NORM ESTIMATES FOR FUNCTIONS OF SEMIGROUPS OF OPERATORS.

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ABSTRACT

In this thesis we study functions of generators of uniformly bounded semigroups of operators on a Hilbert space, $\mathcal{H}$.

A recent paper of V.V.Peller considers polynomials in an operator T whose iterates $\{T^n\}_{n\geq 0}$ form a uniformly bounded discrete semigroup. Upper bounds for the norm of a polynomial in T are obtained and both a representation of the Besov space $B^0_{\infty,1}$ in $B(\mathcal{H})$ and a von-Neumann-type inequality follow.

After studying Peller's methods and results, we use a similar approach to study polynomials in two commuting power-bounded operators and obtain comparable norm estimates. These results require a characterisation of Hankel operators on $H^2(\mathbb{T}^2)$, the Hardy space of functions on the two-dimensional torus. We show that the class of such Hankel operators is isometrically isomorphic to the dual of a quotient of Banach spaces of operator-valued functions, and we investigate conditions for a generalisation of Nehari's Theorem.

Finally, in Chapter 5 we show that analogues of Peller's results hold for functions of the infinitesimal generator of a uniformly bounded, strongly continuous semigroup of operators. This requires a characterisation of the dual space of the injective tensor product $L^1(\mathbb{R}^+)\check{\otimes}L^1(\mathbb{R}^+)$ using conditional expectation operators, and an identification of the class of Hankel-type integral operator kernels with a subspace of the dual of $H^1(\mathbb{R})$.

CONTENTS

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PREFACE

The aim of this thesis is to find upper bounds for the norms of functions of operators on Hilbert spaces. We are interested in the functions of generators of uniformly bounded semigroups of operators, and we show that the results and the methods for studying functions of discrete semigroups have analogous counterparts in the continuous semigroup case. This particular objective is motivated by the recent results of V.V.Peller, producing norm estimates for functions of the generator of a discrete semigroup of operators.

The methods employed throughout the thesis reflect the mixture of matricial, Banach space theory and the theory of function spaces, these elements being connected by harmonic analysis in general and a theorem of Nehari in particular. Consequently, our intermediate results, such as those concerning Hankel-type 4-dimensional arrays and integral operators, may be considered independently of the results for functions of semigroups of operators.

In Chapter 1 we present preliminary results in the theory of tensor products, Schur multipliers and Hankel operators. Although these results are well-known, we give proofs wherever the method of proof is to be generalised later.

The results of von-Neumann [vN] and Sz-Nagy [N2], for contractions and power-bounded invertibles respectively, are described in Chapter 2 when we consider the motivation for Peller's work. We give a full account of the results for power-bounded operators [Pl], hoping to make this complex paper understandable and to prepare the reader for the sequences of results in Chapters 4 and 5.

A result of Ando [An] provides a von-Neumann-type inequality for polynomials of pairs of commuting contractions on a Hilbert space, so it is natural to consider polynomials of commuting power-bounded operators. The results of [P1] have obvious analogues in this case and we show that the operator norm of such operators are bounded above by an expression growing logarithmically in the degree of the polynomial.

The principle difficulty in obtaining this expression is the current absence of a Nehari-type result for Hankel operators on the Hardy space $H^2(\mathbb{T}^2)$ of functions on the two-dimensional torus. Thus, in the form of Chapter 3, we devote a substantial part of the thesis to developing the characterisation of such operators necessary for the application of Peller's methods to polynomials of commuting power-bounded operators. Using Page's characterisation of vectorial Hankel operators ([PA]) and by proving the identification (apparently known to Sarason, [SA]) between a space of trace-class-valued functions, $H^1(\mathcal{C}^1)$ and a quotient of an L^∞ space of bounded operator-valued functions, we show that the class of Hankel operators on $H^2(\mathbb{T}^2)$ is isometrically isomorphic to the dual of a quotient of $H^1(\mathcal{C}^1)$. Although this result does not answer the question of whether an analogue of Nehari's Theorem exists for Hankel operators on $H^2(\mathbb{T}^2)$, we are able to give some new information about the relationship between the class of Hankel operators on $H^2(\mathbb{T}^2)$ and certain tensor products of H^1 spaces.

In Chapter 4 we use the principle results of the previous chapter to obtain our required upper bound on the norm of a polynomial of two commuting power-bounded operators.

Finally, in Chapter 5 we show that analogous results hold for functions of A, the infinitesimal generator of a uniformly bounded, strongly continuous, one-parameter semigroup of operators. The transition from functions on the finite measure spaces $\mathbb{T}$ and $\mathbb{T}^2$ to those functions defined on $\mathbb{R}$ causes more technical difficulty than encountered in previous chapters. For example, in 5.3 we use conditional expectation operators to obtain a suitable description of the dual of the injective tensor product $L^1(\mathbb{R}^+)\check{\otimes}L^1(\mathbb{R}^+)$, and in 5.4 we show that this dual space is isomorphic to the Banach space of pointwise multipliers of integral operator kernels. Whilst a characterisation of the Hankel-type kernels of integral operators is known ([WI]), we give a description of such kernels in terms of linear functionals on $H^1(\mathbb{R})$ and use this result to estimate the norm of f(A) in terms of the norm of f in a space of functions on $\mathbb{R}$. Using techniques similar to those in Chapters 4 and 5, we conclude the thesis by showing that for a Schwartz class function, f, with Fourier transform supported on [1/N,N] for some $N \geq 2$,

$$\| f(A) \| \leq \text{const.} \log N \| f \|_{\infty}.$$

Chapter 1 Introduction.

In this chapter we shall establish some basic notation and definitions, and introduce three themes from the theory of Banach spaces which will be encountered throughout the thesis. The results are mostly well-known and we shall omit proofs whenever appropriate references are available.

Section 1.1 Preliminary Definitions.

Throughout the thesis we shall denote by $\mathcal{H}$, an arbitrary, complex, separable, infinite-dimensional Hilbert space. The norm on $\mathcal{H}$ is denoted by $\| \cdot \|_{\mathcal{H}}$ and the inner product by $(\cdot , \cdot)$. If X is a linear space with norm $\| \cdot \|_X$ then B(X) is the algebra of bounded linear operators on X. If Y is also a linear space, with norm $\| \cdot \|_Y$ then B(X;Y) is the normed linear space of bounded operators from X into Y. In the special case that $Y = \mathbb{C}$, B(X;Y) is the Banach space of bounded linear functionals on X which we denote by X^*. If $x \in X$ and $f \in X^*$ we shall frequently write the value f(x) of f at x as a pair $< x,f >_{x,x^*}$ or simply $< x,f >$.

Furthermore, we shall use the standard notation for the sets of reals ($\mathbb{R}$), non-negative reals ($\mathbb{R}^+$), complex numbers ($\mathbb{C}$), integers ($\mathbb{Z}$), non-negative integers ($\mathbb{Z}^+$) and natural numbers ($\mathbb{N}$). The open unit disc $\{z \in \mathbb{C} : |z| < 1\}$ is denoted by $\mathbb{D}$ and its boundary, the unit circle in $\mathbb{C}$, is denoted by $\mathbb{T}$. Finally, for each $n \in \mathbb{Z}^+$, the set $\{0,1,\ldots,n\}$ will be denoted by $\mathbb{Z}_n$.

The Lebesgue and Hardy Spaces.

If $(\Omega,\mathcal{A},\mu)$ is a σ-finite measure space and if $1 \le p \le \infty$ then $L^p(\Omega,\mathcal{A},\mu)$ (or simply $L^p(\Omega)$) will denote the well-known Lebesgue space of (equivalence classes of μ-a.e. equal) p-integrable, μ-measurable complex-valued functions on Ω with the norm $\| f \|_p = \left(\int_\Omega | f(\omega) |^p \, d\mu(\omega)\right)^{1/p}$. For the case of $p = \infty$, $L^\infty(\Omega,\mathcal{A},\mu)$ will denote the Banach space of (equivalence classes of μ-a.e. equal) μ-essentially bounded measurable complex-valued functions on Ω with the norm $\| f \|_\infty = \operatorname{ess\,sup} \{ |f(\omega)| : \omega \in \Omega \}$. Furthermore, in the special case that Ω is a subset of $\mathbb{Z}$ with the usual counting measure then $L^p(\Omega)$ $(1 \le p < \infty)$ is the sequence space

$$l^p(\Omega) = \left\{ \{\alpha_n\}_{n\in\Omega} : \sum_{n\in\Omega} |\alpha_n|^p < \infty \right\}.$$

Much of our work is concerned with functions on the unit circle $\mathbb{T}$. We denote by m the normalised Lebesgue measure on $\mathbb{T}$. Then for $f \in L^p(\mathbb{T})$ $(1 \le p \le \infty)$, $\hat{f}$ is the Fourier transform of f ([CON,I.5.8]) and the Hardy space $H^p(\mathbb{T})$ is the subset of $L^p(\mathbb{T})$ consisting of those $f \in L^p(\mathbb{T})$ with $\hat{f}(n) = 0$ for all $n < 0$. Note also that we shall use $H^\infty_0(\mathbb{T})$ to denote the subset of $L^\infty(\mathbb{T})$ consisting of those $f \in L^\infty(\mathbb{T})$ with $\hat{f}(n) = 0$ for all $n \le 0$.

The Hilbert space $H^2(\mathbb{T})$ is of particular interest, and it will be assumed throughout that $H^2(\mathbb{T})$ contains the orthonormal basis $\{e_n\}_{n\in\mathbb{Z}^+}$, where for each $n \in \mathbb{Z}$ and $e^{i\theta} \in \mathbb{T}$,

$$e_n(e^{i\theta}) = e^{in\theta}.$$

Note however that in Chapter 5, where confusion is unlikely, we use e_n to denote the function

$$e_n(x) = e^{inx}$$

on $\mathbb{R}$. Then $\{e_n\}_{n\in\mathbb{Z}}$ (respectively $\{e_n\}_{n\in\mathbb{Z}^+}$) is an orthonormal basis for the Hilbert space $L^2(\mathbb{R})$ (respectively, $H^2(\mathbb{R})$).

Occasionally we shall work with $H^p(\mathbb{D})$, the Hardy space of analytic functions on $\mathbb{D}$ for which the function $f_r(e^{i\theta}) = f(re^{i\theta})$ on $\mathbb{T}$ is in $L^p(\mathbb{T})$ for every $0 < r < 1$ and $\sup_{0<r<1} \| f_r \|_p < \infty$. $H^p(\mathbb{D})$ is normed by $\| f \|_p = \sup_{0<r<1} \| f_r \|_p$. Moreover, if $f \in H^p(\mathbb{D})$ then the non-tangential limits of f :

$$f(e^{i\theta}) = \lim_{z\to e^{i\theta}} f(z)$$

exist for a.e. $e^{i\theta} \in \mathbb{T}$ and define a p-integrable function on $\mathbb{T}$. This correspondence allows us to identify $H^p(\mathbb{D})$ with $H^p(\mathbb{T})$ ([HO]).

We shall frequently use the well-known characterisation of the duals of H^p spaces. Indeed, for $1 \le p < \infty$, $H^p(\mathbb{T})^*$ is isometrically isomorphic to the quotient $L^q(\mathbb{T})/H^q_0(\mathbb{T})$ where $\frac{1}{p} + \frac{1}{q} = 1$. The corresponding pairing, for $f \in H^p(\mathbb{T})$ and $g \in L^q(\mathbb{T})$ is

$$< f,g + H^q_0(\mathbb{T}) > = \int_0^{2\pi} f(e^{i\theta})g(e^{i\theta})\, dm(\theta). \qquad (*)$$

When f or g is a polynomial we have

$$< f,g + H^q_0(\mathbb{T}) > = \sum_n \hat{f}(n)\hat{g}(-n).$$

In particular, we note that $H^1(\mathbb{T})^*$ is isometrically isomorphic to $L^\infty(\mathbb{T})/H^\infty_0(\mathbb{T})$ with respect to (*) when $f \in H^1(\mathbb{T})$, $g \in L^\infty(\mathbb{T})$ and $q = \infty$. To avoid confusion we make the following notation formally.

1.1.1 Notation. We shall denote by BMOA the Banach space of analytic functions on $\mathbb{D}$ with power series $\phi(z) = \sum_n \hat{\phi}(n)z^n$ such that $\hat{\phi}(n) = \hat{f}(-n)$ for some $f \in L^\infty(\mathbb{T})$ and all $n \in \mathbb{Z}^+$, normed by

$$\| \phi \|_{BMOA} = \inf \{ \| f \|_\infty : f \in L^\infty(\mathbb{T}),\ \hat{f}(-n) = \hat{\phi}(n) \text{ for all } n \in \mathbb{Z}^+\}.$$

Then BMOA is isometrically isomorphic to the dual of $H^1(\mathbb{D})$ with respect to the pairing

$$< f,\phi > = \lim_{r\to 1} \int_0^{2\pi} f(re^{i\theta})\phi(re^{-i\theta})\, dm(\theta)$$

when $f \in H^1(\mathbb{D})$ and $\phi \in$ BMOA, and

$$< f,\phi > = \sum_n \hat{f}(n)\hat{\phi}(n)$$

when f,ϕ are polynomials on $\mathbb{D}$.

The continuous functions on a Hausdorff space, X, will be denoted by C(X). We are mainly interested in $C(\mathbb{T})$ which, regarded as a normed linear subspace of $L^\infty(\mathbb{T})$, is connected with the predual of $H^1(\mathbb{T})$. Thus we denote by $A_0(\mathbb{T})$ the subspace of $H_0^\infty(\mathbb{T})$ given by

$$A_0(\mathbb{T}) = \left\{ \begin{array}{l} f : f \in C(\mathbb{T}),\ f(e^{i\theta}) = \lim\limits_{z\to e^{i\theta}} \tilde{f}(z) \text{ for some analytic} \\ \qquad\qquad\qquad\qquad\qquad \text{function } \tilde{f} \text{ on } \mathbb{D} \text{ with } \tilde{f}(0) = 0 \end{array} \right\}.$$

Then $A_0(\mathbb{T})$ is a closed subspace of $C(\mathbb{T})$ and the dual space of the quotient $C(\mathbb{T})/A_0(\mathbb{T})$ is isometrically isomorphic to $H^1(\mathbb{T})$ by the natural pairing. For the corresponding functions on $\mathbb{D}$ we shall use the following notation for the predual of $H^1(\mathbb{D})$.

<u>1.1.2 Notation.</u> We shall denote by VMOA the Banach space of analytic functions on $\mathbb{D}$ with power series $\psi(z) = \sum_n \hat{\psi}(n)z^n$ such that $\hat{\psi}(n) = \hat{f}(-n)$ for some $f \in C(\mathbb{T})$ and all $n \in \mathbb{Z}^+$, normed by

$$\| \psi \|_{VMOA} = \inf \{ \| f \|_\infty : f \in C(\mathbb{T}),\ \hat{f}(-n) = \hat{\psi}(n) \text{ for all } n \in \mathbb{Z}^+\}.$$

It is important to note that the use of the notation BMOA and VMOA is for convenience only and does not imply that our results use either the famous identification between the dual of real-valued $H^1(\mathbb{T})$ and the Banach space of real-valued functions of bounded mean oscillation on $\mathbb{T}$ or its corollaries.

Section 1.2 Tensor Product Spaces.

Let X and Y be Banach spaces with norms $\| \cdot \|_X$ and $\| \cdot \|_Y$ respectively. If $x \in X$ and $y \in Y$ then we regard the tensor product $x \otimes y$ as a bilinear functional on $X^* \times Y^*$. For $f \in X^*$, $g \in Y^*$ we have $(x \otimes y)(f,g) = \langle x,f \rangle \langle y,g \rangle$.

The collection of all finite sums of tensors is denoted by $X \otimes Y$, the algebraic tensor product of X,Y. There are many possible norms for the linear space $X \otimes Y$ and of those possible, the following two norms will be used extensively.

1.2.1 Definition. a) The <u>injective tensor norm</u> $\| \alpha \|_{\check{\otimes}}$ of $\alpha \in X \otimes Y$ is the norm of α as a bilinear functional on $X^* \times Y^*$. If $\alpha = \sum_{k=0}^{n} x_k \otimes y_k$ for some $x_k \in X$, $y_k \in Y$ $(0 \le k \le n)$ and $n \in \mathbb{Z}^+$ then

$$\| \alpha \|_{\check{\otimes}} = \sup \left\{ \left| \sum_{k=0}^{n} \langle x_k, f \rangle \langle y_k, g \rangle \right| : \| f \|_{X^*}, \| g \|_{Y^*} \le 1 \right\}.$$

The <u>injective tensor product</u> of X and Y is the completion of $X \otimes Y$ with respect to $\| \cdot \|_{\check{\otimes}}$ and is denoted by $X \check{\otimes} Y$.

b) The <u>projective tensor norm</u> $\| \alpha \|_{\hat{\otimes}}$ of $\alpha \in X \otimes Y$ is given by

$$\| \alpha \|_{\hat{\otimes}} = \inf \left\{ \sum_{k=0}^{n} \| x_k \|_X \| y_k \|_Y : \alpha = \sum_{k=0}^{n} x_k \otimes y_k \text{ for some } x_k \in X, \; y_k \in Y \; (0 \le k \le n), \; n \in \mathbb{Z}^+ \right\}$$

The <u>projective tensor product</u> of X,Y is the completion of $X \otimes Y$ with respect to $\| \cdot \|_{\hat{\otimes}}$ and is denoted by $X \hat{\otimes} Y$.

<u>Remark.</u> We note that any $\alpha \in X \hat{\otimes} Y$ may be written as an infinite sum, $\alpha = \sum_{k \ge 0} x_k \otimes y_k$ $(x_k \in X, \; y_k \in Y, \; k \ge 0)$ and that

$$\| \alpha \|_{\hat{\otimes}} = \inf \left\{ \sum_{k\geq 0} \| x_k \|_X \| y_k \|_Y \ : \ \alpha = \sum_{k\geq 0} x_k \otimes y_k \text{ for some } x_k \in X, \ y_k \in Y \ (k \geq 0). \right\}$$

For further details we refer the reader to [BD, pp 232-235].

Tensor Products of Sequence Spaces.

If $n \geq 1$ and $1 \leq p,q \leq \infty$ then any tensor α in $l^p(\mathbb{Z}^{+n})\check{\otimes}l^p(\mathbb{Z}^{+n})$ can be regarded as a 2n-dimensional array of scalars. More precisely, we define for each $\underline{k} \in \mathbb{Z}^{+n}$ the n-dimensional array $e_{\underline{k}}$ by $e_{\underline{k}}(\underline{l}) = \delta_{\underline{k},\underline{l}}$, where $\delta_{\underline{k},\underline{l}}$ is the (generalised) Kronecker delta function

$$\delta_{\underline{k},\underline{l}} = \begin{cases} 1 & : \ \underline{k} = \underline{l} \in \mathbb{Z}^{+n} \\ 0 & : \ \text{otherwise.} \end{cases}$$

Each co-ordinate functional $e^*_{\underline{k}}(\underline{x}) = x(\underline{k})$ for $x \in l^p(\mathbb{Z}^{+n})$, is in $l^p(\mathbb{Z}^{+n})^*$ so we put

$$\alpha_{\underline{k},\underline{l}} = \alpha(e^*_{\underline{k}}, e^*_{\underline{l}}) \qquad (\underline{k},\underline{l} \in \mathbb{Z}^{+n}).$$

When $\alpha \in l^p(\mathbb{Z}^{+n})\check{\otimes}l^q(\mathbb{Z}^{+n})$ then we put

$$\alpha_{\underline{k},\underline{l}} = \lim_{n\to\infty} \alpha_n(e^*_{\underline{k}}, e^*_{\underline{l}})$$

for some Cauchy sequence $\{\alpha_n\}_{n\geq o}$ converging to α in $l^p(\mathbb{Z}^{+n})\check{\otimes}l^q(\mathbb{Z}^{+n})$.

When $\beta \in l^p(\mathbb{Z}^{+n})\hat{\otimes}l^q(\mathbb{Z}^{+n})$ we can express β as an infinite sum $\sum_{j\geq 0} x_j \otimes y_j$ with $x_j \in l^p(\mathbb{Z}^{+n})$ and $y \in l^q(\mathbb{Z}^{+n})$ (for $j \geq 0$). We then put

$$\beta_{\underline{k},\underline{l}} = \sum_{j\geq 0} x_j(\underline{k}) y_j(\underline{l}) \qquad (\underline{k},\underline{l} \in \mathbb{Z}^{+n})$$

The series converges absolutely for each $\underline{k},\underline{l} \in \mathbb{Z}^{+n}$ and uniquely defines a 2n-dimensional array associated with β.

1.2.2 Notation. a) If $n \geq 1$ and $N \geq 0$ then whenever $\beta = \{\beta_{\underline{k},\underline{l}}\}_{\underline{k},\underline{l}\in\mathbb{Z}^{+n}}$ is a 2n-dimensional array of scalars, we will denote by $P_N^{(n)}\beta$ the following truncated array :

$$(P_N^{(n)}\beta)_{\underline{k},\underline{l}} = \begin{cases} \beta_{\underline{k},\underline{l}} & : \ \underline{k},\underline{l} \in \mathbb{Z}_N^n \\ 0 & : \ \text{otherwise.} \end{cases}$$

b) For $n \geq 1$ we will denote by V_n^2 the collection of all 2n-dimensional arrays, β, for which

$$\sup_{N\geq 0} \| P_N^{(n)}\beta \|_{l^\infty(\mathbb{Z}^{+n})\hat{\otimes}\, l^\infty(\mathbb{Z}^{+n})} < \infty.$$

For $\beta \in V_n^2$ this supremum is denoted by $\| \beta \|_{V_n^2}$.

It is clear that V_n^2 is a linear space over $\mathbb{C}$ and that $\| \cdot \|_{V_n^2}$ is a norm on V_n^2. Moreover, it is easy to show that V_n^2 is complete with respect to $\| \cdot \|_{V_n^2}$. We shall need the following characterisation of $[l^1(\mathbb{Z}^{+n})\check{\otimes} l^1(\mathbb{Z}^{+n})]^*$.

1.2.3 Theorem. If $\beta \in V_n^2$ then the pairing

$$< \alpha,\beta > = \sum \ \alpha_{i,j}\ \beta_{i,j} \tag{$*$}$$

defines a bounded linear functional on $l^1(\mathbb{Z}^{+n})\check{\otimes} l^1(\mathbb{Z}^{+n})$. With respect to this pairing, V_n^2 is isometrically isomorphic to the dual of $l^1(\mathbb{Z}^{+n})\check{\otimes} l^1(\mathbb{Z}^{+n})$.

The proof of 1.2.3 will require the following description of the dual of the finite-dimensional space $l_N^1(\mathbb{Z}^{+n})\check{\otimes} l_N^1(\mathbb{Z}^{+n})$.

1.2.4 Lemma. If $\beta \in l^\infty(\mathbb{Z}_N^n)\hat{\otimes} l^\infty(\mathbb{Z}_N^n)$ then the pairing $(*)$ of 1.2.3 defines a continuous linear functional on $l^1(\mathbb{Z}_N^n)\check{\otimes} l^1(\mathbb{Z}_N^n)$. With respect to this pairing, $l^\infty(\mathbb{Z}_N^n)\hat{\otimes} l^\infty(\mathbb{Z}_N^n)$ is isometrically isomorphic to $[l^1(\mathbb{Z}_N^n)\check{\otimes} l^1(\mathbb{Z}_N^n)]^*$.

<u>Proof.</u> The pairing $\alpha,\beta \rightsquigarrow \langle \alpha,\beta \rangle$ is well-defined and bilinear. Moreover, if $\beta = \sum_{k=0}^{m} x_k \otimes y_k$ for some x_k, $y_k \in l^\infty(\mathbb{Z}_N^n)$ $(k=0,1,\ldots,m)$ and if $\alpha \in l^1(\mathbb{Z}_N^n)\check{\otimes} l^1(\mathbb{Z}_N^n)$ then

$$|\langle \alpha,\beta \rangle| = \Big| \sum_{k=0}^{m} \sum_{\underline{i},\underline{j}\in\mathbb{Z}_N^n} \alpha_{\underline{i},\underline{j}}\, x_k(\underline{i}) y_k(\underline{j}) \Big|$$

$$\leq \| \alpha \|_{\check{\otimes}} \sum_{k=0}^{m} \| x_k \|_\infty \| y_k \|_\infty .$$

Thus, $|\langle \alpha,\beta \rangle| \leq \| \alpha \|_{\check{\otimes}} \| \beta \|_{\hat{\otimes}}$.

Conversely, if $F \in [l^1(\mathbb{Z}_N^n)\otimes l^1(\mathbb{Z}_N^n)]^*$ and $\underline{i},\underline{j} \in \mathbb{Z}_N^n$, we put $\beta(\underline{i},\underline{j}) = F(e_{\underline{i}} \otimes e_{\underline{j}})$. Then $F\alpha = \langle \alpha,\beta \rangle$ for every $\alpha \in l^1(\mathbb{Z}_N^n)\otimes l^1(\mathbb{Z}_N^n)$.

Finally,

$$\| \alpha \|_{\check{\otimes}} = \sup \left\{ \Big| \sum_{\underline{i},\underline{j}} \alpha_{\underline{i},\underline{j}} x(\underline{i}) y(\underline{j}) \Big| \;:\; x,y \in l^\infty(\mathbb{Z}_N^n), \| x \|_\infty \| y \|_\infty \leq 1 \right\}$$

$$= \sup \left\{ | \langle \alpha, x\otimes y \rangle | \;:\; x,y \in l^\infty(\mathbb{Z}_N^n), \| x \|_\infty \| y \|_\infty \leq 1 \right\}$$

$$\leq \sup \left\{ | \langle \alpha,\gamma \rangle | \;:\; \gamma \in l^\infty(\mathbb{Z}_N^n)\hat{\otimes} l^\infty(\mathbb{Z}_N^n), \| \gamma \|_{\hat{\otimes}} \leq 1 \right\}.$$

It now follows that $\| \beta \|_{\hat{\otimes}} \leq \| F \|$.

<u>Proof of 1.2.3.</u> Suppose that $\beta \in l^\infty(\mathbb{Z}^{+n})\otimes l^\infty(\mathbb{Z}^{+n})$ and that $\beta = \sum_{k=0}^{m} \phi_k \otimes \psi_k$ for some $\phi_k, \psi_k \in l^\infty(\mathbb{Z}^{+m})$ $(0 \leq k \leq m)$ with $\sum \| \phi_k \|_\infty \| \psi_k \|_\infty < \| \beta \|_{\hat{\otimes}} + \varepsilon$. Then if $\alpha \in l^1(\mathbb{Z}^{+n})\check{\otimes} l^1(\mathbb{Z}^{+n})$

$$|\langle \alpha,\beta \rangle| = \Big|\sum_{k=0}^{m} \sum_{\underline{i},\underline{j}\in\mathbb{Z}^{+n}} \alpha_{\underline{i},\underline{j}}\phi_k(\underline{i})\psi_k(\underline{j})\Big|$$

$$\le \| \alpha \|_{\check{\otimes}} \sum_{k=0}^{m} \| \phi_k \|_{\infty} \| \psi_k \|_{\infty}$$

$$< \| \alpha \|_{\check{\otimes}} \left(\| \beta \|_{\hat{\otimes}} + \varepsilon \right).$$

Since this holds for any $\varepsilon > 0$ we have

$$|\langle \alpha,\beta \rangle| \le \| \alpha \|_{\check{\otimes}} \ \| \beta \|_{\hat{\otimes}}.$$

Now if $\beta \in V_n^2$ and if $\alpha \in l^1(\mathbb{Z}^{+n})\check{\otimes} l^1(\mathbb{Z}^{+n})$

$$|\langle \alpha,\beta \rangle| = \lim_{N\to\infty} \Big| \sum_{\underline{i},\underline{j}\in\mathbb{Z}^{+n}} \alpha_{\underline{i},\underline{j}} \ \beta_{\underline{i},\underline{j}}\Big|$$

$$= \lim_{N\to\infty} |\langle \alpha, P_N^{(n)}\beta \rangle|$$

$$\le \lim_{N\to\infty} \| \alpha \|_{\check{\otimes}} \| P_N^{(n)}\beta \|_{\hat{\otimes}}$$

$$\le \| \alpha \|_{\check{\otimes}} \| \beta \|_{V_n^2}.$$

Also, if $F \in \left[l^1(\mathbb{Z}^{+n})\check{\otimes} l^1(\mathbb{Z}^{+n}) \right]^*$ we put

$$\beta = \{ F(e_{\underline{i}} \otimes e_{\underline{j}}) \}_{\underline{i},\underline{j}\in\mathbb{Z}^{+n}}$$

so that whenever $\alpha \in l^1(\mathbb{Z}^{+n})\check{\otimes} l^1(\mathbb{Z}^{+n})$

$$F\alpha = \langle \alpha,\beta \rangle.$$

Moreover, for $N > 0$, we have $P_N^{(n)}\beta \in l^{\infty}(\mathbb{Z}^{+n})\otimes l^{\infty}(\mathbb{Z}^{+n})$ and

$$\| P_N^{(n)}\beta \|_{\hat{\otimes}} \le \| P_N^{(n)}\beta \|_{l^{\infty}(\mathbb{Z}_N^n)\hat{\otimes} l^{\infty}(\mathbb{Z}_N^n)}$$

$$= \sup \left\{ |\langle \alpha, P_N^{(n)}\beta \rangle| \ : \alpha \in l^1(\mathbb{Z}_N^n)\otimes l^1(\mathbb{Z}_N^n) \text{ and } \| \alpha \|_{\check{\otimes}} \le 1 \right\}$$

(by 1.2.4)

$$= \sup \left\{ |F\alpha| \ : \alpha \in l^1(\mathbb{Z}_N^n)\otimes l^1(\mathbb{Z}_N^n) \text{ and } \| \alpha \|_{\check{\otimes}} \le 1 \right\}$$

$$\le \sup \left\{ |F\alpha| \ : \alpha \in l^1(\mathbb{Z}^{+n})\check{\otimes} l^1(\mathbb{Z}^{+n}) \text{ and } \| \alpha \|_{\check{\otimes}} \le 1 \right\}$$

$$= \| F \|.$$

So $\beta \in V_n^2$ with $\| \beta \|_{V_n^2} \le \| F \|$ and the proof is complete.

The result known as 'Grothendieck's Inequality' or 'the fundamental theorem of the metric theory of tensor products' ([G]) is of great importance to the study of Banach space geometry and is used extensively in this thesis. It has many different formulations and is stated here in the form given by Lindenstrauss and Pelczyński, [LP, Theorem 2.1].

<u>**1.2.5 Theorem.**</u> There exists a constant, K_G, such that if $\alpha \in l^1(\mathbb{Z}^{+n})\check{\otimes} l^1(\mathbb{Z}^{+n})$ has finitely many non-zero entries and if $\{x(n)\}_{n\in\mathbb{Z}^+}$, $\{y(n)\}_{n\in\mathbb{Z}^+}$ are bounded sequences in $\mathscr{H}$, then

$$\Big|\sum_{i,j} a_{i,j}(x(i),y(j))\Big| \le K_G \, \| \alpha \|_{\check{\otimes}} \sup_{n\in\mathbb{Z}^+} \| x(n) \|_{\mathscr{H}} \sup_{n\in\mathbb{Z}^+} \| y(n) \|_{\mathscr{H}} \, .$$

The following factorisation theorem is a well-known corollary of 1.2.5 and will be used in the sequel.

<u>**1.2.6 Corollary.**</u> Let T be a bounded linear operator from $l^{\infty}(\mathbb{Z}^+)$ into $l^1(\mathbb{Z}^+)$. Then there exists bounded linear operators $A : l^2(\mathbb{Z}^+) \longrightarrow l^1(\mathbb{Z}^+)$ and $B : l^{\infty}(\mathbb{Z}^+) \longrightarrow l^2(\mathbb{Z}^+)$ such that

$$T = AB \text{ and } \| A \| \, \| B \| \le K_G \, \| T \|.$$

<u>Proof.</u> [PI, Corollary 4.4, p 42].

<u>**Tensor Product Duals.**</u>

Finally, we note the general theorems characterising the dual spaces of the injective and projective tensor products of Banach spaces X and Y. For the injective tensor product $X\check{\otimes}Y$ we consider the closed unit balls U_{X^*}, U_{Y^*} of X^*, Y^* with respect to the weak*-

topologies on X^*, Y^* [CON,$\overline{V}$.1.1]. By Alaoglu's Theorem ([CON, $\overline{V}$.3]) U_{X^*} , U_{Y^*} are compact so a tensor $\alpha \in X\check{\otimes}Y$ can be considered as a continuous function on the compact Hausdorff space $U_{X^*} \times U_{Y^*}$. This isometric embedding of $X\check{\otimes}Y$ into $C(U_{X^*} \times U_{Y^*})$ suggests, correctly, that the bounded linear functionals on $X\check{\otimes}Y$ can be represented by integration against Borel measures on $U_{X^*} \times U_{Y^*}$.

The following theorem is due to Grothendieck and appears in [DU, p 231].

__1.2.7 Theorem.__ If μ is a regular Borel measure on $U_{X^*} \times U_{Y^*}$ with variation norm $|\mu|$ then define F_μ on simple tensors $x\otimes y$ in $X\otimes Y$ by

$$F_\mu(x\otimes y) = \int_{U_{X^*} \times U_{X^*}} < x,x^* >< y,y^* > d\mu(x^*,y^*).$$

Then F_μ extends to a continuous linear functional on $X\check{\otimes}Y$ with norm $|\mu|$. Moreover, every $F \in (X\check{\otimes}Y)^*$ has $F = F_\mu$ for some regular Borel measure μ on $U_{X^*} \times U_{Y^*}$ with $|\mu| = \| F \|$.

Secondly, we have the following characterisation of $(X\hat{\otimes}Y)^*$.

__1.2.8 Theorem.__ If $F \in B(X;Y^*)$ then define a linear functional f_F on simple tensors $x\otimes y \in X\otimes Y$ by

$$f_F(x\otimes y) = < y,Fx >.$$

Then f_F extends to a continuous linear functional on $X\hat{\otimes}Y$. Moreover, the mapping of F to f_F is an isometric isomorphism of $B(X;Y^*)$ onto $(X\hat{\otimes}Y)^*$.

__Proof.__ [BD, p 234].

Section 1.3 Hankel Operators on $H^2(\mathbb{T})$.

As we shall see in Chapter 2, the boundedness of Hankel operators on $H^2(\mathbb{T})$ plays a key role in Peller's method for estimating the norm of polynomials of a power-bounded operator. Consequently, much of this thesis is concerned with the boundedness of certain Hankel operators. In this section we briefly describe the well-known results concerning the boundedness of Hankel operators on $H^2(\mathbb{T})$.

1.3.1 Definition. A bounded operator T on $H^2(\mathbb{T})$ is a Hankel operator (with respect to the orthonormal basis $\{e_n\}_{n\in\mathbb{Z}^+}$) if there exists a sequence of scalars $\{\alpha_n\}_{n\in\mathbb{Z}^+}$ such that $(Te_n, e_m) = \alpha_{m+n}$ for all $m,n \in \mathbb{Z}^+$.

Remarks. 1. Our definition ensures that $T \in B(H^2(\mathbb{T}))$ is a Hankel operator if and only if its matrix (with respect to $\{e_n\}_{n\in\mathbb{Z}^+}$) has Hankel form.

2. It is easy to check that T is a Hankel operator if and only if T 'intertwines' the unilateral shift S on $H^2(\mathbb{T})$ and its adjoint : $S^*T = TS$. We use this approach in our definition of vectorial Hankel operators in Section 3.2.

3. It is clear that if $\{\alpha_n\}_{n\in\mathbb{Z}^+}$ is a sequence arising from a Hankel operator T then $\{\alpha_n\}_{n\in\mathbb{Z}^+}$ is square-summable and is therefore the sequence of Fourier Coefficients of some $H^2(\mathbb{T})$ function. The connection between sequences associated with Hankel operators and functions on the unit circle in $\mathbb{C}$ is made explicit by the fundamental theorem of Nehari. We shall prove Nehari's theorem using a lifting theorem of Sz-Nagy and Foias.

1.3.2 Notation. a) If $\phi \in L^{\infty}(\mathbb{T})$, we denote by M_{ϕ} the multiplication operator on $L^2(\mathbb{T})$ defined at $f \in L^2(\mathbb{T})$ by $M_{\phi}f = \phi f$.

b) We denote by J the 'flip' operator on $L^2(\mathbb{T})$ defined at $f \in L^2(\mathbb{T})$ by $(Jf)(e^{i\theta}) = f(e^{-i\theta})$ $(e^{i\theta} \in \mathbb{T})$.

c) For $g \in L^2(\mathbb{T})$ we denote by $g^{\dagger} \in L^2(\mathbb{T})$ the function $g^{\dagger}(e^{i\theta}) = \overline{g(e^{-i\theta})}$ $(e^{i\theta} \in \mathbb{T})$.

d) Let P denote the orthogonal projection of $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$.

Remark. If $\phi \in L^{\infty}(\mathbb{T})$ then $PJM_{\phi}|_{H^2(\mathbb{T})}$ is a Hankel operator on $H^2(\mathbb{T})$ with norm less than or equal to $\| \phi \|_{\infty}$. Indeed, the matrix of M_{ϕ} (with respect to $\{e_n\}_{n\in\mathbb{Z}}$) is the Laurent matrix $\{\hat{\phi}(i-j)\}_{i,j\in\mathbb{Z}}$; the matrix of the operator JM_{ϕ} is $\{\hat{\phi}(-i-j)\}_{i,j\in\mathbb{Z}}$ and thus the operator $PJM_{\phi}|_{H^2(\mathbb{T})}$ has matrix $\{\hat{\phi}(-i-j)\}_{i,j\in\mathbb{Z}^+}$.

1.3.3 Definition. Let T be a contraction on $\mathcal{H}$. Then a minimal isometric (respectively unitary) dilation for T is an isometry (respectively, unitary) V on a Hilbert space $\mathcal{K}$ such that

i) $\mathcal{H}$ is a closed subspace of $\mathcal{K}$ with orthogonal projection $\mathcal{P} : \mathcal{K} \to \mathcal{H}$;

ii) $\mathcal{P}V^n|_{\mathcal{H}} = T^n$ for each $n \geq 0$

and iii) V is minimal in the sense that the smallest reducing subspace for V containing $\mathcal{H}$ is $\mathcal{K}$.

It is well-known that every contraction T on $\mathcal{H}$ has a minimal isometric (respectively, unitary) dilation on some Hilbert space $\mathcal{K}$ ([N1]). This dilation is unique in the sense that if V' is another minimal isometric (respectively, unitary) dilation on $\mathcal{K}'$ say, then there exists a unitary Φ mapping $\mathcal{K}$ onto $\mathcal{K}'$ such that $V' = \Phi V \Phi^*$ and $\Phi(h) = h$ for all $h \in \mathcal{H}$. It is also known, by a theorem of Sz-Nagy and Foias that an operator that intertwines two contractions can be lifted to an operator of the same norm which intertwines the minimal unitary dilations of these contractions. The theorem is as follows.

<u>1.3.4 Theorem.</u> Let T_1, T_2 be contractions on Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ with minimal unitary dilations U_1, U_2 on $\mathcal{K}_1, \mathcal{K}_2$ respectively. If $X \in B(\mathcal{H}_2;\mathcal{H}_1)$ satisfies $T_1X = XT_2$ then there exists $Y \in B(\mathcal{K}_2;\mathcal{K}_1)$ such that $U_1Y = YU_2$, $\| Y \| = \| X \|$ and $X^n = \mathcal{P}Y^n|_{\mathcal{H}_2}$ for all $n \in \mathbb{Z}^+$ where $\mathcal{P}$ is the orthogonal projection of $\mathcal{K}_2$ onto $\mathcal{H}_2$.

<u>Proof.</u> [NF, Theorem 2.3, p 66].

Using this theorem it is straightforward to prove Nehari's theorem.

<u>1.3.5 Theorem.</u> ([NE]). Let $T \in B(H^2(\mathbb{T}))$. Then T is a Hankel operator if and only if $T = PJM_\phi|_{H^2(\mathbb{T})}$ for some $\phi \in L^\infty(\mathbb{T})$. In this case ϕ may be chosen such that $\| \phi \|_\infty = \| T \|_{B(H^2(\mathbb{T}))}$.

<u>Proof.</u> The fact that $PJM_\phi|_{H^2(\mathbb{T})}$ is a Hankel operator for any $\phi \in L^\infty(\mathbb{T})$ has been discussed above. If we suppose that $T \in B(H^2(\mathbb{T}))$ is a Hankel operator, then T must satisfy $S^*T = TS$ where S is the unilateral shift on $H^2(\mathbb{T})$. The minimal unitary dilation of S is the bilateral shift, U on $L^2(\mathbb{T})$. Thus, by 1.3.4 there exists an operator V on $L^2(\mathbb{T})$ such that $U^*V = VU$, $PV^n|_{H^2(\mathbb{T})} = T^n$ for all $n \in \mathbb{Z}^+$ and $\| V \| = \| T \|$.

Now $UJ = JU^*$ so $U(JV) = JU^*V = (JV)U$ and JV commutes with U. It follows that $JV = M_\phi$ for some $\phi \in L^\infty(\mathbb{T})$ and thus that

$$\| T \| = \| V \| = \| JV \| = \| M_\phi \| = \| \phi \|_\infty .$$

Finally, we note that

$$T = PV|_{H^2(\mathbb{T})} = PJM_\phi|_{H^2(\mathbb{T})}, \text{ as required.}$$

<u>Remarks.</u> 1. Nehari's theorem tells us that if $\{\alpha_{i+j}\}_{i,j\in\mathbb{Z}^+}$ is the matrix associated with some Hankel operator on $H^2(\mathbb{T})$ then $\alpha_n = \hat{\phi}(-n)$ for some $\phi \in L^\infty(\mathbb{T})$ and all $n \in \mathbb{Z}^+$.

2. Although the function ϕ is not uniquely determined by the Hankel operator, it is easy to see that ϕ is determined up to the addition of any $\psi \in H^\infty_0(\mathbb{T})$. The norm condition in 1.3.5 ensures that

$$\| PJM_\phi|_{H^2(\mathbb{T})} \|_{B(H^2(\mathbb{T}))} = \| \phi + H^\infty_0 \|_{L^\infty(\mathbb{T})/H^\infty_0(\mathbb{T})}.$$

and it follows that the map $\phi + H^\infty_0(\mathbb{T}) \rightsquigarrow PJM_\phi|_{H^2(\mathbb{T})}$ is an isometric isomorphism of $L^\infty(\mathbb{T})/H^\infty_0(\mathbb{T})$ onto the class of Hankel operators in $B(H^2(\mathbb{T}))$.

By the latter remark and the usual identification of $L^\infty(\mathbb{T})/H^\infty_0(\mathbb{T})$ with $H^1(\mathbb{T})$ [DUR, p 112] we have the following.

<u>**1.3.6 Corollary.**</u> i) The class of Hankel operators on $H^2(\mathbb{T})$ forms a Banach space isometrically isomorphic to the dual of $H^1(\mathbb{T})$.

ii) If T is a Hankel operator on $H^2(\mathbb{T})$ and $\phi \in L^\infty(\mathbb{T})$ is the (non-unique) function given by Theorem 1.3.6 then for all $f,g \in H^2(\mathbb{T})$

$$(Tf,g) = \langle fg^\dagger, \phi + H^\infty_0 \rangle_{H^1(\mathbb{T}),L^\infty(\mathbb{T})/H^\infty_0(\mathbb{T})}.$$

Proof. (i) is clear from the previous remarks. For (ii) we suppose that $T = PJM_\phi|_{H^2(\mathbb{T})}$ for some $\phi \in L^\infty(\mathbb{T})$ and that f,g are polynomials. Then

$$\begin{aligned}
(Tf,g) &= (JM_\phi f,g) \\
&= \sum_{n\in\mathbb{Z}^+} (\phi f)\hat{}\,(-n)\overline{\hat{g}(n)} \\
&= \sum_{n\geq 0}\left(\sum_{m\geq 0} \hat{\phi}(-n-m)\hat{f}(m)\right)\overline{\hat{g}(n)} \\
&= \sum_{k\geq 0}\sum_{l\geq 0} \hat{\phi}(-l)\hat{f}(k)\overline{\hat{g}(l-k)} \\
&= \sum_{l\geq 0} \hat{\phi}(-l)(fg^\dagger)\hat{}\,(l) \\
&= \langle fg^\dagger, \phi + H_0^\infty \rangle_{H^1(\mathbb{T}),L^\infty(\mathbb{T})/H_0^\infty(\mathbb{T})} .
\end{aligned}$$

The result now follows by approximating functions in $H^2(\mathbb{T})$ by sequences of polynomials.

Remark. There are other known methods of proving Nehari's theorem. Nehari's original proof is complicated, reducing the problem first to Hankel matrices with finitely-many non-zero entries. A more recent proof, published in [PAR] and described in [POW2] shows that the matrix $\{\alpha_{i+j}\}_{i,j\in\mathbb{Z}^+}$ can be enlarged step by step to the Laurent matrix $\{\alpha_{i+j}\}_{i,j\in\mathbb{Z}}$ without increasing the norm of the associated operator on $H^2(\mathbb{T})$. Finally, we note that it is possible to deduce Nehari's theorem by first proving that every Hankel operator on $H^2(\mathbb{T})$ defines a unique bounded linear functional on $H^1(\mathbb{T})$ ([POW3]).

Section 1.4 Schur Multipliers.

In this section we shall define the notion of Schur multiplier matrices and we shall show, using a theorem of Bennett that the Banach space of all Schur multipliers of matrices of bounded operators on $H^2(\mathbb{T})$ is isomorphic to the Banach space V_1^2 of 1.2.2. This result is used in [Pl] to relate the V_1^2 norm of a Hankel matrix $\{\hat{\phi}(i+j)\}_{i,j\in\mathbb{Z}^+}$ to a multiplier norm of the function ϕ. Moreover, in Chapters 4 and 5 we shall consider analogous results for Schur multipliers of arrays representing bounded operators on $H^2(\mathbb{T}^2)$ and for pointwise multipliers of kernels of bounded integral operators on $H^2(\mathbb{R})$.

1.4.1 Definitions. a) If A and B are matrices of scalars $(A = \{a_{i,j}\}_{i,j\in\mathbb{Z}^+}$, $B = \{b_{i,j}\}_{i,j\in\mathbb{Z}^+}$ then the Schur product $A \circledast B$ is the matrix $\{a_{i,j}b_{i,j}\}_{i,j\in\mathbb{Z}^+}$.

b) A matrix M is a Schur multiplier of $B(H^2(\mathbb{T}))$ if whenever A is the representing matrix (with respect to $\{e_n\}_{n\in\mathbb{Z}^+}$) of a bounded operator on $H^2(\mathbb{T})$, the Schur product $M \circledast A$ is also the representing matrix of some bounded operator on $H^2(\mathbb{T})$.

c) When M is a Schur multiplier on $B(H^2(\mathbb{T}))$ the map $A \rightsquigarrow M \circledast A$ is (by the Closed Graph Theorem, [CON, III.12.6]) bounded with respect to the operator norm on $B(H^2(\mathbb{T}))$. The multiplier norm $\| M \|_{M(B(H^2(\mathbb{T})))}$ is defined to be the norm of the map $A \rightsquigarrow M \circledast A$.

1.4.2 Lemma. Let $\{\beta_{i,j}\}_{i,j\geq 0}$ be a matrix of positive reals such that $\lim_{N\to\infty} \sum_{i=0}^{N} \beta_{i,N}$ exists, then

$$\lim_{N\to\infty}\sum_{i=0}^{N}\lim_{m\to\infty}\beta_{i,m} \le \lim_{N\to\infty}\sum_{i=0}^{N}\beta_{i,N} \qquad (*)$$

Proof. Denote the right-hand side of (∗) by K. Then for each $N \ge 0$, $\sum_{i=0}^{N}\beta_{i,N} \le K$. For any fixed $N \ge 0$, if $m \ge N$ we have

$$\sum_{i=0}^{N}\beta_{i,m} \le \sum_{i=0}\beta_{i,m} \le K$$

and so,

$$\sum_{i=0}^{N}\lim_{m\to\infty}\beta_{i,m} = \lim_{m\to\infty}\sum_{i=0}^{N}\beta_{i,m} \le K.$$

The result now follows since N was arbitrary.

1.4.3 **Theorem.** Let M be a matrix of scalars, $\{m_{i,j}\}_{i,j\in\mathbb{Z}^+}$. Then the following are equivalent.

i) $M \in V_1^2$;

ii) M is a Schur multiplier on $B(H^2(\mathbb{T}))$

and iii) $M = AB$ for some bounded operators

$A : l^2(\mathbb{Z}^+) \longrightarrow l^\infty(\mathbb{Z}^+)$ and $B : l^1(\mathbb{Z}^+) \longrightarrow l^2(\mathbb{Z}^+)$.

Moreover, when (i), (ii) and (iii) hold

$$\begin{aligned}\| M \|_{M(B(H^2(\mathbb{T})))} &\le \| M \|_{V_1^2} \le K_G \inf\{\| A \| \, \| B \| : M = AB\}\\ &\le K_G\, K \, \| M \|_{M(B(H^2(\mathbb{T})))}\end{aligned}$$

for some constant K independent of M.

Proof. (i) ⇒ (ii). Suppose that M is the matrix $\{f(i)g(j)\}_{i,j\in\mathbb{Z}^+}$ associated with the tensor $f\otimes g \in l^\infty(\mathbb{Z}^+)\otimes l^\infty(\mathbb{Z}^+)$. Then it is easy to show that for any $A \in B(H^2(\mathbb{T}))$

$$M \circledast A = (\mathrm{diag}\ f)\ A\ (\mathrm{diag}\ g)$$

where $(\mathrm{diag}\ f)_{i,j} = f(i)\delta_{i,j}$, $(\mathrm{diag}\ g)_{i,j} = g(i)\delta_{i,j}$ for all $i,j \in \mathbb{Z}^+$. Thus,

$\| M \circledast A \|_{B(H^2(\mathbb{T}))}$

$$\le \| \operatorname{diag} f \|_{B(H^2(\mathbb{T}))} \| A \|_{B(H^2(\mathbb{T}))} \| \operatorname{diag} g \|_{B(H^2(\mathbb{T}))}$$

$$= \| f \|_\infty \| A \|_{B(H^2(\mathbb{T}))} \| g \|_\infty$$

$$= \| f \otimes g \|_{\hat{\otimes}} \| A \|_{B(H^2(\mathbb{T}))} .$$

Similarly when M is the matrix associated with the tensor product $\alpha \in l^\infty(\mathbb{Z}^+) \otimes l^\infty(\mathbb{Z}^+)$ we have

$$\| M \circledast A \|_{B(H^2(\mathbb{T}))} \le \| \alpha \|_{\hat{\otimes}} \| A \|_{B(H^2(\mathbb{T}))} .$$

So if $M \in V_1^2$ and $N \ge 0$, $P_N^{(1)}M$ is a Schur multiplier with

$$\| P_N^{(1)}M \|_{M(B(H^2(\mathbb{T})))} \le \| P_N^{(1)}M \|_{\hat{\otimes}} .$$

Moreover, if $A \in B(H^2(\mathbb{T}))$ with representing matrix $\{a_{i,j}\}_{i,j \in \mathbb{Z}^+}$ and if $f \in H^2(\mathbb{T})$ is a polynomial of degree M,

$$\| (M \circledast A)f \|_2^2 = \sum_{i \ge 0} \Big| \sum_{j \ge 0} m_{i,j} a_{i,j} \hat{f}(j) \Big|^2$$

$$= \lim_{N \to \infty} \sum_{i=0}^{N} \cancel{\lim} \Big| \sum_{j=0}^{M} m_{i,j} a_{i,j} \hat{f}(j) \Big|^2$$

$$\le \lim_{N \to \infty} \sum_{i=0}^{N} \Big| \sum_{j=0}^{N} m_{i,j} a_{i,j} \hat{f}(j) \Big|^2 \quad \cancel{\text{by 1.4.2}}$$

$$= \lim_{N \to \infty} \| (P_N M \circledast A) f \|_2^2$$

$$\le \| M \|_{V_1^2}^2 \| A \|_{B(H^2(\mathbb{T}))}^2 \| f \|_2^2$$

and so M is a Schur multiplier with $\| M \|_{M(B(H^2(\mathbb{T})))} \le \| M \|_{V_1^2}$.

<u>(ii) ⇒ (iii).</u> [BEN1. Thm 6.4, p 619].

