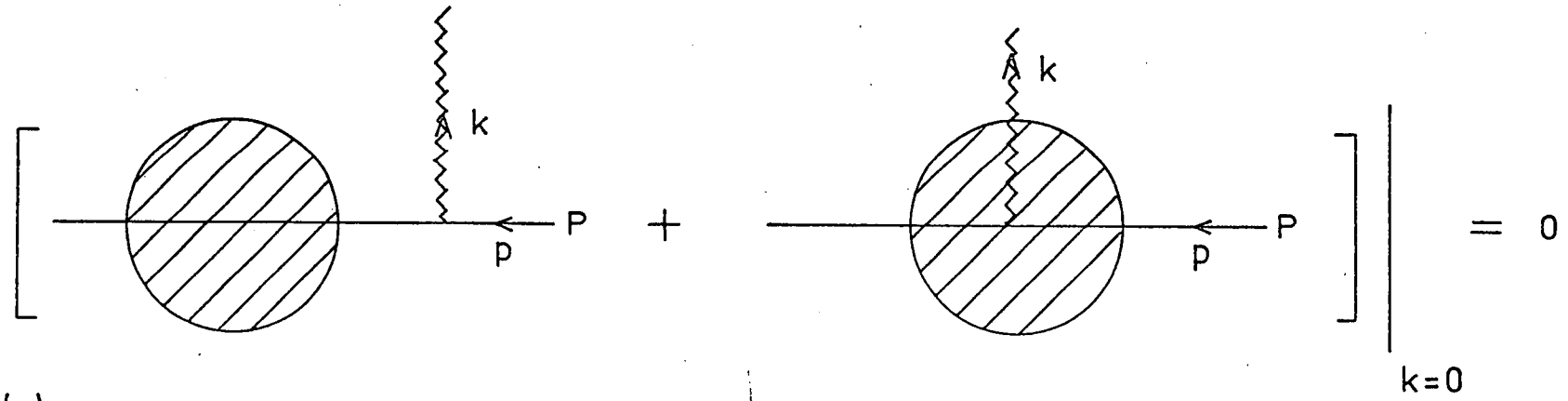
$$\Sigma^*(p) \equiv A + (\not{p}-m)B + (\not{p}-m)\Sigma_f^*(p)(\not{p}-m). \qquad (A.1)$$

Henceforth, we shall consider diagrams where the mass-like contribution A has been absorbed by the propagator (not necessarily renormalised). Exploiting (A.1) the Ward identity

$$\frac{\partial \Sigma^*}{\partial p^\mu}(p) + \Lambda_\mu(p,p) = 0 \qquad (A.2)$$

may be interpreted as Fig. A.1(a). Equivalent to Fig. Al(a) is the expression represented by Fig. A.1(b) as the infrared terms in k can only possibly arise for the diagrams on the left in the limit as $k \to 0$ when the infrared contributions must cancel by Fig. A.1(a). (In fact the limit of $K(k)$ as $k \to 0$ must vanish altogether.)

Firstly we evaluate the contribution of an additional virtual photon added in all possible ways to an amplitude involving self-energy terms. The method follows (59). First consider the additions where both ends of the additional photon do not terminate on the same external electron line or its self-energy terms, i.e. diagrams of the type of Fig. A.2. Applying the identity of Fig. A.1(b) the infrared contributions from the legs on an external line and its self-energy terms cancel except for the last insertion, i.e. the diagrams of Fig. A.2 reduce to those of Fig. A.3 plus some infrared finite factor. Fig. A.2 is evaluated as usual in (59) and the total infrared contribution of Fig. A.2 is



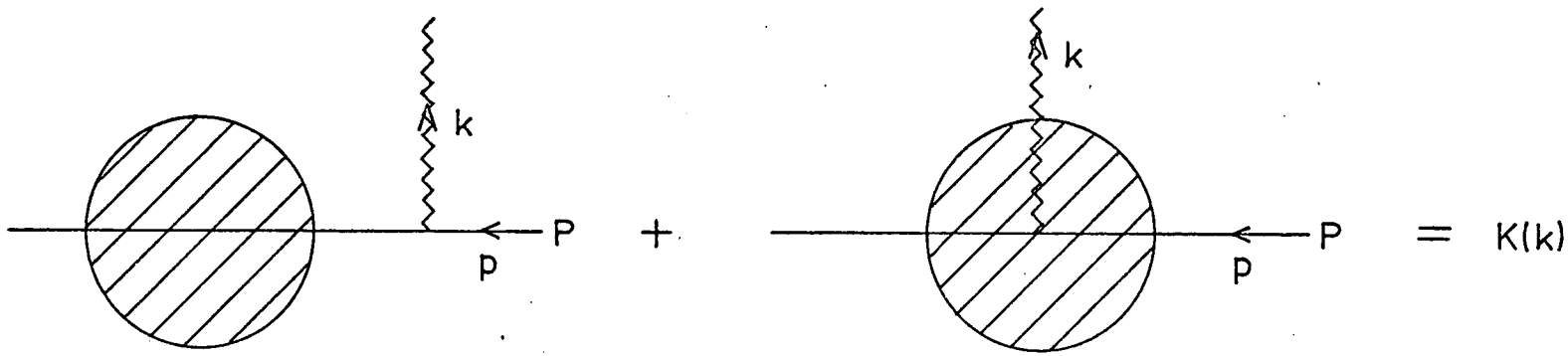
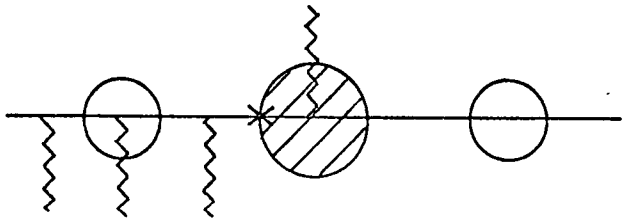


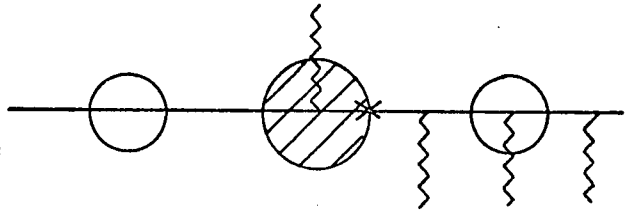
FIG A1 The shading indicates only proper diagrams present; P shows the line is physically polarized; K(k) is infrared finite in k.

(a) Left-leg of virtual photon as shown below line



right-leg to right of X

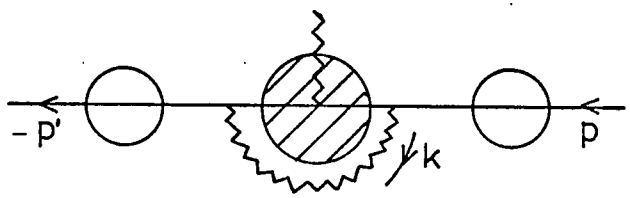
(b) Right-leg of virtual photon as shown below line



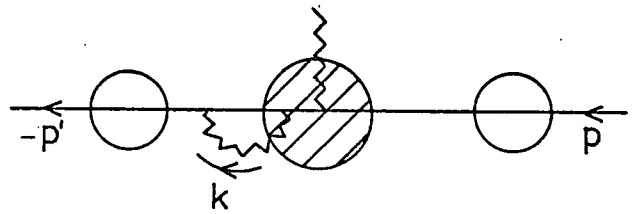
left-leg to left of X

FIG A2 The unshaded blobs represent self-energy terms both proper and improper.

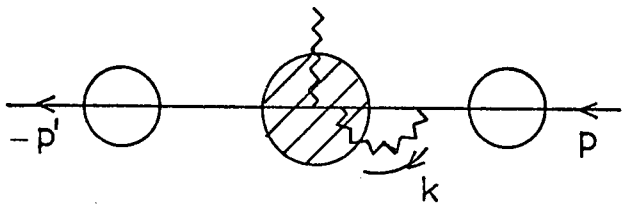
(a)



(b)



(c)



(d)

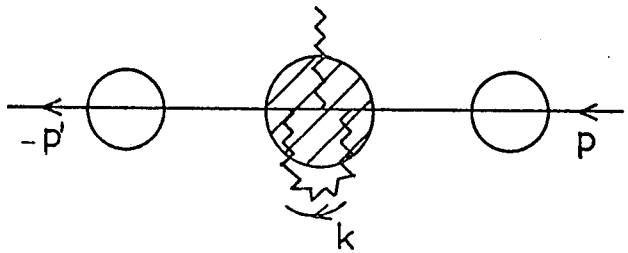
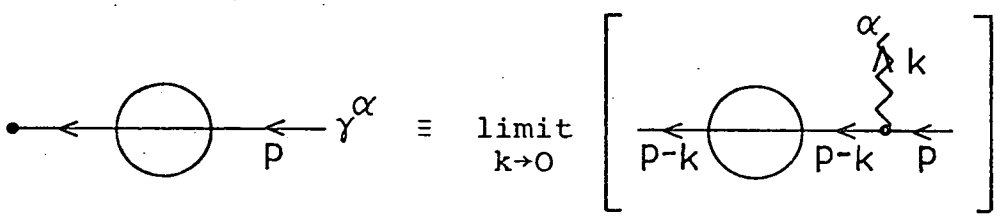


FIG A3

$$B'\bar{u}(p')\Gamma(p',p)u(p) + K(p',p;k) \qquad (A.3)$$

where $B' = \dfrac{ie^2}{(2\pi)^4} \int_{\lambda}^{\Lambda} \dfrac{-(2p'-k).(2p-k)}{(k^2-2p'.k)(k^2-2p.k)}\, d^4k$.

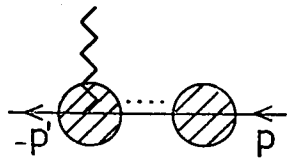
Next we evaluate the contributions of a photon where both ends terminate on the same external line or its self-energy terms. The set of diagrams are of the form of Fig. A.4. Instead consider the equivalent self-energy terms in



(A.4)

By Fig. A.1(a) the right-hand side of (A.4) is equivalent to the terms in Fig. A.5 in the limit as $\ell \to 0$. The photon line is always terminated in the proper self-energy term which involves the additional virtual photon. On applying the identity Fig. A.1(b) to the virtual photon in the diagrams Fig. A.5(a) - (d) we see that, where the additional photon joins more than one proper self-energy part, there is no infrared contribution from it, i.e. diagrams Fig. A.4(a) - (d) have no net infrared contribution. Diagrams, Fig. A.5(e) - (g) are evaluated as

$$(-1)\,\frac{ie^2}{(2\pi)^4} \int_{\lambda}^{\Lambda} \frac{d^4k}{k^2}\, \frac{-(2p'-k).(2p-k)}{(k^2-2p'.k).(k^2-2p.k)}$$



(A.5)

Hence, on using Fig. A.1(b) again and taking the limit as $p \to p'$ we obtain the infrared contribution of diagrams Fig. A.4(e) - (g)

