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Second-Order Logic

Second-order logic is the extension of first-order logic obtaining by introducing quantification of predicate and function variables. A first-order formula, say *Fxy*, may be converted to a second-order formula by replacing *F* by a dyadic relation variable *X*, obtaining *Xxy*. Existential quantification yields $\exists X\, Xxy$, which may be read "there is a relation that *x* bears to *y*". In general, relation variables of all adicities are admissible. Similarly, quantifiable function variables may be introduced.

1. Semantics for Second-Order Logic

A structure, with non-empty domain *D*, for a second-order language includes relation domains $\mathrm{Rel}_n(D)$ and function domains $\mathrm{Func}_n(D)$. In general, $\mathrm{Rel}_n(D) \subseteq \mathbf{P}(D^n)$, where $\mathbf{P}(D^n)$ is the power set of D^n. Similarly, the function domains $\mathrm{Func}_n(D)$ are subsets of the collection of *n*-place total functions on *D*. Such second-order structures are called *Henkin* or *general* structures. If *X* is an *n*-place relation variable, a formula $\exists X\varphi(X)$ is *true* in a Henkin structure M iff there is an *n*-place relation $R \in \mathrm{Rel}_n(D)$ such that $\varphi(X)$ is true in M when *X* has the value *R*. There is a similar definition for formulas of the form $\forall X\varphi(X)$, and for formulas with quantified function variables. A formula φ is a *Henkin semantic consequence* of a set Δ of formulas iff φ is true in all Henkin models of Δ.

The relation domain $\mathrm{Rel}_n(D)$ need not contain *all* subsets of D^n. If $\mathrm{Rel}_n(D) = \mathbf{P}(D^n)$, for each *n*, we say that each relation domain is *full* (similarly for function domains) and that the structure is *full*, *standard* or *principal*. Second-order logic restricted to full structures is called *full* or *standard* second-order logic. A formula φ is a *full semantic consequence* of a set Δ iff φ is true in all full models of Δ. A formula is *valid* iff it is true in all full structures.

With Henkin semantics, the Completeness, Compactness and Löwenheim-Skolem Theorems all hold, because Henkin structures can be reinterpreted as many-sorted first-order structures. This yields Henkin's Completeness Theorem: there exists a deductive system DS such that if φ is a Henkin consequence of axioms Δ, then there is a *deduction* of φ from Δ using the rules of DS. For further details, see Shapiro 1991, Shapiro 2001 or van Dalen 1994.

2. Expressive Power

Following Gottfried Leibniz, we may define "$x = y$" as "any property of *x* is a property of *y*". The corresponding second-order definition $\forall x\forall y(x = y \leftrightarrow \forall X(Xx \rightarrow Xy))$ is valid. In contrast with first-order logic, there are *categorical* second-order theories with infinite models: all full models are isomorphic. For example, let Δ be the theory with axioms $\forall x(s(x) \neq 0)$, $\forall x\forall y(s(x) = s(y) \rightarrow x = y)$ and $\forall X[(X0 \wedge \forall x(Xx \rightarrow Xs(x))) \rightarrow \forall xXx]$. Any full model of Δ is isomorphic to the structure $(\mathbf{N}, 0, S)$, where $\mathbf{N}$ is the set of natural numbers and *S* the successor operation. So, the Löwenheim-Skolem Theorems fail in full second-order logic. Consider the theory $\Delta \cup \{c \neq 0, c \neq s0, c \neq ss0, \ldots\}$, with *c* a constant. This theory has no full model, but any *finite* subset of it has a full model. So, the Compactness Theorem fails too.

Extending Δ with the recursion axioms for addition and multiplication, we obtain the theory PA_2 whose unique full model up to isomorphism is the natural number structure $(\mathbf{N}, 0, S, +, \times)$. Similarly, there is an axiom system whose unique full model up to isomorphism is the ordered field of real numbers, $(\mathbf{R}, 0, 1, +, \times, <)$. More generally, there exist second-order formulas expressing cardinality claims inexpressible in first-order logic. The most striking example concerns the Continuum Hypothesis (CH), which says that there is no cardinal number between $\aleph_0$ and $2^{\aleph_0}$. Results due to Kurt Gödel and Paul Cohen imply that the Continuum Hypothesis is independent of standard axiomatic set theory (ZFC). But there is a second-order formula CH* which is valid just in case CH is true.

If we augment PA_2 with inference rules for the second-order quantifiers and the monadic comprehension scheme $\exists X \forall x(Xx \leftrightarrow \varphi)$, we obtain axiomatic second-order arithmetic, Z_2. (See Simpson 1998 for a detailed investigation of Z_2 and its subsystems.) One may construct a Gödel sentence G, true just in case G is not a theorem of Z_2. Now, all full models of Z_2 are isomorphic to ($\mathbf{N}$, 0, S, +, $\times$). So, an arithmetic sentence φ is true just in case φ is a full semantic consequence of Z_2. G is thus a full semantic consequence of Z_2, but not a theorem of Z_2. The Completeness Theorem therefore fails: there is no sound and complete, recursively axiomatized, deductive system for full second-order logic. Indeed, the set of second-order validities is not recursively enumerable. For further details, see Shapiro 1991, Shapiro 2001 or Enderton 2001.

3. Is Second-Order Logic Logic?

Second-order comprehension has the form $\exists X \forall x_1 \ldots \forall x_n(Xx_1 \ldots x_n \leftrightarrow \varphi)$. Should such existential axioms count as *logical*? Does this violate the *topic-neutrality* of logic? W.V. Quine argued that second-order logic is "set theory in sheep's clothing" because "set theory's staggering existential assumptions are cunningly hidden … in the tacit shift from schematic predicate letter to quantifiable variable" (Quine 1970, p. 68). Another reason for not counting second-order logic as logic is that the full semantic consequence relation does not admit a complete proof procedure.

In reply, George Boolos pointed out that the obvious translation from second-order formulas to first-order set-theoretic formulas does not map valid formulas to set-theoretic theorems. For example, $\exists X \forall y Xy$ is valid, while $\exists x \forall y(y \in x)$ is refutable in axiomatic set theory. Furthermore, $\exists X \exists x \exists y(Xx \wedge Xy \wedge x \neq y)$ is not valid, and so "second-order logic is not committed to the existence of even a two-membered set" (Boolos 1975 (1998), pp. 40-1). Furthermore, first-order logic does have a complete proof procedure but the set of first-order validities is undecidable (Church's Theorem), while the monadic fragment is decidable. So, why is completeness used to draw the line between logic and mathematics rather than decidability?

4. The Interpretation of Second-Order Variables

George Boolos (Boolos 1984, 1985) has provided *monadic* second-order logic with a novel interpretation: the *plural interpretation*. Certain natural language locutions which receive monadic second-order formalizations are perhaps better analysed as instances of plural quantification. For example, the Geach-Kaplan sentence, "Some critics admire only one another", may be formalized as $\exists X(\exists x Xx \wedge \forall x \forall y(Xx \wedge Axy \rightarrow x \neq y \wedge Xy))$. This formula is non-first-orderizable (not equivalent to a first-order formula containing just the predicates A and =). On the usual interpretation, its truth implies the existence of a collection. The plural interpretation reads "There are *some* [critics] such that, for any x and y, if x is *one of them* and admires y, then y is not x and y is *one of them*". Rather than asserting the existence of a collection, this is a plural means of referring to individuals. Second-order logic can also be applied to set theory.

In this context, we can interpret monadic second-order quantification over sets as plural quantification.

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