<u>(iii) ⇒ (i).</u> Suppose that $M = AB$ and that $B : l^1(\mathbb{Z}^+) \to l^2(\mathbb{Z}^+)$, $A : l^1(\mathbb{Z}^+) \to l^2(\mathbb{Z}^+)$ are bounded operators. Let A_i denote the ith row of A ($i \in \mathbb{Z}^+$) and let B_j denote the jth column of B ($j \in \mathbb{Z}^+$). Then for each $i,j \in \mathbb{Z}^+$, $m_{i,j} = (A_i, B_j)$. Thus for any $N \ge 0$ and any $\gamma \in l^1(\mathbb{Z}^+) \check{\otimes} l^1(\mathbb{Z}^+)$

$$|\langle \gamma, P_N M \rangle| = |\sum_{i,j\geq 0} \gamma_{i,j} (P_N M)_{i,j}|$$

$$= |\sum_{i,j=0}^{N} \gamma_{i,j} M_{i,j}|$$

$$= |\sum_{i,j=0}^{N} \gamma_{i,j} (A_i, B_j)|.$$

Now by 1.2.5,

$$|\langle \gamma, P_N M \rangle| \leq K_G \| \gamma \|_{\check{\otimes}} \sup_{0\leq i\leq n} \| A_i \|_2 \sup_{0\leq j\leq n} \| B_j \|_2 .$$

But it is easy to see that the columns of any matrix which defines a bounded map from $l^1(\mathbb{Z}^+)$ to $l^2(\mathbb{Z}^+)$ are in $l^2(\mathbb{Z}^+)$ with norms uniformly bounded by the norm of the map. Since B and A^* map $l^1(\mathbb{Z}^+)$ into $l^2(\mathbb{Z}^+)$ we have, by 1.2.4

$$\| P_N M \|_{\hat{\otimes}} \leq K_G \| A \| \| B \|$$

and hence,

$$\| M \|_{V_1^2} \leq K_G \| A \| \| B \|.$$

Chapter 2 Functions of a Power-Bounded Operator.

Much of this thesis is motivated by the norm estimates for polynomials of power-bounded operators presented in [P1, pp 341-353]. In this chapter we describe the motivation for studying power-bounded operators and give a full account of Peller's results. The results in Sections 2.2 and 2.3, with their proofs, illustrate the methods which will be used extensively in Chapters 4 and 5. Finally, in 2.4 we consider the functional calculii arising from these norm estimates and describe Bennett's solution to one of the questions posed in [P1].

Section 2.1 Motivation.

2.1.1 Definition. Let X be a Banach space and let $T \in B(X)$.

a) We say that T is <u>power-bounded</u> if there exists a constant c such that $\| T^n \| \le c$ for every $n \in \mathbb{N}$.

b) If T is invertible then we say that T is a <u>power-bounded invertible</u> if there exists a constant c such that $\| T^n \| \le c$ for every $n \in \mathbb{Z}$.

We note the following result, known as von-Neumann's inequality.

2.1.2 Theorem. If T is a contraction on $\mathcal{H}$ and ϕ is an analytic polynomial then

$$\| \phi(T) \|_{B(\mathcal{H})} \le \| \phi \|_{\infty}. \tag{*}$$

<u>Proof.</u> See [vN] for the original proof or [NF,p 32] for Sz-Nagy's proof using the minimal unitary dilation of T.

Considering this result as an inequality for contraction operators on some $L^2(\Omega,\mathcal{A},\mu)$, we may look for an extension to contraction operators on $L^p(\Omega,\mathcal{A},\mu)$ for $1 \le p \ne 2$. The analogue of $(*)$ cannot hold for all contractions on $L^p(\Omega,\mathcal{A},\mu)$ $(1 \le p \ne 2)$ since it is known ([F]) that this would imply that $L^p(\Omega,\mathcal{A},\mu)$ is isomorphic to a Hilbert space.

However, we see that for a polynomial $\phi(e^{i\theta}) = \sum_{-N}^{N}\hat{\phi}(n)e^{in\theta}$,

$\| \phi \|_\infty$ is equal to the norm of the operator given by convolution with the sequence $\left\{\hat{\phi}(n)\right\}_{n\in\mathbb{Z}}$ on $l^2(\mathbb{Z})$. For $p \ge 1$ we denote by $N_p(\hat{\phi})$ the norm of convolution with $\hat{\phi}$ on $l^p(\mathbb{Z})$ and we ask whether

$$\| \phi(T) \|_{B(L^p)} \le N_p(\phi) \qquad (**)$$

for every contraction T on $L^p(\Omega,\mathcal{A},\mu)$ and every analytic polynomial, ϕ. For $p \ne 2$ the answer is unknown but it has been shown that $(**)$ holds for certain classes of contractions on $L^p(\Omega,\mathcal{A},\mu)$. ([CRW, Thm 1.5],[P2]).

For power-bounded invertibles on subspaces of $L^p(\Omega,\mathcal{A},\mu)$ $(p > 1)$ we can use the general transference result ([CW, Thm 2.4],[BG, Thm 1.2]) to prove the following.

<u>2.1.3. Theorem.</u> Let T be a power-bounded invertible operator on a closed subspace $\underline{X}$ of $L^p(\Omega,\mathcal{A},\mu)$ $(1 < p < \infty)$ with $\| T^n \| \le c$ for all $n \in \mathbb{Z}$. Then for any trigonometric polynomial ϕ we have

$$\| \phi(T) \|_{B(L^p)} \le c^2 N_p(\hat{\phi}).$$

<u>Proof.</u> [CRW,(2.1), p 52] or [BG, Thm 4.2].

When $p = 2$ we obtain the following corollary.

2.1.4 Corollary. Let T be a power-bounded invertible on $\mathcal{H}$ with $\| T^n \| \leq c$ for all $n \in \mathbb{Z}$. Then for any trigonometric polynomial ϕ we have

$$\| \phi(T) \|_{B(\mathcal{H})} \leq c^2 \| \phi \|_\infty \quad .$$

Alternatively, 2.1.4 can be deduced from a well-known theorem of Sz-Nagy.

2.1.5 Theorem. [N2]. If T is a power-bounded invertible operator on $\mathcal{H}$ with $\| T^n \| \leq c$ for all $n \in \mathbb{Z}$ then T is similar to a unitary operator U on $\mathcal{H}$. Indeed, $T = PUP^{-1}$ for some invertible P with $\| P \|, \| P^{-1} \| \leq c$.

Proof. (Sketch). We use a Banach limit, LIM on $l^\infty(\mathbb{Z})$. ([CON, Thm 7.1, p 85]). The functional LIM is positive, translation invariant and has norm one.

Let T be as in the hypothesis of the theorem and define a bounded sesquilinear form Φ on $\mathcal{H} \times \mathcal{H}$ by

$$\Phi(x,y) = \mathrm{LIM}\Big(\big\{ (T^n x, T^n y) \big\}_{n \in \mathbb{Z}} \Big).$$

Then for $x \in \mathcal{H}$,

$$\begin{aligned} \| x \|^2_{\mathcal{H}} = \| T^{-n} T^n x \|^2_{\mathcal{H}} &\leq c^2 \| T^n x \|^2_{\mathcal{H}} \qquad (***) \\ &= c^2 (T^n x, T^n x) \end{aligned}$$

for all $n \in \mathbb{Z}$, so $\Phi(x,x) \geq \frac{1}{c^2} \| x \|^2_{\mathcal{H}}$. It follows that there exists a positive invertible $A \in B(\mathcal{H})$ with $\Phi(x,y) = (Ax,y)$ for all $x,y \in \mathcal{H}$. The result then follows with $U = A^{1/2}\, T\, A^{-1/2}$.

Proof of Corollary 2.1.4. By writing a power-bounded invertible T as $T = PUP^{-1}$ we have for any trigonometric polynomial ϕ

$$\begin{aligned} \| \phi(T) \| &= \| \phi(PUP^{-1}) \| \\ &= \| P\phi(U)P^{-1} \| \\ &\le \| P \| \| P^{-1} \| \| \phi(U) \| \\ &\le c^2 \| \phi(U) \| . \end{aligned}$$

But $\| \phi(U) \| = \sup \left\{ | \phi(\lambda) | : \lambda \text{ in the spectrum of } U \right\}$

$$= \sup \left\{ | \phi(\lambda) | : \lambda \in \mathbb{T} \right\}$$

$= \| \phi \|_\infty$, as required.

Turning our attention to the power-bounded operator T on $\mathcal{H}$, we seek a bound on the norm of an analytic polynomial in T. A glance at the proofs of 2.1.3 and 2.1.5 indicates that the methods used there require T to be invertible. Firstly the transference method requires the map $n \longrightarrow T^n$ to be a bounded representation of a group rather than a semigroup and secondly, in the proof of 2.1.5 T must be invertible if we are to show (by (***)) that $\phi(x,x) \ge \frac{1}{c^2} \| x \|_{\mathcal{H}}^2$ and thus that A is invertible.

Moreover a counterexample in [FOG] shows that the natural analogue of 2.1.5 does not hold for power-bounded operators. An account of Foguel's counterexample appears in [H] so we restrict the details here to those necessary for a comparison with the analogous example of a continuous semigroup ([PAC]) described in 5.1.

<u>2.1.6 Example.</u> [FOG], [H]. The operator

$$T = \begin{pmatrix} S^* & Q \\ 0 & S \end{pmatrix}$$

on $\mathcal{H} \oplus \mathcal{H}$ where S is the unilateral shift and Q is the projection onto the subspace spanned by $\{ x_{3^j} : j \in \mathbb{Z}^+\}$ is power-bounded and is NOT similar to a contraction.

Proof. We have

$$T^n = \begin{pmatrix} S^{*^n} & \sum_{i=0}^{n-i} (S^*)^{n-i} \; QS^i \\ 0 & S^n \end{pmatrix}$$

for all $n \geq 1$. We prove that $\| \sum_{i=0}^{n} (S^*)^{n-i} QS^i \| \leq 1$ for all $n \geq 1$

so that $\| T^n \| \leq 2$ for all $n \geq 1$.

Note that $\sum_{i=0}^{n} (S^*)^{n-i} QS^i = (S^*)^n \sum_{i=0}^{n} S^i \; QS^i$.

And for $0 \leq i \leq n$ and $m \geq 0$,

$$S^i QS^i x_m = \begin{Bmatrix} x_{m+2i} & \text{if } m+i = 3^j \text{ for some } j \in \mathbb{Z} \\ 0 & \text{otherwise} \end{Bmatrix}$$

Now if $m+i = 3^j$ and $m+i' = 3^{j'}$ for some $0 \leq i < i' \leq n$ and $j < j'$ we have

$$m+i' > 2.3^j = 2m + 2i$$

Hence $m + 2i < i' \leq n$ and so $(S^*)^n \; x_{m+2i} = 0$. It follows that, because of the sparsity of the powers of 3, $\left\{3^j\right\}_{j \in \mathbb{Z}^+}$, there can be at most one value of i between 0 and n for which $(S^*)^n S^i QS^i x_m$ is non-zero. When there <u>is</u> one such value of i, we have $\sum_{i=0}^{n} (S^*)^{n-i} QS^i x_m = x_{m+2i}$ and otherwise $\sum_{i=0}^{n} (S^*)^{n-i} QS^i x_m = 0$. Hence $\| \sum_{i=0}^{n} (S^*)^{n-i} QS^i \| \leq 1$, as required.

To show that T is not similar to a contraction we use the following lemma.

<u>2.1.7 Lemma.</u> [FOG]. For a $\in B(\mathcal{H})$ let $Z(A)$ denote the set of $x \in \mathcal{H}$ for which $A^n x \to 0$ weakly as $n \to \infty$. If A is a contraction then $Z(A) = Z(A^*)$.

If $A \in B(\mathcal{H})$ is similar to a contraction C with $A = PCP^{-1}$ for some invertible p on H then we can show that

$$Z(A) \cap Z(A^*)^{\perp} = P\left(Z\ (C) \cap Z\ (C^*)^{\perp} \right)$$

$$= \{0\} \qquad \text{by 2.1.7}$$

But we can show that $(x_0,0) \in Z(T) \cap Z(T^*)^{\perp}$ as follows. Firstly $(x_0,0) \in Z(T)$ because $T(x_0,0) = (0,0)$. Secondly, if $(f,g) \in Z(T^*)$, we have

$$\begin{aligned}((x_0,0),(f,g)) &= \lim_{j\to\infty} \left((x_0,x_{2.3^{j+1}}),(f,g)\right)\\ &= \lim_{j\to\infty} \left(\left(\sum_{i=0}^{2.3^{j+1}} (S^*)^{2.3^{j+1-i}} QS^i x_0\ ,\ S^{2.3^{j+1}} x_0\right),(f,g)\right)\\ &= \lim_{j\to\infty} \left(T^{2.3^{j+1}} (0,x_0)\ ,\ (f,g)\right)\\ &= \lim_{j\to\infty} \left((0,x_0)\ ,\ (T^*)^{2.3^{j+1}} (f,g)\right)\\ &= 0\ .\end{aligned}$$

Hence $(x_0,0) \in Z(T^*)^{\perp}$ and the proof of 2.1.6 is complete.

Section 2.2 An Estimate for $\| p(T) \|$.

Throughout the rest of this chapter T will denote a fixed power-bounded operator on $\mathcal{H}$ with $\| T^n \| \leq c$ for all $n \in \mathbb{Z}^+$. Also, p will be a fixed polynomial in $e^{i\theta}$, $p(e^{i\theta}) = \sum_{n=0}^{N} \hat{p}(n)e^{in\theta}$. We will produce upper bounds for the operator norm of $p(T) = \sum_{n=0}^{N} \hat{p}(n)T^n$.

The first important step is to associate the polynomial p with an element α of the discrete tensor product $l^1(\mathbb{Z}^+)\check{\otimes}l^1(\mathbb{Z}^+)$. We do this by choosing $\alpha \in l^1(\mathbb{Z}^+)\check{\otimes}l^1(\mathbb{Z}^+)$ for which the diagonals of the matrix $\{\alpha_{i,j}\}_{i,j\in\mathbb{Z}^+}$ sum to the coefficients $\hat{p}(n)$ of p. (Note that we define $\hat{p}(n) = 0$ for $n > N$). This method, with the power-boundedness of T, allows us to use Grothendieck's inequality to estimate $\| p(T) \|$.

<u>2.2.1 Theorem.</u> [P1, Thm 3.1]. Let $\alpha = \{\alpha_{i,j}\}_{i,j\geq 0} \in l^1(\mathbb{Z}^+)\otimes l^1(\mathbb{Z}^+)$ be a finitely non-zero matrix such that $\sum_{i+j=n} \alpha_{i,j} = \hat{p}(n)$ for each $n \in \mathbb{Z}^+$. Then $\| p(T) \| \leq K_G c^2 \| \alpha \|_{\check{\otimes}}$.

Proof. For $x,y \in \mathcal{H}$
$$(p(T)x,y) = \sum_{n\geq 0} \hat{p}(n)(T^n x,y)$$
$$= \sum_{n\geq 0} \Big(\sum_{i+j=n} \alpha_{i,j} \Big) (T^n x,y)$$
$$= \sum_{i,j\geq 0} \alpha_{i,j} \; (T^i x, (T^*)^j y).$$

But $\{T^i x\}_{i\geq 0}$ and $\{(T^*)^j\}_{j\geq 0}$ are bounded sequences in $\mathcal{H}$ so we can apply Grothendieck's inequality (1.2.5) to give
$$| (p(T)x,y)| \leq K_G \, \| \alpha \|_{\check{\otimes}} \; \sup_{i\geq 0} \| T^i x \|_{\mathcal{H}} \; \sup_{j\geq 0} \| (T^*)^j y \|_{\mathcal{H}}$$
$$\leq K_G \, c^2 \, \| \alpha \|_{\check{\otimes}} \; \| x \|_{\mathcal{H}} \; \| y \|_{\mathcal{H}}$$

and the proof is complete.

We note that the choice of such α for a polynomial p is not unique. Thus we shall use the following notation.

2.2.2 Notation. a) Let E denote the subspace of $l^1(\mathbb{Z}^+)\otimes l^1(\mathbb{Z}^+)$ consisting of those matrices $\{\alpha_{i,j}\}_{i,j\in\mathbb{Z}^+}$ with finitely many non-zero terms and with $\sum_{i+j=n} \alpha_{i,j} = 0$ for each $n \in \mathbb{Z}^+$. Let E^- denote the closure of E in $l^1(\mathbb{Z}^+)\check{\otimes} l^1(\mathbb{Z}^+)$.

b) When ϕ is an analytic function on $\mathbb{D}$ with power series
$$\phi(z) = \sum_{n\geq 0} \hat{\phi}(n) z^n,$$
we denote by Γ_ϕ the Hankel matrix $\{\hat{\phi}(i+j)\}_{i,j\in\mathbb{Z}^+}$.

2.2.3 Corollary. Let α be as in 2.2.1. Then
$$\| p(T) \| \leq K_G \, c^2 \, \| \alpha + E^- \|_{(l^1(\mathbb{Z}^+)\check{\otimes} l^1(\mathbb{Z}^+))/E^-}.$$

Proof. By 2.2.1, $\| p(T) \| \leq K_G \, c^2 \| \alpha + \alpha' \|_{\check{\otimes}}$ for any $\alpha' \in E$. The result follows from the definition of the quotient norm.

<u>Remark.</u> Since the equivalence class $\alpha + E^{-}$ is uniquely determined by p we will henceforth refer to $\| p \|_{(l^1\check{\otimes} l^1)/E^-}$, meaning of course, $\| \alpha + E^- \|_{l^1(\mathbb{Z}^+)\check{\otimes} l^1(\mathbb{Z}^+)}$ where α satisfies the conditions of 2.2.1.

The fact that the operator T is the generator of a bounded semigroup $\{ T^n \}_{n\in\mathbb{Z}}$ of operators on $\mathcal{H}$ is essential for the proof of 2.2.1. Having so used the power-boundedness of T, the remaining results in this chapter concern bounds on $\| p \|_{(l^1\check{\otimes} l^1)/E^-}$. Our first expression for $\| p \|_{(l^1\check{\otimes} l^1)/E^-}$ is given by the action of linear functionals on $(l^1(\mathbb{Z}^+)\check{\otimes} l^1(\mathbb{Z}^+))/E^-$.

<u>**2.2.4 Lemma.**</u> [P1, Cor.3.2]. The norm of the polynomial p, considered as an element of $(l^1(\mathbb{Z}^+)\check{\otimes} l^1(\mathbb{Z}^+))/E^-$ is given by

$$\| p \|_{(l^1\check{\otimes} l^1)/E^-} = \sup \left\{ \Big| \sum_{n=0}^{N} \hat{p}(n)\hat{\phi}(n) \Big| : \| \Gamma_\phi \|_{V_1^2} \le 1 \right\}.$$

<u>Proof.</u> By 1.2.3 the dual of $l^1(\mathbb{Z}^+)\check{\otimes} l^1(\mathbb{Z}^+)$ is isometrically isomorphic to V_1^2 so the result will follow when we show that the annihilator of E in V_1^2 consists of the Hankel matrices in V_1^2. Suppose that $\alpha \in E$ and that ϕ is an analytic function on $\mathbb{D}$ such that $\Gamma_\phi \in V_1^2$. then since $\{\alpha_{i,j}\}_{i,j}$ is finitely non-zero we have

$$\begin{aligned} \langle \alpha, \Gamma_\phi \rangle &= \sum_{i,j\ge 0} \alpha_{i,j}\, \hat{\phi}(i+j) \quad \text{(by 1.2.3)} \\ &= \sum_{n\ge 0} \Big(\sum_{i+j=n} \alpha_{i,j} \Big) \hat{\phi}(n) \\ &= 0 \quad \text{since } \alpha \in E. \end{aligned}$$

Conversely, suppose that $\beta = \{ \beta_{i,j} \}_{i,j\in\mathbb{Z}^+} \in V_1^2$ satisfies $< \alpha,\beta > = 0$ wherever $\alpha \in E$. Then $\sum_{n\geq 0} \sum_{i+j=n} \alpha_{i,j} \beta_{i,j} = 0$ for every $\alpha \in E$. If $\delta_{i,j}$ denotes the Kronecker delta function ($\delta_{i,j} = 1$, when i=j and is zero otherwise) then we consider

$$\alpha = \left\{ \delta_{i,k}\delta_{j,l} - \delta_{i,k-1} \delta_{j,l+1} \right\}_{i,j \in \mathbb{Z}^+}$$

for some $k > 0$, $l \geq 0$. Since $\alpha \in E$ we have

$$< \alpha,\beta > = \beta_{k,l} - \beta_{k-1,l+1} = 0$$

and since k,l are arbitrary it follows that there exists a sequence $\{ \beta(n) \}_{n\in\mathbb{Z}^+}$ such that $\beta_{k,l} = \beta(k + l)$ for all $k,l \geq 0$. Thus, $\beta = \Gamma_\phi$ where $\phi(z) = \sum_{n\geq 0} \beta(n)z^n$.

Finally, if $\alpha \in l^1(\mathbb{Z}^+)\check{\otimes}l^1(\mathbb{Z}^+)$ satisfies the hypothesis of 2.2.1 then

$$\begin{aligned} \| p \|_{(l^1\check{\otimes}l^1)/E^-} &= \| \alpha + E^- \|_{(l^1(\mathbb{Z}^+)\check{\otimes}l^1(\mathbb{Z}^+))/E^-} \\ &= \sup\left\{ | < \alpha,\Gamma_\phi >| \ : \ \| \Gamma_\phi \|_{V_1^2} \leq 1 \right\} \\ &= \sup\left\{ | \sum_{i,j\geq 0} \alpha_{i,j} \hat{\phi}(i+j) | \ : \| \Gamma_\phi \|_{V_1^2} \leq 1 \right\} \\ &= \sup\left\{ | \sum_{n=0}^{N} \hat{p}(n)\hat{\phi}(n) | \ : \| \Gamma_\phi \|_{V_1^2} \leq 1 \right\} . \end{aligned}$$

as required.

2.2.5 Definitions.

a) If f and g are analytic functions on $\mathbb{D}$ with power series $\sum_{n\geq 0} \hat{f}(n)z^n$, $\sum_{n\geq 0} \hat{g}(n)z^n$ respectively then we define the <u>convolution</u> $f*g$ to be the analytic function on $\mathbb{D}$ with power series $\sum_{n\geq 0} \hat{f}(n)\hat{g}(n)z^n$.

b) Let X be a Banach space of analytic functions on $\mathbb{D}$. Then an analytic function ϕ on $\mathbb{D}$ is a <u>multiplier</u> of X if the map which sends h to $\phi*h$ is a bounded linear operator on X.

c) We denote the linear space of all multipliers of X by M(X) and define the <u>multiplier norm,</u> $\|\phi\|_{M(X)}$ of $\phi \in M(X)$ to be the operator norm of convolution with ϕ on X.

The following necessary condition on a function ϕ for Γ_ϕ to be in V_1^2 is due to Grahame Bennett. This is a key result for the estimation of $\|p(T)\|$ because it relates the matrix norm $\| \ \|_{V_1^2}$ of the Hankel matrix Γ_ϕ to a (multiplier) norm of the function ϕ. It will shortly enable us to find a bound for $\|p(T)\|$ in terms of the norm of p in a Banach space of functions rather than a matrix norm associated with the sequence $\{\hat{p}(n)\}_{n\in\mathbb{Z}^+}$. Note how the proof of 2.2.6 relies on Nehari's theorem.

<u>2.2.6 **Lemma.**</u> [BEN1, p 632], [P1, Lemma 3.3]. If ϕ is an analytic function on $\mathbb{D}$ such that $\Gamma_\phi \in V_1^2$ then ϕ is a multiplier of $H^1(\mathbb{D})$ with $\|\phi\|_{M(H^1)} \le \|\Gamma_\phi\|_{V_1^2}$.

<u>Proof.</u> Suppose that $\Gamma_\phi \in V_1^2$. By 1.4.3 Γ_ϕ is a Schur multiplier on $B(H^2(\mathbb{T}))$ with $\|\Gamma_\phi\|_{M(B(H^2))} \le \|\Gamma_\phi\|_{V_1^2}$.

We show first that $\phi \in M(BMOA(\mathbb{D}))$. So let $\psi \in BMOA(\mathbb{D})$. Recall that $BMOA(\mathbb{D})$ can be identified with $H^1(\mathbb{T})^*$ so that by the Corollary to Nehari's theorem (1.3.6) Γ_ψ is the matrix of a Hankel operator on $H^2(\mathbb{T})$ with norm equal to $\|\psi\|_{BMOA}$. Moreover,

$$\begin{aligned}\|\Gamma_{\phi*\psi}\|_{B(H^2)} &= \|\Gamma_\phi \circledast \Gamma_\psi\|_{B(H^2)} \\ &\le \|\Gamma_\phi\|_{M(B(H^2))}\, \|\Gamma_\psi\|_{B(H^2)} \\ &\le \|\Gamma_\phi\|_{V_1^2}\, \|\psi\|_{BMOA} .\end{aligned}$$

So $\phi*\psi \in BMOA(\mathbb{D})$ (by 1.3.6 again) and

$$\| \phi*\psi \|_{BMOA} \leq \| \Gamma_\phi \|_{V_1^2} \| \psi \|_{BMOA}$$

as required.

Finally, if $h \in H^1(\mathbb{D})$ is a polynomial then $\phi*h$ is a polynomial and

$$\| \phi*h \|_1 = \sup\left\{ \left| \sum_{n\geq 0} \hat{\phi}(n)\hat{h}(n)\hat{\psi}(n) \right| \ : \ \psi \in BMOA(\mathbb{D}), \| \psi \|_{BMOA} \leq 1 \right\}$$

$$= \sup\left\{ \left| < h, \phi*\psi > \right| \ : \ \psi \in BMOA(\mathbb{D}), \| \psi \|_{BMOA} \leq 1 \right\}$$

$$\leq \| h \| \, \| \Gamma_\phi \|_{V_1^2} \ .$$

The result now follows since the polynomials are dense in $H^1(\mathbb{D})$.

It follows immediately from 2.2.4 and 2.2.6 that we have the following bound on $\| p \|_{(l^1 \check{\otimes} l^1)/E^-}$.

__2.2.7 Corollary.__ [Pl, Cor. 3.4]. The norm $\| p \|_{(l^1 \check{\otimes} l^1)/E^-}$ of the polynomial p considered as an element of $(l^1(\mathbb{Z}^+)\check{\otimes} l^1(\mathbb{Z}^+))/E^-$ is bounded above by

$$\sup\left\{ \left| \sum_{n=0}^{N} \hat{p}(n)\hat{\phi}(n) \right| \ : \phi \in M(H^1), \ \| \phi \|_{M(H^1)} \leq 1 \right\}.$$

__Section 2.3 Further Estimates for $\| p(T) \|$.__

In this section we construct three further bounds on the norm $\| p \|_{(l^1 \check{\otimes} l^1)/E^-}$ of the polynomial p. We consider first the construction of the Banach space $X\hat{*}Y$ for Banach spaces X,Y of analytic functions on $\mathbb{D}$. We give conditions for the dual of $X\hat{*}Y$ to be isometrically isomorphic to M(Y). It then follows from 2.2.7

that $\| p \|_{(l^1 \check{\otimes} l^1)/E^-}$ is dominated by $\| p \|_{VMOA \hat{*} H^1}$. Secondly we define the Besov space $B^0_{\infty,1}(\mathbb{D})$ and show that the "projective convolution" norm $\| \cdot \|_{VMOA \hat{*} H^1}$ is dominated by the Besov norm $\| \cdot \|^0_{\infty,1}$. Lastly, by a simple estimate of $\| p \|^0_{\infty,1}$ we deduce that when $N \geq 2$ is the degree of p we have $\| p(T) \| \leq Kc^2 \log N \| p \|_\infty$ for some constant K independent of p and T.

<u>2.3.1 Definition.</u> Let X,Y be Banach spaces of analytic functions on $\mathbb{D}$. We define the <u>convolution</u> X*Y of X and Y to consist of all finite sums of convolutions $\sum_{n=0}^{N} f_n * g_n$ ($N \geq 0$, $f_n \in X$ and $g_n \in Y$ for $0 \leq n \leq N$). The <u>projective convolution norm</u> $\| \cdot \|_{\hat{*}}$ is defined on X*Y by

$$\| \phi \|_{\hat{*}} = \inf \left\{ \sum_{n=0}^{N} \| f_n \|_x \| g_n \|_y \ : \ \phi = \sum_{n=0}^{N} f_n * g_n \text{ for some } N \geq 0, \ f_n \in X \text{ and } g_n \in Y \ (0 \leq n \leq N) \right\}.$$

The <u>projective convolution</u> of X and Y is the completion of X*Y with respect to $\| \ \|_{\hat{*}}$ and is denoted by $X \hat{*} Y$.

<u>Remarks.</u>

<u>1.</u> Recall that an element of the projective tensor product $X \hat{\otimes} Y$ can be written as an absolutely convergent infinite sum of simple tensors. Similarly we can show that an element α of $X \hat{*} Y$ can be written as an infinite sum $\sum_{n \geq 0} f_n * g_n$ with

$$\sum_{n \geq 0} \| f_n \|_x \| g_n \|_y < \| \alpha \|_{\hat{*}} + \varepsilon \quad \text{say.}$$

<u>2.</u> Moreover, if for each $n \in \mathbb{Z}^+$ the functional $f \rightsquigarrow \hat{f}(n)$ (f

analytic on $\mathbb{D}$) is continuous on X and Y with $|\hat{f}(n)| \le c_1 \| f \|_X$ and $|\hat{g}(n)| \le c_2 \| g \|_Y$ for all $f \in X$, $g \in Y$, $n \in \mathbb{Z}^+$ then the projective convolution $X \hat{*} Y$ is indeed a Banach space of functions.

If $\alpha = \sum_{k\ge 0} f_k * g_k \in X\hat{*}Y$ for some $f_k \in X$, $g_k \in Y$ $(k \ge 0)$ with

$$\sum \| f_k \|_X \| g_k \|_Y < \| \alpha \|_{\hat{*}} + \varepsilon$$

then we put (for $n \in \mathbb{Z}^+$)

$$\hat{\phi}_\alpha(n) = \sum_{k\ge 0} \hat{f}_k(n)\hat{g}_k(n) \text{ and } \bar{\phi}_\alpha(z) = \sum_{n\ge 0} \hat{\phi}_\alpha(n) z^n \text{ for } z \in \mathbb{D}.$$

Then

$$\begin{aligned}
\sum_{n\ge 0}\sum_{k\ge 0} | \hat{f}_k(n)\hat{g}_k(n) z^n | &\le \sum_{n\ge 0}\sum_{k\ge 0} c_2 \| f_k \|_X c_2 \| g_k \|_Y |z|^n \\
&\le c_1 c_2 \sum_{n\ge 0} (\| \alpha \|_{\hat{*}} + \varepsilon) |z|^n \\
&< \infty
\end{aligned}$$

$$\begin{aligned}
\text{so that} \qquad \phi_\alpha(z) &= \sum_{n\ge 0}\left(\sum_{k\ge 0} \hat{f}_k(n)\hat{g}_k(n)\right) z^n \\
&= \sum_{k\ge 0}\sum_{n\ge 0} \hat{f}_k(n)\hat{g}_k(n)\, z^n \\
&= \sum_{k\ge 0} (f_k * g_k)(z) \quad \text{for all } z \in \mathbb{D}.
\end{aligned}$$

<u>3.</u> In fact $X\hat{*}Y$ is just a quotient space of the projective tensor product $X\hat{\otimes}Y$. Let Z be the closure in $X\hat{\otimes}Y$ of the subspace consisting of tensors $\alpha = \sum_{n=0}^{N} f_n \otimes g_n$ for which $\sum_{n=0}^{N} f_n * g_n = 0$. Then it is easy to show that $X\hat{*}Y$ is isometrically isomorphic to $(X\hat{\otimes}Y)/Z$.

2.3.2 Lemma. Suppose that X and Y are Banach spaces of analytic functions on $\mathbb{D}$ which contain the linear space $\mathcal{P}$ of polynomials on $\mathbb{D}$. Suppose that $\mathcal{P}$ is dense in Y and that for any $g = \sum_n \hat{g}(n)z^n \in \mathcal{P}$,

$$\| g \|_Y = \sup \left\{ \left| \sum_{n\geq 0} \hat{f}(n)\hat{g}(n) \right| \ : \ f = \sum_n \hat{f}(n)z^n \in X, \text{ and } \| f \|_X \leq 1 \right\}.$$

Suppose that the functions $\tilde{e}_n(z) = z^n$ on $\mathbb{D}$ satisfy $\| \tilde{e}_n \|_X \leq 1$ and that for $n \in \mathbb{Z}^+$ and $f \in X$, $| \hat{f}(n) | \leq c_1 \| f \|_X$ for all $f \in X$. Then $(X\hat{*}Y)^*$ is isometrically isomorphic to $M(Y)$.

Proof. Let $\phi \in M(Y)$. Define a linear functional Φ on $X*\mathcal{P}$ by

$$\Phi\left(\sum_{k=0}^{N} f_k * g_k\right) = \sum_{k=0}^{N} \sum_{n\geq 0} \hat{f}_k(n)\hat{g}_k(n)\hat{\phi}(n)$$

whenever $f_k \in X$ and $g_k \in Y$ for each $0 \leq n \leq N$. Since $\sum_{k=0}^{N} f_k * g_k = 0$ if and only if $\sum_{n=0}^{N} \hat{f}_k(n)\hat{g}_k(n) = 0$ for each $n \geq 0$ it follows that $\Phi = 0$ if and only if $\phi = 0$. Moreover, if $\alpha = \sum_{k=0}^{N} f_k * g_k \in X*\mathcal{P}$

$$\begin{aligned} | \Phi(\alpha) | &\leq \sum_{k\geq 0}^{N} \left| \sum_{n\geq 0} \hat{f}_k(n) \, (\phi * g_k)\hat{\ }(n) \right| \\ &\leq \sum_{k\geq 0}^{N} \| f_x \|_X \| \phi * g_k \|_Y \\ &\leq \| \phi \|_{M(Y)} \sum_{k=0}^{N} \| f_k \|_X \| g_k \|_Y . \end{aligned}$$

Since this holds for any such representation of α we have

$$| \Phi(\alpha) | \leq \| \phi \|_{M(Y)} \| \alpha \|_{\hat{*}} .$$

It is easy to check that the density of $\mathcal{P}$ in Y ensures that $X*\mathcal{P}$ is dense in $X\hat{*}Y$. Thus Φ extends to a linear functional on $X\hat{*}Y$ with norm dominated by $\| \phi \|_{M(Y)}$.

Conversely we suppose that $F \in (X\hat{*}Y)^*$. For each $n \in \mathbb{Z}^+$ we have $\tilde{e}_n \in Y$ and

$$\| e_n \|_Y = \sup\left\{ \left| \sum_m \hat{f}(m)\hat{\tilde{e}}_n(m) \right| \ : \ \| f \|_X \le 1 \right\}$$

$$= \sup\left\{ | \hat{f}(n) | \ : \ \| f \|_X \le 1 \right\}$$

$$\le c_1 \qquad \text{by hypothesis.}$$

So $\tilde{e}_n * \tilde{e}_n \in X*Y$ and we can put

$$\hat{\tilde{F}}(n) = F(\tilde{e}_n * \tilde{e}_n)$$

$$\text{and} \qquad \tilde{F}(z) = \sum_{n \ge 0} \hat{\tilde{F}}(n) z^n \quad \text{for } z \in \mathbb{D}.$$

Now if $f \in X$ and $g \in \mathcal{P}$ has degree N,

$$F(f*g) = F\left(\sum_{n=0}^{N} \hat{f}(n)\hat{g}(n)(\tilde{e}_n * \tilde{e}_n) \right)$$

$$= \sum_{n=0}^{N} \hat{f}(n)\hat{g}(n)\hat{\tilde{F}}(n)$$

so that when $g \in \mathcal{P}$ has degree N,

$$\| \tilde{F}*g \|_Y = \sup\left\{ \left| \sum_{n=0}^{N} \hat{f}(n)\hat{g}(n)\hat{\tilde{F}}(n) \right| \ : \ \| f \|_X \le 1 \right\}$$

$$= \sup\left\{ | F(f*g) | \ : \ \| f \|_X \le 1 \right\}$$

$$\le \| F \|_{(X\hat{*}Y)^*} \| g \|_Y .$$

Finally, when $g \in Y$ we find a sequence $g_k \in \mathcal{P}$ such that $\| g - g_k \|_Y \to 0$. Then $\{ \tilde{F}*g_k \}_{k \ge 0}$ is a Cauchy sequence in Y converging to some $h \in Y$. It is clear from the hypotheses that the map $\psi \rightsquigarrow \hat{\psi}(n)$ is a norm one linear functional on Y for each $n \in \mathbb{Z}^+$.

Hence

$$\begin{aligned}\hat{h}(n) &= \lim_{n\to\infty} (\tilde{F}*g_k)^{\wedge}(n) \\ &= \lim_{n\to\infty} \hat{\tilde{F}}(n)\hat{g}_k(n) \\ &= \hat{\tilde{F}}(n)\hat{g}(n) \qquad \text{for each } n \in \mathbb{Z}^+\end{aligned}$$

and we have $h = \tilde{F}*g$.

Thus $\tilde{F} \in M(Y)$, $\|\tilde{F}\|_{M(Y)} \leq \|F\|_{(X\hat{*}Y)^*}$ and the proof is complete.

Since $X = VMOA(\mathbb{D})$ and $Y = H^1(\mathbb{D})$ satisfy the conditions of 2.3.2, we have $(VMOA \hat{*} H^1)^*$ isometrically isomorphic to $M(H^1)$ with

$$< \psi*h, \phi > = \sum_{n\geq 0} \hat{\psi}(n)\hat{h}(n)\hat{\phi}(n)$$

when $\psi(z) = \sum_{n\geq 0} \hat{\psi}(n)z^n \in VMOA(\mathbb{D})$, $\phi(z) = \sum \hat{\phi}(n)z^n \in M(H^1)$ and

$h(z) = \sum_{n=0}^{N} \hat{h}(n)z^n \in H^1(\mathbb{D})$ is a polynomial.

<u>2.3.3 Corollary.</u> [P1, Thm. 3.5]. The norm $\|p\|_{(l^1\check{\otimes}l^1)/E^-}$ of the polynomial p considered as an element of $(l^1(\mathbb{Z}^+)\check{\otimes}l^1(\mathbb{Z}^+))/E^-$ satisfies

$$\|p\|_{(l^1\check{\otimes}l^1)/E^-} \leq \|p\|_{VMOA\hat{*}H^1}.$$

<u>Proof.</u> When $p = \psi*h$ for some polynomials $\psi \in VMOA(\mathbb{D})$ and $h \in H^1(\mathbb{D})$ we have for any $\phi \in M(H^1)$

$$\sum_{n=0}^{N}\hat{p}(n)\hat{\phi}(n) = < \psi*h, \phi >.$$

The result now follows from 2.2.7 and 2.3.2.

We will now show that the norm of the polynomial p as an element of $VMOA\hat{*}H^1$ must be less than or equal to the norm of p in the Besov space $B^0_{\infty,1}$.

<u>2.3.4 Definition.</u> For each $N \geq 0$ we define a polynomial W_N on $\mathbb{D}$ by

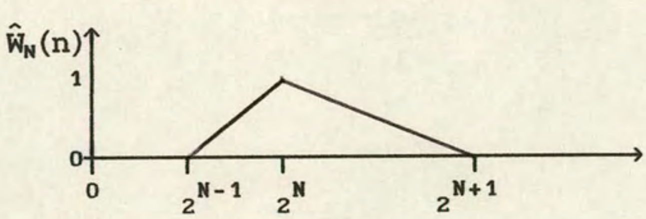
$$\hat{W}_N(n) = \begin{cases} 0 & : n \notin (2^{N-1}, 2^{N+1}) \\ \dfrac{n-2^{N-1}}{2^{N-1}} & : 2^{N-1} \leq n \leq 2^N \\ \dfrac{2^{N+1}-n}{2^N} & : 2^N \leq n \leq 2^{N+1} \end{cases} .$$

We define $W_0(z) = 1 + z$ for $z \in \mathbb{D}$. The Banach space $B^0_{\infty,1}(\mathbb{D})$ consists of analytic functions f on $\mathbb{D}$ for which

$$\| f \|^0_{\infty,1} = \sum_{n\geq 0} \| W_n * f \|_{H^\infty(\mathbb{D})} < \infty.$$

<u>Remarks.</u>

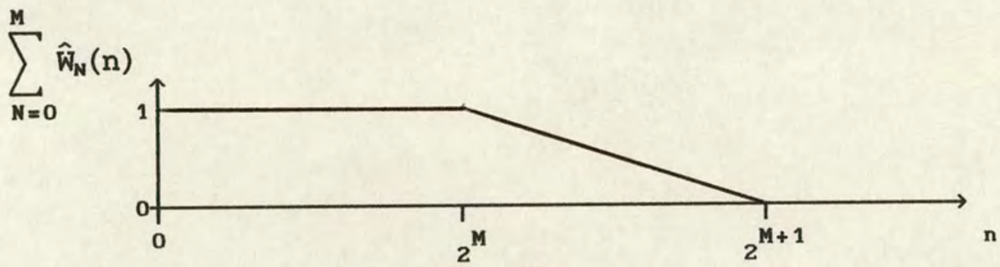
<u>1.</u> For $N > 0$ and $n \in \mathbb{Z}^+$ the value of $\hat{W}_N(n)$ is shown by the graph :



We note that for $M \in \mathbb{N}$ and $n \in \mathbb{Z}^+$

$$\sum_{N=0}^{M} \hat{W}_N(n) = \begin{cases} 1 & : \ 0 \le n \le 2^M \\ \dfrac{2^{M+1}-n}{2^M} & : \ 2^M \le n \le 2^{M+1} \\ 0 & : \ n \ge 2^{M+1} \end{cases}$$

Graphically, the values of $\sum_{N=0}^{M} \hat{W}_N(n)$ are given by



Thus for $f \in B^0_{\infty,1}$, the sum $\sum_{n\ge0} W_n * f$ converges in $H^\infty(\mathbb{D})$ to f and $\| f \|_\infty \le \| f \|^0_{\infty,1}$.

<u>2.</u> The condition $\sum_{n\ge0} \| W_n * f \|_\infty < \infty$ for f to be in $B^0_{\infty,1}$ is an assessment of the convergence of the decomposition $\sum_{n\ge0} W_n * f$ of f with respect to $\{ W_n \}_{N\ge0}$. The $B^0_{\infty,1}$ norm is exactly the $l^1(\mathbb{Z}^+)$ norm of the sequence $\{ \| W_n * f \|_\infty \}_{n\in\mathbb{Z}}$. Further Besov spaces $B^s_{p,q}$ (for $s \in \mathbb{R}$ and for $0 \le p,q \in \mathbb{R}^+$) may be defined by considering the $l^q(\mathbb{Z}^+)$ norm of the sequence $\left\{ z^{ns} \| W_n * f \|_p \right\}_{n\in\mathbb{Z}^+}$.

For the motivation, definitions and properties of the Banach spaces $B^s_{p,q}$ we refer the reader to [T].

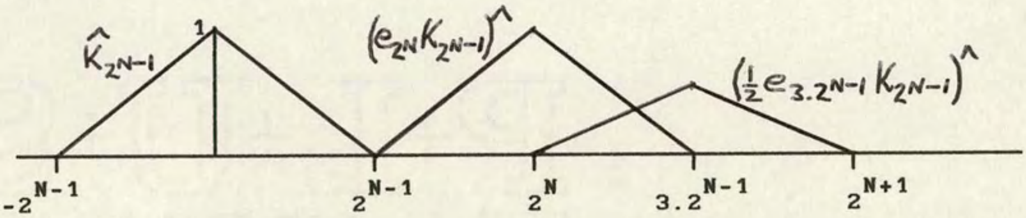
<u>3.</u> If K_N denotes the Fejer Kernel

$$K_N(e^{i\theta}) = \sum_{j=-N}^{N} \frac{N-|j|}{N} e^{ij\theta}$$

(for $N > 0$ and $e^{i\theta} \in \mathbb{T}$) then

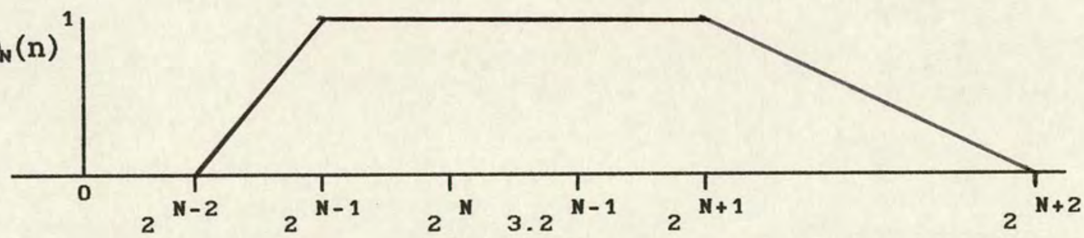
$$W_N(e^{i\theta}) = \left(e_{2^N} + \frac{1}{2} e^{3.2^{N-1} i\theta} \right) K_{2^{N-1}}(e^{i\theta})$$

for $e^{i\theta} \in \mathbb{T}$. This is clear when we consider the coefficients of W_N, $e_{2^N}K_{2^{N-1}}$ and $\frac{1}{2} e_{3.2^{N-1}} K_{2^{N-1}}$.

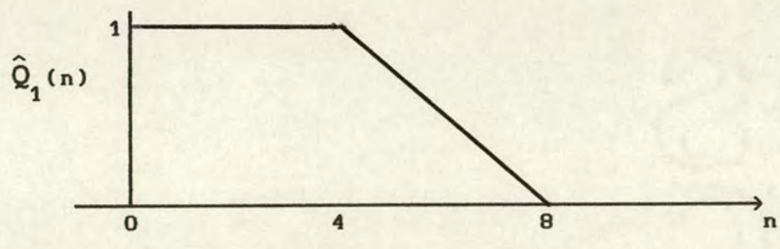


By adding the coefficients of $e_{2^N}K_{2^{N-1}}$ we obtain the coefficients of W_n. Moreover, since $\| K_N \|_1 = 1$ for each $N > 0$ it follows that $\| W_N \|_1 \le \frac{3}{2}$ for all $N > 0$. Note also that $\| W_0 \|_1 \le \| W_0 \|_2 = \sqrt{2} < \frac{3}{2}$.

4. For $N > 0$ we put $Q_N = W_{N-1} + W_N + W_{N+1}$. The coefficients of Q_N (for $N > 1$) are shown by



and the coefficients of Q_1 are shown by



Thus, for $N \ge 1$, $\hat{Q}_N \equiv 1$ on the support of $\hat{W}_N$ and so $Q_N * W_N = W_N$.

2.3.5 Lemma. If $f \in B^0_{\infty,1}(\mathbb{D})$ then $f \in VMOA\hat{*}H^1$ with

$$\| f \|_{\hat{*}} \le \frac{9}{2} \| f \|^0_{\infty,1} .$$

Proof. Let $f \in B^0_{\infty,1}(\mathbb{D})$. Then

$$f = \sum_{n\ge0} f * W_n = f * W_0 * W_0 + \sum_{n\ge1} f * W_n * Q_n .$$

For each $n \ge 0$, $\quad f * W_n \in H^\infty(\mathbb{D}) \subseteq VMOA(\mathbb{D})$

with $\| f*W_n \|_{VMOA(\mathbb{D})} \le \| f \|_\infty$. Also, $Q_n \in H^1(\mathbb{D})$ with

$$\| Q_n \|_1 \le \| W_{n-1} \|_1 + \| W_n \|_1 + \| W_{n+1} \| \le \frac{9}{2} .$$

Hence,

$$\| f*W_0 \|_{VMOA} \| W_0 \|_1 + \sum_{n\ge1} \| f*W_n \|_{VMOA} \| Q_n \|_1 \le \frac{9}{2} \sum_{n\ge0} \| f*W_n \|_\infty$$

$$= \frac{9}{2} \| f \|^0_{\infty\,1},$$

as required.

2.3.6 Corollary. $\| p \|_{(1^1 \check{\otimes} 1^1)/E^-} \le \frac{9}{2} \| p \|^0_{\infty,1}$.

Proof. Immediate from 2.3.3 and 2.3.5.

Remark. It is known (for example in [BEN1, p 632]) that if $\phi \in M(H^1)$ and if ϕ_r $(0 < r < 1)$ denotes the function $\phi(re^{i\theta})$ on $\mathbb{T}$ then

$$\sup_{0<r<1} (1-r) \| \phi_r' \|_1 < \infty .$$

This condition is equivalent to

$$\sup_{n\ge0} \| W_n * \phi \|_1 < \infty ,$$

([ST, p 151]). Thus, $\phi \in B^0_{\infty,1}$. Moreover, the inclusion i of $M(H^1)$ into $B^0_{\infty,1}$ is continuous. For suppose that $\{\phi_n\}_{n\ge0}$ is a sequence in $M(H^1)$ tending to zero and that $\phi \in B^0_{\infty,1}$ has $\| \phi-\phi_n \|^0_{1,\infty} \to 0$ as $n \to \infty$. Then $\| W_N * (\phi-\phi_n) \|_1 \to 0$ as $n \to \infty$ for each $N \ge 0$. So $\hat{\phi}(m) = \lim_{n\to\infty} \hat{\phi}_n(m) = 0$ for every $m \ge 0$, and we conclude that $\phi = 0$. It follows ([CON,III.12.7]) that the graph of i is closed and thus by the Closed Graph Theorem ([CON ,III.12.6]) that i is continuous.

Now if $s \in \mathbb{R}$, $1 \le p \le \infty$ and $1 \le q < \infty$ we have $(B^s_{p,q})^* = B^{-s}_{p',q'}$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$ ([S-T, p171]). Thus, with $s = 0$, $p = \infty$ and $q = 1$ we have $(B^0_{\infty,1})^* = B^0_{1,\infty}$. We conclude that there exists a constant $K > 0$ such that any multiplier ϕ on $H^1(\mathbb{D})$ defines a bounded linear functional on $B^0_{\infty,1}$ with norm not greater than $K \| \phi \|_{M(H^1)}$. Using this fact we could deduce 2.3.6 (with K in place of $\frac{9}{2}$) directly from 2.2.5 without the use of the projective convolution space $VMOA\hat{*}H^1$.

Finally, for the polynomial p we have a simple estimate of $\| p \|^0_{\infty,1}$.

<u>2.3.7 Lemma.</u> If the degree of p is $N \ge 2$ we have

$$\| p \|^0_{\infty,1} \le 9 \,(\log N) \| p \|_\infty .$$

<u>Proof.</u> For any $N \ge 2$ we can choose m such that $2^{m-1} \le N \le 2^m$. Then

$$\sum_{n \ge 0} \| W_n * p \|_\infty = \sum_{n=0}^{M} \| W_n * p \|_\infty$$
$$\le (M+1)\, \frac{3}{2} \| p \|_\infty$$

But $M-1 \le \log_2 N$ so we have

$$\begin{aligned} M+1 &\le (\log_2 N) + 2 \\ &= \log_2 4N \\ &= \frac{\log 4N}{\log 2} \\ &\le \frac{3}{\log 2} \log N. \end{aligned}$$

Hence, $$\| p \|^0_{\infty,1} \le \frac{9}{2\log 2} (\log N) \| p \|_\infty$$
$$\le 9 (\log N) \| p \|_\infty .$$

<u>2.3.8 Corollary.</u> [P1, Cor. 3.9]. There exists a constant $K \le K_G \frac{81}{2}$, independent of p and T such that if p has degree $N \ge 2$ then

$$\| p(T) \| \le K c^2 (\log N) \| p \|_\infty . \qquad (*)$$

<u>Proof.</u> Immediate from 2.2.3, 2.3.6 and 2.3.7.

<u>Remark.</u> We note that the term c^2 in (*) also appears in the bounds on the norm of polynomials in a power-bounded invertible (2.1.3).

It is possible to estimate the norm of p in $VMOA*H^1$ directly rather than use the Besov space $B^0_{\infty,1}$. Since this method will be used in Chapter 4 we will establish the following notation.

<u>2.3.9 Notation.</u> For $N \ge 0$ we denote by h_N the polynomial $h_N(z) = \sum_{n=0}^{N} z^n$ on $\mathbb{D}$ and by D_N the Dirichlet Kernel $D_N(z) = \sum_{n=-N}^{N} z^n$ on $\mathbb{D}$.

We note that when $N \ge 0$ is even,

$$h_N(z) = z^{N/2} D_{N/2}(z)$$

and therefore that $\| h_N \|_1 = \| D_{N/2} \|_1$. The numbers $\| D_N \|_1$ are the Lebesgue constants and are known to satisfy

$$\frac{4}{\pi^2} \log N \le \| D_N \|_1 \le \frac{4}{\pi^2} (\log N) + d$$

for some constant $d > 0$ and all $N \in \mathbb{N}$ ([Z, p 67]).

Thus, if p has degree N we write

$$p = p * h_N \in VMOA * H^1 .$$

We can clearly assume N to be even, so that

$$\begin{aligned}\| p \|_{\hat{*}} &= \| p \|_{VMOA} \| h_N \|_1 \\ &\le \| p \|_\infty \| D_{N/2} \|_1 \\ &\le d' (\log N) \| p \|_\infty\end{aligned}$$

for some constant d′, independent of p. It now follows from 2.2.3 and 2.3.3 that

$$\| p(T) \| \le K_G c^2 d'(\log N) \| p \|_\infty .$$

Remark. The advantage of showing first that $\| p \|_{(l^1 \check{\otimes} l^1)/E^-}$ is dominated by the Besov norm $\| p \|^0_{\infty,1}$ of p will become clear in the next section when we consider the functional calculii arising from the bounds on $\| p(T) \|$.