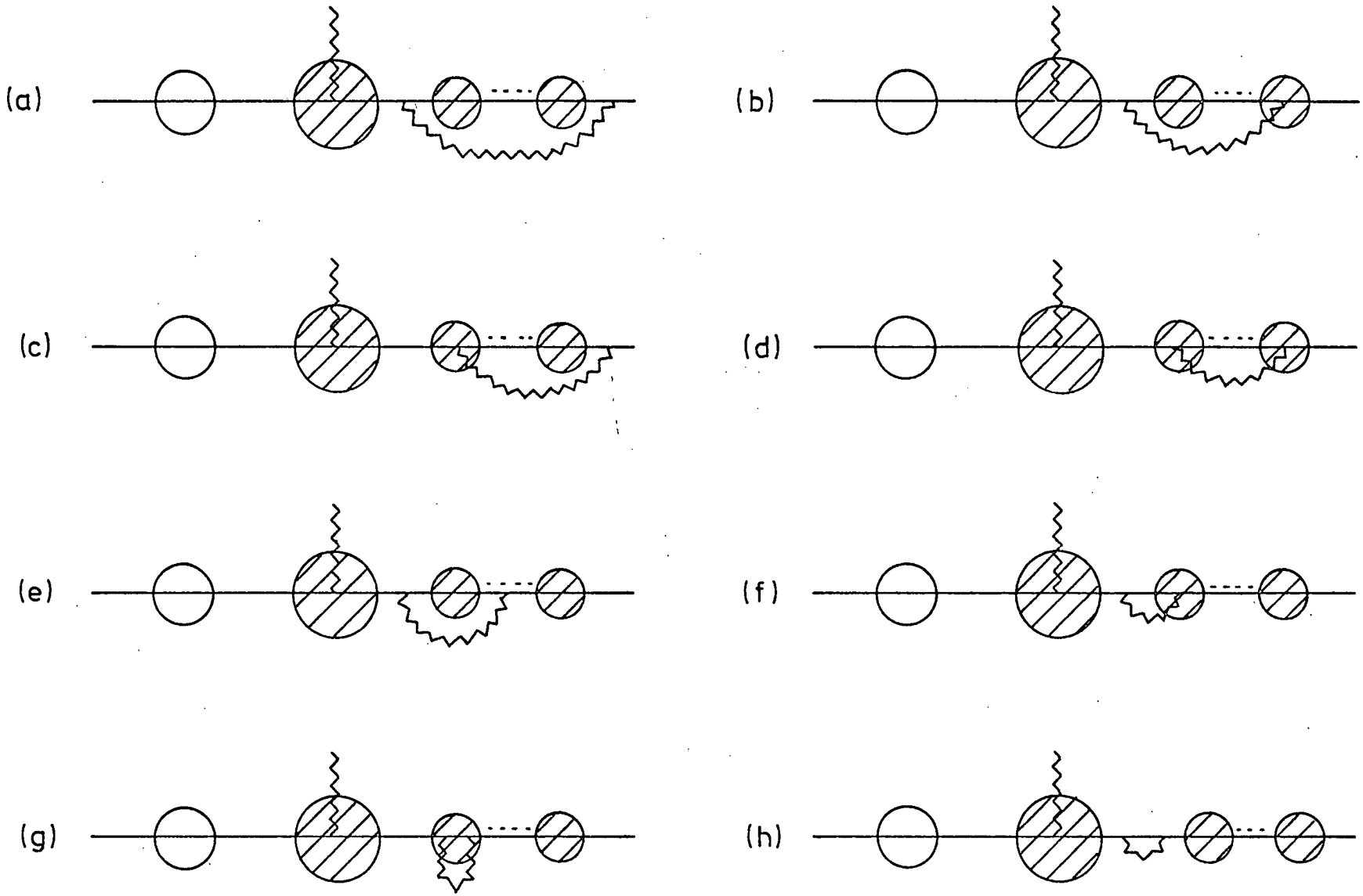


FIG A4

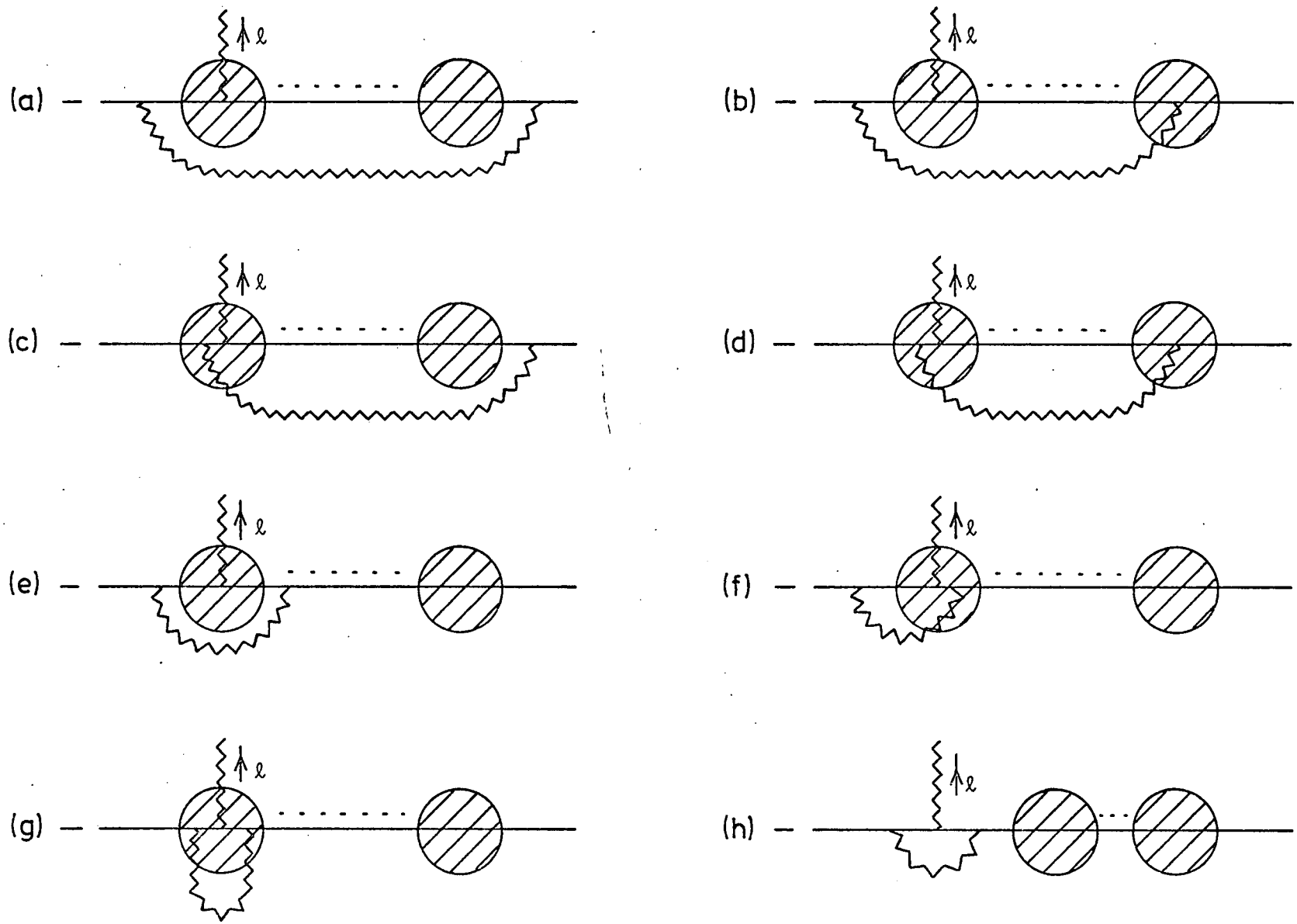
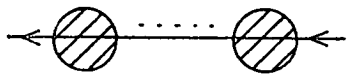
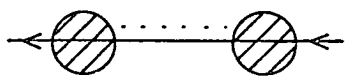


FIG A5 The minus is included to give direct equivalence to Fig. A.4.

$$\frac{ie^2}{(2\pi)^4}\int_{\lambda}^{\Lambda}\frac{d^4k}{k^2}\,\frac{-(2p-k)^2}{(k^2-2p.k)^2}$$  (A.6)

Diagrams Fig. A.5(h) can be immediately evaluated to get the contribution from Fig. A.4(h)

$$\frac{ie^2}{(2\pi)^4}\int_{\lambda}^{\Lambda}\frac{d^4k}{k^2}\,\frac{(2p-k)^2}{(k^2-2p.k)^2}$$  (A.7)

(A.7) differs from (A.6) in sign as it is not necessary to use Fig. A.1(b) again in obtaining (A.7).

There is a contribution (A.6) for each set of proper diagrams in the self-energy terms and a contribution (A.7) for each line connected to the proper diagrams. Hence, the infrared contribution for all diagrams in Fig. A.4 is (A.7). Adding all infrared contributions of Fig. A.4 to (A.3) we obtain (1.12). The factorisation of the self-energy terms could be iterated as usual to give (1.14) as required by consistency as (1.14) is related to (1.13) through the identity Fig. A.1(a).

Appendix B: Solutions for $\overleftrightarrow{\mathcal{R}}_{ij}$.

Here we shall examine the possible solutions to the constraint equations on $\overleftrightarrow{\mathcal{R}}_{ij}$

$$\overleftrightarrow{\mathcal{R}}_{ij}\ \vec{S}_{ja}(A) \equiv \overset{*}{M}_{ib}\ \overleftrightarrow{L}_{1ba} \tag{B.1}$$

$$\overleftrightarrow{\mathcal{R}}_{ij}\ \vec{t}_{ja} \equiv \overset{*}{M}_{ib}\ \overleftrightarrow{L}_{2ba} \tag{B.2}$$

Some solutions of (B.1) can easily be obtained by multiplying on the left by $\overset{*}{t}_{ci}$:

$$\overleftarrow{\tilde{t}}_{ci}\ \overleftrightarrow{\mathcal{R}}_{ij}\ \vec{S}_{ja}(A) \equiv \overleftarrow{\tilde{m}}_{cb}\ \overleftrightarrow{L}_{1ba}$$

$$\therefore\quad \overleftrightarrow{L}_{1ba} \equiv \overset{*}{\tilde{m}}{}^{-1}_{bc}\ \overleftarrow{\tilde{t}}_{ci}\ \overleftrightarrow{\mathcal{R}}_{ij}\ \vec{S}_{ja}(A) \tag{B.3}$$

For most general solution, $\overleftrightarrow{\mathcal{R}}_{ij}$ must satisfy

$$\overleftrightarrow{\mathcal{R}}_{ij}\ \vec{S}_{ja}(A) = \overleftarrow{\tilde{M}}_{ib}\ \overleftarrow{\tilde{m}}{}^{-1}_{bc}\ \overleftarrow{\tilde{t}}_{ck}\ \overleftrightarrow{\mathcal{R}}_{kj}\ \vec{S}_{ja}(A) + \overleftrightarrow{X}_{ia} \tag{B.4}$$

where $\overleftarrow{\tilde{t}}_{ci}\ \overleftrightarrow{X}_{ia} = 0$. (B.5)

But to satisfy (B.1) $\overleftrightarrow{X}_{ia} \equiv \overleftarrow{\tilde{M}}_{ib}\ \overleftrightarrow{Y}_{ba}$

and (B.5) implies $\overleftrightarrow{Y}_{ba} = 0$.

Therefore, most general $\mathcal{R}$ satisfies

$$(I - \overleftarrow{\tilde{M}}\ \overleftarrow{\tilde{m}}{}^{-1}\ \overleftarrow{\tilde{t}})\ \overleftrightarrow{\mathcal{R}}\ \vec{S} = 0 . \tag{B.6}$$

Similarly we require

$$(I - \overleftarrow{\tilde{M}}\ \overleftarrow{\tilde{m}}{}^{-1}\ \overleftarrow{\tilde{t}})\ \overleftrightarrow{\mathcal{R}}\ \vec{t} = 0, \tag{B.7}$$

As $(I - \overleftarrow{\tilde{M}}\,\overleftarrow{\tilde{m}}^{-1}\,\overleftarrow{\tilde{t}})\overleftarrow{\tilde{M}} = O$, $(I - \overleftarrow{\tilde{M}}\,\overleftarrow{\tilde{m}}^{-1}\,\overleftarrow{\tilde{t}})$ cannot be non-singular and neither $\mathcal{R}\,t$ nor $\mathcal{R}\,S$ is zero.

We now consider the eigenvector equation

$$(I - \overleftarrow{\tilde{M}}\,\overleftarrow{\tilde{m}}^{-1}\,\overleftarrow{\tilde{t}})\ \underline{x} = \lambda\,\underline{x} \quad . \tag{B.8}$$

As $\vec{m}$ is non-singular and an $n \times n$ matrix the rank of $\overleftarrow{\tilde{m}}$ is n and the rank of $\overleftarrow{\tilde{M}}$ must be greater than n as $\mathrm{Min}(R(A), R(B)) \geq R(AB)$, where A and B are any matrices. But rank of $\overleftarrow{\tilde{M}}$ is less than n as M is an $m \times n$ matrix. Therefore the rank of $\overleftarrow{\tilde{M}}$ is n. Similarly the rank of $\overleftarrow{\tilde{t}}$ is n as $\overleftarrow{\tilde{t}}$ is an $n \times m$ matrix.

If $\overleftarrow{\tilde{M}}' \equiv \begin{pmatrix} \overleftarrow{X} \\ \hline O \end{pmatrix}$ where X is a non-singular $n \times n$ matrix

$\overleftarrow{\tilde{M}} \equiv P\,\overleftarrow{\tilde{M}}'\,Q$ where P and Q are non-singular.

Let $\overleftarrow{\tilde{t}}' \equiv \overleftarrow{\tilde{t}}\,P^{-1}$ where $\overleftarrow{\tilde{t}}' \equiv (\overleftarrow{Y} \,\vdots\, \overleftarrow{Z})$ and Y is an $n \times n$ matrix

$$\text{Thus} \quad \overleftarrow{\tilde{M}}\,\overleftarrow{\tilde{m}}^{-1}\,\overleftarrow{\tilde{t}} = P \begin{pmatrix} I & \vdots & \overleftarrow{Y}^{-1}\,\overleftarrow{Z} \\ \hline O & \vdots & O \end{pmatrix} P^{-1} \tag{B.9}$$

and (B.8) is equivalent to

$$\left[I - P \begin{pmatrix} I & \vdots & \overleftarrow{Y}^{-1}\,\overleftarrow{Z} \\ \hline O & \vdots & O \end{pmatrix} P^{-1} \right] \underline{x} = \lambda \underline{x} \tag{B.10}$$

or

$$\begin{pmatrix} O & \vdots & -\overleftarrow{Y}^{-1}\,\overleftarrow{Z} \\ \hline O & \vdots & I \end{pmatrix} \underline{y} = \lambda \underline{y} \tag{B.11}$$

where $\underline{y} = P^{-1}\,\underline{x}$.

We have thus obtained the eigenvalue equation

$$\left|\begin{array}{c|c} -\lambda I_n & -\overleftarrow{\underset{\sim}{Y}}^{-1} \quad \overleftarrow{\underset{\sim}{Z}} \\ \hline O & (1-\lambda) I_{m-n} \end{array}\right| = 0$$

$$\text{i.e.} \quad (-\lambda)^n (1-\lambda)^{m-n} = 0 \qquad \text{(B.12)}$$

and $(I - \overleftarrow{\underset{\sim}{M}}\ \overleftarrow{\underset{\sim}{m}}^{-1}\ \overleftarrow{\underset{\sim}{t}})$ has eigenvalues $\lambda = 0$ (nth order degenerate)
and $\lambda = 1$ ($(m-n)^{th}$ order degenerate).

The eigenvectors for $\lambda = 0$ are $\underline{x}_i = P\underline{y}_i$ where

$$\underline{y}_i = \underline{e}_i^m \quad \text{for} \quad 1 \leqslant i \leqslant n$$

and for $\lambda = 1$ are $\underline{x}_i = P\underline{y}_i$ where

$$\underline{y}_i = \underline{e}_i^m \quad \text{for} \quad n+1 \leqslant i \leqslant n$$

with $(\underline{e}_i^m)_j = \delta_{ij}^m$.

For $(\mathcal{R}S)$ and $(\mathcal{R}t)$ to satisfy the conditions $(I - \overleftarrow{\underset{\sim}{M}}\ \overleftarrow{\underset{\sim}{m}}^{-1}\ \overleftarrow{\underset{\sim}{t}})\mathcal{R}S = (I - \overleftarrow{\underset{\sim}{M}}\ \overleftarrow{\underset{\sim}{m}}^{-1}\ \overleftarrow{\underset{\sim}{t}})\mathcal{R}t = 0$ they must be constructed such that their column vectors are linear combinations of the eigenvectors of $(I - \overleftarrow{\underset{\sim}{M}}\ \overleftarrow{\underset{\sim}{m}}^{-1}\ \overleftarrow{\underset{\sim}{t}})$ for eigenvalue $\lambda = 0$,

$$\begin{aligned} &\text{i.e. } j^{th} \text{ column of } (\mathcal{R}S) \equiv \sum_{i=1}^{n} \lambda^S_{ji}\, \underline{x}_i \\ &\text{and} \quad " \quad " \quad " \quad (\mathcal{R}t) \equiv \sum_{i=1}^{n} \lambda^t_{ji}\, \underline{x}_i \ . \end{aligned} \qquad \text{(B.13)}$$

Thus the most general solution to (B.1) and (B.2) is to choose $\mathcal{R}$ to satisfy (B.13) when

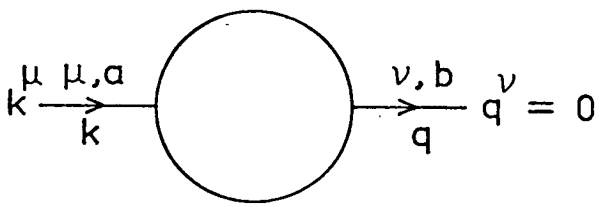
$$\begin{aligned} L_1 &\equiv (\overleftarrow{\underset{\sim}{m}}^{-1}\ \overleftarrow{\underset{\sim}{t}}\ \overleftrightarrow{\mathcal{R}})\ \vec{S} \\ L_2 &\equiv (\overleftarrow{\underset{\sim}{m}}^{-1}\ \overleftarrow{\underset{\sim}{t}}\ \overleftrightarrow{\mathcal{R}})\ \vec{t} \ . \end{aligned} \qquad \text{(B.14)}$$

A particularly simple choice is $\overleftrightarrow{\mathcal{R}}_{jk} \equiv \overleftarrow{\underset{\sim}{M}}_{ja}\ \overleftrightarrow{X}_{ak}$ for any $\overleftrightarrow{X}$.

Appendix C : The Massless Yang-Mills Lagrangian

The massless Yang-Mills fields may be quantized with an arbitrary gauge function as in (50). The S-matrix exhibits the usual gauge invariance for diagrams constructed with a factor (-1) associated with each ghost loop. In particular consider the subset of gauges $C_a \equiv \frac{1}{\alpha} W_a^\mu$. The proof of invariance of the S-matrix, up to the one loop approximation, under variation of α for diagrams with a factor $(-\frac{1}{2})$ associated with each ghost loop, of section c) of Chapter 3 also holds for this set of gauges.

This is easily verified directly as the ghost loops are independent of α . The α dependence of the vertices and propagators mutually cancel. The identity corresponding to (3.37)



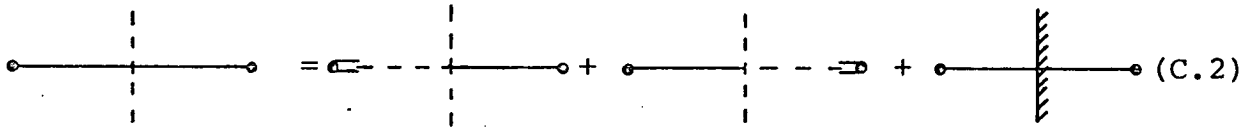
(C.1)

holds for tree diagrams in both approaches and ensures that the α dependent terms $\frac{\alpha^2 k^\mu k^\nu}{(k^2+i\varepsilon)^2}$ in the vector-boson propagator does not contribute in either.

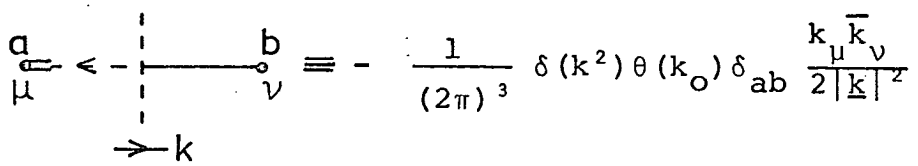
It is imperative to note that neither construction has a manifestly unitary gauge as for no α does the pseudo-scalar ghost become insignificant. If any given α gives rise to explicitly unitary Feynman rules for one formalism, then the other formalism must also be unitary which is of course unacceptable. This contrasts with the situation for the massive theory where both formalisms have an explicitly

unitary form in the limit $\alpha \to 0$. There, it is the formalism which is invariant under variation of α which is the correct one.

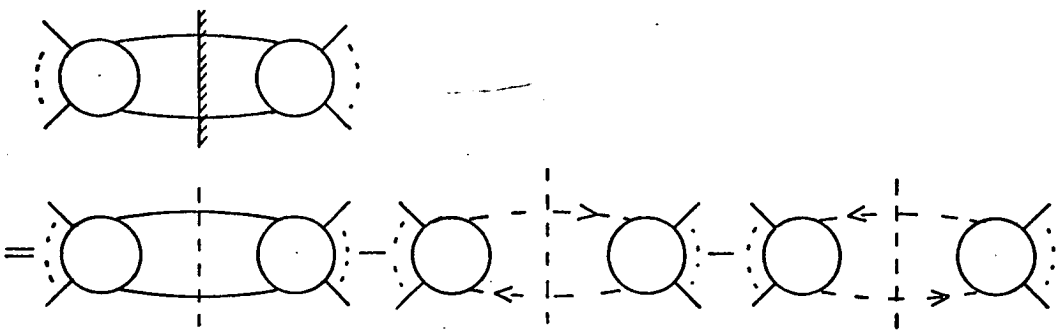
For the massless Yang-Mills Lagrangian to be unitary, it must be proved implicitly for some set of rules. 't. Hooft[27] has done this in the formalism with a loop factor (-1) when $\alpha = 1$. Instead of (3.34) use the relation



(C.2)

where 

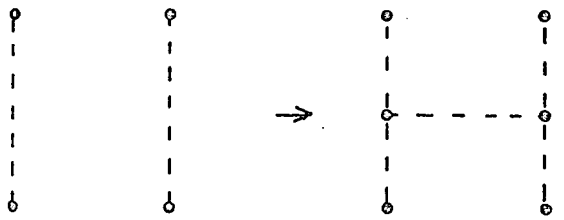
and the physical cut is for the two transverse polarizations only. The two cut line diagrams become

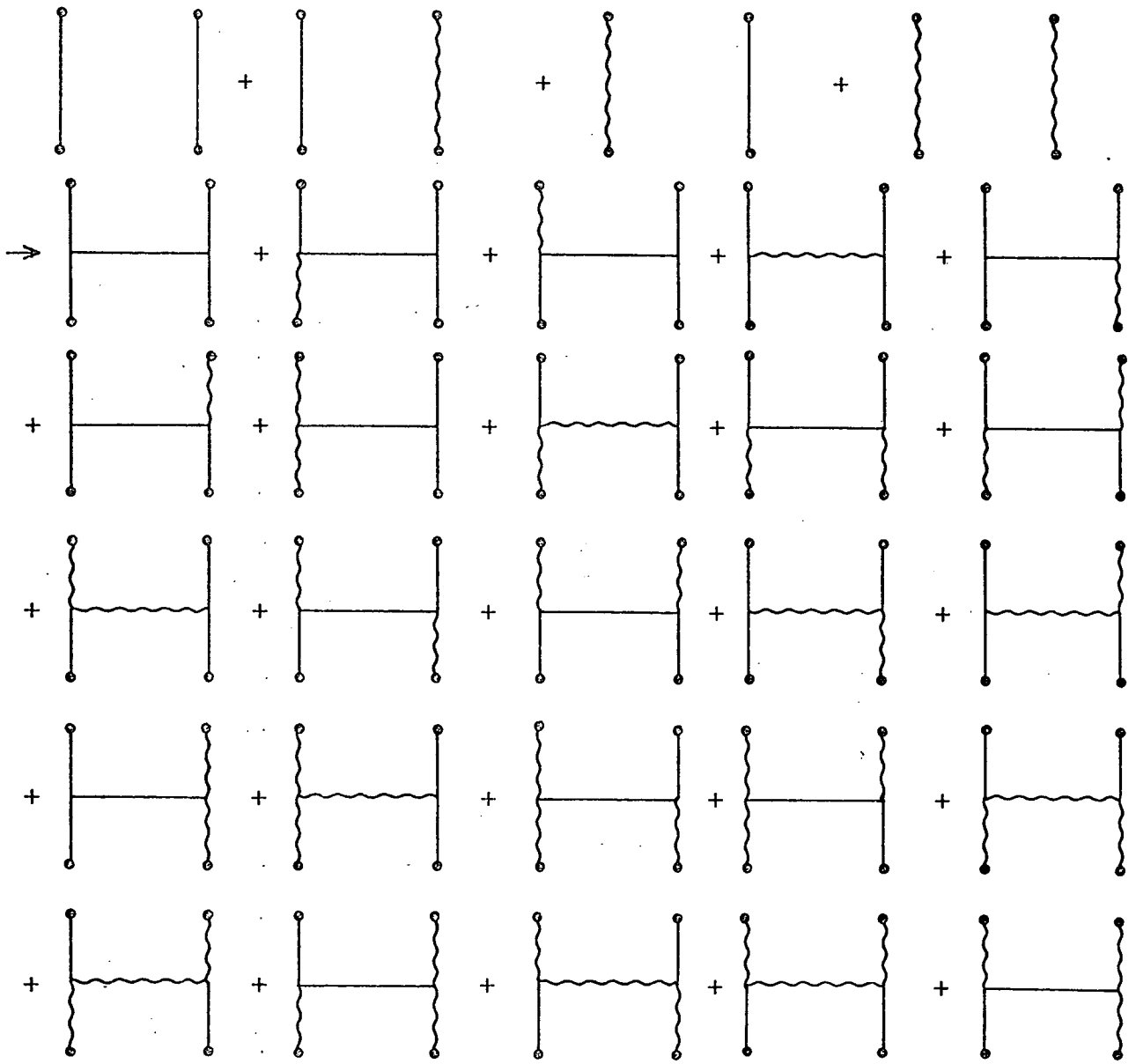


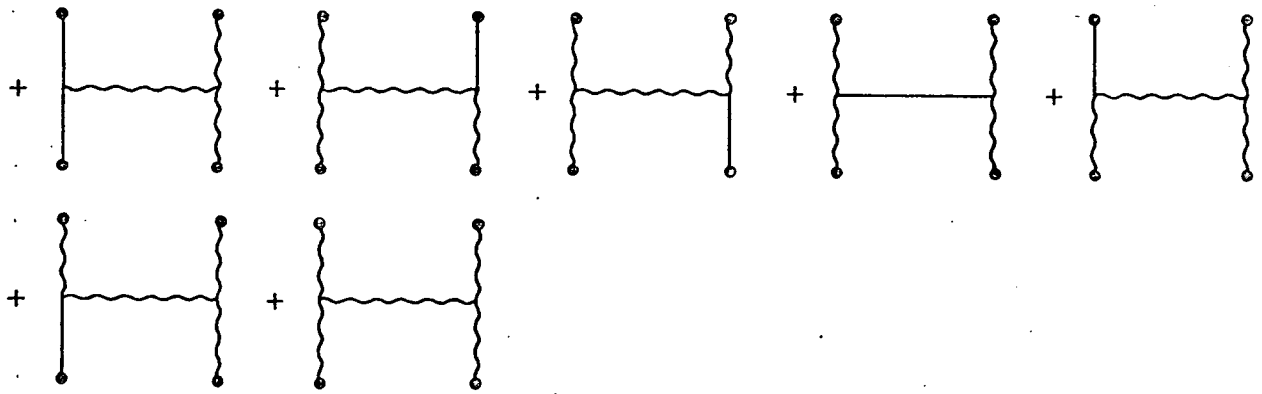
which is the form of the Cutkosky rules for the formalism with loop factor (-1.). The proof is extended to all orders by induction.

Appendix D: The Addition of a Line or Loop to a Duplication Factor Diagram

The most general insertion to be demonstrated is that of a line between two completely independent lines in a scalar configuration

i.e.  where • represents the scalar vertices. Within the context of calculating the duplication factor the replacement to be considered is





All these replacements may be made for all possible attachments of hard and wavy lines to the vertices and so must be evaluated using the rules in Chapter 4, section c). As the mathematics is identical whether the vertex is purely hard-line or an isolated wavy-line tree, the vertices may be labelled n_1, n_2, n_3, n_4 where n_i is the number of legs of the hard-line vertex or the number of hard-line connections to the wavy-line trees. In the above definition the vertices are considered in isolation without any connecting lines. Only wavy-line contributions will actually be used in calculation to avoid repetition. Below all possible vertex attachments are considered; the expression above each diagram is the duplication factor before insertion of the additional line and the expression below, after. R is the contribution to the duplication factor of the rest of the diagram. It is different for each configuration.

$$R\left\{\begin{array}{l} n_1n_2n_3n_4 - n_1n_2n_3 - n_1n_2n_4 \\ -n_1n_3n_4 - n_2n_3n_4 + n_1n_2 + n_1n_3 \\ +n_1n_4 + n_2n_3 + n_2n_4 + n_3n_4 - n_1 \\ -n_2 - n_3 - n_4 + 1 \end{array}\right\}$$

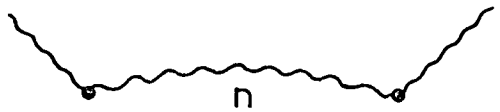
$$R\left\{\begin{array}{l} n\,n_2n_4 - n\,n_2 - n\,n_3 \\ \\ -n_2n_4 + n + n_2 + n_4 - 1 \end{array}\right\}$$

;

$$R\left\{\begin{array}{l} -n_1n_2n_3n_4 + n_1n_2n_3 + n_1n_2n_4 \\ +n_1n_3n_4 + n_2n_3n_4 - n_1n_2 - n_1n_3 \\ -n_1n_4 - n_2n_3 - n_2n_4 - n_3n_4 + n_1 \\ +n_3 + n_3 + n_4 - 1 \end{array}\right\}$$

$$R\left\{\begin{array}{l} -n\,n_2n_4 + n\,n_4 + n\,n_3 \\ \\ +n_2n_4 - n - n_2 - n_4 - 1 \end{array}\right\}$$

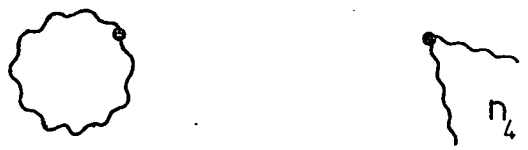
$$R\left\{\begin{array}{l} n\,n_3n_4 - n\,n_3 - n\,n_4 \\ -n_3n_4 + n + n_3 + n_4 - 1 \end{array}\right\}$$



$$R\left\{\begin{array}{l} 2n_1n_2n_4 - 2n_2n_4 - 2n_1n_2 \\ -2n_1n_4 + 2n_1 + 2n_2 + 2n_4 - 2 \end{array}\right\}$$