Section 2.4 Functional Calculii for Power-Bounded Operators.

We have shown in Sections 2.2 and 2.3 that if T is a power-bounded operator of $\mathcal{H}$ we can find constants K_1, K_2 and K_3 such that when p is a polynomial

$$\| p(T) \| \le K_1 \| p \|_{(l^1 \check{\otimes} l^1)/E^-} \le K_2 \| p \|_{VMOA \hat{*} H^1} \le K_3 \| p \|^0_{\infty,1}$$

In this section we shall see that, as spaces of functions, the Banach spaces $(l^1 \check{\otimes} l^1)/E^-$, $VMOA \hat{*} H^1$ and $B^0_{\infty,1}$ are algebras and that in each case multiplication is continuous with respect to the Banach space norm. Consequently the map $p \rightsquigarrow p(T)$ extends to representation of these algebras on $B(\mathcal{H})$.

2.4.1 Definition. Suppose that A is an Banach space of functions on a subset Ω of $\mathbb{C}$ such that pointwise multiplication is a closed continuous operation on A and A contains the polynomials. Then an

A-functional calculus for an operator T on $\mathcal{H}$ is a norm continuous algebra homomorphism π of A into $B(\mathcal{H})$ such that $\pi(z^n) = T^n$ for each $n \geq 0$.

To consider $(l^1(\mathbb{Z}^+)\check{\otimes}l^1(\mathbb{Z}^+))/E^-$ as an algebra we will use Peller's equivalent description of $(l^1(\mathbb{Z}^+)\check{\otimes}l^1(\mathbb{Z}^+))/E^-$ as a Banach space of analytic functions on $\mathbb{D}$.

2.4.2 Definition. Let $\mathcal{L}$ be the Banach space of analytic functions $\phi(z) = \sum_{n\geq 0} \hat{\phi}(n)z^n$ on $\mathbb{D}$ for which there exists $\alpha \in l^1(\mathbb{Z}^+)\check{\otimes}l^1(\mathbb{Z}^+)$ with $\sum_{i+j=n}\alpha_{i,j} = \hat{\phi}(n)$ for every $n \geq 0$. We define, for $\phi \in \mathcal{L}$

$$\| \phi \|_{\mathcal{L}} = \inf \left\{ \| \alpha \|_{\check{\otimes}} \ : \ \alpha \in l^1(\mathbb{Z}^+)\otimes l^1(\mathbb{Z}^+) \text{ has } \sum_{i+j=n}\alpha_{ij} = \hat{\phi}(n) \text{ for every } n \geq 0. \right\}$$

The question of whether $\mathcal{L}$ is closed under pointwise multiplication is posed in [P1] and an affirmative solution is outlined in the review of [P1] by Grahame Bennett.

2.4.3 Theorem. [BEN2]. Pointwise multiplication is a closed continuous operation on $\mathcal{L}$.

Proof. Let f,g be polynomials : $f(z) = \sum_{m=0}^{M} \hat{f}(m)z^m$; $g(z) = \sum_{n=0}^{N} \hat{g}(n)z^n$ for $z \in \mathbb{D}$ and define $\hat{f}(m) = 0$ and $\hat{g}(n) = 0$ for $m > M$ and $n > N$. Throughout the proof all sequences are indexed by $\mathbb{Z}^+$.

Let $\alpha,\beta \in l^1\check{\otimes}l^1$ satisfy $\sum_{i+j=m} \alpha_{i,j} = \hat{f}(m)$ and $\sum_{i+j=n} \beta_{i,j} = \hat{g}(n)$ for all $m,n \in \mathbb{Z}^+$. For each $m,n \in \mathbb{Z}^+$ put $\gamma_{m,n} = \hat{f}(m)\hat{g}(n)$ and we define

$\gamma = \left\{ \gamma_{m,n} \right\}_{m,n\in\mathbb{Z}^+}$. Now we shall estimate $\| \gamma \|_{l^1 \check{\otimes} l^1}$. Note that for $d \geq 0$

$$\sum_{m+n=d} \gamma_{m,n} = \sum_{m+n=d} \sum_{i+j=m} \sum_{k+l=n} \alpha_{i,j} \beta_{k,l}$$

$$= \sum_{i+j+k+l=d} \alpha_{i,j} \beta_{k,l} \qquad (*)$$

$$= \sum_{m+n=d} \sum_{i+k=m} \sum_{j+l=n} \alpha_{i,j} \beta_{k,l} .$$

Let $x,y \in l^\infty$. Then

$$\sum_{m+n=d} \gamma_{m,n} \, x(m) \, y(n) = \sum_{m,n\geq 0} \left(\sum_{i+k=m} \sum_{j+l=n} \alpha_{i,j} \beta_{k,l} \right) x(m) \, y(n)$$

by $(*)$

$$= \sum_{m,n\geq 0} \sum_{i=0}^{m} \sum_{j=0}^{n} \alpha_{i,j} \beta_{m-i,n-j} \; x(m) \, y(n)$$

.

The sum has only finitely many non-zero terms so

$$\sum_{m+n\geq 0} \gamma_{m,n} \, x(m) \, y(n) = \sum_{i,j\geq 0} \sum_{m\geq i} \sum_{n\geq j} \alpha_{i,j} \beta_{m-i,n-j} \; x(m) \, y(n) \qquad (**)$$

$$= \sum_{i,j\geq 0} \alpha_{i,j} \left(\sum_{m,n\geq 0} \beta(m,n) \, x(m+i) \, y(n+j) \right)$$

Let c_0 be the usual Banach space of sequences convergent to zero. Then we may define a bounded operator $\theta(\beta)$ from c_0 into l^1 by $(\theta(\beta)x)(m) = \sum_{n\geq 0} \beta_{m,n} \, x(n)$ (for all $x \in c_0$, $m \in \mathbb{Z}^+$). The conjugate map $\theta(\beta)^*$ is a bounded linear operator from l^∞ to l^1 with norm $\| \theta(\beta)^* \|_\infty$ equal to $\| \beta \|_{\check{\otimes}}$ and $(\theta(\beta)^*x)(m) = \sum_{n\geq 0} \beta_{n,m} \, x(n)$ for all $x \in l^\infty$, $m \in \mathbb{Z}^+$. Using the factorisation theorem (1.2.6) we may write $\theta(\beta)^* = PQ$ where $Q \in B(l^\infty, l^2)$ with norm $\| Q \|_{\infty,2}$, $P \in B(l^2, l^1)$ with norm $\| P \|_{2,1}$ and

$$\| \theta(\beta)^* \|_{\infty,1} \le \| P \|_{2,1} \| Q \|_{\infty,2} \le K_G \| \theta(\beta)^* \|_{\infty,1} .$$

Thus if S^* denotes the backward shift on l^∞ we have (by (**))

$$\Big| \sum_{m,n\ge0} \gamma_{m,n} \, x(m) \, y(n) \Big| = \Big| \sum_{i,j\ge0} \alpha_{i,j} < BS^{*j} x , S^{*i} y >_{l^1,l^\infty} \Big|$$

$$= \Big| \sum_{i,j\ge0} \alpha_{i,j} \, (QS^{*j} x, P^*S^{*i} y)_{l^2} \Big|$$

where $P^* \in B(l^\infty, l^2)$ is the conjugate of P.

But $\{QS^{*j} x\}_{j\ge0}$ is a bounded sequence in the Hilbert space l^2 with

$$\| QS^{*j} x \|_2 \le \| Q \|_{\infty,2} \| x \|_\infty$$

for all $j \ge 0$. Similarly $\{P^*S^{*i}y\}_{i\ge0}$ is a bounded sequence in l^2 with

$$\| P^*S^{*i}y \|_2 \le \| P^* \|_{2,1} \| y \|_\infty .$$

Hence by Grothendieck's inequality (1.2.5)

$$\Big| \sum_{m+n\ge0} \gamma_{m,n} \, x(m) \, y(n) \Big| \le K_G \| \alpha \|_{\check\otimes} \| Q \|_{\infty,2} \| y \|_\infty \| P \|_{2,1} \| x \|_\infty$$

$$\le K_G^2 \| \alpha \|_{\check\otimes} \| \theta(\beta) \|_{\infty,2} \| x \|_\infty \| y \|_\infty$$

$$= K_G^2 \| \alpha \|_{\check\otimes} \| \beta \|_{\check\otimes} \| x \|_\infty \| y \|_\infty$$

Since this holds for any $x,y \in l^\infty$ we have

$$\| \gamma \|_{\check\otimes} \le K_G^2 \| \alpha \|_{\check\otimes} \| \beta \|_{\check\otimes} .$$

Finally we note that since matrices with finitely many non-zero entries are dense in $l^1 \check\otimes l^1$ the polynomials form a dense linear subspace of $\mathcal{L}$. Thus if $f,g \in \mathcal{L}$ we have $fg \in \mathcal{L}$ and

$$\| fg \|_{\mathcal{L}} \le K_G^2 \| f \|_{\mathcal{L}} \| g \|_{\mathcal{L}} .$$

<u>2.4.4 Corollary.</u> Let T be a power-bounded operator on $\mathcal{H}$. The map $p \rightsquigarrow p(T)$ on polynomials extends to an $\mathcal{L}$-functional calculus for T.

Peller has shown in [Pl] that similar results hold for the spaces $VMOA\hat{*}H^1$ and $B^0_{\infty,1}$. The resulting functional calculii for a power-bounded operator T justify the estimation of $\| p(T) \|$ in terms of both $\| p \|_{VMOA\hat{*}H^1}$ and $\| p \|^0_{\infty,1}$ in 2.3. We give statements of these results here and refer the reader to [Pl] for the proofs.

2.4.5 **Theorem.** [Pl, Lemma 3.6]. Pointwise multiplication is submultiplicative on $VMOA\hat{*}H^1$.

2.4.6 **Corollary.** [Pl, Corollary 3.7]. Let T be a power-bounded operator on $\mathcal{H}$. Then the map $p \rightsquigarrow p(T)$ extends to a $VMOA\hat{*}H^1$-functional calculus for T.

Similar results for $B^0_{\infty,1}$ are more straightforward.

2.4.7 **Lemma.** [Pl, p352]. Pointwise multiplication is a closed continuous operation on $B^0_{\infty,1}$.

2.4.8 **Corollary.** [Pl, Thm 3.8]. Let T be a power-bounded operator on $\mathcal{H}$. Then the map $p \rightsquigarrow p(T)$ extends to a $B^0_{\infty,1}$-functional calculus for T.

Chapter 3 Hankel Operators on $H^2(\mathbb{T}^2)$.

In this chapter we develop the theory of Hankel operators on the Hardy space $H^2(\mathbb{T}^2)$ so that Peller's methods for estimating the norm of a polynomial in a power-bounded operator can be applied (in Chapter 4) to polynomials in two commuting power-bounded operators.

The Hankel operators on $H^2(\mathbb{T}^2)$ are, however of independent interest and our progress towards a Nehari-type theorem characterising these operators is a continuation of the work of Page ([PA]) and Power ([POW1]).

Section 3.1 Introduction.

We start the chapter with some notation and definitions.

3.1.1 Notation. a) We denote by $\mathbb{T}^2$ the compact topological group $\{(e^{i\theta},e^{i\phi}) : 0\le \theta,\phi \le 2\pi\}$ with (Haar) measure given by $\iint_E dm(\theta)dm(\phi)$ for E the product of two measurable subsets of $\mathbb{T}$. For each $m,n \in \mathbb{Z}$, $e_{m,n}$ is the function on $\mathbb{T}^2$ given by $e_{m,n}(e^{i\theta},e^{i\phi}) = e^{i(m\theta+n\phi)}$.

b) For $1 \le p \le \infty$, $f \in L^p(\mathbb{T}^2)$ and $m,n \in \mathbb{Z}$ we denote by $\hat{f}(m,n)$ the (m,n)-Fourier coefficient of f :

$$\hat{f}(m,n) = \int_0^{2\pi}\int_0^{2\pi} f(e^{i\theta},e^{i\phi})\, e^{-i(m\theta+n\phi)}\, dm(\theta)dm(\phi) .$$

c) For $1 \le p \le \infty$ we denote by $H^p(\mathbb{T}^2)$ the closed subspace of $L^p(\mathbb{T}^2)$ consisting of those $f \in L^p(\mathbb{T}^2)$ for which $\hat{f}(m,n) = 0$ when $m < 0$ or $n < 0$.

Remarks. 1. The set $\{e_{m,n}\}_{m,n\in\mathbb{Z}}$ forms a complete orthonormal basis for $L^2(\mathbb{T}^2)$ and $H^2(\mathbb{T}^2)$ is the closed linear span of $\{e_{m,n}\}_{m,n\in\mathbb{Z}^+}$.

2. If $T \in B(L^2(\mathbb{T}^2))$ then for each $k,l,m,n \in \mathbb{Z}$ we put

$$T_{k,l,m,n} = (Te_{m,n}, e_{k,l}).$$

Then the array $\{T_{k,l,m,n}\}_{k,l,m,n\in\mathbb{Z}}$ represents T in the sense that for any $f \in L^2(\mathbb{T}^2)$ and $m,n \in \mathbb{Z}$

$$(Tf)\hat{}(k,l) = \sum_{m,n\in\mathbb{Z}} T_{k,l,m,n}\ \hat{f}(m,n) .$$

<u>3.1.2 Definitions.</u> a) A bounded operator T on $H^2(\mathbb{T}^2)$ is a <u>Hankel operator</u> if the array representing T satisfies

$$(Te_{m,n}, e_{k,l}) = \beta_{k+m,l+n} \qquad (k,l,m,n \in \mathbb{Z})$$

for some matrix of scalars $\{\beta_{i,j}\}_{i,j\in\mathbb{Z}^+}$.

b) If ϕ is an analytic function on $\mathbb{D}^2$ with power series $\phi(z_1,z_2) = \sum_{m,n\geq 0} \hat{\phi}(m,n)\ z_1^m\ z_1^n$ then ϕ has the <u>Bounded Hankel Property</u> if the array $\{\hat{\phi}(k+m,l+n)\}_{k,l,m,n\in\mathbb{Z}^+}$ is the representing array of some Hankel operator T_ϕ on $H^2(\mathbb{T}^2)$. In this case we say that $\phi \in$ BHP and we norm the linear space BHP by $\| \phi \|_{BHP} = \| T_\phi \|_{B(H^2(\mathbb{T}^2))}$.

<u>Motivation.</u>

The theorem of Nehari (1.3.5) characterising Hankel operators on $H^2(\mathbb{T})$ shows that the Banach space of analytic functions on $\mathbb{D}$ whose coefficients give rise to bounded Hankel operators is isometrically isomorphic to $L^\infty(\mathbb{T})/H_0^\infty(\mathbb{T})$. Since $L^\infty(\mathbb{T})/H_0^\infty(\mathbb{T})$ is isometrically isomorphic of the dual of $H^1(\mathbb{T})$ we obtain a norm preserving correspondence between the analytic functions on $\mathbb{D}$ giving Hankel operators and the continuous linear functionals on $H^1(\mathbb{T})$. This link between Hankel operators and linear functionals is exploited in the proof of 2.2.6 to show that the Hankel matrix Γ_ϕ is a Schur multiplier only when ϕ is a multiplier of $H^1(\mathbb{D})$.

The following conjecture is the $H^2(\mathbb{T}^2)$ version of Nehari's theorem, and remains undecided.

<u>3.1.3 Conjecture.</u> Let ϕ be an analytic function on $\mathbb{D}^2$ with power series $\phi(z_1,z_2) = \sum_{m,n\geq 0} \hat{\phi}(m,n)\ z_1^m\ z_2^n$. Then $\phi \in$ BHP if and only if there exists $\psi \in L^\infty(\mathbb{T}^2)$ such that

$$\hat{\phi}(m,n) = \hat{\psi}(-m,-n)$$

for all $m,n \in \mathbb{Z}^+$. In this case there exists such a ψ with $\| \psi \|_\infty = \| \phi \|_{BHP}$.

As with Nehari's theorem, the sufficiency part is easy. Briefly, if J denotes the flip operator

$$(Jf)(e^{i\theta},e^{i\phi}) = f\ (e^{-i\theta},e^{-i\phi})$$

$(f \in L^2(\mathbb{T}^2),\ e^{i\theta},e^{i\phi} \in \mathbb{T})$ on $L^2(\mathbb{T}^2)$, if P is the orthogonal projection of $L^2(\mathbb{T}^2)$ onto $H^2(\mathbb{T}^2)$ and if M_ψ is multiplication by some $\psi \in L^\infty(\mathbb{T}^2)$ on $L^2(\mathbb{T}^2)$ then the operator

$$T = PJM_\psi|_{H^2(\mathbb{T}^2)}$$

is a Hankel operator with

$$(Te_{m,n},e_{k,l}) = \hat{\psi}(-k-m,-l-n) \text{ and } \| T \| \leq \| \psi \|_\infty.$$

A corollary of 3.1.3 would be that BHP is isometrically isomorphic to $L^\infty(\mathbb{T}^2)/H^1(\mathbb{T}^2)^\perp$, where $H^1(\mathbb{T}^2)^\perp$ is the annihilator of $H^1(\mathbb{T}^2)$ in $L^\infty(\mathbb{T}^2)$(the dual of $L^1(\mathbb{T}^2)$). Clearly, we would then have BHP isometrically isomorphic to $H^1(\mathbb{T}^2)^*$.

However, in the absence of a proof of the conjecture we shall show instead that BHP is isometrically isomorphic to the dual of $H^1(\mathcal{C}^1)/H^1(B)$ where $\mathcal{C}^1$ is the trace-class of operators on $H^2(\mathbb{T})$;B is the pre-annihilator in $\mathcal{C}^1$ of the class of Hankel operators on $H^2(\mathbb{T})$ and $H^1(B)$, $H^1(\mathcal{C}^1)$ are the H^1-spaces of B-valued and $\mathcal{C}^1$-valued functions on $\mathbb{T}$, respectively.

We will need some preliminary definitions for the study of vector and operator valued functions.

3.1.4 Definitions. a) Let ϕ be a $B(\mathcal{H})$-valued function on $\mathbb{T}$. Then ϕ is weak operator (wo) measurable if the function $e^{i\theta} \rightsquigarrow (\phi(e^{i\theta})x,y)$ is measurable for every $x,y \in \mathcal{H}$.

b) Let X be a Banach space and let f be an X-valued function on $\mathbb{T}$. Then f is

i) a measurable simple function if there exists disjoint measurable subsets of $\mathbb{T}$, $A_1, A_2, \ldots A_N$ and $x_1, x_2, \ldots x_N \in X$ such that $f = \sum_{n=1}^{N} x_n \chi_{A_n}$;

ii) weakly measurable if the function $e^{i\theta} \rightsquigarrow \langle f(e^{i\theta}), y\rangle$ is measurable for every $y \in X^*$

and iii) strongly measurable if there exists a sequence $\{f_n\}_{n\geq 0}$ of measurable simple X-valued functions on $\mathbb{T}$ such that

$$\lim_{n\to\infty} \| f(e^{i\theta}) - f_n(e^{i\theta}) \|_X = 0 \text{ for a.e. } e^{i\theta} \in \mathbb{T}.$$

c) Let X be a Banach space and let f be an X-valued strongly measurable function on $\mathbb{T}$. If there exists a sequence $\{f_n\}_{n\geq 0}$ of measurable simple X-valued functions on $\mathbb{T}$ such that

$$\lim_{n\to\infty} \int_0^{2\pi} \| f(e^{i\theta}) - f_n(e^{i\theta}) \|_X \, dm(\theta) = 0$$

then we say that f is Bochner integrable and we define the Bochner integral of f over $\mathbb{T}$ by $\int_0^{2\pi} f(e^{i\theta})dm(\theta) = \lim_{n\to\infty} \int_0^{2\pi} f_n(e^{i\theta})dm(\theta)$.

Remarks. 1. Pettis' Measurability Theorem ([DU, p 42]) states that a Banach space-valued function which is weakly measurable and has essentially separable range is necessarily strongly measurable.

Thus, every weakly measurable $\mathscr{H}$-valued function is strongly measurable so henceforth we will refer to such functions as measurable functions.

2. If f is strongly measurable and $\int_0^{2\pi} \| f(e^{i\theta}) \|_X dm(\theta) < \infty$ then f is Bochner integrable with

$$\| \int_0^{2\pi} f(e^{i\theta})\, dm(\theta) \|_X \le \int_0^{2\pi} \| f(e^{i\theta}) \|_X dm(\theta) .$$

3.1.5 Notation. a) The Banach space $L_w^\infty(B(\mathscr{H}))$ consists of (equivalence classes of) wo-measurable essentially bounded $B(\mathscr{H})$-valued functions on $\mathbb{T}$ normed by the essential supremum norm

$$\| \phi \|_\infty = \operatorname{ess\,sup} \left\{ \| \phi(e^{i\theta}) \|_{B(\mathscr{H})} : e^{i\theta} \in \mathbb{T} \right\} .$$

b) If $\phi \in L_w^\infty(B(\mathscr{H}))$ then the Fourier series for ϕ : $\sum_{n\in\mathbb{Z}} \hat{\phi}(n)e^{in\theta}$ has coefficients $\hat{\phi}(n) \in B(\mathscr{H})$ defined by

$$(\hat{\phi}(n)x,y) = \int_0^{2\pi} e^{-in\theta}(\phi(e^{i\theta})x,y)\, dm(\theta)$$

(for $x,y \in \mathscr{H}$, $n \in \mathbb{Z}$).

c) The subspace $H_0^\infty(B(\mathscr{H}))$ of $L_w^\infty(B(\mathscr{H}))$ consists of those $\phi \in L_w^\infty(B(\mathscr{H}))$ with $\hat{\phi}(n) = 0$ for each $n \le 0$.

d) For $1 \le p \le \infty$ and a Banach space X, let $L^p(\mathbb{T};X)$ denote the Banach space of (equivalence classes of) strongly measurable X-valued functions on $\mathbb{T}$ for which

$$\| f \|_p = \left(\int_0^{2\pi} \| f(e^{i\theta}) \|_X^p \, dm(\theta) \right)^{1/p} < \infty .$$

e) If $f \in L^p(\mathbb{T};X)$ then the Fourier series for f, $\sum_{n\in\mathbb{Z}} \hat{f}(n)e^{in\theta}$ has coefficients $\hat{f}(n) \in X$ defined by

$$\hat{f}(n) = \int_0^{2\pi} e^{-in\theta} f(e^{i\theta})\, dm(\theta) .$$

for all $n \in \mathbb{Z}$. (The Bochner integral exists since $f \in L^p(\mathbb{T};X)$

ensures that $\int_0^{2\pi} \| f(e^{i\theta}) \|_X \, dm(\theta) < \infty$).

f) For $1 \le p \le \infty$, $H^p(\mathbb{T};X)$ is the closed subspace of $L^p(\mathbb{T};X)$ consisting of those $f \in L^p(\mathbb{T};X)$ with $\hat{f}(n) = 0$ for all $n < 0$.

Remarks. 1. For further details on the X-valued L^p spaces we refer the reader to [DU].

2. We note that $L^2(\mathbb{T};\mathcal{H})$ is a Hilbert space with inner product

$$(f,g)_{L^2(\mathbb{T};\mathcal{H})} = \int_0^{2\pi} (f(e^{i\theta})g(e^{i\theta}))_{\mathcal{H}} \, dm(\theta)$$

($f,g \in L^2(\mathbb{T};\mathcal{H})$). Also, for any $g \in L^2(\mathbb{T};\mathcal{H})$

i) $\sum_{n\in\mathbb{Z}} \hat{f}(n)e^{in\theta}$ converges absolutely in $\| \cdot \|_2$ to f ;

ii) $\| f \|_2^2 = \sum_{n\in\mathbb{Z}} \| \hat{f}(n) \|_{\mathcal{H}}^2$

and iii) $(f,g)_{L^2(\mathbb{T};\mathcal{H})} = \sum_{n\in\mathbb{Z}} (\hat{f}(n),\hat{g}(n))_{\mathcal{H}}$.

3. If $\phi \in L^\infty_w(B(\mathcal{H}))$ then the wo-measurability of ϕ ensures that for each $x \in \mathcal{H}$ the function $e^{i\theta} \rightsquigarrow \phi(e^{i\theta})x$ is a measurable $\mathcal{H}$-valued function. Since $\int_0^{2\pi} \| \phi\, (e^{i\theta})x \|_{\mathcal{H}} \, dm(\theta) \le \| \phi \|_\infty \| x \|_{\mathcal{H}}$ the integral $\int_0^{2\pi} \phi\, (e^{i\theta})x \, dm(\theta)$ exists as an $\mathcal{H}$-valued Bochner integral.

Section 3.2 Vectorial Hankel Operators.

We recall from 1.3 that an operator $T \in B(H^2(\mathbb{T}))$ is a Hankel operator if and only if T intertwines the unilateral shift S and its adjoint S^* : $S^*T = TS$. We use this property as our definition of a vectorial Hankel operator on $H^2(\mathbb{T};\mathcal{H})$.

3.2.1 Definition. a) Define a unilateral shift S_1 (of infinite multiplicity) on $H^2(\mathbb{T};\mathcal{H})$ by

$$(S_1 f)(e^{i\theta}) = e^{i\theta} f(e^{i\theta})$$

for $f \in H^2(\mathbb{T};\mathcal{H})$, $e^{i\theta} \in \mathbb{T}$.

b) The corresponding bilateral shift is denoted by U_1 where $U_1 \in B(L^2(\mathbb{T};\mathcal{H}))$ and

$$(U_1 f)(e^{i\theta}) = e^{i\theta} f(e^{i\theta})$$

for all $f \in L^2(\mathbb{T};\mathcal{H})$ and $e^{i\theta} \in \mathbb{T}$.

3.2.2 Definition. Let $T \in B(L^2(\mathbb{T};\mathcal{H}))$. Then T is a vectorial Hankel operator if $S_1^* T = T S_1$.

Remarks. Recall that an operator A on $H^2(\mathbb{T})$ has a representing matrix of scalars $\{A_{i,j}\}_{i,j\geq 0}$ which satisfies $A_{i,j} = (Ae_j, e_i)$ for $i,j \in \mathbb{Z}^+$ and $(Af)\hat{}(m) = \sum_{m\geq 0} A_{m,n} \hat{f}(n)$ for $m \in \mathbb{Z}^+$ and $f \in H^2(\mathbb{T})$.

Likewise an operator T on $H^2(\mathbb{T};\mathcal{H})$ has a representing matrix of bounded linear operators on $\mathcal{H}$, $\{T_{i,j})_{i,j\in\mathbb{Z}^+}$. Specifically, we define each $T_{i,j}$ by the formula

$$(T_{i,j}\, x, y)_{H^2(\mathbb{T})} = (T(e_j(\cdot)x,\, e_i(\cdot)y))_{H^2(\mathbb{T};\mathcal{H})}$$

for $x,y \in \mathcal{H}$ and $i,j \in \mathbb{Z}^+$. Then for each $f \in H^2(\mathbb{T};\mathcal{H})$ and $m \in \mathbb{Z}^+$ the sum $\sum_{n\geq 0} T_{m,n} \hat{f}(n)$ converges strongly in $\mathcal{H}$ to $(Tf)\hat{}(m)$.

As we would hope, the condition $S_1^* T = T S_1$ on $T \in B(H^2(\mathbb{T};\mathcal{H}))$ is equivalent to the representing matrix of T being of the form $\{T_{i+j})_{i,j\in\mathbb{Z}^+}$ for some sequence of operators $\{T_n\}_{n\in\mathbb{Z}^+}$ on $H^2(\mathbb{T};\mathcal{H})$.

The main result of this section is Page's theorem characterising vectorial Hankel operators on $H^2(\mathbb{T};\mathcal{H})$. The proof follows that of Nehari's theorem given in 1.3 , using the lifting theorem (1.3.4) and a characterisation of the commutant of a bilateral shift on $L^2(\mathbb{T};\mathcal{H})$. We give a full proof of the latter, requiring the following preliminary result.

<u>3.2.3 Lemma.</u> Let Φ be a bounded sesquilinear map of $\mathcal{H}x\mathcal{H}$ into $L^\infty(\mathbb{T})$. Then there exists $\Psi \in L^\infty_w(B(\mathcal{H}))$ such that

$$\Phi(x,y) = (\Psi(\cdot)x,y) \qquad \text{for } x,y \in \mathcal{H},$$

and $$\|\Psi\|_\infty = \|\Phi\| .$$

<u>Proof.</u> Let $\mathcal{D}$ be a countable dense complex rational subspace of $\mathcal{H}$. For each $x,y \in \mathcal{D}$ let $\phi_{x,y}$ be a representative of the equivalence class $\Phi(x,y)$. We shall construct a set of full measure in $\mathbb{T}$ such that for $e^{i\theta}$ in this set the values $\phi_{x,y}(e^{i\theta})$ are sesquilinear in x,y (on $\mathcal{D}x\mathcal{D}$).

By Lebesgue's Differentiation Theorem ([RO, pp 102-103]), when $x,y \in \mathcal{D}$

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_0^\varepsilon \phi_{x,y}(e^{i(\theta+t)})\, dm(t) \tag{$*$}$$

exists and is equal to $\phi_{x,y}(e^{i\theta})$ for a.e. $e^{i\theta} \in \mathbb{T}$. Thus we obtain sets $E_{x,y} \subseteq \mathbb{T}$ with $m(\mathbb{T}\backslash E_{x,y}) = 0$ such that $(*)$ exists and equals $\phi_{x,y}(e^{i\theta})$ for $e^{i\theta} \in E_{x,y}$, $x,y \in \mathcal{D}$. Now put $E = \cap_{x,y\in\mathcal{D}} E_{x,y}$. Then E is measurable, $m(\mathbb{T}\backslash E) = 0$ and moreover $(*)$ exists and equals $\phi_{x,y}(e^{i\theta})$ for $e^{i\theta} \in E$, $x,y \in \mathcal{D}$. For such $e^{i\theta}$,x,y put

$$\begin{aligned} S_{e^{i\theta}}(x,y) &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_0^\varepsilon \phi_{x,y}(e^{i(\theta+t)})\, dm(t) \\ &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_0^\varepsilon \Phi_{x,y}(e^{i(\theta+t)})\, dm(t) . \end{aligned}$$

Since Φ is bounded and sesquilinear we clearly have, for each $e^{i\theta} \in E$, a bounded sesquilinear map $S_{e^{i\theta}}$ on $\mathcal{D}x\mathcal{D}$ with norm not greater than $\|\Phi\|$. We shall now derive from $S_{e^{i\theta}}(e^{i\theta} \in E)$ an operator $\Psi(e^{i\theta})$ on $\mathcal{H}$.

Fix $e^{i\theta} \in E$. For each $x \in \mathfrak{D}$

$$f(e^{i\theta},x)(y) = \overline{S_{e^{i\theta}}(y)} \qquad (y \in \mathfrak{D})$$

defines a bounded linear functional $f(e^{i\theta},x)$ on $\mathfrak{D}$ with $\| f(e^{i\theta},x) \| \leq \| \Phi \| \| x \|_{\mathcal{H}}$. We extend each $f(e^{i\theta},x)$ to $\tilde{f}(e^{i\theta},x)$ on $\mathcal{H}$ with the same norm. By the Riesz Representation Theorem there exists $g(e^{i\theta},x) \in \mathcal{H}$ such that $\tilde{f}(e^{i\theta},x)(y) = (y,g(e^{i\theta},x))$ $(y \in \mathcal{H}, x \in \mathfrak{D})$, and $\| g(e^{i\theta},x) \|_{\mathcal{H}} = \| \tilde{f}(e^{i\theta},x) \| \leq \| \Phi \| \| x \|_{\mathcal{H}}$. The map sending $x \in \mathcal{H}$ to $g(e^{i\theta},x)$ is a bounded linear map of $\mathfrak{D}$ into $\mathcal{H}$ which we extend by continuity to an operator $\Psi(e^{i\theta})$ on $\mathcal{H}$ with $\| \Psi(e^{i\theta}) \| \leq \| \Phi \|$.

For $x,y \in \mathfrak{D}$,

$$\begin{aligned}(\Psi(e^{i\theta})x,y) &= (g(e^{i\theta},x),y) \\ &= \overline{(\tilde{f}(e^{i\theta},x),y)} \\ &= S_{e^{i\theta}}(x,y) \\ &= \phi_{x,y}(e^{i\theta}).\end{aligned}$$

We repeat for each $e^{i\theta} \in E$ and put $\Psi(e^{i\theta}) = 0$ when $e^{i\theta} \in \mathbb{T}\backslash E$ so that $\Psi : \mathbb{T} \longrightarrow B(\mathcal{H})$ and

$$(\Psi(\cdot)\, x,y) = \Phi(x,y) \tag{**}$$

for every $x,y \in \mathfrak{D}$.

To show that (**) holds for any $x,y \in \mathcal{H}$ we approximate $x,y \in \mathcal{H}$ by sequences $\{x_n\}_{n\in\mathbb{Z}^+}$ $\{y_n\}_{n\in\mathbb{Z}^+}$ in $\mathfrak{D}$ and note that

$$\begin{aligned}\| (\Psi(\cdot)x,y) - \Phi(x,y) \|_\infty \leq\ & \| (\Psi(\cdot)x,y) - (\Psi(\cdot)x,y_n) \|_\infty \\ &+ \| (\Psi(\cdot)x,y_n) - (\Psi(\cdot)x_n,y_n) \|_\infty \\ &+ \| (\Psi(\cdot)x_n,y_n) - (\Phi(x_n,y) \|_\infty \\ &+ \| \Phi(x_n,y) - \Phi(x,y) \|_\infty\end{aligned}$$

Each of the four terms tends to zero as $n \longrightarrow \infty$. For example,

$$\begin{aligned}\| (\Psi(\cdot)x,y) - (\Psi(\cdot)x,y_n \|_\infty &= \| (\Psi(\cdot)x,y-y_n \|_\infty \\ &\leq \| \Psi(\cdot) \|_{B(\mathcal{H})} \| x \|_{\mathcal{H}} \| y-y_n \|_{\mathcal{H}} \\ &\leq \| \Phi \| \| x \|_{\mathcal{H}} \| y-y_n \|_{\mathcal{H}} \rightarrow 0 \text{ as } n \longrightarrow \infty\end{aligned}$$

and $\| \Phi(x_n,y) - \Phi(x,y) \| = \| \Phi(x_n - x,y) \|_\infty$

$$\leq \| \Phi \| \, \| x_n - x \|_{\mathscr{H}} \| y \|_{\mathscr{H}} \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

thus $(\Psi(\cdot)x,y) = \Phi(x,y)$ as required.

Finally, using (**) we see that each Ψ is weak-operator measurable and that $\| \Psi \|_\infty = \| \Phi \|$.

<u>3.2.4 Definition.</u> Let $\Phi : T \longrightarrow B(\mathscr{H})$ be a wo-measurable function. then for any $f \in L^2(\mathscr{H})$ the $\mathscr{H}$-valued function $M_\phi f$ is defined a.e. by

$$(M_\phi f)(e^{i\theta}) = \Phi(e^{i\theta})f(e^{i\theta}).$$

The following result is apparently well-known (eg.[PA, p 534]).

<u>3.2.5 Lemma.</u> a) M_Φ is a bounded operator on $L^2(\mathbb{T};\mathscr{H})$ if and only if $\Phi \in L^\infty_w(B(\mathscr{H}))$. In this case $\| M_\Phi \| = \| \Phi \|_\infty$.

b) The commutant of U_1 is $\left\{M_\Phi : \Phi \in L^\infty_w(B(\mathscr{H}))\right\}$.

<u>Proof.</u> <u>(a)</u> Suppose that M_Φ is a bounded operator on $L^2(\mathbb{T};\mathscr{H})$. Consider for fixed $x,y \in \mathscr{H}$ multiplication by $(\Phi(\cdot)(x,y))$ on $L^2(\mathbb{T})$. For $f,g \in L^2(\mathbb{T})$

$$\left| \int_0^{2\pi} (\Phi(e^{i\theta})x,y)f(e^{i\theta})\overline{g(e^{i\theta})}\, dm(\theta) \right|$$

$$= \left| \int_0^{2\pi} (\Phi(e^{i\theta})(f(e^{i\theta})x),g(e^{i\theta})y)\, dm(\theta) \right|$$

$$= \left| (M_\Phi(f(\cdot)x), g(\cdot)y) \right|$$

$$\leq \| M_\Phi \| \, \| f \|_2 \| x \|_{\mathscr{H}} \| g \|_2 \| y \|_{\mathscr{H}} .$$

So multiplication by $(\Phi(\cdot)x,y)$ is a bounded operator on $L^2(\mathbb{T})$ with norm not greater than $\| M_\Phi \| \, \| x \|_{\mathscr{H}} \| y \|_{\mathscr{H}}$. It follows that

$(\Phi(\cdot)(x,y))$ is essentially bounded on $\mathbb{T}$ with $\| (\Phi(\cdot)(x,y)) \|_\infty \leq \| M_\Phi \| \| x \|_{\mathcal{H}} \| y \|_{\mathcal{H}}$.

Thus we may find for each x,y in a countable dense subset $\mathcal{D}$ of $\mathcal{H}$ a set $E_{x,y} \subseteq \mathbb{T}$ with $M(\mathbb{T}\backslash E_{x,y}) = 0$ and

$$|(\Phi(e^{i\theta})x,y)| \leq \| M_\Phi \| \| x \|_{\mathcal{H}} \| y \|_{\mathcal{H}} \tag{**}$$

for every $e^{i\theta} \in E_{x,y}$. Putting $E = \bigcap_{x,y\in\mathcal{D}} E_{x,y}$ we get a measurable subset E of $\mathbb{T}$ such that $M(\mathbb{T}\backslash E) = 0$ and (**) true for every $x,y \in \mathcal{D}$ and every $e^{i\theta} \in E$. Since $\mathcal{D}$ is dense in $\mathcal{H}$ we conclude that $\| \Phi(e^{i\theta}) \|_{B(\mathcal{H})} \leq \| M_\Phi \|$ when $e^{i\theta} \in E$ and hence that Φ is essentially bounded with $\| \Phi \|_\infty \leq \| M_\Phi \|$.

Conversely, suppose that $\Phi \in L^\infty_w(B(\mathcal{H}))$. We show that for $f \in L^2(\mathbb{T};\mathcal{H})$

i) $(\Phi(\cdot)f(\cdot),y)$ is a measurable function on $\mathbb{T}$ for each $y \in \mathcal{H}$ and ii) $\int_0^{2\pi} \| \Phi(e^{i\theta})f(e^{i\theta}) \|_2^2 \, dm(\theta) \leq \| \Phi \|_\infty^2 \| f \|_2^2$.

For (i) we choose a sequence $\{f_n\}_{n\in\mathbb{Z}^+}$ of measurable simple $\mathcal{H}$-valued functions such that $\| f(e^{i\theta}) - f_n(e^{i\theta}) \|_{\mathcal{H}} \longrightarrow 0$ a.e. Then for each $y \in \mathcal{H}$ the functions $(\Phi(\cdot)f_n(\cdot),y)$ are measurable and converge pointwise a.e. to the function $(\Phi(\cdot)f(\cdot),y)$.

Since (ii) is trivial we conclude that M_Φ is a bounded operator on $L^2(\mathbb{T};\mathcal{H})$ with norm dominated by $\| \Phi \|_\infty$.

<u>(b)</u> Suppose that $S \in B(L^2(\mathbb{T};\mathcal{H}))$ commutes with U_1. For $x \in \mathcal{H}$ denote by c_x the function on $\mathbb{T}$ defined by $c_x(e^{i\theta}) = x$ for $e^{i\theta} \in \mathbb{T}$. Let f be a trigonometric polynomial $f(e^{i\theta}) = \sum_{-N}^{N} \hat{f}(n) e^{in\theta}$. Then

$$S(f(\cdot)x) = S\left(\sum_{-N}^{N} \hat{f}(n)xe_n\right)$$

$$= S\left(\sum_{-N}^{N} \hat{f}(n)U_1^n(c_x)\right)$$

$$= \sum_{-N}^{N} \hat{f}(n)U_1^n S(c_x)$$

since S commutes with U_1 , and so for a.e. $e^{i\theta} \in \mathbb{T}$

$$S(f(\cdot)x)(e^{i\theta}) = \sum_{-N}^{N} \hat{f}(n)\left[U_1^n S(c_x)\right](e^{i\theta})$$

$$= \sum_{-N}^{N} \hat{f}(n)e^{in\theta}\ S(c_x)(e^{i\theta})$$

$$= f(e^{i\theta})S(c_x)(e^{i\theta}).$$

Now fix $x,y \in \mathscr{H}$ and put

$$\Psi_{x,y}(e^{i\theta}) = (S(c_x)(e^{i\theta}),y)\ .$$

For any trigonometric polynomial f and any $g \in L^2(\mathbb{T})$

$$(S(f(\cdot)x),g(\cdot)y) = \int_0^{2\pi} (f(e^{i\theta})S(c_x)(e^{i\theta}),g(e^{i\theta})y)\ dm(\theta)$$

$$= \int_0^{2\pi} (S(c_x)(e^{i\theta}),y)f(e^{i\theta})\overline{g(e^{i\theta})}\ dm(\theta)$$

$= (M'_{\Psi_{x,y}} f,g)$ where $M'_{\Psi_{x,y}}$ denotes multiplication by $\Psi_{x,y}$ on $L^2(\mathbb{T})$.

Since S is bounded we have

$$|(M'_{\Psi_{x,y}} f,g)| \le \| S \| \| f \|_2 \| x \|_{\mathscr{H}} \| g \|_2 \| y \|_{\mathscr{H}}\ .$$

It follows by a standard argument using a.e. convergence that $M'_{\Psi_{x,y}}$ is bounded on $L^2(\mathbb{T})$ with norm dominated by $\| S \|\ \| x \|_{\mathscr{H}} \| y \|_{\mathscr{H}}$

Thus $\Psi_{x,y}$ is essentially bounded with $\| \Psi_{x,y} \|_\infty \le \| S \|\ \| x \|_{\mathscr{H}} \| y \|_{\mathscr{H}}$

We apply 3.2.3 to the sesquilinear form $x,y \rightsquigarrow \Psi_{x,y}$ to get $\Psi \in L_w^\infty(B(\mathcal{H}))$ with $(\Phi(\cdot)x,y) = \Psi_{x,y}$ for all $x,y \in \mathcal{H}$ and $\|\Phi\|_\infty \le \|S\|$

Now if $f \in L^2(\mathbb{T};\mathcal{H})$ is a trigonometric polynomial, $f(e^{i\theta}) = \sum_{-N}^{N} \hat{f}(n)e^{in\theta}$ for some $\hat{f}(n) \in \mathcal{H}$ $(-N \le n \le N)$ then $Sf = \sum_{-N}^{N} U_1^n S(c_{\hat{f}(n)})$.

So if g is also a trigonometric polynomial in $L^2(\mathcal{H})$, $g(e^{i\theta}) = \sum_{-M}^{M} \hat{g}(m)e^{im\theta}$ with $\hat{g}(m) \in \mathcal{H}$ $(-M \le m \le M)$ then

$$
\begin{aligned}
(Sf,g) &= \Big(\sum_{-N}^{N} U_1^n S(c_{\hat{f}(n)}),\ \sum_{-M}^{M} U_1 \ S(c_{\hat{g}(m)})\Big) \\
&= \sum_{-N}^{N}\sum_{-M}^{M} \big(U_1^n S(c_{\hat{f}(n)}),\ U_1 S(c_{\hat{g}(m)})\big) \\
&= \sum_{-N}^{N}\sum_{-M}^{M} \int_0^{2\pi} e^{in\theta} \big(S(c_{\hat{f}\ (n)})(e^{i\theta}), e^{im\theta}\ \hat{g}(m) \big)\, dm(\theta) \\
&= \sum_{-N}^{N}\sum_{-M}^{M} \int_0^{2\pi} e^{i(n-m)\theta} \big(S(c_{\hat{f}(n)})(e^{i\theta}),\ \hat{g}(m) \big)\, dm(\theta) \\
&= \sum_{-N}^{N}\sum_{-M}^{M} \int_0^{2\pi} e^{i(n-m)\theta} \big(\Phi\, (e^{i\theta})\hat{f}(n),\ \hat{g}(m) \big)\, dm(\theta) \\
&= \int_0^{2\pi} \Phi(e^{i\theta})\Big(\Big[\sum_{-N}^{N} \hat{f}(n)e^{in\theta} \Big],\Big[\sum_{-M}^{M} \hat{g}(m)e^{im\theta} \Big]\Big)\, dm(\theta) \\
&= (M_\Phi f,g)
\end{aligned}
$$

Since the trigonometric polynomials are dense in $L^2(\mathbb{T};\mathcal{H})$ we conclude that $S = M_\Phi$.

Conversely, that any M_Φ commutes with U_1 is shown by :

$$
\begin{aligned}
(M_\Phi U_1)(f)(e^{i\theta}) &= \Phi(e^{i\theta})\ (e^{i\theta}\ f\ (e^{i\theta})) \\
&= e^{i\theta}\ \Phi(e^{i\theta})f(e^{i\theta}) \\
&= (U_1 M_\Phi f)(e^{i\theta})
\end{aligned}
$$

(for $f \in L^2(\mathbb{T};\mathcal{H})$, $\Phi \in L_w^\infty(B(\mathcal{H}))$ and a.e. $e^{i\theta} \in \mathbb{T}$).

3.2.6 Definition. a) The 'flip' operator J on $L^2(\mathbb{T};\mathcal{H})$ is defined at $f \in L^2(\mathbb{T};\mathcal{H})$ by $(Jf)(e^{i\theta}) = f(e^{-i\theta})$ $(e^{i\theta} \in \mathbb{T})$.

b) The orthogonal projection of $L^2(\mathbb{T};\mathcal{H})$ onto $H^2(\mathbb{T};\mathcal{H})$ is denoted by P.

Remark. If $\Phi \in L^\infty_w(B(\mathcal{H}))$ and $T = PJM_\Phi|_{H^2(\mathbb{T};\mathcal{H})}$ then T is a vectorial Hankel operator on $H^2(\mathbb{T};\mathcal{H})$. To see this we can examine the (m,n)-th entry of the representing matrix of operators for T :

For $x,y \in \mathcal{H}$, $m,n \in \mathbb{Z}^+$

$$\begin{aligned}
(T(e_n(\cdot)x),e_m(\cdot)y) &= (JM_\Phi(e_n(\cdot)x),e_m(\cdot)y) \\
&= \int_0^{2\pi} (e^{-in\theta}\,\Phi(e^{-i\theta})x, e^{im\theta}\,y)\,dm(\theta) \\
&= \int_0^{2\pi} e^{-i(m+n)\theta}\,(\Phi(e^{-i\theta})x,y)\,dm(\theta) \\
&= \int_0^{2\pi} e^{i(m+n)\theta}\,(\Phi(e^{i\theta})x,y)\,dm(\theta) \\
&= (\hat{\Phi}(-m-n)x,y) .
\end{aligned}$$

Thus the (m,n)-th entry is $\hat{\Phi}(-m,-n)$ and so the matrix has Hankel form.

3.2.7 Theorem. [PA, Thm 5, p 534] . Let $T \in B(H^2(\mathbb{T};\mathcal{H}))$. Then T is a vectorial Hankel operator if and only if $T = PJM_\Phi|_{H^2(\mathbb{T};\mathcal{H})}$ for some $\Phi \in L^\infty_w(B(\mathcal{H}))$. In this case we can choose Φ with $\|\Phi\|_\infty = \|T\|$.

Proof. Note that the minimal unitary dilation of S_1 is the bilateral shift U_1. Now if T intertwines S_1 and S_1^* then by the Nagy-Foias lifting theorem (1.3.4) there exists an operator V on $L^2(\mathbb{T};\mathcal{H})$ such that $U_1^*V = VU_1$, $\|V\| = \|T\|$ and $T = PV|_{H^2(\mathbb{T};\mathcal{H})}$. It follows that JV commutes with U_1 and so by 3.2.5(b), $JV = M_\Phi$ for some $\Phi \in L^\infty_w(B(\mathscr{H}))$. Thus $\|\Phi\|_\infty = \|M_\Phi\| = \|JV\| = \|V\| = \|T\|$ and $T = PJM_\Phi|_{H^2(\mathbb{T};\mathscr{H})}$.

It is clear from the remarks above that the theorem is equivalent to the following result.

<u>3.2.8 Corollary.</u> Let $T \in B(H^2(\mathbb{T};\mathscr{H}))$ have representing matrix $\{T_{i,j}\}_{i,j\in\mathbb{Z}^+}$. Then T is a vectorial Hankel operator if and only if there exists $\Phi \in L^\infty_w(B(\mathscr{H}))$ with $T_{m,n} = \hat{\Phi}(-m-n)$ for each $m,n \in \mathbb{Z}^+$. In this case Φ may be chosen with $\|\Phi\|_\infty = \|T\|$.

<u>Remark.</u> Since the operator $PJM_\Phi|_{H^2(\mathbb{T};\mathscr{H})}$ $(\Phi \in L^\infty_w(B(\mathscr{H})))$ depends only on the coefficients $\hat{\Phi}(-m)$ $(m \in \mathbb{Z}^+)$ of Φ we see that every vectorial Hankel operator on $H^2(\mathbb{T};\mathscr{H})$ is associated with a unique equivalence class $\Phi + H^\infty_0(B(\mathscr{H}))$ in $L^\infty_w(B(\mathscr{H}))/H^\infty_0(B(\mathscr{H}))$. The norm condition of 3.2.7 ensures that this correspondence is an isometric isomorphism of the class of vectorial Hankel operators onto $L^\infty_w(B(\mathscr{H}))/H^\infty_0(B(\mathscr{H}))$.

Section 3.3 The Predual of $L^\infty_w(B(\mathscr{H}))/H^\infty_0(B(\mathscr{H}))$.

In this section we show that the Banach space $L^\infty_w(B(\mathscr{H}))/H^\infty_0(B(\mathscr{H}))$ is the dual of an H^1space of operator valued functions on $\mathbb{T}$. Using the results of the previous section we can then identify the class of vectorial Hankel operators on $H^2(\mathbb{T};\mathscr{H})$ with the dual of an H^1 space.

We have recently found that a theorem equivalent to our principle result (3.3.9) concerning the predual of $L^\infty_w(B(\mathscr{H}))$ appears in [SA]. The following representation theorem (3.3.1) was proved independently of Sarason's work and gives an alternative method of finding the predual of $L^{\infty}_{w}(B(\mathscr{H}))$. Further details on the equivalence between 3.3.9 and Sarason's result appear in the remarks following 3.3.9.

<u>3.3.1 Theorem.</u> Let $F : L^1(\mathbb{T}) \longrightarrow B(\mathscr{H})$ be a bounded linear operator. Then there exists a unique $\Psi_F \in L^{\infty}_{w}(B(\mathscr{H}))$ such that

$$(Fh)x = \int_0^{2\pi} h(e^{i\theta})\Psi_F(e^{i\theta})x \; dm(\theta)$$

whenever $h \in L^1(\mathbb{T})$ and $x \in \mathscr{H}$. Moreover, $\| \Phi_F \|_{\infty} = \| F \|$ and the map $R_1 : B(L^1;B(\mathscr{H})) \longrightarrow L^{\infty}_{w}(B(\mathscr{H}))$ defined at $f \in B(L^1;B(\mathscr{H}))$ by $R_1(F) = \Psi_F$ is an isometric isomorphism.

The proof of 3.3.1 requires the following simple lemma.

<u>3.3.2 Lemma.</u> Let f be a measurable $\mathscr{H}$-valued Bochner integrable function on $\mathbb{T}$. Then for $x \in \mathscr{H}$

$$\int_0^{2\pi} (f(e^{i\theta}),x) \; dm(\theta) = (\int_0^{2\pi} f(e^{i\theta}) \; dm(\theta),x)$$

<u>Proof.</u> Either by approximating f by a sequence of measurable simple $\mathscr{H}$-valued functions or by applying a general result of Hille ([HI,p 44],[DU,Thm 6,p 47]) to the linear functional $F_y = (y,x)$ on $\mathscr{H}$.

<u>Proof of 3.3.1.</u> For each $x,y \in \mathscr{H}$ we define a bounded linear functional $\Phi(x,y)$ on $L^1(\mathbb{T})$ by

$$\Phi(x,y)h = ((Fh)x,y) \qquad (h \in L^1(\mathbb{T}))$$

Then $\| \Phi(x,y) \| \leq \| F \| \| x \|_{\mathscr{H}} \| y \|_{\mathscr{H}}$ and we identify this functional with an element of $L^{\infty}(\mathbb{T})$, also denoted by $\Phi(x,y)$.

The map $\Phi : (x,y) \rightsquigarrow \Phi(x,y)$ is a sesquilinear map of $\mathscr{H}\times\mathscr{H}$ into

$L^\infty(\mathbb{T})$, bounded by $\| F \|$. We apply 3.2.3 to get $\Psi_F \in L_w^\infty(B(\mathscr{H}))$ with $(\Psi_F(\cdot)x,y) = \Phi(x,y)$ for all $x,y \in \mathscr{H}$ and

$$\| \Psi_F\|_\infty \le \| F \| \tag{$*$}$$

Since Φ_F is wo-measurable it follows that if $h \in L^1(\mathbb{T})$ and $x \in \mathscr{H}$ the function $h(\cdot)\Psi_F(\cdot)x$ is measurable and that $\int_0^{2\pi} h(e^{i\theta})\Psi_F(e^{i\theta})x \, dm(\theta)$ exists as an $\mathscr{H}$-valued Bochner integral with norm dominated by $\| h \|_1 \| F \| \| x \|_{\mathscr{H}}$.