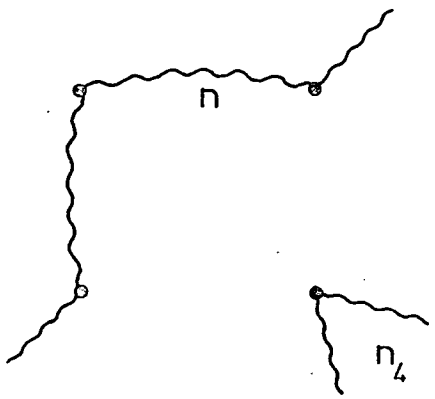
;

$$R\left\{\begin{array}{l} -n\,n_3n_4 + n\,n_3 + n\,n_4 \\ +n_3n_4 - n - n_3 - n_4 + 1 \end{array}\right\}$$



$$R\left\{\begin{array}{l} -2n_1n_2n_4 + 2n_2n_4 + 2n_1n_2 \\ +2n_1n_4 - 2n_1 - 2n_2 - 2n_4 + 2 \end{array}\right\}$$

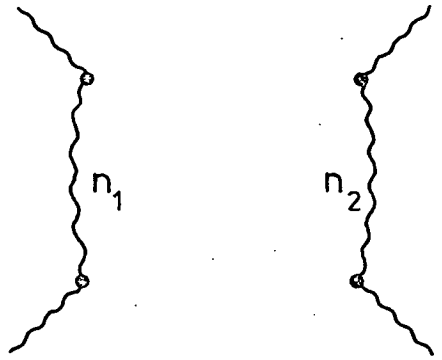
$R\{n\,n_4 - n - n_4 + 1\}$



$R\{-n\,n_4 + n + n_4 - 1\}$

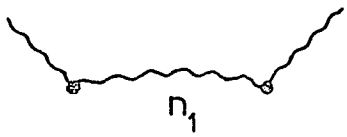
;

$R\{n_1 n_2 - n_1 - n_2 + 1\}$



$R\{-n_1 n_2 + n_1 + n_2 - 1\}$

$R\{n_1 n_2 - n_1 - n_2 + 1\}$



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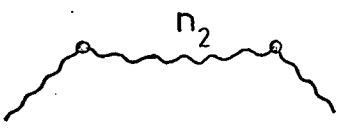
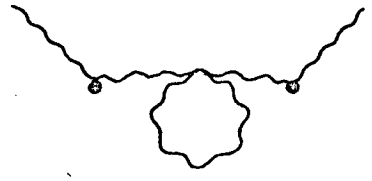


$R\{-n_1 n_2 + n_1 + n_2 - 1\}$

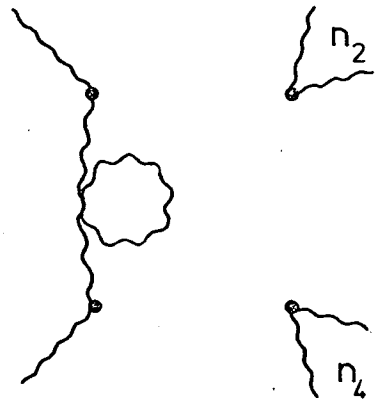
$R\{2n_3 n_4 - 2n_3 - 2n_4 + 2\}$



;

$R\{-2n_3 n_4 + 2n_3 + 2n_4 - 2\}$

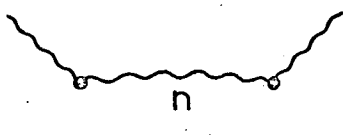
$R\{2n_2 n_4 - 2n_2 - 2n_4 + 2\}$



;

$R\{-2n_2 n_4 + 2n_2 + 2n_4 - 2\}$

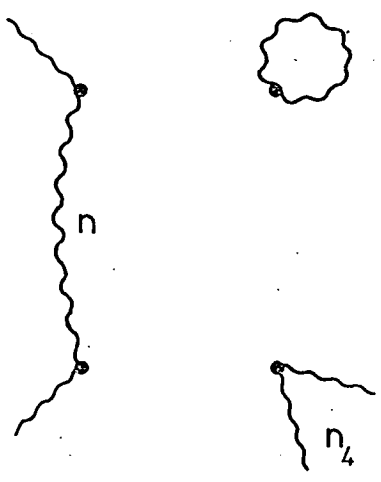
$R\{2n\,n_4 - 2n - 2n_4 + 2\}$



;

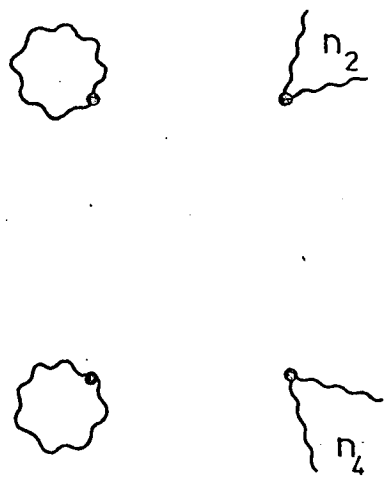
$R\{-2n\,n_4 + 2n + 2n_4 - 2\}$

$R\{2n\,n_4 - 2n - 2n_4 + 2\}$



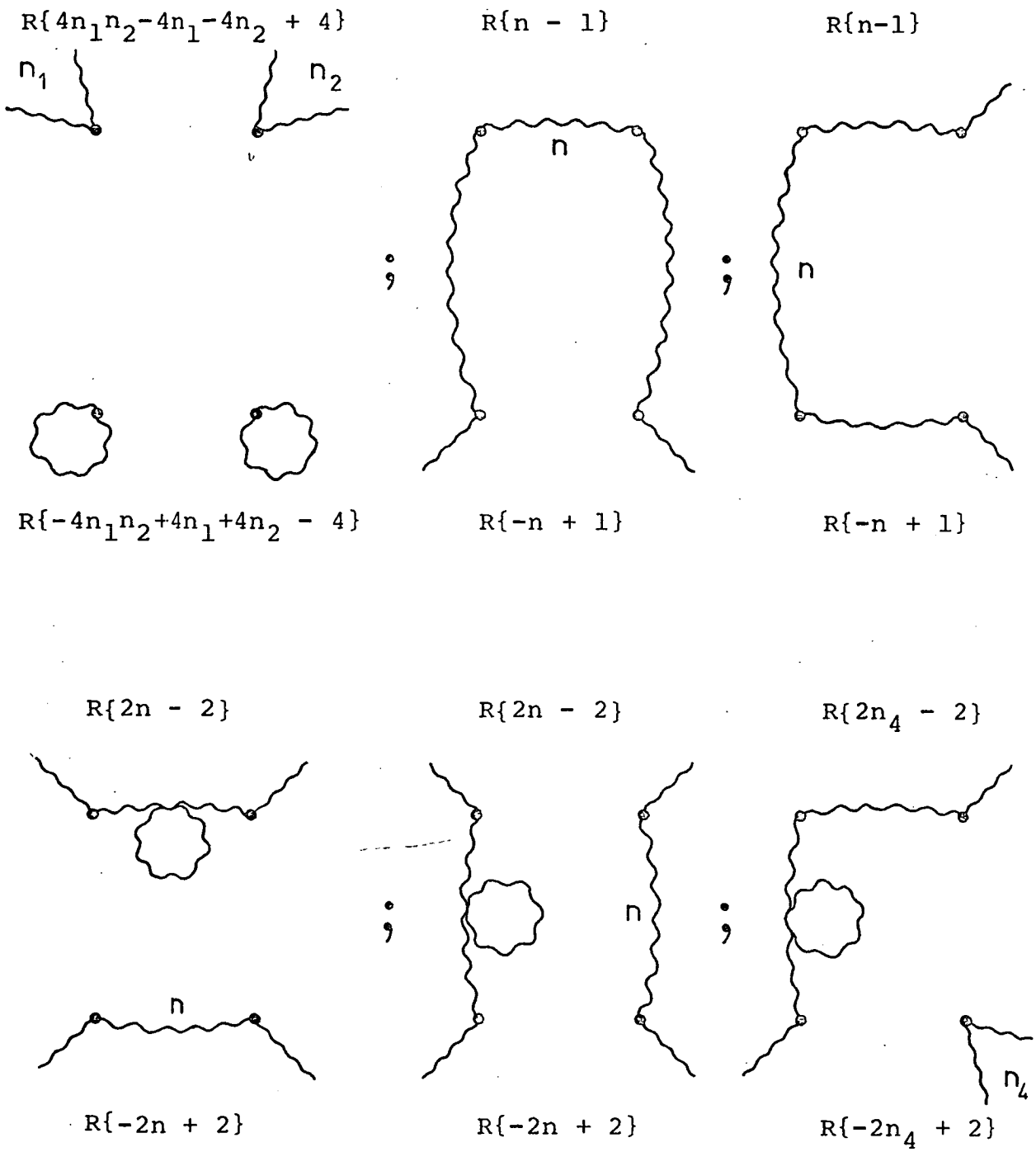
$R\{-2n\,n_4 + 2n + 2n_4 - 2\}$

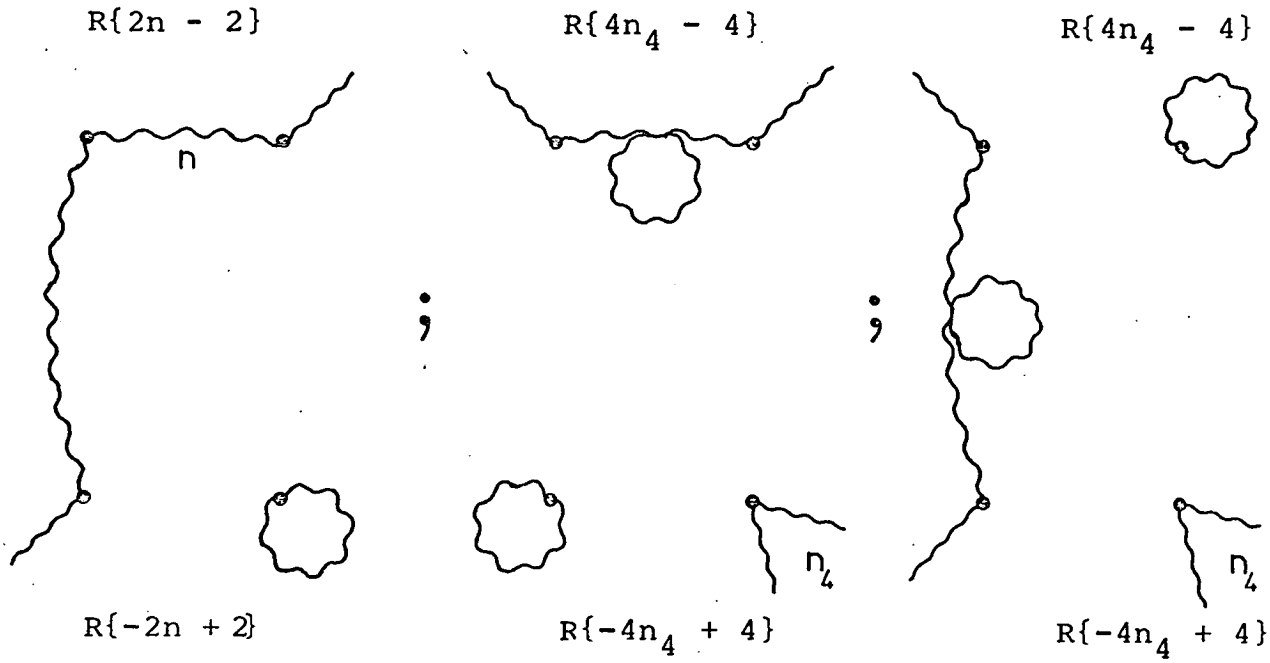
$R\{4n_2 n_4 - 4n_2 - 4n_4 + 4\}$

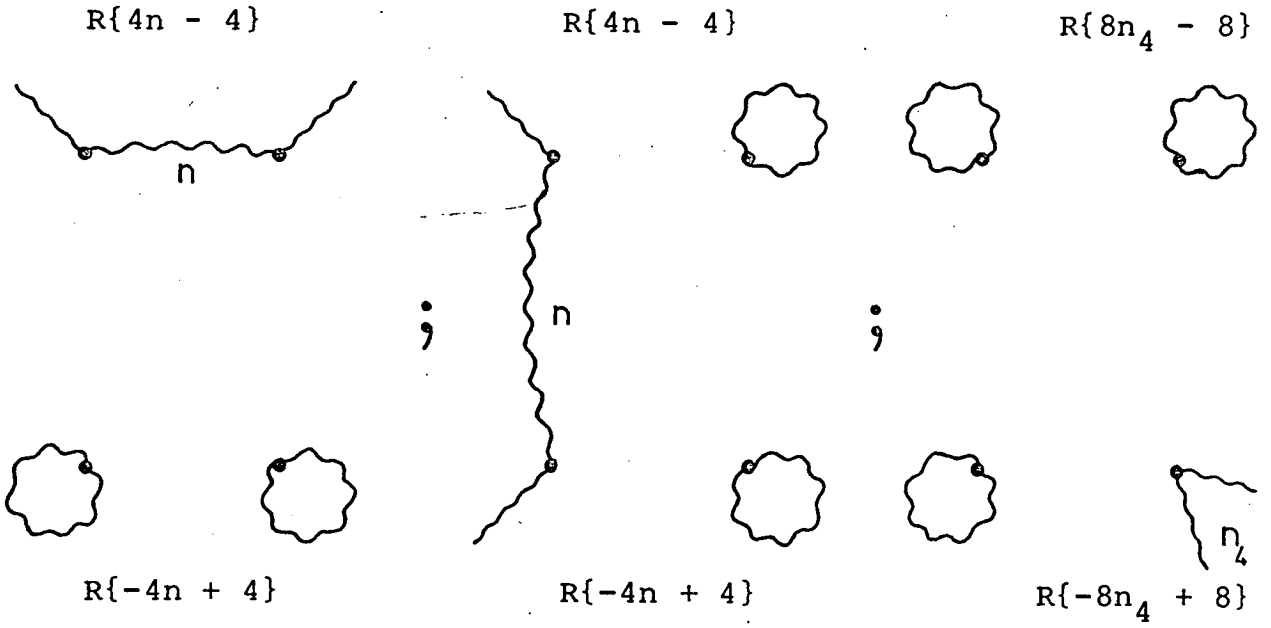


;

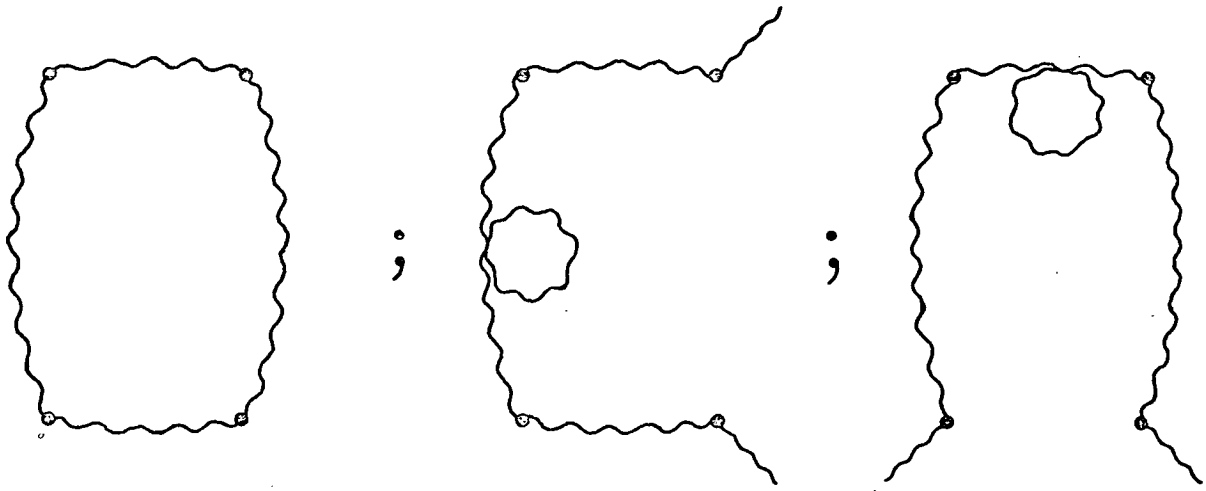
$R\{-4n_2 n_4 + 4n_2 + 4n_4 - 4\}$


R{4n1n2-4n1-4n2 + 4}
n1
n2
R{n - 1}
n
R{n-1}
n
R{-4n1n2+4n1+4n2 - 4}
R{-n + 1}
R{-n + 1}
R{2n - 2}
n
R{2n - 2}
n
R{2n4 - 2}
n4
R{-2n + 2}
R{-2n + 2}
R{-2n4 + 2}


R{2n - 2}
n
R{-2n + 2}
;
R{4n4 - 4}
R{-4n4 + 4}
;
R{4n4 - 4}
n4
R{-4n4 + 4}

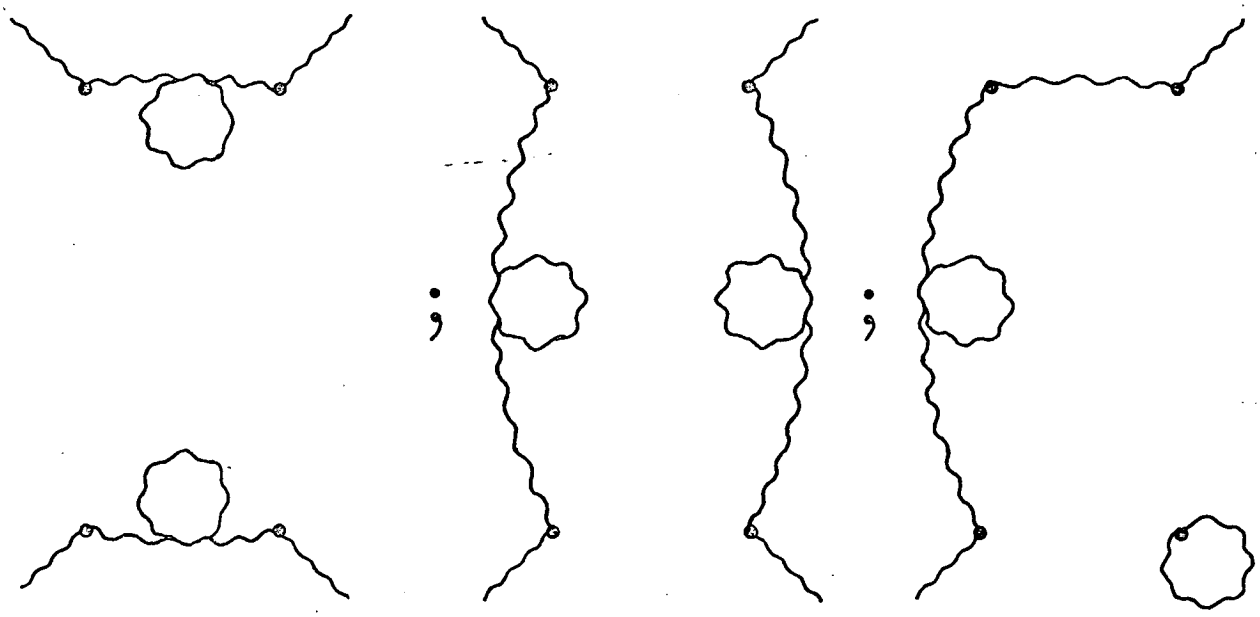

R{4n - 4}
n
R{-4n + 4}
;
R{4n - 4}
n
R{-4n + 4}
;
R{8n4 - 8}
n4
R{-8n4 + 8}

R{2} R{2} R{2}

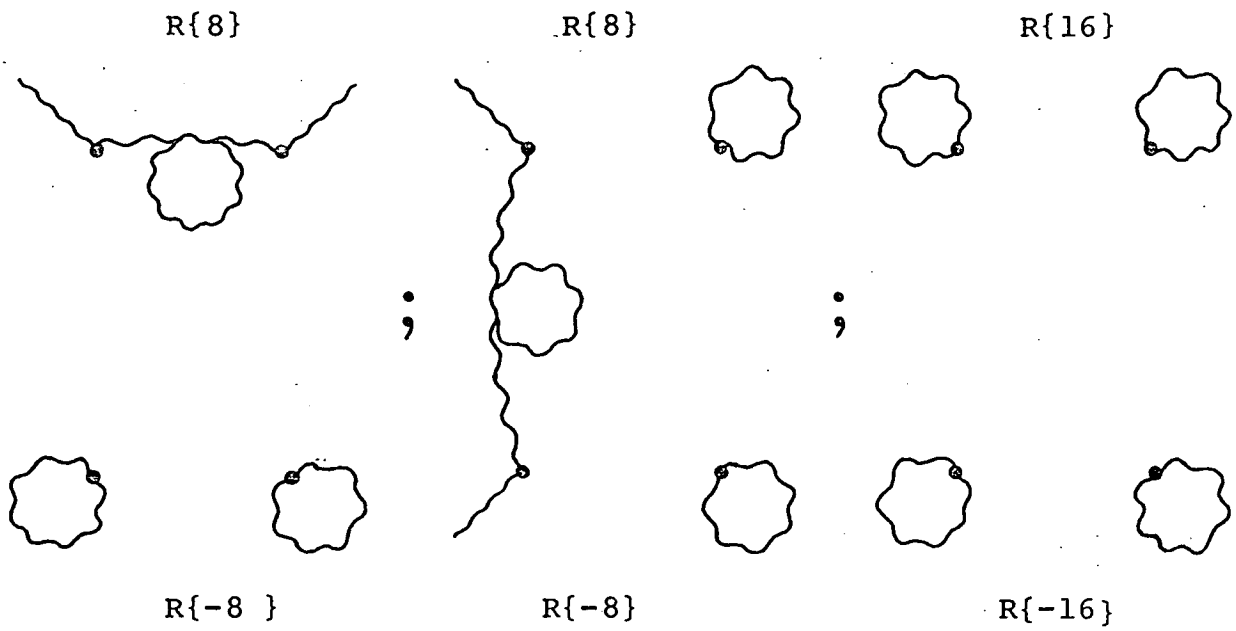


R{-2} R{-2} R{-2}

R{4} R{4} R{4}

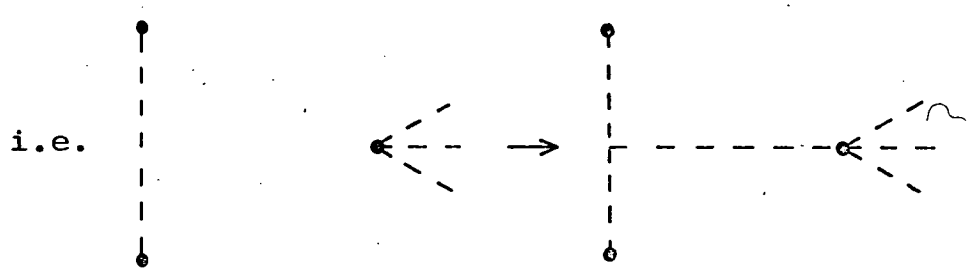


R{-4} R{-4} R{-4}

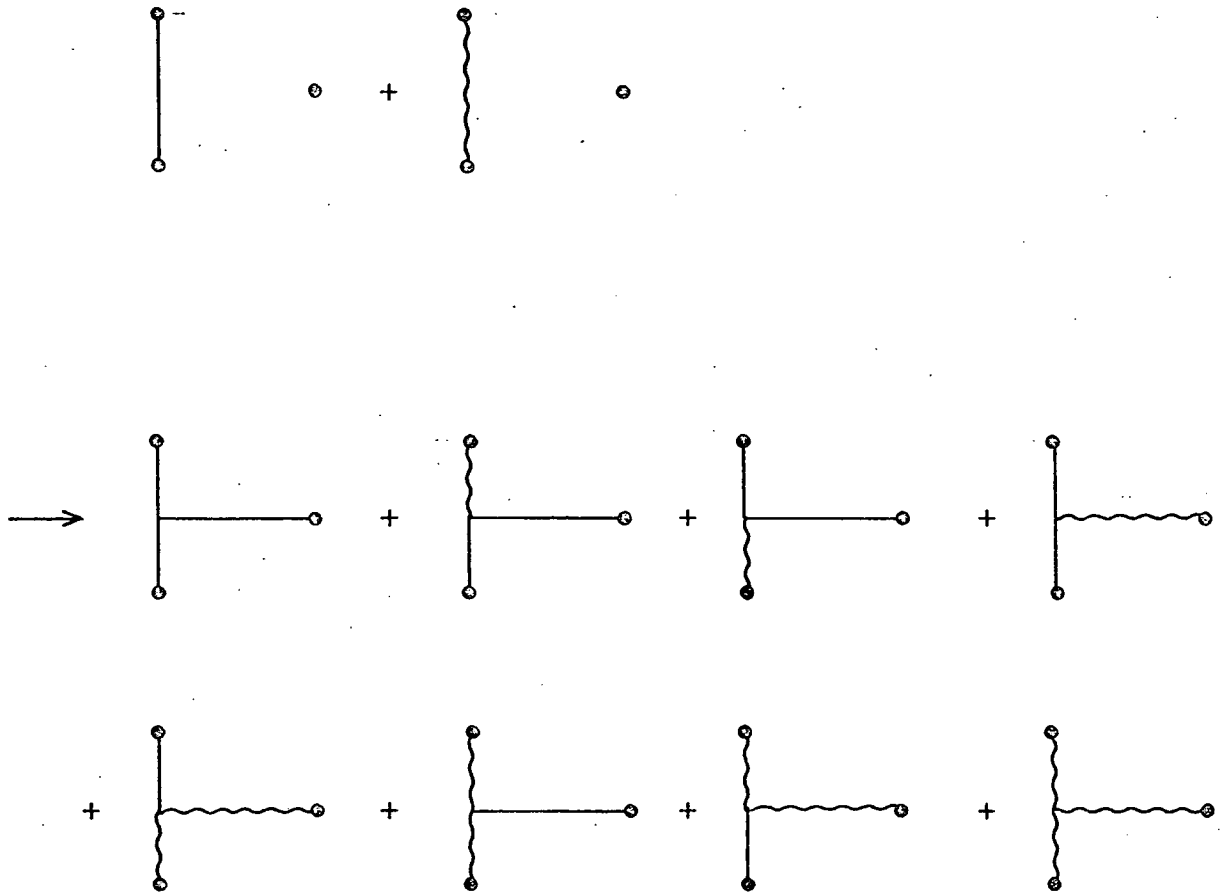


In each possibility the inclusion of the extra line only causes the duplication factor to acquire a multiple (-1). Hence the total duplication factor only changes by that amount.