Now by 3.3.2, for $y \in \mathscr{H}$ we have

$$\begin{aligned}
\left(\int_0^{2\pi} h(e^{i\theta})\Psi_F(e^{i\theta})x \, dm(\theta), y\right) &= \int_0^{2\pi} h(e^{i\theta})(\Psi_F(e^{i\theta})x,y) \, dm(\theta) \\
&= \int_0^{2\pi} h(e^{i\theta})\Phi(x,y)(e^{i\theta}) \, dm(\theta) \\
&= ((Fh)x,y).
\end{aligned}$$

So $\int_0^{2\pi} h(e^{i\theta})\Psi_F(e^{i\theta})x \, dm(\theta) = (Fh)x$. Moreover, since $\| (Fh)x \| \le \| \Psi_F \|_\infty \| h \|_1 \| x \|_{\mathscr{H}}$ we have $\| F \| \le \| \Psi_F\|_\infty$ and by $(*)$, $\| F \| = \| \Psi_F\|_\infty$.

To see that Ψ_F is unique we note that for each $n \in \mathbb{Z}$ and $x,y \in \mathscr{H}$

$$\begin{aligned}
((Fe_n)x,y) &= \int_0^{2\pi} e^{in\theta}(\Psi_F(e^{i\theta})x,y) \, dm(\theta) \\
&= (\hat{\Psi}_F(-n)x,y).
\end{aligned}$$

So $Fe_n = \hat{\Psi}_F(-n)$, which determines Ψ_F uniquely.

Lastly we note that any $\Psi \in L_w^\infty(B(\mathscr{H}))$ clearly defines a bounded linear operator F_Ψ from $L^1(\mathbb{T})$ to $B(\mathscr{H})$ by

$$(F_\Psi h)x = \int_0^{2\pi} h(e^{i\theta})\Psi(e^{i\theta})x \, dm(\theta)$$

for $h \in L^1(\mathbb{T})$, $x \in \mathscr{H}$. Then $\Psi = R_1(F_\Psi)$ and $\| F_\Psi\| \le \| \Psi \|_\infty$. Thus R_1 is an isometric isomorphism.

3.3.3 <u>Definition.</u> The <u>trace class</u> of operators on $\mathcal{H}$ consists of those $T \in B(\mathcal{H})$ for which $\sum_{n\geq 0} |(T\phi_n,\phi_n)| < \infty$ for every orthonormal set $\{\phi_n\}_{n\in\mathbb{Z}^+}$ in $\mathcal{H}$. We denote the trace class by $\mathcal{C}^1$ and for $T \in \mathcal{C}^1$ define the trace of T by $\mathrm{tr}(T) = \sum_{n\geq 0} (Tx_n,x_n)$ where $\{x_n\}_{n\in\mathbb{Z}^+}$ is any orthonormal basis of $\mathcal{H}$. For $T\in B(\mathcal{H})$ we denote by $|T|$ the positive operator $(T^*T)^{1/2}$ and for $T \in \mathcal{C}^1$ we define $\| T \|_1 = \mathrm{tr}(|T|)$.

<u>Remark.</u> The von-Neumann-Schatten p-classes of operators $\mathcal{C}^p$ ($1 \leq p < \infty$) determined by the norms

$$\| T \|_p = \mathrm{tr}(|T|^p)^{1/p}$$

form a sequence of Banach spaces of compact operators, originally investigated by von-Neumann and Schatten in 1948 ([vNS]). For the proof that tr(T) is independent of the choice of orthonormal basis (when $T \in \mathcal{C}^1$!) and that $\| \cdot \|_1$ is a norm on $\mathcal{C}^1$ making it a Banach space, and for the rich theory of $\mathcal{C}^p$ classes which follows we refer the reader to [RI].

We note however two results that will be of particular use in the sequel.

<u>3.3.4 Notation.</u> For $x,y \in \mathcal{H}$, let $x\tilde{\otimes} y \in B(\mathcal{H})$ be defined by $(x\tilde{\otimes}y)(z) = (z,y)x \quad (z \in \mathcal{H})$.

3.3.5 <u>Theorem.</u> i) Let $T \in B(\mathcal{H})$. If $S \in \mathcal{C}^1$ then $ST \in \mathcal{C}^1$ and $|\mathrm{tr}(ST)| \leq \| T \|_{B(\mathcal{H})} \| S \|_1$. Moreover the map $f_T(S) = \mathrm{tr}(ST)$ on $\mathcal{C}^1$ is a continuous linear functional with norm $\| T \|_{B(\mathcal{H})}$. The map $T \rightsquigarrow f_T$ is an isometric isomorphism of $B(\mathcal{H})$ onto $(\mathcal{C}^1)^*$.

ii) The class $\mathcal{F}$ of finite rank operators on $\mathcal{H}$ forms a dense subspace of $\mathcal{C}^1$. Moreover, if $S \in \mathcal{C}^1$ there exist $x_n,y_n \in \mathcal{H}$ ($n \in \mathbb{Z}^+$)

such that $S = \sum_{n\geq 0} x_n \widehat{\otimes} y_n$ in $\mathcal{C}^1$ and $\| S \|_1 = \sum_{n\geq 0} \| x_n\|_{\mathcal{H}} \| y_n\|_{\mathcal{H}}$.

Proof. i) See [RI, Thm 2.1.6 p 81, Lemma 2.3.3 p 87 and Thm 2.3.12 p 99].

ii) See [RI, p 85].

3.3.6 Corollary. The Banach space $B(L^1(\mathbb{T});B(\mathcal{H}))$ is isometrically isomorphic to the dual of $L^1(\mathbb{T})\hat{\otimes}\mathcal{C}^1$. The map $R_2 : B(L^1(\mathbb{T});B(\mathcal{H})) \longrightarrow (L^1(\mathbb{T})\hat{\otimes}\mathcal{C}^1)^*$ defined at $F \in B(L^1(\mathbb{T});B(\mathcal{H}))$ by

$$R_2(F)\left(\sum_{n=0}^{N} h_N\otimes S_n\right) = \sum_{n=0}^{N} \mathrm{tr}(S_nF(h_n))$$

(for $h_n \in L^1(\mathbb{T})$, $S_n \in \mathcal{C}^1$, $0 \leq n \leq N$, $N \geq 0$) is an isometric isomorphism.

Proof. By 1.2.8 $(L^1(\mathbb{T})\hat{\otimes}\mathcal{C}^1)^*$ is isometrically isomorphic to $B(L^1(\mathbb{T});(\mathcal{C}^1)^*)$, which is isometrically isomorphic to $B(L^1(\mathbb{T});B(\mathcal{H}))$ by 3.3.5(i). In the notation of 1.2.8 we have $R_2(F)\left(\sum_{n=0}^{N} h_n\otimes S_n\right) = f_F\left(\sum_{n=0}^{N} h_N\otimes S_n\right)$ for all $F \in B(L^1(\mathbb{T});B(\mathcal{H}))$, $h_n \in L^1(\mathbb{T})$, $S_n \in \mathcal{C}^1$ $(0 \leq n \leq N)$, $N \geq 0$, so it is clear that R_2 is the corresponding isometric isomorphism.

We will require the following general result for the projective tensor product of $L^1(\mathbb{T})$ with a Banach space, X.

3.3.7 Theorem. Let X be a Banach space. Define an operator J from $L^1(\mathbb{T})\otimes X$ into $L^1(\mathbb{T};X)$ on simple tensors by $J(f\otimes x) = f(\cdot)x$ (f $\in$

$L^1(\mathbb{T})$, $x \in X$). Then J extends to an isometric isomorphism of $L^1(\mathbb{T})\hat{\otimes}X$ onto $L^1(\mathbb{T};X)$.

<u>Proof.</u> [DU, p 228].

<u>3.3.8 Corollary.</u> There is a unique isometric isomorphism R_3 of $L^1(\mathbb{T};\mathcal{C}^1)^*$ onto $(L^1(\mathbb{T})\hat{\otimes}\mathcal{C}^1)^*$ such that when $\Phi \in L^1(\mathbb{T};\mathcal{C}^1)^*$, $f \in L^1(\mathbb{T})$ and $S \in \mathcal{C}^1$

$$R_3(\Phi)\ (f\otimes S) = \Phi(f(\cdot)S)$$

<u>Proof.</u> Apply 3.3.7 with $X = \mathcal{C}^1$ and then put $R_3 = J^*$ for the result.

We now combine results 3.3.1, 3.3.6 and 3.3.8 to give the following characterisation of $L^\infty_w\ (B(\mathcal{H}))$.

<u>3.3.9 Corollary.</u> The map $R = R_3^{-1}\ R_2\ R_1^{-1}$ is an isometric isomorphism of $L^\infty_w\ (B(\mathcal{H}))$ onto $L^1(\mathbb{T},\mathcal{C}^1)^*$. Moreover, if $f \in L^1(\mathbb{T},\mathcal{C}^1)$ and $\Phi \in L^\infty_w\ (B(\mathcal{H}))$ then

$$R(\Phi)f = \int_0^{2\pi} \mathrm{tr}\ (f(e^{i\theta})\Phi(e^{i\theta}))\ dm(\theta).$$

<u>Proof.</u> The map R is clearly an isometric isomorphism of $L^\infty_w\ (B(\mathcal{H}))$ onto $L^1(\mathbb{T};\mathcal{C}^1)^*$. Let $h \in L^1(\mathbb{T})$, $S \in \mathcal{C}^1$ and $\Phi \in L^\infty_w\ (B(\mathcal{H}))$. Then

$$\begin{aligned}(R_3^{-1}\ R_2\ R_1^{-1}\Phi)(h(\cdot)S) &= (R_2\ R_1^{-1}\Phi)(h\otimes S)\\ &= \mathrm{tr}(S(R_1^{-1}\Phi)(h))\\ &= \sum_{n\geq 0} (S(R_1^{-1}\Phi)(h)x_n, x_n)\\ &= \sum_{n\geq 0} \left(\int_0^{2\pi} h(e^{i\theta})\Phi(e^{i\theta})x_n\ dm(\theta),\ S^*x_n \right) \qquad (*)\end{aligned}$$

$$= \sum_{n\geq 0} \int_0^{2\pi} (h(e^{i\theta})\Phi(e^{i\theta})x_n, S^* x_n)\ dm(\theta)$$

(by 3.3.2).

But if $f_N(e^{i\theta}) = \sum_{n=0}^{N} (h(e^{i\theta})S\Phi(e^{i\theta})x_n, x_n)$

then $f_N \to \sum_{n\geq 0} ((h(\cdot)S\Phi(\cdot)x_n, x_n)$ pointwise a.e on $\mathbb{T}$ and

$$|f_N(e^{i\theta})| \leq \sum_{n\geq 0} |(h(e^{i\theta})S\Phi(e^{i\theta})x_n, x_n)|$$

$$= |h(e^{i\theta})| \sum_{n\geq 0} |(S\Phi(e^{i\theta})x_n, x_n)|$$

$$\leq |h(e^{i\theta})|\ \| S\Phi(e^{i\theta})\|_1$$

$$\leq |h(e^{i\theta})|\ \| S \|_1\ \| \Phi \|_\infty .$$

So by Dominated Convergence and $(*)$

$$(R_3^{-1} R_2 R_1^{-1}\Phi)(h(\cdot)S) = \int_0^{2\pi} \sum_{n\geq 0} (h(e^{i\theta})S\Phi(e^{i\theta})x_n, x_n)\ dm(\theta)$$

$$= \int_0^{2\pi} \mathrm{tr}\ (h(e^{i\theta})S\Phi(e^{i\theta})\ dm(\theta).$$

Now fix $f \in L^1(\mathbb{T};\mathcal{C}^1)$ and $\Phi \in L_w^\infty(B(\mathcal{H}))$. We approximate f by a sequence $\{f_n\}_{n\in\mathbb{Z}^+}$ of measurable simple $\mathcal{C}^1$-valued functions such that $\| f(e^{i\theta})-f_n(e^{i\theta}) \|_1 \to 0$ as $n \to \infty$ for a.e. $e^{i\theta} \in \mathbb{T}$. By the wo-measurability of Φ we see that for each $x \in \mathcal{H}$ the function $(f_n(\cdot)\Phi(\cdot)x,x)$ is measurable. Since $(f_n(\cdot)\Phi(\cdot)x,x) \to (f(\cdot)\Phi(\cdot)x,x)$ pointwise a.e on $\mathbb{T}$, $(f(\cdot)\Phi(\cdot)x,x)$ is measurable. Thus, $\mathrm{tr}(f(\cdot)\Phi(\cdot))$ $= \sum_{n\geq 0} (f(\cdot)\Phi(\cdot)x_n, x_n)$ is measurable.

Since finite sums of tensors are dense in $L^1(\mathbb{T})\hat{\otimes}\mathcal{C}^1$ and the map J of 3.3.7 is an isometric isomorphism, we can find $h_n \in L^1(\mathbb{T})$ and $S_n \in \mathcal{C}^1$ $(n \geq 0)$ such that

$$\sum_{n=0}^{N} h_n(\cdot)S_n \longrightarrow f$$

in $L^1(\mathbb{T};\mathcal{C}^1)$ as $N \to \infty$. Then

$$\left| \int_0^{2\pi} \operatorname{tr}\,(f(e^{i\theta})\Phi(e^{i\theta}))\,dm(\theta) - \int_0^{2\pi} \operatorname{tr}\left(\left(\sum_{n=0}^{N} h_n(e^{i\theta})S_n\right)\Phi(e^{i\theta})\right)\,dm(\theta)\right|$$

$$\le \int_0^{2\pi} \left| \operatorname{tr}\left(\left(f(e^{i\theta}) - \sum_{n=0}^{N} h_n(e^{i\theta})S_n\right)\Phi(e^{i\theta})\right) \right| dm(\theta)$$

$$\le \int_0^{2\pi} \| f(e^{i\theta}) - \sum_{n=0}^{N} h_n(e^{i\theta})S_n \|_1 \, \| \Phi(e^{i\theta}) \|_\infty \, dm(\theta)$$

$$\le \| f - \sum_{n=0}^{N} h_n(\cdot)S_n \|_{L^1(\mathbb{T};\mathcal{C}^1)} \, \| \Phi \|_\infty$$

$$\longrightarrow 0 \text{ as } N \longrightarrow \infty .$$

Hence,

$$(R_3^{-1} \; R_2 \; R_1^{-1}\Phi)(f) = \lim_{N\to\infty} (R_3^{-1} \; R_2 \; R_1^{-1}\Phi)\left(\sum_{n=o}^{N} h_n(\cdot)S_n\right)$$

$$= \lim_{N\to\infty} \int_0^{2\pi} \operatorname{tr}\left(\left(\sum_{n=0}^{N} h_n(e^{i\theta})S_n\right)\Phi(e^{i\theta})\right) dm(\theta)$$

$$= \int_0^{2\pi} \operatorname{tr}\,(f(e^{i\theta})\Phi(e^{i\theta}))\,dm(\theta) \quad ,$$

as required.

Remarks. 1. As noted at the start of this section a theorem equivalent to 3.3.9 is known. It is stated without proof in [SA, p 197] that when $L^1_w(\mathbb{T};\mathcal{C}^1)^*$ denotes the Banach space of wo-measurable $\mathcal{C}^1$-valued functions f on $\mathbb{T}$ for which

$$\| f \|_1 = \int_0^{2\pi} \| f(e^{i\theta}) \|_1 \, dm(\theta) \; < \infty$$

then $L^1_w(\mathbb{T};\mathcal{C}^1)^*$ is isometrically isomorphic to $L^\infty_w(B(\mathcal{H}))$ by the formula given in 3.3.9. Since $\mathcal{C}^1$ is separable we have by a theorem of Dunford ([HI, Thm 3.3.1(2), p 38]) that every wo-measurable $\mathcal{C}^1$-valued function is strongly measurable and thus that $L^1_w(\mathbb{T};\mathcal{C}^1)^* = L^1(\mathbb{T};\mathcal{C}^1)$. Hence Sarason's result is equivalent to 3.3.9.

2. If X is a Banach space then we may ask whether $L^1(\mathbb{T};X)^*$ is isometrically isomorphic to $L^\infty(\mathbb{T};X^*)$ by the natural identification. Working as we have done above we find that this is equivalent to the question of whether every bounded linear operator F from $L^1(\mathbb{T})$ to X^* can be represented by an X^*-valued Bochner integral

$$Fh = \int_0^{2\pi} h(e^{i\theta})g(e^{i\theta})\, dm(\theta)$$

for some $g \in L^\infty(\mathbb{T};X^*)$. That is, whether $B(L^1;X^*)$ is isometrically isomorphic in a natural way to $L^\infty(\mathbb{T};X^*)$.

In fact the solution is given by considering the continuous X^*-valued vector measures of bounded variation on $\mathbb{T}$. We say that a Banach space Y has the Radon-Nikodým Property (with respect to the measure space $(\mathbb{T},dm)$) if a (generalised) Radon-Nikodým Theorem holds for Y :

"Every continuous vector measure of bounded variation $G : \mathbb{T} \longrightarrow Y$ can be represented as

$$G(E) = \int_E g(e^{i\theta})\, dm(\theta)$$

for some $g \in L^1(\mathbb{T},X)$ and every measurable subset E of $\mathbb{T}$.".

Indeed, we find [DU, p 63] that every $T \in B(L^1;X)$ is representable (ie. $Tf = \int fg$ for some $g \in L^\infty(\mathbb{T};X)$) if and only if X has the Radon-Nikodým Property. Equivalently, $B(L^1;X)$ is isometrically isomorphic in a natural way to $L^\infty(\mathbb{T};X)$ if and only if X has the Radon-Nikodým Property, and again, equivalently [DU, p 98], $L^1(\mathbb{T};X)^*$ is isometrically isomorphic in a natural way to $L^\infty(\mathbb{T};X^*)$ if and only if X^* has the Radon-Nikodým Property.

Note that $L^\infty(\mathbb{T};B(\mathcal{H}))$ is the Banach space of essentially bounded strongly measurable functions $g : \mathbb{T} \longrightarrow B(\mathscr{H})$ where we mean "strongly measurable" in the sense that g can be approximated by a sequence of measurable simple $B(\mathscr{H})$-valued functions. This contrasts with what we might call a strong-operator measurable function g for which $g(\cdot)x$ is a measurable $\mathscr{H}$-valued function for each $x \in \mathscr{H}$. By Pettis' Measurability Theorem a $B(\mathscr{H})$-valued function is strong-operator measurable if and only if it is weak-operator measurable.

Although every strongly measurable $g \in L^{\infty}(\mathbb{T};B(\mathscr{H}))$ will define a bounded linear operator F_g from $L^1(\mathbb{T})$ to $B(\mathscr{H})$ by $F_g f = \int_0^{2\pi} f(e^{i\theta})g(e^{i\theta})\, dm(\theta)$, the space $L^{\infty}(\mathbb{T};B(\mathscr{H}))$ is too small to ensure that every bounded $F : L^1(\mathbb{T}) \longrightarrow B(\mathscr{H})$ can be represented in this way. Consistent with this is the observation that $B(\mathscr{H})$ does not have the Radon-Nikodým Property. If a Banach space has the Radon-Nikodým Property then so must any of its closed subspaces . But the Banach space c_0 of sequences convergent to zero does not have the Radon-Nikodým Property ([DU, p 60]), whilst we can identify c_0 with the closed subspace of $B(\mathscr{H})$ consisting of operators with matrices of the form $\{\alpha_i \delta_{i,j}\}_{i,j\in\mathbb{Z}^+}$ $(\{\alpha_n\} \in c_0)$.

Thus the dual of $L^1(\mathbb{T};\mathscr{C}^1)$ is not $L^{\infty}(\mathbb{T};B(\mathscr{H}))$ but the larger space $L^{\infty}_w(B(\mathscr{H}))$ of <u>weak-operator</u> measurable functions.

Having found the predual of $L^{\infty}_w(B(\mathscr{H}))$ we shall find the predual of its quotient $L^{\infty}_w(B(\mathscr{H}))/H^{\infty}_0(B(\mathscr{H}))$.

<u>3.3.10 Lemma.</u> The image of $H^{\infty}_0(B(\mathscr{H}))$ under the map R of 3.3.9 is the annihilator $H^1(\mathbb{T};\mathscr{C}^1)^{\perp}$ of $H^1(\mathbb{T};\mathscr{C}^1)$ in $L^1(\mathbb{T};\mathscr{C}^1)^*$.

Proof. Since the polynomials $\sum_{m=0}^{M} e_m(\cdot)S_m$ $(S_m \in \mathcal{C}^1$ for $0 \le m \le M)$ form a dense subspace of $H^1(\mathbb{T};\mathcal{C}^1)$ we will consider only the action of linear functionals at $e_m(\cdot)S$ for $S \in \mathcal{C}^1$ and $m \in \mathbb{Z}^+$.

If $\Phi \in L^{\infty}_{w}(B(\mathcal{H}))$ then by equation (*) in the proof of 3.3.9 we have for $s \in \mathcal{C}^1$ and $m \in \mathbb{Z}^+$

$$(R\Phi)(e_m(\cdot)S) = \sum_{n\geq 0}\int_0^{2\pi} e^{im\theta}(\Phi(e^{i\theta})x_n,S^*x_n)\ dm(\theta)$$

$$= \sum_{n\geq 0}(\hat{\Phi}(-m)x_n,S^*x_n).$$

Thus, if $\Phi \in H^{\infty}_{0}(B(\mathcal{H}))$ then $\hat{\Phi}(-m) = 0$ for all $m \in \mathbb{Z}^+$ and therefore $(R\Phi)(e_m(\cdot)S) = 0$ for all $m \in \mathbb{Z}^+$. Conversely, if $(R\Phi(e_m(\cdot)S) = 0$ for all $m \in \mathbb{Z}^+$ and $S \in \mathcal{C}^1$ then by (**), using $S = x_l \tilde{\otimes} x_k \in \mathcal{C}^1$ $(k,l \in \mathbb{Z}^+)$ we obtain

$$(\hat{\Phi}(-m)x_l,x_k) = 0 \quad \text{for all } k,l,m \in \mathbb{Z}^+.$$

Hence $\hat{\Phi}(-m) = 0$ for $m \in \mathbb{Z}^+$ and $\Phi \in H^{\infty}_{0}(B(\mathcal{H}))$.

3.3.11 Corollary. The Banach space $L^{\infty}_{w}(B(\mathcal{H}))/H^{\infty}_{0}(B(\mathcal{H}))$ is isometrically isomorphic to $H^1(\mathbb{T};\mathcal{C}^1)^*$.

Proof. By 3.3.10 and the standard identification of the dual of a subspace of a Banach space.

The application of 3.3.11 to the class of vectorial Hankel operators on $H^2(\mathbb{T};\mathcal{H})$ is the following corollary.

3.3.12 Corollary The class of vectorial Hankel operators on $H^2(\mathbb{T};\mathcal{H})$ is isometrically isomorphic to $H^1(\mathbb{T};\mathcal{C}^1)^*$. If T is a vectorial Hankel operator on $H^2(\mathbb{T};\mathcal{H})$ with representing matrix of operators $\{T_{i+j}\}_{i,j\in\mathbb{Z}^+}$ and if $h \in H^1(\mathbb{T};\mathcal{C}^1)$ is a polynomial :

$$h = \sum_{n=0}^{N} e_n(\cdot)S_n \qquad (S_n \in \mathcal{C}^1)$$

then

$$\langle h,T\rangle = \sum_{n=0}^{N} \operatorname{tr}(S_nT_n).$$

Proof. That the class of vectorial Hankel operators on $H^2(\mathbb{T};\mathcal{H})$ is isometrically isomorphic to $H^1(\mathbb{T};\mathcal{C}^1)^*$ follows immediately from 3.3.11 and the remark after 3.2.8. Let T be a vectorial Hankel operator on $H^2(\mathbb{T},\mathcal{C}^1)$ and let $\Phi \in L^\infty_w\ (B(\mathcal{H}))$ be a function given by 3.2.8 satisfying $\hat{\Phi}(-n) = T_n$ for each $n \geq 0$. Then if $h = \sum_{n=0}^{N} e_n(\cdot)S_n$ is a polynomial in $H^1(\mathbb{T};\mathcal{C}^1)$, by working as in the proof of 3.3.10 we have

$$\begin{aligned}\langle h,T\rangle &= (R\Phi)(h)\\ &= \sum_{n=0}^{N}\sum_{m\geq 0} (\ \hat{\Phi}(-n)x_m, S_n^*\ x_m\ )\\ &= \sum_{n=0}^{N} \operatorname{tr}(S_n\hat{\Phi}(-n))\\ &= \sum_{n=0}^{N} \operatorname{tr}(S_nT_n), \text{ as required.}\end{aligned}$$

To end the section we consider the relationship between $H^1(\mathbb{T})\hat{\otimes}\mathcal{C}^1$ and $H^1(\mathbb{T};\mathcal{C}^1)$. Although we know that $L^1(\mathbb{T})\hat{\otimes}\mathcal{C}^1$ is isomorphic to $L^1(\mathbb{T};\mathcal{C}^1)$ we find that the subtle difference between the $H^1(\mathbb{T})\hat{\otimes}\mathcal{C}^1$ and $L^1(\mathbb{T})\hat{\otimes}\mathcal{C}^1$ norms prohibits the identification of $H^1(\mathbb{T})\hat{\otimes}\mathcal{C}^1$ with $H^1(\mathbb{T};\mathcal{C}^1)$.

3.3.13 Corollary. Let $cl^{L^1\hat{\otimes}\mathcal{C}^1}(H^1\otimes\mathcal{C}^1)$ denote the closure of $H^1(\mathbb{T})\otimes\mathcal{C}^1$ in $L^1(\mathbb{T})\hat{\otimes}\mathcal{C}^1$. Then $cl^{L^1\hat{\otimes}\mathcal{C}^1}(H^1\otimes\mathcal{C}^1)$ is isometrically isomorphic to $H^1(\mathbb{T};\mathcal{C}^1)$.

Proof. We show that the isometry J of Theorem 3.3.7 maps $cl^{L^1\hat{\otimes}\mathcal{C}^1}(H^1\otimes\mathcal{C}^1)$ onto $H^1(\mathbb{T};\mathcal{C}^1)$. Firstly, if $\alpha = \sum_{n=0}^{N} f_n\otimes S_n \in H^1(\mathbb{T})\otimes\mathcal{C}^1$, with $f_n \in H^1(\mathbb{T})$, $S_n \in \mathcal{C}^1$, $(0 \le n \le N)$ then $J\alpha = \sum_{n=0}^{N} f_n(\cdot)S_n \in H^1(\mathbb{T};\mathcal{C}^1)$ so $J(H^1(\mathbb{T})\otimes\mathcal{C}^1) \subseteq H^1(\mathbb{T};\mathcal{C}^1)$. Since J is continuous and $H^1(\mathbb{T};\mathcal{C}^1)$ is a closed subspace of $L^1(\mathbb{T};\mathcal{C}^1)$ we have

$$J(cl^{L^1\hat{\otimes}\mathcal{C}^1}(H^1\otimes\mathcal{C}^1)) \subseteq H^1(\mathbb{T};\mathcal{C}^1).$$

Secondly, if g is a polynomial in $H^1(\mathbb{T};\mathcal{C}^1)$, say $g = \sum_{n=0}^{N} e_n(\cdot)S_n$ $(S_n \in \mathcal{C}^1)$ then $g = J\left(\sum_{n=0}^{N} e_n\otimes S_n\right) \in J(H^1\otimes\mathcal{C}^1)$. As in the classical theory, convolution of $f\in L^1(\mathbb{T};X)$ with Fejér's kernels gives a sequence of trigonometric polynomials converging to f in $L^1(\mathbb{T};X)$. [KAT. 2.3,2.5 p11/12]. Thus, the polynomials are dense in $H^1(\mathcal{C}^1)$. Since J is an isometry it follows that J maps $cl^{L^1\hat{\otimes}\mathcal{C}^1}(H^1\otimes\mathcal{C}^1)$ onto $H^1(\mathbb{T};\mathcal{C}^1)$.

Remarks. 1. If $\alpha = \sum_{n=0}^{N} f_n\otimes S_n \in H^1(\mathbb{T})\otimes\mathcal{C}^1$ for some $f_n \in H^1(\mathbb{T})$, $S_n \in \mathcal{C}^1$ $(0 \le n \le N)$ then there are two natural projective tensor norms for α.

We denote by $\|\cdot\|_{\hat{\otimes}1}$ the $H^1(\mathbb{T})\hat{\otimes}\mathcal{C}^1$ norm :

$$\|\alpha\|_{\hat{\otimes}1} = \inf\left\{\sum_{n=0}^{N}\|f_n\|_1\|S_n\|_1 : \alpha = \sum_{n=0}^{N} f_n\otimes S_n \text{ for some } f_n \in H^1(\mathbb{T}),\ S_n \in \mathcal{C},\ N \ge 0\right\}$$

and by $\|\cdot\|_{\hat{\otimes}2}$ the $L^1(\mathbb{T})\hat{\otimes}\mathcal{C}^1$ norm :

$$\|\alpha\|_{\hat{\otimes}2} = \inf\left\{\sum_{n=0}^{N}\|f_n\|_1\|S_n\|_1 : \alpha = \sum_{n=0}^{N} f_n\otimes S_n \text{ for some } f_n \in L^1(\mathbb{T}),\ S_n \in \mathcal{C},\ N \ge 0\right\}.$$

Clearly, $\| \alpha \|_{\hat{\otimes}_2} \le \| \alpha \|_{\hat{\otimes}_1}$.

2. Using Lemma 3.3.5 we can identify $\mathcal{C}^1$ and $\mathcal{H}\hat{\otimes}\mathcal{H}$ by the isometry T defined by

$$T\left(\sum_{k\geq 0} x_k \tilde{\otimes} y_k\right) = \sum_{k\geq 0} x_k \otimes y_k .$$

when $\{x_k\}_{k\in\mathbb{Z}^+}$, $\{y_k\}_{k\in\mathbb{Z}^+}$ are sequences in $\mathcal{H}$ with

$$\sum_{k\geq 0} \| x_k \|_{\mathcal{H}} \| y_k \|_{\mathcal{H}} < \infty.$$

This allows us to prove results for $H^1(\mathbb{T})\hat{\otimes}\mathcal{H}\hat{\otimes}\mathcal{H}$ and $L^1(\mathbb{T})\hat{\otimes}\mathcal{H}\hat{\otimes}\mathcal{H}$ in place of $H^1(\mathbb{T})\hat{\otimes}\mathcal{C}^1$ and $L^1(\mathbb{T})\hat{\otimes}\mathcal{C}^1$.

3.3.14 Lemma. Let $i : H^1(\mathbb{T}) \rightarrow L^1(\mathbb{T})$ be the inclusion of $H^1(\mathbb{T})$ into $L^1(\mathbb{T})$ and let I_0 denote the identity operator on $\mathcal{C}^1$, $I_0(S) = S$ for all $S \in \mathcal{C}^1$. Then the map $i\otimes I_0$ defined at a simple tensor $h\otimes S$ of $H^1(\mathbb{T})\otimes\mathcal{C}^1$ by

$$(i\otimes I_o)(h\otimes S) = h\otimes S$$

extends to a continuous injection of $H^1(\mathbb{T})\hat{\otimes}\mathcal{C}^1$ into $L^1(\mathbb{T})\hat{\otimes}\mathcal{C}^1$.

Proof. It is clear that $i\otimes I_0$ extends linearly to a continuous map of $H^1(\mathbb{T})\otimes\mathcal{C}^1$ into $L^1(\mathbb{T})\otimes\mathcal{C}^1$. If $\alpha \in H^1(\mathbb{T})\hat{\otimes}\mathcal{C}^1$ then there exists $h_n \in H^1(\mathbb{T})$ and $S_n \in \mathcal{C}^1$ $(n \geq 0)$ such that $\alpha = \sum_{n\geq 0} h_n\otimes S_n$ and $\sum_{n\geq 0} \| h_n \|_1 \| S_n \|_1 < \infty$ so that $i\otimes I_0$ extends by continuity to $H^1(\mathbb{T})\hat{\otimes}\mathcal{C}^1$ with $(i\otimes I_0)\left(\sum_{n\geq 0} h_n\otimes S_n\right) = \sum_{n\geq 0} h_n\otimes S_n \in L^1(\mathbb{T})\otimes\mathcal{C}^1$.

We must prove that $i\otimes I_0$ is injective. To do this we denote by X_n the finite dimensional subspace of $\mathcal{H}$ spanned by $\{x_0, x_1, \ldots . x_n\}$ $(n \geq 1)$ and by p_n the orthogonal projection of $\mathcal{H}$ onto X_n. We put $P_n = p_n\otimes p_n$ on $\mathcal{H}\hat{\otimes}\mathcal{H}$ so that

$$P_n\left(\sum_{k\geq 0} \phi_k\otimes\psi_k\right) = \sum_{k\geq 0} p_n(\phi_k)\otimes p_n(\psi_k)$$

for all $\sum_{k\geq 0} \phi_k \otimes \psi_k \in \mathscr{H}\hat{\otimes}\mathscr{H}$ and $n \geq 1$. Then $P_n\beta \in X_n\hat{\otimes}X_n$ for all $n \geq 1$ and $\beta \in \mathscr{H}\hat{\otimes}\mathscr{H}$.

It is easy to check that each P_n is a norm 1 projection of $\mathscr{H}\hat{\otimes}\mathscr{H}$ onto $X_n\hat{\otimes}X_n$ and that for each $\beta \in \mathscr{H}\hat{\otimes}\mathscr{H}$, $\| P_n\beta - \beta \|_{\mathscr{H}\hat{\otimes}\mathscr{H}} \longrightarrow 0$ as $n \longrightarrow \infty$. Thus, if $\alpha \in H^1(\mathbb{T})\hat{\otimes}\mathscr{H}\hat{\otimes}\mathscr{H}$,

$$\| (I|_{H^1}\otimes P_n)\alpha - \alpha \|_{\hat{\otimes}_1} \longrightarrow 0 \qquad (*)$$

as $n \longrightarrow \infty$.

Let I_n denote the identity operator on $X_N\hat{\otimes}X_n$ ($n \geq 1$). Then since $X_n\otimes X_n$ is finite-dimensional, $i\otimes I_n$ is an injection of $H^1(\mathbb{T})\hat{\otimes}X_n\hat{\otimes}X_n$ into $L^1(\mathbb{T})\hat{\otimes}X_n\hat{\otimes}X_n$.

The result now follows since the diagram :

$$\begin{tikzcd}
H^1(\mathbb{T})\hat{\otimes}\mathscr{H}\hat{\otimes}\mathscr{H} \arrow[r, "i\otimes I_0"] \arrow[d, "I|_{H^1}\otimes P_n"] & L^1(\mathbb{T})\hat{\otimes}\mathscr{H}\hat{\otimes}\mathscr{H} \arrow[d, "I\otimes P_n"] \\
H^1(\mathbb{T})\hat{\otimes}X_n\hat{\otimes}X_n \arrow[r, "i\otimes I_n"] & L^1(\mathbb{T})\hat{\otimes}X_n\hat{\otimes}X_n
\end{tikzcd}$$

is commutative for each $n \geq 1$. Indeed, if $\alpha \in H^1(\mathbb{T})\hat{\otimes}\mathscr{H}\hat{\otimes}\mathscr{H}$ is such that $(I\otimes I_0)(\alpha) = 0$ then $(i\otimes I_n)(I|_{H^1}\otimes P_n)(\alpha) = 0$ for all $n \geq 1$. Thus, since $i\otimes I_n$ is injective, $(I|_{H^1}\otimes P_n)(\alpha) = 0$ for all $n \geq 1$ and consequntly, $\alpha = 0$ by $(*)$.

<u>3.3.15 Theorem.</u> The projective tensor product $H^1(\mathbb{T})\hat{\otimes}\mathscr{C}^1$ is NOT closed in $L^1(\mathbb{T})\hat{\otimes}\mathscr{C}^1$.

<u>Proof.</u> The following example of a sequence in $H^1(\mathbb{T})\otimes\mathscr{C}^1$ which is bounded in $\|\cdot\|_{\hat{\otimes}2}$ but unbounded in $\|\cdot\|_{\hat{\otimes}1}$ was communicated to me by Sten Kaisjer.

Let $y_0 \in \mathcal{H}$ have $\| y_0 \| = 1$ and define $\beta : H^1(\mathbb{T}) \otimes \mathcal{H} \otimes \mathcal{H} \longrightarrow \mathbb{C}$ On simple tensors by

$$\beta(h \otimes x \otimes y) = \sum_{k \geq 0} \hat{h}(2^k)(x, x_k)(y, y_0).$$

Then,

$$| \beta(h \otimes x \otimes y) | = \sum_{k \geq 0} | \hat{h}(2^k) | \, | (x, x_k) | \, | (y, y_0) |$$

$$\leq \left(\sum_{k \geq 0} | \hat{h}(2^k) |^2 \right)^{1/2} \left(\sum_{k \geq 0} | (x, x_k) |^2 \right)^{1/2} \| y \|_{\mathcal{H}}$$

$$= \left(\sum_{k \geq 0} | \hat{h}(2^k) |^2 \right)^{1/2} \| x \|_{\mathcal{H}} \| y \|_{\mathcal{H}} .$$

By Paley's Inequality ([DUR, p 104]) there exists a constant c such that whenever $f \in H^1(\mathbb{T})$, $\left(\sum_{k \geq 0} |\hat{f}(2^k)|^2 \right)^{1/2} \leq c \| f \|_1$. Hence,

$$| \beta(h \otimes x \otimes y) | \leq c \| h \|_1 \| x \|_{\mathcal{H}} \| y \|_{\mathcal{H}} .$$

If we extend β linearly to $H^1(\mathbb{T}) \otimes \mathcal{H} \otimes \mathcal{H}$ we obtain a $\| \cdot \|_{\hat{\otimes}_1}$-continuous linear functional which we may then extend to a continuous $\tilde{\beta}$ on $H^1(\mathbb{T}) \hat{\otimes} \mathcal{H} \hat{\otimes} \mathcal{H}$.

Now let Y_0 be the 1-dimensional subspace of $\mathcal{H}$ generated by y_0. Clearly, if $\beta \in L^1(\mathbb{T}) \otimes \mathcal{H} \otimes Y_0$ then $\| \beta \|_{\hat{\otimes}_2} \leq \| \beta \|_{L^1 \hat{\otimes} \mathcal{H} \hat{\otimes} Y_0}$ and so $L^1(\mathbb{T}) \hat{\otimes} \mathcal{H} \hat{\otimes} Y_0$ is a subspace of $L^1(\mathbb{T}) \hat{\otimes} \mathcal{H} \hat{\otimes} \mathcal{H}$. Also, $\mathcal{H} \hat{\otimes} Y_0$ is isometrically isomorphic to $\mathcal{H}$ and the set $\{x_k \otimes y_0\}_{k \geq 0}$ is a complete orthonormal basis for the Hilbert space $\mathcal{H} \hat{\otimes} Y_0$.

We now define the required sequence by

$$\alpha_n = \sum_{k=1}^{n} \frac{1}{k} \, e_{2^k} \otimes x_k \otimes y_0 \qquad (n \geq 1).$$

Then for each $n \geq 1$, $\alpha_n \in H^1(\mathbb{T}) \otimes \mathcal{H} \otimes Y_0$ and moreover,

$$
\begin{aligned}
\| \alpha_n \|_{\hat{\otimes}2} \le \| \alpha_n \|_{H^1(\mathbb{T})\hat{\otimes}\mathscr{H}\otimes Y_0} & \\
&= \| J\alpha_n \|_{L^1(\mathscr{H}\hat{\otimes}Y_0)} \qquad \text{by } 3.3.7 \\
&= \Big\| \sum_{k=1}^{n} \frac{1}{k}\, e_{2^k}(\cdot)(x_k\otimes y_0) \Big\|_{L^1(\mathscr{H}\hat{\otimes}Y_0)} \\
&= \int_0^{2\pi} \Big\| \sum_{k=1}^{n} \frac{1}{k}\, e^{2^k i\theta}(x_k\otimes y_0) \Big\|_{\mathscr{H}\otimes Y_0} dm(\theta) \\
&= \int_0^{2\pi} \left(\sum_{k=1}^{n} \frac{1}{k^2} \right)^{1/2} dm(\theta) \\
&\le \pi/\sqrt{6}
\end{aligned}
$$

Yet

$$
\begin{aligned}
\| \alpha_n \|_{\hat{\otimes}1} &= \frac{|\,\beta(\alpha_n)\,|}{\| \beta \|} \\
&= \frac{\sum_{k=1}^{n} 1/k}{\| \beta \|} \\
&\longrightarrow \infty \text{ as } n \longrightarrow \infty.
\end{aligned}
$$

Hence $\{\alpha_n\}_{n\ge1}$ is bounded in $\| \cdot \|_{\hat{\otimes}2}$ but unbounded in $\| \cdot \|_{\hat{\otimes}1}$ and the proof is complete.

Remark. We conclude from Lemma 3.3.14 and Theorem 3.3.15 that $H^1(\mathbb{T})\hat{\otimes}\mathscr{C}^1$ is isometrically imbedded onto a strict subspace of $H^1(\mathbb{T};\mathscr{C}^1)$.

3.3.16 Corollary. The dual space of $H^1(\mathbb{T};\mathscr{C}^1)$ is isometrically isomorphic to the Banach space of $\| \cdot \|_{\hat{\otimes}2}$-continuous linear functionals on $H^1(\mathbb{T})\hat{\otimes}\mathscr{C}^1$.

Proof. Immediately from 3.3.14.

Remark. The functional β constructed in the proof of 3.3.15 demonstrates that not every $\|\cdot\|_{\hat{\otimes}1}$-continuous linear functional on $H^1(\mathbb{T})\hat{\otimes}\mathcal{C}^1$ is $\|\cdot\|_{\hat{\otimes}2}$-continuous. It follows that $H^1(\mathbb{T};\mathcal{C}^1)^*$ is embedded as a proper subspace of $(H^1(\mathbb{T})\hat{\otimes}\mathcal{C}^1)^*$.

Section 3.4 Hankel Operators on $H^2(\mathbb{T}^2)$.

We shall show in this section how the results concerning vectorial Hankel operators on $H^2(\mathbb{T};\mathcal{H})$ can be translated and extended to give descriptions of the Hankel operators on $H^2(\mathbb{T}^2)$. We deduce the main theorem of this chapter which identifies the class of Hankel operators on $H^2(\mathbb{T}^2)$ with the dual of a quotient of $H^1(\mathbb{T};\mathcal{C}^1)$ and we give conditions under which it could be proved that this dual space is isometrically isomorphic to $L^\infty(\mathbb{T}^2)/H^1(\mathbb{T}^2)^\perp$.

Recall definition 3.1.2(a) that an operator $T \in B(H^2(\mathbb{T}^2))$ is Hankel if its representing array $\{\beta_{k,l,m,n}\}_{k,l,m,n\in\mathbb{Z}^+}$ has Hankel form : $\{\beta_{k+m,l+n}\}_{k,l,m,n\in\mathbb{Z}^+}$ for some matrix of scalars $\{\beta_{i,j}\}_{i,j\in\mathbb{Z}^+}$. Again we can give an equivalent definition in terms of unilateral shifts.

3.4.1 Notation. We denote by s_1,s_2 the unilateral shifts on $H^1(\mathbb{T}^2)$ defined at $f \in H^2(\mathbb{T}^2)$ by $(s_1f)(e^{i\theta},e^{i\phi}) = e^{i\theta}f(e^{i\theta},e^{i\phi})$ and $(s_2f)(e^{i\theta},e^{i\phi}) = e^{i\phi}f(e^{i\theta},e^{i\phi})$ for $(e^{i\theta},e^{i\phi}) \in \mathbb{T}^2$.

Remark. It is easy to check that $T \in B(H^2(\mathbb{T}^2))$ is a Hankel operator if and only if T intertwines with both s_1,s_1^* and s_2,s_2^* : $s_1^*T = Ts_1$ and $s_2^*T = Ts_2$. However, an attempt at proving conjecture 3.1.3 by the method used for 3.2.7 fails because we do not have the required lifting theorem. It is known that if A_1, A_2 are commuting isometries on $\mathscr{H}$ and $T \in B(\mathscr{H})$ commutes with A_1 and A_2 then T cannot necessarily be lifted to an operator V of the same norm as T which commutes with the minimal unitary dilations of A_1,A_2. It is not known whether the theorem holds for pairs of doubly commuting isometries, ie. isometries A_1,A_2 with $A_1A_2 = A_2A_1$ and $A_1A_2^* = A_2^*A_1$. In particular, it is not known whether an operator which commutes with s_1,s_2 can necessarily be lifted to an operator of the same norm on $L^2(\mathbb{T}^2)$ which commutes with the corresponding bilateral shifts u_1,u_2. For theorems concerning the dilation of families of commuting contractions/isomertries see [NF1, pp 22,23] and for a discussion of the above lifting theorems and problems see [POW1, pp 50-52].

<u>3.4.2 Notation.</u> We denote by Ψ the isometric isomorphism of $H^2(\mathbb{T}^2)$ onto $H^2(H^2(\mathbb{T}))$ defined at $f \in H^2(\mathbb{T}^2)$ by $(\Psi f)(e^{i\theta})(e^{i\phi}) = f(e^{i\theta},e^{i\phi})$ for $e^{i\theta},e^{i\phi} \in \mathbb{T}$.

<u>3.4.3 Lemma.</u> Let $T \in B(H^2(H^2(\mathbb{T})))$. Then the following are equivalent

a) $T = \Psi H\Psi^*$ for some Hankel operator $H \in B(H^2(\mathbb{T}^2))$;

b) T intertwines with S_1,S_1^* and S_2,S_2^* on $H^2(H^2(\mathbb{T}))$ where $(S_1f)(e^{i\theta}) = e^{i\theta}f(e^{i\theta})$ and $(S_2f)(e^{i\theta})(e^{i\phi}) = e^{i\phi}f(e^{i\theta})(e^{i\phi})$ for $f \in H^2(H^2(\mathbb{T}))$ and $e^{i\theta},e^{i\phi} \in \mathbb{T}$;

c) The representing matrix of operators for T is of the form $\{T_{m+n}\}_{m,n\in\mathbb{Z}^+}$ for some sequence of Hankel operators $\{T_n\}_{n\in\mathbb{Z}^+}$ on $H^2(\mathbb{T})$.

Proof. (a) $\Leftrightarrow$ (b). Clearly $S_1 = \Psi s_1 \Psi^*$ and $S_2 = \Psi s_2 \Psi^*$. Now if $T = \Psi H \Psi^*$ for some Hankel H on $H^2(\mathbb{T})$ then $S_1^* T = (\Psi s_1^* \Psi^*)(\Psi H \Psi^*) = \Psi(s_1^* H)\Psi^* = \Psi(H s_1)\Psi^* = (\Psi H \Psi^*)(\Psi s_1 \Psi^*) = T S_1$. Similarly $S_2^* T = T S_2$ and (b) holds.

Conversely, if $S_1^* T = T S_1$ then $s_1^*(\Psi^* T \Psi) = (\Psi^* S_1^* \Psi)(\Psi^* T \Psi) = \Psi^*(S_1^* T)\Psi = \Psi^*(T S_1)\Psi = (\Psi^* T \Psi)(\Psi^* S_1 \Psi) = (\Psi^* T \Psi) s_1$. Similarly $s_2^*(\Psi^* T \Psi) = (\Psi^* T \Psi) s_2$ and we conclude that $(\Psi^* T \Psi)$ is a Hankel operator H, say, on $H^2(\mathbb{T}^2)$. Thus $T = \Psi H \Psi^*$ as required.

(a) $\Leftrightarrow$ (c). Put $H = \Psi^* T \Psi$. Suppose that T is represented by the matrix of operators $\{T_{i,j}\}_{i,j \in \mathbb{Z}^+}$ and that H is represented by the array $\{\beta_{k,l,m,n}\}_{k,l,m,n \in \mathbb{Z}^+}$. Then

$$(T_{m,k} e_n, e_l) = (Te_m(\cdot) e_n, e_k(\cdot)\, e_l) = (He_{m,n}, e_{k,l}) = \beta_{k,l,m,n}$$

for all $k,l,m,n \in \mathbb{Z}^+$. Thus (a) holds $\Leftrightarrow$ $\beta_{k,l,m,n} = \beta_{k+m,l+n}$ for some matrix $\{\beta_{i,j}\}_{i,j \in \mathbb{Z}^+}$ $\Leftrightarrow$ $((T_{m,k} e_n, e_l)$ depends only on $k+m, l+n$ $\Leftrightarrow$ (c) holds.

Using 3.4.3 we can apply the theorem of Page (Cor. 3.2.8) to Hankel operators on $H^2(\mathbb{T}^2)$.

3.4.4 Theorem. Let $T \in B(H^2(\mathbb{T}^2))$. Then T is a Hankel operator if and only if there exists $\Phi \in L^\infty_w(B(H^2(\mathbb{T})))$ such that $\hat{\Phi}(-j)$ is a Hankel operator for each $j \in \mathbb{Z}^+$ and

$$(Te_{m,n}, e_{k,l}) = (\hat{\Phi}(-k-m)e_n, e_l)$$

for all $k,l,m,n, \in \mathbb{Z}^+$. In this case we may choose Φ with $\| \Phi \|_\infty = \| T \|$.

Remarks. 1. Theorem 3.4.4 shows that there is a norm preserving correspondence between the class of Hankel operators on $H^2(\mathbb{T}^2)$ and the subspace of $L^\infty_w(B(H^2(\mathbb{T})))/H^\infty_0(B(H^2(\mathbb{T})))$ consisting of those Φ +

$H_0^\infty(B(H^2(\mathbb{T})))$ where $\hat{\Phi}(-j)$ is Hankel for each $j \in \mathbb{Z}^+$. As shown in [POW1, pp 57,58], if there exists a constant K such that for any such Φ we could find $\Phi' \in L_w^\infty(B(H^2(\mathbb{T})))$ such that $\| \Phi' \|_\infty \le K \| \Phi \|_\infty$, $\Phi - \Phi' \in$
$H_0^\infty(B(H^2(\mathbb{T})))$ and $\Phi'(j)$ Hankel for <u>every</u> $j \in \mathbb{Z}$ then Conjecture 3.1.3 holds.

<u>2.</u> Let X denote the closed subspace of $B(L^1(\mathbb{T});B(H^2(\mathbb{T})))$ consisting of those $F \in B(L^1(\mathbb{T});B(H^2(\mathbb{T})))$ for which $H^1(\mathbb{T}) \subseteq \operatorname{Ker} F$. From Theorem 3.3.1 (with $\mathcal{H} = H^2(\mathbb{T})$) we see that the subspace

$$\{\Phi + H_0^\infty(B(H^2(\mathbb{T}))) : \hat{\Phi}(-j) \text{ is Hankel for } j \in \mathbb{Z}^+\}$$

of $L_w^\infty(B(H^2(\mathbb{T})))/H_0^\infty(B(H^2(\mathbb{T})))$ is isometrically isomorphic to the subspace

$$\{ F + X : F \in B(L^1(\mathbb{T});B(H^2(\mathbb{T}))),\ F \text{ is Hankel-valued on } H^1(\mathbb{T})\}$$

of $B(L^1(\mathbb{T});B(H^2(\mathbb{T})))/X$. Suppose that $\Phi \in L^\infty(B(H^2(\mathbb{T})))$ has $\hat{\Phi}(-j)$ Hankel for $j \in \mathbb{Z}^+$ and that $F_\Phi \in B(L^1(\mathbb{T});B(H^2(\mathbb{T})))$ is the operator with $R_1 F_\Phi = \Phi$. Then finding $\Phi' \in L^\infty(B(H^2(\mathbb{T})))$ with $\Phi' - \Phi \in H_0^\infty(B(H^2(\mathbb{T})))$, $\hat{\Phi}(j)$ Hankel for every $j \in \mathbb{Z}$ and $\| \Phi' \|_\infty \le K \| \Phi \|_\infty$ is equivalent to finding $F'_\Phi \in B(L^1(\mathbb{T});B(H^2(\mathbb{T})))$ with $F'_\Phi - F_\Phi \in X$, F'_Φ Hankel-valued on $L^1(\mathbb{T})$ and $\| F'_\Phi \| \le K \| F_\Phi \|$.

<u>3.</u> If there exists a bounded projection π of $B(H^2(\mathbb{T}))$ onto the class of Hankel operators on $H^2(\mathbb{T})$ then for any $F \in B(L^1(\mathbb{T}); B(H^2(\mathbb{T})))$ we have $\pi F - F \in X$, πF Hankel-valued on $L^1(\mathbb{T})$ and $\| \pi F \| \le \| \pi \| \, \| F \|$. Thus, the existence of such a projection ensures that Conjecture 3.1.3 holds. This is a slight improvement on the result ([POW1, p 58]) that Conjecture 3.1.3 holds if there exists a projection of $B(H^2(\mathbb{T}))$ onto the class of Hankel operators on $H^2(\mathbb{T})$ which is continuous with respect to the weak operator topology on $B(H^2(\mathbb{T}))$. Unfortunately, the question of the existence of such a

projection (weak operator continuous or otherwise) remains unsolved.

3.4.5 Notation. a) Throughout the rest of the chapter $\mathcal{C}^1$ will denote the trace class of operators on $H^2(\mathbb{T})$.

b) We will denote by B the pre-annihilator in $\mathcal{C}^1$ of the class of Hankel operators on $H^2(\mathbb{T})$:

$B = \{ S \in \mathcal{C}^1 \ ; \ tr(ST) = 0$ for every Hankel operator $T \in B(H^2(\mathbb{T}))\}$.

3.4.6 Lemma. Let $S \in \mathcal{C}^1$. Then $S \in B$ if and only if $\sum_{i+j=n}(Se_j, e_i) = 0$ for every $n \in \mathbb{Z}^+$.

Proof. By 3.3.5(ii) there exist sequences $\{f_k\}_{k\in\mathbb{Z}^+}$, $\{g_k\}_{k\in\mathbb{Z}^+}$ in $H^2(\mathbb{T})$ such that $\| S - \sum_{k=0}^{N} (f_k\tilde{\otimes}g_k)\| \longrightarrow 0$ as $N \longrightarrow \infty$ and $\sum_{k\geq 0}\|f_k\|_2\|g_k\|_2 = \| S \|_1$. Note that for each $i,j \in \mathbb{Z}^+$, $(Se_j, e_i) = \sum_{k\geq 0}((f_k\tilde{\otimes}g_k)e_j, e_i) = \sum_{k\geq 0}(e_j, g_k)(f_k, e_i) = \sum_{k\geq 0} \hat{f}_k(i)\hat{g}_k(j)$.

Also, if T is a Hankel operator on $H^2(\mathbb{T})$ there exists by 1.3.6 $\phi \in L^\infty(\mathbb{T})$ such that $(Tf,g) = \langle fg^\dagger, \phi + H_0^\infty(\mathbb{T})\rangle_{H^1(\mathbb{T}),L^\infty(\mathbb{T})/H_0^\infty(\mathbb{T})}$ for all $f,g \in H^2(\mathbb{T})$. Note that it is easy to show that $tr((f\tilde{\otimes}g)T) = (Tf,g)$ for all $f,g \in H^2(\mathbb{T})$.

$$
\begin{aligned}
\text{Now } tr(ST) &= \sum_{k\geq 0} tr((f_k\tilde{\otimes}g_k)T) \\
&= \sum_{k\geq 0}(Tf_k, g_k) \\
&= \sum_{k\geq 0}\langle f_k g_k^\dagger , \phi + H_0^\infty(\mathbb{T})\rangle \\
&= \langle \sum_{k\geq 0} f_k g_k^\dagger , \phi + H_0^\infty(\mathbb{T})\rangle
\end{aligned}
$$

since $\sum_{k\geq 0} f_k g_k^\dagger$ converges in $H^1(\mathbb{T})$.

Thus, $\mathrm{tr}(ST) = 0$ for all Hankel T on $H^2(\mathbb{T})$ if and only if $\sum_{k\geq 0} f_k g_k^\dagger = 0$ in $H^1(\mathbb{T})$. But for $n \in \mathbb{Z}^+$,

$$\left(\sum_{k\geq 0} f_k g_k^\dagger\right)^{\wedge}(n) = \sum_{k\geq 0} (f_k g_k^\dagger)^{\wedge}(n)$$

$$= \sum_{k\geq 0} \sum_{i+j=n} \hat{f}_k(i)\overline{\hat{g}_k(j)}$$

$$= \sum_{i+j=n} \sum_{k\geq 0} \hat{f}_k(i)\overline{\hat{g}_k(j)}$$

$$= \sum_{i+j=n} (Se_j, e_i) .$$

So $\mathrm{tr}\,(ST) = 0$ for all Hankel T on $H^2(\mathbb{T})$ if and only if $\sum_{i+j=n} (Se_j, e_i)$

$= 0$ for all $n \in \mathbb{Z}^+$.

<u>3.4.7 Theorem.</u> The class of Hankel operators on $H^2(\mathbb{T}^2)$ is isometrically isomorphic to the dual of $H^1(\mathbb{T};\mathcal{C}^1)/H^1(\mathbb{T};B)$. If T is a Hankel operator on $H^2(\mathbb{T}^2)$ and $\Phi \in L^\infty(B(H^2(\mathbb{T})))$ is an operator-valued function as in Theorem 3.4.4, and if $h \in H^1(\mathbb{T})$, $f,g \in H^2(\mathbb{T})$ then

$$\langle h(\cdot)(f\tilde{\otimes}g) + H^1(\mathbb{T};B) , T\rangle = \int_0^{2\pi} h(e^{i\theta})(\Phi(e^{i\theta})f,g)\, dm(\theta).$$

<u>Proof.</u> By Lemma 3.4.3 we may identify the class of Hankel operators on $H^2(\mathbb{T}^2)$ with the subspace of the class of vectorial Hankel operators on $H^2(\mathbb{T};H^2(\mathbb{T}))$ consisting of those S with a representing matrix of Hankel operators $\{S_{i+j}\}_{i,j\in\mathbb{Z}^+}$. We show that the isometric isomorphism of 3.3.12 mapping the class of vectorial

Hankel operators on $H^2(\mathbb{T};H^2(\mathbb{T}))$ onto $H^1(\mathbb{T};\mathcal{C}^1)^*$ maps the subspace onto the annihilator of $H^1(\mathbb{T};B)$ in $H^1(\mathbb{T};\mathcal{C}^1)^*$. The result then follows from the standard identification of the dual of a quotient space.

Suppose that S is a vectorial Hankel operator on $H^2(\mathbb{T};H^2(\mathbb{T}))$ with a representing matrix of Hankel operators $\{S_{i+j}\}_{i,j\in\mathbb{Z}^+}$. Then if $h = \sum_{n=0}^{N} e_n T_n$ is a polynomial in $H^1(\mathbb{T};B)$

$$<h,S> = \sum_{n=0}^{N} tr(S_n T_n) = 0 .$$

It follows that S annihilates $H^1(\mathbb{T};B)$.

Conversely, any vectorial Hankel operator S on $H^2(\mathbb{T};H^2(\mathbb{T}))$ which annihilates $H^1(\mathbb{T};B)$ must have $tr(ST) = <e_n(\cdot)T,S> = 0$ for every $n \in \mathbb{Z}^+$ and every $T \in B$. Thus $S_n \in B^\perp$ for every $n \in \mathbb{Z}^+$. But the class of Hankel operators on $H^2(\mathbb{T})$ is weak*-closed and is therefore equal to $B^\perp$. Thus S_n is Hankel for all $n \in \mathbb{Z}^+$, as required.

Finally, for $h \in H^1(\mathbb{T})$, $f,g \in H^2(\mathbb{T})$ and a Hankel operator $T \in B(H^2(\mathbb{T}^2)$, if $\phi \in L^\infty_w(B(H^2(\mathbb{T})))$ is as in Theorem 3.4.4 we have

$$\begin{aligned} <h(\cdot)(f\tilde{\otimes}g) + H^1(\mathbb{T};B),T> &= R(\Phi)(h(\cdot)(f\tilde{\otimes}g)) \\ &= \int_0^{2\pi} tr(h(e^{i\theta})(f\tilde{\otimes}g)\Phi(e^{i\theta}))\, dm(\theta) \quad \text{by 3.3.9} \\ &= \int_0^{2\pi} h(e^{i\theta})tr((f\tilde{\otimes}g)\Phi(e^{i\theta}))\, dm(\theta). \end{aligned}$$

But for each $e^{i\theta} \in \mathbb{T}$, $tr((f\tilde{\otimes}g)\Phi(e^{i\theta})) = (\Phi(e^{i\theta})f,g)$ so

$$<h(\cdot)(f\tilde{\otimes}g) + H^1(\mathbb{T};B),T> = \int_0^{2\pi} h(e^{i\theta})(\Phi(e^{i\theta})f,g)\, dm(\theta).$$

We now show that an expression for the action of T on an element $h(\cdot)(f\tilde{\otimes}g) + H^1(\mathbb{T};B)$ of $H^1(\mathbb{T};\mathcal{C}^1)/H^1(\mathbb{T};B)$ is useful because finite sums of elements of this form constitute a dense linear subspace of $H^1(\mathbb{T};\mathscr{C}^1)/H^1(\mathbb{T};B)$.

<u>3.4.8 Lemma.</u> If p is a polynomial of degree N in $H^1(\mathbb{T};\mathscr{C}^1)$ then there exist $f_n, g_n \in H^2(\mathbb{T})$ for $0 \le n \le N$ such that

$$p + H^1(\mathbb{T};B) = \sum_{n=0}^{N} e_n(\cdot)(f_n \tilde{\otimes} g_n) + H^1(\mathbb{T};B).$$

<u>Proof.</u> Suppose that $p = \sum_{n=0}^{N} e_n(\cdot)\hat{p}(n)$ for some $\hat{p}(n) \in \mathscr{C}^1$ for $0 \le n \le N$. For each $0 \le n \le N$ we may find sequences $\{f_{n,k}\}_{k\ge0}, \{g_{n,k}\}_{k\ge0}$ in $H^2(\mathbb{T})$ such that $\hat{p}(n) = \sum_{k\ge0} f_{n,k}\tilde{\otimes}g_{n,k}$ and

$$\| \hat{p}(n) \|_1 = \sum_{k\geq0} \| f_{n,k} \|_2 \| g_{n,k} \|_2 .$$

Fix $n \le N$. Then $\sum_{k\ge0} \| f_{n,k} \, g_{n,k}^{\dagger} \|_1 \le \sum \| f_{n,k} \|_2 \| g_{n,k}^{\dagger} \| < \infty$ so $\sum_{k\ge0} f_{n,k} \, g_{n,k}^{\dagger}$ converges in $H^1(\mathbb{T})$ to an $H^1(\mathbb{T})$ function which we factorise as $f_n g_n^{\dagger}$ for some $f_n, g_n \in H^2(\mathbb{T})$. We show that $\hat{p}(n) - f_n\tilde{\otimes}g_n \in B$. Let T be a Hankel operator on $H^2(\mathbb{T})$. Then

$$\begin{aligned} tr(\hat{p}(n)T) &= \sum_{k\ge0} tr((f_{n,k}\tilde{\otimes}g_{n,k})T) \\ &= \sum_{k\ge0} (Tf_{n,k}, g_{n,k}). \end{aligned}$$

By 1.3.6 there exists $\phi \in L^{\infty}(\mathbb{T})$ such that for all $f,g \in H^2(\mathbb{T})$

$$< (Tf,g) = <f_{\bullet}g^{\dagger}, \phi + H_0^{\infty}(\mathbb{T}) >_{H^1(\mathbb{T}), L^{\infty}(\mathbb{T})/H^{\infty}(\mathbb{T})} .$$

Thus

$$\begin{aligned} \operatorname{tr}\,(\hat{p}(n)T) &= \sum_{k\geq 0} \langle f_{n,k} g^{\dagger}_{n,k},\ \phi + H^{\infty}_{0}(\mathbb{T})\rangle \\ &= \langle \sum_{k\geq 0} f_{n,k} g^{\dagger}_{n,k},\ \phi + H^{\infty}_{0}(\mathbb{T})\rangle \\ &= \langle f_n g^{\dagger}_n,\ \phi + H^{\infty}_{0}(\mathbb{T})\rangle \\ &= (Tf_n, g_n) \\ &= \operatorname{tr}((f_n \tilde{\otimes} g_n)T). \end{aligned}$$

Hence $\operatorname{tr}(\hat{p}(n) - (f_n\tilde{\otimes}g_n)T) = 0$, as required.

Since this holds for each $0 \leq n \leq N$ we obtain $f_n, g_n \in H^2(\mathbb{T})$ for $0 \leq n \leq N$ such that

$$\sum_{n=0}^{N} e_n(\cdot)(f_n\tilde{\otimes}g_n - \hat{p}(n)) \in H^1(\mathbb{T};B)$$

and therefore $p + H^1(\mathbb{T};B) = \sum_{n=0}^{N} e_n(\cdot)(f_n\tilde{\otimes}g_n) + H^1(\mathbb{T};B)$.

To end the section we note that Theorem 3.4.7 may be rephrased in terms of functions in BHP.

<u>3.4.9 Corollary.</u> Let ϕ be an analytic function on $\mathbb{D}^2$ with power-series $\phi(z_1,z_2) = \sum_{m,n\geq 0} \hat{\phi}(m,n)z_1^m z_2^n$. Then ϕ = BHP if and only if there exists $f \in [H^1(\mathbb{T};\mathcal{C}^1)/H^1(\mathbb{T};B)]^*$ with

$$f(e_j(\cdot)(e_n \tilde{\otimes} e_l) + H^1(\mathbb{T};B)) = \hat{\phi}(j,l+n)$$

for all $j,l,n \geq 0$. In this case $\| f \| = \| \phi \|_{BHP}$.

<u>Section 3.5 The Symbol of a Hankel Operator on $H^2(\mathbb{T}^2)$.</u>

To complete this chapter we shall show in this section how a symbol of a Hankel operator on $H^2(\mathbb{T}^2)$ defines a bounded linear functional on $H^1(\mathbb{T};\mathcal{C}^1)/H^1(\mathbb{T};B)$. Our motivation is the fact that when $\phi \in L^{\infty}(\mathbb{T})$ is a symbol of some Hankel operator S on $H^2(\mathbb{T})$, the action of S as a linear functional on $H^1(\mathbb{T})$ may be written explicitly in terms of the symbol ϕ :

$$< h,S > = \int_0^{2\pi} h(e^{i\theta})\phi(e^{i\theta})\, dm(\theta). \qquad (*)$$

($h \in H^1(\mathbb{T})$). Having shown in 3.4.7 that the class of Hankel operators on $H^2(\mathbb{T}^2)$ is isometrically isomorphic to the dual of $H^1(\mathbb{T};\mathcal{C}^1)/H^1(\mathbb{T};B)$ we shall show how a $H^2(\mathbb{T}^2)$ Hankel operator acts on $H^1(\mathbb{T};\mathcal{C}^1)/H^1(\mathbb{T};B)$ via an expression analogous to (*).

We note first that if T is a Hankel operator on $H^2(\mathbb{T}^2)$ with representing array $\{\beta_{k+m,l+n}\}_{k,l,m,n\in\mathbb{Z}^+}$ then by considering $Te_{0,0}$ we see that the matrix $(\beta_{k,l}\}_{k,l\in\mathbb{Z}^+}$ is square summable. Thus, the series $\sum_{k,l\geq 0} \beta_{k,l}e_{-k,-l}$ is the Fourier series of a function in $\overline{H^2(\mathbb{T}^2)} = \{ \bar{f} : f \in H^2(\mathbb{T}^2)\}$.

3.5.1 Definition. Let $\beta \in L^1(\mathbb{T}^2)$. We say that β is a symbol of the Hankel operator H on $H^2(\mathbb{T}^2)$ if for each $k,l,m,n, \in \mathbb{Z}^+$

$$(He_{m,n},e_{k,l}) = \hat{\beta}(-k-m,-l-n).$$

The remark preceding the definition ensures that every Hankel operator on $H^2(\mathbb{T}^2)$ has a symbol, so suppose that H is a fixed Hankel operator on $H^2(\mathbb{T}^2)$ with a symbol, β. For $0 < r,\ s < 1$ we will denote by $\beta_{r,s}$ the continuous function on $\mathbb{T}^2$ defined at $(e^{i\theta},e^{i\phi})$ by

$$\beta_{r,s}(e^{i\theta},e^{i\phi}) = \sum \hat{\beta}(m,n)r^{|m|}s^{|n|}e^{im\theta}e^{in\phi}.$$

Using Theorem 3.4.4 we find $\phi_\beta \in L^{\infty}(B(H^2(\mathbb{T})))$ such that $\hat{\phi}_\beta(-j)$ is a Hankel operator for each $j \geq 0$, $\| \phi_\beta \|_{\infty} = \| H \|$ and $(\hat{\phi}_\beta(-j)e_n,e_l) = \hat{\beta}(-j,-l-n)$ for each $j,l,n \geq 0$. Then by Theorem 3.3.1 we may find a bounded linear operator $F_\beta : L^1(\mathbb{T}) \longrightarrow B(H^2(\mathbb{T})$ such that, for $h \in L^1(\mathbb{T})$ and $f \in H^2(\mathbb{T})$

$$(F_\beta h)(f) = \int_0^{2\pi} h(e^{i\theta})\phi_\beta(e^{i\theta})f \, dm(\theta).$$

We will prove the following result.

<u>3.5.2 Theorem.</u> If $h \in L^1(\mathbb{T})$ and $f,g \in H^2(\mathbb{T})$

$$((F_\beta h)f,g) = \lim_{r,s\to 1}\int_0^{2\pi}\int_0^{2\pi} \beta_{r,s}(e^{i\theta},e^{i\phi})h(e^{i\theta})f(e^{i\phi})g^\dagger(e^{i\phi}) \, dm(\theta)dm(\phi).$$

The proof of 3.5.2 is divided into three lemmas. Note that each $\beta_{r,s}$ belongs to $L^\infty(\mathbb{T}^2)$ and so defines a bounded multiplication operator on $L^2(\mathbb{T}^2)$. We may thus construct Hankel operators $H_{r,s}$ on $H^2(\mathbb{T}^2)$ in the usual way and deduce from Theorem 3.4.4 the existence of (non-unique) $\phi_{r,s} \in L^\infty(B(H^2(\mathbb{T})))$ with

$$(\hat{\phi}_{r,s}(-j)e_n,e_l) = \hat{\beta}_{r,s}(-j,-l-n)$$

for each $j,l,n \geq 0$. The first lemma uses the original Φ_β to construct such $\phi_{r,s}$.

<u>3.5.3 Lemma.</u> Let $\beta \in L^1(\mathbb{T}^2)$ and $\phi_\beta \in L^\infty(B(H^2(\mathbb{T})))$ be as above. Then for each $0 < r,\ s < 1$ there exists $\phi_{r,s} \in L^\infty(B(H^2(\mathbb{T})))$ such that for each $j,l,n \geq 0$

$$(\hat{\phi}_{r,s}(-j)e_n,e_l) = \hat{\beta}_{r,s}(-j,-l-n)$$

<u>Proof.</u> Let P_r denote the Poisson Kernel

$$P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta+r^2}$$

(for $0 < r < 1$ and $0 \leq \theta \leq 2\pi$). Then for each $0 < r < 1$, $0 \leq \theta \leq 2\pi$ and $f \in H^2(\mathbb{T})$, the function

$$e^{it} \rightsquigarrow P_r(\theta-t)\phi_\beta(e^{it})f$$

is a weakly-measurable $H^2(\mathbb{T})$-valued function on $\mathbb{T}$ and thus has separable range. We may therefore define $(P_r*\Phi_\beta)(e^{i\theta}) \in B(H^2(\mathbb{T}))$

for $e^{i\theta} \in \mathbb{T}$ by

$$(P_r * \Phi_\beta)(e^{i\theta})f = \int_0^{2\pi} P_r(\theta - t)\phi_\beta(e^{it})f \, dm(t).$$

Then $P_r * \phi_\beta \in L^\infty(B(H^2(\mathbb{T})))$ with $\| P_r * \phi_\beta \|_\infty \leq \| P_r \|_1 \| \phi_\beta \|_\infty = \| \phi_\beta \|_\infty$.

For each $n \in \mathbb{Z}$, $0 < r < 1$ and $f, g \in H^2(\mathbb{T})$

$$\begin{aligned}
((P_r * \Phi_\beta)\hat{\ }(n)f, g) &= \int_0^{2\pi} e^{-in\theta}((P_r * \Phi_\beta)(e^{i\theta})f, g) \, dm(\theta) \\
&= \int_0^{2\pi} e^{-in\theta}\left(\int_0^{2\pi} P_r(\theta - t)\phi_\beta(e^{it})f \, dm(t), g \right) dm(\theta) \\
&= \int_0^{2\pi} e^{-in\theta}\left[\int_0^{2\pi} P_r(\theta - t)(\phi_\beta(e^{it})f, g) dm(t) \right] dm(\theta)
\end{aligned}$$

by 3.3.2

$$= \int_0^{2\pi}\left(\int_0^{2\pi} e^{-n(\theta - t)} P_r(\theta - t) \, dm(\theta) \right) e^{-int}(\phi_\beta(e^{it})f, g) dm(t)$$

by Fubini's Theorem

$$= \hat{P}_r(n)(\hat{\phi}_\beta(n)f, g).$$

So,

$$((P_r * \phi_\beta)\hat{\ }(n)f, g) = r^{|n|}(\hat{\phi}_\beta(n)f, g). \qquad (*)$$

Now for $f \in H^2(\mathbb{T})$ we define $P_r * f$ in the usual way by

$$(P_r * f)(e^{i\theta}) = \int_0^{2\pi} P_r(\theta - t)f(e^{it}) \, dm(t).$$

Then $P_r * f \in H^2(\mathbb{T})$ with $\| P_r * f \|_2 \leq \| f \|_2$ and

$$(P_r * f)\hat{\ }(n) = r^n \hat{f}(n) \qquad (**)$$

for each $0 < r < 1$, $n \geq 0$.

We define $\phi_{r,s} : \mathbb{T} \longrightarrow B(H^2(\mathbb{T}))$ by

$$\phi_{r,s}(e^{i\theta})f = P_s * [(P_r * \phi_\beta)(e^{i\theta})(P_s * f)]$$

($e^{i\theta} \in \mathbb{T}$, $f \in H^2(\mathbb{T})$). Then each $\phi_{r,s}$ is wo-measurable and $\| \phi_{r,s}(e^{i\theta})f \|_2 \leq \| \phi_\beta \|_\infty \| f \|_2$ for each $e^{i\theta} \in \mathbb{T}$, $f \in H^2(\mathbb{T})$.

Moreover, if $j, l, n \geq 0$ and $0 < r, s < 1$

$$(\hat{\phi}_{r,s}(-j)e_n,e_1) = \int_0^{2\pi} e^{ij\theta}(\phi_{r,s}(e^{i\theta})e_n,e_1)\ dm(\theta)$$

$$= \int_0^{2\pi} e^{ij\theta}(P_s*[(P_r*\phi_\beta)(e^{i\theta})(P_s*e_n)],e_1)\ dm(\theta)$$

$$= \int_0^{2\pi} e^{ij\theta}((P_r*\phi_\beta)(e^{i\theta})(P_s*e_n),(P_s*e_1))\ dm(\theta)$$

$$= S^{n+1}((P_r*\phi_\beta)\hat{}(-j)e_n,e_1) \quad \text{by } (**)$$

$$= S^{n+1}r^j(\hat{\phi}_\beta(-j)e_n,e_1) \qquad \text{by } (*)$$

$$= S^{n+1}r^j\hat{\beta}(-j,-1-n)$$

as required.

Now for each of the $\phi_{r,s}$ in 3.5.3 $(0 < r,s < 1)$ we define $F_{r,s} \in B(L^1(\mathbb{T});B(H^2(\mathbb{T})))$ by

$$(F_{r,s}h)(f) = \int_0^{2\pi} h(e^{i\theta})\phi_{r,s}(e^{i\theta})f\ dm(\theta)$$

$(h \in L^1(\mathbb{T}),\ f \in H^2(\mathbb{T}))$.

The second lemma shows that at each $h \in L^1(\mathbb{T})$, $F_{r,s}h$ converges to $F_\beta h$ in the weak operator topology on $B(H^2(\mathbb{T}))$ as $r,s \to 1$ from below.

<u>3.5.4 Lemma.</u> If $F_{r,s}$ $(0 < r,s < 1)$ and F_β are as above and if $h \in L^1(\mathbb{T})$ and $f,g \in H^2(\mathbb{T})$ then

$$((F_{r,s}h)f,g) \to ((F_\beta h)f,g)$$

as $r,s \to 1$ from below.

<u>Proof.</u> From the definition of $F_{r,s}$ and F_β and from lemma 3.3.2 if is clear that for each $h \in L^1(\mathbb{T})$, $f,g \in H^2(\mathbb{T})$ and $0 < r,s < 1$

$$((F_\beta h)f,g) = \langle h,(\phi_\beta(\cdot)f,g)\rangle_{L^1,L^\infty}$$

and

$$((F_{r,s}h)f,g) = \langle h,(\phi_{r,s}(\cdot)f,g)\rangle_{L^1,L^\infty}.$$

Let $h \in L^1(\mathbb{T})$ and let $f,g \in H^2(\mathbb{T})$. Then for $0 < r,s < 1$

$$\begin{aligned}
&\langle h,(\phi_{r,s}(\cdot)f,g)\rangle - \langle h,(\phi_\beta(\cdot)f,g)\rangle \\
&\qquad = \langle h,\ P_r*(\phi_\beta(\cdot)(P_s*f),P_s*g)\rangle - \langle h,(\phi_\beta(\cdot)f,g)\rangle \\
&\qquad = \langle h,\ P_r*(\phi_\beta(\cdot)(P_s*f),P_s*g)\rangle - \langle h,P_r*(\phi_\beta(\cdot)f,g)\rangle \\
&\qquad\qquad + \langle h,P_r*(\phi_\beta(\cdot)f,g)\rangle - \langle h,(\phi_\beta(\cdot)f,g)\rangle
\end{aligned}$$

For the latter two terms, we see that for any $\varepsilon > 0$ there exists $0 < r_0 < 1$ such that for all $r_0 \le r < 1$

$$|\langle h,P_r*(\phi_\beta(\cdot)f,g)\rangle - \langle h,(\phi_\beta(\cdot)f,g)\rangle| < \varepsilon.$$

For the first two terms we have

$$\begin{aligned}
&|\langle h,\ P_r*(\phi_\beta(\cdot)(P_s*f),P_s*g)\rangle - \langle h,P_r*(\phi_\beta(\cdot)f,g)\rangle| \\
&\le \| h \|_1 \| P_r*(\phi_\beta(\cdot)(P_s*f),P_s*g) - P_r*(\phi_\beta(\cdot)f,g) \|_\infty \\
&\le \| h \|_1 \| (\phi_\beta(\cdot)(P_s*f),P_s*g) - (\phi_\beta(\cdot)f,g) \|_\infty \\
&\le \| h \|_1 \Big[\| (\phi_\beta(\cdot)(P_s*f-f),P_s*g) \|_\infty + \| (\phi_\beta(\cdot)f,(P_s*g-g)) \|_\infty \Big] \\
&\le \| h \|_1 \Big[\| \phi_\beta \|_\infty \| (P_s*f-f) \|_2 \| P_s*g) \|_2 \\
&\qquad\qquad + \| \phi_\beta \|_\infty \| f \|_2 \| P_s*g-g \|_2 \Big]
\end{aligned}$$

$\longrightarrow 0$ as $s \longrightarrow 1$ from below, uniformly in r.

Thus,

$$|\langle h,(\phi_{r,s}(\cdot)f,g)\rangle - \langle h,(\phi_\beta(\cdot)f,g)\rangle| \longrightarrow 0$$

as $s,r \longrightarrow 1$ from below and the result follows.

Lastly, we give an expression for the action of $F_{r,s}$ on $L^1(\mathbb{T})$.

<u>3.5.5 Lemma.</u> Let $F_{r,s}$ $(0 < r,s < 1)$ and $\beta_{r,s}$ be as above. Then for $h \in L^1(\mathbb{T})$ and $f,g \in H^2(\mathbb{T})$

$$((F_{r,s}h)f,g) = \int_0^{2\pi}\int_0^{2\pi} \beta_{r,s}(e^{i\theta},e^{i\phi})h(e^{i\theta})f(e^{i\phi})g^{\dagger}(e^{i\phi})\,dm(\theta)dm(\phi).$$

Proof. Since h,f and g can be approximated by polynomials, we need only to establish the formula for $h = e_j$, $f = e_l$ and $g = e_n$ for some $j \in \mathbb{Z}$, $l,n \in \mathbb{Z}^+$. In this case we have

$$\begin{aligned}((F_{r,s}e_j)e_l,e_n) &= \int_0^{2\pi} e^{ij\theta}(\phi_{r,s}(e^{i\theta})e_l,e_n)\,dm(\theta)\\ &= (\hat{\phi}_{r,s}(-j)e_l,e_n)\\ &= \hat{\beta}_{r,s}(-j,-l-n) \quad \text{by 3.5.3}\end{aligned}$$

and

$$\int_0^{2\pi}\int_0^{2\pi} \beta_{r,s}(e^{i\theta},e^{i\phi})e^{ij\theta}e^{il\phi}e^{in\phi}\,dm(\theta)dm(\phi) = \hat{\beta}_{r,s}(-j,-l-n).$$

as required.

Proof of 3.5.2. Fix $h \in L^1(\mathbb{T})$ and $f,g \in H^2(\mathbb{T})$. By lemmas 3.5.3, 3.5.4 there exists a family $\{F_{r,s}\}_{0<r,s<1}$ of bounded linear operators from $L^1(\mathbb{T})$ to $B(H^2(\mathbb{T}))$such that

$$\begin{aligned}((F_\beta h)f,g) &= \lim_{r,s\to 1} (F_{r,s}f,g)\\ &= \lim_{r,s\to 1} \int_0^{2\pi}\int_0^{2\pi} \beta_{r,s}(e^{i\theta},e^{i\phi})h(e^{i\theta})f(e^{i\phi})g^{\dagger}(e^{i\phi})\,dm(\theta)dm(\phi)\end{aligned}$$

by 3.5.5

Our principal result now follows easily.

3.5.6 Corollary. Let H be a Hankel operator on $H^2(\mathbb{T}^2)$ with symbol β. let $\beta_{r,s}$ be as defined above, let $h \in H^1(\mathbb{T})$ and let $f,g \in H^2(\mathbb{T})$. Then the action of H as a linear functional on the element $h(\cdot)(f\tilde{\otimes}g) + H^1(\mathbb{T};B)$ of $H^1(\mathbb{T};\mathcal{C}^1)/H^1(\mathbb{T};B)$ is given by

$< h(\cdot)(f\tilde{\otimes}g) + H^1(\mathbb{T};B),\ H >$

$$= \lim_{r,s\to 1} \int_0^{2\pi}\int_0^{2\pi} \beta_{r,s}(e^{i\theta},e^{i\phi})h(e^{i\theta})f(e^{i\phi})g^{\dagger}(e^{i\phi})\ dm(\theta)dm(\phi).$$

<u>Proof.</u> Immediately from Theorems 3.4.7 and 3.5.2.

Finally, to complete our study of Hankel operators on $H^2(\mathbb{T}^2)$ we shall note the relationships between the Banach spaces $H^1(\mathbb{T}^2)^*$, BHP and $(H^1(\mathbb{T})\hat{\otimes}H^1(\mathbb{T}))^*$.

<u>1.</u> <u>$H^1(\mathbb{T}^2)^*$ is continuously embedded in BHP.</u>

Any bounded linear functional F on $H^1(\mathbb{T}^2)$ may be represented by some $\phi_F \in L^\infty(\mathbb{T}^2)$ with $\| \phi_F \|_\infty = \| F \|$. We can form a Hankel operator from ϕ in the usual way : $H_\phi = PJ'M_\phi|_{H^2(\mathbb{T}^2)}$ where M_ϕ is multiplication by ϕ on $L^2(\mathbb{T}^2)$, J' is the 'flip' operator on $L^2(\mathbb{T}^2)$, $(J'f)(e^{i\theta},e^{i\phi}) = f(e^{-i\theta},e^{-i\phi})$, and P is the orthogonal projection of $L^2(\mathbb{T}^2)$ onto $H^2(\mathbb{T}^2)$. Then H_ϕ has representing array $\{\phi(-\underline{m}-\underline{n})\}_{\underline{m},\underline{n}\in\mathbb{Z}^{+2}}$, so the analytic function ϕ_F with power-series $\sum_{\underline{0}\le\underline{j}} \hat{\phi}(-\underline{j})(z_1,z_2)^{\underline{j}}$ is in BHP with $\| \phi_F \|_{BHP} \le \| \phi \|_\infty = \| F \|$. The map $F \rightsquigarrow \phi_F$ is then a well-defined contraction from $H^1(\mathbb{T}^2)^*$ into BHP.

<u>2.</u> <u>BHP is continuously embedded in $(H^1(\mathbb{T})\hat{\otimes}H^1(\mathbb{T}))^*$.</u>

Suppose that $\phi \in$ BHP. Then the Hankel operator with representing array $\{\phi(\underline{i} + \underline{j})\}_{\underline{i},\underline{j}\in\mathbb{Z}^{+2}}$ has (by 3.4.4) an associated (non-unique) function $\Phi \in L^\infty(B(H^2(\mathbb{T})))$ with $\| \Phi \|_\infty = \| \phi \|_{BHP}$ and $\hat{\Phi}(-n)$ Hankel for each $n \ge 0$. The corresponding operator $F : L^1(\mathbb{T}) \longrightarrow B(H^2(\mathbb{T}))$ (as in 3.3.1) is Hankel-valued on $H^1(\mathbb{T})$ so the restriction F' to $H^1(\mathbb{T})$ may be regarded (via Nehari's Theorem) as a bounded linear operator from $H^1(\mathbb{T})$ into $H^1(\mathbb{T})^*$. But $B(H^1(\mathbb{T});H^1(\mathbb{T})^*)$ is isometrically isomorphic to $(H^1(\mathbb{T})\hat{\otimes}H^1(\mathbb{T}))^*$, so F' defines a continuous linear functional f on $H^1(\mathbb{T})\hat{\otimes}H^1(\mathbb{T})$ with

$$\begin{aligned}\| f \| = \| F' \|_{B(H^1;H^{1*})} &\le \| F \|_{B(L^1;B(H^2))} \\ &= \| \Phi \|_\infty \\ &= \| \phi \|_{BHP} .\end{aligned}$$

<u>3.</u> It is easy to show that $H^1(\mathbb{T}^2)$ is isometrically isomorphic to $H^1(\mathbb{T};H^1(\mathbb{T}))$. By arguing as in 3.3.13 we can show that for any Banach space X, $H^1(\mathbb{T};X)$ is isometrically isomorphic to the closure of $H^1(\mathbb{T})\otimes X$ in $L^1(\mathbb{T})\hat{\otimes}X$. Thus, $H^1(\mathbb{T}^2)$ is isometrically isomorphic to the closure of $H^1(\mathbb{T})\otimes H^1(\mathbb{T})$ in $L^1(\mathbb{T})\hat{\otimes}H^1(\mathbb{T})$.

Now let $\| \cdot \|_{\hat{\otimes}1}$ denote the $H^1(\mathbb{T})\hat{\otimes}H^1(\mathbb{T})$ norm on $H^1(\mathbb{T})\otimes H^1(\mathbb{T})$ and let $\| \cdot \|_{\hat{\otimes}2}$ denote the $L^1(\mathbb{T})\hat{\otimes}H^1(\mathbb{T})$ norm on $H^1(\mathbb{T})\otimes H^1(\mathbb{T})$. Then, using $\hookrightarrow$ to denote a continuous embedding, we have from 1. and 2. that

$$(H^1(\mathbb{T})\otimes H^1(\mathbb{T}), \| \cdot \|_{\hat{\otimes}2})^* \hookrightarrow BHP \hookrightarrow (H^1(\mathbb{T})\otimes H^1(\mathbb{T}), \| \cdot \|_{\hat{\otimes}1})^*.$$

<u>4.</u> We do not know whether BHP is isomorphic to either $(H^1(\mathbb{T})\otimes H^1(\mathbb{T}), \| \cdot \|_{\hat{\otimes}2})^*$ or $(H^1(\mathbb{T})\otimes H^1(\mathbb{T}), \| \cdot \|_{\hat{\otimes}1})^*$ but we can show that

$(H^1(\mathbb{T})\otimes H^1(\mathbb{T}), \| \cdot \|_{\hat{\otimes}2})^*$ is strictly contained in $(H^1(\mathbb{T})\otimes H^1(\mathbb{T}), \| \cdot \|_{\hat{\otimes}1})^*$. To achieve this we shall amend the example in 3.3.15 to produce a $\| \cdot \|_{\hat{\otimes}1}$-continuous linear functional on $H^1(\mathbb{T})\otimes H^1(\mathbb{T})$ which is NOT $\| \cdot \|_{\hat{\otimes}2}$-continuous.

Define a functional β at a simple element $h\otimes g$ of $H^1(\mathbb{T})\otimes H^1(\mathbb{T})$ by

$$\beta(f\otimes g) = \sum_{k\ge 0} \hat{h}(2^k)g(2^k) .$$

Then

$$| \beta(h\otimes g) | = \left(\sum_{k\geq 0} |\hat{h}(2^k)|^2 \right)^{1/2} \left(\sum_{k\geq 0} | g(2^k) |^2 \right)^{1/2}$$

$$\leq c^2 \| h \|_1 \| g \|_1$$

by Paley's Inequality ([DUR],p 104) for some constant c, independent of h,g.

Clearly β extends linearly to a $\| \cdot \|_{\hat{\otimes}1}$-continuous linear functional on $H^1(\mathbb{T})\otimes H^1(\mathbb{T})$.

Define $$\alpha_n = \sum_{k=1}^{n} \frac{1}{k} e_{2^k}\otimes e_{2^k} \qquad \text{(for } n \geq 1\text{)}.$$

Then $\beta(\alpha_n) = \displaystyle\sum_{k=1}^{n} \frac{1}{k} \longrightarrow \infty$ as $n \longrightarrow \infty$ whilst

$$\begin{aligned} \| \alpha_n \|_{\hat{\otimes}2} &= \| \sum_{k=1}^{n} \frac{1}{k} e_{2^k}(\cdot)e_{2^k} \|_{L^1(\mathbb{T};H^1(\mathbb{T}))} \\ &= \int_0^{2\pi} \| \sum_{k=1}^{n} \frac{1}{k} e^{2^k i\theta} e_{2^k} \|_{H^1(\mathbb{T})} \, dm(\theta) \\ &\leq \int_0^{2\pi} \| \sum_{k=1}^{n} \frac{1}{k} e^{2^k i\theta} e_{2^k} \|_{H^2(\mathbb{T})} \, dm(\theta) \\ &= \left(\sum^{n} \frac{1}{k^2} \right)^{1/2} \\ &\leq \pi/\sqrt{6} \qquad \text{for each } n \geq 1. \end{aligned}$$

Hence, β is not $\| \cdot \|_{\hat{\otimes}2}$-continuous, as required.

Chapter 4 Norm Estimates for Polynomials in Two Commuting Power Bounded Operators.

In this chapter we will employ the methods of Chapter 2 and the main results of Chapter 3 to produce upper bounds on the norm of $p(S,T)$ when p is a polynomial and S,T are commuting power-bounded operators on $\mathcal{H}$.

Throughout the chapter S,T are fixed power-bounded operators and c,d are constants such that $\| S^n \| \le c$ and $\| T^n \| \le d$ for every $n \ge 0$. We regard the commuting pair (S,T) as the generator of a uniformly bounded semigroup $\{S^m T^n\}_{m,n\in\mathbb{Z}^+}$ and the map $(m,n) \rightsquigarrow S^m T^n$ as a bounded representation of the semigroup $\mathbb{Z}^{+2}$.

We show that, using the characterisation of Hankel operators on $H^2(\mathbb{T}^2)$ given in Chapter 3, Peller's methods can be extended to consider functions of (S,T). Indeed our principle result is that there exists a constant K such that for any polynomial $p(e^{i\theta},e^{i\phi})$ of degree $N \ge 2$

$$\| p(S,T) \| \le K\, c^2 d^2\, (\log N)^2\, \| p \|_{L^\infty(\mathbb{T}^2)} .$$

Section 4.1 A First Estimate of $\| p(S,T) \|$.

4.1.1 Notation. For $\underline{k} = (k_1,k_2) \in \mathbb{Z}^{+2}$ and $\underline{j} = (j_1,j_2) \in \mathbb{Z}^{+2}$ we write $\underline{k} \le \underline{j}$ when $k_1 \le j_1$ and $k_2 \le j_2$. If $\underline{z} = (z_1,z_2) \in \mathbb{C}^2$ then $\underline{z}^{\underline{k}} = z_1^{k_1} z_2^{k_2} \in \mathbb{C}$ and $(S,T)^{\underline{k}} = S^{k_1} T^{k_2} \in B(\mathcal{H})$.

Now let p be a fixed polynomial in $(e^{i\theta},e^{i\phi})$

$$p(e^{i\theta},e^{i\phi}) = \sum_{\underline{0}\le\underline{k}\le\underline{N}} \hat{p}(\underline{k})(e^{i\theta},e^{i\phi})^{\underline{k}}$$

for some $\underline{N} = (N_1,N_2) \in \mathbb{Z}^{+2}$. Let N denote the degree of p, namely $\max(N_1,N_2)$.

We will need the following (presumably known) application of Grothendieck's Inequality.

<u>4.1.2 Lemma.</u> Suppose that $\alpha = \{\alpha_{\underline{i},\underline{j}}\}_{\underline{i},\underline{j}\in\mathbb{Z}^{+2}} \in l^1(\mathbb{Z}^{+2})\check{\otimes} l^1(\mathbb{Z}^{+2})$ has $\alpha_{\underline{i},\underline{j}} = 0$ whenever $\underline{i} \nleq \underline{M}$ or $\underline{j} \nleq N$. Let $\{x_{\underline{i}}\}_{\underline{i}\in\mathbb{Z}^{+2}}$,$\{y_{\underline{i}}\}_{\underline{i}\in\mathbb{Z}^{+2}}$ be matrices with entries in $\mathscr{H}$ such that $\| x_{\underline{i}}\|_{\mathscr{H}} \le a$ and $\| y_{\underline{i}}\|_{\mathscr{H}} \le b$ for each $\underline{i} \in \mathbb{Z}^{+2}$. Then

$$\Big| \sum_{\underline{i},\underline{j}\in\mathbb{Z}^{+2}} \alpha_{\underline{i},\underline{j}} (x_{\underline{i}},y_{\underline{j}}) \Big| \le K_G \| \alpha \|_{\check{\otimes}}\, ab.$$

<u>Proof.</u> Suppose that $\underline{M} = (M_1,M_2)$ and $\underline{N} = (N_1,N_2)$. We can rearrange the array α to give a matrix $\tilde{\alpha}$ with $\tilde{\alpha}_{p(M_2+1)+q,r(N_2+1)+s} = \alpha_{(p,q),(r,s)}$ for every $\underline{0} \le (p,q) \le \underline{M}$ and $\underline{0} \le (r,s) \le \underline{N}$. We put $\tilde{\alpha}_{i,j} = 0$ when $i > M_1(M_2+1) + M_2$ or $j > N_1(N_2+1) + N_2$.

Then $\tilde{\alpha} \in l^1(\mathbb{Z}^+)\check{\otimes} l^1(\mathbb{Z}^+)$ and if $\phi,\psi \in l^\infty(\mathbb{Z}^+)$

$$\Big| \sum_{i,j\ge 0}\alpha_{i,j}\phi(i)\psi(j) \Big|$$

$$= \Big| \sum_{p,q,r,s\ge 0} \tilde{\alpha}_{p(M_2+1)+q,r(N_2+1)+s}\ \phi(p(M_2+1)+q)\psi(r(N_2+1)+s) \Big|$$

$$= \Big| \sum_{\substack{\underline{0}\le(p,q)\\ \underline{0}\le(r,s)}} \alpha_{(p,q),(r,s)}\ \phi(p(M_2+1)+q)\psi(r(N_2+1)+s) \Big|$$

By considering the supremum over all ϕ,ψ with $\| \phi \|_\infty$, $\| \psi \|_\infty \le 1$ we see that

$$\| \tilde{\alpha} \|_{l^1(\mathbb{Z}^+)\check{\otimes} l^1(\mathbb{Z}^+)} \le \| \alpha \|_{l^1(\mathbb{Z}^{+2})\check{\otimes} l^1(\mathbb{Z}^{+2})} .$$

If $\{x_{\underline{i}}\}_{\underline{i}\in\mathbb{Z}^{+2}}$ and $\{y_{\underline{i}}\}_{\underline{i}\in\mathbb{Z}^{+2}}$ are matrices with entries in $\mathcal{H}$ we can rearrange similarly to give sequences $\{\tilde{x}_n\}_{n\geq 0}, \{\tilde{y}_n\}_{n\geq 0}$ with $\tilde{x}_{p(m_2+1)+q} = x_{(p,q)}$ and $\tilde{y}_{r(N_2+1)+s} = y_{(r,s)}$ for every $\underline{0} \leq (p,q)$ and $\underline{0} \leq (r,s)$. Thus

$$\Big| \sum_{\underline{i},\underline{j}\in\mathbb{Z}^{+2}} \alpha_{\underline{i},\underline{j}} (x_{\underline{i}}, y_{\underline{j}}) \Big|$$

$$= \Big| \sum_{p,q,r,s\geq 0} \alpha_{(p,q),(r,s)} (x_{(p,q)}, y_{(r,s)}) \Big|$$

$$= \Big| \sum_{i,j\geq 0} \tilde{\alpha}_{i,j} (\tilde{x}_i, \tilde{y}_j) \Big|$$

$$\leq K_G \, \| \tilde{\alpha} \|_{\check{\otimes}} \sup_{i\geq 0} \| \tilde{x}_i \|_{\mathcal{H}} \sup_{j\geq 0} \| \tilde{y}_j \|_{\mathcal{H}}$$

by Grothendieck's Inequality, 1.2.5

$$\leq K_G \, \| \alpha \|_{\check{\otimes}} \, ab \, .$$

Note that we can find a tensor $\alpha \in l^1(\mathbb{Z}^{+2})\hat{\otimes} l^1(\mathbb{Z}^{+2})$ with an array $\{\alpha_{\underline{i},\underline{j}}\}_{\underline{i},\underline{j}\in\mathbb{Z}^{+2}}$ of finitely many non-zero entries such that

$$\sum_{\underline{i}+\underline{j}=\underline{k}} \alpha_{\underline{i},\underline{j}} = \hat{p}(\underline{k}) \qquad (*)$$

for all $\underline{k} \in \mathbb{Z}^{+2}$. We can now use the power-boundedness of S,T and the fact that they commute to prove a result analogous to 2.2.1.

<u>4.1.3 Theorem.</u> If p is related to $\alpha \in l^1(\mathbb{Z}^{+2})\check{\otimes} l^1(\mathbb{Z}^{+2})$ by $(*)$ then

$$\| p(S,T) \| \leq K_G \, c^2 d^2 \, \| \alpha \|_{\check{\otimes}} \, .$$

Proof. For any $x, y \in \mathcal{H}$

$$| (p(S,T)x,y)| = \Big| \sum_{\underline{0 \le k \le N}} \sum_{\underline{i}+\underline{j}=\underline{k}} \alpha_{\underline{i},\underline{j}} ((S,T)^{\underline{k}} x, y) \Big|$$

$$= \Big| \sum_{\underline{i},\underline{j} \ge \underline{0}} \alpha_{\underline{i},\underline{j}} ((S,T)^{\underline{i}} x, (S^*,T^*)^{\underline{j}} y) \Big|$$

(since S,T commute)

$$\le K_G \| \alpha \|_{\check{\otimes}} c^2 d^2 \| x \|_{\mathcal{H}} \| y \|_{\mathcal{H}}$$

by 4.1.2 since S,T are power-bounded by c,d.

The tensor α is not uniquely determined by the polynomial p so we will associate p with an equivalence class of tensors.

4.1.4 Notation. We denote by $E^{(2)}$ the linear subspace of $l^1(\mathbb{Z}^{+2})\check{\otimes}l^1(\mathbb{Z}^{+2})$ consisting of those $\alpha \in l^1(\mathbb{Z}^{+2})\check{\otimes}l^1(\mathbb{Z}^{+2})$ with array $\{\alpha_{\underline{i},\underline{j}}\}_{\underline{i},\underline{j} \in \mathbb{Z}^{+2}}$ having finitely many non-zero entries and satisfying

$$\sum_{\underline{i}+\underline{j}=\underline{n}} \alpha_{\underline{i},\underline{j}} = 0 \quad \text{for every } \underline{n} \in \mathbb{Z}^{+2}.$$

We denote the closure of $E^{(2)}$ in $l^1(\mathbb{Z}^{+2})\check{\otimes}l^1(\mathbb{Z}^{+2})$ by $\overline{E^{(2)}}$.

By arguing as in 2.2.3 we obtain the following corollary.

4.1.5 Corollary. If p is related to $\alpha \in l^1(\mathbb{Z}^{+2})\check{\otimes}l^1(\mathbb{Z}^{+2})$ by (*) then

$$\| p(S,T) \| \le K_G c^2 d^2 \| \alpha + \overline{E^{(2)}} \|_{(l^1(\mathbb{Z}^{+2})\check{\otimes}l^1(\mathbb{Z}^{+2}))/\overline{E^{(2)}}} .$$

By 1.2.3 the dual of $l^1(\mathbb{Z}^{+2})\check{\otimes}l^1(\mathbb{Z}^{+2})$ is isometrically isomorphic to V_2^2 with respect to the pairing $<\alpha,\beta> = \sum_{\underline{o}\leq\underline{i},\underline{j}} \alpha_{\underline{i},\underline{j}}\, \beta_{\underline{i},\underline{j}}$

($\alpha \in l^1(\mathbb{Z}^{+2})\check{\otimes} l^1(\mathbb{Z}^{+2})$, $\beta \in V_2^2$). Moreover, it is clear that the annihilator of $E^{(2)}$ in V_2^2 consists of those arrays in V_2^2 that are of Hankel form :

$$E^{(2)\perp} = \left\{ \{\beta(\underline{i} + \underline{j})\}_{\underline{i},\underline{j}\in\mathbb{Z}^{+2}} \in V_2^2 : \{\beta(\underline{n})\}_{\underline{n}\in\mathbb{Z}^{+2}} \text{ is a matrix of scalars} \right\}$$

4.1.6 Notation. If ϕ is an analytic function on $\mathbb{D}^2$ with power series

$$\phi(z_1,z_2) = \sum_{\underline{n}\in\mathbb{Z}^{+2}} \hat{\phi}(\underline{n})(z_1,z_2)^{\underline{n}}$$

then we denote by $\Gamma_\psi^{(2)}$ the Hankel-type array defined by

$$\Gamma_\psi^{(2)}(\underline{i},\underline{j}) = \hat{\phi}(\underline{i} + \underline{j}) \qquad (\underline{i},\underline{j} \in \mathbb{Z}^{+2}).$$

Note that when $\sum_{\underline{i}+\underline{j}=\underline{n}} \alpha_{\underline{i},\underline{j}} = \hat{p}(\underline{n})$ for all $\underline{n} \in \mathbb{Z}^{+2}$ and when $\Gamma_\psi^{(2)}$ $\in V_2^2$

$$< \alpha, \Gamma_\psi^{(2)} > = \sum_{\underline{n}\in\mathbb{Z}^{+2}} \hat{p}(\underline{n})\hat{\phi}(\underline{n}).$$

Moreover, since

$$\| \alpha + \overline{E^{(2)}} \| = \sup \{ \ | < \alpha,\beta > | \ : \beta \in E^{(2)\perp}, \| \beta \|_{V_2^2} \leq 1 \}$$

we obtain the following inequality.

4.1.7 Corollary.

$$\| p(S,T) \| \leq K_G\, c^2 d^2 \sup \left\{ \ | \sum_{\underline{n}\in\mathbb{Z}^{+2}} \hat{p}(\underline{n})\hat{\psi}(\underline{n}) \ | \ : \| \Gamma_\psi^{(2)} \|_{V_2^2} \leq 1 \right\}.$$

Section 4.2 Further Estimates of $\| p(S,T) \|$.

In this section we use results from chapter 3 to show how the quantity

$$\sup \left\{ \left| \sum_{\underline{0} \le \underline{n} \le \underline{N}} \hat{p}(\underline{n})\hat{\phi}(\underline{n}) \right| : \| \Gamma_{\phi}^{(2)} \|_{V_2^2} \le 1 \right\}$$

is related to a projective tensor product norm associated with the polynomial, p. As a corollary we deduce a bound on $\| p(S,T) \|$ in terms of $\| p \|_{\infty}$ and the degree of p.

We start by extending the notion of Schur multiplication from matrices to arrays indexed by $\underline{i},\underline{j} \in \mathbb{Z}^{+2}$.