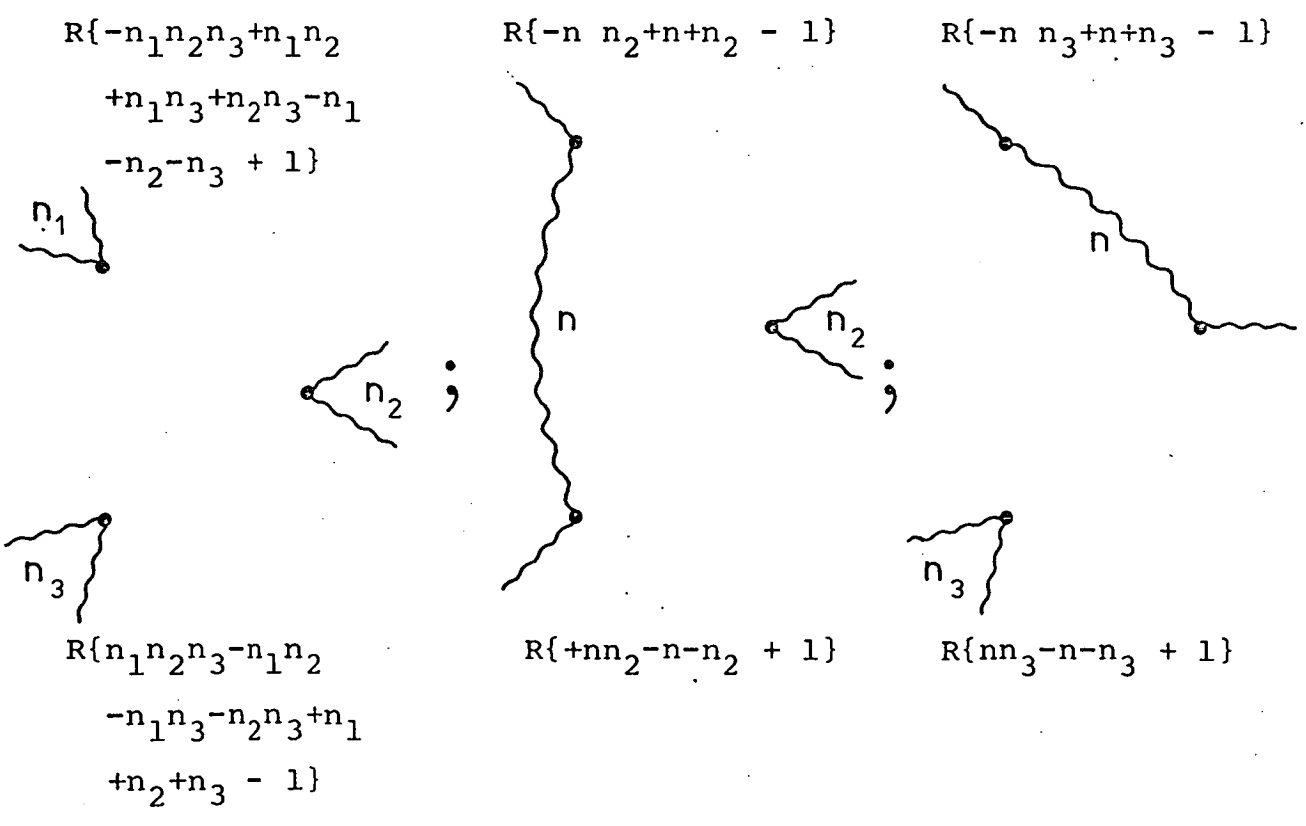
Another insertion could be that of a line between a vertex and an independent line

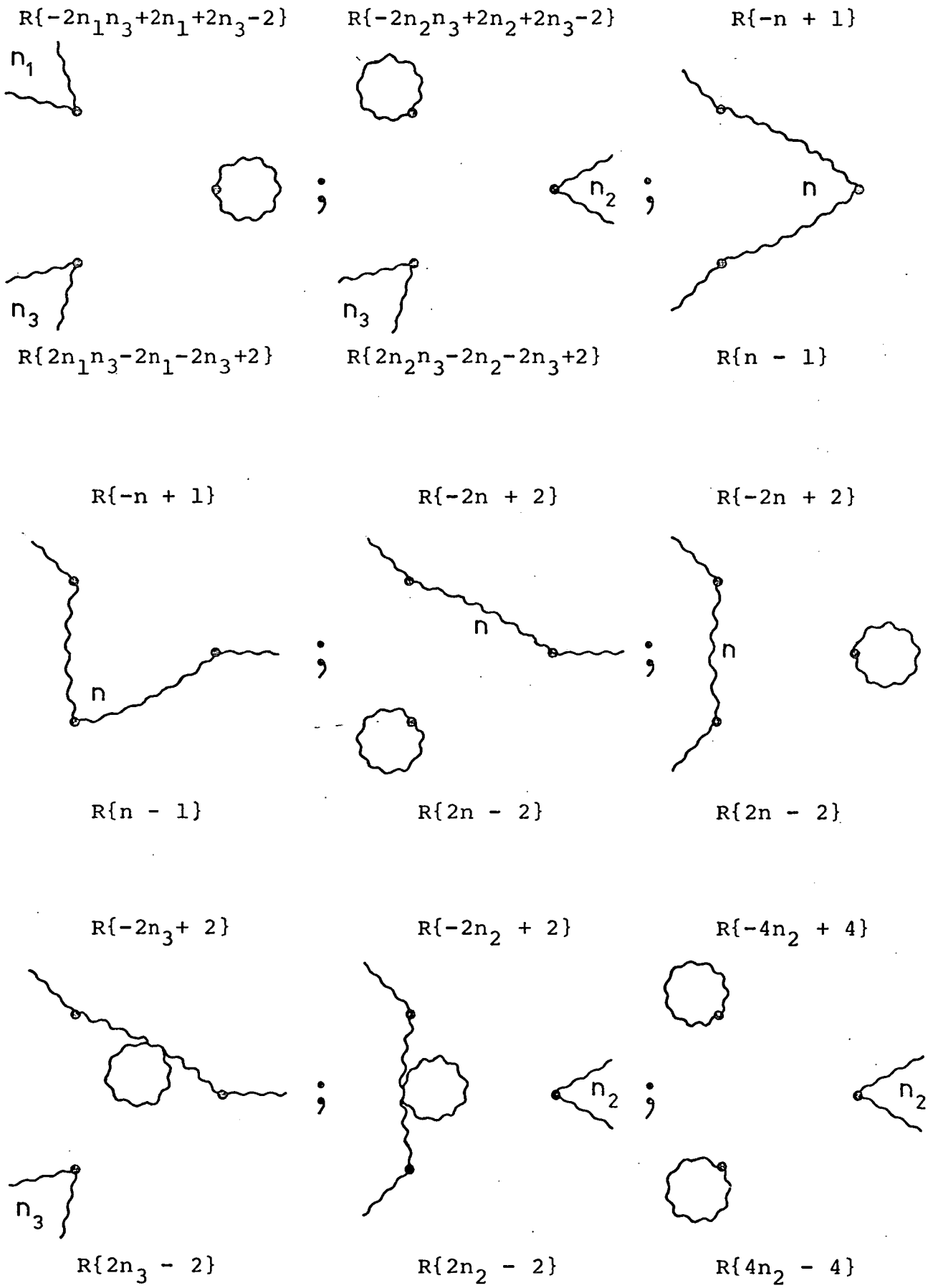


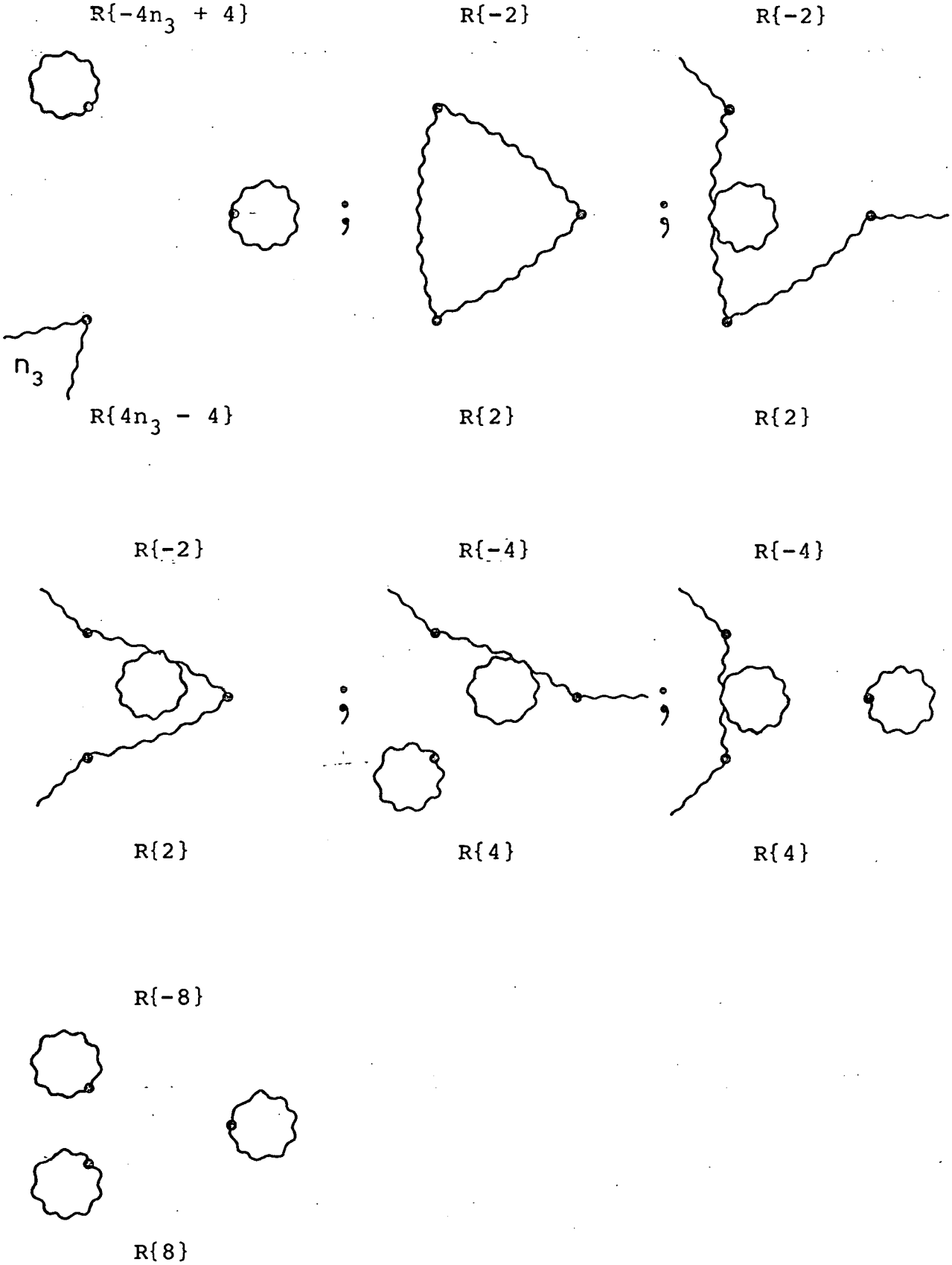
for which the corresponding replacement in calculating the duplication factor is



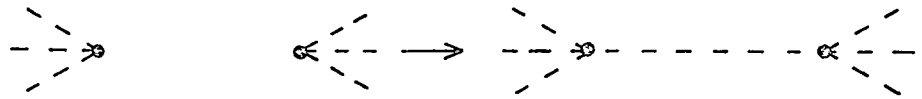
in the situations

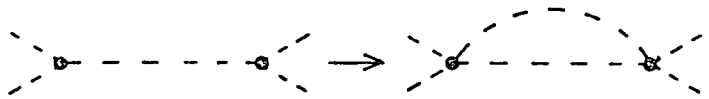



R{-2n1n3+2n1+2n3-2}
n1
n3
R{2n1n3-2n1-2n3+2}
;
R{-2n2n3+2n2+2n3-2}
n2
n3
R{2n2n3-2n2-2n3+2}
;
R{-n + 1}
n
R{n - 1}
R{-n + 1}
n
R{n - 1}
;
R{-2n + 2}
n
R{2n - 2}
;
R{-2n + 2}
n
R{2n - 2}
R{-2n3+ 2}
n3
R{2n3 - 2}
;
R{-2n2 + 2}
n2
R{2n2 - 2}
;
R{-4n2 + 4}
n2
R{4n2 - 4}

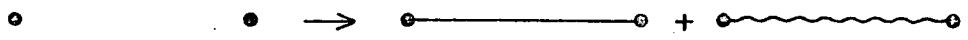

R{-4n3 + 4}
R{-2}
R{-2}
n3
R{4n3 - 4}
R{2}
R{2}
R{-2}
R{-4}
R{-4}
R{2}
R{4}
R{4}
R{-8}
R{8}

The inclusion of a line between two vertices

i.e. 

which obviously includes 

requires the replacement



in the following situations:

$R\{n_1n_2-n_1-n_2 + 1\}$ $R\{n - 1\}$ $R\{2\}$

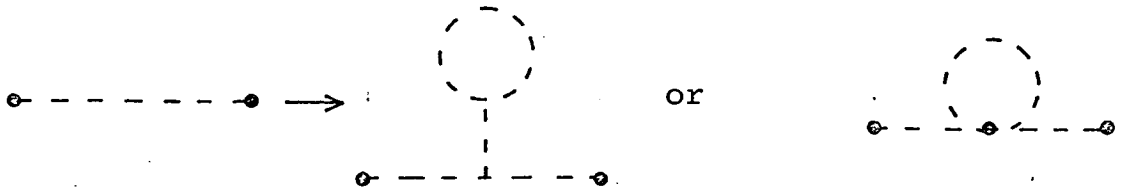
$R\{-n_1n_2+n_1+n_2 - 1\}$ $R\{-n + 1\}$ $R\{-2\}$