4.2.1 Definitions. a) If $A = \{a_{\underline{i},\underline{j}}\}_{\underline{i},\underline{j} \in \mathbb{Z}^{+2}}$ and $B = \{b_{\underline{i},\underline{j}}\}_{\underline{i},\underline{j} \in \mathbb{Z}^{+2}}$ are 4-dimensional arrays of scalars then the Schur product $A \circledast B$ of A,B is formed by pointwise multiplication of the entries :

$$A \circledast B = \{a_{\underline{i},\underline{j}} \, b_{\underline{i},\underline{j}}\}_{\underline{i},\underline{j} \in \mathbb{Z}^{+2}}.$$

b) An array $A = \{a_{\underline{i},\underline{j}}\}_{\underline{i},\underline{j} \in \mathbb{Z}^{+2}}$ is a Schur multiplier on $B(H^2(\mathbb{T}^2))$ if whenever $T = \{t_{\underline{i},\underline{j}}\}_{\underline{i},\underline{j} \in \mathbb{Z}^{+2}}$ is the representing array (with respect to $\{e_{\underline{m}}\}_{\underline{m} \in \mathbb{Z}^{+2}}$) of a bounded linear operator on $H^2(\mathbb{T}^2)$ the Schur product $A \circledast T$ is also the representing array of some bounded linear operator on $H^2(\mathbb{T}^2)$.

c) If A is a Schur multiplier on $B(H^2(\mathbb{T}^2))$ then the multiplier norm $\| A \|_{M(B(H^2(\mathbb{T}^2)))}$ of A is the operator norm of Schur multiplication by A on (representing arrays of operators in) $B(H^2(\mathbb{T}^2))$.

4.2.2 Lemma. If $M \in V_2^2$ then M is a Schur multiplier on $B(H^2(\mathbb{T}^2))$ with multiplier norm dominated by $\| M \|_{V_2^2}$.

Proof. If $x \in l^{\infty}(\mathbb{Z}^{+2})$ then we denote by diag x the array $\{x(\underline{m})\delta_{\underline{m},\underline{n}}\}_{\underline{m},\underline{n}\in\mathbb{Z}^{+2}}$. If $x,y \in l^{\infty}(\mathbb{Z}^{+2})$ it is easy to show that when $\beta = x\otimes y$ and A is the representing array of a bounded operator on $H^2(\mathbb{T}^2)$

$$\beta \circledast A = (\text{diag } x)\ A\ (\text{diag } y).$$

As in the proof of 1.4.3 we extend to finite sums of tensors of the form $\beta = \sum_{k=0}^{n} x_k\otimes y_k$ ($x_k, y_k \in l^{\infty}(\mathbb{Z}^{+2})$, $0 \le k \le n$). Then,

$$\| \beta \circledast A \| = \| \sum_{k=0}^{n} (\text{diag } x_k\ )\ A\ (\text{diag } y_k\ ) \|$$

$$\le \| A \|_{B(H^2(\mathbb{T}^2))} \sum_{k=0}^{n} \| x_k \|_{\infty} \| y_k \|_{\infty} .$$

Hence, if $M \in V_2^2$ and $N > 0$, $P_N^{(2)}M$ is a Schur multiplier on $B(H^2(\mathbb{T}^2))$ with

$$\| P_N^{(2)}M \|_{M(B(H^2(\mathbb{T}^2)))} \le \| P_N^{(2)}M \|_{l^{\infty}(\mathbb{Z}^{+2})\hat{\otimes} l^{\infty}(\mathbb{Z}^{+2})}$$

$$\le \| M \|_{V_2^2} \quad \text{by definition.}$$

Moreover, if $h \in H^2(\mathbb{T}^2)$ is a polynomial and $A = \{a_{\underline{i},\underline{j}}\}_{\underline{i},\underline{j}\in\mathbb{Z}^{+2}}$ is the representing array of a bounded linear operator on $H^2(\mathbb{T}^2)$

$$\| (M \circledast A)h \|_2^2 = \sum_{\underline{i}\in\mathbb{Z}^{+2}} | \sum_{\underline{j}\in\mathbb{Z}^{+2}} M_{\underline{i},\underline{j}}\ a_{\underline{i},\underline{j}}\ \hat{h}(\underline{j}) |^2$$

$$= \lim_{n\to\infty} \sum_{\underline{0}\le\underline{i}\le(n,n)} | \lim_{m\to\infty} \sum_{\underline{0}\le\underline{j}\le(m,m)} M_{\underline{i},\underline{j}}\ a_{\underline{i},\underline{j}}\ \hat{h}(\underline{j}) |^2 .$$

Then, by two applications of 1.4.2

$$\| (M \circledast A)h \|_2^2 \le \lim_{n\to\infty} \sum_{\underline{0}\le\underline{i}\le(n,n)} | \sum_{\underline{0}\le\underline{j}\le(n,n)} M_{\underline{i},\underline{j}}\ a_{\underline{i},\underline{j}}\ \hat{h}(\underline{j}) |^2$$

$$= \lim_{n\to\infty} \| ((P_n M) \circledast A)h \|_2^2$$

$$\le \| M \|_{V_2^2}^2 \| A \|_{B(H^2(\mathbb{T}^2)))}^2 \| h \|_2^2 .$$

Since the polynomials form a dense subspace of $H^2(\mathbb{T}^2)$ it follows that $\| M \circledast A \|_{B(H^2(\mathbb{T}^2))} \le \| M \|_{V_2^2} \| A \|_{B(H^2(\mathbb{T}^2))}$ and hence that $M \in M(B(H^2(\mathbb{T}^2)))$ with $\| M \|_{M(B(H^2(\mathbb{T}^2)))} \le \| M \|_{V_2^2}$.

We will use 4.2.2 to prove the appropriate analogue of 2.2.6. Recall (3.1.2) that BHP is the Banach space of analytic functions ψ on $\mathbb{D}^2$ for which the Hankel array $\Gamma_\psi^{(2)}$ is the representing array of a bounded linear operator on $H^2(\mathbb{T}^2)$. We must now define a Banach space of multipliers on BHP.

<u>**4.2.3 Definitions.**</u> a) An analytic function ϕ on $\mathbb{D}^2$ with power series $\phi(z_1,z_2) = \sum_{m,n\ge 0} \hat{\phi}(m,n)z_1^m z_2^n$ is a <u>multiplier on BHP</u> if for every $\psi \in$ BHP (with power series $\psi(z_1,z_2) = \sum_{m,n\ge 0} \hat{\psi}(m,n)z_1^m z_2^n$) the series $\sum_{m,n\ge 0} \hat{\phi}(m,n)\hat{\psi}(m,n)z_1^m z_2^n$ is the power series of a BHP function, denoted by $\phi * \psi$.

b) If ϕ is a multiplier on BHP then the <u>multiplier norm</u>, $\| \phi \|_{M(BHP)}$ is the operator norm of the map $\psi \rightsquigarrow \phi * \psi$ on BHP.

c) The normed linear space of all multipliers on BHP with the norm $\| \cdot \|_{M(BHP)}$ is denoted by M(BHP).

<u>**4.2.4 Lemma.**</u> If ϕ is an analytic function on $\mathbb{D}^2$ such that $\Gamma_\phi^{(2)} \in V_2^2$ then ϕ is a multiplier on BHP with $\| \phi \|_{M(BHP)} \le \| \Gamma_\phi^{(2)} \|_{V_2^2}$.

<u>Proof.</u> Suppose that $\Gamma_\phi^{(2)} \in V_2^2$ and $\psi \in$ BHP. Then, by 4.2.2, $\Gamma_\phi^{(2)}$ is a Schur multiplier on $B(H^2(\mathbb{T}^2))$ and, by definition, $\Gamma_\psi^{(2)}$ is the representing array (with respect to $\{e_{\underline{m}}\}_{\underline{m}\in\mathbb{Z}^{+2}}$) of a bounded Hankel

operator on $H^2(\mathbb{T}^2)$.

Thus, $\Gamma^{(2)}_{\phi*\psi} = \Gamma^{(2)}_{\phi} \circledast \Gamma^{(2)}_{\psi}$ is the representing array of a bounded Hankel operator on $H^2(\mathbb{T}^2)$ with norm dominated by $\| \Gamma^{(2)}_{\phi} \|_{V_2^2} \| \psi \|_{BHP}$. Hence $\phi * \psi \in$ BHP and

$$\| \phi * \psi \|_{BHP} \le \| \Gamma^{(2)}_{\phi} \|_{V_2^2} \| \psi \|_{BHP}$$

as required.

<u>Remark.</u> To complete the analogy with 2.2.6 we should define the notion of a multiplier on $H^1(\mathcal{E}^1)/H^1(B)$ and further deduce that any multiplier on BHP is a multiplier on its predual $H^1(\mathcal{E}^1)/H^1(B)$. However, since it is not clear how to define multipliers on $H^1(\mathcal{E}^1)/H^1(B)$ (and since $H^1(\mathcal{E}^1)/H^1(B)$ itself has no obvious predual) it is preferable to work with the multipliers on BHP. Thus we state the following corollary.

<u>4.2.5 Corollary</u>.

$$\| p(S,T) \| \le K_G \, c^2 d^2 \sup \left\{ \left| \sum_{\underline{n} \in \mathbb{Z}^{+2}} \hat{p}(\underline{n}) \hat{\phi}(\underline{n}) \right| : \| \phi \|_{M(BHP)} \le 1 \right\}.$$

<u>Proof.</u> By 4.1.7 and 4.2.4.

The identification of BHP with a dual space (by Corollary 3.4.9) allows each multiplier of BHP to be considered as a linear functional on a projective tensor product space.

<u>4.2.6 Theorem.</u> Let ϕ be a multiplier on BHP. Then there exists a linear functional Φ on $[H^1(\mathcal{E}^1)/H^1(B)] \hat{\otimes}$ BHP with $\| \Phi \| \le \| \phi \|_{M(BHP)}$

and

$$\Phi([h + H^1(B)] \otimes \psi) = < h + H^1(B), \phi * \psi >_{H^1(\mathscr{C}^1)/H^1(B),BHP} \qquad (*)$$

for all $h \in H^1(\mathscr{C}^1)$, $\psi \in BHP$.

Proof. Let $\mathscr{D}$ denote the dense subspace of $H^1(\mathscr{C}^1)$ consisting of the $\mathscr{C}^1$-valued analytic polynomials. Let $\mathscr{D} + H^1(B)$ denote the subspace $\{ p + H^1(B) : p \in \mathscr{D} \}$ of $H^1(\mathscr{C}^1)/H^1(B)$. Clearly $\mathscr{D} + H^1(B)$ is dense in $H^1(\mathscr{C}^1)/H^1(B)$.

Now let ϕ be a multiplier on BHP. We define Φ on a simple tensor $[p + H^1(B)] \otimes \psi$ $(p \in \mathscr{D}, \psi \in BHP)$ by

$$\Phi([p + H^1(B)] \otimes \psi) = < p + H^1(B), \phi * \psi >_{H^1(\mathscr{C}^1)/H^1(B),BHP}$$

Now extend Φ linearly to $[\mathscr{D} + H^1(B)] \otimes BHP$ by setting

$$\Phi\left(\sum_{n=0}^{N}[p_n + H^1(B)] \otimes \psi_n\right) = \sum_{n=0}^{N}\Phi([p_n + H^1(B)] \otimes \psi_n)$$

$(p_n \in \mathscr{D}, \psi_n \in BHP, 0 \le n \le N)$

To show that Φ is well-defined we will require the following.

Claim. If $h_n \in H^1(\mathbb{T})$, $f_n, g_n \in H^2(\mathbb{T})$ and $\psi_n \in BHP$ (for $0 \le n \le N$) are such that

$$\alpha \equiv \sum_{n=0}^{N}[h_n(\cdot)(f_n\tilde{\otimes}g_n) + H^1(B)] \otimes \psi_n = 0$$

then

$$\sum_{n=0}^{N} \hat{h}_n(i)(f_n g_n^{\dagger})\hat{}(j)\hat{\psi}(i,j) = 0$$

for each $i,j \ge 0$.

Proof of Claim. Note that $\alpha = 0$ if and only if $\alpha(F,G) = 0$ for all $F \in (H^1(\mathscr{C}^1)/H^1(B))^*$ and $G \in (BHP)^*$. Fix $i,j \ge 0$. Then the function

$e_{i,j}$ on $\mathbb{T}^2$ is in BHP and therefore defines a functional on $H^1(\mathscr{C}^1)/H^1(B)$. Indeed by 3.5.6

$$< h_n(\cdot)(f_n\tilde{\otimes}g_n) + H^1(B),\ e_{i,j} > = \hat{h}_n(i)(f_n g_n^{\dagger})\hat{\ }(j)$$

Moreover, since $e_i(\cdot)(e_j\tilde{\otimes}\ e_0) + H^1(B)$ is in $H^1(\mathscr{C}^1)/H^1(B)$ we have an element $[e_i(\cdot)(e_j\tilde{\otimes}\ e_0) + H^1(B)]'$ in its double dual, BHP^*. For $\psi \in$ BHP and $i,j \in \mathbb{Z}^{+2}$

$$< \psi,\ [e_i(\cdot)(e_j\tilde{\otimes}\ e_0) + H^1(B)]' >_{BHP,BHP^*}$$

$$= <e_i(\cdot)(e_j\tilde{\otimes}\ e_0) + H^1(B), \psi >_{H^1(\mathscr{C}^1)/H^1(B),BHP}$$

$$= \hat{\psi}(i,j)$$

by 3.4.9.

Hence,

$$\sum_{n=0}^{N}\hat{h}_n(i)(f_n g_n^{\dagger})\hat{\ }(j)\hat{\psi}(i,j)$$

$$= \sum < h_n(\cdot)(f_n\tilde{\otimes}g_n) + H^1(B),\ e_{i,j} > \quad <\psi_n,\ [\ e_i(\cdot)(e_j\tilde{\otimes}\ e_0) + H^1(B)]'>$$

$$= \alpha\ (e_{i,j}, [e_i(\cdot)(e_j\tilde{\otimes}\ e_0) + H^1(B)]')$$

$$= 0$$

and the claim is proved.

We now show that Φ is well-defined. Suppose that $p_n \in \mathscr{D}$ and $\psi_n \in$ BHP are such that $\sum_{n=0}^{N} [\ p_n + H^1(B)] \otimes \psi_n = 0$. By 3.4.8 we may choose $f_{n,m}, g_{n,m} \in H^2(\mathbb{T})$ $(m = 0,1,\ldots,M_n)$ such that

$$p_n + H^1(B) = \sum_{m=0}^{M_n} e_m(\cdot)(f_{n,m}\ \tilde{\otimes}\ g_{n,m}) + H^1(B).$$

Then,

$$\Phi\left(\sum_{n=0}^{N}[\,p_n + H^1(B)] \otimes \psi_n\right)$$

$$= \sum_{n=0}^{N} < \sum_{m=0}^{M_n} e_m(\cdot)(f_{n,m} \tilde{\otimes} g_{n,m}) + H^1(B), \phi * \psi_n >$$

$$= \sum_{n=0}^{N}\sum_{m=0}^{M_n}\sum_{i,j\geq 0} \hat{e}_m(i)(f_{n,m}\, g^{\dagger}_{n,m})\hat{\ }(j)\hat{\phi}(i,j)\hat{\psi}_n(i,j)$$

$$= \sum_{i,j\geq 0}\hat{\phi}(i,j)\sum_{n=0}^{N}\sum_{m=0}^{M_n} \hat{e}_m(i)(f_{n,m}\, g^{\dagger}_{n,m})\hat{\ }(j)\hat{\psi}_n(i,j)$$

$= 0$ by the above claim.

We now show that Φ is continuous with respect to the $[H^1(\mathcal{C}^1)/H^1(B)]\hat{\otimes}\mathrm{BHP}$ norm on $[\mathcal{D} + H^1(B)]\otimes\mathrm{BHP}$. For $p_n \in \mathcal{D}$ and $\psi_n \in \mathrm{BHP}$ $(0 \leq n \leq N)$

$$\left|\, \Phi\left(\sum_{n=0}^{N}[\,p_n + H^1(B)] \otimes \psi_n\right)\right|$$

$$\leq \sum_{n=0}^{N} |< p_n + H^1(B), \phi * \psi_n >|$$

$$\leq \|\, \phi \,\|_{M(\mathrm{BHP})} \sum_{n=0}^{N} \|\, p_n + H^1(B)\,\|_{H^1(\mathcal{C}^1)/H^1(B)} \|\, \psi_n \,\|_{\mathrm{BHP}}$$

as required.

Since $[\mathcal{D} + H^1(B)]\otimes\mathrm{BHP}$ is dense in $[H^1(\mathcal{C}^1)/H^1(B)]\hat{\otimes}\mathrm{BHP}$ we extend Φ by continuity to a functional Φ on $[H^1(\mathcal{C}^1)/H^1(B)]\hat{\otimes}\mathrm{BHP}$ with $\|\, \Phi \,\| \leq \|\, \phi \,\|_{M(\mathrm{BHP})}$.

Finally, if $h \in H^1(\mathcal{C}^1)$ and $\psi \in \mathrm{BHP}$ we approximate h by a sequence $\{p_n\}_{n\geq 0}$ in $\mathcal{D}$ satisfying $\|\, h-p_n \,\|_{H^1(\mathcal{C}^1)} \longrightarrow 0$ as $n \longrightarrow \infty$.

Then

$$\Phi([h + H^1(B)] \otimes \psi) = \lim_{n\to\infty} \Phi([p_n + H^1(B)] \otimes \psi)$$
$$= \lim_{n\to\infty} < p_n + H^1(B), \phi * \psi >$$
$$= < h + H^1(B), \phi * \psi >.$$

<u>4.2.7 Corollary.</u> If there exists $h \in H^1(\mathbb{T}), f,g \in H^2(\mathbb{T})$ and $\psi \in$ BHP such that

$$\hat{p}(m,n) = \hat{h}(m)(fg^{\dagger})\hat{~}(n)\hat{\psi}(m,n)$$

for all $m,n \geq 0$, then

$$\| p(S,T) \| \leq K_G c^2 d^2 \| h \|_1 \| f \|_2 \| g \|_2 \| \psi \|_{BHP} .$$

<u>Proof.</u> If we can find such $h,f,g,$and ψ then for any multiplier on BHP, ϕ and its associated functional Φ

$$\Big| \sum_{m,n=0}^{N_1,N_2} \hat{p}(m,n)\hat{\phi}(m,n) \Big|$$
$$= \Big| \sum_{m,n\geq 0} \hat{h}(m)(fg^{\dagger})\hat{~}(n)\hat{\psi}(m,n)\hat{\phi}(m,n) \Big|$$
$$= | \Phi ([h(\cdot)(f\tilde{\otimes}g) + H^1(B)] \otimes \psi) | \quad \text{by 4.2.6}$$
$$\leq \| \phi \|_{M(BHP)} \| [h(\cdot)(f\tilde{\otimes}g) + H^1(B)] \otimes \psi \|_{\hat{\otimes}}$$
$$= \| \phi \|_{M(BHP)} \| h(\cdot)(f\tilde{\otimes}g) + H^1(B)\|_{H^1(\mathcal{C}^1)/H^1(B)} \| \psi \|_{BHP}$$
$$\leq \| \phi \|_{M(BHP)} \| h \|_1 \| f \|_2 \| g \|_2 \| \psi \|_{BHP} .$$

The result now follows from 4.2.5.

<u>4.2.8 Corollary.</u> There exists a constant K such that for any polynomial

$$p = \sum_{\underline{0}\leq\underline{n}\leq(N_1,N_2)} \hat{p}(\underline{n})e_{\underline{n}} \tag{*}$$

with $N_1, N_2 \geq 2$ and for any commuting pair of power-bounded operators S,T with $\| S^n \| \leq c$ and $\| T^n \| \leq d$ for all $n \geq 0$ we have

$$\| p(S,T) \| \leq K\, c^2 d^2 \log N_1 \log N_2 \| p \|_{L^\infty(\mathbb{T}^2)} .$$

<u>Proof.</u> As in 2.3.7 we denote by h_N the polynomial $h_N(z) = \sum_{n=0}^{N} z^n$ on $\mathbb{D}$ $(N \geq 0)$ and note that for $N \geq 2$ $\| h_N \|_1 \leq d' \log N$ for some constant d' independent of N.

Let p be as in $(*)$ and factorise $h_{N_2} = fg^\dagger$ for some $f,g \in H^2(\mathbb{D})$ with $\| h_{N_2} \|_1 = \| f \|_2 \| g^\dagger \|_2$. Then

$$\hat{p}(m,n) = \hat{h}_{N_1}(m)(fg^\dagger)\hat{\ }(n)\hat{p}(m,n)$$

for all $m,n \geq 0$.

Let P be the projection of $L^2(\mathbb{T}^2)$ onto $H^2(\mathbb{T}^2)$, let J be the operator on $L^2(\mathbb{T}^2)$ defined at $h \in L^2(\mathbb{T}^2)$ by $(Jh)\hat{\ }(m,n) = \hat{h}(-m,-n)$ and let M_{Jp} denote multiplication by Jp on $L^2(\mathbb{T}^2)$. Then $PJM_{Jp}|_{H^2(\mathbb{T}^2)}$ is a Hankel operator on $H^2(\mathbb{T}^2)$ with representing array $\{\hat{p}(\underline{i} + \underline{j})\}_{\underline{i},\underline{j} \in \mathbb{Z}^{+2}}$ Thus, $p \in BHP$ with $\| p \|_{BHP} = \| PJM_{Jp}|_{H^2(\mathbb{T}^2)} \|_{B(H^2(\mathbb{T}^2))} \leq \| Jp \|_\infty = \| p \|_\infty$.

Hence, by 4.2.7

$$\begin{aligned} \| p(S,T) \| &\leq K\, c^2 d^2 \| h_{N_1} \|_1 \| f \|_2 \| g \|_2 \| p \|_{BHP} \\ &\leq K_G\, c^2 d^2\, d'(\log N_1) d'(\log N_2) \| p \|_\infty \\ &= K\, c^2 d^2 (\log N_1)(\log N_2) \| p \|_\infty \end{aligned}$$

when $K = K_G(d')^2$.

Chapter 5 Functions of a Bounded Continuous Semigroup of Operators.

We have seen how power-bounded operators generate bounded discrete semigroups of operators on $\mathcal{H}$ and moreover we have produced norm estimates for polynomials in such generators. In this chapter we consider the generator of a bounded, strongly continuous, one parameter semigroup of operators and we show that similar norm estimates can be found for a certain class of functions of this generator. The methods used are similar to those of chapters 2 and 4. They require characterisations of the dual of $L^1(\mathbb{R}^+)\check{\otimes}L^1(\mathbb{R}^+)$, (5.3.10) and of the class of bounded integral operators with Hankel-type kernel, $K(x+y)$, (5.4.13).

We start with some basic definitions from the theory of semigroups.

Section 5.1 Definitions and Motivation.

5.1.1 Definition. A C_0-semigroup $\mathcal{T}$ on $\mathcal{H}$ is a family of bounded operators $\{T(t) : t \in \mathbb{R}^+\}$ satisfying

i) $T(s)T(t) = T(s+t)$ for all $s,t \in \mathbb{R}^+$;

ii) $T(0) = I$

and iii) $T(t)x$ is a continuous $\mathcal{H}$-valued function of t for each $x \in \mathcal{H}$.

5.1.2 Definition. A C_0-semigroup $\mathcal{T}$ on $\mathcal{H}$ is bounded if there exists a constant $c \geq 1$ such that $\| T(t) \| \leq c$ for all $t \in \mathbb{R}^+$. A bounded C_0-semigroup $\mathcal{T}$ is contractive if $\| T(t) \| \leq 1$ for all $t \in \mathbb{R}^+$.

5.1.3 Definition. Let $\mathcal{T}$ be a C_0-semigroup on $\mathcal{H}$. The infinitessimal generator A of $\mathcal{T}$ is a linear operator defined at $x \in$

$\mathscr{H}$ by

$$Ax = -i \lim_{t \to 0} \frac{T(t)x - x}{t} \tag{$*$}$$

whenever this limit exists.

Remarks.

1. Many authors prefer to use "uniformly bounded" instead of "bounded" in 5.1.2.

2. The factor $-i$ in $(*)$ is unconventional. We choose this definition so that a C_0-semigroup may be expressed as $\{e^{itA} : t \in \mathbb{R}^+\}$ rather than the usual $\{e^{tA} : t \in \mathbb{R}^+\}$. The motivation for this choice is the use of the Fourier rather than the Laplace transform later in the chapter.

A simple adaptation of the well-known Hille-Yosida-Phillips Generation Theorem ([GOLD, p 20],[DS1, pp 624,626]) describes the generators of bounded semigroups.

5.1.4 Theorem. A linear operator A on $\mathscr{H}$ is the generator of a bounded C_0-semigroup $\mathcal{T}$ on $\mathscr{H}$ if and only if

i) A is closed and densely defined on $\mathscr{H}$

and ii) there exists $c \geq 1$ such that for all $\lambda > 0$ the operator $-i-A$ is invertible and $\| (-i\lambda - A)^{-n} \| \leq \frac{c}{\lambda^n}$ for all $n \geq 0$.

In this case we have $\| T(t) \| \leq c$ for all $t \in \mathbb{R}^+$ and $T(t) = e^{itA}$ in the sense that

$$T(t)x = \lim_{h \to 0} e^{itA_h} x$$

where $x \in \mathscr{H}$ and $A_h = -i \dfrac{T(h) - I}{h} \in B(\mathscr{H})$ for each $h > 0$.

Remark The generator of a C_0-semigroup $\mathbb{T} = \{T(t) : t \in \mathbb{R}^+\}$ is a bounded operator when and only when $\mathbb{T}$ is uniformly continuous. That is, $T(t)$ is continuous on t, with respect to the norm topology on $B(\mathcal{H})$. ([DS1. p 621]).

Packel's Counterexample.

In Chapter 2 we noted that a counterexample by Foguel showed that not every power-bounded operator on $\mathcal{H}$ is similar to a contraction. Before proceeding with our norm estimates for functions of the generator of a bounded C_0-semigroup it is important to note that not every such generator is similar to the generator of some contractive C_0-semigroup. Indeed, a counterexample by E.W.Packel ensures that this is the case ([PAC]). The construction of the required C_0-semigroup and the method of showing that its generator is not similar to the generator of a contractive C_0-semigroup closely follows Foguel's ideas and Halmos' interpretation.

Section 5.2 A Bound on $\| f(A) \|$.

Throughout the rest of this chapter we suppose that A is the infinitesimal generator of a bounded C_0-semigroup $\mathbb{T}$ on $\mathcal{H}$. Let $c \geq 1$ denote a uniform bound on $\| T(t) \|$ $(t \geq 0)$. Lebesgue measure on $\mathbb{R}^n$

$(n \geq 1)$ will be denoted by μ.

Let $f \in L^1(\mathbb{R})$ and let $\hat{f}$ denote its Fourier transform. If $\hat{f}$ is integrable the inversion formula gives

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(s)e^{ist}\, d\mu(s)$$

for a.e. $t \in \mathbb{R}$. We use this as our motivation for the definition of $f(A)$.

5.2.1 Definition. If $f \in L^1(\mathbb{R})$ is such that the support of $\hat{f}$, $\operatorname{supp}\hat{f}$ is a compact subset of $\mathbb{R}^+$ then we define $f(A) \in B(\mathcal{H})$ by

$$f(A)x = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \hat{f}(s)e^{isA}(x)\, d\mu(s) \qquad (*)$$

$(x \in \mathcal{H})$.

Note that the integral $(*)$ exists as a Bochner integral since the strong continuity of the semi-group $\mathbb{T}$ implies the weak measurability of the function $t \rightsquigarrow T(t)x$ $(x \in \mathcal{H})$ and

$$\int_{0}^{\infty} \| \hat{f}(s)e^{isA}(x) \|_{\mathcal{H}}\, d\mu(s) \leq c\| f \|_1 \| x \|_{\mathcal{H}}\, \mu(\operatorname{supp}\hat{f}) < \infty.$$

To find our first bound on $\| f(A) \|$ we will use the tensor product spaces $L^1(\mathbb{R}^+)\check{\otimes}L^1(\mathbb{R}^+)$ and $L^1(\mathbb{R}^+)\hat{\otimes}L^1(\mathbb{R}^+)$. By 3.3.7, the projective tensor product $L^1(\mathbb{R}^+)\hat{\otimes}L^1(\mathbb{R}^+)$ is isometrically isomorphic to $L^1(\mathbb{R}^+;L^1(\mathbb{R}^+))$ which is in turn isometrically isomorphic to $L^1(\mathbb{R}^+\times\mathbb{R}^+)$.

5.2.2 Notation. Let $\beta \in L^1(\mathbb{R}^+)\hat{\otimes}L^1(\mathbb{R}^+)$. We denote the image of β under the isometric isomorphism of $L^1(\mathbb{R}^+)\hat{\otimes}L^1(\mathbb{R}^+)$ onto $L^1(\mathbb{R}^+\times\mathbb{R}^+)$ by $\widetilde{\beta}$.

Note that when $\beta = \sum_{n=0}^{N} f_n \otimes g_n \in L^1(\mathbb{R}^+) \otimes L^1(\mathbb{R}^+)$ $(f_n, g_n \in L^1(\mathbb{R}^+),\ 0 \le n \le N)$ we have

$$\tilde{\beta}(s,t) = \sum_{n=0}^{N} f_n(s) g_n(t)$$

for a.e. $s,t \in \mathbb{R}^+$ and the injective tensor norm of β is given by

$$\| \beta \|_{\check{\otimes}} = \sup \left\{ \left| \int_0^\infty \int_0^\infty \tilde{\beta}(s,t) \phi(s) \psi(t)\, d\mu(s)\, d\mu(t) \right| : \begin{array}{l} \phi, \psi \in L^\infty(\mathbb{R}^+), \\ \| \phi \|_\infty, \| \psi \|_\infty \le 1 \end{array} \right\}$$

Using these facts we prove the following version of Grothendieck's Inequality.

<u>5.2.3 Theorem.</u> Let $\beta \in L^1(\mathbb{R}^+) \hat{\otimes} L^1(\mathbb{R}^+)$ and let F,G be essentially bounded measurable $\mathcal{H}$-valued functions on $\mathbb{R}^+$. Then

$$\left| \int_0^\infty \int_0^\infty \tilde{\beta}(s,t)(F(s),G(t))\, d\mu(s)\, d\mu(t) \right| \le K_G \| \beta \|_{\check{\otimes}} \| F \|_\infty \| G \|_\infty$$

where K_G denotes Grothendieck's constant.

<u>Proof.</u> We first consider the case when $\beta \in L^1(\mathbb{R}^+) \otimes L^1(\mathbb{R}^+)$. Suppose that F,G are simple measurable $\mathcal{H}$-valued functions :

$$F = \sum_{i=1}^{n} x_i \chi_{\sigma i} \qquad G = \sum_{j=1}^{m} y_j \chi_{\tau j}$$

for some collections of disjoint measurable subsets of $\mathbb{R}^+$, $\{\sigma_i\}_{i=1}^n$ and $\{\tau_j\}_{j=1}^m$, and some x_i, $y_j \in \mathcal{H}$ $(0 \le i \le n,\ 0 \le j \le m)$. Then

$$\left| \int_0^\infty \int_0^\infty \tilde{\beta}(s,t)(F(s),G(t))\, d\mu(s)\, d\mu(t) \right| \tag{$*$}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} \left(\int_{\tau_j} \int_{\sigma_i} \tilde{\beta}(s,t)\, d\mu(s)\, d\mu(t) \right) (x_i, y_j).$$

Put $\alpha_{i,j} = \int_{\tau_j}\int_{\sigma_i} \tilde{\beta}(s,t)d\mu(s)d\mu(t)$ for each $1 \le i \le n$ and $1 \le j \le m$.

Then for any scalars s_i, t_j $(0 \le i \le n,\ 0 \le j \le m)$ with $|s_i|, |t_j| \le 1$ for all $0 \le i \le n$, $0 \le j \le m$ we have

$$\begin{aligned}|\sum_{i,j} \alpha_{i,j} s_i t_j| &= |\int_0^\infty \int_0^\infty \tilde{\beta}(s,t)\left(\sum_{i=1}^n s_i \chi_{\sigma_i}(s)\right)\left(\sum_{j=1}^m t_j \chi_{\tau_j}(t)\right) d\mu(s)d\mu(t)| \\ &\le \| \beta \|_{\check{\otimes}} \|\sum_{i=1}^n s_i \chi_{\sigma_i}\|_\infty \|\sum_{j=1}^m t_j \chi_{\tau_j}\|_\infty \\ &< \| \beta \|_{\check{\otimes}} .\end{aligned}$$

Hence by Grothendieck's inequality (1.2.5)

$$|\sum_{i=1}^n \sum_{j=1}^m \alpha_{i,j}(x_i, y_j)| \le K_G \| \beta \|_{\check{\otimes}} \sup_{1\le i\le n} \| x_i \|_{\mathscr{H}} \sup_{1\le j\le m} \| y_j \|_{\mathscr{H}}$$

and from $(*)$

$$| \int_0^\infty \int_0^\infty \tilde{\beta}(s,t)(F(s),G(t))\, d\mu(s)\, d\mu(t) | \le K_G \| \beta \|_{\check{\otimes}} \| F \|_\infty \| G \|_\infty.$$

Now suppose that $F \in L^\infty(\mathbb{R}^+;\mathscr{H})$. By the strong measurability of F there exists a sequence of measurable simple functions $\{F_n\}_{n\ge0}$ which converges pointwise a.e. to F. We can adapt each F_n to get a sequence $\{F'_n\}_{n\ge0}$ which further satisfies

$$\| F'_n \|_\infty \le \| F \|_\infty \quad \text{for all } n \ge 0.$$

To do this we suppose that $F_n = \sum_{i=0}^N x_i \chi_{\sigma_i}$ for some $N > 0$, $x_i \in \mathscr{H}$ for $0 \le i \le N$ and some collection of disjoint measurable subsets of $\mathbb{R}^+$, $\{\sigma_i\}_{i=0}^N$. We define for each $0 \le i \le N$

$$y_i = \begin{cases} x_i & : \| x_i \|_{\mathcal{H}} \le \underset{s\in\sigma_i}{\operatorname{ess\,sup}} \| F(s) \|_{\mathcal{H}} \\ \dfrac{x_i}{\| x_i \|_{\mathcal{H}}} \underset{s\in\sigma_i}{\operatorname{ess\,sup}} \| F(s) \|_{\mathcal{H}} & : \text{otherwise} \end{cases}$$

and put $F'_n = \sum_{i=0}^{N} y_i \chi_{\sigma_i}$. Then $\| F'_n \|_\infty \le \| F \|_\infty$ and it is easy to show that $\| y_i - F(s) \|_{\mathcal{H}} \le \| x_i - F(s) \|_{\mathcal{H}}$ for each $0 \le i \le N$ and a.e. $s \in \sigma_i$. It follows that $\| F'_n(s) - F(s) \|_{\mathcal{H}} \le \| F_n(s) - F(s) \|_{\mathcal{H}}$ for a.e. $s \in \mathbb{R}^+$ and therefore that $F'_n \longrightarrow F$ pointwise a.e. on $\mathbb{R}^+$.

Suppose also that $G \in L^\infty(\mathbb{R}^+;\mathcal{H})$ and that $\{G'_n\}_{n\ge 0}$ is a sequence of measurable simple $\mathcal{H}$-valued functions convergent pointwise a.e. to G with $\| G'_n \|_\infty \le \| G \|_\infty$ for each $n \ge 0$. Then by dominated convergence

$$\left| \int_0^\infty \int_0^\infty \tilde{\beta}(s,t)(F(s),G(t))\, d\mu(s)\, d\mu(t) \right|$$

$$= \lim_{n\to\infty} \left| \int_0^\infty \int_0^\infty \tilde{\beta}(s,t)(F'_n(s),G'_n(t))\, d\mu(s)\, d\mu(t) \right|$$

$$\le \lim_{n\to\infty} K_G \| \beta \|_{\check{\otimes}} \| F'_n \|_\infty \| G'_n \|_\infty \quad \text{since } F'_n,\ G'_n \text{ are all simple}$$

$$\le K_G \| \beta \|_{\check{\otimes}} \| F \|_\infty \| G \|_\infty .$$

Finally, suppose that $\beta \in L^1(\mathbb{R}^+)\hat{\otimes}L^1(\mathbb{R}^+)$ and $0 \ne F,G \in L^\infty(\mathbb{R}^+;\mathcal{H})$. For any $\varepsilon > 0$ we may choose $\beta' \in L^1(\mathbb{R}^+)\otimes L^1(\mathbb{R}^+)$ such that

$$\| \beta - \beta' \|_{\hat{\otimes}} < \frac{\varepsilon}{K_G \| F \|_\infty \| G \|_\infty}$$

Then

$$\left| \int_0^\infty \int_0^\infty \tilde{\beta}(s,t)(F(s),G(t))\, d\mu(s)\, d\mu(t) \right|$$

$$\leq \left| \int_0^\infty \int_0^\infty (\tilde{\beta}(s,t) - \tilde{\beta}'(s,t))(F(s),G(t))\, d\mu(s)\, d\mu(t) \right|$$
$$+ \left| \int_0^\infty \int_0^\infty \tilde{\beta}'(s,t)(F(s),G(t))\, d\mu(s)\, d\mu(t) \right|$$

$$\leq \| F \|_\infty \| G \|_\infty \| \tilde{\beta} - \tilde{\beta}' \|_1 + K_G \| \beta' \|_{\check{\otimes}} \| F \|_\infty \| G \|_\infty$$

$$= \| F \|_\infty \| G \|_\infty \| \beta - \beta' \|_{\hat{\otimes}} + K_G \| \beta' - \beta + \beta \|_{\check{\otimes}} \| F \|_\infty \| G \|_\infty$$

$$< \left(\frac{\varepsilon}{K_G} + \varepsilon \right) + K_G \| \beta \|_{\check{\otimes}} \| F \|_\infty \| G \|_\infty$$

and the result follows.

We can now use 5.2.3 and the method of [Pl, Thm 3.1] to produce a bound on $\| f(A) \|$.

<u>5.2.4 Theorem.</u> Let $f \in H^1(\mathbb{R})$ be such that supp $\hat{f} \subseteq [0,N]$ for some $N > 0$. Suppose that $\beta_f \in L^1(\mathbb{R}^+)\hat{\otimes}L^1(\mathbb{R}^+)$ satisfies

$$\hat{f}(t) = \int_0^t \tilde{\beta}_f(s,t-s)\, d\mu(s) \qquad (*)$$

for a.e. $t \in \mathbb{R}^+$. Then

$$\| f(A) \| \leq \frac{1}{\sqrt{2\pi}} K_G\, c^2 \| \beta_f \|_{\check{\otimes}} .$$

<u>Proof.</u> For each $x,y \in \mathcal{H}$

$$(f(A)x,y) = \frac{1}{\sqrt{2\pi}} \int_0^N \hat{f}(t)(T(t)x,y)\, d\mu(t)$$
$$= \frac{1}{\sqrt{2\pi}} \int_0^N \left(\int_0^t \tilde{\beta}_f(s,t-s)\, d\mu(s) \right) (T(t)x,y)\, d\mu(t)$$
$$= \frac{1}{\sqrt{2\pi}} \int_0^N \int_0^{N-r} \tilde{\beta}_f(s,r)(T(s+r)x,y)\, d\mu(s)\, d\mu(r)$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty \int_0^\infty \tilde{\beta}_f(s,r)(T(r)x,\ T(s)^*y)\ d\mu(s)\ d\mu(r).$$

Hence by 5.2.2, since $T(r)x$ and $T(s)y$ are bounded measurable $\mathcal{H}$-valued functions of r,s respectively,

$$|(f(A)x,y)| \le \frac{1}{\sqrt{2\pi}} K_G \| \beta_f \|_{\check{\otimes}} \sup_{r\ge 0} \| T(r)x \|_{\mathcal{H}} \sup_{s\ge 0} \| T(s)^*y \|_{\mathcal{H}}$$

$$\le \frac{1}{\sqrt{2\pi}} K_G \| \beta_f \|_{\check{\otimes}}\ c^2 \| x \|_{\mathcal{H}} \| y \|_{\mathcal{H}}$$

and the proof is complete.

Note that the condition (∗) in 5.2.4 is easily satisfied since we may take

$$\tilde{\beta}(s,t) = \frac{\hat{f}(s+t)}{s+t}$$

for $s,t \in \mathbb{R}^+$, $s+t \ne 0$. In this case

$$\int_0^t \tilde{\beta}(s,t-s)\ d\mu(s) = \int_0^t \frac{\hat{f}(t)}{t}\ d\mu(s)$$

$$= \hat{f}(t) \quad \text{for all } t \in \mathbb{R}^+$$

and $$\int_0^\infty \int_0^\infty | \tilde{\beta}(s,t) |\ d\mu(s)\ d\mu(t) = \int_0^\infty \left(\int_0 | \tilde{\beta}(s,s-t) |\ d\mu(s) \right) d\mu(t)$$

$$= \int_0^\infty \left(\int_0^t \left| \frac{\hat{f}(t)}{t} \right| d\mu(s) \right) d\mu(t)$$

$$= \int_0^\infty | \hat{f}(t) |\ d\mu(t)$$

$$< \infty.$$

Moreover, since (∗) in 5.2.4 may be satisfied by different β_f's with different norms, we make the following definition.

<u>5.2.5 Notation.</u> Let E denote the subspace of $L^1(\mathbb{R}^+)\otimes L^1(\mathbb{R}^+)$ consisting of those $\alpha \in L^1(\mathbb{R}^+)\otimes L^1(\mathbb{R}^+)$ for which $\int_0^t \tilde{\alpha}(s,t-s)\ d\mu(s) =$

0 for a.e. $t \in \mathbb{R}^+$.

Clearly, if $\beta_f \in L^1(\mathbb{R}^+)\hat{\otimes}L^1(\mathbb{R}^+)$ satisfies (*) in 5.2.4 then so does $\beta_f + \alpha$ for any $\alpha \in E$. Thus if E^- denotes the closure of E in $L^1(\mathbb{R}^+)\check{\otimes}L^1(\mathbb{R}^+)$ we have the following corollary immediately.

5.2.6 Corollary. If f and β_f are as in 5.2.4 then

$$\| f(A) \| \le \frac{1}{\sqrt{2\pi}} K_G\, c^2 \,\| \beta_f + E^- \|_{(L^1(\mathbb{R}^+)\check{\otimes}L^1(\mathbb{R}^+))/E^-}$$

Section 5.3 The Dual of $L^1(\mathbb{R}^+)\check{\otimes}L^1(\mathbb{R}^+)$.

For $\beta \in L^1(\mathbb{R}^+)\check{\otimes}L^1(\mathbb{R}^+)$ and the subspace E of $L^1(\mathbb{R}^+)\check{\otimes}L^1(\mathbb{R}^+)$ defined in 5.2.5 we note that

$$\| \beta + E^- \|_{(L^1(\mathbb{R}^+)\check{\otimes}L^1(\mathbb{R}^+))/E^-} = \sup\left\{ |\, F\beta \,| \ : \ F \in E^{\perp}\cap(L^1(\mathbb{R}^+)\check{\otimes}L^1(\mathbb{R}^+))^*,\ \| F \| \le 1 \right\}.$$

In this section we show that $(L^1(\mathbb{R}^+)\check{\otimes}L^1(\mathbb{R}^+))^*$ is isometrically isomorphic to a subspace of $L^\infty(\mathbb{R}^+\times\mathbb{R}^+)$ and we identify the annihilator of E with those functions in this subspace which for almost every $t \in \mathbb{R}^+$ are almost everywhere constant on the diagonal from (0,t) to (t,0). Thus when $f \in L^1(\mathbb{R})$ has supp $\hat{f} \subseteq [0,N]$ for some $N > 0$ and β_f satisfies the condition (*) of 5.2.4 we obtain the equality

$$\| \beta + E^- \|_{(L^1(\mathbb{R}^+)\check{\otimes}L^1(\mathbb{R}^+))/E^-} = \sup\left\{ \left|\, \int_0^N \hat{f}(t)h(t)\, d\mu(t) \,\right| \ : \ \gamma_h \in E^{\perp},\ \| \gamma_h \|_{(L^1(\mathbb{R}^+)\check{\otimes}L^1(\mathbb{R}^+))^*} \le 1 \right\}$$

where $\gamma_h(s,t) = h(s+t)$ for all $s,t \in \mathbb{R}^+$.

This important step allows us to show that the operator norm of f(A) is bounded by an expression involving the integral of $\hat{f}$ against functions in a known subspace of $L^\infty(\mathbb{R}^+)$.

The characterisation of $(L^1(\mathbb{R}^+)\check{\otimes}L^1(\mathbb{R}^+))^*$ forms the greatest part of this section and is of independent interest. The only known characterisation (1.2.7) does not shed light on the problem of describing $\| \beta_f + E^- \|$ in terms of the function f.

Our strategy for characterising the dual of $L^1(\mathbb{R}^+)\check{\otimes}L^1(\mathbb{R}^+)$ is to use conditional expectation operators on an element g of $L^\infty(\mathbb{R}^+\times\mathbb{R}^+)$ to produce a sequence of finite-dimensional matrices. We show that if the norms of these matrices in $l^\infty(\mathbb{Z}^+)\hat{\otimes}l^\infty(\mathbb{Z}^+)$ have a uniform bound then g defines a bounded linear functional on $L^1(\mathbb{R}^+)\check{\otimes}L^1(\mathbb{R}^+)$ with norm equal to the supremum of the projective tensor norms of these matrices.

__5.3.1 Notation.__ a) For $N \in \mathbb{Z}$ and $i \in \mathbb{Z}^+$ we denote by I_N^i the dyadic interval $\left[\frac{i}{2^N} , \frac{i+1}{2^N} \right)$.

b) If $N \in \mathbb{Z}$ and if $M \in \mathbb{Z}^+$ is such that $M2^N \in \mathbb{Z}^+$ then we denote by $\mathcal{F}_{N,M}$ the σ-algebra of subsets of $[0,M)$ generated by $\{ I_N^i\}_{i=0}^{N2^M-1}$.

c) If $N \in \mathbb{Z}$ we denote by $\mathcal{F}_N$ the σ-algebra of subsets of $\mathbb{R}^+$ generated by $\{ I_N^i\}_{i\in\mathbb{Z}^+}$.

d) For $1 \le p \le \infty$ and $0 < N < \infty$ we denote by L_N^p the Banach space $L^p([0,N), \mathcal{F}_{N,N}, \mu)$ and by l_N^p the Banach space $l^p(\mathbb{Z}_N)$.

e) If $(X,\mathcal{A},\nu)$ is a measure space we define $L^\infty(X,\mathcal{A},\nu)\otimes_{pt}L^\infty(X,\mathcal{A},\nu)$ to be the linear space of (ν-a.e.-equivalence classes of) functions $g \in L^\infty(X\times X,\mathcal{A}\times\mathcal{A},\nu\times\nu)$ for which there exists $M \ge 0$ and $\phi_k,\psi_k \in L^\infty(X,\mathcal{A},\nu)$ $(0 \le k \le M)$ such that

$$g(s,t) = \sum_{k=0}^{M} \phi_k(s)\psi_k(t) \quad \text{for a.e. } s,t \in \mathbb{R}^+. \qquad (*)$$

f) For $g \in L^\infty(X,\mathcal{A},\nu)\otimes_{pt}L^\infty(X,\mathcal{A},\nu)$ we put

$$\| g \|_{\hat{\otimes}_{pt}} = \inf \left\{ \sum_{k=0}^{M} \| \phi_k \|_\infty \| \psi_k \|_\infty : M \geq 0,\ \phi_k, \psi_k \in L^\infty(X,\mathcal{A},\nu) \quad (0 \leq k \leq M) \text{ satisfy } (*) \right\}.$$

g) Let $L^\infty(X,\mathcal{A},\nu)\hat{\otimes}_{pt}L^\infty(X,\mathcal{A},\nu)$ denote the completion of $L^\infty(X,\mathcal{A},\nu)\otimes_{pt}L^\infty(X,\mathcal{A},\nu)$ with respect to the norm $\| \cdot \|_{\hat{\otimes}_{pt}}$.

<u>5.3.2 Lemma.</u> For each $N > 0$, $(L^1_N \check{\otimes} L^1_N)^*$ is isometrically isomorphic to $L^\infty_N \hat{\otimes}_{pt} L^\infty_N$ with respect to the pairing

$$< \alpha,g > = \int_0^N \int_0^N \tilde{\alpha}(s,t)g(s,t)\, d\mu(s)\, d\mu(t)$$

when $\alpha \in L^1_N\check{\otimes}L^1_N$ and $g \in L^\infty_N\hat{\otimes}_{pt}L^\infty_N$.

<u>Proof.</u> Fix $N > 0$ and let $m = N2^N-1$. If $f \in L^p_N$ for some $p \geq 1$ then f must be a.e.-constant on I^i_N for each $0 \leq i \leq m$. The sequence $\{f_i\}^m_{i=0}$ determined by

$$f(s) = 2^{N/p}f_i \quad \text{for a.e. } s \in I^i_N \text{ and each } 0 \leq i \leq m$$

is in l^p_N and this construction gives an isometric isomorphism of L^p_N onto l^p_m which we will denote by T_p.

Let $S = T_1\otimes T_1$, the corresponding isometric isomorphism of $L^1_N\check{\otimes}L^1_N$ onto $l^1_m\check{\otimes}l^1_m$. Then for $\alpha \in L^1_N\otimes L^1_N$ and $0 \leq i,j \leq m$ we have $(S\alpha)_{i,j} = 2^{-2N}\,\tilde{\alpha}(s,t)$ for a.e. $(s,t) \in I^i_N \times I^j_N$.

By Lemma 1.2.4, $(l^1_m \check{\otimes} l^1_m)^*$ is isometrically isomorphic to $l^\infty_m\hat{\otimes}l^\infty_m$ with respect to the pairing $< \alpha,\beta > = \sum_{i,j=0}^{m} \alpha_{i,j}\ \beta_{i,j}$ when $\alpha = \{\alpha_{i,j}\}^m_{i,j=0} \in l^1_m\check{\otimes}l^1_m$ and $\beta = \{\beta_{i,j}\}^m_{i,j=0} \in l^\infty_m\hat{\otimes}l^\infty_m$. It follows that $(L^1_N\check{\otimes}L^1_N)^*$ is isometrically isomorphic to $L^\infty_N\hat{\otimes}_{pt}L^\infty_N$ with respect to the pairing $< \alpha,g > = < S\alpha, Tg >$ when $\alpha \in L^1_N\check{\otimes}L^1_N$ and $g \in L^\infty_N\hat{\otimes}_{pt}L^\infty_N$. Moreover,

$$\int_0^N \int_0^N \tilde{\alpha}(s,t)g(s,t)\, d\mu(s)\, d\mu(t)$$

$$= \sum_{i,j=0}^{m} \int_{I_N^i} \int_{I_N^i} \tilde{\alpha}(s,t)g(s,t)\, d\mu(s)\, d\mu(t)$$

$$= \sum_{i,j=0}^{m} \int_{I_N^i} \int_{I_N^i} 2^{2N}(S\alpha)_{i,j}(Tg)_{i,j}\, d\mu(s)\, d\mu(t)$$

$$= \langle S\alpha, Tg \rangle$$

$$= \langle \alpha, g \rangle$$

and the Lemma is proved.

<u>**5.3.3 Definition.**</u> Let $(X,\mathcal{A},\nu)$ be a measure space. A <u>decomposition</u> of $(X,\mathcal{A},\nu)$ is a sequence $\{\mathcal{A}_j\}_{j\in\mathbb{Z}}$ of sub-σ-algebras of $\mathcal{A}$

i) $\mathcal{A}_j \subseteq \mathcal{A}_{j+1}$ for each $j \in \mathbb{Z}$;

ii) if $E \in \bigcap_{j\in\mathbb{Z}} \mathcal{A}_j$ and $\nu(E) < \infty$ then either $\nu(E) = 0$ or $\nu(X\backslash E) = 0$;

iii) $\bigcup_{j\in\mathbb{Z}} \mathcal{A}_j$ generates $\mathcal{A}$

and iv) if $n \in \mathbb{Z}$, $F \in \mathcal{A}$ and $\nu(F) < \infty$ then there exists a countable family $\{U_j\}_{j\geq 1}$ in $\mathcal{A}_n$ such that $F \subseteq \bigcup_{j\geq 1} U_j$ and $\nu(U_j) < \infty$ for each $j \geq 1$.

The definition of conditional expectation operators is based on the following lemma.

<u>**5.3.4 Lemma.**</u> Let $1 \leq p \leq \infty$ and let $\{\mathcal{A}_j\}_{j\in\mathbb{Z}}$ be a decomposition of a measure space $(X,\mathcal{A},\nu)$. Then for any $f \in L^p(X,\mathcal{A},\nu)$ and any $n \in \mathbb{Z}$ there is an essentially unique, locally integrable function g_n that is $\mathcal{A}_n$-measurable, vanishes off a σ-finite set and satisfies

$$\int_{\Omega} g_n \, d\nu = \int_{\Omega} f \, d\nu$$

for every set Ω of finite measure in $\mathcal{A}_n$.

Proof. [EG, p 77].

5.3.5 Definition. If $(X,\mathcal{A},\nu),\{\mathcal{A}_n\}_{n\in\mathbb{Z}}$ and f are as in 5.3.4 then the function g_n is the conditional expectation of f given $\mathcal{A}_n$. The operator $\mathbb{E}_n$, mapping f to g_n, defined on $L^p(X,\mathcal{A},\nu)$ for each $1 \le p < \infty$ is called the conditional expectation operator relative to $\mathcal{A}_n$.

We note the essential features of conditional expectation operators necessary for their use in this section.