$R\{2n_2 - 2\}$ $R\{4\}$ $R\{2\}$

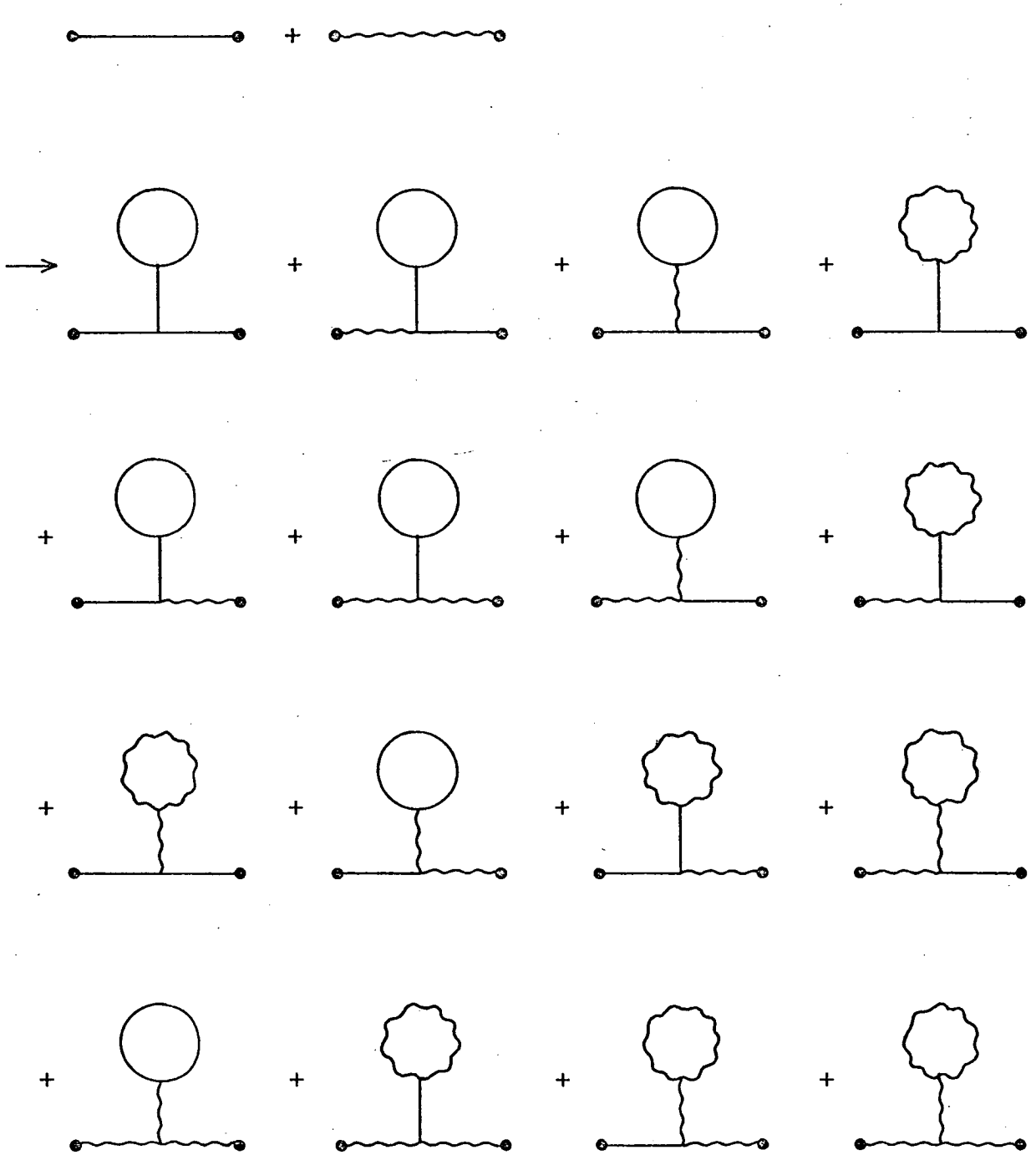
$R\{-2n_2 + 2\}$ $R\{-4\}$ $R\{-2\}$

In both the above methods of adding a line to a configuration the duplication factor only changes by (-1).

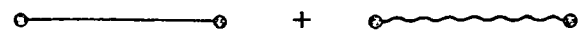
We now consider the attachment of an additional loop. It may be attached to a line between independent vertices by

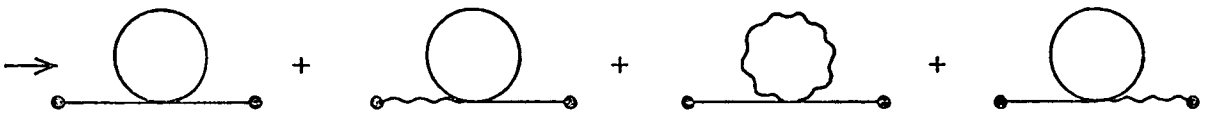


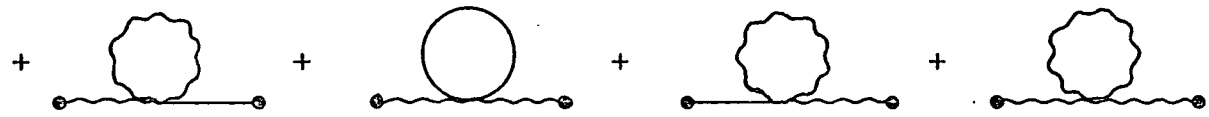
The replacements for the first are



The replacements for the second are







In the following situations the expression below is the duplication factor with either of the above replacements (they are the same in all cases).

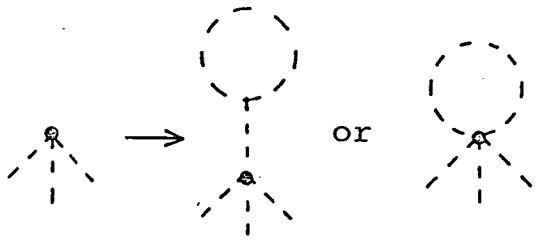
$R\{-n_1n_2+n_1+n_2-1\}$ $R\{-2n_2+2\}$ $R\{-n+1\}$

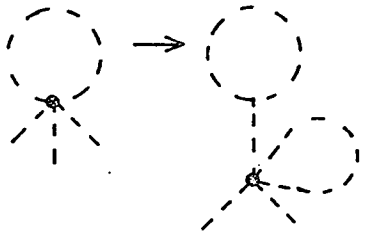
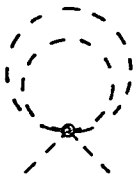
$R\{n_1n_2-n_1-n_2+1\}$ $R\{2n_2-2\}$ $R\{n-1\}$

$R\{-2\}$ $R\{-4\}$ $R\{-2\}$

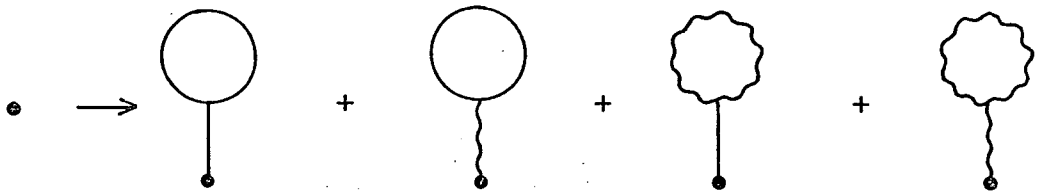
$R\{2\}$ $R\{4\}$ $R\{2\}$

The loop may also be added to a vertex, i.e.

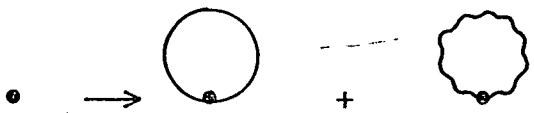


which includes  or  etc.

The replacements for the first are

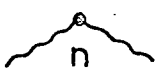


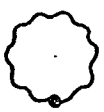
and for the second



For these there are only two possibilities:

R{n - 1} R{2}

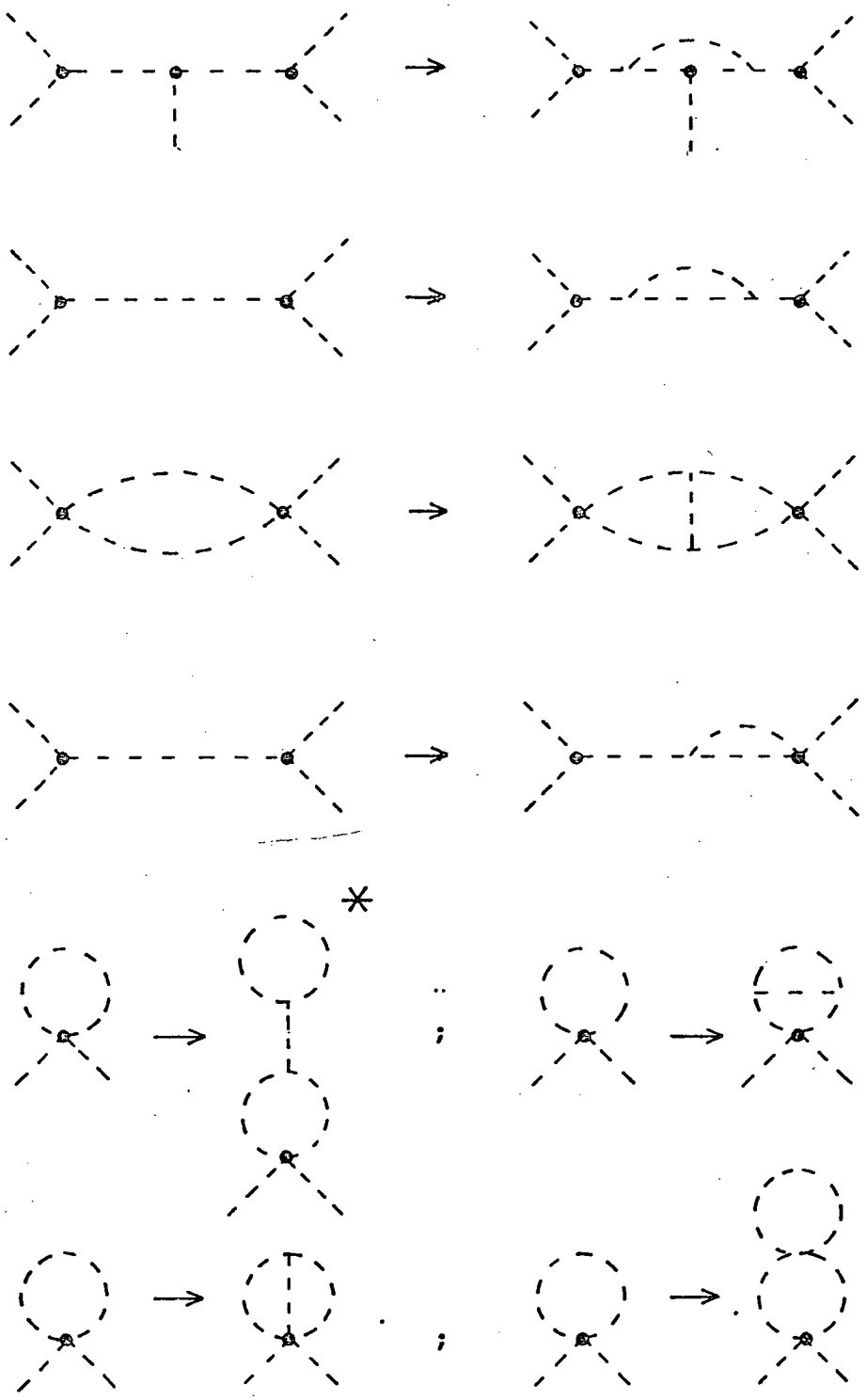


and 

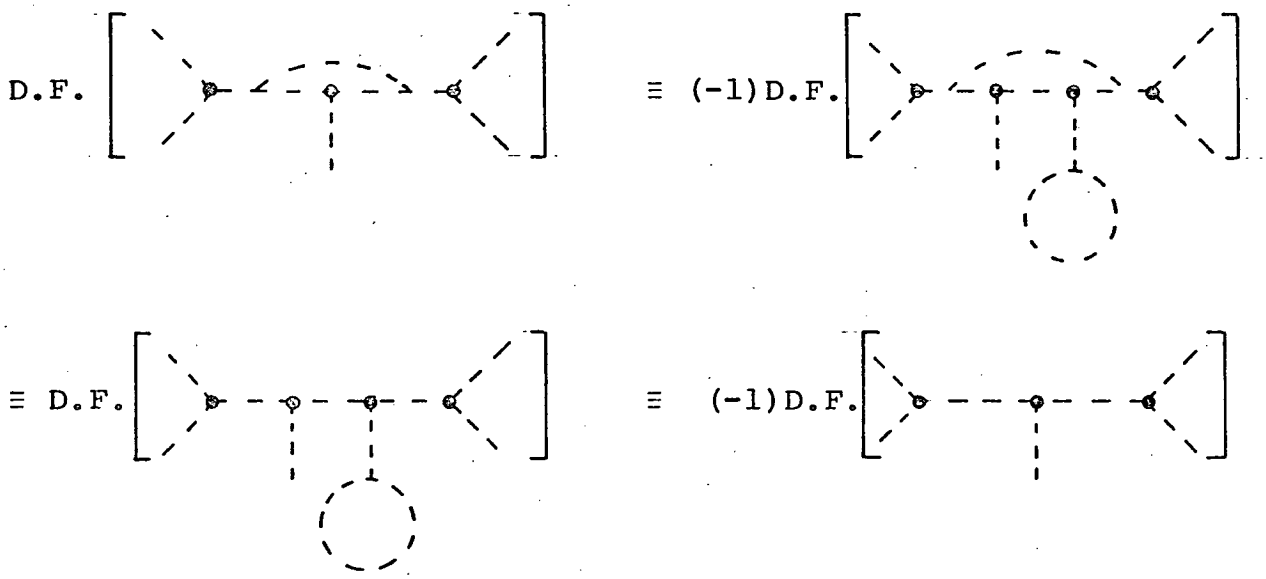
R{-n + 1} R{-2}

Again for the addition of a loop the only change in the duplication factor is the multiple (-1).

The types of additions still to be considered are:



These may be verified directly or deduced from additions already demonstrated. (The addition * is required to justify those following it). For example



In each addition the duplication factor (D.F.) only changes by (-1).

Hence any insertion of a line or loop only causes the duplication factor for the amended diagram to be (-1) times the duplication factor for the original.