5.3.6 Lemma. For each $n \in \mathbb{Z}$ and $1 \le p < \infty$

a) $\mathbb{E}_n$ is a contractive, positive linear operator on $L^p(X,\mathcal{A},\nu)$;

b) $\mathbb{E}_m\mathbb{E}_n f = \mathbb{E}_m f$ whenever $m \le n$ and $f \in L^p(X,\mathcal{A},\nu)$;

c) if $f \in L^p(X,\mathcal{A}_n,\nu)$ then $\mathbb{E}_n f = f$;

d) if $f \in L^1(X,\mathcal{A},\nu)$ and $g \in L^\infty(X,\mathcal{A},\nu) \cap L^p(X,\mathcal{A},\nu)$

then
$$\int_X (\mathbb{E}_n f)g \, d\nu = \int_X f(\mathbb{E}_n g) \, d\nu = \int_X (\mathbb{E}_n f)(\mathbb{E}_n g) \, d\nu$$

and e) if $f \in L^p(X,\mathcal{A},\nu)$ then $\mathbb{E}_n f \longrightarrow f$ in $L^p(X,\mathcal{A},\nu)$ as $n \longrightarrow \infty$.

Proof. [EG, pp 78,84].

We now define a particular sequence of conditional expectation operators on $L^p(\mathbb{R}^{+k})$ for $1 \le p < \infty$ and $k \in \mathbb{N}$.

5.3.7 Definition. For $k \in \mathbb{N}$, $1 \le p < \infty$, $f \in L^p(\mathbb{R}^{+k})$ and $N \in \mathbb{Z}$ we define $\mathbb{E}_N^{(k)} f$ at $\underline{x} \in \mathbb{R}^{+k}$ by

$$(\mathbb{E}_N^{(k)} f)(\underline{x}) = 2^{kN} \int\limits_{\prod_{j=1}^{k} I_N^{i_j}} f(\underline{y})\, d\mu(\underline{y}) \qquad (**)$$

when $\underline{x} \in \prod_{j=1}^{k} I_N^{i_j}$ for some $i_1, i_2, \ldots. i_k \ge 0$.

Thus, the value of $\mathbb{E}_N^{(k)} f$ at $\underline{x} \in \mathbb{R}^{+k}$ is given by the average of f over the dyadic "cube" containing the point $\underline{x}$. It is clear that $\left\{\prod_{j=1}^{k} \mathcal{F}_N\right\}_{N\in\mathbb{Z}}$ is a decomposition of $\left(\mathbb{R}^{+k}, \prod_{j=1}^{k} \mathcal{B}, \mu\right)$ and that if $N \in \mathbb{Z}$, $1 \le p < \infty$ and $f \in L^p(\mathbb{R}^{+k})$ then $\mathbb{E}_N^{(k)} f$ is the conditional expectation of f given $\prod_{j=1}^{k} \mathcal{F}_N$.

Note that when $f \in L^\infty(\mathbb{R}^{+k})$, $(**)$ defines an element $\mathbb{E}_N^{(k)} f$ of $L^\infty(\mathbb{R}^{+k})$ with $\| \mathbb{E}_N^{(k)} f \|_\infty \le \| f \|_\infty$. In this case, the sequence $\{\mathbb{E}_N^{(k)} f\}_{N\ge 0}$ does not necessarily converge uniformly to f as $N \to \infty$, but when $g \in L^1(\mathbb{R}^{+k})$

$$\int\limits_{\mathbb{R}^{+k}} (\mathbb{E}_N^{(k)} f) g \, d\mu = \int\limits_{\mathbb{R}^{+k}} f(\mathbb{E}_N^{(k)} g)\, d\mu$$

and it follows that $\mathbb{E}_N^{(k)} f$ converges to f in the weak*-topology on $L^\infty(\mathbb{R}^{+k})$.

5.3.8 Notation. For $k,N > 0$ we denote by $P_N^{(k)}$ the operator on $L^p(\mathbb{R}^{+k})$ $(1 \le p \le \infty)$ given by

$$(P_N^{(k)} f)(\underline{x}) = \begin{cases} f(\underline{x}) & : \underline{x} \in \prod_{j=1}^{k} [0,N) \\ 0 & : \text{otherwise} \end{cases}$$

$(\underline{x} \in \mathbb{R}^{+k})$.

The properties stated in Lemma 5.3.6 hold for $P_N^{(k)} \mathbb{E}_N^{(k)}$ on $L^p(\mathbb{R}^{+k})$ $(1 \le p < \infty)$ except (c) which applies only to $f \in$

$L^p\left(\prod_{j=1}^{k}[0,N), \prod_{j=1}^{k}\mathcal{F}_{N,N}, \mu\right)$. We may regard the operator $P_N^{(1)}\mathbb{E}_N^{(1)}$ as a contractive linear mapping of $L^p(\mathbb{R}^+)$ onto L_N^p. Moreover, $P_N^{(2)}\mathbb{E}_N^{(2)}$ maps $L^\infty(\mathbb{R}^+\times\mathbb{R}^+)$ onto $L_N^\infty\hat{\otimes}_{pt}L_N^\infty$. Thus, in the spirit of 1.2.2 we will use the following notation.

5.3.9 Notation. Let $\mathcal{V}^2$ denote the linear space of essentially bounded measurable functions g on $\mathbb{R}^+\times\mathbb{R}^+$ for which

$$\| g \|_{\mathcal{V}^2} = \sup_{N>0} \| P_N^{(2)}\mathbb{E}_N^{(2)} g \|_{\hat{\otimes}_{pt}} < \infty .$$

Clearly, $\mathcal{V}^2$ is a Banach space with the norm $\| \cdot \|_{\mathcal{V}^2}$ and we now show that it is a dual space.

5.3.10 Theorem. The dual space of the injective tensor product $L^1(\mathbb{R}^+)\check{\otimes}L^1(\mathbb{R}^+)$ is isometrically isomorphic to $\mathcal{V}^2$. The pairing is defined for $\alpha \in L^1(\mathbb{R}^+)\otimes L^1(\mathbb{R}^+)$ and $g \in \mathcal{V}^2$ by

$$< \alpha,g > = \int_0^\infty \int_0^\infty \tilde{\alpha}(s,t)g(s,t)\, d\mu(s,t)$$

Proof. Suppose first that $g \in L^\infty(\mathbb{R}^+)\otimes_{pt}L^\infty(\mathbb{R}^+)$ and that $\phi_k,\psi_k \in L^\infty(\mathbb{R}^+)$ $(0 \le k \le m)$ are such that

$$g(s,t) = \sum_{k=0}^{m} \phi_k(s)\psi_k(t) \tag{$*$}$$

for a.e. $s,t \in \mathbb{R}^+$. Then for any $\alpha \in L^1(\mathbb{R}^+)\otimes L^1(\mathbb{R}^+)$

$$< \alpha,g > = \int_0^\infty \int_0^\infty \tilde{\alpha}(s,t)\left(\sum_{k=0}^{m} \phi_k(s)\psi_k(t)\right) d\mu(s,t)$$

$$= \sum_{k=0}^{m} \int_0^\infty \int_0^\infty \tilde{\alpha}(s,t)\, \phi_k(s)\psi_k(t)\, d\mu(s)\, d\mu(t)$$

by Fubini's Theorem

$$= \sum_{k=0}^{m} \alpha(\phi_k,\psi_k).$$

Thus,

$$|\langle \alpha, g \rangle| \le \| \alpha \|_{\check{\otimes}} \sum_{k=0}^{m} \| \phi_k \|_\infty \| \psi_k \|_\infty$$

and since this holds for any $\{\phi_k\}$, $\{\psi_k\}$ satisfying (*) for a.e. s,t $\in \mathbb{R}^+$ we have

$$|\langle \alpha, g \rangle| \le \| \alpha \|_{\check{\otimes}} \| g \|_{\hat{\otimes}_{pt}}.$$

Now if $g \in \mathcal{V}^2$, since $P_N^{(2)} \mathbb{E}_N^{(2)} \tilde{\alpha} \longrightarrow \tilde{\alpha}$ in $L^1(\mathbb{R}^+ \times \mathbb{R}^+)$ as $N \longrightarrow \infty$,

$$\begin{aligned}
\langle \alpha, g \rangle &= \int_0^\infty \int_0^\infty \tilde{\alpha}(s,t) g(s,t)\, d\mu(s,t) \\
&= \lim_{N \to \infty} \int_0^\infty \int_0^\infty (P_N^{(2)} \mathbb{E}_N^{(2)} \tilde{\alpha})(s,t) g(s,t)\, d\mu(s,t) \\
&= \lim_{N \to \infty} \int_0^\infty \int_0^\infty \tilde{\alpha}(s,t) (P_N^{(2)} \mathbb{E}_N^{(2)} g)(s,t)\, d\mu(s,t) \\
&= \lim_{N \to \infty} \langle \alpha, P_N^{(2)} \mathbb{E}_N^{(2)} g \rangle.
\end{aligned}$$

So, since $P_N^{(2)} \mathbb{E}_N^{(2)} g \in L^\infty(\mathbb{R}^+) \otimes_{pt} L^\infty(\mathbb{R}^+)$

$$\begin{aligned}
|\langle \alpha, g \rangle| &\le \lim_{N \to \infty} \| \alpha \|_{\check{\otimes}} \| P_N^{(2)} \mathbb{E}_N^{(2)} g \|_{\hat{\otimes}_{pt}} \\
&\le \| \alpha \|_{\check{\otimes}} \| g \|_{\mathcal{V}^2}.
\end{aligned}$$

Since $L^1(\mathbb{R}^+) \hat{\otimes} L^1(\mathbb{R}^+)$ is a subspace of $L^1(\mathbb{R}^+) \check{\otimes} L^1(\mathbb{R}^+)$ it follows that $(L^1(\mathbb{R}^+) \check{\otimes} L^1(\mathbb{R}^+))^*$ is a subspace of $(L^1(\mathbb{R}^+) \hat{\otimes} L^1(\mathbb{R}^+))^*$ and therefore is a subspace of $L^\infty(\mathbb{R}^+ \times \mathbb{R}^+)$. So given $F \in (L^1(\mathbb{R}^+) \check{\otimes} L^1(\mathbb{R}^+))^*$ there exists a unique $g_F \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^+)$ such that

$$F\alpha = \int_0^\infty \int_0^\infty \tilde{\alpha}(s,t) g_F(s,t)\, d\mu(s,t)$$

for every $\alpha \in L^1(\mathbb{R}^+) \otimes L^1(\mathbb{R}^+)$.

We show that $g_F \in \mathcal{V}^2$ with $\| g_F \|_{\mathcal{V}^2} \le \| F \|$. For each $N \ge 0$ we have

$$\| P_N^{(2)} \mathbb{E}_N^{(2)} g \|_{\hat{\otimes}_{pt}} \le \| P_N^{(2)} \mathbb{E}_N^{(2)} g \|_{L_N^\infty \hat{\otimes}_{pt} L_N^\infty}$$

and by 5.3.2

$$\| P_N^{(2)}\mathbb{E}^{(2)}g \|_{L_N^\infty \hat{\otimes}_{pt} L_N^\infty}$$

$$\leq \sup\left\{\left|\int_0^\infty \int_0^\infty \tilde{\alpha}(s,t)(P_N^{(2)}\mathbb{E}_N^{(2)}g)(s,t)d\mu(s,t)\right| \ : \alpha \in L_N^1 \otimes L_N^1, \ \| \alpha \|_{\check{\otimes}} \leq 1\right\}$$

$$= \sup\left\{\left|\int_0^\infty \int_0^\infty \tilde{\alpha}(s,t)g(s,t)d\mu(s,t)\right| \ : \alpha \in L_N^1 \otimes L_N^1, \ \| \alpha \|_{\check{\otimes}} \leq 1\right\}$$

since $P_N^{(2)}\mathbb{E}_N^{(2)}\tilde{\alpha} = \tilde{\alpha}$

$$= \sup\left\{ \ | \ F(\alpha)| \ : \alpha \in L_N^1 \otimes L_N^1, \ \| \alpha \|_{\check{\otimes}} \leq 1 \right\}$$

$\leq \| F \|$.

Hence, $\| g \|_{\mathcal{V}^2} \leq \| F \|$, as required.

We can now describe the annihilator of E in $(L^1(\mathbb{R}^+)\check{\otimes}L^1(\mathbb{R}^+))^*$ in terms of $L^\infty(\mathbb{R}^+\times\mathbb{R}^+)$ functions.

<u>5.3.11 Lemma.</u> If $g \in L^\infty(\mathbb{R}^+\times\mathbb{R}^+)$ is such that

$$\int_0^\infty \int_0^\infty \tilde{\alpha}(s,t)g(s,t) \ d\mu(s,t) = 0 \qquad (*)$$

for all $\alpha \in E$ then there exists $h \in L^\infty(\mathbb{R}^+)$ such that for a.e. $t \in \mathbb{R}^+$ and a.e. $0 \leq s \leq t$

$$g(s,t-s) = h(t).$$

<u>Proof.</u> Suppose that $g \in L^\infty(\mathbb{R}^+\times\mathbb{R}^+)$ is real-valued and satisfies $(*)$ whenever $\alpha \in E$ is real-valued. For $t > 0$ put

$$A_g(t) = \frac{1}{t}\int_0^t g(s,t-s) \ d\mu(s).$$

Fix $N \in \mathbb{N}$ and define

$$\tilde{\alpha}_N(s,t) = \begin{cases} g(s,t) - A_g(s+t) & : \quad 0 \le s+t \le N \\ 0 & : \quad s+t > N \end{cases}$$

Then $\tilde{\alpha}_N \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^+)$ and is real-valued. Also,

$$\begin{aligned}
\int_0^t \tilde{\alpha}_N(s,t-s)d\mu(s) &= \int_0^t g(s,t-s)\, d\mu(s) - \int_0^t A_g(t)\, d\mu(s) \\
&= \int_0^t g(s,t-s)\, d\mu(s) - \int_0^t g(s,t-s)\, d\mu(s) \\
&= 0 \quad \text{for all } t > 0.
\end{aligned}$$

Hence,

$$\begin{aligned}
0 &= \int_0^\infty \int_0^\infty \tilde{\alpha}_N(s,t)g(s,t)\, d\mu(s,t) \\
&= \int_0^\infty \int_0^t \tilde{\alpha}_N(s,t-s)g(s,t-s)\, d\mu(s,t) \\
&= \int_0^N \int_0^t \Big(g(s,t-s) - A_g(t)\Big)\Big(g(s,t-s) - A_g(t) + A_g(t)\Big)\, d\mu(s,t) \\
&= \int_0^N \int_0^t \Big(g(s,t-s) - A_g(t)\Big)^2\, d\mu(s,t) + \\
&\qquad\qquad \int_0^N \int_0^t \Big(g(s,t-s) - A_g(t)\Big)\, A_g(t)\, d\mu(s,t) \\
&= \int_0^N \int_0^t \Big(g(s,t-s) - A_g(t)\Big)^2\, d\mu(s,t) + 0
\end{aligned}$$

and we conclude that $g(s,t-s) = A_g(t)$ for a.e. $t \le N$ and a.e. $s \le t$. Since this holds for each $N \in \mathbb{N}$, the proof is complete in the real-valued case.

Finally, if complex-valued g satisfies $(*)$ for all $\alpha \in E$ then we apply the above proof to the real and imaginary parts of g.

<u>5.3.12 Notation.</u> For $h \in L^\infty(\mathbb{R}^+)$ we denote by γ_h the essentially bounded measurable function on $\mathbb{R}^+ \times \mathbb{R}^+$ defined by $\gamma_h(s,t) = h(s+t)$.

<u>5.3.13 Corollary.</u> The annihilator of E in $(L^1(\mathbb{R}^+)\check{\otimes}L^1(\mathbb{R}^+))^*$ is isometrically isomorphic to $\{ \gamma_h \in \mathcal{V}^2 : h \in L^\infty(\mathbb{R}^+) \}$.

<u>Proof.</u> Immediately from 5.3.10 and 5.3.11.

To complete the section we have the following bound on $\| f(A) \|$.

<u>5.3.14 Corollary.</u> If $f \in L^1(\mathbb{R})$ has supp $\hat{f} \subseteq [0,N]$ for some $N > 0$ then

$$\| f(A) \| \le \frac{1}{\sqrt{2\pi}} K_G\, c^2 \sup\left\{ \left| \int_0^N \hat{f}(t)h(t)\, d\mu(t) \right| : h \in L^\infty(\mathbb{R}^+), \| \gamma_h \|_{\mathcal{V}^2} \le 1 \right\}$$

<u>Proof.</u> From 5.2.6 we deduce that

$$\| f(A) \| \le \frac{1}{\sqrt{2\pi}} K_G\, c^2 \sup\left\{ |F_{\beta_f}| : F \in E^\perp \cap (L^1(\mathbb{R}^+)\check{\otimes}L^1(\mathbb{R}^+))^*, \| F \| \le 1 \right\}$$

for any $\beta_f \in L^1(\mathbb{R}^+)\otimes L^1(\mathbb{R}^+)$ satisfying (*) of 5.2.4. Now by 5.3.10 we have

$$\| f(A) \| \le \frac{1}{\sqrt{2\pi}} K_G\, c^2 \sup\left\{ \left| \int_0^\infty \int_0^\infty \tilde{\beta}_f(s,t) g(s,t)\, d\mu(s,t) \right| : \begin{array}{l} g \in E^\perp \cap \mathcal{V}^2, \\ \| g \|_{\mathcal{V}^2} \le 1 \end{array} \right\}$$

But by 5.3.13 any $g \in E^\perp \cap \mathcal{V}^2$ is of the form $g = \gamma_h$ for some $h \in L^\infty(\mathbb{R}^+)$ and

$$\begin{aligned} &\int_0^\infty \int_0^\infty \tilde{\beta}_f(s,t)\gamma_h(s,t)\, d\mu(s,t) \\ &= \int_0^\infty \int_0^\infty \tilde{\beta}_f(s,t)\gamma_h(s,t)\, d\mu(s)\, d\mu(t) \quad \text{by Fubini's theorem} \\ &= \int_0^N \int_0^\infty \tilde{\beta}_f(s,t-s)h(t)\, d\mu(s)\, d\mu(t) \\ &= \int_0^N \hat{f}(t)h(t)\, d\mu(t) \quad \text{since } \beta_f \text{ satisfies (*) of 5.2.4.} \end{aligned}$$

Hence,

$$\| f(A) \| \le \frac{1}{\sqrt{2\pi}} K_G c^2 \sup\left\{ \left|\int_0^N \hat{f}(t)h(t)\, d\mu(t)\right| : h \in L^{\infty}(\mathbb{R}^+),\ \| \gamma_h \|_{\gamma^2} \le 1 \right\}$$

as required.

Section 5.4 Integral Operators on $H^2(\mathbb{R})$.

This section concerns bounded integral operators on $H^2(\mathbb{R})$ and in particular, those integral operators with a Hankel-type kernel γ_h (where h is a measurable function on $\mathbb{R}^+$). The kernel of a bounded integral operator on $H^2(\mathbb{R})$ is analogous to the matrix associated with a bounded operator on $H^2(\mathbb{T})$. So, motivated by this analogy we make the following definition.

5.4.1 Definition. Let k be a measurable complex-valued function on $\mathbb{R}^+ \times \mathbb{R}^+$. If for every $f \in H^2(\mathbb{R})$

i) $\displaystyle\int_0^{\infty} |k(x,y)\hat{f}(y)|\, d\mu(y) < \infty$ for a.e. $x \in \mathbb{R}^+$

and ii) $\displaystyle\int_0^{\infty} \left|\int_0^{\infty} k(x,y)\hat{f}(y)\right|^2 d\mu(x) < \infty$

then we define the integral operator T_k associated with the kernel k at $f \in H^2(\mathbb{R})$ by

$$(T_k f)\hat{}\,(x) = \int_0^{\infty} k(x,y)\hat{f}(y)\, d\mu(y) \quad (\text{for } x \in \mathbb{R}^+)$$

Since the Fourier transform $f \rightsquigarrow \hat{f}$ maps $H^2(\mathbb{R})$ onto $L^2(\mathbb{R}^+)$ and (ii) is satisfied for every $f \in H^2(\mathbb{R})$ we have defined a linear operator from $H^2(\mathbb{R})$ into $H^2(\mathbb{R})$. Moreover, T_k is a closed operator ([HS, p 15]) and it follows by the Closed Graph Theorem that T_k is bounded.

5.4.2 Definitions. a) A measurable function on $\mathbb{R}^+\times\mathbb{R}^+$ is a bounded kernel if it is the kernel of a bounded integral operator. We denote the linear space of all bounded kernels by BK and for $k \in$ BK we set $\| k \|_{BK} = \| T_k \|_{B(H^2(\mathbb{R}))}$.

b) A kernel k is of Hankel-type if there is a measurable function h on $\mathbb{R}^+$ such that $k = \gamma_h$ a.e. We denote by BHK the linear space of measurable functions h on $\mathbb{R}^+$ for which $\gamma_h \in$ BK and for $h \in$ BHK we set $\| h \|_{BHK} = \| \gamma_h \|_{BK}$.

There are no known necessary and sufficient conditions for a measurable function k on $\mathbb{R}^+\times\mathbb{R}^+$ to be the kernel of a bounded integral operator and there is no Nehari-type result giving a simple characterisation of BHK. We do not attempt such a theorem here but show instead that BHK is isometrically isomorphic to a subspace of $H^1(\mathbb{R})^*$ and use this result in section 5.5. Characterisation questions in the theory of integral operators such as "when is a bounded operator unitarily equivalent to an integral operator ?" and "when is a bounded operator an integral operator ?" are discussed in [HS]. The answer to the second is known but does not appear in [HS]. The following theorem is a special case of those proved by Buhvalov ([BU], 1974), Schep ([SCH], 1977) and Lessner ([LE], 1978).

5.4.3 Theorem. Let $T \in B(H^2(\mathbb{R}))$. Then $T = T_k$ for some $k \in$ BK if and only if, for any sequence $\{g_n\}_{n\in\mathbb{Z}^+}$ in $H^2(\mathbb{R})$ satisfying

(i) $\| g_n \|_2 \longrightarrow 0$ as $n \longrightarrow \infty$

and (ii) $| \hat{g}_n | \le | \hat{g} |$ for some $g \in H^2(\mathbb{R})$,

we have $(Tg_n)\hat{}(x) \longrightarrow 0$ as $n \longrightarrow \infty$ for a.e. $x \in \mathbb{R}^+$.

An account of Schep's proof is given by Zaanen in his review of

[HS]. ([ZA]).

The notion of Schur multipliers of matrices of bounded operators on $H^2(\mathbb{T})$ has an obvious analogy in the theory of integral operators.

<u>5.4.4 Definitions.</u> a) Let $g \in L^{\infty}(\mathbb{R}^+\times\mathbb{R}^+)$. We say that g is a <u>multiplier on</u> BK if pointwise multiplication by g is a bounded linear operator on BK. The linear space of all multipliers on BK is denoted by M(BK) and for $g \in M(BK)$ we define $\| g \|_{M(BK)}$ to be the operator norm of multiplication by g on BK.

b) Similarly, $h \in L^{\infty}(\mathbb{R}^+)$ is a <u>multiplier on</u> BHK if pointwise multiplication by h is a bounded operator on BHK. The linear space of all bounded multipliers on BHK is denoted by M(BHK) and for $h \in M(BHK)$ we define $\| h \|_{M(BHK)}$ to be the operator norm of multiplication by h on BHK.

We shall show that if $g \in \mathcal{V}^2$, the dual of $L^1(\mathbb{R}^+)\check{\otimes}L^1(\mathbb{R}^+)$, then g is a multiplier on BK. The proof of this result requires the following lemma which shows that if k is a bounded kernel there is a sequence of measurable subsets of finite measure of $\mathbb{R}^+\times\mathbb{R}^+$, increasing to a set of full measure, such that k is integrable on each subset in this sequence. This lemma is a special case of Lemma 7.4 in [HS] and its proof is a clarification of that given in [HS].

<u>5.4.5 Lemma.</u> Let $k \in BK$. There exists an increasing sequence $\{X_n\}_{n\geq 1}$ of measurable subsets of $\mathbb{R}^+$ satisfying

i) $\mu(X_n) < \infty$ for each $n \geq 1$;

ii) $\mu(\mathbb{R}^+\setminus \bigcup_{n\geq 1} X_n) = 0$

and iii) $\displaystyle\int_{X_n\times[0,m]} |\ k(x,y)\ |\ d\mu(x,y) < \infty$ for every $n,m \ge 1$.

Proof. We show first that if $Z \subseteq \mathbb{R}^+$ has $\mu(Z) < \infty$ then for any $\varepsilon > 0$ there is a subset Z' of Z such that $\nu(Z\backslash Z') < \varepsilon$ and

$$\int_{Z'\times[0,m]} |\ k(x,y)\ |\ d\mu(x,y) < \infty \text{ for every } m \ge 1.$$

Since $\mathfrak{X}_{[0,m]} \in L^2(\mathbb{R}^+)$ and $k \in BK$ we have

$$\int_0^m |\ k(x,y)\ |\ d\mu(y) < \infty \quad \text{for a.e. } x \in \mathbb{R}^+.$$

We define for each $m,n \ge 1$

$$Z_{m,n} = \left\{ x \in Z : \int_0^m |\ k(x,y)\ |\ d\mu(y) < n \right\}.$$

Then for each $m \ge 1$, $\{Z_{m,n}\}_{n\ge1}$ is an increasing sequence of measurable subsets on Z with $\mu(Z\backslash \bigcup_{n\ge1} Z_{m,n}) = 0$. By choosing a sequence of integers $\{n_m\}_{m\ge1}$ such that

$$\mu(Z\backslash Z_{m,n_m}) < \varepsilon/2^m$$

for every m, and putting

$$Z' = \bigcap_{m\ge1} Z_{m,n_m}$$

we have

$$\int_0^m |\ k(x,y)\ |\ d\mu(y) < n_m$$

for every $x \in Z'$ and every $m \ge 1$.

To complete the proof we define a sequence of measurable subsets $\{Z_n\}_{n\ge1}$ as follows. Let $Z_1 \subset [0,1]$ be such that $\mu([0,1]\backslash Z_1) < 1$ and $\displaystyle\int_{Z_1\times[0,m]} |\ k(x,y)\ |\ d\mu(x,y) < \infty$ for every $m \ge 1$. Let $Z_2 \subset [0,2]\backslash Z_1$ be such that $\mu([0,2]\backslash(Z_1 \cup Z_2)) < \frac{1}{2}$ and $\displaystyle\int_{Z_2\times[0,m]} |\ k(x,y)\ |\ d\mu(x,y) < \infty$ for every $m \ge 1$. Let $Z_3 \subset [0,3]\backslash(Z_1 \cup Z_2)$ be such that

$\mu([0,3]\setminus(Z_1 \cup Z_2 \cup Z_3)) < \frac{1}{3}$ and $\int_{Z_3\times[0,m]} |\ k(x,y)\ |\ d\mu(x,y) < \infty$ for every $m \geq 1$.

This process gives a sequence $\{Z_n\}_{n\geq 1}$ with $Z_n \subset [0,N]\setminus \bigcup_{k=1}^{n-1} Z_k$, $\mu([o,N]\setminus \bigcup_{k=1}^{n} Z_k) < \frac{1}{n}$ and $\int_{Z_n\times[0,m]} |k(x,y)|\ d\mu(x,y) < \infty$ for every $m,n \geq 1$.

$$\begin{aligned}\text{Moreover, } \mu(\mathbb{R}^+\setminus \bigcup_{k\geq 1} Z_k) &= \lim_{n\to\infty} \mu([0,n]\setminus \bigcup_{k\geq 1} Z_k) \\ &\leq \overline{\lim_{n\to\infty}}\, \mu([0,n]\setminus \bigcup_{k=1}^{n} Z_k) \\ &\leq \lim_{n\to\infty} \frac{1}{n} \\ &= 0.\end{aligned}$$

The result now follows by defining, for each $n \geq 1$, $X_n = \bigcup_{k=1}^{n} Z_k$.

<u>**5.4.6 Theorem.**</u> If $g \in \mathcal{V}^2$ then $g \in M(BK)$ with $\| g \|_{M(BK)} \leq \| g \|_{\mathcal{V}^2}$.

<u>Proof.</u> Note first that for $g \in L^{\infty}(\mathbb{R}^+\times\mathbb{R}^+)$, $k \in BK$ and $f \in H^2(\mathbb{R})$, $\int_0^{\infty} |g(s,t)k(s,t)\hat{f}(t)|\ d\mu(t) \leq \| g \|_{\infty} \int_0^{\infty} |k(s,t)\hat{f}(t)|\ d\mu(t) < \infty$ for a.e. $s \in \mathbb{R}^+$.

Now suppose that g is a simple element of $L^{\infty}(\mathbb{R}^+)\otimes_{pt} L^{\infty}(\mathbb{R}^+)$, say $g(s,t) = \phi(s)\psi(t)$ for some $\phi,\psi \in L^{\infty}(\mathbb{R}^+)$ and a.e. $s,t \in \mathbb{R}^+$. We denote by $L_{\bar{\phi}}$ and L_{ψ} the Laurent operators on $H^2(\mathbb{R})$ defined at $f \in H^2(\mathbb{R})$ by $(L_{\bar{\phi}}f)\hat{} = \bar{\phi}\hat{f}$ and $(L_{\psi}f)\hat{} = \psi\hat{f}$.

If $k \in BK$ and $f,h \in H^2(\mathbb{R})$ then

$$\begin{aligned}&\int_0^{\infty} \left(\int_0^{\infty} g(s,t)k(s,t)\hat{f}(t)\ d\mu(t) \right) \overline{\hat{h}(s)}\ d\mu(s) \\ &= \int_0^{\infty} \left(\int_0^{\infty} k(s,t)(\psi\hat{f})(t)\ d\mu(t) \right) \overline{(\bar{\phi}\hat{h})(s)}\ d\mu(s) \\ &= \left((T_k L_{\psi} f)\hat{},(L_{\bar{\phi}}h)\hat{} \right).\end{aligned}$$

Thus,

$$\begin{aligned}|(T_{gk}f,h)| &= |(T_k L_\psi f,\ L_{\bar\phi} h)| \\ &\le \| T_k \| \| L_\psi \| \| f \|_2 \| L_{\bar\phi} \| \| h \|_2 \\ &= \| k \|_{BK} \| \psi \|_\infty \| \phi \|_\infty \| f \|_2 \| h \|_2.\end{aligned}$$

Similarly when $g(s,t) = \sum_{r=0}^{n} \phi_r(s)\psi_r(t)$ for some $\phi_r, \psi_r \in L^\infty(\mathbb{R}^+)$ $(0 \le r \le n)$ we have

$$|(T_{gk}f,h)| \le \| k \|_{BK} \left(\sum_{r=0}^{n} \| \phi_r \|_\infty \| \psi_r \|_\infty \right) \| f \|_2 \| h \|_2.$$

whenever $k \in BK$ and $f,h \in H^2(\mathbb{R})$. It follows that $g_k \in BK$ with $\| gk \|_{BK} \le \| g \|_{\hat\otimes_{pt}} \| k \|_{BK}$.

Now suppose that $g \in \mathcal{V}^2$ so that for each $N > 0$ we have $P_N^{(2)}\mathbb{E}_N^{(2)} g \in L^\infty(\mathbb{R}^+)\otimes_{pt} L^\infty(\mathbb{R}^+)$ with

$$\| P_N^{(2)}\mathbb{E}_N^{(2)} g \|_{\hat\otimes_{pt}} \le \| g \|_{\mathcal{V}^2} < \infty.$$

Let $k \in BK$ and let $\{X_n\}_{n\ge1}$ be a sequence of subsets of $\mathbb{R}^+$ given by Lemma 5.4.5. Then we define :

$$\mathcal{D}_1 = \left\{ f \in H^2(\mathbb{R}) : \hat f \text{ is bounded, supp } \hat f \subseteq [0,m] \text{ for some } m > 0 \right\}$$

and

$$\mathcal{D}_2 = \left\{ f \in H^2(\mathbb{R}) : \hat f \text{ is bounded, supp } \hat f \subseteq X_n \text{ for some } n \in \mathbb{N} \right\}$$

Let $f \in \mathcal{D}_1$ and $h \in \mathcal{D}_2$. We denote by Φ, the function $(s,t) \rightsquigarrow k(s,t)\hat f(t)\overline{\hat h(s)}$. Then for some $n \in \mathbb{N}$ and $m > 0$

$$\begin{aligned}\int_{\mathbb{R}^+\times\mathbb{R}^+} | \Phi(s,t) | \, d\mu(s,t) &= \int_{X_n\times[0,m]} | \Phi(s,t) | \, d\mu(s,t) \\ &\le \int_{X_n\times[0,m]} | k(s,t) | \, d\mu(s,t) \sup_{[0,m]} |\hat f| \sup_{X_n} |\hat h| \\ &< \infty.\end{aligned}$$

Hence $\Phi \in L^1(\mathbb{R}^+\times\mathbb{R}^+)$ and by 5.3.6 (e), $P_N^{(2)}\mathbb{E}_N^{(2)}\Phi \longrightarrow \Phi$ in $L^1(\mathbb{R}^+\times\mathbb{R}^+)$ as $N \longrightarrow \infty$. Moreover, since $g \in L^{\infty}(\mathbb{R}^+\times\mathbb{R}^+)$ the integrals

$$\int_{\mathbb{R}^+\times\mathbb{R}^+} | g(s,t)(P_N^{(2)}\mathbb{E}_N^{(2)}\Phi)(s,t)| \, d\mu(s,t)$$

$$\text{and} \qquad \int_{\mathbb{R}^+\times\mathbb{R}^+} | g(s,t)\Phi(s,t)| \, d\mu(s,t)$$

are finite and

$$\int_{\mathbb{R}^+\times\mathbb{R}^+} | g(s,t)(P_N^{(2)}\mathbb{E}_N^{(2)}\Phi)(s,t)| \, d\mu(s,t) \longrightarrow \int_{\mathbb{R}^+\times\mathbb{R}^+} | g(s,t)\Phi(s,t)| \, d\mu(s,t) \qquad (*)$$

as $N \longrightarrow \infty$.

The Parseval property of conditional expectation operators (5.3.6(d)) gives

$$\int_{\mathbb{R}^+\times\mathbb{R}^+} g(s,t)(P_N^{(2)}\mathbb{E}_N^{(2)}\Phi)(s,t) \, d\mu(s,t)$$

$$= \int_{\mathbb{R}^+\times\mathbb{R}^+} (P_N^{(2)}\mathbb{E}_N^{(2)}g)(s,t)\Phi(s,t) \, d\mu(s,t)$$

$$= \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \left((P_N^{(2)}\mathbb{E}_N^{(2)}g)(s,t)\Phi(s,t) \, d\mu(t)\right) d\mu(s)$$

by Fubini's theorem,

$$= (T_{P_N^{(2)}\mathbb{E}_N^{(2)}gk} \, f \, , \, h \,).$$

Also,

$$\int_{\mathbb{R}^+\times\mathbb{R}^+} \Phi(s,t) \, d\mu(s,t) = \int_{\mathbb{R}^+} \left(\int_{\mathbb{R}^+} \Phi(s,t) \, d\mu(t) \right) d\mu(s)$$

by Fubini's theorem

$$= (T_{gk} f,h).$$

Thus, by $(*)$, as $N \rightarrow \infty$,

$$(T_{P_N^{(2)}\mathbb{E}_N^{(2)}gk} \, f \, , \, h \,) \longrightarrow (T_{gk} f,h).$$

And since for every $N > 0$

$$|(T_{P_N^{(2)}\mathbb{E}_N^{(2)}gk}\, f\, ,\, h\,)| \le \| P_N^{(2)}\mathbb{E}_N^{(2)}g \|_{\hat{\otimes}_{pt}} \| k \|_{BK} \| f \|_2 \| h \|_2$$

$$\le \| g \|_{\mathcal{V}^2} \| k \|_{BK} \| f \|_2 \| h \|_2$$

we have

$$|(T_{gk}f,h)| \le \| g \|_{\mathcal{V}^2} \| k \|_{BK} \| f \|_2 \| h \|_2 \,. \qquad (**)$$

Finally, because $\bigcup_{n\ge1} X_n$ is a set of full measure in $\mathbb{R}^+$, the sets $\mathcal{D}_1$ and $\mathcal{D}_2$ are dense on $H^2(\mathbb{R})$. By approximating $f \in H^2(\mathbb{R})$ by a sequence in $\mathcal{D}_1$ and approximating $h \in H^2(\mathbb{R})$ by a sequence in $\mathcal{D}_2$ we obtain (**) for any $f,h \in H^2(\mathbb{R})$. We conclude that $gk \in BK$ with $\| gk \|_{BK} \le \| g \|_{\mathcal{V}^2} \| k \|_{BK}$.

To complete the description of multipliers on BK we show that, as in the discrete case (1.4.3) the converse is true.

<u>5.4.7 Theorem.</u> There exists a constant $d > 0$ such that if $g \in M(BK)$ then $g \in \mathcal{V}^2$ with $\| g \|_{\mathcal{V}^2} \le d \| g \|_{M(BK)}$.

<u>Proof.</u> Let $g \in M(BK)$, let $N > 0$ and put $m = N2^N-1$. Recall from 5.3.1 that for $i \ge 0$, $I_N^i = \left[\dfrac{i}{2^N}\, ,\, \dfrac{i+1}{2^N}\right)$, so that $P_N^{(2)}\mathbb{E}_N^{(2)}g$ is constant on each square $I_N^i \times I_N^j$ $(i,j \in \mathbb{Z}^+)$ and is zero on $I_N^i \times I_N^j$ when $i > m$ or $j > m$. We may choose a sequence $\{\xi_i\}_{i\in\mathbb{Z}^+}$ with $\xi_i \in I_N^i$ for each $i \in \mathbb{Z}^+$ and $(P_N^{(2)}\mathbb{E}_N^{(2)}g)(s,t) = (P_N^{(2)}\mathbb{E}_N^{(2)}g)(\xi_i,\xi_j)$ for every $(s,t) \in I_N^i \times I_N^j$ and every $i,j \in \mathbb{Z}^+$.

We show first that the matrix

$$g_N = \left\{(P_N^{(2)}\mathbb{E}_N^{(2)}g)(\xi_i,\xi_j)\right\}_{i,j\in\mathbb{Z}^+}$$

is a Schur multiplier on $B(l^2(\mathbb{Z}^+))$ with

$$\| g_N \|_{M(B(l^2(\mathbb{Z}^+)))} \le \| g \|_{M(BK)} \,.$$

Let $S = \{S_{i,j}\}_{i,j\in\mathbb{Z}^+}$ be the matrix of a bounded operator on $l^2(\mathbb{Z}^+)$ (with respect to the usual orthonormal basis). Then we form its inflation $\tilde{S}$ on $\mathbb{R}^+\times\mathbb{R}^+$ by defining

$$\tilde{S}(s,t) = S_{i,j}$$

when $(s,t) \in I_N^i \times I_N^j$ $(i,j \in \mathbb{Z}^+)$. Similarly for a vector $x \in l^2(\mathbb{Z}^+)$ we define $\tilde{x}$ on $\mathbb{R}^+$ by

$$\tilde{x}(s) = x(i)$$

when $s \in I_N^i$ $(i \in \mathbb{Z}^+)$. Then $\tilde{x} \in L^2(\mathbb{R}^+)$ with $\| \tilde{x} \|_2 = 2^{-N/2}\| x \|_2$. Also, if $f \in H^2(\mathbb{R})$ then the sequence

$$f_N = \left\{(\mathbb{E}_N^{(2)}\hat{f})(\xi_i)\right\}_{i\in\mathbb{Z}^+}$$

is in $l^2(\mathbb{Z}^+)$ with $\| f_N \|_2 \le 2^{N/2}\| f \|_2$.

Now if $f,g \in H^2(\mathbb{R})$ have compactly supported Fourier transforms we find that for the integral operator, $T_{\tilde{S}}$

$$(T_{\tilde{S}}f,g) = 2^{-2N}(Sf_N,g_N)$$

and so

$$|(T_{\tilde{S}}f,g)| \le 2^{-2N}\| S \|_{B(l^2(\mathbb{Z}^+))} \| f_N \|_2 \| g_N \|_2$$

$$\le 2^{-2N}\| S \|_{B(l^2(\mathbb{Z}^+))} \| f \|_2 \| g \|_2$$

Thus, $\tilde{S} \in BK$ with $\| \tilde{S} \|_{BK} \le 2^{-2N}\| S \|_{B(l^2(\mathbb{Z}^+))}$.

A calculation using the Parseval property of conditional expectation operators shows that when $x,y \in l^2(\mathbb{Z}^+)$ are finitely non-zero sequences

$$\sum_{i,j=0}^{m} g_N(i,j)S_{i,j}x(j)\overline{y(i)} = 2^{2N}(T_{g\tilde{S}}(P_N\tilde{x})^{\check{}},(P_N\tilde{y})^{\check{}})$$

and therefore

$$\left| \sum_{i,j=0}^{m} g_N(i,j)S_{i,j}x(i)\overline{y(j)} \right|$$

$$\le 2^{2N}\| g \|_{M(BK)} \| \tilde{S} \|_{BK} \| \tilde{x} \|_2 \| \tilde{y} \|_2$$

$$\leq 2^{2N} \| g \|_{M(BK)} 2^{-N} \| \tilde{\tilde{S}} \|_{B(l^2(\mathbb{Z}^+))} 2^{-N/2} \| \tilde{x} \|_2 \, 2^{-N/2} \| \tilde{y} \|_2$$

$$= \| g \|_{M(BK)} \| S \|_{B(l^2(\mathbb{Z}^+))} \| x \|_2 \| y \|_2 .$$

Hence,

$$\| g_N \|_{M(B(l^2(\mathbb{Z}^+)))} \leq \| g \|_{M(BK)} \qquad (*)$$

as required.

The proof now follows that of Theorem 1.4.3. Any Schur multiplier A on $B(l^2(\mathbb{Z}^+))$ factors through $l^2(\mathbb{Z}^+)$ ([BEN1. Thm 6.4]). That is, there exists $B \in B(l^2;l^\infty)$ and $C \in B(l^1;l^2)$ such that $A = BC$ and $\| B \| \| C \| \leq d' \| A \|_{M(B(l^2(\mathbb{Z}^+)))}$ (d' independent of A). It follows that such A defines a bounded linear functional on $l^1(\mathbb{Z}_m)\check{\otimes}l^1(\mathbb{Z}_m)$ since for any $\alpha \in l^1(\mathbb{Z}_m)\check{\otimes}l^1(\mathbb{Z}_m)$

$$|\langle \alpha,A \rangle| = \Big| \sum_{i,j=0}^{m} \alpha_{i,j} A_{i,j} \Big|$$

$$= \Big| \sum_{i,j=0}^{m} \alpha_{i,j} (B_{i,\cdot}, \overline{C_{\cdot,j}}) \Big|$$

$$\leq K_G \| \alpha \|_{\check{\otimes}} \sup_{i\in\mathbb{Z}^+} \| B_{i,\cdot} \|_2 \sup_{j\in\mathbb{Z}^+} \| C_{\cdot,j} \|_2$$

$$\leq K_G \| \alpha \|_{\check{\otimes}} \| B \| \| C \|$$

$$\leq K_G d' \| \alpha \|_{\check{\otimes}} \| A \|_{M(B(l^2(\mathbb{Z}^+)))} .$$

Thus we have, by 1.2.4, that $A \in l^\infty(\mathbb{Z}_m)\hat{\otimes}l^\infty(\mathbb{Z}_m)$ with

$$\| A \|_{\hat{\otimes}} \leq K_G d' \| A \|_{M(B(l^2(\mathbb{Z}^+)))} .$$

Applying this to the matrix g_N and putting $d = K_G d'$ gives

$$\| P_N^{(2)} \mathbb{E}_N^{(2)} g \|_{l^\infty \hat{\otimes}_{pt} l^\infty} \leq d \| g_N \|_{M(B(l^2(\mathbb{Z}^+)))}$$

$$\leq d \| g \|_{M(BK)} \quad \text{by } (*).$$

But N was chosen arbitrarily so $g \in \mathcal{V}^2$ with $\| g \|_{\mathcal{V}^2} \leq d \| g \|_{M(BK)}$.

We now return to the main theme, the annihilator of the subspace E^- of $L^1(\mathbb{R}^+)\check{\otimes}L^1(\mathbb{R}^+)$. Recall that Corollary 5.3.13 stated that the annihilator of E^- is isometrically isomorphic to the subspace of $\mathcal{V}^2$ consisting of Hankel-type functions γ_h.

<u>5.4.8 Corollary.</u> If h is a measurable function on $\mathbb{R}^+$ such that $\gamma_h \in \mathcal{V}^2$ then $h \in M(BHK)$ with $\| h \|_{M(BHK)} \le \| \gamma_h \|_{\mathcal{V}^2}$.

<u>Proof.</u> If $\gamma_h \in \mathcal{V}^2$ then by theorem 5.4.6, $\gamma_h \in M(BK)$ with multiplier norm less than or equal to its $\mathcal{V}^2$ norm. Thus if $\phi \in BHK$ we have

$$\begin{aligned} \| h\phi \|_{BHK} &= \| \gamma_{h\phi} \|_{BK} \quad \text{(by definition)} \\ &= \| \gamma_h \gamma_\phi \|_{BK} \\ &\le \| \gamma_h \|_{M(BK)} \| \gamma_\phi \|_{BK} \\ &\le \| \gamma_h \|_{\mathcal{V}^2} \| \gamma_\phi \|_{BK} \\ &= \| \gamma_h \|_{\mathcal{V}^2} \| \phi \|_{BHK} \quad \text{(by definition).} \end{aligned}$$

The application of 5.4.8 to functions of the infinitesimal generator of a bounded C_0-semigroup is the following.

<u>5.4.9 Corollary.</u> Let $f \in H^1(\mathbb{R}^+)$ have supp $\hat{f} \subseteq [0,N]$ for some $N > 0$. Then

$$\| f(A) \| \le \frac{1}{\sqrt{2\pi}} K_G c^2 \sup \left\{ \left| \int_0^N \hat{f}(t)h(t)\, d\mu(t) \right| : \| h \|_{M(BHK)} \le 1 \right\}$$

<u>Proof.</u> By Corollary 5.3.14, if f is as in the hypothesis

$$\| f(A) \| \le \frac{1}{\sqrt{2\pi}} K_G c^2 \sup \left\{ \left| \int_0^N \hat{f}(t)h(t)\, d\mu(t) \right| : \| \gamma_h \|_{\mathcal{V}^2} \le 1 \right\}$$

The result now follows from Corollary 5.4.8.

To complete this section we show that BHK is isometrically isomorphic to a subspace of $H^1(\mathbb{R}^+)^*$. The idea for the proof arises from the alternative proof of Nehari's theorem given in [POW2, p 431]. We can show that when S is a bounded Hankel operator on $H^2(\mathbb{T})$ and when $f,g \in H^2(\mathbb{T})$ are such that $fg^\dagger \in H^2(\mathbb{T})$ we have

$$(Sf,g) = (S1,fg^\dagger)$$

(where 1 is the function that is constantly 1 on $\mathbb{T}$). Thus for $h \in H^1(\mathbb{T}) \cap H^2(\mathbb{T})$, with Riesz factorisation $h = fg^\dagger$

$$\begin{aligned} |(S1,h)| &= |(Sf,g)| \\ &\leq \| S \| \, \| f \|_2 \, \| g \|_2 \\ &= \| S \| \, \| h \|_1 \end{aligned}$$

and the map $h \rightsquigarrow (S1,h)$ is a bounded linear functional on $H^1(\mathbb{T}) \cap H^2(\mathbb{T})$. By extending this functional to $H^1(\mathbb{T})$ and using the identification of $H^1(\mathbb{T})$ with $L^\infty(\mathbb{T})/H_0^\infty(\mathbb{T})$ we find a (non-unique) $\phi \in L^\infty(\mathbb{T})$ such that for $f,g \in H^2(\mathbb{T})$ with $fg^\dagger \in H^1(\mathbb{T}) \cap H^2(\mathbb{T})$

$$(Sf,g) = \int_0^{2\pi} \phi(e^{i\theta})(fg^\dagger)(e^{i\theta}) \, dm(\theta).$$

When $fg^\dagger$ is a polynomial we have

$$(Sf,g) = \sum_n \hat{\phi}(n)(fg^\dagger)\hat{}\,(n).$$

In the context of integral operators on $H^2(\mathbb{R})$ we would like to show that when $k \in$ BHK and f,g are in some dense subspace of $H^2(\mathbb{R})$

$$(T_k f,g) = \int_{\mathbb{R}^+} k(u)(fg^\dagger)\hat{}\,(u) \, d\mu(u)$$

where $g^\dagger(x) = \overline{g(-x)}$ for all $x \in \mathbb{R}$.

Then the map $h \rightsquigarrow \int_{\mathbb{R}^+} k(u)\hat{h}(u) \, d\mu(u)$ is a bounded linear functional on some dense subspace of $H^1(\mathbb{R})$. Since $\mathbb{R}^+$ has infinite measure, the function that is constantly 1 on $\mathbb{R}^+$ is not in $H^2(\mathbb{R})$ and the arguments for the existence of the above integrals are more delicate. However, we can show first that for $\delta > 0$ and for $x \in \mathbb{R}^+$ the integrals of $|\gamma_k(x,\cdot)|$ over $[n\delta,(n+1)\delta]$ form a square-summable sequence, with l^2-norm bounded by some constant independent of x.

5.4.10 Lemma. If $k \in BHK$ and $\delta > 0$ then there exists a constant M_δ such that

$$\sum_{n\geq 1} \left(\int_{n\delta}^{(n+1)\delta} |k(x+y)| \, d\mu(y) \right)^2 \leq M_\delta$$

for every $x \in \mathbb{R}^+$.

Proof. Let $k \in BHK$ and $\delta > 0$. We show first that

$$\sum_{n\geq 1} \left(\int_{n\delta}^{(n+1)\delta} |k(x+y)| \, d\mu(y) \right)^2 < \infty.$$

Note that, since $k \in BHK$ we have, for every $f \in H^2(\mathbb{R})$

$$\int_0^\infty |k(x+y)\hat{f}(y)| \, d\mu(y) < \infty \tag{$*$}$$

for a.e $x \in \mathbb{R}^+$. We can choose suitable $f \in H^2(\mathbb{R})$ and $x \in \mathbb{R}^+$ to show that $\sum_{n\geq 1} \left(\int_{(n+\frac{1}{4})\delta}^{(n+1)\delta} |k(y)| \, d\mu(y) \right) |\alpha_n| < \infty$ for any $\{\alpha_n\} \in l^2(\mathbb{Z}^+)$ and conclude that

$$\sum_{n\geq 1} \left(\int_{(n+\frac{1}{4})\delta}^{(n+1)\delta} |k(y)| \, d\mu(y) \right)^2 < \infty. \tag{$**$}$$

Indeed, if $\{\alpha_n\} \in l^2(\mathbb{Z}^+)$ we choose $f \in H^2(\mathbb{R})$ such that

$$\hat{f} = \sum_{n\geq 1} \alpha_n \chi_{[n\delta, (n+1)\delta)}$$

and we can choose $0 < x < \delta/4$ such that $(*)$ holds. But

$$\begin{aligned}\int_0^\infty |k(x+y)\hat{f}(y)| \, d\mu(y) &= \int_0^\infty |k(y)\hat{f}(y-x)| \, d\mu(y) \\ &= \sum_{n\geq 1} \int_{n\delta+x}^{(n+1)\delta+x} |k(y)||\alpha_n| \, d\mu(y) \\ &\geq \sum_{n\geq 1} \left(\int_{(n+\frac{1}{4})\delta}^{(n+1)\delta} |k(y)| \, d\mu(y) \right) |\alpha_n| .\end{aligned}$$

Hence, $\sum_{n\geq 1} \left(\int_{(n+\frac{1}{4})\delta}^{(n+1)\delta} |k(y)| \, d\mu(y) \right) |\alpha_n| < \infty$ and $(**)$ holds.