Appendix E : The Canonical Quantization of (6.4)

We wish to canonically quantize the free Hamiltonian

$$\mathcal{H}_{FREE} \equiv \tfrac{1}{2}\underline{\Pi}^k_W \cdot \underline{\Pi}^k_W - \frac{1}{\alpha M}\underline{\Pi}^k_W \cdot \partial^k \underline{\Pi}_\chi + \tfrac{1}{2}\partial^j \underline{W}^k \cdot (\partial^j \underline{W}^k - \partial^k \underline{W}^j) + \tfrac{1}{2}M^2 \underline{W}^k \cdot \underline{W}^k$$

$$- \tfrac{1}{2}\frac{1}{\alpha^2}\underline{\Pi}_\chi \cdot \underline{\Pi}_\chi - \tfrac{1}{2}\beta \underline{\chi} \cdot \underline{\chi} - \alpha M \underline{W}^k \cdot \partial^k \underline{\chi} \qquad (E.2)$$

with the interactions

$$\mathcal{H}_{INT} \equiv -g(\partial^j \underline{W}^k) \cdot (\underline{W}^j \times \underline{W}^k) + \frac{g^2}{4}(\underline{W}^j \times \underline{W}^k) \cdot (\underline{W}^j \times \underline{W}^k)$$

$$- \frac{g}{\alpha M}\underline{\Pi}^k_W \cdot (\underline{\Pi}_\chi \times \underline{W}^k) \; . \qquad (E.2)$$

It is necessary to split $\mathcal{H}_{FREE}$ such that

$$\mathcal{H}_{FREE} = H_o + H_1 \qquad (E.3)$$

with

$$H_o \equiv \tfrac{1}{2}\underline{\Pi}^k_W \cdot \underline{\Pi}^k_W - \frac{1}{\alpha M}\underline{\Pi}^k_W \cdot \partial^k \underline{\Pi}_\chi + \tfrac{1}{2}\partial^j \underline{W}^k \cdot (\partial^j \underline{W}^k - \partial^k \underline{W}^j) + \tfrac{1}{2}M^2 \underline{W}^k \cdot \underline{W}^k$$

$$- \tfrac{1}{2}\frac{1}{\alpha^2}\underline{\Pi}_\chi \cdot \underline{\Pi}_\chi - \tfrac{1}{2}\alpha^2 M^2 \underline{\chi} \cdot \underline{\chi} - \alpha M \underline{W}^k \cdot \partial^k \underline{\chi} \qquad (E.4)$$

$$H_1 \equiv -\tfrac{1}{2}(\beta - \alpha^2 M^2)\underline{\chi} \cdot \underline{\chi} \qquad (E.5)$$

H_1 is added to the interaction Hamiltonian (E.2) and we quantize H_o.

The equations of motion for H_o are

$$\dot{\underline{W}}^k = \underline{\Pi}^k_W - \frac{1}{\alpha M}\partial^k \underline{\Pi}_\chi \qquad (E.6)$$

$$\dot{\underline{\chi}} = -\frac{1}{\alpha^2}\underline{\Pi}_\chi + \frac{1}{\alpha M}\partial^k \underline{\Pi}^k_W \qquad (E.7)$$

$$\dot{\underline{\Pi}}_\chi = \alpha^2 M^2 \underline{\chi} - \alpha M \partial^k \underline{W}^k \qquad (E.8)$$

$$\dot{\underline{\Pi}}^k_W = -M^2 \underline{W}^k + \alpha M \partial^k \underline{\chi} + (\partial^2 \delta^{k\ell} - \partial^k \partial^\ell)\underline{W}^\ell \qquad (E.9)$$

We choose the operator expansion

$$W_a^k(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3\underline{q}}{2\omega_{\underline{q}}} \{e^{-iq.x} A_a^k(q) + e^{iq.x} A_a^{+k}(q)\} \quad (E.10)$$

$$\Pi_{W_a}^k(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3\underline{q}}{2\omega_{\underline{q}}} \{e^{-iq.x} B_a^k(q) + e^{iq.x} B_a^{+k}(q)\} \quad (E.11)$$

$$\chi_a(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3\underline{q}}{2\omega_{\underline{q}}} \{e^{-iq.x} C_a(q) + e^{iq.x} C_a^{+}(q)\} \quad (E.12)$$

$$\Pi_{\chi_a}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3\underline{q}}{2\omega_{\underline{q}}} \{e^{-iq.x} D_a(q) + e^{iq.x} D_a^{+}(q)\} \quad (E.13)$$

where $q_o = \omega_{\underline{q}} = \sqrt{\underline{q}^2 + M^2}$. For consistency the expansions (E.9) - (E.13) must satisfy the equations of motion (E.6) - (E.9). To do so we require

$$B_a^k(q) = -i\omega_{\underline{q}} A_a^k(q) + \frac{iq^k}{\alpha M} D_a(q) \quad (E.14)$$

$$C_a(q) = \frac{-i\omega_{\underline{q}}}{\alpha^2 M^2} D_a(q) + \frac{1}{\alpha M} iq^k A_a^k(q) \quad (E.15)$$

The normal quantization can now be carried out requiring

$$[\Pi_{W_a}^j(x), W_b^k(y)] \Big|_{x_o=y_o} = -i \delta^{jk} \delta_{ab} \delta^3(\underline{x} - \underline{y}) \quad (E.16)$$

$$[\Pi_{\chi_a}(x), \chi_b(y)] \Big|_{x_o=y_o} = -i \delta_{ab} \delta^3(\underline{x} - \underline{y}) \quad (E.17)$$

and all other equal time commutation relations zero. To satisfy the equal time commutation relation the operators must obey the commutation relations

$$[A_a^j(q), A_b^{+k}(q')] = 2\omega_{\underline{q}} \delta^{jk} \delta_{ab} \delta^3(\underline{q} - \underline{q}') \quad (E.18)$$

$$[D_a(q), D_b^{+}(q')] = -2\alpha^2 M^2 \omega_{\underline{q}} \delta_{ab} \delta^3(\underline{q} - \underline{q}') \quad (E.19)$$

with all others zero.

The complete commutation relations for the fields with (E.18) and (E.19) are

$$\left[W_a^k(x),\ W_b^\ell(y)\right] = \frac{1}{(2\pi)^3}\int d^4q\ e^{-iq.(x-y)}\,\delta(q^2-M^2)\,\varepsilon(q)\,\delta^{\ell k}\delta_{ab}$$

$$\left[\Pi_{W_a}^k(x),\ W_b^\ell(y)\right] = \frac{1}{(2\pi)^3}\int d^4q\ e^{-iq.(x-y)}\,\delta(q^2-M^2)\,\varepsilon(q)\,(-iq_o)\,\delta^{\ell k}\delta_{ab}$$

$$\left[\Pi_{W_a}^k(x),\ \Pi_{W_b}^\ell(y)\right] = \frac{1}{(2\pi)^3}\int d^4q\ e^{-iq.(x-y)}\,\delta(q^2-M^2)\,\varepsilon(q)\,(q_o^2\,\delta^{k\ell} - q^k q^\ell)\,\delta_{ab}$$

$$\left[\Pi_\chi^a(x),\ \Pi_\chi^b(y)\right] = \frac{1}{(2\pi)^3}\int d^4q\ e^{-iq(x-y)}\,\delta(q^2-M^2)\,\varepsilon(q)\,(-\alpha^2M^2)\,\delta_{ab}$$

$$\left[\Pi_\chi^a(x),\ \chi^b(y)\right] = \frac{1}{(2\pi)^3}\int d^4q\ e^{-iq(x-y)}\,\delta(q^2-M^2)\,\varepsilon(q)\,(-iq^o)\,\delta_{ab}$$

$$\left[\chi^a(x),\ \chi^b(y)\right] = \frac{1}{(2\pi)^3}\int d^4q\ e^{-iq(x-y)}\,\delta(q^2-M^2)\,\varepsilon(q)\,\left(\frac{1}{\alpha^2}\right)\delta_{ab}$$

$$\left[W_a^k(x),\ \chi^b(y)\right] = \frac{1}{(2\pi)^3}\int d^4q\ e^{-iq(x-y)}\,\delta(q^2-M^2)\,\varepsilon(q)\,\frac{(-iq^k)}{\alpha M}\,\delta_{ab}$$

$$\left[\Pi_{W_a}^k(x),\Pi_\chi^b(y)\right] = \frac{1}{(2\pi)^3}\int d^4q\ e^{-iq(x-y)}\ (q^2-M^2)\,\varepsilon(q)\,(-\alpha M iq^k)\,\delta_{ab}$$

$$\left[W_a^k(x),\Pi_{\chi_b}(y)\right] = \left[\Pi_{W_a}^k(x),\ \chi_a(y)\right] = 0$$

These have the corresponding time ordered vacuum expectation values:-

$$\langle 0|T\{W_a^k(x)W_b^\ell(y)\}|0\rangle \equiv \frac{-1}{(2\pi)^4 i}\int d^4q\, e^{-iq(x-y)}\,\frac{\delta^{k\ell}}{q^2-M^2+i\varepsilon}\,\delta_{ab}$$

$$\langle 0|T\{\Pi_{W_a}^k(x)W_b^\ell(y)\}|0\rangle \equiv \frac{-1}{(2\pi)^4 i}\int d^4q\, e^{-iq(x-y)}\,\frac{-iq_o\delta^{k\ell}}{q^2-M^2+i\varepsilon}\,\delta_{ab}$$

$$\langle 0|T\{\Pi_{W_a}^k(x)\Pi_{W_b}^\ell(y)\}|0\rangle \equiv \frac{-1}{(2\pi)^4 i}\int d^4q\, e^{-iq(x-y)}\,\frac{q_o^2\delta^{k\ell}-q^kq^\ell}{q^2-M^2+i\varepsilon}\,\delta_{ab}$$

$$\langle 0|T\{\Pi_{\chi_a}(x)\Pi_{\chi_b}(y)\}|0\rangle \equiv \frac{-1}{(2\pi)^4 i}\int d^4q\, e^{-iq(x-y)}\,\frac{-\alpha^2M^2}{q^2-M^2+i\varepsilon}\,\delta_{ab}$$

$$\langle 0|T\{\Pi_{\chi_a}(x)\chi_b(y)\}|0\rangle \equiv \frac{-1}{(2\pi)^4 i}\int d^4q\, e^{-iq(x-y)}\,\frac{-iq_o}{q^2-M^2+i\varepsilon}\,\delta_{ab}$$

$$\langle 0|T\{\chi_a(x)\chi_b(y)\}|0\rangle \equiv \frac{-1}{(2\pi)^4 i}\int d^4q\, e^{-iq(x-y)}\,\frac{1/\alpha^2}{q^2-M+i\varepsilon}\,\delta_{ab}$$

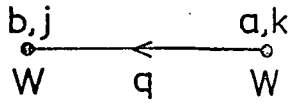
$$\langle 0|T\{W_a^k(x)\chi_b(y)\}|0\rangle \equiv \frac{-1}{(2\pi)^4 i}\int d^4q\, e^{-iq(x-y)}\,\frac{-iq^k/\alpha M}{q^2-M^2+i\varepsilon}\,\delta_{ab}$$

$$\langle 0|T\{\Pi_{W_a}^k(x)\Pi_{\chi_b}(x)\}|0\rangle \equiv \frac{-1}{(2\pi)^4 i}\int d^4q\, e^{-iq(x-y)}\,\frac{-i\alpha Mq^k}{q^2-M^2+i\varepsilon}\,\delta_{ab}$$

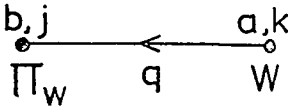
We thus have the propagators of Fig. E.1 which on absorbing the bilinear vertex of H_1

$$-(\beta-M^2\alpha^2)\delta_{ab}$$

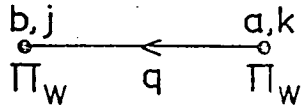
generate the Feynman rules of Fig. 42.



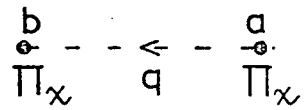
$$\frac{-\delta^{kj}}{q^2-M^2+i\varepsilon} \quad ;$$



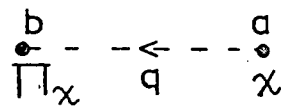
$$\frac{iq_o\delta^{kj}}{q^2-M^2+i\varepsilon} \quad ;$$



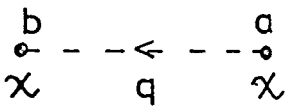
$$\frac{-q_o^2\delta^{kj}+q^kq^j}{q^2-M^2+i\varepsilon}\ \delta_{ab} \quad ;$$



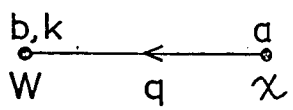
$$\frac{\alpha^2M^2}{q^2-M^2+i\varepsilon}\ \delta_{ab} \quad ;$$



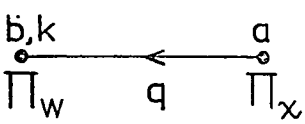
$$\frac{iq_o}{q^2-M^2+i\varepsilon}\ \delta_{ab} \quad ;$$



$$\frac{-1/\alpha^2}{q^2-M^2+i\varepsilon}\ \delta_{ab} \quad ;$$



$$\frac{iq/\alpha M}{q^2-M^2+i\varepsilon}\ \delta_{ab} \quad ;$$



$$\frac{i\alpha Mq^k}{q^2-M^2+i\varepsilon}\ \delta_{ab} \quad .$$

FIG E1

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68. The notation is that of (58).

ABSTRACT OF THESIS

Name of Candidate WILLIAM E. LEITHEAD

Address DEPARTMENT OF PHYSICS, JAMES CLERK MAXWELL BLDG., MAYFIELD ROAD.

Degree DOCTOR OF PHILOSOPHY *Date* SEPTEMBER, 1976

Title of Thesis WARD IDENTITIES AND VECTOR-BOSON FIELD THEORIES

The object of this thesis is to investigate, using Ward identities, two aspects of vector-boson field theories.

The first is to examine, in detail, how the renormalisation counter-terms for gauge field theories are accommodated without destroying the symmetry or corresponding Ward identities. In Chapter One the wave function and coupling constant renormalisations are studied and in Chapter Two the mass renormalisations. The conclusion is that, although there is complete freedom of choice of subtraction points for the wave function and coupling constant, the mass renormalisations are not so clear and may be restricted depending on the theory.

The second topic is the massive Yang-Mills Lagrangian. In Chapter Three, we investigate the Ward identities, and their implications, for the tree approximation. In Chapter Four, we develope the Ward identities to all orders. The massive Yang-Mills Lagrangian is shown to be identical to a Lagrangian with transverse vector-boson propagators and a compensating scalar Lagrangian with an infinite series of interactions. The Lagrangian is identical to that of Boulware which was developed in the path integral formalism. The Ward identity approach we use is shown to be equivalent to Veltman's in Chapter Five. Furthermore, it is shown that it is the S-matrices which are identical. In Chapter Six, other possible equivalent formalisms of the massive Yang-Mills Lagrangian are

Use other side if necessary.

investigated. The formalism of Hsu & Sudarshan is shown to be for mixed spin-one spin-zero fields and not pure spin-one fields as required. Finally a formulation is discussed which, in conjunction with the dimensional regularisation scheme of 't. Hooft and Veltman, generates the identical S-matrix from Feynman rules which are re-normalisable according to power-counting.