Similarly, when we choose any $\{\alpha_n\} \in l^2(\mathbb{Z}^+)$ and choose $f \in H^2(\mathbb{R})$ such that

$$\hat{f} = \sum_{n\geq 1} \alpha_n \chi_{[n\delta,(n+1)\delta)}$$

there must exist $0 < x < \delta/4$ such that

$$\int_0^\infty \left| k(x+\tfrac{\delta}{2}+y)\hat{f}(y) \right| d\mu(y) < \infty.$$

Then

$$\begin{aligned}\int_0^\infty \left| k(x+\tfrac{\delta}{2}+y)\hat{f}(y) \right| d\mu(y) &= \int_0^\infty \left| k(\tfrac{\delta}{2}+y)\hat{f}(y-x) \right| d\mu(y) \\ &= \sum_{n\geq 1} \int_{(n-1)\delta+x}^{n\delta+x} |k(\tfrac{\delta}{2}+y)|\ |\alpha_n|\ d\mu(y) \\ &\geqslant \sum_{n\geq 1} \left(\int_{(n-\frac{3}{4})\delta}^{n\delta} |k(\tfrac{\delta}{2}+y)|\ d\mu(y) \right) |\alpha_n|\end{aligned}$$

and we conclude that

$$\sum_{n\geq 1} \left(\int_{(n-\frac{3}{4})\delta}^{n\delta} |k(\tfrac{\delta}{2}+y)|\ d\mu(y) \right)^2 < \infty.$$

Thus,

$$\sum_{n\geq 1} \left(\int_{(n-\frac{1}{4})\delta}^{(n+\frac{1}{2})\delta} |k(y)|\ d\mu(y) \right)^2 < \infty. \qquad (***)$$

But for each $n \geq 1$

$$\int_{n\delta}^{(n+1)\delta} |k(y)|\ d\mu(y) \leq \int_{(n-\frac{1}{4})\delta}^{(n+\frac{1}{2})\delta} |k(y)|\ d\mu(y) + \int_{(n+\frac{1}{4})\delta}^{(n+1)\delta} |k(y)|\ d\mu(y)$$

so,

$$\begin{aligned}\sum_{n\geq 1} \left(\int_{n\delta}^{(n+1)\delta} |k(y)|\ d\mu(y) \right)^2 &\leq 2\sum_{n\geq 1} \left(\int_{(n-\frac{1}{4})\delta}^{(n+\frac{1}{2})\delta} |k(y)|\ d\mu(y) \right)^2 \\ &\quad + 2\sum_{n\geq 1} \left(\int_{(n+\frac{1}{4})\delta}^{(n+1)\delta} |k(y)|\ d\mu(y) \right)^2 \\ &< \infty \quad \text{by } (**) \text{ and } (***).\end{aligned}$$

Finally, we put

$$M'_\delta = \sum_{n\geq 1} \left(\int_{n\delta}^{(n+1)\delta} |k(y)| \ d\mu(y) \right)^2$$

and choose any $x \in \mathbb{R}^+$. Write $x = m\delta + \varepsilon$ for some $m \in \mathbb{Z}^+$ and $0 \leq \varepsilon < \delta$. Then for each $n \gg 1$

$$\begin{aligned} [n\delta+x,(n+1)\delta+x] &= [(n+m)\delta+\varepsilon,(n+m+1)\delta+\varepsilon] \\ &\subseteq [(n+m)\delta,(n+m+2)\delta] \\ &= [(n+m)\delta,(n+m+1)\delta]\cup[(n+m+1)\delta,(n+m+2)\delta] \end{aligned}$$

so,

$$\begin{aligned} \sum_{n\geq 1} \left(\int_{n\delta}^{(n+1)\delta} |k(x+y)| \ d\mu(y) \right)^2 &= \sum_{n\geq 1} \left(\int_{n\delta+x}^{(n+1)\delta+x} |k(y)| \ d\mu(y) \right)^2 \\ &\leq \sum_{n\geq 1} \left(\int_{(n+m)\delta}^{(n+m+1)\delta} |k(y)| \ d\mu(y)| + \right. \\ &\qquad \left. \int_{(n+m+1)\delta}^{(n+m+2)\delta} |k(y)| \ d\mu(y)| \right)^2 \\ &\leq 4M'_\delta \ . \end{aligned}$$

Putting $M_\delta = 4M'_\delta$ completes the proof.

We will use 5.4.10 to show that for "well-behaved" $f,g \in H^2(\mathbb{R})$ the inner product $(T_k f,g^\dagger)$ depends only on the product fg. Indeed, when $f \in H^2(\mathbb{R})$ has supp $\hat{f} \subseteq [\delta,\infty)$ for some $\delta > 0$ and has $\hat{f}(x) \leq \frac{c}{x}$ for some constant $c \geq 0$ and all $x > 0$, and when $g \in H^2(\mathbb{R})$ has $\hat{g} \in L^1(\mathbb{R}^+)$ we can deduce from 5.4.10 that

$$\int_{\mathbb{R}^+} \int_{\mathbb{R}^+} | \ k(x+y)\hat{f}(y)\hat{g}(x) \ | \ d\mu(y)d\mu(x) < \infty. \qquad (*)$$

Then the existence of the double integral

$$\int_{\mathbb{R}^+\times\mathbb{R}^+} k(x+y)\hat{f}(y)\hat{g}(x) \ d\mu(x,y)$$

enables us to show that $(T_k f, g^\dagger) = \int_{\mathbb{R}^+} k(u)(fg)\hat{}(u)\, d\mu(u)$.

Note that $k \in BHK$ satisfies (*) for <u>every</u> $f,g \in H^2(\mathbb{R})$ if and only if k is an absolutely bounded kernel in the sense that $|k| \in BHK$ ([HS]).

<u>5.4.11 Lemma.</u> Let $k \in BHK$ and suppose that $f,g \in H^2(\mathbb{R})$ are such that

i) $\text{supp}\, \hat{f} \subseteq [\delta,\infty)$ for some $\delta > 0$;

ii) there exists $c > 0$ such that $f(x) \leq \frac{c}{x}$ for all $x > 0$

and iii) $\hat{g} \in L^1(\mathbb{R}^+)$.

Then

$$(T_k f, g^\dagger) = \int_{\mathbb{R}^+} k(u)(fg)\hat{}(u)\, d\mu(u).$$

<u>Proof.</u> We have

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}^+} |\, k(x+y)\hat{f}(y)\hat{g}(x)\,|\, d\mu(y)d\mu(x)$$

$$= \int_{\mathbb{R}^+} \left(\sum_{n\geq 1} \int_{n\delta}^{(n+1)\delta} |k(x+y)\hat{f}(y)|\, d\mu(y) \right) |\hat{g}(x)|\, d\mu(x)$$

$$\leq \int_{\mathbb{R}^+} \left(\sum_{n\geq 1} \int_{n\delta}^{(n+1)\delta} |k(x+y)|\, \frac{c}{n\delta}\, d\mu(y) \right) |\hat{g}(x)|\, d\mu(x).$$

But the sequence $\left\{\int_{n\delta}^{(n+1)\delta} |k(x+y)|\, d\mu(y)\right\}_{n\geq 1}$ is square-summable so by Cauchy-Schwartz and 5.4.10

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}^+} |\, k(x+y)\hat{f}(y)\hat{g}(x)\,|\, d\mu(y)d\mu(x)$$

$$\leq \int_{\mathbb{R}^+} \left(\sum_{n\geq 1} \left(\int_{n\delta}^{(n+1)\delta} |k(x+y)|\, d\mu(y) \right)^2 \right)^{1/2} \left(\sum_{n\geq 1} \frac{c^2}{n^2\delta^2} \right)^{1/2} |\hat{g}(x)|\, d\mu(x)$$

$$\leq M_\delta^{1/2} \left(\sum_{n\geq 1} \frac{c^2}{\delta^2 n^2} \right)^{1/2} \int |\hat{g}(x)|\, d\mu(x)$$

where M_δ is as in 5.4.10. Finally, the assumption that $\hat{g} \in L^1(\mathbb{R}^+)$ ensures that

$$\int_{\mathbb{R}^+} \int_{\mathbb{R}^+} | k(x+y)\hat{f}(y)\hat{g}(x) | \, d\mu(y)d\mu(x) < \infty$$

and consequently that

$$\int_{\mathbb{R}^+\times\mathbb{R}^+} | k(x+y)\hat{f}(y)\hat{g}(x)| \, d\mu(x,y) < \infty. \qquad (**)$$

Recall that $g^\dagger(x) = \overline{g(-x)}$ for all $x \in \mathbb{R}$ so that $(g^\dagger)\hat{}\,(x) = \overline{\hat{g}(x)}$ for all $x \in \mathbb{R}^+$. Thus, by Fubini's Theorem and (**),

$$\begin{aligned}(T_k f,g^\dagger) &= ((T_k f)\hat{}\,,(g^\dagger)\hat{}\,) \\ &= \int_{\mathbb{R}^+} \left(\int_{\mathbb{R}^+} k(x+y)\hat{f}(y) \, d\mu(y)\right)\hat{g}(x) \, d\mu(x) \\ &= \int_{\mathbb{R}^+\times\mathbb{R}^+} k(x+y)\hat{f}(y)\hat{g}(x) \, d\mu(x,y).\end{aligned}$$

The measurable transformation $u = x+y$, $v = y$ now gives

$$\begin{aligned}(T_k f,g^\dagger) &= \int_{\mathbb{R}^+\times\mathbb{R}^+} k(u)\hat{f}(v)\hat{g}(u-v) \, d\mu(u,v) \\ &= \int_{\mathbb{R}^+} k(u) \left(\int_{\mathbb{R}^+} \hat{f}(v)\hat{g}(u-v) \, d\mu(v) \right) d\mu(u) \\ &\qquad\qquad \text{by Fubini's Theorem} \\ &= \int_{\mathbb{R}^+} k(u)(\hat{f}*\hat{g})(u) \, d\mu(u) \\ &= \int_{\mathbb{R}^+} k(u)(fg)\hat{}\,(u) \, d\mu(u) \quad \text{as required.}\end{aligned}$$

To complete the proof that BHK is isometrically isomorphic to a subspace of $H^1(\mathbb{R})^*$ we will require the following notation.

<u>5.4.12 Notation.</u> a) We denote by $\mathcal{S}$ the Schwartz class of infinitely differentiable functions $f : \mathbb{R} \to \mathbb{C}$ for which

$$\sup_{x\in\mathbb{R}} (1 + |x|)^m |(D^n f)(x)| < \infty$$

for all $m,n \in \mathbb{Z}^+$. ([BL, p 131]).

b) For $t > 0$, P_t will denote the Poisson Kernel on $\mathbb{R}$ given by $P_t(x) = K \dfrac{t}{t^2+|x|^2}$, where K is chosen (independently of r) such that

$$\int_{\mathbb{R}} P_r(t)\, d\mu(t) = 1$$

c) Let f be the analytic extension to the upper half-plane of some $\tilde{f} \in H^p(\mathbb{R})$ $(1 \leq p \leq \infty)$ and let $\alpha = \{a_n\}_{\substack{n\in\mathbb{Z}^+ \\ n\geq 0}}$ be the zeros of f with $\alpha_n \neq i$ for all $n \in \mathbb{Z}^+$. Then we denote by B_α, the Blaschke product on the upper half-plane,

$$B_\alpha(z) = \frac{z-i}{z+i} \prod_n \frac{|\alpha_n^2+1|}{\alpha_n^2+1} \frac{z-\alpha_n}{z-\overline{\alpha_n}}$$

[DUR, p 191].

d) For $f \in H^2(\mathbb{R})$ and $\delta > 0$ we denote by f_δ the $H^2(\mathbb{R})$ function defined at $x \in \mathbb{R}$ by $f_\delta(x) = e^{i\delta x}f(x)$.

<u>5.4.13 Theorem.</u> The Banach space BHK is isometrically isomorphic to a subspace of $H^1(\mathbb{R})^*$.

Before the proof of 5.4.13 we give a sketch of its underlying strategy. We will show first that when $h \in H^1(\mathbb{R})\cap\mathscr{S}$, the canonical factorisation of $h = fg$ for some $f,g \in H^2(\mathbb{R})$ with $\| f \|_2 = \| g \|_2 = \| h \|_1^{1/2}$ must have $f,g \in H^1(\mathbb{R})$. Since $\hat{f},\hat{g}$ are then bounded we can show that for $t,\delta > 0$ P_t*f_δ and P_t*g satisfy the conditions (i),(ii) and (iii) of 5.4.11. Thus by 5.4.11, when $k \in$ BHK

$$(T_k(P_t*f_\delta),(P_t*g)^\dagger) = \int_{\mathbb{R}^+} k(u)[(P_t*f_\delta)(P_t*g)]\hat{\ }(u)\, d\mu(u).$$

Then since $(P_t*f_\delta)(P_t*g) = P_t*f_\delta g$ and since $P_t*f_\delta \longrightarrow f$ and $P_t*g \longrightarrow g$ as $t,\delta \longrightarrow 0$ we obtain

$$(T_k f,g) = \lim_{t,\delta\to 0} \int_{\mathbb{R}^+} k(u)(P_t*f_\delta g)\hat{\ }(u)\, d\mu(u) \qquad (*)$$

The right-hand side of (*) depends only on the product fg = h so we can take this expression as our definition of a <u>linear</u> functional F_k at $h \in H^1(\mathbb{R})\cap\mathcal{S}$. Then $(*)$ ensures that $\| F_k \| \leq \| k \|_{BHK}$ on the dense subspace $H^1(\mathbb{R})\cap\mathcal{S}$ of $H^1(\mathbb{R})$ so we extend F_k to $\tilde{F}_k$ on $H^1(\mathbb{R})$ with $\| \tilde{F}_k \| \leq \| k \|_{BHK}$. Finally we show that $\| k \|_{BHK} \leq \| \tilde{F}_k \|$ and the proof is complete.

<u>Proof of 5.4.13.</u> Let $k \in BHK$ and let $h \in H^1(\mathbb{R})\cap\mathcal{S}$. Assuming that h is not identically zero we let $\alpha = \{\alpha_n\}_{n\geq 1}$ denote the zeros of $\tilde{h}$, the analytic extension of h to the upper half-plane. Then the function $\tilde{h}/B_\alpha$ has an analytic square root $(\tilde{h}/B_\alpha)^{1/2}$ and by setting $\tilde{f} = B_\alpha(h/B_\alpha)^{1/2}$ and $\tilde{g} = (h/B_\alpha)^{1/2}$ we obtain the factorisation $h = fg$ with $f,g \in H^2(\mathbb{R})$ and $\| f \|_2 = \| g \|_2 = \| h \|_1^{1/2}$. ([HO]). Moreover, since $|B_\alpha(x)| = 1$ for a.e. $x \in \mathbb{R}$ we have

$$|f(x)| = |g(x)| = |h(x)|^{1/2} \qquad (**)$$

for a.e. $x \in \mathbb{R}$. But $h \in \mathcal{S}$ ensures that $|h|^{1/2} = O(1/x^2)$ as $|x| \longrightarrow \infty$ and consequently that $\int_{\mathbb{R}^+} |h(x)|^{1/2}\, d\mu(x) < \infty$. Thus, by $(**)$ we see that $f,g \in H^1(\mathbb{R})$.

We claim that for each $t,\delta > 0$

i) $\operatorname{supp}\, [(P_t*f_\delta)^\wedge] \subseteq [\delta,\infty)$;

ii) $(P_t*f_\delta)^\wedge = O(1/x)$

and iii) $(P_t*g)^\wedge \in L^1(\mathbb{R}^+)$.

Part (i) is due to the fact that

$$(P_t*f_\delta)^\wedge(x) = e^{-tx}\, \hat{f}(x-\delta) \qquad (***)$$

for $x \in \mathbb{R}^+$. For part (ii) we note that $f \in H^1(\mathbb{R})$ implies that $\hat{f}$ is bounded and hence by $(***)$ that
$$|(P_t*f_\delta)^\wedge(x)| \leq \| \hat{f}_\delta \|_\infty e^{-tx} = O(1/x).$$

Part (iii) follows similarly since $g \in H^1(\mathbb{R})$ ensures that

$$\int_{\mathbb{R}^+} |(P_t*g)\hat{}(x)|\ d\mu(x) = \int_{\mathbb{R}^+} |e^{-tx}\hat{g}(x)\ |\ d\mu(x)$$
$$\le \|\ \hat{g}\ \|_\infty \int_{\mathbb{R}^+} e^{-tx}\ d\mu(x)$$
$$< \infty.$$

Hence by 5.4.11, for each $t,\delta > 0$

$$(T_k(P_t*f_\delta),(P_t*g)^\dagger) = \int_{\mathbb{R}^+} k(u)[(P_t*f_\delta)(P_t*g)]\hat{}(u)\ d\mu(u).$$

Now for $\delta > 0$ let $\tilde{f}_\delta$ and $\tilde{g}$ denote the analytic extensions to the upper half-plane of f_δ and g respectively. Then for $t > 0$ and $x \in \mathbb{R}$ $(P_t*f_\delta)(x) = \tilde{f}_\delta(x+it)$ and $(P_t*g)(x) = \tilde{g}(x+it)$

so that

$$[(P_t*f_\delta)(P_t*g)](x) = \tilde{f}_\delta(x+it)\tilde{g}(x+it)$$
$$= (\tilde{f}_\delta\tilde{g})(x+it)$$
$$= (P_t*(f_\delta g))(x).$$

Thus, since $P_t*f_\delta \longrightarrow f$ in $H^2(\mathbb{R})$ as $t,\delta \longrightarrow 0$ and since $P_t*g \longrightarrow g$ in $H^2(\mathbb{R})$ as $t \longrightarrow 0$, we have

$$(T_kf,g^\dagger) = \lim_{t,\delta\to0} (T_k(P_t*f_\delta),(P_t*g))$$
$$= \lim_{t,\delta\to0} \int_{\mathbb{R}^+} k(u)[P_t*(f_\delta g)]\hat{}(u)d\mu(u).$$

Now for any $h \in H^1(\mathbb{R})\cap\mathcal{S}$ with factorisation $h = fg$ for some $f,g \in H^2(\mathbb{R})$ we define

$$F_kh = \lim_{t,\delta\to0} \int_{\mathbb{R}^+} k(u)(P_t*(f_\delta g))\hat{}(u)\ d\mu(u).$$

Since the integral is independent of the choice of factorisation we see that F_k is a <u>linear</u> functional on $H^1(\mathbb{R})\cap\mathcal{S}$ and moreover that

$$|F_kh| = |\ (T_kf,g^\dagger)|$$
$$\le \|\ k\ \|_{BHK}\ \|\ f\ \|_2\ \|\ g\ \|_2.$$

By choosing f,g with $\|\ f\ \|_2 = \|\ g\ \|_2 = \|\ h\ \|_1^{1/2}$ we have

$$|F_kh| \le \|\ k\ \|_{BHK}\ \|\ h\ \|_1$$

and hence $\|\ F_k\ \| \le \|\ k\ \|_{BHK}$ on $H^1(\mathbb{R})\cap\mathcal{S}$.

Since $\mathcal{S}$ is dense in $L^2(\mathbb{R}^+)$ and the Fourier transform maps $\mathcal{S}$ onto itself ([ZE, 7.3]) we know by considering the inverse Fourier transform on $L^2(\mathbb{R}^+)$ that $\mathcal{S}\cap H^2(\mathbb{R})$ is dense in $H^2(\mathbb{R})$. Thus if $h \in H^1(\mathbb{R})$ we can factorise $h = fg$ ($f,g \in H^2(\mathbb{R})$) and approximate f,g by sequences $\{f_n\}_{n\in\mathbb{Z}^+}, \{g_n\}_{n\in\mathbb{Z}^+}$ in $H^2(\mathbb{R})\cap\mathcal{S}$. The sequence $\{f_n g_n\}_{n\in\mathbb{Z}^+}$ is then in $H^1(\mathbb{R})\cap\mathcal{S}$ and converges to h in $H^1(\mathbb{R})$. Hence, $H^1(\mathbb{R})\cap\mathcal{S}$ is dense in $H^1(\mathbb{R})$ and we may continuously extend F_k to $\widetilde{F}_k$ on $H^1(\mathbb{R})$ with

$$\| \widetilde{F}_k \| = \| F_k \| \le \| k \|_{BHK}.$$

To show that $\| \widetilde{F}_k \| = \| k \|_{BHK}$ we note that whenever $f,g \in H^2(\mathbb{R})\cap\mathcal{S}$, the product $f.g^\dagger \in H^1(\mathbb{R})\cap\mathcal{S}$ and so

$$\begin{aligned} |(T_k f,g)| &= |F_k(fg^\dagger)| \\ &\le \| F_k \| \, \| fg^\dagger \|_1 \\ &\le \| F_k \| \, \| f \|_2 \, \| g \|_2. \end{aligned}$$

Thus, $\| k \|_{BHK} \le \| F_k \| = \| \widetilde{F}_k \|$, as required.

Section 5.5 A Final Estimate of $\| f(A) \|_{B(\mathcal{H})}$.

To complete this chapter we will find a final upper bound on the norm of the operator $f(A)$ for a generator of a uniformly bounded C_0-semigroup, A and for f in a class of infinitely differentiable functions on $\mathbb{R}$. Indeed, we will show that for a Schwartz class function f whose Fourier transform is compactly supported away from zero in $\mathbb{R}^+$, the operator norm of $f(A)$ is bounded by the product of a constant (depending only on the generator A), a logarithmic term and the supremum norm of f on $\mathbb{R}$. As in the discrete case, the logarithmic term depends on the support of $\hat{f}$.

Throughout the section we will use the notation of 5.4.12 for the Scwartz class of functions, $\mathcal{S}$, the Poisson Kernels $\{P_t\}_{t>0}$ and the functions f_δ for $\delta > 0$. We will also require the following.

<u>5.5.1 Notation.</u> We will denote by X the image in $H^1(\mathbb{R})^*$ of BHK under the isometric isomorphism of 5.4.13.

We will define, in a distributional sense, the convolution of an $H^2(\mathbb{R})$ function with a linear functional in X and consequently a normed linear space of multipliers on X. Using these definitions we can show that a compactly supported multiplier on BHK is the Fourier transform of some multiplier on X having equal multiplier norm. This allows us to replace the estimate of $\| f(A) \|$ involving multipliers on BHK (5.4.9) by one involving multipliers on X.

Following Peller's method as described in Chapter 2 we define a projective convolution space $H^1(\mathbb{R})\hat{\circledast}X$ and show that any multiplier g on X naturally defines a linear functional Φ_g on $H^1(\mathbb{R})\hat{\otimes}X$ with norm less than or equal to its multiplier norm. Finally, using the Besov space $B^0_{\infty,1}(\mathbb{R})$, we show that any $f \in \mathcal{S}$ with supp $\hat{f} \subseteq [1/N,N]$ for some $N > 1$ can be considered as an element of $H^1(\mathbb{R})\hat{\circledast}X$ and that there exists constants c_1,c_2 such that for any such f we have

$$\| f \|_{\hat{\otimes}} \le c_1c_2 \log N \| f \|_\infty .$$

Consequently, when $f \in \mathcal{S}$ has supp $f \subseteq [1/N,N]$ for some $N > 1$ and when g is a multiplier on X with norm less than or equal to 1,

$$\left| \int_{1/N}^{N} \hat{f}(t)\hat{g}(t)\, d\mu(t) \right| = |\Phi_g(f)| \le \| f \|_{\hat{\otimes}} \le c_1c_2 \log N \| f \|_\infty$$

and the final bound on $\| f(A) \|$ follows.

<u>5.5.2 Definitions.</u> a) Suppose that $g \in H^2(\mathbb{R})$ and $F \in H^1(\mathbb{R})^*$. If there exists a constant d such that $|F(g*h)| \le d \| h \|_1$ for all $h \in H^1(\mathbb{R})\cap\mathcal{S}$ then we define $g*F$ at $h \in H^1(\mathbb{R})\cap\mathcal{S}$ by $(g*F)(h) = F(g*h)$.

Since $g*F$ is continuous on $H^1(\mathbb{R})\cap\mathcal{S}$ we will also denote by $g*F$ its norm-preserving extension to $H^1(\mathbb{R})$.

b) Let $g \in H^2(\mathbb{R})$. If, for each $F \in X$, $g*F$ exists and is in X then g is a <u>multiplier on</u> X.

c) The linear space of all multipliers on X is denoted by M(X) and is normed by

$$\| g \|_{M(X)} = \sup \left\{ \| g*F \|_{H^1(\mathbb{R})^*} : F \in X, \| F \| \le 1 \right\}$$

<u>Remarks</u> <u>1.</u> Note that in 5.5.2(a) the convolution $g*h$ of $g \in H^2(\mathbb{R})$ with $h \in H^1(\mathbb{R})\cap\mathcal{S}$ exists since $h \in H^2(\mathbb{R})$.

<u>2.</u> By the Closed Graph Theorem ([CON, III.12.6]), if $g*F \in X$ for all $F \in X$ then convolution with g is a continuous operator on X and the norm $\| g \|_{M(X)}$ is finite.

<u>5.5.3 Lemma.</u> Let $h \in L^\infty(\mathbb{R}^+)$ have supp $h \subseteq [0,N]$ for some $N > 1$. Then $h \in M(BHK)$ if and only if $h = \hat{g}$ for some $g \in M(X)$ and in this case $\| h \|_{M(BHK)} = \| g \|_{M(X)}$.

<u>Proof.</u> Suppose first that $h \in M(BHK)$. Since $h \in L^2(\mathbb{R}^+)$, it is clear that $h = \hat{g}$ for some $g \in H^2(\mathbb{R})$. Now if $\phi \in H^1(\mathbb{R})\cap\mathcal{S}$ then $\phi \in H^2(\mathbb{R})$ and so for any $t,\delta > 0$, $P_t*\phi_\delta \in H^2(\mathbb{R})$ and $P_t*\phi_\delta*g \in H^1(\mathbb{R})$. Moreover,

$$\left.\begin{aligned} (P_t*\phi_\delta)\hat{}\, h &= (P_t*\phi_\delta)\hat{}\,\hat{g} \\ &= (P_t*\phi_\delta*g)\hat{} \\ &= (P_t*(\phi*g)_\delta)\hat{}\,. \end{aligned}\right\} \qquad (*)$$

Let $k \in$ BHK. Let $\tilde{F}_k, \tilde{F}_{hk}$ denote the images of k and hk respectively, under the isometric isomorphism of 5.4.13. Then for any $\phi \in H^1(\mathbb{R})\cap\mathcal{S}$, since $g \in H^2(\mathbb{R})$ we have $g*\phi \in \mathcal{S}$ and

$$\tilde{F}_k(g*\phi) = \lim_{t,\delta\to 0} \int_{\mathbb{R}^+} k(u)(P_t*(g*\phi)_\delta)\hat{\ }(u)\, d\mu(u)$$

By (*),

$$\tilde{F}_k(g*\phi) = \lim_{t,\delta\to 0} \int_{\mathbb{R}^+} k(u)(P_t*\phi_\delta)\hat{\ }(u)h(u)\, d\mu(u)$$

$$= \tilde{F}_{hk}(\phi).$$

Hence, by 5.4.13,

$$|F_k(g*\phi)| = |F_{hk}(\phi)|$$
$$\leq \| h \|_{M(BHK)} \| k \|_{BHK} \| \phi \|_1.$$

Thus we see that $g*F_k$ exists and equals F_{hk} and consequently that $g \in M(X)$ with $\| g \|_{M(X)} = \| h \|_{M(BHK)}$.

Now suppose that $h = \hat{g}$ for some $g \in M(X)$. We must show that for all $k \in$ BHK, hk is a bounded Hankel Kernel.

Let $k \in$ BHK. By 5.4.5 there exists an increasing sequence $\{X_n\}_{n\geq 1}$ of measurable subsets of $\mathbb{R}^+$ satisfying

i) $\mu(X_n) < \infty$ for each $n \geq 1$;

ii) $\mu(\mathbb{R}^+ \setminus \bigcup_{n\geq 1} X_n) = 0$

and iii) $\int_{X_n \times [0,m]} |k(x,y)|\, d\mu(x,y) < \infty$ for every $n,m \geq 1$.

Now if $f_1 \in H^2(\mathbb{R})\cap\mathcal{S}$ has $\hat{f}_1$ bounded and supp $\hat{f}_1 \subseteq [0,m]$ for some $m \geq 1$, and if $f_2 \in H^2(\mathbb{R})\cap\mathcal{S}$ has $\hat{f}_2$ bounded and supp $\hat{f}_2 \subseteq X_n$ for some $n \geq 1$, then by working as in the proof of 5.4.6 we have

$$\int_{\mathbb{R}^+\times\mathbb{R}^+} |k(x+y)\hat{f}_1(y)\hat{f}_2(x)|\, d\mu(x,y) < \infty.$$

Similarly if $t,\delta > 0$

$$\int_{\mathbb{R}^+\times\mathbb{R}^+} |k(x+y)(P_t*(f_1)_\delta)\hat{\ }(y)(P_t*f_2)\hat{\ }(x)|\, d\mu(x,y) < \infty.$$

Since $\hat{g} \in L^\infty(\mathbb{R}^+)$ it then follows that for $t,\delta > 0$

$$\int_{\mathbb{R}^+\times\mathbb{R}^+} |\hat{g}(x+y)k(x+y)\hat{f}_1(y)\hat{f}_2(x)|\, d\mu(x,y) \leq \infty \qquad (**)$$

and

$$\int_{\mathbb{R}^+\times\mathbb{R}^+} |\hat{g}(x+y)k(x+y)(P_t*(f_1)_\delta)\hat{\ }(y)(P_t*g)\hat{\ }(x)|\, d\mu(x,y) < \infty.$$

But $(P_t*(f_1)_\delta)\hat{\ }$ and $(P_t*f_2)\hat{\ }$ converge pointwise a.e. to f_1 and f_2 respectively as $t,\delta \longrightarrow 0$, so by dominated convergence

$$\int_{\mathbb{R}^+\times\mathbb{R}^+} \hat{g}(x+y)k(x+y)(P_t*(f_1)_\delta)\hat{\ }(y)(P_t*g)\hat{\ }(x)\, d\mu(x,y) < \infty.$$

$$\longrightarrow \int_{\mathbb{R}^+\times\mathbb{R}^+} \hat{g}(x+y)k(x+y)\hat{f}_1(y)\hat{f}_2(x)\, d\mu(x,y) \le \infty \text{ as } t,\delta \longrightarrow 0.$$

Hence we have

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}^+} \hat{g}(x+y)k(x+y)\hat{f}_1(y)\hat{f}_2(x)\, d\mu(y)\, d\mu(x)$$

$$= \int_{\mathbb{R}^+\times\mathbb{R}^+} \hat{g}(x+y)k(x+y)\hat{f}_1(y)\hat{f}_2(x)\, d\mu(x,y)$$

by (**) and Fubini's Theorem.

$$= \lim_{t,\delta\to 0} \int_{\mathbb{R}^+\times\mathbb{R}^+} \hat{g}(x+y)k(x+y)(P_t*(f_1)_\delta)\hat{\ }(y)(P_t*f_2)\hat{\ }(x)\, d\mu(x,y)$$

$$= \lim_{t,\delta\to 0} \int_{\mathbb{R}^+\times\mathbb{R}^+} \hat{g}(u)k(u)(P_t*(f_1)_\delta)\hat{\ }(v)(P_t*f_2)\hat{\ }(u-v)\, d\mu(u,v)$$

by the transformation $u = x+y$, $v = y$

$$= \lim_{t,\delta\to 0} \int_{\mathbb{R}^+} \hat{g}(u)k(u)[(P_t*(f_1)_\delta)\hat{\ }*(P_t*f_2)\hat{\ }](u)\, d\mu(u)$$

$$= \lim_{t,\delta\to 0} \int_{\mathbb{R}^+} \hat{g}(u)k(u)[(P_t*(f_1)_\delta)(P_t*f_2)]\hat{\ }(u)\, d\mu(u).$$

But for each $t,\delta > 0$

$$(P_t*(f_1)_\delta)(P_t*f_2) = P_t*((f_1)_\delta f_2)$$
$$= P_t*(f_1f_2)_\delta.$$

Thus,

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}^+} \hat{g}(x+y)k(x+y)\hat{f}_1(y)\hat{f}_2(x)\, d\mu(y)\, d\mu(x)$$

$$= \lim_{t,\delta\to 0} \int_{\mathbb{R}^+} \hat{g}(u)k(u)(P_t*(f_1f_2)_\delta)\hat{\ }(u)\, d\mu(u)$$

$$= \lim_{t,\delta\to 0} \int_{\mathbb{R}^+} k(u)(P_t*(f_1f_2)_\delta*g)\hat{}(u)\, d\mu(u)$$

$$= \tilde{F}_k(g*(f_1f_2)) \qquad \text{by the definition of } \tilde{F}_k$$

$$= (g*\tilde{F}_k)(f_1f_2).$$

Hence,

$$\left| \int_{\mathbb{R}^+}\int_{\mathbb{R}^+} \hat{g}(x+y)k(x+y)\hat{f}_1(y)\hat{f}_2(x)\, d\mu(y)\, d\mu(x)\right|$$

$$\le \| g \|_{M(X)} \| k \|_{BHK} \| f_1 \|_2 \| f_2 \|_2$$

and since arbitrary $f_1', f_2' \in H^2(\mathbb{R})$ can be approximated by such f_1, f_2 we conclude that $\hat{g}k \in BHK$ with $\| gk \|_{BHK} \le \| g \|_{M(X)} \| k \|_{BHK}$.

We now have the following upper bound on $\| f(A) \|$.

<u>5.5.4 Corollary.</u> Let $f \in H^1(\mathbb{R})$ have supp $\hat{f} \subseteq [0,N]$ for some $N > 0$. Then

$$\| f(A) \| \le \frac{1}{\sqrt{2\pi}} K_G c^2 \sup \left\{ \left| \int_0^N \hat{f}(t)\hat{g}(t)\, d\mu(t) \right| : \| g \|_{M(X)} \le 1 \right\}$$

<u>Proof.</u> We have from 5.4.9

$$\| f(A) \| \le \frac{1}{\sqrt{2\pi}} K_G c^2 \sup \left\{ \left| \int_0^N \hat{f}(t)h(t)\, d\mu(t) \right| : \| h \|_{M(BHK)} \le 1 \right\}.$$

Since we only need to consider the supremum over $h \in M(BHK)$ with $\| h \|_{M(BHK)} \le 1$ and supp $h \subseteq [0,N]$ for some $N > 0$, the result follows from 5.5.3.

The next estimate of $\| f(A) \|$ will require the following 'projective convlution' space of linear functionals on $H^1(\mathbb{R})$.

5.5.5 Definition. We define the linear space $H^1(\mathbb{R})*X$ to consist of all finite sums of convolutions $F = \sum_{n=-m}^{m} h_n * F_n$ where $h_n \in H^1(\mathbb{R})$ and $F_n \in X$ ($M \in \mathbb{Z}^+$, $-m \le n \le m$).

For such F we define the projective norm

$$\| F \|_{\hat{*}} = \inf \left\{ \sum_{n=-m}^{m} \| h_n \|_1 \| F_n \|_{H^1(\mathbb{R})^*} : F = \sum_{n=-m}^{m} h_n * F_n \text{ in } H^1(\mathbb{R})^*, \quad h_n \in H^2(\mathbb{R}),\ F_N \in X \right\}$$

We denote by $H^1(\mathbb{R})\hat{*}X$, the completion of $H^1(\mathbb{R})*X$ with respect to the norm $\| \cdot \|_{\hat{\otimes}}$.

Remarks. 1. For $h, h' \in H^1(\mathbb{R})$ and $F \in X$ we have $h*h' \in H^1(\mathbb{R})$ with $\| h*h' \|_1 \le \| h \|_1 \| h' \|_1$, so that $|F(h*h')| \le \| F \| \| h \|_1 \| h' \|_1$. Thus, $h*F \in H^1(\mathbb{R})^*$ as required.

2. It is easy to check that $\| \cdot \|_{\hat{*}}$ is a well-defined norm on $H^1(\mathbb{R})*X$.

5.5.6 Lemma. Let $g \in M(X)$. Define Φ_g on a simple element $h*F$ of $H^1(\mathbb{R})*X$ by

$$\Phi_g(h*F) = (g*F)(h).$$

and on $\sum_{n=-m}^{m} h_n * F_n \in H^1(\mathbb{R})*X$ by

$$\Phi_g\left(\sum_{n=-m}^{m} h_n * F_n \right) = \sum_{-m}^{m} \Phi_g(h_n * F_n).$$

Then Φ_g is continuous with respect to $\| \cdot \|_{\hat{*}}$ and extends to a linear functional $\tilde{\Phi}_g$ on $H^1(\mathbb{R})\hat{*}X$ with $\| \tilde{\Phi}_g \|_{(H^1(\mathbb{R})\hat{\otimes}X)^*} \le \| g \|_{M(X)}$.

Proof. For $h \in H^1(\mathbb{R})$ and $F \in X$

$$\begin{aligned} |\Phi_g(h*F)| &= |(g*F)(h)| \\ &\le \| g*F \|_{H^1(\mathbb{R}^+)^*} \| h \|_1 \\ &\le \| g \|_{M(X)} \| F \|_{H^1(\mathbb{R})^*} \| h \|_1 . \end{aligned}$$

Then by the definition of $\| \cdot \|_{\hat{\otimes}}$ it is clear that Φ_g is continuous on $H^1(\mathbb{R})*X$ with $\| \Phi_g \| \le \| g \|_{M(X)}$. We must check that Φ_g is well-defined. Note first that if $\{g_k\}_{k\in\mathbb{Z}^+}$ is a sequence in $H^2(\mathbb{R})$ converging in $\| \cdot \|_2$ to g then for any $h \in H^2(\mathbb{R})$, $g_k*h \longrightarrow g*h$ in $H^1(\mathbb{R})$ as $k \longrightarrow \infty$. Now suppose that $h_n \in H^1(\mathbb{R})\cap\mathscr{S}$ and $F_n \in X$ $(-m \le n \le m)$ are such that $\sum_{n=-m}^{m} h_n*F_n = 0 \in H^2(\mathbb{R})*X$. Then by definition of h_n*F_n,

$$\sum_{n=-m}^{m} F_n(h*h_n) = 0 \tag{$*$}$$

for each $h \in H^1(\mathbb{R})\cap\mathscr{S}$.

Now

$$\begin{aligned} \Phi_g\left(\sum_{n=-m}^{m} h_n*F_n\right) &= \sum_{n=-m}^{m} F_n(g*h_n) \\ &= \lim_{k\to\infty} \sum_{n=-m}^{m} F_n(g_k*h_n) \end{aligned}$$

where g_k $(k \in \mathbb{Z}^+)$ is the kth partial sum $\sum_{j=0}^{k} \hat{g}(j)e_j$. Moreover, for each $k \in \mathbb{Z}^+$, $g_k \in H^1(\mathbb{R})\cap\mathscr{S}$, so $\sum_{n=-m}^{m} F_n(g_k*h_n) = 0$ by $(*)$. Hence,

$$\Phi_g\left(\sum_{n=-m}^{m} h_n*F_n\right) = 0$$

as required.

Finally, we extend Φ_g by continuity to give the required $\tilde{\Phi}_g$.

To produce the final bound on $\| f(A) \|$ we will show that if $f \in \mathcal{S}$ with supp $\hat{f} \subseteq [0,N]$ for some $N > 0$ then f may be considered as an element of $H^1(\mathbb{R})\hat{*}X$ with $\| f \|_{\hat{*}}$ bounded above by $c_1 \log N \| f \|_\infty$ for some constant c_1, independent of f. As an intermediate stage we will show that such f must be in the Besov space $B^0_{\infty,1}(\mathbb{R})$. For the background, motivation and full definition of the scale $B^s_{p,q}(\mathbb{R})$ $(-\infty \leq -p,q \leq \infty,\ s \in \mathbb{R}^+)$ we refer the reader to [T, pp 38,46 ; p 238]. Here we follow the more succinct definitions given in [BL, pp 135,146].

<u>5.5.7 Lemma.</u> There exists $\phi \in \mathcal{S}$ satisfying

i) supp $\phi = [\frac{1}{2},2]$;

ii) $\phi > 0$ on $(\frac{1}{2},2)$

and iii) $\displaystyle\sum_{k=-\infty}^{\infty} \phi(x/2^k) = 1$ for every $x > 0$.

<u>Proof.</u> Let f be any Schwartz class function satisfying (i) and (ii). Then if $F(x) = \displaystyle\sum_{k=-\infty}^{\infty} \phi(x/2^k)$ for $x > 0$, we have, for each $x > 0$

$$F(x) = f(x/2^k) + f(x/2^{k-1}) > 0$$

for some $k \in \mathbb{Z}$.

Put $\phi = f/F$. Then $\phi \in \mathcal{S}$ and satisfies (i) and (ii). Moreover, for $x > 0$,

$$\sum_{k=-\infty}^{\infty} \phi(x/2^k) = \sum_{k=-\infty}^{\infty} \frac{f(x/2^k)}{\sum_{j=-\infty}^{\infty} f(x/2^{j+k})}$$

$$= \frac{\sum_{k=-\infty}^{\infty} f(x/2^k)}{\sum_{j=-\infty}^{\infty} f(x/2^j)}$$

$$= 1$$

and so (iii) holds.

<u>**5.5.8 Notation.**</u> a) For $f \in L^{\infty}(\mathbb{R})$, let $R(f)$ denote the restriction to $H^1(\mathbb{R})$ of the linear functional on $L^1(\mathbb{R})$ whose value at $g \in L^1(\mathbb{R})$ is given by $\int_{\mathbb{R}} g(x)f(-x)\, d\mu(x)$.

b) For each $n \in \mathbb{Z}$ we denote by ψ_n the unique Schwartz class function satisfying

$$\hat{\psi}_n(x) = \phi(x/2^n) \quad \text{for all } x > 0.$$

<u>**5.5.9 Definition.**</u> The (homogeneous) <u>Besov space</u> $B^0_{\infty,1}(\mathbb{R})$ consists of those $f \in \mathcal{S}$ for which $\operatorname{supp} \hat{f} \in \mathbb{R}^+$ and

$$\sum_{n\in\mathbb{Z}} \| f*\psi_n \|_{\infty} < \infty.$$

We define a norm on $B^0_{\infty,1}(\mathbb{R})$ by

$$\| f \|^0_{\infty,1} = \sum_{n\in\mathbb{Z}} \| f*\psi_n \|_{\infty}.$$

We note that $B^0_{\infty,1}(\mathbb{R})$ is a Banach space with respect to $\| \cdot \|^0_{\infty,1}$ and that for $f \in B^0_{\infty,1}(\mathbb{R})$ $\sum_{n\in\mathbb{Z}} f*\psi_n$ converges in $L^{\infty}(\mathbb{R})$ to f.

5.5.10 Lemma. Let $f \in \mathcal{S}$ have supp $\hat{f} \subseteq [\frac{1}{N},N]$ for some $N > 1$. Then $f \in B^0_{\infty,1}(\mathbb{R})$ with $\| f \|^0_{\infty,1} \leq c_1 \log N \| f \|_\infty$ for some constant c_1 independent of f.

Proof. Since supp $\hat{\psi}_n \subseteq [2^{n-1},2^{n+1}]$ for all $n \in \mathbb{Z}$, we have $f*\psi_n = 0$ whenever $|n| > \log_2 n + 1$.

Note that, for each $n \in \mathbb{Z}$ and $x > 0$, $\psi_n(x) = 2^n\psi_0(2^n x)$ so that

$$\begin{aligned} \| \psi_n \|_1 &= 2^n \int_{\mathbb{R}} \psi_0(2^n x)\, dx \\ &= \| \psi_0 \|_1 . \end{aligned}$$

Thus, if we put $m = (\log_2 N)+1$,

$$\begin{aligned} \sum_{n\in\mathbb{Z}} \| f*\psi_n \|_\infty &= \sum_{n=-m}^{m} \| f*\psi_n \|_\infty \\ &\leq \sum_{n=-m}^{m} \| f \|_\infty \| \psi_n \|_1 \\ &= (2m+1) \| f \|_\infty \| \psi_0 \|_1 \\ &\leq c_1 \log N \| f \|_\infty \end{aligned}$$

for some constant c_1.

5.5.11 Lemma. If $f \in B^0_{\infty,1}(\mathbb{R})$ then $R(f) \in H^1(\mathbb{R})\hat{*}X$ with

$$\| R(f) \|_{\hat{*}} \leq c_2 \| f \|^0_{\infty,1}$$

for some constant c_2 independent of f.

Further, if f also satisfies supp $\hat{f} \subseteq [\frac{1}{N},N]$ for some $N > 1$ then $R(f) \in H^1(\mathbb{R})*X$.

Proof. Firstly, for each $n \in \mathbb{Z}$, we put

$$Q_n = \psi_{n-1} + \psi_n + \psi_{n+1} .$$

Then $\psi_n = \psi_n*Q_n$ for every $n \in \mathbb{Z}$ and $\| Q_n \|_1 \leq 3 \| \psi_0 \|_1$.

Now let $f \in B^0_{\infty,1}(\mathbb{R})$ so that

$$f = \sum_{n\in\mathbb{Z}} f*\psi_n = \sum_{n\in\mathbb{Z}} (f*\psi_n)*Q_n \tag{$*$}$$

$$= \sum_{n\in\mathbb{Z}} Q_n*(f*\psi_n).$$

For each $n \in \mathbb{Z}$ and $h \in H^1(\mathbb{R})$

$$< h,R(Q_n*(f*\psi_n)) > = \int_{\mathbb{R}} h(t)(Q_n*(f*\psi_n))(-t)\, d\mu(t)$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} h(t)Q_n(s)(f*\psi_n)(-t-s)\, d\mu(s)\, d\mu(t)$$

$$= \int_{\mathbb{R}\times\mathbb{R}} h(t)Q_n(s)(f*\psi_n)(-t-s)\, d\mu(s,t)$$

by Fubini's Theorem.

By the transformation of u=t and v=s+t,

$$< h,R(Q_n*(f*\psi_n)) > = \int_{\mathbb{R}\times\mathbb{R}} h(u)Q_n(v-u)(f*\psi_n)(-v)\, d\mu(u,v)$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} h(u)Q_n(v-u)(f*\psi_n)(-v)\, d\mu(v)\, d\mu(u)$$

by Fubini's Theorem

$$= \int_{\mathbb{R}} (h*Q_n)(v)(f*\psi_n)(-v)\, d\mu(v)$$

$$= < h*Q_n, R(f*\psi_n) >$$

$$= < h, Q_n*R(f*\psi_n) > .$$

Hence,

$$R(Q_n*(f*\psi_n)) = Q_n*R(f*\psi_n)$$

and we have from ($*$) that

$$Rf = \sum_{n\in\mathbb{Z}} R(Q_n*(f*\psi_n)) \tag{$**$}$$

$$= \sum_{n\in\mathbb{Z}} Q_n*R(f*\psi_n).$$

Now since $f*\psi_n \in H^1(\mathbb{R})\cap H^\infty(\mathbb{R})$ and has compactly supported

Fourier transform, its Fourier transform must be in BHK. Indeed we claim the following.

Claim. $(f*\psi_n)^\wedge \in$ BHK and the image of $(f*\psi_n)^\wedge$ under the isometric isomorphism of BHK onto X is $R(f*\psi_n)$.

Proof of Claim. Let $g_1 \in H^2(\mathbb{R})$. Then for every $x \in \mathbb{R}^+$

$$\int_{\mathbb{R}^+} |(f*\psi_n)^\wedge(x+y)\hat{g}_1(y)| \, d\mu(y)$$

$$\leq \int_{2^{n-1}-x}^{2^{n+1}-x} \|(f*\psi_n)^\wedge\|_\infty \, |\hat{g}_1(y)| \, d\mu(y)$$

$$\leq \| f*\psi_n \|_1 \, (3.2^{n-1})^{1/2} \, \| g_1 \|_2$$

$$< \infty.$$

Moreover, if we also have $g_2 \in H^2(\mathbb{R})$ then

$$\left| \int_{\mathbb{R}^+}\int_{\mathbb{R}^+} (f*\psi_n)^\wedge(x+y)\hat{g}_1(y)\hat{g}_2(x) \, d\mu(y) \, d\mu(x) \right|$$

$$\leq \|(f*\psi)^\wedge\|_\infty \int_0^{2n+1} \int_0^{2n+1} |\hat{g}_1(y)\hat{g}_2(x)| \, d\mu(y)d\mu(x)$$

$$\leq \| f*\psi_n \|_1 \, (2n+1) \, \| g_1 \|_2 \, \| g_2 \|_2$$

and we conclude that $(f*\psi_n)^\wedge \in$ BHK with

$$\| (f*\psi_n)^\wedge \|_{BHK} \leq \| f*\psi_n \|_1 (2n+1).$$

Now let $\tilde{F}_{(f*\psi_n)^\wedge}$ denote the image of $(f*\psi_n)^\wedge$ under the isometric isomorphism of 5.4.13. Then if $h \in H^1(\mathbb{R})\cap\mathcal{S}$ has supp $\hat{h} \subseteq [2^{n-1}, 2^{n+1}]$

$$\begin{aligned}\tilde{F}_{(f*\psi_n)^\wedge}(h) &= \lim_{t,\delta\to 0} \int_{\mathbb{R}^+} (f*\psi_n)^\wedge(x)(P_t*h_\delta)^\wedge(x) \, d\mu(x) \\ &= \lim_{t,\delta\to 0} \int_{\mathbb{R}} (f*\psi_n)(-x)(P_t*h_\delta)(x) \, d\mu(x) \\ &= \lim_{t,\delta\to 0} \langle P_t*h_\delta, R(f*\psi_n) \rangle \\ &= \langle h, R(f*\psi_n) \rangle\end{aligned}$$

because $P_t * h_\delta \longrightarrow h$ in $H^1(\mathbb{R})$. Thus, $\widetilde{F}_{(f*\psi_n)^\wedge} = R(f*\psi_n)$ and the claim is proved.

It follows from the claim and (**) that $R(f) \in H^1(\mathbb{R})\hat{*}X$ with

$$\| R(f) \|_{\hat{*}} \le \sum_{n\in\mathbb{Z}} \| Q_n \|_1 \| R(f*\psi_n) \|_{H^1(\mathbb{R})^*}$$

$$\le 3 \| \phi_0 \|_1 \sum_{n\in\mathbb{Z}} \| f*\psi_n \|_\infty$$

$$= 3 \| \phi_0 \|_1 \| f \|^0_{\infty,1}$$

Finally, if we further suppose that supp $\hat{f} \subseteq [\frac{1}{N},N]$ for some $N > 1$ then by putting $m = \log_2 N + 1$ we have

$$f = \sum_{n=-m}^{m} f*\psi_n$$

$$= \sum_{n=-m}^{m} Q_n*(f*\psi_n)$$

and so $R(f) = \sum_{n=-m}^{m} Q_n *R(f*\psi_n) \in H^1(\mathbb{R})*X$.

<u>5.5.12 Theorem.</u> There exists a constant $k > 0$ such that whenever $f \in \mathcal{S}$ has supp $\hat{f} \subseteq [\frac{1}{N},N]$ for some $N > 1$,

$$\| f(A) \| \le k\, c^2 K_G \log N \| f \|_\infty.$$

<u>Proof.</u> Let $f \in \mathcal{S}$ have supp $\hat{f} \subseteq [\frac{1}{N},N]$ for some $N > 1$. Recall from 5.5.4 that

$$\| f(A) \| \le \frac{1}{\sqrt{2\pi}} K_G c^2 \sup \left\{ \left| \int_0^N \hat{f}(t)\hat{g}(t)\, d\mu(t) \right| : \| g \|_{M(X)} \le 1 \right\}$$

By 5.5.10 $f \in B^0_{\infty,1}(\mathbb{R})$ with $\| f \|^0_{\infty,1} \le c_1 \log N \| f \|_\infty$ and then by 5.5.11, $R(f) \in H^1(\mathbb{R})\hat{*}X$ with $\| R(f) \|_{\hat{*}} \le c_2c_1(\log N) \| f \|_\infty$. Indeed, since supp $\hat{f} \subseteq [\frac{1}{N},N]$ we have $R(f) = \sum_{n=-m}^{m} Q_n *R(f*\psi_n) \in H^1(\mathbb{R})*X$ where

$m = \log_2 N + 1$.

Now let $g \in M(X)$ and let $\tilde{\Phi}_g \in (H^1(\mathbb{R})*X)^*$ be the linear functional defined in 5.5.6. Then

$$\int_0^N \hat{f}(t)\hat{g}(t)\, d\mu(t) = \int_0^N \left(\sum_{n=-m}^{m} \hat{Q}_n(t)(f*\psi_n)\hat{\ }(t) \right) \hat{g}(t)\, d\mu(t)$$

$$= \sum_{n=-m}^{m} \int_0^N (f*\psi_n)\hat{\ }(t)(g*Q_n)\hat{\ }(t)\, d\mu(t)$$

$$= \sum_{n=-m}^{m} \int_{\mathbb{R}} (f*\psi_n)(-t)(g*Q_n)(t)\, d\mu(t)$$

$$= \sum_{n=-m}^{m} < g*Q_n, R(f*\psi_n) >$$

$$= \sum_{n=-m}^{m} < Q_n, g*R(f*\psi_n) >$$

$$= \sum_{n=-m}^{m} \tilde{\Phi}_g(Q_n, R(f*\psi_n))$$

$$= \tilde{\Phi}_g\left(\sum_{n=-m}^{m} Q_n*(f*\psi_n) \right)$$

$$= \tilde{\Phi}_g(R(f)).$$

Thus when $g \in M(X)$ has $\| g \|_{M(X)} \le 1$ we have

$$\begin{aligned} \left| \int_0^N \hat{f}(t)\hat{g}(t)\, d\mu(t) \right| &\le \| \tilde{\Phi}_g \| \, \| R(f) \|_{\hat{*}} \\ &\le \| g \|_{M(X)} \, \| R(f) \|_{\hat{*}} \\ &\le \| R(f) \|_{\hat{*}} \\ &\le c_2 c_1 (\log N) \, \| f \|_\infty \end{aligned}$$

and so by 5.5.4

$$\| f(A) \| \le \frac{1}{\sqrt{2\pi}} K_G\, c^2 c_2 c_1 (\log N) \, \| f \|_\infty .$$

Putting $k = \dfrac{1}{\sqrt{2\pi}} c_2 c_1$ completes the proof.

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